ON THE STABILITY OF FUNCTIONAL INEQUALITIES WITH CAUCHY–JENSEN MAPPINGS

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Abstract. In this paper, we investigate the generalized Hyers–Ulam stability of the following functional inequality

$$\|f(ax+by+cz)-2bf(y)-2cf(z)\| \le \|f(ax-by-cz)\| + \phi(x,y,z)$$

associated with Cauchy–Jensen additive mappings. As a result, we obtain that if a mapping satisfies the functional inequalities with perturbation which satisfies certain conditions then there exists a Cauchy–Jensen additive mapping near the mapping.

1. Introduction

In 1940, S. M. Ulam [16] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

In 1941, D. H. Hyers [7] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y)-f(x)-f(y)\| \le \epsilon$$

for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \le \epsilon.$$

In 1978, Th. M. Rassias [13] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

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Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$.
Then the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$.

In 1991, Z. Gajda [3] following the same approach as in Th. M. Rassias [13], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [3], as well as by Th. M. Rassias and P. Šemrl [14] that one cannot prove a Th. M. Rassias' type theorem when $p = 1$. The inequality (1.1) that was introduced for the first time by Th. M. Rassias [13] provided a lot of influence in the development of a generalization of the Hyers–Ulam stability concept. This new concept of stability is known as *generalized Hyers–Ulam stability* or *Hyers–Ulam–Rassias stability* of functional equations (cf. the books of P. Czerwik [1], D. H. Hyers, G. Isac and Th. M. Rassias [8]).

P. Găvruta [6] provided a further generalization of Th. M. Rassias' Theorem. During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers–Ulam stability to a number of functional equations and mappings (see [9],[12]–[14]).

Gilányi[4] and Rätz[15] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \tag{1.3}$$

then f satisfies the Jordan–von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

Gilányi [5] and Fechner [2] proved the generalized Hyers–Ulam stability of the functional inequality (1.3).

Now, we consider the following functional inequality

$$\|f(ax + by + cz) - 2bf(y) - 2cf(z)\| \leq \|f(ax - by - cz)\| + \phi(x, y, z), \tag{1.4}$$

which is associated with Jordan–von Neumann type Cauchy–Jensen additive functional equations, where the function ϕ is a perturbing term of the functional inequality

$$\|f(ax + by + cz) - 2bf(y) - 2cf(z)\| \leq \|f(ax - by - cz)\|.$$

The purpose of this paper is to prove that if f satisfies one of the inequality (1.4) which satisfies certain conditions, then we can find a Cauchy–Jensen additive mapping near f and thus we prove the generalized Hyers–Ulam stability of the functional inequality (1.4).

2. Stability of functional inequality (1.4)

Throughout this paper, let G be a normed vector space and Y a Banach space. First, we consider solutions of the functional inequality (1.4) with perturbing term zero.

Lemma 2.1. *Let $f : G \to Y$ be a mapping with $f(0) = 0$ such that*

$$\|f(ax + by + cz) - 2bf(y) - 2cf(z)\| \leq \|f(ax - by - cz)\| \tag{2.1}$$

for all $x, y, z \in G$, where $abc \neq 0$. Then f is Cauchy–Jensen additive.

Proof. By setting $y := \frac{ax}{b}$ and $z := 0$ in (2.1), we get $\|f(2ax) - 2bf(\frac{ax}{b})\| \leq \|f(0)\| = 0$, which implies

$$f(2x) - 2bf(\frac{x}{b}) = 0 \tag{2.2}$$

for all $x \in G$. Similarly, we have

$$f(2x) - 2cf(\frac{x}{c}) = 0 \tag{2.3}$$

for all $x \in G$.

Also by letting $x := 0$, $y := \frac{-cxx}{b}$ and $z := x$ in (2.1), we get

$$\|bf(\frac{-cx}{b}) + cf(x)\| \leq \|f(0)\| = 0 \tag{2.4}$$

for all $x \in G$. These three equalities (2.2), (2.3) and (2.4) lead to

$$2cf(-x) = f(-2cx) = 2bf(\frac{-cx}{b}) = -2cf(x)$$

for all $x \in G$. Therefore the mapping f is odd.

