On Modified Hyperperfect Numbers
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Abstract. We introduce the so-called modified hyperperfect numbers, and completely determine their form in the case of classical divisors, unitary, bi-unitary, and e-divisors, respectively.


1. Introduction.

Let \( d \) be a positive divisor of the integer \( n > 1 \). If \( (d, \frac{n}{d}) = 1 \), then \( d \) is called a unitary divisor of \( n \). If the greatest common unitary divisor of \( d \) and \( \frac{n}{d} \) is 1, then \( d \) is called a bi-unitary divisor of \( n \). If \( n = p_1^{a_1} \cdots p_r^{a_r} > 1 \) is the prime factorization of \( n \), a divisor \( d \) of \( n \) is called an exponential divisor (or e-divisor, for short), if \( d = p_1^{b_1} \cdots p_r^{b_r} > 1 \) with \( b_i | a_i \) \((i=1, r)\). For the history of these notions, as well as the connected arithmetical functions, see e.g. [1], [2]. In what follows \( \sigma(n), \sigma^*(n), \sigma^{**}(n), \sigma_e(n) \) will denote the sum of divisors, -unitary divisors, -bi-unitary divisors, and e-divisors, respectively. It is well-known that a positive integer \( m \) is called \( n \)-hyperperfect (HP-for short), if

\[(1). \quad m = 1 + n \left[ \sigma(m) - m - 1 \right] \]

For \( n = 1 \) one has \( \sigma(m) = 2m \), i.e. the 1-HP numbers coincide with the classical perfect numbers. For results on HP-numbers, see [2].

Let \( f : \mathbb{N}^* \to \mathbb{N}^* = \{1, 2, ..., \} \) be an arithmetical function. Then \( m \) will be called \( f \)-n-hyperperfect number, if

\[(2). \quad m = 1 + n \left[ f(m) - m - 1 \right] \]

for some integer \( n \geq 1 \). For \( f(m) = \sigma(m) \) one obtains the HP-numbers, while for \( f(m) = \sigma^*(m) \), we get the unitary hyperperfect numbers (UHP) introduced by P. Hagis [2]. When \( f(m) = \sigma^{**}(m) \), we get the bi-unitary hyperperfect numbers (BHP), introduced by the first author ([3]). For \( f(m) = \sigma_e(m) \), we get the e-hyperperfect numbers (e-HP), introduced also by the first author [4].

2. Modified hyperperfect numbers.

In what follows, \( m \) will be called a modified \( f \)-n-hyperperfect number, if

\[(3). \quad m = n \left[ f(m) - m \right] \]

For \( f(m) = \sigma(m) \), we get the modified hyperperfect numbers (MHP). Since for \( n = 1 \) one has in (3) \( f(m) = 2m \), one obtains again a generalization of f-perfect numbers. First we prove:

**Theorem 1.** All MHP numbers are the classical perfect numbers, as well as the prime numbers.

**Proof.** Since (3) implies \( n|m \), put \( m = kn \), giving \( k = f(kn) - kn \), so

\[(3'). \quad f(kn) = k(n + 1) \]
For \( f \equiv \sigma \) this gives

(4). \( \sigma (kn) = k (n + 1) \)

For \( n = 1 \), (4) gives \( \sigma (k) = 2k \), so \( m = k \) is the classical perfect number.

For \( k = 1 \), relation (4) implies \( \sigma (n) = n + 1 \), which is possible only for \( n = p \) (prime), since \( \sigma (n) \geq n + 1 \), with equality if \( n \) has only two distinct divisors - namely 1 and \( n \), so \( n \) is a prime. Thus \( m = p \) is a modified p-hyperperfect number.

Assume now that, \( k > 1, n > 1 \) in (4). Then it is well known that \( \sigma (kn) > k \sigma (n) \) (see e.g. [2]). Since \( \sigma (n) \geq n + 1 \) for all \( n > 1 \), we can infer that \( \sigma (kn) > k (n + 1) \), in contradiction with (4).

For the case of unitary divisors, one can state:

**Theorem 2.** All UMHP-numbers are the unitary perfect numbers, as well as, the prime powers.

**Proof.** (3') now becomes

(5). \( \sigma^* (kn) = k (n + 1) \)

For \( n = 1 \) we get \( \sigma^* (k) = 2k \), i.e. \( k \) is a unitary perfect number.

