Multidimensional Integration via Dimension Reduction and Generators

P. Cerone

School of Comp. Sci. & Math., Victoria University, PO Box 14428, Melbourne City, VIC 8001, Australia.

Abstract. An iterative approach is used to represent multidimensional integrals in terms of lower dimensional integrals and function evaluations. The procedure is quite general utilising one dimensional identities as the *generator* to procure multidimensional identities. Bounds are obtained from the identities. Both weighted and unweighted integrals are considered.

Keywords: Multidimensional integrals, Dimension Reduction, Recursive method, Ostrowski, trapezoidal and three point generators, a *priori* bounds.
PACS: 02.60.-x

1. INTRODUCTION

We firstly present one-dimensional identities which may be used as *generators* for higher dimensional results.

For $f:[a,b]\to\mathbb{R}$ we define the Ostrowski and Trapezoidal functionals by

$$\mathscr{S}(f;c,x,d):=f(x)-\mathscr{M}(f;c,d) \tag{1}$$

and

$$T(f;c,x,d):=\left(\frac{x-c}{d-c}\right)f(c)+\left(\frac{d-x}{d-c}\right)f(d)-\mathscr{M}(f;c,d), \tag{2}$$

respectively, where

$$\mathscr{M}(f;c,d):=\frac{1}{d-c}\int_c^d f(u)\,du,\ \text{the integral mean.} \tag{3}$$

The following identities may be easily shown to hold for f of bounded variation, by an integration by parts argument of the Riemann-Stieltjes integrals and so

$$\mathscr{S}(f;c,x,d)=\int_c^d p(x,t,c,d)\,df(t),\ \ p(x,t,c,d)=\begin{cases}\frac{t-c}{d-c}, & t\in[c,x]\\ \frac{t-d}{d-c}, & t\in(x,d]\end{cases} \tag{4}$$

and

$$T(f;c,x,d)=\int_c^d q(x,t,c,d)\,df(t),\ \ q(x,t,c,d)=\frac{t-x}{d-c},\ x,t\in[c,d]. \tag{5}$$

The book [10] is devoted to Ostrowski type results involving (1) and numerous generalisations. See also [1], [11] and [14].

Further, define the three point functional $\mathfrak{T}(f;a,\alpha,x,\beta,b)$ which involves the difference between the integral mean and, a weighted combination of a function evaluated at the end points and an interior point. Namely, for $a\le\alpha<x<\beta\le b$,

$$\mathfrak{T}(f;a,\alpha,x,\beta,b):=\left(\frac{\alpha-a}{b-a}\right)f(a)+\left(\frac{\beta-\alpha}{b-a}\right)f(x)+\left(\frac{b-\beta}{b-a}\right)f(b)-\mathscr{M}(f;a,b), \tag{6}$$

which for f of bounded variation, the identity

$$\mathfrak{T}(f;a,\alpha,x,\beta,b)=\int_a^b r(x,t)\,df(t),\ \ r(x,t)=\begin{cases}\frac{t-\alpha}{b-a}, & t\in[a,x]\\ \frac{t-\beta}{b-a}, & t\in(x,b]\end{cases} \tag{7}$$

may easily be shown to be valid.

Further, if $f(t)$ is assumed to be absolutely continuous for t over its respective interval, then $df(t) = f'(t)\,dt$ and the Riemann-Stieltjes integrals in (4), (5) and (7) are equivalent to Riemann integrals.

In the current work, weighted *generators* are used to obtain identities involving multidimensional integrals. The identities allow *a priori* bounds on the error. Ostrowski generators are utilised to procure multidimensional results. For applications to weighted trapezoidal and three point generators, the reader is referred to the full paper, [6]. The results of Cerone [3] and [4] are recaptured if the weights are taken to be identically one.

2. WEIGHTED MULTIDIMENSIONAL OSTROWSKI IDENTITIES AND BOUNDS FROM AN ITERATIVE APPROACH

The following theorem uses an iterative approach to extend a weighted Ostrowski functional identity to multidimensions. Firstly, we will require some notation.

