Some Generalizations of Ostrowski Inequalities and Their Applications to Numerical Integration and Special Means

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Dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in Mathematics

Centre for Advanced Studies
in Pure and Applied Mathematics,
Bahauddin Zakariya University,
Multan, Pakistan.

JULY 2010

Acknowledgements

All Thanks to Almighty Allah Who blessed me with the ability to do this work and Hazrat Muhammad (Peace be upon him) who gave us the spirit to learn. It is the blessings of Allah and His Prophet (Peace be upon him) that have enabled me to make my efforts a success.

During the last few years of my Ph. D. studies, I have interacted with so many people who have greatly facilitated my academic career; I am greatly indebted to all of them since with their sincere and utmost support I have been able to make my aspirations a reality.

A teacher undoubtedly plays an important role in the life of a student. I am greatly thankful to my principle supervisor Prof. Dr. Nazir Ahmad Mir for granting me sincerest and supreme guidance at every stage which evoked in me an insight to carry out my research work efficiently. Without his affectionate, generous and distinguished supervision during the research work and in compilation of this dissertation, I would not have been able to complete this work.

I am highly obliged to my second supervisor Dr. Nusrat Yasmin for her valuable instructions and supervision in preparation of this dissertation. Her help and cooperation should specially be mentioned and appreciated.

I am very thankful to Prof. Dr. Muhammad Anwar Chaudhry, Director CASPAM, Bahauddin Zakariya University, Multan for providing me with all the research facilities and a healthy research environment in the Centre; for endowing me with his sincere advices in all the official and academic matters.

Special thanks to Prof. Dr. Muhammad Anwar Kamal for his continuous encouragement during my postgraduate studies at the university. He was always willing to take time to provide me with his valuable advices and suggestions on all the academic matters.

My thanks are for all the teachers of CASPAM for their time to time exchange of knowledge enhancing ideas and healthy discussions. They provided me with all the opportunities to seek their guidance and assistance.

I am wholeheartedly thankful to all of my sincere friends especially to Nikhat Bano and Tahira Nawaz who were always there to motivate and help me. I wish them best of luck for their Ph. D. studies and in all the tasks lying ahead of them.

I have a lot of gratitude towards all my Ph. D. fellows principally to Mr. Muhammad Athar Kharal, Mr. Muhammad Fahim, Mr. Muhammad Raza, Mr. Siffat Hussain and Mr. Farooq Ahmad for their good wishes and fellowship.

Acknowledgements are gratefully extended to Higher Education Commission of Pakistan for its financial support and for creating research culture in Pakistan. I would also like to express my gratitude towards the authorities of Bahauddin Zakariya University, Multan for granting me study leave during the entire period of my studies.

I owe special gratitude to all dear ones of my family; my grandmother, my sisters, my little nieces; Rida, Mahnoor, Maryam and Mehreen for their love and prayers and most importantly to my parents & elder sister. Abu, Ammi and Shehla ji without your backing and support in the times of despair and hardship I would not be where I am today. Before signing off, I feel it necessary to express my heartfelt love and adoration to my loving father (Late Zafar Ahmed Siraj) for endowing me with the confidence and belief to achieve every success in life.

Fiza Zafar

Abstract

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Submitted to the Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan on March 05, 2009 in partial fulfillment of the requirement for the degree of Doctor in Philosophy of Mathematics.

Keywords: Ostrowski inequality, Grüss inequality, Čebyšev inequality, Numerical Integration, Special Means, Random variable, Probability Density Functions, Cumulative Distribution Function, Nonlinear Equations, Iterative Methods.

2000 Mathematics Subject Classification: 26D10; 26D15; 26D20; 41A55; 60E15; 34A34; 26C10; 65H05.

In the last few decades, the field of mathematical inequalities has proved to be an extensively applicable field. It is applicable in the following manner:

- Integral inequalities play an important role in several other branches of mathematics and statistics with reference to its applications.
- The elementary inequalities are proved to be helpful in the development of many other branches of mathematics.

The development of inequalities has been established with the publication of the books by G. H. Hardy, J. E. Littlewood and G. Polya [47] in 1934, E. F. Beckenbach and R. Bellman [13] in 1961 and by D. S. Mitrinović, J. E. Pečarić and A. M. Fink [64] & [65] in 1991. The publication of later has resulted to bring forward some new integral inequalities involving functions with bounded derivatives that measure bounds on the deviation of functional value from its mean value namely, Ostrowski inequality [69]. The books by D. S. Mitrinović, J. E. Pečarić and A. M. Fink have also brought to focus integral inequalities which establish a connection between the integral of the product of two functions and the product of the integrals of the two functions namely, inequalities of Grüss [46] and Čebyšev type (see [64], p. 297).

These type of inequalities are of supreme importance because they have immediate applications in Numerical integration, Probability theory, Information theory and Integral operator theory. The monographs presented by S. S. Dragomir and Th. M. Rassias [36] in 2002 and by N. S. Barnett, P. Cerone and S. S. Dragomir [8] in 2004 can well justify this statement. In these monographs, separate aspects of applications of inequalities of Ostrowski-Grüss and Čebyšev type were established.

The main aim of this dissertation is to address the domains of establishing inequalities of Ostrowski-Grüss and Čebyšev type and their applications in Statistics, Numerical integration and Non-linear analysis. The tools that are used are Peano kernel approach, the most classical and extensively used approach in developing such integral inequalities, Lebesgue and Riemann-Stieltjes integrals, Lebesgue spaces, Korkine's identity [52], the classical Čebyšev functional, Pre-Grüss and Pre-Čebyšev inequalities proved in [60].

This dissertation presents some generalized Ostrowski type inequalities. These inequalities are being presented for nearly all types of functions i.e., for higher differentiable functions, bounded functions, absolutely continuous functions, (l, L)-Lipschitzian functions, monotonic functions and functions of bounded variations. The inequalities are then applied to composite quadrature rules, special means, probability density functions, expectation of a random variable, beta random variable and to construct iterative methods for solving non-linear equations.

The generalizations to the inequalities are obtained by introducing arbitrary parameters in the Peano kernels involved. The parameters can be so adjusted to recapture the previous results as well as to obtain some new estimates of such inequalities.

The Ostrowski type inequalities for twice differentiable functions have been extensively addressed by N. S. Barnett et al. and Zheng Liu in [9] and [59]. We have presented some perturbed inequalities of Ostrowski type in $L_p(a, b)$, $p \geq 1, p = \infty$ which generalize and refine the results of [9] and [59].

In the past few years, Ostrowski type inequalities are developed for functions in higher spaces i.e., for L-Lipschitzian functions and (l, L)-Lipschitzian functions. We, in here, have obtained Ostrowski type inequality for n- differentiable (l, L)-Lipschitzian functions, a generalizations of such inequalities for L-Lipschitzian functions and (l, L)-Lipschitzian functions.

The first inequality of Ostrowski-Grüss type was presented by S. S. Dragomir and S. Wang in [39]. In this dissertation, some improved and generalized Ostrowski-Grüss type inequalities are further generalized for the first and twice differentiable functions in $L_2(a,b)$. Some generalizations of Ostrowski-Grüss type inequality in terms of upper and lower bounds of the first and twice differentiable functions are also given. The inequalities are then applied to probability density functions, special means, generalized beta random variable and composite quadrature rules.

In the recent past, many researchers have used Čebyšev type functionals to obtain some new product inequalities of Ostrowski-, Čebyšev-, and Grüss type. We, in here, have also taken into account this domain to present some generalizations and improvements of such inequalities. The generalizations are obtained for first differentiable absolutely continuous functions with first derivatives in $L_p(a,b)$, $p > 1$ and for twice differentiable functions in $L_\infty(a,b)$. A product inequality is also given for monotonic non-decreasing functions. The inequalities are then applied to the expectation of a random variable.

In [3], G. A. Anastassiou has extended Čebyšev-Grüss type inequalities on $\mathbb{R}^N$ over spherical shells and balls. We have extended this inequality for n-dimensional Euclidean space over spherical shells and balls on $L_p[a,b]$, $p > 1$.

Some weighted Ostrowski type inequalities for a random variable whose probability density functions belong to $\{L_p(a,b), p = \infty, p > 1\}$ are presented as weighted extensions of the results of [10] and [33]. Ostrowski type inequalities are also applied to obtain various tight bounds for the random variables defined on a finite intervals whose probability density functions belong to $\{L_p(a,b), p = \infty, p > 1\}$.

This dissertation also describes the applications of specially derived Ostrowski type inequalities to obtain some two-step and three-step iterative methods for solving non-linear equations.

Some Ostrowski type inequalities for Newton-Cotes formulae are also presented in a generalized or optimal manner to obtain one-point, two-point and four-point Newton-Cotes formulae of open as well as closed type.

The results presented here extend various inequalities of Ostrowski type upto their year of publication.

Contents

List of Tables

Chapter 1

Introduction

G. H. HARDY, J. E. LITTLEWOOD and G. POLYA in their book titled "INEQUALITIES", in the preface to the first edition of 1934, say:

". ., Historical and bibliographical questions are particularly troublesome in a subject like this, which has applications in every part of mathematics but has never been developed systematically. It is often really difficult to trace the origin of a familiar inequality. It is quite likely to occur first as an auxiliary proposition, often without explicit statement, in a memoir on geometry or astronomy; it may have been rediscovered. Many years later, by half a dozen different authors; and no accessible statement of it may be quite complete... . We have done our best to be accurate and have given all references we can, but we have never undertaken systematic bibliographical research. We follow the common practice, when a particular inequality is habitually associated with a particular mathematician's name; we speak of the inequalities of Schwartz, Hölder and Jensen, though all these inequalities can be traced further back; . . . "

In 1938, a Ukrainian mathematician Alexander Markowich Ostrowski (1893-1986) discovered an inequality through his paper [69]. Since then this inequality is stated after the name of A. Ostrowski as Ostrowski inequality. Following the subsequent idea of general and particular inequalities as given by A. M. Fink in his essay "On history of inequalities", this is a particular inequality which holds for a class of functions with bounded derivatives. This inequality is recalled and addressed several times in many books and research papers in view of its generalizations and refinements, yet we would say that a comprehensive overview is still required to follow the course of its advancements systematically.

We, in the following section would attempt to bring to light the history and advancements of Ostrowski Inequality.

1.1 Ostrowski type inequalities- a historical overview

Inequalities have proved to be an exalted and applicable tool for the development of many branches of Mathematics. It's importance has increased noticeably during the past few decades and it is now dealt as an independent branch of Mathematics. Many research groups are working to create an awareness of the theory of inequalities and their applicability in sciences, e.g., Research Group of Mathematical Inequalities and Applications (RGMIA). This field is active and experiencing a tremendous boost with the passage of time in theory as well as in applications. One element that particularly intensifies its importance is its applications in various fields. Uptill now, a vast number of research papers and books have been dedicated to inequalities and their numerous applications. The theory being presented through this literature has not only brought forward some new results but it would also be helpful in creating new insights in the years to come.

This subject paved its way towards attracting attention since the publication of the classical books by G. H. Hardy, J. E. Littlewood and G. Polya [47] (1934), E. F. Beckenbach and R. Bellman [13] (1961), D. S. Mitrinović [66] (1970) and by D. S. Mitrinović, J. E. Pečarić and A. M. Fink [64] (1991). These monographs covered comprehensive literature of the classical and new inequalities upto their year of publication followed by a series of monographs by G. V. Milovanović [41] and B. G. Pachpatte [75] and by D. Bainov and P. Simeonov [7] on Integral inequalities and their applications.

Integral inequalities that establish bounds on the physical quantities are of supreme importance in the sense that these types of inequalities are not only useful in nonlinear analysis, numerical integration, approximation theory, probability theory, stochastic analysis, statistics, information theory, and integral operator theory but also have applications in the areas of physics, technology and biological sciences.

Among this type, there are many inequalities measuring the deviation of the average of a function over an interval from a linear combination of the values of the function and some of its derivatives. A chapter namely, "Integral inequality involving functions with bounded derivatives" on similar type of inequalities was presented by D. S. Mitrinović et al. [65] (1991) in their book which had drawn the attention of the research world towards a special domain of the theory of inequalities “Ostrowski Inequality” which estimates the deviation of the values of a function from its mean value. The chapter was based on the classical papers by A. Ostrowski [69], G. V. Milovanović and J. E. Pecarić [62] and A. M. Fink [42]. These research papers had in the true sense laid down the foundation stone of the further development of Ostrowski type inequalities which would be defined in the sequel.