Letting $z = \frac{ax-by}{c}$ in (2.1), we get

$$\|f(2ax) - 2bf(y) - 2cf(\frac{ax - by}{c})\| \leq \|f(0)\| = 0$$

for all $x, y \in G$. It follows from the equalities (2.3) and (2.4) that $2cf(\frac{ax}{c}) + 2bf(\frac{-by}{c}) - 2cf(\frac{ax-by}{c}) = 0$, that is, $f(u) - f(v) - f(u - v) = 0$ for all $u, v \in G$, as desired. □

We prove the generalized Hyers–Ulam stability of a functional inequality (1.4) associated with a Jordan–von Neumann type 3-variable Cauchy–Jensen additive functional equation.

Theorem 2.2. *Assume that a mapping $f : G \to Y$ with $f(0) = 0$ satisfies the functional inequality*

$$\|f(ax + by + cz) - 2bf(y) - 2cf(z)\| \leq \|f(ax - by - cz)\| + \phi(x, y, z) \tag{2.5}$$

and that the map $\phi : G \times G \times G \to [0, \infty)$ *satisfies the condition*

$$\Phi(x,y,z) := \sum_{j=1}^{\infty} 2^j \phi(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}) < \infty$$

for all $x, y, z \in G$. *Then there exists a unique Cauchy–Jensen additive mapping* $A : G \to Y$, *defined by* $A(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$, *such that*

$$\|A(ax+by+cz) - 2bA(y) - 2cA(z)\| \le \|A(ax-by-cz)\|, \tag{2.6}$$
$$\|A(x) - f(x)\| \le \frac{1}{4|c|}\left[\Phi(\frac{cx}{a}, \frac{-cx}{b}, 2x) + \Phi(\frac{cx}{a}, 0, x) + \Phi(0, \frac{-cx}{b}, x)\right]$$

for all $x, y, z \in G$.

Proof. Letting $y := 0$ and $z := \frac{ax}{c}$ in (2.5), we get

$$\left\|f(2ax) - 2cf(\frac{ax}{c})\right\| \le \phi(x, 0, \frac{ax}{c}) \tag{2.7}$$

for all $x \in G$. By letting $x := 0$, $y := -y$ and $z := \frac{by}{c}$ in (2.5), one obtains

$$\left\|2bf(-y) + 2cf(\frac{by}{c})\right\| \le \phi(0, -y, \frac{by}{c}) \tag{2.8}$$

for all $x \in G$. Replacing z by $\frac{ax-by}{c}$ in (2.5), we get

$$\left\|f(2ax) - 2bf(y) - 2cf(\frac{ax-by}{c})\right\| \le \phi(x, y, \frac{ax-by}{c}).$$

And then substitute $y := -y$ in the last inequality to obtain that

$$\left\|f(2ax) - 2bf(-y) - 2cf(\frac{ax+by}{c})\right\| \le \phi(x, -y, \frac{ax+by}{c}) \tag{2.9}$$

for all $x \in G$.

It follows from (2.7), (2.8) and (2.9) that

$$2|c|\left\|f(\frac{ax}{c}) + f(\frac{by}{c}) - f(\frac{ax+by}{c})\right\|$$
$$\le \phi(x, -y, \frac{ax+by}{c}) + \phi(x, 0, \frac{ax}{c}) + \phi(0, -y, \frac{by}{c}),$$

which yields the Cauchy difference

$$\|f(x) + f(y) - f(x+y)\| \tag{2.10}$$
$$\le \frac{1}{2|c|}\left[\phi(\frac{cx}{a}, \frac{-cy}{b}, x+y) + \phi(\frac{cx}{a}, 0, x) + \phi(0, \frac{-cy}{b}, y)\right]$$

for all $x \in G$.

Now it follows from (2.10) that

$$\|2f(x) - f(2x)\| \leq \frac{1}{2|c|}\left[\phi(\frac{cx}{a}, \frac{-cx}{b}, 2x) + \phi(\frac{cx}{a}, 0, x) + \phi(0, \frac{-cx}{b}, x)\right],$$

and so

$$\begin{aligned}
&\|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m})\| \\
&\leq \sum_{j=l+1}^{m} \|2^j f(\frac{x}{2^j}) - 2^{j-1} f(\frac{x}{2^{j-1}})\| \qquad (2.11)\\
&\leq \frac{1}{4|c|}\sum_{j=l+1}^{m} 2^j \left[\phi(\frac{cx}{a2^j}, \frac{-cx}{b2^j}, \frac{2x}{2^j}) + \phi(\frac{cx}{a2^j}, 0, \frac{x}{2^j}) + \phi(0, \frac{-cx}{b2^j}, \frac{x}{2^j})\right]
\end{aligned}$$

