Accurate Approximations of the Riemann-Stieltjes Integral with (l, L)-Lipschitzian Integrators

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Abstract. Grüss-type inequalities for the Riemann-Stieltjes integral with (l,L)-Lipschitzian integrators and applications for the Čebyšev functional are given. Sharp inequalities complementing results of Čebyšev, Grüss, Ostrowski and Lupaş are given.

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1. INTRODUCTION

In order to accurately approximate the Riemann-Stieltjes integral, S.S. Dragomir and I. Fedotov introduced in [9] the following error functional $D(f;u) := \int_a^b f(t)\,du(t) - [u(a) - u(b)] \cdot \frac{1}{b-a}\int_a^b f(t)\,dt$ provided the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist. In the same paper, the authors have shown that

$$|D(f;u)| \le \frac{1}{2} \cdot L(M-m)(b-a), \tag{1}$$

provided that u is $L-$Lipschitzian, i.e., $|u(t) - u(s)| \le L|t-s|$ for any $t,s \in [a,b]$ and f is Riemann integrable and bounded below by m and above by M. The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller quantity. In the follow-up paper [10], the same authors established a different result, namely

$$|D(f;u)| \le \frac{1}{2}K(b-a)\bigvee_a^b(u), \tag{2}$$

provided that u is of bounded variation and f is $K-$Lipschitzian with a constant $K > 0$. Here $\frac{1}{2}$ is also best possible.

In [7], by the use of the following representation

$$D(f;u) = \int_a^b \Phi_u(t)\,df(t), \tag{3}$$

where

$$\Phi_u(t) := \frac{1}{b-a}\left[(t-a)u(b) + (b-t)u(a)\right] - u(t), \quad t \in [a,b], \tag{4}$$

the author has established the following inequality as well:

$$|D(f;u)| \le \frac{1}{2}L(b-a)[u(b) - u(a) - K(u)] \le \frac{1}{2}L(b-a)[u(b) - u(a)], \tag{5}$$

where $K(u) := \frac{4}{(b-a)^2}\int_a^b \left(t - \frac{a+b}{2}\right)u(t)\,dt\,(\ge 0)$, u is monotonic nondecreasing and f is $L-$Lipschitzian, and

$$|D(f;u)| \le [u(b) - u(a) - Q(u)] \cdot \bigvee_a^b(f) \le [u(b) - u(a)] \cdot \bigvee_a^b(f), \tag{6}$$

where $Q(u) := \frac{1}{b-a}\int_a^b u(t)\,\mathrm{sgn}\left(t - \frac{a+b}{2}\right)dt\,(\ge 0)$, and f is of bounded variation, The constant $\frac{1}{2}$ in (5) and the first inequality in (6) are sharp.

The main aim of the present paper is to provide other bounds for $D(f;u)$ in the case where the integrator u is $(l,L)-$Lipschitzian (see Definition 1). Natural applications for the Čebyšev functional that complement the classical results due to Čebyšev, Grüss, Ostrowski and Lupaş are also given.

2. SHARP BOUNDS FOR (l,L)-LIPSCHITZIAN INTEGRATORS

We say that a function $v : [a,b] \to \mathbb{R}$ is $K-$Lipschitzian with $K > 0$ if $|v(t) - v(s)| \le K|t-s|$ for any $t,s \in [a,b]$. The following lemma may be stated:

Lemma 1 *Let $u : [a,b] \to \mathbb{R}$ and $l, L \in \mathbb{R}$ with $L > l$. The following statements are equivalent:*

(i) The function $u - \frac{l+L}{2} \cdot e$, where $e(t) = t$, $t \in [a,b]$ is $\frac{1}{2}(L-l)-$Lipschitzian;

(ii) We have the inequalities

$$l \le \frac{u(t)-u(s)}{t-s} \le L \quad \text{for each } t,s \in [a,b] \quad \text{with } t \ne s; \tag{7}$$

(iii) We have the inequalities

$$l(t-s) \le u(t)-u(s) \le L(t-s) \quad \text{for each } t,s \in [a,b] \quad \text{with } t > s. \tag{8}$$

Following [13], we can introduce the definition of (l,L)-Lipschitzian functions:

Definition 1 *The function $u:[a,b]\to\mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) from Lemma 1 is said to be (l,L)-Lipschitzian on $[a,b]$. If $L>0$ and $l=-L$, then $(-L,L)-$Lipschitzian means L-Lipschitzian in the classical sense.*

Theorem 1 *If $u:[a,b]\to\mathbb{R}$ is (l,L)-Lipschitzian on $[a,b]$, then*

$$|\Phi_u(t)| \le \frac{(L-l)(b-t)(t-a)}{b-a} \le \frac{1}{4}(L-l)(b-a) \text{ for each } t\in[a,b]. \tag{9}$$

The inequalities are sharp and the constant $\frac{1}{4}$ is best possible.

Proof: First of all, let us observe that

$$\Phi_u(t) = \Phi_{u-\frac{l+L}{2}e}(t) \quad \text{for each } t\in[a,b]. \tag{10}$$

Now, if $v:[a,b]\to\mathbb{R}$ is $K-$Lipschitzian, then by the definition of Φ_v we have

$$|\Phi_v(t)| \le \frac{(b-t)|v(t)-v(a)|+(t-a)|v(b)-v(t)|}{b-a} \le \frac{2K(b-t)(t-a)}{b-a}, \text{ for any } t\in[a,b]. \tag{11}$$

Now, applying (11) for $v=u-\frac{l+L}{2}e$ which is $\frac{1}{2}(L-l)-$Lipschitzian, we deduce $\left|\Phi_{u-\frac{l+L}{2}e}(t)\right| \le \frac{(L-l)(b-t)(t-a)}{b-a}$, $t\in[a,b]$ which together with (10) produces the first inequality in (9). The second inequality in (9) is obvious.

Consider the function $u:[a,b]\to\mathbb{R}$, $u(t)=\left|t-\frac{a+b}{2}\right|$. Then u is $(-1,1)-$Lipschitzian, $u(a)=u(b)=\frac{b-a}{2}$, $u\left(\frac{a+b}{2}\right)=0$ and introducing these values in (9) for $t=\frac{a+b}{2}$, we obtain an equality with both terms $\frac{1}{2}(b-a)$. □

Corollary 1 *With the assumptions of Theorem 1, we have the inequality:*

$$\left|\frac{u(a)+u(b)}{2}-u\left(\frac{a+b}{2}\right)\right| \le \frac{1}{4}(L-l)(b-a). \tag{12}$$

The constant $\frac{1}{4}$ is best possible.

Theorem 2 *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is (l,L)-Lipschitzian and f is of bounded variation, then*

$$|D(f;u)| \le \frac{1}{4}(L-l)(b-a)\bigvee_a^b(f). \tag{13}$$

The constant $\frac{1}{4}$ is best possible in (13).

Proof: We use the following representation of the Grüss type functional $D(f;u)$ obtained in [7] (see also [5]):

$$D(f;u) = \int_a^b \Phi_u(t)\,df(t). \tag{14}$$

It is well known that if $p:[\alpha,\beta]\to\mathbb{R}$ is continuous and $v:[\alpha,\beta]\to\mathbb{R}$ is of bounded variation, then the Riemann-Stieljtes integral $\int_\alpha^\beta p(t)\,dv(t)$ exists and $\left|\int_\alpha^\beta p(t)\,dv(t)\right| \le \sup_{t\in[\alpha,\beta]}|p(t)|\bigvee_\alpha^\beta(v)$. Applying this property we then have $|D(f;u)| = \left|\int_a^b \Phi_u(t)\,df(t)\right| \le \sup_{t\in[a,b]}|\Phi_u(t)|\bigvee_a^b(f) \le \frac{1}{4}(L-l)(b-a)\bigvee_a^b(f)$ and (13) is obtained.