Ostrowski type inequalities add up to the literature of inequalities in the sense that they have immediate applications in Numerical Integration and Probability Theory. In 1998, S. S. Dragomir and S. Wang [40] presented a new proof to the classical Ostrowski's inequality and for the first time applied it to the estimation of error bounds for some special means and for some numerical quadrature rules. It is with the same viewpoint, the two monographs [36, 8] were written in 2002 and 2004 by the members of RGMIA to present some selected results on Ostrowski type inequalities and their applications. In [36], one may find results for univariate and multivariate real functions and their natural applications in the error analysis of numerical quadrature for both simple and multiple integrals as well as for the Riemann-Stieltjes integral and the intention of [8] was to establish applications in Probability Theory & Statistics by obtaining various tight bounds for the expectation, variance and moments of continuous random variables defined over a finite interval as an evident application of Ostrowski type inequalities.

In the last few years, the researchers, in an attempt of obtaining sharp bounds of this inequality in terms of variety of Lebesgue spaces involving, at most, the first derivative have been able to construct some new inequalities, for example, inequalities of Ostrowski-Grüss type, Ostrowski-Čebyšev type, etc. The key role in obtaining these inequalities has been played by Peano kernels, Hölder's inequality, Grüss inequality, Čebyšev functional, Korkine's identity, pre-Grüss inequality and pre-Čebyšev inequality.

Historically, the first step was taken by S. S. Dragomir and S. Wang [39] in 1997 to construct an inequality of Ostrowski-Grüss type- a perturbed version of Ostrowski inequality by the use of Grüss inequality. This domain was then addressed by many authors in the coming years, for example, in 1999 P. Cerone et al. [21] extended this inequality for twice differentiable mappings, in 2000 M. Matić et al. [61] generalized and improved this inequality for n-differentiable mappings which was improved by N. S. Barnett et al. [12] by the use of Čebyšev functional and later in 2001 by X. L. Cheng in his paper [23]. Recently, a number of authors have worked to obtain tighter estimates of this inequality by the use of Euler type and generalized Euler type identities. These identities are also used to develop some higher order Ostrowski type inequalities and some efficient quadrature rules of Gauss-Legendre type. The Gauss type quadrature rules have also been developed in the research papers [99, 95, 96] by N. Ujević. N. Ujević, however, has obtained these quadrature rules as a consequence of constructing optimal quadrature rules by minimizing their error bound in the sense of inequalities.

Another inequality of Ostrowski type was constructed by connecting Ostrowski inequality with an inequality due to P. L. Čebyšev (see [64], p. 297). This inequality is named in literature as Ostrowski-Čebyšev type inequality. These type of inequalities have also been developed by applying certain convexity assumptions on the underlying function.

An obvious extension towards the generalization of Ostrowski type inequalities was to use weighted integrals, hence giving rise to weighted Ostrowski type inequalities. The weighted version of Ostrowski inequality was first presented in 1983 by J. E. Pečarić and B. Savić in ([84], Teorema 8, p. 190) which was rediscovered in ([35], Theorem 2.1) in 1999. D. S. Mitrinović et al. [65] have reported a weighted multi-dimensional analogue of the Ostrowski inequality in the first partial derivatives of the mapping. In [36], a chapter namely "Product inequalities and weighted quadrature" had been devoted to report the further advancements of weighted Ostrowski type inequalities.

An evident step towards the generalization of Ostrowski inequality was to give its multi-dimensional analogue. As mentioned above in [65], D. S. Mitrinović et al. have reported a weighted multi-dimensional version of the Ostrowski inequality in the first partial derivatives of the mapping involved. However, an optimal upper bound on the deviation of a multi-dimensional function from its averages i.e., a multivariate Ostrowski inequality was presented by G. A. Anastassiou [1] in 1997 as a generalization to classical Ostrowski inequality. In 1998, N. S. Barnett and S. S. Dragomir [11] gave an Ostrowski type inequality for double integrals while an n-dimensional analogue of Ostrowski inequality was established by S. S. Dragomir et al. [32] in 1999 for mappings of Hölder type. Furthermore, B. G. Pachpatte in 2000, in his article [73] presented an Ostrowski type inequality for three independent variables. Later, in 2001, B. G. Pachpatte had given an Ostrowski type inequality for two independent variables [74]. This topic was revisited by G. A. Anastassiou in 2002 in [2]. This domain was also addressed in view of Ostrowski-Grüss type inequalities by N. Ujević in 2003 in [104]. Recently, in [3], G. A. Anastassiou has presented Čebyšev-Grüss type inequalities on $\mathbb{R}^N$ over spherical shells and balls which are inequalities of multivariate type in spherical coordinate system.

It is impossible to list all the work dealing with the estimates of Ostrowski type inequalities due to its wide range of generalizations, extensions and variations. Moreover, such estimates are considered not only on Lebesgue spaces but also for functions that are of bounded variation, convex, Hölder continuous and Lipschitzian or absolutely continuous and for differentiable function of higher order. Results related to Ostrowski type Inequalities for twice differentiable mappings with derivatives in different Lebesgue spaces $L_p[a,b]$ $(1 \leq p \leq \infty)$ are discussed in [9].

The books and research papers mentioned above provide an extensive amount of literature on Ostrowski type inequalities which may be helpful for new researchers in exploring noteworthy results of this field.

1.2 Some significant results

We would now like to state and summarize some significant results, concepts of this area and some fundamental inequalities of our interest. We start with the following:

1.2.1 Ostrowski Inequality

In 1938, A. Ostrowski has proved an inequality involving function with bounded derivative which was named as Ostrowski inequality [69] (see also [65] p. 468). The result is given as follows:

Theorem 1.1 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on (a,b) and let, on (a,b), $|f'(x)| \leq M$. Then, for every $x \in [a,b]$,*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_\infty , \tag{1.1}$$

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

The interpretation of Ostrowski inequality can be described in two ways as follows:

1. Estimation of deviation of functional value from its Average value.
2. The estimate of approximating area under the curve by a rectangle.

G. V. Milovanović and J. E. Pecarić [62] (see inequalities also [65], pp. 468-469) proved:

Theorem 1.2 *Let $f(x)$ be $n\,(>1)$ times differentiable function such that $\left|f^{(n)}(x)\right| \leq M$ for $x \in (a,b)$. Then, for every $x \in [a,b]$*

$$\begin{aligned} &\left| \frac{1}{n} \left(f(x) + \sum_{k=1}^{n-1} F_k \right) - \frac{1}{b-a} \int_a^b f(y)\,dy \right| \\ \leq\ & \frac{M}{n(n+1)!} \frac{(x-a)^{n+1} + (b-x)^{n+1}}{b-a}, \end{aligned} \tag{1.2}$$

where F_k is defined by

$$\begin{aligned} F_k &= F_k(f; n; x, a, b) \\ &= \frac{n-k}{k!} \frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a}. \end{aligned} \tag{1.3}$$

A. M. Fink [42] generalized the above result as:

Theorem 1.3 *Let $f^{(n-1)}(t)$ be absolutely continuous on $[a,b]$ with $f^{(n)} \in L_p(a,b)$ then*

$$\begin{aligned} &\left| \frac{1}{n} \left(f(x) + \sum_{k=1}^{n-1} F_k \right) - \frac{1}{b-a} \int_a^b f(y)\,dy \right| \\ \leq\ & K(n,p,x) \left\| f^{(n)} \right\|_p , \end{aligned} \tag{1.4}$$

where

$$K(n,p,x) = \frac{1}{n!} \frac{\left[(x-a)^{n+\frac{1}{p'}} + (b-x)^{n+\frac{1}{p'}} \right]^{\frac{1}{p'}}}{b-a} B\left((n-1)p' + 1, p' + 1\right)^{\frac{1}{p'}} , \tag{1.5}$$

if $1 < p \leq \infty$ *and*

$$K(n,1,x) = \frac{(n-1)^{n-1}}{n^n n!} \frac{\max\{(x-a)^n, (b-x)^n\}}{b-a},$$

where $B(x,y)$ *is the beta function.*

For $p > 1$, these are best possible in the sense that for each x there is an f for which equality holds. For $p = 1$, equality holds for no function. The proof of this theorem follows by applying Hölder's inequality on the remainder $\frac{1}{(n-1)!} \int\limits_y^x (x-t)^{n-1} f^{(n)}(t)\,dt$.

1.2.2 Peano Kernel

Let $f(x)$ have a continuous $(n+1)\,st$ derivative in $[a,b]$ and let a linear functional $F(f)$ of f be approximated by a linear functional $E(f)$ such that $E(f)$ vanishes when f is any polynomial of degree n or less. Then,

$$E(f) = \int\limits_a^b f^{(n+1)}(t)\,K(t)\,dt,$$

where

$$K(t) = \frac{1}{n!} E_x\left[(x-t)_+^n\right],$$

and

$$(x-t)_+^n = \begin{cases} (x-t)^n, & x \geq t \\ 0, & x < t. \end{cases}$$

The notations E_x means the linear functional E is applied to the x variable in its argument $\left[(x-t)_+^n\right]$. The function $K(t)$ is called Peano kernel for the linear functional E. It is also called an efficient function for E.

1.2.3 Čebyšev Functional

For two measurable functions $f,\ g : [a,b] \to \mathbb{R}$, define the functional,

$$T(f,g;a,b) := \frac{1}{b-a}\int\limits_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int\limits_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int\limits_a^b g(x)\,dx\right), \tag{1.6}$$

which in literature is called the Čebyšev functional, provided the integrals in (1.6) exist.

1.2.4 Grüss inequality

In [46], G. Grüss proved:

Theorem 1.4 *Let f and g be two functions defined and integrable on $[a,b]$.Further, let*

$$\varphi \leq f(x) \leq \phi, \; \gamma \leq g(x) \leq \Gamma, \tag{1.7}$$

for each $x \in [a,b]$, where $\varphi, \phi, \gamma, \Gamma$ are given real constants. Then,

$$|T(f,g)| \leq \frac{1}{4}(\phi - \varphi)(\Gamma - \gamma), \tag{1.8}$$

where the constant $\frac{1}{4}$ is the best possible.

1.2.5 Pre-Grüss inequality

The following Pre-Grüss inequality was established by M. Matić et al. in [60].

Theorem 1.5 *Let $g, h : [a,b] \to \mathbb{R}$ be two integrable functions. If*

$$\alpha \leq g(t) \leq A, \tag{1.9}$$

for all $t \in [a,b]$ for some constants α and A, then

$$|T(g,h)| \leq \frac{1}{2}(A - \alpha)\sqrt{T(h,h)}. \tag{1.10}$$

The proof follows by combining Grüss inequality with

$$T^2(g,h) \leq T(g,g)\,T(h,h), \tag{1.11}$$

which is valid (see [83], p. 209). In [36], the term premature Grüss inequality was used for pre-Grüss inequality that the result was obtained by not fully completing the proof of the Grüss inequality. It has been further mentioned that the premature Grüss inequality is completed if one of the functions, g or h, is explicitly known.

1.2.6 Čebyšev Inequality

In ([64], p. 297), it has been stated that the first conversion of the Čebyšev inequality is due to P. L. Čebyšev. In 1882, he proved that:

Theorem 1.6 *Let f and g be absolutely continuous functions on $[a,b]$ and if $f^{'}$ and $g^{'}$ are the functions bounded on $[a,b]$. Then*

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \left\|f^{'}\right\|_{\infty} \left\|g^{'}\right\|_{\infty}, \tag{1.12}$$

is valid with equality if and only if $f^{'}$ and $g^{'}$ are constants.

1.2.7 Pre-Čebyšev Inequality

The following Pre-Čebyšev inequality was proved by M. Matić et al. in [60].

Theorem 1.7 *Let g be absolutely continuous functions on $[a,b]$ and g and h are integrable on $[a,b]$, then*

$$|T(g,h)| \leq \frac{1}{\sqrt{12}}(b-a) \sup_{t\in[a,b]} \left|g^{'}(t)\right| \sqrt{T(h,h)}. \tag{1.13}$$

The proof follows by combining Čebyšev inequality with

$$T^2(g,h) \leq T(g,g)\,T(h,h), \tag{1.14}$$

which is valid (see [83], p. 209). In ([8], Remark 60) it has been stated that the Pre-Čebyšev inequality provides better estimates than would be obtained using the classical Čebyšev inequality.