For \( k = 1 \) we get \( \sigma^* (n) = n + 1 \), which is true only for \( n = p^a \) (prime power), by \( \sigma^* (n) = \prod_p (p^n + 1) \).

Let us now assume that, \( n, k > 1 \). Since \( k (n + 1) = kn + k \), and \( k \) is not only a divisor, but a unitary one of \( kn \), one can write \( (k, \frac{n}{k}) = 1 \), i.e. \( (k, n) = 1 \).

But then, \( \sigma^* \) being multiplicative, \( \sigma^* (kn) = \sigma^* (k) \sigma^* (n) \geq (k + 1) (n + 1) > k (n + 1) \) for \( k > 1, n > 1 \). This contradicts (5), so Theorem 2 is proved.

For bi-unitary divisors we can state:

**Theorem 3.** All BMHP-numbers are 6, 60, 90; as well as all primes or squares of primes.

**Proof.** (3') now is

(6). \( \sigma^{**} (kn) = k (n + 1) \)

Where \( \sigma^{**} (n) \) denotes the sum of bi-unitary divisors of \( n \). For \( n = 1 \) we get \( \sigma^{**} (k) = 2k \), so by a result of Ch. Wall (see [2]) one can write \( k \in \{ 6, 60, 90 \} \).

For \( k = 1 \) we get \( \sigma^{**} (n) = n + 1 \), which is possible only for \( n = p \) or \( n = p^2 \) (\( p \)-prime). This is well-known, but we note that it follows also from \( \sigma^* (p^n) = \sigma (p^n) \), if \( a \) is odd (\( p \)-prime), \( \sigma^{**} (p^n) = \sigma (p^n) - p^{n/2} \) if \( a \) is even; and the multiplicativity of \( \sigma^{**} \).

Let now \( k > 1, n > 1 \). Then \( kn \not\equiv k, kn \not\equiv n \), and \( (k, \frac{n}{k})_* = (k, n)_* = 1 \) where \( (k, n)_* \) denotes the greatest common unitary divisors of \( k \) and \( n \). Since \( k \not\equiv n \), by \( (k, n)_* = 1 \), and \( n \) is also a divisor of \( n \), but not a bi-unitary one, by (5) (i.e. \( \sigma^{**} (kn) = kn + k \) - which means that the only bi-unitary divisors of \( kn \) are \( kn \) and \( k \)). But then \( (n, \frac{n}{k})_* \not\equiv 1 \), i.e. \( (n, k)_* \not\equiv 1 \), in contradiction with \( (k, n)_* = 1 \).

Finally, the case of e-divisors is contained in:

**Theorem 4.** All modified exponentially \( n \)-hyperperfect numbers \( m \) are given by \( m = kn \), where \( k = p_1 p_2 \ldots p_r \), \( n = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r} \), with \( p_1, p_2, \ldots, p_r \) distinct primes, and \( q \) an arbitrary primes; as well as the e-perfect numbers.

**Proof.** We have to study the equation:

(7). \( \sigma_e (kn) = k (n + 1) \)
For $n = 1$ we have $\sigma_e(k) = 2k$, i.e. the e-perfect numbers.

For $k = 1$ we get $\sigma_e(n) = n + 1$. For $n = \text{squarefree}$, one has $\sigma_e(n) = n$, while for $n \neq \text{squarefree}$, by the above lemma, $\sigma_e(n) > n + 1$, giving a contradiction.

**Lemma.** If $n$ is not squarefree, $n > 1$ then $\sigma_e(n) > n + n/q^{a-1}$, where $q^a \mid n$ and $a \geq 2$. There is equality only for $p_1 \ldots p_r q^a$ with $p_1, \ldots, p_r, q$ distinct primes, and $p$ an arbitrary prime.

**Proof.** Let $n = p_1^{a_1} \ldots p_r^{a_r}$, where $(\exists) a \in \{a_1, \ldots, a_r\}$ with $a \geq 2$. Thus $\sigma_e(n) \geq p_1^{a_1} \ldots p_r^{a_r} (q^a + q) = n + p_1^{a_1} \ldots p_r^{a_r}$ since $\sigma_e(q^a) \geq q^{a + 1} + q^a$ with equality only if $n$ is squarefree, by the above lemma, $\sigma_e(n) = n + n/q^{a-1}$ if $a = \text{prime}$, while $\sigma_e(q^b) \geq q^b$, with equality only for $b = 1$.

