On a bi-Pexider functional equation and its stability

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Abstract. In this paper, we prove the stability of a bi-Pexider functional equation

$$f(x+y,z+w) = f_1(x,z) + f_2(x,w) + f_3(y,z) + f_4(y,w).$$

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INTRODUCTION

Throughout this paper, let X and Y be vector spaces. A mapping $g : X \to Y$ is called a Cauchy mapping (respectively a Jensen mapping) if g satisfies the functional equation $g(x+y) = g(x)+g(y)$ (respectively $2g(\frac{x+y}{2}) = g(x)+g(y)$). For given mappings $f, f_1, f_2, f_3, f_4 : X \times X \to Y$, we define

$$\begin{aligned}
&Df(x,y,z,w) := f(x+y,z+w) - f(x,z) - f(x,w) - f(y,z) - f(y,w),\\
&D^1 f(x,y,z,w) := 2f(x+y,\frac{z+w}{2}) - f(x,z) - f(x,w) - f(y,z) - f(y,w),\\
&D^2 f(x,y,z,w) := 2f(\frac{x+y}{2},z+w) - f(x,z) - f(x,w) - f(y,z) - f(y,w),\\
&D^3 f(x,y,z,w) := 4f(\frac{x+y}{2},\frac{z+w}{2}) - f(x,z) - f(x,w) - f(y,z) - f(y,w),\\
&D(f, f_1, f_2, f_3, f_4)(x,y,z,w)\\
&\qquad := f(x+y,z+w) - f_1(x,z) - f_2(x,w) - f_3(y,z) - f_4(y,w)
\end{aligned}$$

for all $x,y,z,w \in X$. A mapping $f : X \times X \to Y$ is called a biadditive (Cauchy-Jensen, Jensen-Cauchy, bi-Jensen, bi-Pexider, respectively) mapping if f satisfies the functional equations $Df = 0(D^1 f = 0, D^2 f = 0, D^3 f = 0$, and $D(f, f_1, f_2, f_3, f_4) = 0$, respectively).

In 2006, Park and Bae [5] obtained the generalized Hyers-Ulam stability of Cauchy-Jensen mapping mapping. In 2007, Jun and Lee ([2], [3]) improved the Park and Bae's results.

THE SOLUTION OF A BI-PEXIDER EQUATION

Theorem 2.1 *Let* $f, f_1, f_2, f_3, f_4 : X \times X \to Y$ *be mappings satisfying*

$$D(f, f_1, f_2, f_3, f_4)(x,y,z,w) = 0 \tag{1}$$

for all $x,y,z,w \in X$. *Then* f *is a bi-Jensen mapping and there exist a bi-additive mapping* $F : X \times X \to Y$ *and additive mappings* $A_1, A_2 : X \to Y$ *satisfying the equations*

$$\begin{aligned}
f(x,y) &= F(x,y) + A_1(x) + A_2(y) + f(0,0) \\
F(x,y) &= f(x,y) - f(x,0) - f(0,y) + f(0,0) \qquad (2) \\
&= f_1(x,y) - f_1(x,0) - f_1(0,y) + f_1(0,0) \\
&= f_2(x,y) - f_2(x,0) - f_2(0,y) + f_2(0,0) \\
&= f_3(x,y) - f_3(x,0) - f_3(0,y) + f_3(0,0) \\
&= f_4(x,y) - f_4(x,0) - f_4(0,y) + f_4(0,0) \\
A_1(x) &= f(x,0) - f(0,0), \qquad (3) \\
A_2(y) &= f(0,y) - f(0,0) \qquad (4)
\end{aligned}$$

for all $x,y \in X$.

Example 2.2 The mappings $f, f_1, f_2, f_3, f_4 : X \times X \to Y$ defined by

$$\begin{aligned}
f(x,y) &= xy + 2x + 2y + 2 \\
f_1(x,y) &= xy + x + x^2 + y + y^2 + 1 \\
f_2(x,y) &= xy + x - x^2 + y - \sqrt{y} - 1 \\
f_3(x,y) &= xy + x - \sqrt{x} + y - y^2 \\
f_4(x,y) &= xy + x + \sqrt{x} + y + \sqrt{y} + 2
\end{aligned}$$

satisfy the conditions of Theorem 2.1 but $f_1, f_2, f_3, f_4 : X \times X \to Y$ are not bi-Jensen mappings.