1.2.8 Čebyšev-Grüss type Inequality

In 1970, A. Ostrowski [70] proved the following combination of Čebyšev and Grüss inequalities:

Theorem 1.8 *Let $f:[a,b]\to\mathbb{R}$ be integrable on $[a,b]$ with $-\infty \leq m \leq f \leq M \leq \infty$ and let g be absolutely continuous function on $[a,b]$ and $g^{'} \in L_{\infty}(a,b)$. Then*

$$|T(f,g)| \leq \frac{1}{8}(b-a)(M-m)\left\|g^{'}\right\|_{\infty}. \tag{1.15}$$

1.2.9 Ostrowski-Grüss type inequality

In 1997, S. S. Dragomir and S. Wang [39], by the use of the Grüss inequality proved the following Ostrowski- Grüss type integral inequality.

Theorem 1.9 *Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I^0 of I, and let $a, b \in I^0$ with $a < b$. If $\gamma \leq f^{'}(x) \leq \Gamma$, $x \in [a, b]$ for some constants $\gamma, \Gamma \in \mathbb{R}$, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \leq \frac{1}{4} (b-a)(\Gamma - \gamma), \tag{1.16}$$

for all $x \in [a, b]$.

1.2.10 Mid-point Inequality

The classical midpoint inequality states that

$$\left| (b-a) f(\frac{a+b}{2}) - \int_a^b f(t)\, dt \right| \leq \frac{1}{24} (b-a)^3 \left\| f'' \right\|_\infty, \tag{1.17}$$

where the mapping $f : [a, b] \to \mathbb{R}$ is assumed to be twice continuously differentiable on the interval (a, b) and the second derivative be bounded on (a, b), that is,

$$\left\| f'' \right\|_\infty = \sup_{t \in (a,b)} \left| f''(t) \right| < \infty.$$

1.2.11 Trapezoid Inequality

The classical trapezoid inequality states that

$$\left| \frac{(b-a)}{2} [f(a) + f(b)] - \int_a^b f(t)\, dt \right| \leq \frac{1}{12} (b-a)^3 \left\| f'' \right\|_\infty, \tag{1.18}$$

where the mapping $f : [a, b] \to \mathbb{R}$ is assumed to be twice continuously differentiable on the interval (a, b) and the second derivative be bounded on (a, b), that is,

$$\left\| f'' \right\|_\infty = \sup_{t \in (a,b)} \left| f''(t) \right| < \infty.$$

1.2.12 Simpson's Inequality

The following inequality is known in literature as Simpson's inequality:

$$\left| \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2} \right) + f(b) \right] - \int_a^b f(t)\, dt \right| \leq \frac{1}{2880} \left\| f^{(4)} \right\|_\infty (b-a)^5, \tag{1.19}$$

where the mapping $f:[a,b]\to\mathbb{R}$ is assumed to be four times continuously differentiable on the interval (a,b) and the fourth derivative be bounded on (a,b), that is,

$$\left\|f^{(4)}\right\|_\infty=\sup_{t\in(a,b)}\left|f^{(4)}(t)\right|<\infty.$$

1.2.13 Korkine's Identity

The Korkine's identity (see [52]) & ([64], p. 296) is defined as

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\\ =\;&\frac{1}{2(b-a)^2}\int_a^b\int_a^b(f(t)-f(s))(g(t)-g(s))\,dtds,\end{aligned}\tag{1.20}$$

provided that $f,g:[a,b]\to\mathbb{R}$ are measurable and all the involved integrals exist.

1.2.14 Diaz-Metcalf inequality

The Diaz-Metcalf inequality presented by J. B. Diaz and F. T. Metcalf in [27] or (see [65], p. 83) is stated as:

Theorem 1.10 *If f is continuously differentiable on (a,b) and suppose $f(t_1)=f(t_2)$ for $a\le t_1\le t_2\le b$, then*

$$\begin{aligned}&\int_a^b[f(t)-f(t_1)]^2\,dt\\ \le\;&\frac{4}{\pi^2}\max\left\{(t_1-a)^2,(b-t_2)^2,\left(\frac{t_2-t_1}{2}\right)^2\right\}\int_a^b\left(f'(x)\right)^2dx.\end{aligned}\tag{1.21}$$

1.2.15 Special Means

Let us recall the following means:

(a) The Arithmetic Mean

$$A=A(a,b)=\frac{a+b}{2};\,a,b\ge 0.$$

(b) The Geometric Mean

$$G=G(a,b)=\sqrt{ab}\;;\;a,b\ge 0.$$

(c) The Harmonic Mean

$$H = H(a,b) = \frac{2}{\frac{1}{a} + \frac{1}{b}} \ ; \ a,b > 0.$$

(d) The Logarithmic Mean

$$L = L(a,b) = \begin{cases} a, & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a}, & \text{if } a \neq b \end{cases} \ ; \ a,b > 0.$$

(e) The Identric Mean

$$I = I(a,b) = \begin{cases} a, & \text{if } a = b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } a \neq b. \end{cases} \ ; \ a,b > 0.$$

(f) The p-logarithmic Mean

$$L_p = L_p(a,b) = \begin{cases} a, & \text{if } a = b \\ \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if } a \neq b, \end{cases}$$

where $p \in \mathbb{R} \setminus \{-1, 0\}$, $a, b > 0$.

The following inequality holds in literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$ and $L_0 = I$ and $L_{-1} = L$.

1.3 Objective of the Thesis

The problem of generalization/extension of integral inequalities that present bounds on the physical quantities i.e., inequalities of Ostrowski-Grüss and Čebyšev type is aimed to be overseen in this thesis in the following manner:

1. By estimating some more bounds on the deviations through other norms while the classical approach refers to the supremum norm i.e., for functions with bounded derivatives.

2. By estimating bounds on the deviations in terms of higher differentiable functions while the classical inequalities were assumed to involve first differentiable functions only.

3. Consideration of random variables, therefore, generating obvious applications of these inequalities in probability theory.

4. Consideration of special means with reference to its applications in finding direct relations of these means.

5. By switching to multiplicative framework, hence, developing some new multivariate and product inequalities of Ostrowski, Grüss and Čebyšev type.

6. By directing towards other classes of functions; for functions that are of: bounded variation, monotonic and lipschitzian or absolutely continuous, hence, paving way to move in larger domains, for example, from absolutely continuous functions to function of bounded variation and from Lipschitzian functions to (l, L)-Lipschitzian functions.

7. By defining such integral inequalities in Euclidean domain i.e., moving from intervals to rectangles, shells, balls and spheres.

8. The deduction of the optimal error bound for such inequalities under the assumptions under consideration.

9. Extending the variations of Ostrowski-Grüss and Čebyšev inequalities; hence, working on to obtain some more inequalities of Ostrowski-Grüss, Ostrowski-Čebyšev and Čebyšev-Grüss type.

10. Presenting new applications of these inequalities in non-linear analysis, i.e., construction of quadrature based iterative methods for solving non-linear equations in single variable.

This dissertation is oriented towards generalizing some results on Ostrowski type inequalities keeping in view the above mentioned goals. The directions in which the generalized versions of Ostrowski type inequalities are intended to be used are its applications in numerical integration and special means, applications to cumulative distribution functions, expectations for random variables and some new applications in solving non-linear equations will also be given.

1.4 Thesis Overview

The dissertation presents some generalization of the Ostrowski type inequality and its applications. To obtain these generalizations, the approach being taken into account is to modify the Peano kernel involved by introducing arbitrary parameters or by introducing weight functions.

The dissertation comprises nine chapters.

Chapter 2 covers the generalization of Ostrowski type inequalities for twice differentiable functions in the domain of usual Lebesgue spaces $L_p(a,b)$, $p=\infty,1$, $p>1$. It also incorporates generalized Ostrowski type inequalities for n-times differentiable and (l,L)- Lipschitzian mappings. Applications of the obtained inequalities in numerical integration and special means are also presented.

In Section 2.1, a general form of integral inequality of Ostrowski type for twice differentiable function whose first derivative is absolutely continuous and second derivative is bounded is presented. In Section 2.2, some perturbed generalizations of inequalities of Ostrowski type involving functions whose first derivative is absolutely continuous and second derivative belongs to $L_p(a,b)$, $p=1,\infty$, $p>1$ are presented. Section 2.3 contains a generalized Ostrowski type inequality for (l,L) Lipschitzian mappings. Applications to composite quadrature rules are also given.

Chapter 3 includes some extensions of Ostrowski-Grüss type inequalities for first and twice differentiable functions and their applications to quadrature rules, special means, probability density functions and beta random variable.

In Section 3.1, an Ostrowski-Grüss type inequality involving functions whose first derivative belongs to $L_2(a,b)$ is obtained. In Section 3.2, a generalized Ostrowski-Grüss type inequality for twice differentiable function in terms of lower and upper bound of the second derivative is established. Later, in Section 3.3, a similar inequality is obtained with second derivative in $L_2(a,b)$. In Section 3.4, the estimates of first inequality of Ostrowski-Grüss type are presented in terms of the upper and lower bounds of the first derivative and later in terms of the Euclidean norm of second derivative by using Diaz-Metcalf inequality.

Chapter 4 presents some generalized product inequalities of Ostrowski-Čebyšev type and their applications for the expectation of a random variable. It also includes Čebyšev-Grüss type inequality for spherical shells and balls calculated by using spherical coordinate system in the sense of multivariate inequalities.

In Section 4.1, a product inequality of Čebyšev type is obtained for functions which are absolutely continuous with first derivatives in $L_p(a,b)$, $p>1$. In Section 4.2, some product inequalities of Čebyšev type have been developed for twice differentiable functions whose first derivatives are absolutely continuous and second derivatives belong to $L_\infty(a,b)$. In Section 4.3, an integral inequality involving the product of two functions and its applications to probability density functions is presented. Section 4.4 presents extension of Čebyšev-Grüss type inequalities for $L_p[a,b]$, $p>1$ on n-dimensional Euclidean space over spherical shells and balls, thus, obtaining multivariate inequalities of Čebyšev-Grüss type are obtained by working in $\mathbb{R}^n$.

Chapter 5 incorporates some Ostrowski type inequalities for Newton-Cotes formulae in an optimal or generalized manner. The error inequalities thus obtained generate one-point, two-point and four-point Newton-Cotes formulae as special cases.

In Section 5.1, a family of four-point quadrature rule of closed type is developed which recaptures Gauss two-point, Simpson's $\frac{3}{8}$ and Lobatto four-point quadrature rule with error bounds in terms of twice differentiable functions. The optimal case is also addressed. In Section 5.2, a two-point quadrature rule is developed for functions of bounded variations and for L-Lipschitzian functions which can generate Newton-Cotes formulae of open as well as closed type as special cases.

Chapter 6 comprises some weighted Ostrowski type inequalities for a random variable whose probability density functions belongs to the usual Lebesgue spaces $L_p[a,b]$, $p = \infty, 1$, $p > 1$ which generalize some previous results by including weighted integrals.

In Section 6.1, some weighted Ostrowski type inequalities for probability density functions, expectation of a random variable and for beta random variable are obtained. The estimates are presented in terms of the $\|.\|_\infty$-norm of the probability density function. In Section 6.2, some weighted Ostrowski type inequalities are developed for a random variable whose probability density function belong to $L_p[a,b]$, $p > 1$.

Chapter 7 describes applications of Ostrowski type inequalities to probability density function, expectation of a random variable and generalized beta random variable. The inequalities obtained in this chapter are improvements of some previous inequalities of this domain.

Chapter 8 contains some new applications of Ostrowski type inequalities in constructing iterative algorithms for solving non-linear equations. Some generalized and computationally efficient iterative algorithms are presented in this chapter.

Chapter 9 takes into account a critical analysis of the generalizations and improvements obtained in this dissertation and some future extensions.

Chapter 2

Generalization of Ostrowski type inequalities for differentiable mappings

We, in this chapter, present some Ostrowski type inequalities for twice differentiable functions.

2.1 A generalized integral inequality for twice differentiable mappings

In this section, a general form of integral inequality of Ostrowski type for twice differentiable mappings whose second derivatives are bounded and first derivatives are absolutely continuous is established. The generalized integral inequality points some better bounds than some already presented bounds. The inequality is then applied to numerical integration and special means.

