APPROXIMATION OF THE SUM OF RECIPROCAL OF IMAGINARY PARTS OF ZETA ZEROS

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Abstract. In this paper, we approximate \( \gamma_n \), where \( 0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \cdots \) are consecutive ordinates of nontrivial zeros \( \rho = \beta + i\gamma \) of \( \zeta(s) \); the Riemann zeta function. Then we obtain explicit bounds for the summation \( \sum_{0 < \gamma \leq T} \frac{1}{\gamma} \).

1. Introduction

The Riemann zeta-function is defined for \( \text{Re}(s) > 1 \) by

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},
\]

and extended by analytic continuation to the complex plane with one singularity at \( s = 1 \); in fact a simple pole with residues 1. This was one of the results which B. Riemann obtained in his only paper on the theory of numbers [10], another one is functional equation which stated symmetrically as follows:

\[
\pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma \left( \frac{1-s}{2} \right) \zeta(1-s),
\]

where \( \Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt \) is a meromorphic function of the complex variable \( s \), with simple poles at \( s = 0, -1, -2, \cdots \) (see [8]). Riemann made a number of wonderful conjectures. For example, he guessed that the number \( N(T) \) of zeros \( \rho \) of \( \zeta(s) \) with \( 0 < \Im(\rho) \leq T \) and \( 0 \leq \Re(\rho) \leq 1 \), satisfies the following relation:

\[
N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).
\]

This conjecture of Riemann proved by H. von Mangoldt more than 30 years later [4, 7]. Some immediate corollaries of above approximate formula, which is known as Riemann-van Mangoldt formula, are

\[
A(T) = \sum_{0 < \gamma \leq T} \frac{1}{\gamma} = O(\log^2 T),
\]

and \( \gamma_n \sim \frac{2\pi n}{\log n} \) when \( n \to \infty \), where \( 0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \cdots \) are consecutive ordinates of nontrivial zeros \( \rho = \beta + i\gamma \) of \( \zeta(s) \), which follow by partial summation from Riemann-van Mangoldt formula and using the obvious inequality \( N(\gamma_n - 1) < n \leq N(\gamma_n + 1) \), respectively [7]. In this paper, we make some explicit approximation of \( \gamma_n \) and \( A(T) \).

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In 1941, Rosser [11] introduced the following approximation of $N(T)$:

\begin{equation}
|N(T) - F(T)| \leq R(T) \quad (T \geq 2),
\end{equation}

where

\begin{equation}
F(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8},
\end{equation}

and

\begin{equation}
R(T) = 0.137 \log T + 0.443 \log \log T + 1.588.
\end{equation}

In this paper, using Rosser’s result, we approximate $\gamma_n$, and then $A(T)$, explicitly. Using (2.1) and $N(\gamma_n) = n$, we have:

\[(F - R)(\gamma_n) \leq n \leq (F + R)(\gamma_n).\]

Both of the functions $(F \pm R)(T)$ are increasing for $T \geq 14$, thus,

\[(F + R)^{-1}(n) \leq \gamma_n \leq (F - R)^{-1}(n),\]

holds for every $n \geq 1$. Unfortunately, finding an explicit formula for the inverses $(F \pm R)^{-1}(T)$ isn’t possible and we must replace error term $R$ by another one. Let

\[Y(T) = \frac{25}{1447} T.\]

For every $T \geq 14$, we have $R(T) \leq Y(T)$, and the functions $(F \pm Y)(T)$ are increasing for $T \geq 18$. Since $\gamma_2 \approx 21.02$, we obtain:

\[(F + Y)^{-1}(n) \leq \gamma_n \leq (F - Y)^{-1}(n) \quad (n \geq 2).\]