for all $x \in G$ and for all nonnegative integers m and l with $m > l$. It means that for any $x \in G$ a sequence $\{2^m f(\frac{x}{2^m})\}$ is Cauchy in Y. Since Y is complete, the sequence $\{2^m f(\frac{x}{2^m})\}$ converges. So one can define a mapping $A : G \to Y$ by $A(x) := \lim_{m\to\infty} 2^m f(\frac{x}{2^m})$ for all $x \in G$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.11), we get the approximation (2.6) of f by A.

Next, we claim that the mapping $A : G \longrightarrow Y$ is Cauchy–Jensen additive satisfying the functional inequality (2.1). In fact, it follows easily from (2.5) and the condition of ϕ that

$$\begin{aligned}
&\|A(ax + by + cz) - 2bA(y) - 2cA(z)\| \\
&= \lim_{m\to\infty} 2^m \left\| f\Big(\frac{ax + by + cz}{2^m}\Big) - 2bf\Big(\frac{y}{2^m}\Big) - 2cf\Big(\frac{z}{2^m}\Big)\right\| \\
&\leq \lim_{m\to\infty} 2^m \left[\left\| f\Big(\frac{ax - by - cz}{2^m}\Big)\right\| + \phi\Big(\frac{x}{2^m}, \frac{y}{2^m}, \frac{z}{2^m}\Big)\right] \\
&= \|A(ax - by - cz)\|.
\end{aligned}$$

Thus the mapping $A : G \longrightarrow Y$ is Cauchy–Jensen additive by Lemma 2.1.

Now, let $T : G \longrightarrow Y$ be another Cauchy–Jensen additive mapping satisfying (2.6). Then we obtain

$$\begin{aligned}
&\|2^n f(\frac{x}{2^n}) - T(x)\| = 2^n\|f(\frac{x}{2^n}) - T(\frac{x}{2^n})\| \\
&\leq \frac{1}{4|c|}2^n \left[\Phi(\frac{cx}{a2^n}, \frac{-cx}{b2^n}, \frac{2x}{2^n}) + \Phi(\frac{cx}{a2^n}, 0, \frac{x}{2^n}) + \Phi(0, \frac{-cx}{b2^n}, \frac{x}{2^n})\right] \\
&\leq \frac{1}{4|c|}\sum_{j=n+1}^{\infty} 2^j \left[\phi(\frac{cx}{a2^j}, \frac{-cx}{b2^j}, \frac{2x}{2^j}) + \phi(\frac{cx}{a2^j}, 0, \frac{x}{2^j}) + \phi(0, \frac{-cx}{b2^j}, \frac{x}{2^j})\right],
\end{aligned}$$

which tends to zero as $n \to \infty$. So we can conclude that $A(x) = T(x)$ for all $x \in G$. This proves the uniqueness of A. □

Theorem 2.3. *Assume that a mapping $f : G \to Y$ with $f(0) = 0$ satisfies the inequality* (2.5) *and that the map $\phi : G \times G \times G \to [0, \infty)$ satisfies the condition*

$$\Phi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \phi(2^j x, 2^j y, 2^j z) < \infty$$

for all $x, y, z \in G$. Then there exists a unique Cauchy–Jensen additive mapping $A : G \to Y$, defined by $A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$, such that

$$\|A(ax + by + cz) - 2bA(y) - 2cA(z)\| \le \|A(ax - by - cz)\|,$$
$$\|A(x) - f(x)\| \le \frac{1}{4|c|} \left[\Phi(\frac{cx}{a}, \frac{-cx}{b}, 2x) + \Phi(\frac{cx}{a}, 0, x) + \Phi(0, \frac{-cx}{b}, x) \right] \quad (2.12)$$

for all $x, y, z \in G$.