2.1.1 Introduction

In recent years, a number of authors have worked on the generalizations of Ostrowski's inequality. For example, this topic is considered in [34, 59, 61, 100].

In [22], P. Cerone, S. S. Dragomir and J. Roumeliotis, established an integral inequality of Ostrowski type for mappings with bounded second derivatives. A similar inequality has been established by S. S. Dragomir and N. S. Barnett in [31]. In [38], S. S. Dragomir and A. Sofo, pointed out an integral inequality of Ostrowski type similar in a sense to that of [22] or [31]. However, this inequality contains a minor mistake. The corrected version [63] of the inequality is given in the form of the following theorem:

Theorem CDR. Let $g : [a, b] \longrightarrow \mathbb{R}$ be a mapping whose first derivative is absolutely continuous on $[a, b]$ and assume that the second derivative $g^{''} \in L_\infty(a, b)$. Then, we have the inequality

$$\left| \int_a^b g(t)dt - \frac{1}{2}\left[(b-a)\left(g(x) + \frac{g(a)+g(b)}{2}\right) - (b-a)\left(x - \frac{a+b}{2}\right) g^{'}(x)\right]\right|$$
$$\leq \left\| g^{''} \right\|_\infty \left(\frac{1}{3}\left| x - \frac{a+b}{2}\right|^3 + \frac{(b-a)^3}{48} \right), \tag{2.1}$$

for all $x \in [a, b]$.

The main aim of this section is to point out a generalization of (2.1). It turns out that this generalization can give better results than the estimations based on (2.1).

2.1.2 Main Results

We establish here a general form of integral inequality (2.1) and apply it to numerical integration and special means. The inequality is given in the form of the following theorem:

Theorem 2.1 *Let $g : [a, b] \longrightarrow \mathbb{R}$ be a mapping whose first derivative is absolutely continuous on $[a, b]$ and assume that the second derivative $g^{''} \in L_\infty(a, b)$. Then, we have the inequality*

$$\left| \frac{1}{(b-a)} \int_a^b g(t)dt - \frac{1}{2}\left[(1-h)\, g(x) + (1+h)\left(\frac{g(a)+g(b)}{2}\right)\right.\right.$$
$$\left.\left. - (1-h)\,(x - \frac{a+b}{2}) g^{'}(x) - h\frac{b-a}{4}\left(g^{'}(b) - g^{'}(a)\right)\right]\right|$$
$$\leq \left\| g^{''} \right\|_\infty \frac{1}{(b-a)} \left[\frac{1}{3}\left| x - \frac{a+b}{2}\right|^3 + \frac{(b-a)^3}{48}\Psi(h) \right] \tag{2.2}$$

or equivalently,

$$\left|\int_a^b g(t)dt - \frac{(b-a)}{2}\left[(1-h)\,g(x) + (1+h)\left(\frac{g(a)+g(b)}{2}\right)\right.\right.$$
$$\left.\left. - (1-h)\left(x - \frac{a+b}{2}\right)g'(x) - h\frac{b-a}{4}\left(g'(b) - g'(a)\right)\right]\right|$$
$$\leq \left\|g''\right\|_\infty \left[\frac{1}{3}\left|x - \frac{a+b}{2}\right|^3 + \frac{(b-a)^3}{48}\Psi(h)\right], \tag{2.3}$$

for all $x \in [a + h\frac{b-a}{2}, b - h\frac{b-a}{2}]$,

$$\text{where } \Psi(h) = (1-h)\left[2(1-h)^2 - 1\right] + 2h, \; h \in [0,1].$$

Proof. Let us start with the following integral identity,

$$f(x) = \frac{1}{(1-h)}\left[\frac{1}{(b-a)}\int_a^b f(t)dt - \frac{h}{2}(f(a) + f(b))\right]$$
$$+\frac{1}{(b-a)(1-h)}\int_a^b p(x,t)f'(t)dt.$$

This implies

$$(1-h)\,f(x) = \frac{1}{(b-a)}\int_a^b f(t)dt - \frac{h}{2}(f(a) + f(b))$$
$$+\frac{1}{b-a}\int_a^b p(x,t)f'(t)dt, \tag{2.4}$$

for all $x \in [a + h\frac{b-a}{2}, b - h\frac{b-a}{2}]$, $h \in [0,1]$ provided f is absolutely continuos on $[a,b]$ and the kernel $p : [a,b]^2 \longrightarrow \mathbb{R}$ defined in [34] is given by:

$$p(x,t) = \begin{cases} t - \left(a + h\frac{b-a}{2}\right), & if\ t \in [a,x] \\ t - \left(b - h\frac{b-a}{2}\right), & if\ t \in (x,b]. \end{cases}$$

A simple proof using the integration by parts can be found in [34]. We choose in (2.4),

$$f(x) = \left(x - \frac{a+b}{2}\right)g'(x),$$

to get

$$\begin{aligned}
&(1-h)\left(x-\frac{a+b}{2}\right)g^{'}(x) \\
=\ &\frac{1}{(b-a)}\int_a^b\left(t-\frac{a+b}{2}\right)g^{'}(t)dt-\frac{h}{4}(b-a)\left(g^{'}(b)-g^{'}(a)\right) \\
&+\frac{1}{b-a}\int_a^b p(x,t)\left[g^{'}(t)+\left(t-\frac{a+b}{2}\right)g^{''}(t)\right]dt. \qquad (2.5)
\end{aligned}$$

Integrating by parts, we have

$$\int_a^b\left(t-\frac{a+b}{2}\right)g^{'}(t)dt=(b-a)\left(\frac{g(a)+g(b)}{2}\right)-\int_a^b g(t)dt. \qquad (2.6)$$

Also,

$$\int_a^b p(x,t)g^{'}(t)dt=(1-h)(b-a)g(x)+h\frac{b-a}{2}(g(a)+g(b))-\int_a^b g(t)dt. \qquad (2.7)$$

Using (2.6) and (2.7) in (2.5), we get:

$$\begin{aligned}
&(1-h)\left(x-\frac{a+b}{2}\right)g^{'}(x) \\
=\ &\frac{(1+h)}{2}(g(a)+g(b))-h\frac{b-a}{4}\left(g^{'}(b)-g^{'}(a)\right) \\
&+(1-h)g(x)-\frac{2}{b-a}\int_a^b g(t)dt, \\
&+\frac{1}{b-a}\int_a^b p(x,t)\left(t-\frac{a+b}{2}\right)g^{''}(t)dt.
\end{aligned}$$

or

$$\begin{aligned}
\frac{1}{b-a}\int_a^b g(t)dt\ =\ &\frac{(1+h)}{4}(g(a)+g(b))-h\frac{b-a}{8}\left(g^{'}(b)-g^{'}(a)\right) \\
&+\frac{(1-h)}{2}g(x)-\frac{(1-h)}{2}\left(x-\frac{a+b}{2}\right)g^{'}(x). \\
&+\frac{1}{2(b-a)}\int_a^b p(x,t)\left(t-\frac{a+b}{2}\right)g^{''}(t)dt,
\end{aligned}$$

for all $x \in [a+h\frac{b-a}{2}, b-h\frac{b-a}{2}]$. This implies

$$\begin{aligned}
& \left| \frac{1}{b-a}\int_a^b g(t)dt \right. \\
& -\frac{1}{2}\left[(1-h)\,g(x)+(1+h)\left(\frac{g(a)+g(b)}{2}\right)\right. \\
& \left.\left. -(1-h)\left(x-\frac{a+b}{2}\right)g^{'}(x)-h\frac{b-a}{4}\left(g^{'}(b)-g^{'}(a)\right)\right]\right| \\
= & \left|\frac{1}{2(b-a)}\int_a^b p(x,t)\left(t-\frac{a+b}{2}\right)g^{''}(t)dt\right|, \\
\leq & \frac{1}{2(b-a)}\int_a^b |p(x,t)|\left|t-\frac{a+b}{2}\right|\left|g^{''}(t)\right|dt.
\end{aligned} \tag{2.8}$$

Obviously, we have

$$\begin{aligned}
& \int_a^b |p(x,t)|\left|t-\frac{a+b}{2}\right|\left|g^{''}(t)\right|dt \\
\leq & \left\|g^{''}\right\|_\infty \int_a^b |p(x,t)|\left|t-\frac{a+b}{2}\right|dt,
\end{aligned} \tag{2.9}$$

where

$$\left\|g^{''}\right\|_\infty = \sup_{t\in(a,b)}\left|g^{''}(t)\right| < \infty.$$

Also,

$$I = \int_a^b |p(x,t)|\left|t-\frac{a+b}{2}\right|dt$$

or

$$I = \int_a^x \left|t-\left(a+h\frac{b-a}{2}\right)\right|\left|t-\frac{a+b}{2}\right|dt + \int_x^b \left|t-\left(b-h\frac{b-a}{2}\right)\right|\left|t-\frac{a+b}{2}\right|dt. \tag{2.10}$$

We have two cases:

a) For $x \in \left[a+h\frac{b-a}{2}, \frac{a+b}{2}\right]$, we obtain:

$$\begin{aligned}
I = & \int_a^{a+h\frac{b-a}{2}} \left(a+h\frac{b-a}{2}-t\right)\left(\frac{a+b}{2}-t\right)dt \\
& + \int_{a+h\frac{b-a}{2}}^x \left[t-\left(a+h\frac{b-a}{2}\right)\right]\left(\frac{a+b}{2}-t\right)dt
\end{aligned}$$

$$+\int_{x}^{\frac{a+b}{2}}\left(b-h\frac{b-a}{2}-t\right)\left(\frac{a+b}{2}-t\right)dt$$

$$+\int_{\frac{a+b}{2}}^{b-h\frac{b-a}{2}}\left(b-h\frac{b-a}{2}-t\right)\left(t-\frac{a+b}{2}\right)dt$$

$$+\int_{b-h\frac{b-a}{2}}^{b}\left[t-\left(b-h\frac{b-a}{2}\right)\right]\left(t-\frac{a+b}{2}\right)dt.$$

After some simple calculations, we obtain

$$I=\frac{2}{3}\left(\frac{a+b}{2}-x\right)^{3}+\frac{(b-a)^{3}}{24}\left[3h+2(1-h)^{3}-1\right], \tag{2.11}$$

for all $x\in\left[a+h\frac{b-a}{2},\frac{a+b}{2}\right]$.

b) For $x\in\left[\frac{a+b}{2},b-h\frac{b-a}{2}\right]$, we take

$$\begin{aligned} I &= \int_{a}^{a+h\frac{b-a}{2}}\left(a+h\frac{b-a}{2}-t\right)\left(\frac{a+b}{2}-t\right)dt \\ &\quad+\int_{a+h\frac{b-a}{2}}^{\frac{a+b}{2}}\left[t-\left(a+h\frac{b-a}{2}\right)\right]\left(\frac{a+b}{2}-t\right)dt \\ &\quad+\int_{\frac{a+b}{2}}^{x}\left[t-\left(a+h\frac{b-a}{2}\right)\right]\left(t-\frac{a+b}{2}\right)dt \\ &\quad+\int_{x}^{b-h\frac{b-a}{2}}\left(b-h\frac{b-a}{2}-t\right)\left(t-\frac{a+b}{2}\right)dt \\ &\quad+\int_{b-h\frac{b-a}{2}}^{b}\left[t-\left(b-h\frac{b-a}{2}\right)\right]\left(t-\frac{a+b}{2}\right)dt. \end{aligned}$$

$$I=\frac{2}{3}\left(x-\frac{a+b}{2}\right)^{3}+\frac{(b-a)^{3}}{24}\left[3h+2(1-h)^{3}-1\right], \tag{2.12}$$

for all $x\in\left[\frac{a+b}{2},b-h\frac{b-a}{2}\right]$.