Now, we are able to find inverses $(F \pm Y)^{-1}(T)$; considering Lambert $W$ function $W(x)$, defined by $W(x)e^{W(x)} = x$ for $x \in [-e^{-1}, +\infty)$, for every $n \geq 2$ we yield that:

\begin{equation}
\frac{1}{4} W \left( \frac{(8n - 7)\pi}{(8n - 7)e^{-1 + \frac{60}{147}\pi}} \right) \leq \gamma_n \leq \frac{1}{4} W \left( \frac{(8n - 7)\pi}{\frac{1}{8} (8n - 7)e^{-1 - \frac{50}{147}\pi}} \right),
\end{equation}

which holds also for $n = 1$. To make some explicit bounds, independent of Lambert $W$ function, we use the following bounds

\[
\log x - \log \log x < W(x) < \log x,
\]

which the left hand side holds true for $x > 41.19$ and the right hand side holds true for $x > e$ [5]. Thus, we obtain:

\begin{equation}
\gamma_n < \frac{2\pi(n - \frac{7}{8})}{\log(n - \frac{7}{8}) - \log \left( (1 + \frac{60}{147}\pi) - (1 + \frac{50}{147}\pi) \right)},
\end{equation}

which holds for $(n - \frac{7}{8})e^{-(1 + \frac{50}{147}\pi)} > 41.19$ or equivalently for $n > \frac{7}{8} + 41.19e^{1 + \frac{50}{147}\pi} \approx 326.83$, and by computation for $13 \leq n \leq 326$, too. Also, we obtain:

\begin{equation}
\frac{2\pi(n - \frac{7}{8})}{\log(n - \frac{7}{8}) - (1 - \frac{50}{147}\pi)} < \gamma_n,
\end{equation}

which holds for $(n - \frac{7}{8})e^{-(1 - \frac{50}{147}\pi)} > e$ or equivalently for $n > \frac{7}{8} + e^{-2 - \frac{50}{147}\pi} \approx 3.41$, and by computation for $n = 1$ and $n = 3$, too.
3. Approximation of $A(T)$

We note that:

$$A(T) = G(N) = \sum_{n=1}^{N} \frac{1}{\gamma_n},$$

in which

$$N = \max\{n : \gamma_n \leq T\} = N(T).$$

Now, we are ready to make explicit bounds for $A(T)$.

3.1. Upper Bound. Consider (2.6) and the following inequality:\footnote{We generate this numerical inequality, because the inequality (2.6) isn’t true for $n = 2$. To compute the value of $N_0$, which is best possible value, we used numerical data concerning zeros of $\zeta(s)$, due to A. Odlyzko [9] and the following program in Maple software worksheet:}

$$G(N_0) < \frac{4}{\pi} \sum_{n=1}^{N_0} \frac{\log \left( \frac{1}{8} (8n - 7) e^{-1 + \frac{50}{3\pi} \pi} \right)}{8n - 7}$$

For every $N \geq N_0$, we have

$$G(N) < \frac{4}{\pi} \sum_{n=1}^{N} \frac{\log \left( \frac{1}{8} (8n - 7) e^{-1 + \frac{50}{3\pi} \pi} \right)}{8n - 7}$$

$$= \frac{4}{\pi} \sum_{n=1}^{N} \frac{\log(8n - 7)}{8n - 7} + c_1 \Psi \left( N + \frac{1}{8} \right) - c_1 \Psi \left( \frac{1}{8} \right),$$

where $c_1 = \frac{25}{147} - \frac{1+3 \log 2}{2 \pi} \approx -0.3200403161$, and $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ with $\Gamma(x) = \int_{0}^{\infty} e^{-t} t^{x-1} dt$, is digamma function [8]. For every $x \geq 1$, it is known that

$$(3.1) \quad \log \left( x - \frac{1}{2} \right) < \Psi(x) \leq \log \left( x - 1 + e^{-c} \right),$$

where $c \simeq 0.5772156649$ is Euler constant [1]. In other hand, we have:

$$\sum_{n=1}^{N} \frac{\log(8n - 7)}{8n - 7} \leq \sum_{n=2}^{N} \frac{\log(8n)}{8(n - 1)} = \frac{\log 8}{8} H(N - 1) + \frac{1}{8} \sum_{n=2}^{N} \frac{\log n}{n - 1},$$

restart:
with(stats):
N:=9996:
x:=array(1..N):
fp:=fopen("zeros1.txt",READ):
g:=0:
for i from 1 by 1 to N do g:=g+1/describe[mean](fscanf(fp,"%f",x[i])) end do:
fclose(fp):
G(N):=g;