The sharpness of the constant $\frac{1}{4}$, can be proved on choosing $u,f:[a,b]\to\mathbb{R}$, $u(t)=\left|t-\frac{a+b}{2}\right|$ and $f(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$. The details are omitted. □

Theorem 3 *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is $(l,L)-$Lipschitzian and f is $K-$Lipschitzian on $[a,b]$, then*

$$|D(f;u)| \le \frac{1}{6}K(L-l)(b-a)^2. \tag{15}$$

Proof: It is known that, if $p:[\alpha,\beta]\to\mathbb{R}$ is Riemann integrable and $v:[a,b]\to\mathbb{R}$ is $L-$Lipschitzian, then the Riemann-Stieltjes integral $\int_\alpha^\beta p(t)\,dv(t)$ exists and $\left|\int_\alpha^\beta p(t)\,dv(t)\right| \le L\int_a^b|v(t)|\,dt$. If we apply this property to the integral $\int_a^b \Phi_u(t)\,df(t)$ and use the identity (14), we then have

$$|D(f;u)| = \left|\int_a^b \Phi_u(t)\,df(t)\right| \le K\int_a^b|\Phi_u(t)|\,dt \le \frac{K(L-l)}{b-a}\int_a^b(b-t)(t-a)\,dt = \frac{1}{6}K(L-l)(b-a)^2$$

and the inequality (15) is proved. □

Remark 1 *It is an open problem whether or not the constant $\frac{1}{6}$ is the best possible constant in (15).*

Theorem 4 *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is (l,L) −Lipschitzian and f is monotonic nondecreasing, then*

$$|D(f;u)|\leq 2\cdot\frac{L-l}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\,dt\leq\begin{cases}\frac{1}{2}(L-l)\max\{|f(a)|,|f(b)|\}(b-a);\\ \frac{1}{(q+1)^{\frac{1}{q}}}(L-l)\|f\|_p(b-a)^{\frac{1}{q}} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ (L-l)\|f\|_1,\end{cases}\tag{16}$$

where $\|f\|_p:=\left(\int_a^b|f(t)|^p\,dt\right)^{\frac{1}{p}}$, $p\geq 1$ are the Lebesgue norms. The constants 2 and $\frac{1}{2}$ are best possible in (16).

Proof: It is well known that if $p:[\alpha,\beta]\to\mathbb{R}$ is continuous and $v:[\alpha,\beta]\to\mathbb{R}$ is monotonic nondecreasing, then the Riemann-Stieltjes integral $\int_\alpha^\beta p(t)\,dv(t)$ exists and $\left|\int_\alpha^\beta p(t)\,dv(t)\right|\leq\int_\alpha^\beta|p(t)|\,dv(t)$. Then, on applying this property for the integral $\int_a^b\Phi_u(t)\,df(t)$, we have

$$|D(f;u)|=\left|\int_a^b\Phi_u(t)\,df(t)\right|\leq\int_a^b|\Phi_u(t)|\,df(t)\leq\frac{L-l}{b-a}\int_a^b(b-t)(t-a)\,df(t),\tag{17}$$

where, for the last inequality, we have used the inequality (9).

Integrating by parts in the Riemann-Stieltjes integral, we have $\int_a^b(b-t)(t-a)\,df(t)=2\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\,dt$, which together with (17) produces the first inequality in (16).

The last part follows on utilising the Hölder inequality, namely

$$\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\,dt\leq\begin{cases}\sup_{t\in[a,b]}|f(t)|\int_a^b\left|t-\frac{a+b}{2}\right|dt\\ \left(\int_a^b|f(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_a^b\left|t-\frac{a+b}{2}\right|^q dt\right)^{\frac{1}{q}}\\ \qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \sup_{t\in[a,b]}\left|t-\frac{a+b}{2}\right|\int_a^b|f(t)|\,dt\end{cases}\leq\begin{cases}\frac{1}{4}\max\{|f(a)|,|f(b)|\}(b-a)^2;\\ \frac{1}{2}\cdot\frac{1}{(q+1)^{\frac{1}{q}}}\|f\|_p(b-a)^{1+\frac{1}{q}}\\ \qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{2}\|f\|_1(b-a).\end{cases}$$

□

3. APPLICATIONS FOR THE ČEBYŠEV FUNCTIONAL

For two Lebesgue integrable functions, $f,g:[a,b]\to\mathbb{R}$ with fg an integrable function, consider the *Čebyšev functional* $C(\cdot,\cdot)$ defined by

$$C(f,g):=\frac{1}{b-a}\int_a^b f(t)g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt.\tag{18}$$

In 1934, Grüss [12] showed that

$$|C(f,g)|\leq\frac{1}{4}(M-m)(N-n),\tag{19}$$

provided m,M,n,N are real numbers with the property

$$-\infty<m\leq f\leq M<\infty,\quad -\infty<n\leq g\leq N<\infty\quad\text{a.e. on }[a,b].\tag{20}$$

The constant $\frac{1}{4}$ is best possible in (18) in the sense that it cannot be replaced by a smaller quantity.