Proof. We get by (2.10)

$$\begin{aligned} &\|\frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x)\| \qquad (2.13)\\ &\le \sum_{j=l}^{m-1} \|\frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x)\| \\ &\le \frac{1}{4|c|} \sum_{j=l}^{m-1} \frac{1}{2^j} \left[\Phi(\frac{2^j cx}{a}, \frac{-2^j cx}{b}, 2^{j+1} x) + \Phi(\frac{2^j cx}{a}, 0, 2^j x) + \Phi(0, \frac{-2^j cx}{b}, 2^j x) \right] \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in G$. It means that a sequence $\{\frac{1}{2^m} f(2^m x)\}$ is Cauchy sequence in Y for all $x \in G$. Since Y is complete, the sequence $\{\frac{1}{2^m} f(2^m x)\}$ converges. So one can define a mapping $A : G \to Y$ by $A(x) := \lim_{m\to\infty} \frac{1}{2^m} f(2^m x)$ for all $x \in G$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.13), we get (2.12).

The remaining proof goes through by the similar argument to Theorem 2.2. □

Corollary 2.4. *Assume that there exists a nonnegative numbers δ such that a mapping $f : G \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|f(ax + by + cz) - 2bf(y) - 2cf(z)\| \le \|f(ax - by - cz)\| + \delta$$

for all $x, y, z \in G$.
Then there exists a unique Cauchy–Jensen additive mapping $A : G \to Y$ such that

$$\|A(ax + by + cz) - 2bA(y) - 2cA(z)\| \le \|A(ax - by - cz)\|,$$
$$\|f(x) - A(x)\| \le \frac{3\delta}{2|c|} \quad (2.14)$$

for all $x, y, z \in G$.

We recall that a subadditive function is a function $\phi : A \to B$, having a domain A and a codomain $(B, \leq)$ that are both closed under addition, with the following property:

$$\phi(x+y) \leq \phi(x) + \phi(y), \ \forall x, y \in A.$$

Now we say that a function $\phi : A \to B$ is *contractively subadditive* if there exists a constant L with $0 < L < 1$ such that

$$\phi(x+y) \leq L[\phi(x) + \phi(y)], \ \forall x, y \in A.$$

Then ϕ satisfies the following properties $\phi(2x) \leq 2L\phi(x)$ and so $\phi(2^n x) \leq (2L)^n \phi(x)$. Similarly, we say that a function $\phi : A \to B$ is *expansively superadditive* if there exists a constant L with $0 < L < 1$ such that

$$\phi(x+y) \geq \frac{1}{L}[\phi(x) + \phi(y)], \ \forall x, y \in A.$$

Then ϕ satisfies the following properties $\phi(x) \leq \frac{L}{2}\phi(2x)$ and so $\phi(\frac{x}{2^n}) \leq (\frac{L}{2})^n \phi(x)$.

Theorem 2.5. *Assume that a mapping $f : G \to Y$ with $f(0) = 0$ satisfies the inequality* (2.5) *and that the map $\phi : G \times G \times G \to [0, \infty)$ is expansively superadditive with a constant L. Then there exists a unique Cauchy–Jensen additive mapping $A : G \to Y$, defined by $A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$, such that*

$$\|A(ax+by+cz) - 2bA(y) - 2cA(z)\| \leq \|A(ax-by-cz)\|, \tag{2.15}$$

$$\|f(x) - A(x)\| \leq \frac{L}{4|c|(1-L)}\left[\phi(\frac{cx}{a}, \frac{-cx}{b}, 2x) + \phi(\frac{cx}{a}, 0, x) + \phi(0, \frac{-cx}{b}, x)\right]$$

for all $x, y, z \in G$.

Proof. We observe by the contractively superadditive condition that for any $x, y, z \in G$ $\phi(\frac{(x,y,z)}{2^n}) \leq (\frac{L}{2})^n \phi(x, y, z)$. Thus it follows from (2.10) and (2.11) that

$$\begin{aligned}
&\|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m})\| \\
&\leq \sum_{j=l+1}^{m} \|2^j f(\frac{x}{2^j}) - 2^{j-1} f(\frac{x}{2^{j-1}})\| \\
&\leq \frac{1}{4|c|}\sum_{j=l+1}^{m} 2^j \left[\phi(\frac{cx}{a2^j}, \frac{-cx}{b2^j}, \frac{2x}{2^j}) + \phi(\frac{cx}{a2^j}, 0, \frac{x}{2^j}) + \phi(0, \frac{-cx}{b2^j}, \frac{x}{2^j})\right] \\
&\leq \frac{1}{4|c|}\sum_{j=l+1}^{m} L^j \left[\phi(\frac{cx}{a}, \frac{-cx}{b}, 2x) + \phi(\frac{cx}{a}, 0, x) + \phi(0, \frac{-cx}{b}, x)\right]
\end{aligned}$$

for all $x \in G$ and for all nonnegative integers m and l with $m > l$. It means that a sequence $\{2^m f(\frac{x}{2^m})\}$ is Cauchy sequence for all $x \in G$. Since Y is complete, the sequence $\{2^m f(\frac{x}{2^m})\}$ converges. So one can define a mapping $A : G \to Y$ by $A(x) := \lim_{m\to\infty} 2^m f(\frac{x}{2^m})$ for all $x \in G$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in the last inequality, we get (2.15).