G.A. Pirayesh helped me to write above program, which I deem my duty to thank him for his kind helps.
where \( H(N) = \sum_{n=1}^{N} \frac{1}{n} \), and for every \( N \geq 1 \), we have \( H(N) \leq c + \log(N - 1 + e^{1-c}) \) (see [1]). Also, we have:

\[
\sum_{n=2}^{N} \frac{\log n}{n-1} < \int_{1}^{N} \frac{\log t}{t-1} dt = -\text{dilog}(N),
\]

where \( \text{dilog}(x) \) is Dilogarithm function, defined by \( \text{dilog}(x) = \int_{1}^{x} \frac{\log t}{t-1} dt \) for \( x > 0 \) (see [16]). It is known that [6] for every \( x > 1 \), the inequalities

\[
\mathcal{D}(x, N) < \text{dilog}(x) < \mathcal{D}(x, N) + \frac{1}{xN}
\]

holds true for all \( N \in \mathbb{N} \), with

\[
\mathcal{D}(x, N) = -\frac{1}{2} \log^2 x - \frac{\pi^2}{6} + \sum_{n=1}^{N} \frac{1}{n} + \frac{1}{n} \log x.
\]

Therefore, we have

\[
(3.2) \quad -\frac{1}{2} \log^2 x - \frac{\pi^2}{6} + \frac{1 + \log x}{x} < \text{dilog}(x),
\]

and using this, we obtain

\[
\mathcal{G}(N) < \frac{1}{4 \pi} \log^2 N + \left( \frac{\log 8}{2 \pi} + c_1 \right) \log N + \left( \frac{c \log 8}{2 \pi} + \frac{\pi}{12} - c_1 \Psi \left( \frac{1}{8} \right) \right) + E_1(N),
\]

where

\[
E_1(N) = \frac{\log 8}{2 \pi} \log \left( 1 + \frac{e^{1-c} - 2}{N} \right) + c_1 \log \left( 1 - \frac{3}{8N} \right) - \frac{1 + \log N}{2 \pi N} < -\frac{\log N}{2 \pi N}.
\]

Thus

\[
(3.3) \quad \mathcal{G}(N) < \frac{1}{4 \pi} \log^2 N + c_2 \log N + c_3 - \frac{1}{2 \pi} \log \frac{N}{N},
\]

for every \( N \geq 9996 \), with \( c_2 = \frac{\log 8}{2 \pi} + c_1 \approx 0.0109130841 \) and \( c_3 = \frac{c \log 8}{2 \pi} + \frac{\pi}{12} - c_1 \Psi \left( \frac{1}{8} \right) \approx -2.231824968 \). Also, it holds true for 4905 \( \leq N \leq 9995 \), by computation. Remembering \( N = n(T) \), and using (2.1), we obtain the following explicit upper bound:

\[
(3.4) \quad \mathcal{A}(T) < \frac{1}{4 \pi} \log^2 (F(T) + R(T)) + c_2 \log (F(T) + R(T)) + c_3 - \frac{1}{2 \pi} \log \frac{F(T) + R(T)}{F(T) + R(T)} \quad (N(T) \geq 4905).
\]

### 3.2. Lower Bound

Consider (2.5), which holds true for \( n \geq 13 \), and \( \mathcal{G}(12) \approx 0.3731710458 \). For every \( N \geq 13 \) we have

\[
\mathcal{G}(N) > \mathcal{G}(12) + \frac{4}{\pi} \sum_{n=13}^{N} \log(n - \frac{7}{8}) - \log \left( \log(n - \frac{7}{8}) - (1 + \frac{50}{177 \pi}) \right) - (1 + \frac{50}{177 \pi})
\]