Another lesser known inequality, even though it was derived in 1882 by Čebyšev [1], under the assumption that f',g' exist and are continuous in $[a,b]$ is given by

$$|C(f,g)|\leq\frac{1}{12}\|f'\|_\infty\|g'\|_\infty(b-a)^2,\tag{21}$$

where $\|f'\|_\infty:=\sup_{t\in[a,b]}|f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case. We notice that the Čebyšev inequality (21) also holds if $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous on $[a,b]$ and $f',g'\in L_\infty[a,b]$.

In 1970, Ostrowski [15] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results, namely

$$|C(f,g)|\leq\frac{1}{8}(b-a)(M-m)\|g'\|_\infty,\tag{22}$$

provided f satisfies (20) while g is absolutely continuous and $f',g'\in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (22).

Finally, let us recall that in 1973, Lupaş [14], proved the following inequality in terms of the Euclidean norm:

$$|C(f,g)|\leq\frac{1}{\pi^2}(b-a)\|f'\|_2\|g'\|_2,\tag{23}$$

provided that f,g are absolutely continuous and $f',g'\in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is best possible.

For other results on the Čebyšev functional, see [2], [3], [5], [6], [8] and [11].

Now, assume that $g:[a,b]\to\mathbb{R}$ is Lebesgue integrable on $[a,b]$ and $-\infty<m\le g(t)\le M<\infty$ for a.e. $t\in[a,b]$. Then the function $u(t):=\int_a^t g(s)\,ds$ is (m,M) −Lipschitzian on $[a,b]$ and

$$\tilde{\Phi}_g(t):=\Phi_u(t)=\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds,\quad t\in[a,b]. \tag{24}$$

On utilising the Theorem 1 we can state the following result that provides a sharp bound for $\tilde{\Phi}_g(t)$ in (24).

Proposition 1 *If $g:[a,b]\to\mathbb{R}$ is Lebesgue integrable on $[a,b]$ and $-\infty<m\le g(s)\le M<\infty$ for a.e. $s\in[a,b]$, then*

$$\left|\tilde{\Phi}_g(t)\right|\le\frac{(M-m)(b-t)(t-a)}{b-a}\le\frac{1}{4}(M-m)(b-a), \tag{25}$$

for a.e. $t\in[a,b]$. The first inequality is sharp. The constant $\frac{1}{4}$ is best possible.

The inequality is obvious by (13). The sharpness follows on choosing $t=\frac{a+b}{2}$ and $g(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$ in (25). The details are omitted.

The following result for the Čebyšev functional can be stated:

Proposition 2 *If $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is Lebesgue integrable and satisfies the bounds*

$$-\infty<m\le g\le M<\infty\quad \text{a.e. on } [a,b], \tag{26}$$

then

$$|C(f,g)|\le\frac{1}{4}(M-m)\bigvee_a^b(f). \tag{27}$$

The constant $\frac{1}{4}$ is best possible.

The following result can be stated as well.

Proposition 3 *Assume that $g:[a,b]\to\mathbb{R}$ is as in Proposition 2. If $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then*

$$|C(f,g)|\le 2\cdot\frac{(M-m)}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\,dt\le\begin{cases}\frac{1}{2}(M-m)\max\{|f(a)|,|f(b)|\};\\ \frac{1}{(q+1)^{\frac{1}{q}}}(M-m)\|f\|_p(b-a)^{-\frac{1}{p}} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ (M-m)\frac{1}{b-a}\|f\|_1.\end{cases} \tag{28}$$

The constants 2 and $\frac{1}{2}$ are best possible.

The proof of the inequalities in (28) are obvious from (16). The sharpness of the constants follows on choosing $f(t)=g(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$, $t\in[a,b]$.

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