On an inequality of Klamkin

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Abstract. Connections of an inequality of Klamkin with Stolarsky means and convexity are shown. An application to arithmetical functions is given.

Keywords and phrases: Means and their inequalities, log-convex functions, arithmetical functions.
AMS Subject Classification: 11A25, 11N37, 26A51, 26D99.

1 Introduction

In 1974 M. S. Klamkin [3] proved the following result: Let x be a non-negative real number, and m, n integers with $m \geq n \geq 1$. Then

$$(m+n)(1+x^m) \geq 2n\frac{1-x^{m+n}}{1-x^n}. \tag{1}$$

We note that for $x = 1$, the right side of (1) is understood as $\lim\limits_{x\to 1}$, when the inequality becomes an equality. Also, for $x = 0$ (1) becomes $m + n \geq 2n$, which is true. For $m = n$ there is equality in (1). In fact, it can be shown that for all real numbers $m > n > 0$, and all $x > 0$, (1) holds true with strict inequality (see the solutions of (1) in [1]).

Assume now that $x = a \geq 1$, $m = p$, $n = q$, where $p \geq q \geq 0$ are real numbers. Then, since

$$(1 + a^p)(1 - a^q) = a^p - a^q + 1 - a^{p+q},$$

after some transformations, (1) becomes equivalent to

$$(p - q)(a^{p+q} - 1) \geq (p + q)(a^p - a^q). \tag{2}$$

In the case of $p - q \leq 1$, a weaker result than (2) appears in the famous monograph by D. S. Mitronović [4] (3.6.26, page 276).

For certain arithmetical applications of Klamkin's inequality, see [5].

In what follows we will point out some surprising connections of inequality (2) (i.e., in fact (1)) with certain special means of two arguments. Also, a new application of (1) will be given.

2 Stolarsky means

Let $m, n > 0$ and put $p + q = m$, $p - q = n$. Then $p = \dfrac{m+n}{2}$, $q = \dfrac{m-n}{2}$ and (2) gives

$$\frac{a^m - 1}{a^n - 1} > \frac{m}{n} a^{(m-n)/2}. \tag{3}$$

By letting $a = \dfrac{x}{y}$ $(x > y > 0)$, relation (3) may be written also as

$$\left(\frac{x^m - y^n}{x^n - yn} \cdot \frac{n}{m} \right)^{1/(m-n)} > \sqrt{xy}. \tag{4}$$

If $n = 1$, the expression on the left side of (4) is called as the **Stolarsky mean** of x and y. Put

$$S(m) = S(x, y, m) = \left(\frac{x^m - y^m}{x - y} \cdot \frac{1}{m} \right)^{1/(m-1)}.$$

It is not difficult to see that S can be defined also for all real numbers $m \notin \{0, 1\}$, while for $m = 0$, and $m = 1$, by the limits

$$\lim_{m\to 0} S(x, y, m) = \frac{x - y}{\ln x - \ln y}$$

and

$$\lim_{m\to 1} S(x, y, m) = \frac{1}{e}(y^y/x^x)^{1/(y-x)} \quad (y \neq x),$$

the definition of S can be extended to all real numbers m.

Let

$$L(x, y) = \frac{x - y}{\ln x - \ln y}, \quad I(x, y) = \frac{1}{e}(y^y/x^x)^{1/(y-x)} \quad (x \neq y),$$
$$L(x, x) = I(x, x) = x.$$

These means are known as the **logarithmic** and **identric means** of x and y (see e.g. [8] for their properties). Stolarsky [10] has proved that S is a strictly increasing function of m. Therefore $S(-1) < S(0) < S(1) < S(2)$, giving

$$\sqrt{xy} < L(x, y) < I(x, y) < \frac{x + y}{2}. \tag{5}$$

Since $S(-1) < S(0) < S(m)$ for $m > 0$, we get

$$\sqrt{xy} < L(x, y) < \left(\frac{x^m - y^m}{m(x - y)}\right)^{1/(m-1)}, \tag{6}$$

which is an improvement of (4), when $n = 1$.

3 Main results

We shall prove that the following refinement of (4) holds true:

Theorem 1.

$$\sqrt{xy} < (L(x^{m-n}, y^{m-n}))^{1/(m-n)} < \left(\frac{x^m - y^m}{x^n - y^n} \cdot \frac{n}{m}\right)^{1/(m-n)} \quad (m > n). \tag{7}$$

Proof. Put $f(x) = \dfrac{a^x - 1}{x}$ $(x > 0)$, where $a > 1$; and let $\varphi(p) = \dfrac{f(p+q)}{f(p-q)}$ $(p > q > 0)$, where q is fixed. We first show that φ is strictly increasing function. Since

$$\varphi'(p) = \frac{f'(p+q)f(p-q) - f'(p-q)f(p+q)}{f^2(p-q)},$$

it will be sufficient to prove that $\dfrac{f'(p+q)}{f(p+q)} > \dfrac{f'(p-q)}{f(p-q)}$. Since $p+q > p-q$, this will follow, if $f'/f = g$ is an increasing function. By

$$g'(t) = (f'(t)/f(t))' = \frac{f''(t)f(t) - (f'(t))^2}{f^2(t)},$$

it will be sufficient to show that f is strictly log-convex (i.e. $\ln f$ is strictly convex).