The remaining proof goes through by the similar argument to Theorem 2.2. □

Corollary 2.6. *Assume that there exist a nonnegative numbers* θ *and a real* $p > 1$ *such that a mapping* $f : G \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$\|f(ax + by + cz) - 2bf(y) - 2cf(z)\| \leq \|f(ax - by - cz)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in G$.
Then there exists a unique Cauchy–Jensen additive mapping $A : G \to Y$ *such that*

$$\|A(ax + by + cz) - 2bA(y) - 2cA(z)\| \leq \|A(ax - by - cz)\|,$$
$$\|f(x) - A(x)\| \leq \frac{\theta\|x\|^p}{2|c|(2^p - 2)}\left[\frac{2|c|^p}{|a|^p} + \frac{2|c|^p}{|b|^p} + 2^p + 2\right]$$

for all $x, y, z \in G$.

Theorem 2.7. *Assume that a mapping* $f : G \to Y$ *with* $f(0) = 0$ *satisfies the inequality* (2.5) *and that the map* $\phi : G \times G \times G \to [0, \infty)$ *is contractively subadditive with a constant* L. *then there exists a unique Cauchy–Jensen additive mapping* $A : G \to Y$, *defined by* $A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$, *such that*

$$\|A(ax + by + cz) - 2bA(y) - 2cA(z)\| \leq \|A(ax - by - cz)\|, \tag{2.16}$$
$$\|f(x) - A(x)\| \leq \frac{1}{4|c|(1 - L)}\left[\phi(\frac{cx}{a}, \frac{-cx}{b}, 2x) + \phi(\frac{cx}{a}, 0, x) + \phi(0, \frac{-cx}{b}, x)\right]$$

for all $x, y, z \in G$.

Proof. We get by (2.10) and (2.11)

$$\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\| \leq \sum_{j=l}^{m-1} \|\frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1}x)\| \tag{2.17}$$
$$\leq \frac{1}{2|c|}\sum_{j=l}^{m-1}\frac{1}{2^{j+1}}\left[\phi(\frac{2^j cx}{a}, \frac{-2^j cx}{b}, 2^{j+1}x) + \phi(\frac{2^j cx}{a}, 0, 2^j x) + \phi(0, \frac{-2^j cx}{b}, 2^j x)\right]$$
$$\leq \frac{1}{4|c|}\sum_{j=l}^{m-1} L^j\left[\phi(\frac{cx}{a}, \frac{-cx}{b}, 2x) + \phi(\frac{cx}{a}, 0, x) + \phi(0, \frac{-cx}{b}, x)\right]$$

for all nonnegative integers m and l with $m > l$ and all $x \in G$. It means that a sequence $\{\frac{1}{2^m}f(2^m x)\}$ is Cauchy sequence for all $x \in G$. Since Y is complete, the sequence $\{\frac{1}{2^m}f(2^m x)\}$ converges. So one can define a mapping $A : G \to Y$ by $A(x) := \lim_{m\to\infty} \frac{1}{2^m}f(2^m x)$ for all $x \in G$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.17), we get (2.16).

The remaining proof goes through by the similar argument to Theorem 2.5. □

Corollary 2.8. *Assume that there exist a nonnegative numbers θ, δ and a real $p < 1$ such that a mapping $f : G \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|f(ax + by + cz) - 2bf(y) - 2cf(z)\| \leq \|f(ax - by - cz)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in G$.
Then there exists a unique Cauchy–Jensen additive mapping $A : G \to Y$ such that

$$\|A(ax + by + cz) - 2bA(y) - 2cA(z)\| \leq \|A(ax - by - cz)\|,$$
$$\|f(x) - A(x)\| \leq \frac{\theta\|x\|^p}{2|c|(2 - 2^p)}\left[\frac{2|c|^p}{|a|^p} + \frac{2|c|^p}{|b|^p} + 2^p + 2\right]$$

for all $x, y, z \in G$.