**Corollary.** $\sigma_e(n) \geq n + \gamma(n)$ for $n \neq \text{squarefree}$, where $\gamma(n) = \prod_{p|n} p = \text{product of distinct prime divisors of } n$.

a). Now, suppose that $(n, k) = 1$. Since $\sigma_e$ is multiplicative, (7) becomes $\sigma_e(n) \sigma_e(k) = k(n + 1)$. If $k > 1$ is squarefree, then $\sigma_e(k) = k$, so this is $\sigma_e(n) = n + 1$, which is impossible. If $k$ is not squarefree, but $n$ is squarefree, then $\sigma_e(n) = n$, so (7) becomes $n \sigma_e(k) = k(n + 1)$. Since $(n, k) = 1$ and $(n, n + 1) = 1$, this is again impossible.

By summarizing, if $(n, k) = 1$ for $n > 1, k > 1$, the equation is unsolvable.

b). Let $(n, k) > 1$. Writing $n = p_1^{a_1} \ldots p_r^{a_r} q_1^{b_1} \ldots q_s^{b_s}$, $k = p_1^{a_1'} \ldots p_r^{a_r'} q_1^{c_1} \ldots q_t^{c_t}$, where $p_1, q_j, \gamma_k$ are distinct prime, and $a_i, b_j, c_k$ are nonnegative integers $(1 \leq i \leq r, 1 \leq j \leq s, 1 \leq k \leq t)$. Since by (7) written in the form $\sigma_e(kn) = kn + k - k$ is an e-divisor of $nk = p_1^{a_1 + a_1'} \ldots p_r^{a_r + a_r'} q_1^{c_1} \ldots q_t^{c_t}$, we must have $b_1 = \ldots = b_s = 0$.

Also $a_1' \mid (a_1 + a_1')$, ..., $a_r' \mid (a_r + a_r')$, i.e. $a_1 = (m_1 - 1) a_1'$, ..., $a_r = (m_r - 1) a_r'$, with $m_i$ $(1 \leq i \leq r)$ positive integers.

We note that, since $a_1 \geq 1, \ldots, a_r \geq 1$, we have $m_1 > 1, \ldots, m_r > 1$. Since $\gamma_1^{c_1} \neq k$ is also an e-divisor of $nk$, we must have $c_1 = 0$. Similarly, $c_2 = \ldots = c_r = 0$. Thus, $n = p_1^{(m_1 - 1)a_1'} \ldots p_r^{(m_r - 1)a_r'}$, $k = p_1^{a_1'} \ldots p_r^{a_r'}$; $nk = p_1^{m_1 a_1'} \ldots p_r^{m_r a_r'}$.

Remark that by $m_1 > 1$, if one assumes $a_1' > 1$, then $1 | m_1 a_1'$ implies that $p_1 \ldots p_r$ is also an e-divisor of $nk$, with $p_1 \ldots p_k \neq k$. Thus we must have necessarily $a_1' = \ldots = a_r' = 1$, so $k = p_1 \ldots p_r$ and $nk = p_1^{m_1} \ldots p_r^{m_r}$. Since $n > 1$, at least one of $m_1, \ldots, m_r$ is greater than 1. Put $m_1 > 1$. Then, if at least one of $m_2, \ldots, m_r$ is greater than 1, then $p_1 p_2 \ldots p_r^{m_r} \neq k$ is another e-divisor of $nk$. If $m_1 > 1$ is not a prime, then $m_1$ can have also a divisor $1 < a < m_1$ so $p_1 p_2 \ldots p_r$ will be another e-divisor, contradiction. Thus $a = q$ = prime, which finishes the proof of Theorem 4.

For results and/or open problems on perfect, unitary perfect, e-perfect numbers; as well as on hyperperfect or unitary hyperperfect numbers, see the monographs [1], [2].

**References.**


[5]. Mihály Bencze: On perfect numbers, Studia Mathematica, Univ. Babes-