On Ky Fan's Inequality

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Abstract

Suppose that $\lambda_1, \lambda_2, \cdots, \lambda_n > 0$ with $\sum_{i=1}^{n} \lambda_i = 1$, and $x_1, \cdots, x_n$ belong to $(0, \frac{1}{2}]$. Let

$$A_n = \sum_{i=1}^{n} \lambda_i x_i, \qquad G_n = \prod_{i=1}^{n} x_i^{\lambda_i},$$

be the arithmetic and geometric means of $x_1, \cdots, x_n$, and similarly

$$A'_n = \sum_{i=1}^{n} \lambda_i (1 - x_i), \qquad G'_n = \prod_{i=1}^{n} (1 - x_i)^{\lambda_i},$$

the arithmetic and geometric means of $1 - x_1, \cdots, 1 - x_n$, respectively.
The Ky Fan's inequality states that

$$\frac{A'_n}{G'_n} \leq \frac{A_n}{G_n},$$

with equality holding if and only if $x_1 = \cdots = x_n$.

In this talk, first using some techniques of inequality theory, such as, Maclaurin's and gradient methods, Dinghas identity and binomial expansion, we introduce some new proofs of Ky Fan's inequality, and then using the concept of convexity, we refine Ky Fan's inequality and give some converses for it.

Two interesting refinements of Ky Fan's inequality are as follows:

$$\left(\frac{A'_n}{G'_n}\right)^{A'_n+G'_n} \leq \left(\frac{A_n}{G_n}\right)^{A_n+G_n},$$

and

$$\left(\frac{A'_n}{G'_n}\right)^{A_n-G_n} \leq \left(\frac{A_n}{G_n}\right)^{A'_n-G'_n},$$

with equality holding if and only if $x_1 = \cdots = x_n$.