Using (2.9), (2.10) (2.11) and (2.12) in (2.8), we obtain

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b g(t)dt-\frac{1}{2}\left[(1-h)\,g(x)+(1+h)\left(\frac{g(a)+g(b)}{2}\right)\right.\right.\\
&\left.\left.-(1-h)\left(x-\frac{a+b}{2}\right)g'(x)-h\frac{b-a}{4}\left(g'(b)-g'(a)\right)\right]\right|\\
\leq\;&\frac{\left\|g''\right\|_\infty}{2(b-a)}\left[\frac{2}{3}\left|x-\frac{a+b}{2}\right|^3+\frac{(b-a)^3}{24}\left[3h+2(1-h)^3-1\right]\right],\\
=\;&\frac{\left\|g''\right\|_\infty}{(b-a)}\left[\frac{1}{3}\left|x-\frac{a+b}{2}\right|^3+\frac{(b-a)^3}{48}\Psi(h)\right],
\end{aligned}$$

for all $x\in[a+h\frac{b-a}{2},b-h\frac{b-a}{2}]$,

$$\begin{aligned}
\text{where }\Psi(h)\;&=\;3h+2(1-h)^3-1,\\
&=\;(1-h)\left[2(1-h)^2-1\right]+2h,\;\;h\in[0,1].
\end{aligned}$$

■

Remark 2.1 *Choosing $h=0$ in (2.3) gives us the inequality (2.1).*

Remark 2.2 *In (2.1), if we investigate the estimates for the end points $x=a$, $x=b$ and the midpoint $x=\frac{a+b}{2}$, we find that the midpoint gives us the best estimate, so that from inequality (2.3), we have:*

$$\begin{aligned}
&\left|\int_a^b g(t)dt-\frac{(b-a)}{2}[(1-h)\,g\left(\frac{a+b}{2}\right)\right.\\
&\left.+(1+h)\left(\frac{g(a)+g(b)}{2}\right)-h\frac{b-a}{4}\left(g'(b)-g'(a)\right)\right]\right|\\
\leq\;&\left\|g''\right\|_\infty\frac{(b-a)^3}{48}\Psi(h).
\end{aligned}\tag{2.13}$$

Remark 2.3 *If we investigate $\Psi(h)$ for different values of $h\in[0,1]$, we find that*

$$\Psi(h)<1,\;\textit{for}\;0<h<\frac{6}{10}\tag{2.14}$$

and it is minimum for $h=\frac{3}{10}$. Thus, for the specified range of h as mentioned in (2.14), our result gives us better estimate than as given in [22] i.e.,

$$\frac{\Psi(h)}{48}<\frac{1}{48},\textit{for}\;0<h<\frac{6}{10}.$$

The special cases of (2.13) are given in the form of following remark.

Remark 2.4 *(i) Choosing $h = \frac{3}{10}$ in the inequality (2.13) gives us the best estimate:*

$$\left| \int_a^b g(t)dt - \frac{(b-a)}{20}\left[7g(\frac{a+b}{2}) + \frac{13}{2}(g(a)+g(b)) - \frac{3}{4}(b-a)\left(g^{'}(b) - g^{'}(a)\right)\right]\right| \leq \frac{293}{24000}(b-a)^3 \left\|g^{''}\right\|_{\infty}, \quad (2.15)$$

which has a better estimate than the three-point quadrature inequalities presented in [9] and [59] for $\|.\|_{\infty}$ norm

(ii) If we choose $h = 1$ in the inequality (2.13), we get a perturbed trapezoid inequality as follows:

$$\left| \int_a^b g(t)dt - (b-a)\left(\frac{g(a)+g(b)}{2}\right) - \frac{b-a}{8}\left(g^{'}(b) - g^{'}(a)\right)\right| \leq \left\|g^{''}\right\|_{\infty} \frac{(b-a)^3}{24}, \quad (2.16)$$

which has a better estimate than the perturbed trapezoid inequalities presented in [9] and [59] for $\|.\|_{\infty}$ norm.

2.1.3 Applications in Numerical Integration

We may use the inequality (2.2), to get the estimates of composite quadrature rules with smaller error than that which may be obtained by the classical results.

Theorem 2.2 *Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a partition of the interval $[a,b]$, $h_i = x_{i+1} - x_i$, $\delta \in [0,1]$, $x_i + \delta\frac{h_i}{2} \leq \xi_i \leq x_{i+1} - \delta\frac{h_i}{2}$, $i = 0, \cdots, n-1$, then*

$$\int_a^b g(t)\ dt = S(g, g^{'}, I_n, \xi, \delta) + R(g, g^{'}, I_n, \xi, \delta),$$

where

$$S(g, g^{'}, I_n, \xi, \delta) = \frac{1}{2}\sum_{i=0}^{n-1}\left[(1-\delta)\, g(\xi_i) + (1+\delta)\left(\frac{g(x_i) + g(x_{i+1})}{2}\right)\right.$$
$$\left. - (1-\delta)\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) g^{'}(\xi_i) - \frac{\delta}{4}h_i\left(g^{'}(x_{i+1}) - g^{'}(x_i)\right)\right] h_i \quad (2.17)$$

and

$$\begin{aligned} & \left|R(g,g^{'},I_n,\xi,\delta)\right| \\ \leq\ & \left\|g^{''}\right\|_{\infty}\left[\sum_{i=0}^{n-1}\left(\frac{1}{3}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|^3+\frac{h_i^3}{48}\Psi\left(\delta\right)\right)\right], \\ =\ & \left\|g^{''}\right\|_{\infty}\left[\frac{1}{3}\sum_{i=0}^{n-1}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|^3+\frac{\Psi\left(\delta\right)}{48}\sum_{i=0}^{n-1}h_i^3\right], \end{aligned} \tag{2.18}$$

where $\Psi\left(\delta\right)=\left(1-\delta\right)\left[2\left(1-\delta\right)^2-1\right]+2\delta,\ \delta\in\left[0,1\right].$

Proof. Applying inequality (2.12) on $\xi_i\in\left[x_i+\delta\frac{h_i}{2},x_{i+1}-\delta\frac{h_i}{2}\right]$ and summing over i from 0 to $n-1$ and using triangular inequality, we get (2.18). ■

Remark 2.5 *Choosing $\delta=0$ gives us as a special case [38], the corrected version of estimates of composite quadrature rules.*

Corollary 2.1 *For $\xi_i=\frac{x_i+x_{i+1}}{2}$, $\left(i=0,\cdots n-1\right)$, then we have the following quadrature rule:*

$$\begin{aligned} & S(g,g^{'},I_n,\delta) \\ =\ & \frac{1}{2}\sum_{i=0}^{n-1}[\left(1-\delta\right)g\left(\frac{x_i+x_{i+1}}{2}\right)+\left(1+\delta\right)\left(\frac{g\left(x_i\right)+g\left(x_{i+1}\right)}{2}\right) \\ & -\frac{\delta}{4}h_i\left(g^{'}\left(x_{i+1}\right)-g^{'}\left(x_i\right)\right)]h_i \end{aligned} \tag{2.19}$$

and

$$\left|R(g,g^{'},I_n,\delta)\right|\leq\frac{\Psi\left(\delta\right)}{48}\left\|g^{''}\right\|_{\infty}\sum_{i=0}^{n-1}h_i^3,\ \ \delta\in\left[0,1\right]. \tag{2.20}$$

Remark 2.6 *(i) If we choose $\delta=0$ in (2.19) and (2.20), $\left(i=0,\cdots n-1\right)$, then*

$$\bar{S}(g,I_n)=\frac{1}{2}\sum_{i=0}^{n-1}\left[g\left(\frac{x_i+x_{i+1}}{2}\right)+\frac{g\left(x_i\right)+g(x_{i+1}}{2}\right]h_i \tag{2.21}$$

and

$$\bar{R}(g,I_n)\leq\frac{\left\|g^{''}\right\|_{\infty}}{48}\sum_{i=0}^{n-1}h_i^3. \tag{2.22}$$

It may be noted that $\bar{S}(g,I_n)$ is an arithmetic mean of the midpoint and trapezoidal quadrature rules.

(ii) If we choose $\delta = \frac{3}{10}$ *in (2.19) and (2.20),* $(i = 0, \cdots n-1)$, *then*

$$S(g, g^{'}, I_n) = \frac{1}{20}\sum_{i=0}^{n-1}\left[7g\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{13}{2}g(x_i) + g(x_{i+1})\right]h_i - \frac{3}{80}\sum_{i=0}^{n-1}\left[g^{'}(x_{i+1}) - g^{'}(x_i)\right]h_i \quad (2.23)$$

and

$$R(g, I_n) \le \frac{293}{24000}\left\|g^{''}\right\|_{\infty}\sum_{i=0}^{n-1}h_i^3, \quad (2.24)$$

which is a perturbed composite three point quadrature inequality of Simpson's type.

(iii) If we choose $\delta = 1$ *in (2.19) and (2.20),* $(i = 0, \cdots n-1)$, *then*

$$S(g, g^{'}, I_n) = \frac{1}{2}\sum_{i=0}^{n-1}(g(x_i) + g(x_{i+1}))h_i - \frac{1}{8}\sum_{i=0}^{n-1}\left[g^{'}(x_{i+1}) - g^{'}(x_i)\right]h_i \quad (2.25)$$

and

$$R(g, I_n) \le \frac{1}{24}\left\|g^{''}\right\|_{\infty}\sum_{i=0}^{n-1}h_i^3, \quad (2.26)$$

which is a perturbed composite trapezoid inequality.

2.1.4 Applications for some Special Means

The inequality (2.2) may be written as

$$\left|\frac{(1-h)}{2}g(x) + \frac{(1+h)}{2}\left(\frac{g(a)+g(b)}{2}\right) - \frac{(1-h)}{2}(x - A(a,b))g^{'}(x) - h\frac{b-a}{8}\left(g^{'}(b) - g^{'}(a)\right) - \frac{1}{b-a}\int_a^b g(t)dt\right| \le \frac{\left\|g^{''}\right\|_{\infty}}{(b-a)}\left[\frac{1}{3}|x - A(a,b)|^3 + \frac{(b-a)^3}{48}\Psi(h)\right], \quad (2.27)$$

where $\Psi(h) = (1-h)\left[2(1-h)^2 - 1\right] + 2h,\ h \in [0,1]$.

Choosing $h=0$ gives us as a special case, the corrected version of the inequality in [38] as follows:

$$\left|\frac{1}{2}\left[g(x)+\left(\frac{g(a)+g(b)}{2}\right)-(x-A(a,b))\,g^{'}(x)\right]-\frac{1}{b-a}\int_a^b g(t)dt\right|$$
$$\leq \left\|g^{''}\right\|_\infty\left[\frac{1}{3(b-a)}\left|x-A(a,b)\right|^3+\frac{(b-a)^2}{48}\right]. \qquad (2.28)$$

We may now apply (2.27), to deduce some inequalities for special means using some particular mappings. The results of the special means are therefore as follows:

Example 1 *Consider* $g(t)=\ln t,\ g:(0,\infty)\to\mathbb{R},$ *then*

$$\begin{aligned}\frac{1}{b-a}\int_a^b g(t)dt &= \ln I(a,b),\\ \frac{g(a)+g(b)}{2} &= \ln G(a,b),\\ g^{'}(b)-g^{'}(a) &= -\frac{b-a}{G^2(a,b)}\end{aligned}$$

and

$$\left\|g^{''}\right\|_\infty=\sup_{t\in(a,b)}\left\|g^{''}(t)\right\|=\frac{1}{a^2}.$$

From (2.27), we have:

$$\left|(1-h)\ln x+(1+h)\ln G(a,b)-(1-h)\left(1-\frac{A(a,b)}{x}\right)\right.$$
$$\left.+h\frac{(b-a)^2}{4}\frac{1}{G^2(a,b)}-2\ln I(a,b)\right|$$
$$\leq \frac{2}{a^2}\left(\frac{1}{3(b-a)}\left|x-A(a,b)\right|^3+\frac{(b-a)^2}{48}\Psi(h)\right).$$

from which we obtain the estimate at the centre $x=\frac{a+b}{2}=A(a,b)$, *so that*

$$\left|(1-h)\ln A(a,b)+(1+h)\ln G(a,b)+h\frac{(b-a)^2}{4}\frac{1}{G^2}-2\ln I(a,b)\right|$$
$$\leq \frac{(b-a)^2}{24a^2}\Psi(h)$$

or

$$\left|\ln\left(\frac{A^{(1-h)}G^{(1+h)}}{I^2}\right)+h\frac{(b-a)^2}{4}\frac{1}{G^2(a,b)}\right|\leq\frac{(b-a)^2}{24a^2}\Psi(h),$$

from which we obtain the best estimate if we choose $h = \frac{3}{10}$,*that is*

$$\left| \ln \left(\frac{A^{\frac{7}{10}} G^{\frac{13}{10}}}{I^2} \right) + \frac{3\,(b-a)^2}{40} \frac{1}{G^2(a,b)} \right| \leq \frac{293}{12000a^2}\,(b-a)^2 \,.$$

