RESEARCH GROUP IN MATHEMATICAL INEQUALITIES AND APPLICATIONS

PROBLEM CORNER

Problem 14, (2009), Solution No. 1

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Solution. Let

$$\varphi(t) = \omega'(t) + \omega(t) = -\frac{1}{(e^t-1)^2} + \frac{1-t}{t^2} + \frac{1}{2},$$

we have

$$\lim_{t\to 0} \varphi(t) = \lim_{t\to 0} \frac{(1-t)(t^2+t^3+\frac{7}{12}t^4+o(t^4)) - t^2}{t^2(t^2+o(t^2))} + \frac{1}{2} = \frac{1}{12} > 0.$$

If we show that $\varphi'(t) > 0$ for $t > 0$, we get $\varphi(t) > 0$ and the desired inequality

$$\omega''(t) + 2\omega'(t) + \omega(t) = \varphi'(t) + \varphi(t) > 0.$$

So we show that

$$\varphi(t)' = \frac{2e^t}{(e^t-1)^3} + \frac{t-2}{t^3} > 0.$$

This is equivalent to $f(t) > 0$, where

$$f(t) = 2t^3e^t + (t-2)(e^t-1)^3.$$

We have

$$f'(t) = (2t^3 + 6t^2)e^t + (3te^t - 5e^t - 1)(e^t - 1)^2,$$
$$f''(t) = e^t(2t^3 + 12t^2 + 12t + (9te^t - 12e^t - 3t)(e^t - 1)).$$

Let $g(t) = 2t^3 + 12t^2 + 12t + (9te^t - 12e^t - 3t)(e^t - 1)$. We have

$$g'(t) = 6t^2 + 24t + (18te^{2t} - 12te^t) - 15(e^{2t} - 1),$$
$$g''(t) = 12t + (36te^{2t} - 12te^t) - 12(e^{2t} - 1) - 12(e^t - 1),$$
$$g'''(t) = (72te^{2t} - 12te^t) + 12(e^{2t} - 1) - 24(e^t - 1).$$

Since $g'''(t) > (72te^{2t} - 72te^{2t}) + 24(e^t - 1) - 24(e^t - 1) = 0$ and $g''(0) = 0$, we get $g''(t) > 0$. In a similar way, we get $g'(t) > 0$ and $g(t) > 0$. Since $f''(t) > 0$ and $f'(0) = 0$, we get $f'(t) > 0$. In a similar way, we get $f(t) > 0$. □

References

[1] Chao-Ping Chen, Problem 14, (2009), *Research Group In Mathematical Inequalities And Applications*.