Lemma. *The function f is strictly log-convex.*

Proof. After certain simple computations (which we omit here), it follows that

$$f'(t) = \frac{ta^t \ln t - (a^t - 1)}{t^2},$$

$$f''(t) = \frac{t^2 a^t \ln^2 a - 2ta^t \ln a + 2a^t - 2}{t^3},$$

and

$$f''(t)f(t) - (f'(t))^2 = \frac{a^{2t} - 2a^t - t^2 a^t \ln^2 a + 1}{t^4}$$

$$= \frac{(a^t - 10\sqrt{a^t}\ln a^t)(a^t - 1 + \sqrt{a^t}\ln a^t)}{t^4}.$$

Put $a^t = h$. Then $h - 1 - \sqrt{h}\ln h > 0$, since $\dfrac{h-1}{\ln h} > \sqrt{h}$ by $L(h,1) > \sqrt{h}$ (left side of (5)). This proves the log-convexity property of f for $a > 1$.

Since φ is strictly increasing, one can write

$$\varphi(p) > \lim_{p \to q, p > q} f(p+q)/f(p-q) = \frac{a^{2q} - 1}{2q \ln a}.$$

Write $p+q=m$, $p-q=n$, $a=\dfrac{x}{y}$, and the right side of (7) follows.

For the left side of (7) remark that again by the left side of (5) one has

$$L(x^{m-n}, y^{m-n}) > \sqrt{x^{m-n}y^{m-n}} = (xy)^{(m-n)/2},$$

which implies the desired inequality.

Remark. φ being strictly increasing, it follows also that

$$\varphi(p) < \lim_{p\to\infty} \frac{a^{p+q}-1}{a^{p-q}-1} \cdot \frac{p-q}{p+q} = a^{2q},$$

i.e.

$$(p-q)(a^{p+q}-1) \le (p+q)a^{2q}(a^{p-q}-1), \tag{8}$$

which is complementary to (2).

4 Arithmetical applications

A divisor d of N is called **unitary divisor** of the positive integer $N>1$, if $(d, N/d) = 1$. For $k \ge 0$, let $\sigma_k(N)$ resp. $\sigma_k^*(N)$ denote the sum of kth powers of divisors, resp. unitary divisors of N. Remark that $\sigma_0(N) = d(N)$, $\sigma_0^*(N) = d^*(N)$ are the number of these divisors of N. It is well-known that (see e.g. [3], [9]) if the prime factorization of N is

$$N = \prod_{i=1}^{r} p_i^{a_i}$$

(p_i distinct primes, $a_i \ge 1$ integers), then

$$\begin{aligned} &\sigma_k(N) = \prod_{i=1}^{r}(p_i^{k(a_i+1)}-1)/(p_i^k-1),\ d(N) = \prod_{i=1}^{r}(a_i+1), \\ &\sigma_k^*(N) = \prod_{i=1}^{r}(p_i^{ka_i}+1),\ d^*(N) = 2^r (= 2^{\omega(N)}), \end{aligned} \tag{9}$$

where $\omega(r) = r$ denotes the number of distinct prime divisors of N.

Write now (1), and a reverse of it (see [1]) in the form

$$2n\frac{x^{m+n}-1}{x^n-1} \le (m+n)(1+x^m) \le 2m\frac{x^{m+n}-1}{x^n-1}, \tag{10}$$

where $x > 1$, $m \ge n \ge 1$. Put $n = k$, $m = ka_i$, $x = p_i$ $(i = 1, 2, \dots, r)$. Writing (10), after term-by-term multiplication, we get

$$2^{\omega(N)}\sigma_k(N) \le d(N)\sigma_k^*(N) \le 2^{\omega(N)}\beta(N)\sigma_k(N), \tag{11}$$

where $\beta(N) = \prod_{i=1}^{r} a_i$ (for this, and the other functions, too, see e.g. [6], [9]). The left side of (11) appears also in [5]. Now, remarking that

$$2^{\omega(N)}\beta(N) = \prod_{i=1}^{r}(2a_i) \le \prod_{i=1}^{r} 2^{a_i} = 2^{\Omega(N)},$$

where $\Omega(N)$ denotes the total number of prime factors of N (we have used the classical inequality $2^{a-1} \ge a$ for all $a \ge 1$), relation (11) implies also

$$2^{\omega(N)} \le \frac{d(N)\sigma_k^*(N)}{\sigma_k(N)} \le 2^{\Omega(N)}. \tag{12}$$

Theorem 2. *The normal order of magnitude of*

$$\log(d(N)\sigma_k^*(N)/\sigma_k(N))$$

is $(\log 2)(\log\log N)$.

Proof. Let P be a property in the set of positive integers and set $a_p(n) = 1$ if n has the property P; $a_p(n) = 0$, otherwise. Let $A_p(x) = \sum_{n \le x} a_p(n)$. If $A_p(x) \sim x$ $(x \to \infty)$ we say that the property P holds for almost all natural numbers. We say that the normal order of magnitude of the arithmetical function $f(n)$ is the function $g(n)$, if for each $\varepsilon > 0$, the inequality $|f(n) - g(n)| < \varepsilon g(n)$ holds true for almost all positive integers n.

By a well-known result of Hardy and Ramanujan (see e.g. [2], [4], [6]), the normal order of magnitude of $\omega(N)$ and $\Omega(N)$ is $\log\log N$. By (12) we can write

$$(1-\varepsilon)(\log\log N) < \omega(N) \le \frac{1}{\log 2}\log d(N)\sigma_k^*(N)/\sigma_k(N)$$
$$\le \Omega(N) < (1+\varepsilon)\lg\lg N$$

for almost all N, so Theorem 2 follows.

Acknowledgements. The author thanks Professor Klamkin for sending him a copy of [1] and for his interest in applications of his inequality.

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