For $h = 0$*, we have*

$$\begin{aligned} & |\ln A(a,b) + \ln G(a,b) - 2\ln I(a,b)| \\ \leq\ & \frac{(b-a)^2}{24a^2}. \end{aligned}$$

Example 2 *Consider* $g(x) = \frac{1}{t}, g : (0, \infty) \to (0, \infty)$ *, then*

$$\begin{aligned} \frac{1}{b-a} \int_a^b g(t)dt &= L^{-1}(a,b) \\ \frac{g(a)+g(b)}{2} &= \frac{A(a,b)}{G^2(a,b)}, \\ g^{'}(b) - g^{'}(a) &= \frac{2\,(b-a)}{H\,(a,b)\,G^2\,(a,b)} \end{aligned}$$

and

$$\left\| g^{''} \right\|_\infty = \sup_{t\in(a,b)} \left\| g^{''}(t) \right\| = \frac{2}{a^3}.$$

From (2.27), we have

$$\begin{aligned} & |\frac{(1+h)\ A(a,b)}{2G^2(a,b)} + \frac{(1-h)}{2x} \left(2 - \frac{A(a,b)}{x} \right) \\ & -h\frac{(b-a)^2}{4} \frac{1}{H\,(a,b)\,G^2\,(a,b)} - L^{-1}(a,b)| \\ \leq\ & \frac{2}{a^3} \left(\frac{1}{3\,(b-a)} \left| x - A\,(a,b) \right|^3 + \frac{(b-a)^2}{48} \Psi\,(h) \right) \end{aligned}$$

and the estimate at the centre point $x = \frac{a+b}{2} = A(a,b)$*, so that*

$$\begin{aligned} & \left| \frac{(1+h)\ A(a,b)}{2G^2(a,b)} + \frac{(1-h)}{2A(a,b)} - h\frac{(b-a)^2}{4} \frac{1}{H\,(a,b)\,G^2\,(a,b)} - L^{-1}(a,b) \right| \\ \leq\ & \frac{(b-a)^2}{24a^3} \Psi\,(h)\,, \end{aligned}$$

which becomes best by choosing $h = \frac{3}{10}$ *in the above inequality,*

$$\begin{aligned} & \left| \frac{13\ A(a,b)}{20G^2(a,b)} + \frac{7}{20A(a,b)} - \frac{3\,(b-a)^2}{40} \frac{1}{H\,(a,b)\,G^2\,(a,b)} - L^{-1}(a,b) \right| \\ \leq\ & \frac{293}{12000a^3}\,(b-a)^2\,. \end{aligned}$$

Also for $h=0$, we have

$$\left|\frac{A(a,b)}{2G^2(a,b)}+\frac{1}{2A(a,b)}-L^{-1}(a,b)\right|\leq\frac{(b-a)^2}{24a^3}.$$

Example 3 *Consider $g(t)=t^p, g:(0,\infty)\longrightarrow(0,\infty)$, where $p\in\mathbb{R}\backslash\{-1,0\}$ then for $a<b$*

$$\begin{aligned}\frac{1}{b-a}\int_a^b g(t)dt &= L_p^p(a,b),\\ \frac{g(a)+g(b)}{2} &= A(a^p,b^p),\\ g'(b)-g'(a) &= p(p-1)(b-a)L_{p-2}^{p-2}(a,b)\end{aligned}$$

and

$$\left\|g''\right\|_\infty=|p(p-1)|\begin{cases}b^{p-2} & \text{if } p\in[2,\infty)\\ a^{p-2} & \text{if } p\in(-\infty,2]\backslash\{-1,0\}.\end{cases}$$

From (2.27), we obtain

$$\begin{aligned}&\left|\frac{(1-h)}{2}x^{p-1}\left[(1-p)x+pA(a,b)\right]+\frac{(1+h)}{2}A(a^p,b^p)\right.\\ &\left.-h\frac{(b-a)^2}{8}p(p-1)L_{p-2}^{p-2}(a,b)-L_p^p(a,b)\right|\\ \leq\ & |p(p-1)|\,\delta_p(a,b)\left(\frac{1}{3(b-a)}\left|x-A(a,b)\right|^3+\frac{(b-a)^2}{48}\Psi(h)\right),\end{aligned}$$

where

$$\delta_p(a,b)=\begin{cases}b^{p-2} & \text{if } p\in[2,\infty)\\ a^{p-2} & \text{if } p\in(-\infty,2]\backslash\{-1,0\}.\end{cases}$$

At $x=\frac{a+b}{2}=A(a,b)$, we get

$$\begin{aligned}&\left|\frac{(1-h)\ A^p(a,b)}{2}+\frac{(1+h)}{2}A(a^p,b^p)\right.\\ &\left.-h\frac{(b-a)^2}{8}p(p-1)L_{p-2}^{p-2}(a,b)-L_p^p(a,b)\right|\\ \leq\ & |p(p-1)|\ \delta_p(a,b)\frac{(b-a)^2}{48}\Psi(h)\end{aligned}$$

or

$$\begin{aligned}&\left|(1-h)A^p(a,b)+(1+h)A(a^p,b^p)\right.\\ &\left.-h\frac{(b-a)^2}{4}p(p-1)L_{p-2}^{p-2}(a,b)-2L_p^p(a,b)\right|\\ \leq\ & |p(p-1)|\,\delta_p(a,b)\frac{(b-a)^2}{24}\Psi(h).\end{aligned}$$

which gives us the best estimate at $h = \frac{3}{10}$,

$$\begin{aligned} & \left| \frac{7}{10} A^p(a,b) + \frac{13}{10} A(a^p, b^p) \right. \\ & \left. -3\frac{(b-a)^2}{10} p(p-1) L_{p-2}^{p-2}(a,b) - 2L_p^p(a,b) \right| \\ \leq & \ |p(p-1)| \, \delta_p(a,b) \frac{293(b-a)^2}{12000}. \end{aligned}$$

Moreover, at $h = 0$

$$\begin{aligned} & \left| A^p(a,b) + A(a^p, b^p) - 2L_p^p(a,b) \right| \\ \leq & \ |p(p-1)| \, \delta_p(a,b) \frac{(b-a)^2}{24}. \end{aligned}$$

2.2 Some new perturbed Ostrowski type inequalities

In this section, some new perturbed Ostrowski type inequalities are presented by working with twice differentiable functions whose first derivatives are absolutely continuous and the second derivatives belong to $\{L_i(a,b) : i = \infty, 1, p\}$, $p > 1$, the usual Lebesgue spaces which refine and generalize some previous inequalities of this domain.

2.2.1 Introduction

Recently, in [59], Zheng Liu established some more Ostrowski type inequalities for twice differentiable mappings.

In this section, we present some new perturbed Ostrowski type inequalities for twice differentiable mappings which generalize and refine the inequalities presented in [19, 20, 22, 38] and ([9], Theorem 20). The inequalities presented in [38] and ([9], Theorem 20), however, contained minor mistakes. The corrected versions [63, 86] of the inequalities are given in the form of the following theorems:

Theorem 2.3 *Let* $g : [a,b] \longrightarrow \mathbb{R}$ *be a mapping whose first derivative is absolutely continuous on* $[a,b]$ *and assume that the second derivative* $g'' \in L_\infty(a,b)$. *Then, we* *have the inequality*

$$\left|\int_a^b g(t)\,dt - \frac{1}{2}(b-a)\left[g(x) + \frac{g(a)+g(b)}{2} - \left(x - \frac{a+b}{2}\right)g'(x)\right]\right|$$
$$\leq \left(\frac{(b-a)^3}{48} + \frac{1}{3}\left|x - \frac{a+b}{2}\right|^3\right)\left\|g''\right\|_\infty, \tag{2.29}$$

for all $x \in [a,b]$.

Theorem 2.4 *Let* $g : [a,b] \longrightarrow \mathbb{R}$ *be a mapping whose first derivative is absolutely continuous on* $[a,b]$. *If we assume that the second derivative* $g'' \in L_p(a,b)$, $1 < p < \infty$, *then we have the inequality:*

$$\left|\int_a^b g(t)\,dt - \frac{1}{2}(b-a)\left[g(x) + \frac{g(a)+g(b)}{2} - \left(x - \frac{a+b}{2}\right)g'(x)\right]\right|$$
$$\leq \frac{1}{2}\left(\frac{b-a}{2}\right)^{2+\frac{1}{q}}\left\|g''\right\|_p$$
$$\times \begin{cases} \left[B(q+1,q+1) + B_{x_1}(q+1,q+1) + \Psi_{x_2}(q+1,q+1)\right]^{\frac{1}{q}}, \\ \qquad for\ x \in \left[a, \frac{a+b}{2}\right] \\ \left[B(q+1,q+1) + \Psi_{x_3}(q+1,q+1) + B_{x_4}(q+1,q+1)\right]^{\frac{1}{q}}, \\ \qquad for\ x \in \left(\frac{a+b}{2}, b\right], \end{cases} \tag{2.30}$$

where $\frac{1}{p} + \frac{1}{q} = 1,\ p > 1,\ q > 1,$ *and* $B(.,.)$ *is the Beta function of Euler given by*

$$B(l,s) = \int_0^1 t^{l-1}(1-t)^{s-1}\,dt,\ l,\ s > 0.$$

Moreover,

$$B_r(l,s) = \int_0^r t^{l-1}(1-t)^{s-1}\,dt,$$

is the incomplete Beta function,

$$\Psi_r(l,s) = \int_0^r t^{l-1}(1+t)^{s-1}\,dt,$$

is a real positive valued integral,

$$x_1 = \frac{2(x-a)}{b-a},\ x_2 = 1 - x_1,$$
$$x_3 = x_1 - 1,\ x_4 = 2 - x_1,$$

and

$$\left\|g''\right\|_p = \left(\int_a^b \left|g''(t)\right|^p dt\right)^{\frac{1}{p}}.$$

If $g'' \in L_1(a,b)$, *then*

$$\begin{aligned}&\left|\int_a^b g(t)\,dt - \frac{1}{2}\left[g(x) + \frac{g(a)+g(b)}{2} - \left(x - \frac{a+b}{2}\right)g'(x)\right](b-a)\right| \\ \leq\ & \frac{\left\|g''\right\|_1}{8}(b-a)^2, \qquad (2.31)\end{aligned}$$

where

$$\left\|g''\right\|_1 = \left(\int_a^b \left|g''(t)\right| dt\right).$$

Moreover, the special cases of the inequalities presented in the following subsection are comparable with those presented in [59] and in some cases present some new and better estimations.

