Oriented Bond Percolation and Phase Transitions: an Analytic Approach

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Abstract. An analytic approach is introduced for the determination of rigorous lower bounds for the critical probability of bond percolation in an oriented lattice. This is illustrated by an example, the oriented square lattice in two dimensions.

Keywords: oriented bond percolation, critical probability, phase transition, positive term power series
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1. INTRODUCTION

Percolation problems on infinite lattices are the subject of a large mathematical literature, including several books. See, for example, the consolidated accounts of Smythe and Wierman [19], Kesten [15], Durrett [8], Hughes [13, 14], Grimmett [9, 10] and Bollobás and Riordan [4].

The literature relates to bond graphs and also to site and mixed bond-and-site graph processes. With bond graphs, each bond is, independently of every other, open with probability p and closed with probability $1-p$. When there exists a connected path of open bonds from the origin to infinity, percolation is said to occur. The probability $\theta(p)$ of percolation is a non-decreasing function of p.

A common phenomenon is for there to exist a critical value $p = p_{cb} \in (0,1)$ such that

$$\theta(p) \begin{cases} = 0 & \text{for } 0 \leq p < p_{cb} \\ > 0 & \text{for } p_{cb} < p \leq 1 \end{cases} .$$

When this phenomenon occurs, the process is said to undergo a phase transition at $p = p_{cb}$. The exact determination of the critical probability p_{cb} is usually difficult and has been achieved for relatively few graph configurations. Much effort has gone into finding good upper and lower bounds for critical probabilities: see, for example, [1], [5], [17], [20], [21], [22]. Work in this area is characterized by subtle and intricate probabilistic arguments and sometimes also heavy computation.

The problem has proved more difficult in oriented graphs. In the bond case, each bond has an orientation and paths are required to proceed in the direction of that orientation on each link. For a discussion of oriented percolation see [7] and [11].

The graph $\vec{\mathbb{L}}^2$, the single-quadrant oriented square lattice in two dimensions, is considered in an oriented bond or site percolation model by Durrett [7], Liggett [16] and Balister, Bollobas and Stacey [1]. Durrett [7] derived the rigorous upper bound $p_{cb} \leq 0.84$ for the critical probability. Balister, Bollóbas and Stacey improved this to $p_{cb} \leq 0.6863$ in 1993 [1] and to $p_{cb} \leq 0.6735$ in 1994 [2]. In 1995 Liggett [16] derived $p_{cb} \leq 2/3$, currently the best published upper bound. The arguments involved are quite involved.

There are few published rigorous results available for lower bounds. Hammersley [12] derived $p_{cb} \geq 0.5176$ in 1957 and Dhar [6] $p_{cb} \leq 0.6298$ in 1982.

An elegant analytic approach was introduced by Bishir in 1963 [3] to derive a lower bound for the critical probability for site percolation on the oriented lattice $\vec{\mathbb{L}}^2$. The ideas have been taken up by Pearce and Fletcher [18], and used very recently to give a considerable improvement on known exact results for site percolation on the related lattice $\vec{\mathbb{L}}^2_{alt}$.

In this article we provide an example, that of $\vec{\mathbb{L}}^2$, showing how a modification of this approach may be used to derive a rigorous lower bound for critical probabilities in a bond-percolation setting. Given the effort researchers have expended on this problem, it would be remarkable if we obtained the best result to date and we do not. However the bound, $2-\sqrt{2} \approx 0.586$, is surprisingly good considering the relative simplicity of the derivation.

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