Let $I^n = \prod_{i=1}^n [a_i, b_i] = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$. Further, let $f : I^n \to \mathbb{R}$ and define operators $F_i(f)$ and $\lambda_{i,w_i}(f)$ by

$$F_i(f) := f(t_1, \ldots, t_{i-1}, x_i, t_{i+1}, \ldots, t_n) \text{ where } x_i \in [a_i, b_i] \tag{8}$$

and

$$\lambda_{i,w_i}(f) := \frac{1}{W_i} \int_{a_i}^{b_i} w_i(t_i)\, f(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n)\, dt_i, \tag{9}$$

where $w_i(t_i)$ are positive weight functions for $t_i \in [a_i, b_i]$, $i = 1, 2, \ldots, n$ satisfying

$$W_i = \int_{a_i}^{b_i} w_i(t_i)\, dt_i > 0. \tag{10}$$

That is, $F_i(f)$ evaluates $f(\cdot)$ in the ith variable at $x_i \in [a_i, b_i]$ and $\lambda_{i,w_i}(f)$ is the weighted integral mean of $f(\cdot)$ in the ith variable. Assuming that $f(\cdot)$ is absolutely continuous in the ith variable $t_i \in [a_i, b_i]$, we have

$$\mathscr{L}_{i,w_i}(f) = \frac{1}{W_i} \int_{a_i}^{b_i} P_i(x_i, t_i) \frac{\partial f}{\partial t_i} dt_i = (F_i - \lambda_{i,w_i})(f), \tag{11}$$

for $i = 1, 2, \ldots, n$, where

$$\frac{P_i(x_i, t_i)}{W_i} = \begin{cases} \frac{\int_{a_i}^{t_i} w_i(s) ds}{W_i}, & t_i \in [a_i, x_i] \\ \frac{-\int_{t_i}^{b_i} w_i(s) ds}{W_i}, & t_i \in (x_i, b_i]. \end{cases} \tag{12}$$

Thus (11) – (12) is ostensibly equivalent to a weighted Montgomery identity which reduces to (4) for $w_i(t_i) \equiv 1$ and $f(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n)$ absolutely continuous with $t_i \in [a_i, b_i]$.

Theorem 1 *Let $f : I^n \to \mathbb{R}$ be absolutely continuous in such a manner that the partial derivatives of order one with respect to every variable exist. Then*

$$\begin{aligned} E_n(f) \quad = \quad & f(x_1, x_2, \ldots, x_n) - \sum_{i=1}^n \frac{1}{W_i} \int_{a_i}^{b_i} w_i(t_i)\, f(x_1, x_2, \ldots, x_{i-1}, t_i, x_{i+1}, \ldots, x_n)\, dt_i \\ & + \sum_{i<j}^n \frac{1}{W_i W_j} \int_{a_j}^{b_j} \int_{a_i}^{b_i} w_i(t_i)\, w_j(t_j)\, f(x_1, \ldots, x_{i-1}, t_i, x_{i+1}, \ldots, t_j, \ldots, x_n)\, dt_i dt_j \\ & - \cdots - \frac{(-1)^n}{W^*} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \prod_{i=1}^n w_i(t_i)\, f(t_1, \ldots, t_n)\, dt_1 \ldots dt_n := \tau_n(\mathbf{a}, \mathbf{x}, \mathbf{b}), \end{aligned} \tag{13}$$

where

$$E_n(f) = \frac{1}{W^*} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \prod_{i=1}^n P_i(x_i, t_i) \frac{\partial^n f}{\partial t_n \ldots \partial t_1} dt_1 \ldots dt_n, \tag{14}$$

$$W^* = \prod_{i=1}^n W_i, \tag{15}$$

with W_i given by (10) and $P_i(x_i, t_i)$ is given by (12).

Remark 1 *The result given by (13) may be utilised to approximate the weighted n− dimensional integral in terms of lower dimensional integrals and a function evaluation* $f(x_1,x_2,\ldots,x_n)$ *where* $x_i \in [a_i,b_i]$, $i=1,2,\ldots,n$. *Specifically, there are* $\binom{n}{0}$ *function evaluations,* $\binom{n}{1}$ *single integral evaluations in each of the axes,* $\binom{n}{2}$ *double integral evaluations and so on, and, of course,* $\binom{n}{n}$ *n−dimensional integral evaluations. This results from the fact that from (8) – (11)*

$$E_n(f) = \left(\prod_{i=1}^{n} \mathscr{L}_{i,w_i}\right)(f) = \left(\prod_{i=1}^{n} (F_i - \lambda_{i,w_i})\right)(f). \tag{16}$$

The above procedure of utilising a one-dimensional identity as the *generator* to recursively obtain a multidimensional identity which is quite general, may be extended to utilising other one-dimensional identities.

