A REFINEMENT OF MANDL’S INEQUALITY

MEHDI HASSANI

Abstract. In this short note, we prove that \( \frac{n}{2} p_n - \sum_{i=1}^{n} p_i > 0.01659n^2 \) holds for every \( n \geq 9 \). This is a refinement of Mandl’s inequality which asserts \( \frac{n}{2} p_n - \sum_{i=1}^{n} p_i > 0 \), for those values of \( n \).

1. Introduction and Refinement

As usual, let \( p_n \) be the \( n \)th prime. The Mondl’s conjecture (see [1] and [2]) asserts that for every \( n \geq 9 \), we have:

\[
\frac{n}{2} p_n - \sum_{i=1}^{n} p_i > 0.
\]

To prove Mandl’s inequality, Dusart ([1], page 50) uses the following inequality

\[
\int_{2}^{p_n} \pi(t)dt \geq c + \frac{p_n^2}{2 \log p_n} \left(1 + \frac{3}{2 \log p_n}\right) \quad (n \geq 109),
\]

in which

\[
c = 35995 - 3Li(599^2) + \frac{599^2}{\log 599} \approx -47.06746,
\]

and

\[
Li(x) = \lim_{\epsilon \to 0} \left( \int_{0}^{1-\epsilon} \frac{dt}{\log t} + \int_{1+\epsilon}^{x} \frac{dt}{\log t} \right),
\]

is logarithmic function, and base of all logarithms are \( e \). Note that one can gets (1.1), using the following known bound [1]:

\[
\pi(t) \geq \frac{t}{\log t} \left(1 + \frac{1}{\log t}\right) \quad (t \geq 599).
\]

Also, for using (1.1) to prove Mandl’s inequality, we note that:

\[
\int_{2}^{p_n} \pi(t)dt = \sum_{i=2}^{n} (p_i - p_{i-1})(i-1) = \sum_{i=2}^{n} (ip_i - (i-1)p_{i-1}) - \sum_{i=2}^{n} p_i = np_n - \sum_{i=1}^{n} p_i.
\]

Therefore, we have:

\[
n p_n - \sum_{i=1}^{n} p_i \geq c + \frac{p_n^2}{2 \log p_n} \left(1 + \frac{3}{2 \log p_n}\right) \quad (n \geq 109).
\]

Now, we use the following bound ([1], page 36):

\[
\frac{x}{\log x} \left(1 + \frac{1.2762}{\log x}\right) \geq \pi(x) \quad (x \geq 2).
\]
Considering this bound and (1.2), for every $n \geq 109$, we obtain:

$$np_n - \sum_{i=1}^{n} p_i \geq c + \frac{p_n^2}{2 \log p_n} \left( \frac{0.2238}{\log p_n} \right) + \frac{p_n^2}{2 \log p_n} \left( 1 + \frac{1.2762}{\log p_n} \right)$$

$$\geq c + 0.1119 \frac{p_n^2}{\log^* p_n} + \frac{p_n}{2} \pi(p_n) = c + 0.1119 \frac{p_n^2}{\log^* p_n} + \frac{n}{2} p_n.$$

So, for every $n \geq 109$, we have:

(1.3) $$\frac{n}{2} p_n - \sum_{i=1}^{n} p_i \geq c + 0.1119 \frac{p_n^2}{\log^* p_n}.$$

In other hand, we have the following bounds for $p_n$ ([3], page 69):

$$n \log n \leq p_n \leq n (\log n + \log \log n) \quad (n \geq 6).$$

Combining these bounds with (1.3), for every $n \geq 109$, we yield that:

$$\frac{n}{2} p_n - \sum_{i=1}^{n} p_i \geq c + \frac{0.1119 (n \log n)^2}{\log^* (n (\log n + \log \log n))}.$$

But, for every $n \geq 89$, we have $c + \frac{0.0119 (n \log n)^2}{\log^* (n (\log n + \log \log n))} > 0$, and so, we obtain the following inequality for every $n \geq 89$:

$$\frac{n}{2} p_n - \sum_{i=1}^{n} p_i \geq \frac{(n \log n)^2}{10 \log^* (n (\log n + \log \log n))}.$$

This holds also for $10 \leq n \leq 88$. Thus, considering $\log(n (\log n + \log \log n)) < 2 \log n + 1$, we yield that:

$$\frac{n}{2} p_n - \sum_{i=1}^{n} p_i \geq \frac{n^2 \log n}{10 \left( \frac{2 \log n + 1}{2 \log 9 + 1} \right)^2},$$

which holds for every $n \geq 9$. This is a refinement of Mandl’s inequality, with quadratic and logarithms terms. Now, for every $n \geq 9$, we note that:

$$\frac{n}{2} p_n - \sum_{i=1}^{n} p_i \geq \frac{n^2 \log n}{10 \left( \frac{2 \log 9 + 1}{2 \log 9 + 1} \right)^2} > 0.01659 n^2.$$

**References**


Institute for Advanced, Studies in Basic Sciences, P.O. Box 45195-1159, Zanjan, Iran. E-mail address: mhassani@iasbs.ac.ir