THE STABILITY OF A BI-PEXIDER EQUATION

Theorem 3.1 *Let* $f, f_1, f_2, f_3, f_4 : X \times X \to Y$ *be mappings satisfying*

$$\|D(f, f_1, f_2, f_3, f_4)(x,y,z,w)\| \leq \varepsilon \tag{5}$$

for all $x,y,z,w \in X$. *Then there exists a unique bi-Jensen mappings* $F : X \times X \to Y$ *satisfying*

$$\|f(x,y) - F(x,y)\| \leq 14\varepsilon \tag{6}$$

for all $x,y \in X$. *The map* F *is given by*

$$F(x,y) = \lim_{n\to\infty}\left(\frac{f(2^n x, 2^n y)}{4^n} + \frac{f(2^n x, 0)}{2^n} + \frac{f(0, 2^n y)}{2^n}\right) + f(0,0)$$

for all $x,y \in X$.

Theorem 3.2 *Let* $f, f_1, f_2, f_3, f_4 : X \times X \to Y$ *be mappings satisfying (5) for all* $x,y,z,w \in X$*. Then there exists a unique bi-Jensen mapping* $F : X \times X \to Y$ *satisfying*

$$\|f(x,y) - F(x,y)\| \leq 8\varepsilon \tag{7}$$

for all $x,y \in X$*. The map* F *is given by*

$$\begin{aligned} F(x,y) &= \lim_{n\to\infty} \frac{f(2^n x,y) + f(0,2^n y)}{2^n} + f(0,0) \\ &= \lim_{n\to\infty} \frac{f(x,2^n y) + f(2^n x,0)}{2^n} + f(0,0) \end{aligned}$$

for all $x,y \in X$.

Theorem 3.3 *Let* $f, f_1, f_2, f_3, f_4 : X \times X \to Y$ *be mappings satisfying (5) for all* $x,y,z,w \in X$*. Then there exists a unique bi-Jensen mapping* $F : X \times X \to Y$ *satisfying*

$$\|f(x,y) - F(x,y)\| \leq 6\varepsilon \tag{8}$$

for all $x,y \in X$*. The map* F *is given by*

$$\begin{aligned} F(x,y) &= \lim_{n\to\infty} \frac{f(2^n x,y) + f(0,2^n y)}{2^n} + f(0,0) \\ &= \lim_{n\to\infty} \frac{f(x,2^n y) + f(2^n x,0)}{2^n} + f(0,0) \end{aligned}$$

for all $x,y \in X$.

Remark 3.4 *Let* $f : X \times X \to Y$ *be mappings satisfying*

$$\|D^1(x,y,z,w)\| \leq \varepsilon$$

for all $x,y,z,w \in X$*. Then there exists a unique Cauchy-Jensen mapping* $F : X \times X \to Y$ *satisfying*

$$\|f(x,y) - F(x,y)\| \leq \varepsilon$$

for all $x,y \in X$*. The map* F *is given by*

$$F(x,y) = \lim_{n\to\infty} \frac{f(2^n x,y)}{2^n} + f(0,0)$$

for all $x,y \in X$.

Remark 3.5 *Let* $f : X \times X \to Y$ *be mappings satisfying*

$$\|D^1(x,y,z,w)\| \leq \varepsilon$$

for all $x,y,z,w \in X$*. Then there exists a unique Cauchy-Jensen mapping* $F : X \times X \to Y$ *satisfying*

$$\|f(x,y) - F(x,y)\| \leq \frac{2}{3}\varepsilon$$

for all $x,y \in X$*. The map* F *is given by*

$$F(x,y) = \lim_{n\to\infty} \frac{f(2^n x,y) + f(0,2^j y)}{2^n} + f(0,0)$$

for all $x,y \in X$.

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