Theorem 2 *Let* $f : I^n \to \mathbb{R}$ *be absolutely continuous in a manner that the partial derivatives of order one with respect to every variable exist. Then*

$$W^* |\tau_n(\mathbf{a},\mathbf{x},\mathbf{b})| \le \begin{cases} \prod_{i=1}^{n}\left(\int_{a_i}^{b_i} |x_i - t_i| w_i(t_i)\,dt_i\right)\left\|\frac{\partial^n f}{\partial t_n \ldots \partial t_1}\right\|_\infty, & \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \in L_\infty[I^n]; \\ \left(\prod_{i=1}^{n} P_i(q)\right)^{\frac{1}{q}} \left\|\frac{\partial^n f}{\partial t_n \ldots \partial t_1}\right\|_p, & \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \in L_p[I^n], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \prod_{i=1}^{n} \theta_i \left\|\frac{\partial^n f}{\partial t_n \ldots \partial t_1}\right\|_1, & \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \in L_1[I^n], \end{cases} \tag{17}$$

where $\tau_n(\mathbf{a},\mathbf{x},\mathbf{b})$ *is as defined by (13),*

$$P_i(q) = \int_{a_i}^{x_i}\left(\int_{a_i}^{t_i} w_i(s)\,ds\right)^q dt_i + \int_{x_i}^{b_i}\left(\int_{t_i}^{b_i} w_i(s)\,ds\right)^q dt_i, \tag{18}$$

$$\theta_i = \frac{1}{2}\int_{a_i}^{b_i} w_i(s)\,ds + \frac{1}{2}\left|\int_{a_i}^{x_i} w_i(s)\,ds - \int_{x_i}^{b_i} w_i(s)\,ds\right|. \tag{19}$$

Remark 2 *The expression for* $\tau_n(\mathbf{a},\mathbf{x},\mathbf{b})$ *may be written in a less explicit form which is perhaps more appealing. Namely,*

$$\tau_n(\mathbf{a},\mathbf{x},\mathbf{b}) = f(x_1,x_2,\ldots,x_n) + \sum_{k=1}^{n-1}(-1)^k \sum_k \mathscr{M}_k + (-1)^n \mathscr{M}_n, \tag{20}$$

where $\mathscr{M}_k$ *represents the integral means in k variables with the remainder being evaluated at their respective interior point and* $\sum_k \mathscr{M}_k$ *is a sum over all* $\binom{n}{k}$, *k−dimensional integral means. Here*

$$\mathscr{M}_n = \frac{1}{W^*}\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\prod_{i=1}^{n} w_i(t_i)\, f(t_1,\ldots,t_n)\,dt_1\ldots dt_n$$

and

$$\begin{aligned}\sum_1 \mathscr{M}_1 \quad = \quad & \frac{1}{W_1}\int_{a_1}^{b_1} w_1(t_1)\, f(t_1,x_2,\ldots,x_n)\,dt_1 + \frac{1}{W_2}\int_{a_2}^{b_2} w_2(t_2)\, f(x_1,t_2,x_3,\ldots,x_n)\,dt_2 \\ & + \cdots + \frac{1}{W_n}\int_{a_n}^{b_n} w_n(t_n)\, f(x_1,x_2,\ldots,x_{n-1},t_n)\,dt_n.\end{aligned}$$

It should be noted that (20) may be written as

$$\tau_n(\mathbf{a},\mathbf{x},\mathbf{b}) = \sum_{k=0}^{n}(-1)^k \sum_k \mathscr{M}_k \tag{21}$$

if we define the degenerate 0th integral mean $\mathscr{M}_0 = f(x_1,x_2,\ldots,x_n)$.

