AN EQUIVALENT FOR PRIME NUMBER THEOREM

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Abstract. In this short research report, we show that \( \Psi(p_n) \sim \log n \), when \( n \to \infty \) is equivalent with Prime Number Theorem, in which \( \Psi(x) = \frac{d}{dx} \log \Gamma(x) \) is digamma function.

1. Introduction and Main Result

As usual, let \( \mathbb{P} \) be the set of all primes and \( \pi(x) = \#\mathbb{P} \cap [2, x] \). Also, for \( x > 0 \) define digamma function \( \Psi(x) \) to be \( \Psi(x) = \frac{d}{dx} \log \Gamma(x) \), in which \( \Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt \) is well-known gamma function [1].

In [4], it is shown that for every \( x \geq 3299 \), we have

\[
\frac{x}{\Psi(x) - A} < x < \frac{x}{\Psi(x) - B},
\]

in which \( A = \frac{3298}{3299} - \frac{1}{4 \log 3299} \approx 0.969 \) and \( B = 2 + \frac{151}{100 \log 7} - \gamma \approx 2.199. \) So, for every \( x \geq 3299 \), we obtain

\[
\frac{x}{\pi(x)} + A < x < \frac{x}{\pi(x)} + B,
\]

and by putting \( x = p_n, \) \( n \)th prime, for \( n \geq 463 \) we yield that

\[
\frac{p_n}{n} + A < \Psi(p_n) < \frac{p_n}{n} + B.
\]

In other hand, we have [3] the following sharp bounds for \( p_n \), which holds for every \( n \geq 27076 \)

\[
\log n + \log_2 n - 1 + \frac{\log_2 n - 2.25}{\log n} \leq \frac{p_n}{n} \leq \log n + \log_2 n - 1 + \frac{\log_2 n - 1.8}{\log n},
\]

in which \( \log_2 = \log \log \) and base of all logarithms is \( e \). Considering this inequality with (1.1), for every \( n \geq 27076 \) we obtain

\[
\log n + \log_2 n + A - 1 + \frac{\log_2 n - 2.25}{\log n} < \Psi(p_n) < \log n + \log_2 n + B - 1 + \frac{\log_2 n - 1.8}{\log n}.
\]

This inequality is very strong form of an equivalent of Prime Number Theorem (PNT), which asserts \( \pi(x) \sim \frac{x}{\log x} \) and is equivalent with \( p_n \sim n \log n \) (see [2]). In this note, we show that \( \Psi(p_n) \sim \log n \), when \( n \to \infty \) is another equivalent for PNT. To do this, first suppose PNT. Thus, we have \( p_n = n \log n + o(n \log n) \). Also, (1.1) yields that \( \Psi(p_n) = \frac{\log n}{n} + O(1) \). Therefore, we have

\[
\Psi(p_n) = \frac{n \log n + o(n \log n)}{n} + O(1) = \log n + o(\log n).
\]
Conversely, suppose \( \Psi(p_n) = \log n + o(\log n) \). By solving (1.1) according to \( p_n \), we obtain
\[
n\Psi(p_n) - Bn < p_n < n\Psi(p_n) - An.
\]
Therefore, we have
\[
p_n = n\Psi(p_n) + O(n) = n\left(\log n + o(\log n)\right) + O(n) = n \log n + o(n \log n),
\]
which, this is PNT.

References


