A geometric mean in the Furuta inequality

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First of all, we cite the Furuta inequality [3]:

The Furuta inequality
If $A \geq B \geq 0$, then for each $r \geq 0$,

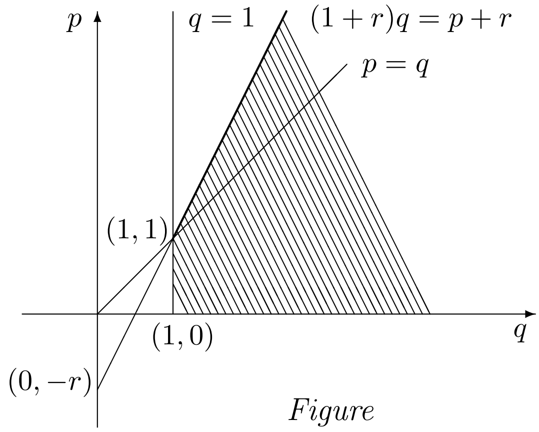
(i) $$\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(B^{\frac{r}{2}} B^p B^{\frac{r}{2}}\right)^{\frac{1}{q}}$$

and

(ii) $$\left(A^{\frac{r}{2}} A^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with

$$(1+r)q \geq p+r.$$



Figure

Afterwards, Ando [1] proposed a variant of the Furuta inequality, which is extended to a two variable version as follows:

For $A, B > 0$, $A \gg B$, i.e., $\log A \geq \log B$, if and only if

$$\left(A^{\frac{r}{2}} A^p A^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \geq \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{r}{p+r}}$$

It is represented in terms of the monotonicity of an operator function in the following way, [2]

Theorem A *For $A, B > 0$, $A \gg B$ if and only if for each $s \geq 0$, $F(t,r) = A^{-r} \sharp_{\frac{s+r}{t+r}} B^t$ is an increasing function of both $t \geq s$ and $r \geq 0$, where $\sharp_\alpha$ is the α-geometric mean.*

Recently Uchiyama [5] discussed some extensions of the Furuta inequality by using the operator means established by Kubo-Ando. For this, he paid his attention to the Jensen inequality for operator concave functions.

Theorem B *If $A \le B \,!_\mu\, C$ for $A, B, C > 0$, then*

$$B^s \nabla_\mu C^s \le A^{-r} \sharp_{\frac{s+r}{t+r}} (B^t \nabla_\mu C^t)$$

for $r \ge 0$ and $t \ge s \ge 0$, where $!_\mu$ and ∇_μ are μ-harmonic and arithmetic means respectively.

Very recently, we found the following result in [4] which is based on Theorem A.

Theorem C *Suppose that $A, B, C > 0$ and $r, s \ge 0$. If $A^t \ll B^t \nabla_\mu C^t$ for all $t \ge 0$, then*

$$f(t) = A^{-r} \sharp_{\frac{s+r}{t+r}} (B^t \nabla_\mu C^t)$$

is an increasing function of $t \ge s$. On the other hand, if $A^t \ll B^t \,!_\mu\, C^t$ for all $t \ge 0$, then

$$h(t) = A^{-r} \sharp_{\frac{s+r}{t+r}} (B^t \,!_\mu\, C^t)$$

is a decreasing function of $t \ge s$.

In this talk, we discuss Theorem C and related inequalities. We begin with the following lemma.

Lemma 1 *For $B, C > 0$ and $\mu \in [0, 1]$, $\log(B^t \nabla_\mu C^t)^{1/t}$ converges to $\mu \log B + (1-\mu)\log C$ decreasingly as $t \searrow 0$. Consequently there exists*

$$s - \lim (B^t \nabla_\mu C^t)^{1/t} = e^{\mu \log B + (1-\mu)\log C}.$$

Definition 1 *For $B, C > 0$ and $\mu \in [0, 1]$,*

$$B \diamondsuit_\mu C = e^{\mu \log B + (1-\mu)\log C}$$

is said to be the μ-chaotically geometric mean of B and C.

Theorem 2 *For $B, C > 0$ and $\mu \in [0,1]$, the following statements are mutually equivalent:*

(1) $A \ll B \diamondsuit_\mu C$.

(2) $B^s \nabla_\mu C^s \leq A^{-r} \natural_{\frac{s+r}{t+r}} (B^t \nabla_\mu C^t)$ for $r \geq 0$ and $t \geq s \geq 0$.

(3) For each $r, s \geq 0$, $f(t) = A^{-r} \natural_{\frac{s+r}{t+r}} (B^t \nabla_\mu C^t)$ is an increasing function of $t \geq s$.

Related to Theorem B, we have the following results.

Theorem 3 *Suppose that $A, B, C > 0$ satisfy $A \ll (B^{t_0} \nabla_\mu C^{t_0})^{1/t_0}$ for some t_0. If $t_0 \geq 0$, then*

$$B^s \nabla_\mu C^s \leq A^{-r} \natural_{\frac{s+r}{t+r}} (B^t \nabla_\mu C^t)$$

for all $r \geq 0$ and $t \geq s \geq 0$ with $t \geq t_0$. On the other hand, if $t_0 < 0$, then

$$(B^t \,!_\mu\, C^t)^{\frac{s}{t}} \leq A^{-r} \natural_{\frac{s+r}{t+r}} (B^t \,!_\mu\, C^t)$$

for all $r \geq 0$ and $-t_0 \geq t \geq s \geq 0$.

References

[1] T.Ando, *On some operator inequalities*, Math. Ann., 279(1987), 157-159.

[2] M.Fujii, T.Furuta and E.Kamei, *Furuta's inequality and application to Ando's theorem*, Linear Alg. and Appl., 179(1993), 161-169.

[3] T.Furuta, *$A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$*, Proc. Amer. Math. Soc., 101(1987), 85-88.

[4] T.Furuta and E.Kamei, *An extension of Uchiyama's result associated with an order preserving operator inequality*, preprint.

[5] E.Kamei, *A satellite to Furuta's inequality*, Math. Japon., 33(1988), 883-886.

[6] M.Uchiyama, *An operator inequality related to Jensen's inequality*, preprint.