\[
= \frac{4}{\pi} \sum_{n=13}^{N} \left\{ \frac{\log(8n - 7)}{8n - 7} - \frac{\log \left( 147 \log(n - \frac{7}{8}) - 147 - 50\pi \right)}{8n - 7} \right\}
\]

\[
+ c_4 \Psi \left( N + \frac{1}{8} \right) + c_5,
\]
where
\[ c_4 = \frac{4}{\pi} \left( -\frac{3}{8} \log 2 + \frac{1}{8} \log 147 - \frac{1}{8} - \frac{25}{588} \pi \right) \approx 0.1340756439, \]
and
\[ c_5 = G(12) - c_4 \left( \frac{13236224754014816}{12208333676768925} + \Psi \left( \frac{1}{8} \right) \right) + \frac{6618112377007408}{12208333676768925} \pi \approx 1.769772. \]

Easily, we have:
\[ N \sum_{n=13}^{N} \log \left( 8n - 7 \right) - \frac{1}{8} \sum_{n=13}^{N} \log \left( n - \frac{7}{8} \right) + \frac{1}{8} \sum_{n=13}^{N} \frac{1}{n - \frac{7}{8}}, \]
and
\[ N \sum_{n=13}^{N} \frac{1}{n - \frac{7}{8}} > \int_{13 - \frac{7}{8}}^{N + 1 - \frac{7}{8}} \frac{\log t}{t} \ dt = \frac{1}{2} \log^2 \left( N + \frac{1}{8} \right) + c_6, \]
with
\[ c_6 = -\frac{9}{2} \log^2 2 + 3 \log 2 \log 97 - \frac{1}{2} \log^2 97 \approx -3.113184782, \]
and
\[ N \sum_{n=13}^{N} \frac{1}{n - \frac{7}{8}} = \Psi \left( N + \frac{1}{8} \right) + c_7, \]
with
\[ c_7 = - \left( \frac{13236224754014816}{12208333676768925} + \Psi \left( \frac{1}{8} \right) \right) \approx -2.45346877. \]

In other hand, we have:
\[ N \sum_{n=13}^{N} \log \frac{\left( 147 \log \left( n - \frac{7}{8} \right) - 147 - 50 \pi \right)}{8n - 7} < \sum_{n=13}^{N} \log \left( \frac{147 \log \left( n - \frac{7}{8} \right)}{8n - 7} \right) \]
\[ = \frac{1}{8} \log 147 \Psi \left( N + \frac{1}{8} \right) + c_8 \]
\[ + \sum_{n=13}^{N} \log \frac{\log (8n - 7) - \log 8}{8n - 7}, \]
where
\[ c_8 = -\frac{1}{8} \log 147 \left( \frac{13236224754014816}{12208333676768925} + \Psi \left( \frac{1}{8} \right) \right) \approx -1.530482008. \]

Also, we have:
\[ N \sum_{n=13}^{N} \log \frac{\log (8n - 7) - \log 8}{8n - 7} < \sum_{n=13}^{N} \log \frac{\log (8n - 7)}{8n - 7} < \int_{8(12)^{-7}}^{8N^{-7}} \frac{\log \log t}{t} \ dt \]
\[ = (\log \log (8N - 7) - 1) \log (8N - 7) + c_9, \]
where
\[ c_9 = - \log \log 89 \log 89 + \log 89 \approx -2.251270867. \]
Therefore, combining all of above inequalities and considering (3.2), for every $N \geq 13$, we obtain:

$$
G(N) > \frac{1}{4\pi} \log^2 \left( N + \frac{1}{8} \right) - \frac{4}{\pi} \left( \log \log(8N - 7) - 1 \right) \log(8N - 7)
+ c_{10} \log \left( N - \frac{7}{8} + e^{-c} \right) + c_{11},
$$

with

$$
c_{10} = \frac{3\log 2 - \log 147}{2\pi} + c_4 \approx -0.3292229701,
$$

and

$$
c_{11} = \frac{c_6 + (3\log 2)c_7}{2\pi} - \frac{4(c_8 + c_9)}{\pi} + c_5 \approx 5.277388010.
$$

REFERENCES


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