RESEARCH GROUP IN MATHEMATICAL INEQUALITIES AND APPLICATIONS

PROBLEM CORNER

Problem 1, (2008), Solution No. 1

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Problem 1. For $-1 < A \leq 1$ and $0 < \theta \leq 2\pi$, find a lower bound of $\Re\left(\frac{1}{1+Ae^{i\theta}}\right)$.

Solution. We prove that

$$\inf_{\theta\in(0,2\pi]} \Re\left(\frac{1}{1+Ae^{i\theta}}\right) = \frac{1}{1+|A|} = \begin{cases} \frac{1}{1-A}, & \text{if } A < 0 \\ \frac{1}{1+A}, & \text{if } A \geq 0. \end{cases}$$

A calculation shows that

$$\Re\left(\frac{1}{1+Ae^{i\theta}}\right) = \frac{1+A\cos\theta}{1+2A\cos\theta+A^2}.$$

Let $f : [-1,1] \to \mathbb{R}$ be the function defined by $f(x) = \frac{1+Ax}{1+2Ax+A^2}$. We have that

$$f'(x) = \frac{A(1+A^2-2)}{(1+2Ax+x^2)^2}.$$

If $A > 0$ we get that $f'(x) < 0$, and hence, f decreases. It follows that $f(x) \geq f(1) = \frac{1}{1+A}$.

On the other hand, if $A < 0$ we get that $f'(x) > 0$, and hence, f increases. It follows that $f(x) \geq f(-1) = \frac{1}{1-A}$, and the problem is solved.