The following approximation of f by A has much simpler upper bound than that of (2.6).

Theorem 2.9. *Assume that a mapping $f : G \to Y$ with $f(0) = 0$ satisfies the functional inequality*

$$\|f(ax + by + cz) - 2bf(y) - 2cf(z)\| \leq \|f(ax - by - cz)\| + \phi(x, y, z) \qquad (2.18)$$

and that the map $\phi : G \times G \times G \to [0, \infty)$ satisfies the condition

$$\Phi(x, y, z) := \sum_{j=1}^{\infty}|\lambda|^j \phi(\frac{x}{\lambda^j}, \frac{y}{\lambda^j}, \frac{z}{\lambda^j}) < \infty$$

for all $x, y, z \in G$, where $\lambda := 2(b+c) \neq 0$. Then there exists a unique Cauchy–Jensen additive mapping $A : G \to Y$, defined by $A(x) = \lim_{n\to\infty} \lambda^n f(\frac{x}{\lambda^n})$, such that

$$\|A(ax + by + cz) - 2bA(y) - 2cA(z)\| \leq \|A(ax - by - cz)\|, \qquad (2.19)$$
$$\|A(x) - f(x)\| \leq \frac{1}{|\lambda|}\Phi\left(\frac{b + c}{a}x, x, x\right)$$

for all $x, y, z \in G$.

Proof. Replacing (x, y, z) by $\left(\frac{b+c}{a}x, x, x\right)$ in (2.18), we get

$$\|f(\lambda x) - \lambda f(x)\| \leq \phi\left(\frac{b + c}{a}x, x, x\right). \qquad (2.20)$$

Now it follows from (2.20) that

$$\begin{aligned}\|\lambda^l f(\frac{x}{\lambda^l}) - \lambda^m f(\frac{x}{\lambda^m})\| &\leq \sum_{j=l}^{m-1}\|\lambda^j f(\frac{x}{\lambda^j}) - \lambda^{j+1} f(\frac{x}{\lambda^{j+1}})\| \\ &\leq \frac{1}{|\lambda|}\sum_{j=l}^{m-1}\lambda^{j+1}\phi\left(\frac{b + c}{a}\frac{x}{\lambda^{j+1}}, \frac{x}{\lambda^{j+1}}, \frac{x}{\lambda^{j+1}}\right)\end{aligned}$$

for all $x \in G$ and for all nonnegative integers m and l with $m > l$.

The rest of proof is similar to the corresponding part of Theorem 2.2. □

Theorem 2.10. *Assume that a mapping $f : G \to Y$ with $f(0) = 0$ satisfies the inequality* (2.5) *and that the map $\phi : G \times G \times G \to [0, \infty)$ satisfies the condition*

$$\Phi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{|\lambda|^j} \phi(\lambda^j x, \lambda^j y, \lambda^j z) < \infty$$

for all $x, y, z \in G$, where $\lambda := 2(b+c) \neq 0$. Then there exists a unique Cauchy–Jensen additive mapping $A : G \to Y$, defined by $A(x) := \lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x)$, such that

$$\|A(ax + by + cz) - 2bA(y) - 2cA(z)\| \leq \|A(ax - by - cz)\|,$$
$$\|A(x) - f(x)\| \leq \frac{1}{|\lambda|} \Phi\left(\frac{b+c}{a} x, x, x\right)$$

for all $x, y, z \in G$.

Corollary 2.11. *Assume that there exists a nonnegative numbers δ such that a mapping $f : G \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|f(ax + by + cz) - 2bf(y) - 2cf(z)\| \leq \|f(ax - by - cz)\| + \delta$$

for all $x, y, z \in G$, where $0 < |\lambda := 2(b + c)| \neq 1$ Then there exists a unique Cauchy–Jensen additive mapping $A : G \to Y$ such that

$$\|A(ax + by + cz) - 2bA(y) - 2cA(z)\| \leq \|A(ax - by - cz)\|,$$
$$\|f(x) - A(x)\| \leq \frac{\delta}{\Big|2|b + c| - 1\Big|} \tag{2.21}$$

for all $x, y, z \in G$.

We observe that the best approximation between (2.14) and (2.21) of f by A is determined by constants b, c.

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