2.2.2 Main Results

The following theorem holds:

Theorem 2.5 *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping whose first derivative is absolutely continuous on* $[a,b]$ *and assume that the second derivative* $f'' \in L_p(a,b)$, $1 \leq p \leq \infty$. *Then, we have the inequality:*

$$\begin{aligned}&\left|\int_a^b f(t)\,dt - (b-a)\left[(1-2h)f(x) + h(f(a)+f(b))\right.\right. \\ &\left.\left. - (1-2h)\left(x - \frac{a+b}{2}\right)f'(x)\right] + \frac{h^2}{2}(b-a)^2\left(f'(b) - f'(a)\right)\right| \\ \leq\ & E(p,x,h)\left\|f''\right\|_p, \qquad (2.32)\end{aligned}$$

for all $x \in [a,b]$ *and* $h \in [0,1]$, *where*

$$\begin{aligned}& E(p,x,h) \\ =\ & \frac{1}{2(2q+1)^{\frac{1}{q}}}\left[2h^{2q+1}(b-a)^{2q+1} + (x-a-h(b-a))^{2q+1}\right. \\ & \left. + (b-x-h(b-a))^{2q+1}\right]^{\frac{1}{q}}, \qquad (2.33)\end{aligned}$$

if $p > 1,\ q \geq 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and for* $p = 1$

$$E(1, x, h) = \begin{cases} \frac{1}{2}\left[\left(\frac{1}{2} - h\right)(b-a) + \left|x - \frac{a+b}{2}\right|\right]^2, & 0 \leq h < \frac{1}{2} \\ \frac{1}{2}h^2 (b-a)^2, & \frac{1}{2} \leq h \leq 1. \end{cases} \tag{2.34}$$

Proof. Let us define the piece-wise continuous mapping $K(.,.;h) : [a,b]^2 \to \mathbb{R}$ for $h \in [0,1]$ as:

$$K(x,t;h) = \begin{cases} \frac{1}{2}(t - a - h(b-a))^2, & if\ t \in [a,x] \\ \frac{1}{2}(t - b + h(b-a))^2, & if\ t \in (x,b]. \end{cases} \tag{2.35}$$

Integrating by parts, we obtain

$$\begin{aligned} & \int_a^b K(x,t;h) f''(t)\, dt \\ = & \int_a^x \frac{(t-a-h(b-a))^2}{2} f''(t)\, dt + \int_x^b \frac{(t-b+h(b-a))^2}{2} f''(t)\, dt \\ = & (1-2h)(b-a)\left(x - \frac{a+b}{2}\right) f'(x) + \frac{h^2}{2}(b-a)^2 \left(f'(b) - f'(a)\right) \\ & -(1-2h)(b-a) f(x) - h(b-a)(f(a)+f(b)) + \int_a^b f(t)\, dt, \end{aligned} \tag{2.36}$$

which results into the following integral identity:

$$\begin{aligned} \int_a^b f(t)\, dt = & (1-2h)(b-a) f(x) + h(b-a)(f(a)+f(b)) \\ & -(1-2h)(b-a)\left(x - \frac{a+b}{2}\right) f'(x) \\ & -\frac{h^2}{2}(b-a)^2\left(f'(b) - f'(a)\right) + \int_a^b K(x,t;h) f''(t)\, dt. \end{aligned} \tag{2.37}$$

Applying modulus on both sides of (2.36), we get

$$\Bigg| \int_a^b f(t)\, dt - (b-a)\Big[(1-2h) f(x) + h(f(a)+f(b)) \\ -(1-2h)\left(x - \frac{a+b}{2}\right) f'(x)\Big] + \frac{h^2}{2}(b-a)^2\left(f'(b) - f'(a)\right)\Bigg|$$

$$= \left| \int_a^b K(x,t;h) f''(t)\,dt \right|. \tag{2.38}$$

For fixed x, by applying Hölder's inequality on the right hand side of (2.38) for $\frac{1}{p}+\frac{1}{q}=1$ with $p>1,\ q\geq 1$, we obtain

$$\left| \int_a^b f(t)\,dt - (b-a)\left[(1-2h)f(x) + h(f(a)+f(b))\right.\right.$$
$$\left.\left. -(1-2h)\left(x-\frac{a+b}{2}\right)f'(x)\right] + \frac{h^2}{2}(b-a)^2\left(f'(b)-f'(a)\right)\right|$$
$$\leq \left\|f''\right\|_p \left(\int_a^b |K(x,t;h)|^q\,dt\right)^{\frac{1}{q}}. \tag{2.39}$$

Now simple calculation leads to

$$\int_a^b |K(x,t;h)|^p\,dt$$
$$= \frac{1}{2^q(2q+1)}\left[2h^{2q+1}(b-a)^{2q+1} + (x-a-h(b-a))^{2q+1}\right.$$
$$\left. + (b-x-h(b-a))^{2q+1}\right]. \tag{2.40}$$

Using (2.40) in (2.39), we get the required inequality (2.32) with $E(p,x,h)$ defined by (2.33).

For $p=1$, (2.38) gives

$$\left| \int_a^b f(t)\,dt - (b-a)\left[(1-2h)f(x) + h(f(a)+f(b))\right.\right.$$
$$\left.\left. -(1-2h)\left(x-\frac{a+b}{2}\right)f'(x)\right] + \frac{h^2}{2}(b-a)^2\left(f'(b)-f'(a)\right)\right|$$
$$\leq \sup_{t\in[a,b]} |K(x,t;h)| \left\|f''\right\|_1, \tag{2.41}$$

and it can be easily calculated that

$$\sup_{t\in[a,b]} |K(x,t;h)| = \begin{cases} \frac{1}{2}\left[\left(\frac{1}{2}-h\right)(b-a) + \left|x-\frac{a+b}{2}\right|\right]^2, & 0\leq h<\frac{1}{2} \\ \frac{1}{2}h^2(b-a)^2, & \frac{1}{2}\leq h\leq 1. \end{cases} \tag{2.42}$$

Combining (2.41) and (2.42), completes the proof for $1\leq p\leq\infty$. ■

We now state the inequality (2.32) explicitly for $p=\infty$ in the form of the following theorem:

Theorem 2.6 *Let f be as in Theorem 2.5, then*

$$\left| \int_a^b f(t)\,dt - (b-a)\left[(1-2h)f(x) + h(f(a)+f(b)) - (1-2h)\left(x-\frac{a+b}{2}\right)f'(x)\right] + \frac{h^2}{2}(b-a)^2\left(f'(b)-f'(a)\right) \right|$$
$$\leq \left[\frac{(b-a)^3}{24}\left(1-6h+12h^2\right) + \frac{(b-a)}{2}(1-2h)\left(x-\frac{a+b}{2}\right)^2\right]\left\|f''\right\|_\infty . \quad (2.43)$$

Remark 2.7 *By choosing $h=0$ in (2.43), we get the inequality*

$$\left| \int_a^b f(t)\,dt - (b-a)\left[f(x) - \left(x-\frac{a+b}{2}\right)f'(x)\right] \right|$$
$$\leq \left[\frac{(b-a)^3}{24} + \frac{1}{2}(b-a)\left(x-\frac{a+b}{2}\right)^2\right]\left\|f''\right\|_\infty , \quad (2.44)$$

which is exactly ([22], Theorem 2.1). Thus, Theorem 2.6 is a generalization of ([22], Theorem 2.1).

Remark 2.8 *By choosing $h=\frac{1}{4}$ in (2.43), we get the inequality*

$$\left| \int_a^b f(t)\,dt - \frac{(b-a)}{2}\left[f(x) + \frac{f(a)+f(b)}{2} - \left(x-\frac{a+b}{2}\right)f'(x)\right] + \frac{1}{32}(b-a)^2\left(f'(b)-f'(a)\right) \right|$$
$$\leq \left[\frac{(b-a)^3}{96} + \frac{1}{4}(b-a)\left(x-\frac{a+b}{2}\right)^2\right]\left\|f''\right\|_\infty . \quad (2.45)$$

It can be observed that the left hand side of (2.45) is a perturbation of left hand side of inequality (2.29). Moreover, (2.45) provides better estimations than (2.29) for $x=\frac{a+b}{2}$. Therefore, (2.45) can be regarded as its refinement.

Corollary 2.2 *Let f be as in Theorem 2.5, then the following holds:*

$$\left| \int_a^b f(t)\,dt - (b-a)\left[(1-2h)f\left(\frac{a+b}{2}\right) + h(f(a)+f(b))\right] + \frac{h^2}{2}(b-a)^2\left(f'(b)-f'(a)\right) \right|$$
$$\leq \frac{(b-a)^3}{24}\left(1-6h+12h^2\right)\left\|f''\right\|_\infty , \quad (2.46)$$

for all $h\in[0,1]$.

Proof. Putting $x = \frac{a+b}{2}$ in (2.43), we get the desired inequality (2.46). ∎

The following special cases of (2.46) hold:

Remark 2.9 *(i) For $h = 0$, (2.46) recaptures the classical midpoint inequality as follows*

$$\left| \int_a^b f(t)\,dt - (b-a)\, f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^3}{24} \left\| f'' \right\|_\infty . \tag{2.47}$$

(ii) For $h = \frac{1}{2}$, (2.46) gives

$$\left| \int_a^b f(t)\,dt - \frac{(b-a)}{2}\left(f(a) + f(b)\right) + \frac{(b-a)^2}{8}\left(f'(b) - f'(a)\right) \right| \leq \frac{(b-a)^3}{24} \left\| f'' \right\|_\infty , \tag{2.48}$$

which is a perturbed trapezoid inequality and it is not difficult to see that it is better than the classical trapezoid inequality. Moreover, it is also better than the perturbed trapezoid inequalities presented in [22] and [59] for $f'' \in L_\infty(a,b)$.

(iii) For $h = \frac{1}{4}$, (2.46) gives

$$\left| \int_a^b f(t)\,dt - \frac{(b-a)}{4}\left(f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right) + \frac{(b-a)^2}{32}\left(f'(b) - f'(a)\right) \right| \leq \frac{(b-a)^3}{96} \left\| f'' \right\|_\infty , \tag{2.49}$$

which is a new perturbed averaged trapezoid-midpoint rule and it is better than the simple average midpoint-trapezoid inequality presented in [38] and [59] for $f'' \in L_\infty(a,b)$.

(iv) For $h = \frac{1}{6}$, (2.46) gives

$$\left| \int_a^b f(t)\,dt - \frac{(b-a)}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) + \frac{(b-a)^2}{72}\left(f'(b) - f'(a)\right) \right| \leq \frac{(b-a)^3}{72} \left\| f'' \right\|_\infty , \tag{2.50}$$

which is a new perturbed variant of Simpson's inequality for twice differentiable function f. However, the simple Simpson's inequality established in [59] for $f'' \in L_\infty(a,b)$ is better than (2.50).

Corollary 2.3 *Let f be as in Theorem 2.5, then we have a family of perturbed trapezoid inequality as follows:*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}(f(a)+f(b)) + \frac{(b-a)^2}{2}\left(h^2-h+\frac{1}{2}\right)\left(f'(b)-f'(a)\right)\right|$$

$$\leq \frac{(b-a)^3}{6}\left(1-3h+3h^2\right)\left\|f''\right\|_\infty, \tag{2.51}$$

for all $h \in [0,1]$.

Proof. The inequality (2.51) can be easily obtained by choosing $x = a$ and $x = b$ in (2.43), summing up the resultant inequalities, using the triangular inequality and dividing them by 2. ■

Remark 2.10 *It may be observed that (2.20) can presents some better perturbed trapezoid inequalities as compared to the classical trapezoid rule for the range $\left(\frac{1}{2}-\frac{1}{6}\sqrt{3}, \frac{1}{2}+\frac{1}{6}\sqrt{3}\right)$ of h.*

The explicit representation of the inequality (2.32) for $p = 2$ is given as:

Corollary 2.4 *Let f be as in Theorem 2.5, then*

$$\left|\int_a^b f(t)\,dt - (b-a)\left[(1-2h)f(x) + h(f(a)+f(b))\right.\right.$$

$$\left.\left. -(1-2h)\left(x-\frac{a+b}{2}\right)f'(x)\right] + \frac{h^2}{2}(b-a)^2\left(f'(b)-f'(a)\right)\right|$$

$$\leq \frac{(b-a)^{\frac{5}{2}}}{2}\left[\frac{1}{80}\left(32h^5+(1-2h)^5\right) + \frac{1}{2}(1-2h)^3\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right.$$

$$\left. +(1-2h)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^4\right]^{\frac{1}{2}}\left\|f''\right\|_2, \tag{2.52}$$

for all $x \in [a,b]$ and $h \in [0,1]$.

Remark 2.11 *By choosing $h = 0$ in (2.52), we get the inequality*

$$\left| \int_a^b f(t)\,dt - (b-a)\left[f(x) - \left(x - \frac{a+b}{2} \right) f'(x) \right] \right|$$
$$\leq \frac{(b-a)^{\frac{5}{2}}}{2} \left[\frac{1}{80} + \frac{1}{2} \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^4 \right]^{\frac{1}{2}} \left\| f'' \right\|_2, \tag{2.53}$$

which is exactly ([19], Corollary 2.2). Thus, (2.52) is a generalization of ([19], Corollary 2.2).