Remark 3 *For proofs of the above results, the reader is referred to Cerone [6]. Cerone [3] using an iterative approach from using the Montgomery identity as a* ***generator*** *obtained an unweighted version of the results in Theorems 1 and*

2 for which the tightest bounds occur when we choose to sample at the mid-points of the respective intervals, namely, for $x_i = \frac{a_i+b_i}{2}$*. This is not the case for the weighted results depicted in Theorem 2. If we let*

$$m(c,d) = \int_c^d w(s)\,ds \quad \text{and} \quad M(c,d) = \int_c^d s w(s)\,ds.$$

then we observe that, for example,

$$\int_a^b |x-t|\,w(t)\,dt = \int_a^x (x-t)\,w(t)\,dt + \int_x^b (t-x)\,w(t)\,dt = x\,[m(a,x) - m(x,b)] + M(x,b) - M(a,x)\,.$$

Thus we see that the bound is simplified, although not necessarily globally minimised, at the median $x = x^*$*, where* $m(a,x^*) = m(x^*,b)$*.*

In Cerone [6], the bounds resulting from multidimensional integrals using weighted trapezoidal generators are shown to be similar to those presented in Theorem 2 above, emanating from Ostrowski. This was shown to be true in general for one dimensional integrals in Cerone [5].

3. CONCLUDING REMARKS

Weighted rules of Ostrowski type have been investigated in the current work as generators for multidimensional integration. This results in product form weight functions in the multidimensional integral. The current Ostrowski-based generators may be developed for weighted trapezoidal and three point generators and the reader is referred to Cerone [6] for futher details. The procedure developed in [3] and [4] may also be used to include higher order formulae involving the behaviour of higher derivatives for its bounds. Multidimensional results based on an *m* branched Peano kernel producing function evaluations at $m+1$ points are also possible using the methodology. Finally, we are not restricted to using the same identity in each of the directions but may use different ones as long as we are able to justify this.

REFERENCES

1. G.A. Anastassiou, Ostrowski type inequalities, *Proc. Amer. Math. Soc.,* **123**(12) (1995), 3775-3781.
2. N.S. Barnett and S.S. Dragomir, An Ostrowski type inequality for double integrals and applications for cubature formulae, *Soochow J. Math*., **27**(1) (2001), 1-10.
3. P. Cerone, Approximate multidimensional integration through dimension reduction via the Ostrowski functional, *Nonlinear Funct. Anal. & Applics.,* **8**(3) (2003), 313-333.
4. P. Cerone, Multidimensional integration via trapezoidal and three point generators, *J. Korean Math. Soc.,* **40**(2) (2000), 251-272.
5. P. Cerone, On relationships between Ostrowski, trapezoidal and Chebychev identities and inequalities, *Soochow J. Math*., **28**(3) (2002), 311-328.
6. P. Cerone, Multidimensional integration via dimension reduction, *RGMIA Res. Rep. Coll.,* **10**(Supp.) (2007), Article 9. [ONLINE] `http://rgmia.vu.edu.au/v10(E).html`
7. P. Cerone and S.S. Dragomir, On some inequalities arising from Montgomery's identity, *J. of Computational Analysis and Applications,* **5**(4) (2003), 341-367.
8. P. Cerone and S.S. Dragomir, Midpoint type rules from an inequalities point of view, *Handbook of Analytic-Computational Methods in Applied Mathematics,* G.A. Anastassiou (Ed), CRC Press, New York (2000), 135-200.
9. P. Cerone, S.S. Dragomir and J. Roumeliotis, Some Ostrowski type inequalities for $n-$time differentiable mappings and applications, *Demonstratio Mathematica,* **32**(4) (1999), 133-138.
10. S.S. Dragomir and T.M. Rassias (Ed.), *Ostrowski type inequalities and applications in numerical integration*, In press, Kluwer Academic Publishers.
11. A.M. Fink, Bounds on the derivation of a function from its averages, *Czech. Math. Journal,* **42** (1992), No. 117, 289-310.
12. G. Hanna, P. Cerone and J. Roumeliotis, An Ostrowski type inequality in two dimensions using the three point rule, *ANZIAM*, **42**(E) (2000), C671-C689.
13. G. Hanna, S.S. Dragomir and P. Cerone, A general Ostrowski type inequality for double integrals, *Tamkang J. Math.,* **33**(4) (2002), 319-333.
14. A. Ostrowski, Uber die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel,* **10** (1938), 226-227.