Remark 2.12 *By choosing $h = \frac{1}{4}$ in (2.52), we get the inequality*

$$\left| \int_a^b f(t)\,dt - \frac{(b-a)}{2}\left[f(x) + \frac{f(a)+f(b)}{2} - \left(x - \frac{a+b}{2} \right) f'(x) \right] \right.$$
$$\left. + \frac{1}{32}(b-a)^2 \left(f'(b) - f'(a) \right) \right|$$
$$\leq \frac{(b-a)^{\frac{5}{2}}}{2} \left[\frac{1}{1280} + \frac{1}{16} \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 + \frac{1}{2} \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^4 \right]^{\frac{1}{2}} \left\| f'' \right\|_2. \tag{2.54}$$

It can be observed that the left hand side of (2.54) is a perturbation of left hand side of inequality (2.30). Moreover, (2.54) is better than inequality ([9], Corollary 11) for $x = \frac{a+b}{2}$. Therefore, (2.54) can comparatively present better estimations than ([9], Corollary 11).

Corollary 2.5 *Let f be as in Theorem 2.5, then the following holds:*

$$\left| \int_a^b f(t)\,dt - (b-a)\left[(1-2h) f\left(\frac{a+b}{2} \right) + h\left(f(a) + f(b) \right) \right] \right.$$
$$\left. + \frac{h^2}{2}(b-a)^2 \left(f'(b) - f'(a) \right) \right|$$
$$\leq \frac{(b-a)^{\frac{5}{2}}}{8\sqrt{5}} \left(32h^5 + (1-2h)^5 \right)^{\frac{1}{2}} \left\| f'' \right\|_2, \tag{2.55}$$

for all $h \in [0,1]$.

Proof. Putting $x = \frac{a+b}{2}$ in (2.52), we get the desired inequality (2.55). ■

The following special cases of (2.55) hold:

Remark 2.13 *(i) For $h = 0$ in (2.55), we recapture the following midpoint inequality for $f'' \in L_2(a,b)$*

$$\left| \int_a^b f(t)\,dt - (b-a) f\left(\frac{a+b}{2}\right) \right| \le \frac{(b-a)^{\frac{5}{2}}}{8\sqrt{5}} \left\| f'' \right\|_2 . \tag{2.56}$$

(ii) For $h = \frac{1}{2}$, (2.55) gives

$$\left| \int_a^b f(t)\,dt - \frac{(b-a)}{2}\left(f(a) + f(b)\right) + \frac{(b-a)^2}{8}\left(f'(b) - f'(a)\right) \right| \le \frac{(b-a)^{\frac{5}{2}}}{8\sqrt{5}} \left\| f'' \right\|_2 , \tag{2.57}$$

which is a perturbed trapezoid inequality and it is not difficult to see that (2.57) is comparable with the best bound of inequality of this type established in [19] and [59].

(iii) For $h = \frac{1}{4}$, (2.55) gives

$$\left| \int_a^b f(t)\,dt - \frac{(b-a)}{4}\left(f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right) + \frac{(b-a)^2}{32}\left(f'(b) - f'(a)\right) \right| \le \frac{(b-a)^{\frac{5}{2}}}{32\sqrt{5}} \left\| f'' \right\|_2 , \tag{2.58}$$

which is a new perturbed averaged trapezoid-midpoint rule for $f'' \in L_2(a,b)$ and is better than the simple average midpoint-trapezoid inequality presented in [59].

(iv) For $h = \frac{1}{6}$, (2.55) gives

$$\left| \int_a^b f(t)\,dt - \frac{(b-a)}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) + \frac{(b-a)^2}{72}\left(f'(b) - f'(a)\right) \right| \le \frac{11}{72\sqrt{55}} (b-a)^{\frac{5}{2}} \left\| f'' \right\|_2 , \tag{2.59}$$

which is a new perturbed variant of Simpson's inequality for twice differentiable function f for $f'' \in L_2(a,b)$ and is better than the simple Simpson's inequality presented in [59].

Corollary 2.6 *Let f be as in Theorem 2.5, then we have a family of perturbed trapezoid inequality*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}(f(a)+f(b)) + \frac{(b-a)^2}{2}\left(h^2 - h + \frac{1}{2}\right)\left(f'(b) - f'(a)\right)\right|$$

$$\leq \frac{(b-a)^{\frac{5}{2}}}{2\sqrt{5}}\left(h^5 + (1-h)^5\right)^{\frac{1}{2}}\left\|f''\right\|_2, \tag{2.60}$$

for all $h \in [0,1]$.

Proof. The inequality (2.60) can be easily obtained by choosing $x = a$ and $x = b$ in (2.52), summing up the resultant inequalities, using the triangular inequality and dividing them by 2. ■

The following special cases of perturbed Ostrowski type inequality (2.32) for $p = 1$ hold:

Remark 2.14 *By choosing $h = 0$ in (2.32) and (2.34), we get the inequality*

$$\left|\int_a^b f(t)\,dt - (b-a)\left[f(x) - \left(x - \frac{a+b}{2}\right)f'(x)\right]\right|$$

$$\leq \frac{1}{2}\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^2\left\|f''\right\|_1. \tag{2.61}$$

which is exactly ([20], Theorem 2.1). Thus, the inequality (2.32) together with (2.34) generalizes ([20], Theorem 2.1).

Remark 2.15 *By choosing $h = \frac{1}{4}$ in (2.32) and (2.34), we get the inequality*

$$\left|\int_a^b f(t)\,dt - \frac{(b-a)}{2}\left[f(x) + \frac{f(a)+f(b)}{2} - \left(x - \frac{a+b}{2}\right)f'(x)\right]\right.$$

$$\left. + \frac{1}{32}(b-a)^2\left(f'(b) - f'(a)\right)\right|$$

$$\leq \frac{1}{2}\left[\frac{b-a}{4} + \left|x - \frac{a+b}{2}\right|\right]^2\left\|f''\right\|_1. \tag{2.62}$$

It can be observed that the left hand side of (2.62) is a perturbation of left hand side of inequality (2.31). Moreover, (2.62) is better than (2.31) for $x = \frac{a+b}{2}$. Therefore, (2.62) can comparatively present better and refined estimations than (2.31).

Corollary 2.7 *Let f be as in Theorem 2.5, then the following holds:*

$$\left| \int_a^b f(t)\,dt - (b-a)\left[(1-2h) f\left(\frac{a+b}{2}\right) + h\left(f(a)+f(b)\right)\right] + \frac{h^2}{2}(b-a)^2\left(f'(b)-f'(a)\right)\right|$$

$$\leq \begin{cases} \frac{(b-a)^2}{8}(1-2h)^2, & 0 \leq h < \frac{1}{2} \\ \frac{1}{2}h^2(b-a)^2, & \frac{1}{2} \leq h \leq 1 \end{cases} \left\| f'' \right\|_1, \qquad (2.63)$$

for all $h \in [0,1]$.

Proof. Putting $x = \frac{a+b}{2}$ in (2.32) and (2.34), we get the desired inequality (2.63). ■

Remark 2.16 *(i) By choosing $h = 0$ in (2.63), we recapture the midpoint inequality for $f'' \in L_1(a,b)$ as follows*

$$\left| \int_a^b f(t)\,dt - (b-a) f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{8}\left\| f'' \right\|_1. \qquad (2.64)$$

(ii) By choosing $h = \frac{1}{4}$ in (2.63), we get

$$\left| \int_a^b f(t)\,dt - \frac{(b-a)}{4}\left(f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right) + \frac{(b-a)^2}{32}\left(f'(b)-f'(a)\right)\right| \leq \frac{(b-a)^2}{32}\left\| f'' \right\|_1, \qquad (2.65)$$

which is a new perturbed average trapezoid-midpoint rule with $f'' \in L_1(a,b)$.

(iii) By choosing $h = \frac{1}{6}$ in (2.63), we obtain

$$\left| \int_a^b f(t)\,dt - \frac{(b-a)}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) + \frac{(b-a)^2}{72}\left(f'(b)-f'(a)\right)\right| \leq \frac{(b-a)^2}{18}\left\| f'' \right\|_1, \qquad (2.66)$$

which is a new perturbed variant of Simpson's inequality for twice differentiable function f with $f'' \in L_1(a,b)$.

Corollary 2.8 *Let f be as in Theorem 2.5, then we have a family of perturbed trapezoid inequality*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}(f(a)+f(b)) + \frac{(b-a)^2}{2}\left(h^2-h+\frac{1}{2}\right)\left(f'(b)-f'(a)\right)\right|$$

$$\leq \frac{(b-a)^2}{2}\left\|f''\right\|_1 \begin{cases} (1-h)^2, 0\leq h\leq \frac{1}{2}, \\ h^2 \quad , \frac{1}{2}<h\leq 1, \end{cases} \tag{2.67}$$

for all $h \in [0,1]$.

Proof. The inequality (2.67) can be easily obtained by choosing $x = a$ and $x = b$ in (2.32) and (2.34), summing up the resultant inequalities, using the triangular inequality and dividing them by 2. ■

Remark 2.17 *For $h = \frac{1}{2}$ in (2.67), we have the following perturbed trapezoid inequality:*

$$\left|\int_a^b f(t)\,dt - \frac{(b-a)}{2}(f(a)+f(b)) + \frac{(b-a)^2}{8}\left(f'(b)-f'(a)\right)\right|$$

$$\leq \frac{(b-a)^2}{8}\left\|f''\right\|_1, \tag{2.68}$$

which has been obtained in [17].

2.2.3 Applications in Numerical Integration

We may use Theorem 2.5 to get the estimates of composite quadrature rules with smaller error than that which may be obtained by the classical results.

Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $h_i = x_{i+1} - x_i$, $\delta \in [0,1]$, $x_i \leq \zeta_i \leq x_{i+1}$, $i = 0, \cdots, n-1$, be a sequence of intermediate points, then the following theorems hold:

Theorem 2.7 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping whose first derivative is absolutely continuous on $[a,b]$ and assume that the second derivative $f'' \in L_\infty(a,b)$. Then, we have the following quadrature formula:*

$$\int_a^b f(t)\ dt = A(f, f', I_n, \zeta, \delta) + R_1(f, f', I_n, \zeta, \delta)$$

where

$$A(f,f^{'},I_n,\zeta,\delta)=(1-2\delta)\sum_{i=0}^{n-1}h_i f(\zeta_i)+\delta\sum_{i=0}^{n-1}h_i\left(f(x_i)+f(x_{i+1})\right)$$

$$-(1-2\delta)\sum_{i=0}^{n-1}h_i\left(\zeta_i-\frac{x_i+x_{i+1}}{2}\right)f^{'}(\zeta_i), \tag{2.69}$$

and the remainder satisfies the estimation:

$$\left|R_1(f,f^{'},I_n,\zeta,\delta)\right| \leq \left\|f^{''}\right\|_{\infty}\left[\frac{1}{24}\left(1-6\delta+12\delta^2\right)\sum_{i=0}^{n-1}h_i^3 + \frac{(1-2\delta)}{2}\sum_{i=0}^{n-1}h_i\left(\zeta_i-\frac{x_i+x_{i+1}}{2}\right)^2\right], \tag{2.70}$$

for all $\delta\in[0,1]$.

Proof. Applying inequality (2.43) on $\zeta_i\in[x_i,x_{i+1}]$ $(i=0,...,n-1)$ and summing over i from 0 to $n-1$ and using triangular inequality, we get (2.70). ■

Theorem 2.8 *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping whose first derivative is absolutely continuous on* $[a,b]$ *and assume that the second derivative* $f^{''}\in L_p(a,b)$, $p>1$. *Then, we have the following quadrature formula:*

$$\int_a^b f(t)\ dt = A(f,f^{'},I_n,\zeta,\delta)+R_2(f,f^{'},I_n,\zeta,\delta)$$

where

$$A(f,f^{'},I_n,\zeta,\delta)=(1-2\delta)\sum_{i=0}^{n-1}h_i f(\zeta_i)+\delta\sum_{i=0}^{n-1}h_i\left(f(x_i)+f(x_{i+1})\right)$$

$$-(1-2\delta)\sum_{i=0}^{n-1}h_i\left(\zeta_i-\frac{x_i+x_{i+1}}{2}\right)f^{'}(\zeta_i),$$

and the remainder satisfies the estimation:

$$\left|R_2(f,f^{'},I_n,\zeta,\delta)\right| \leq \left\|f^{''}\right\|_{p}\frac{1}{2\left(2q+1\right)^{\frac{1}{q}}}\left[\sum_{i=0}^{n-1}\left(2\delta^{2q+1}h_i^{2q+1} + \left(\zeta_i-x_i-\delta h_i\right)^{2q+1} + \left(x_{i+1}-\zeta_i-\delta h_i\right)^{2q+1}\right)\right]^{\frac{1}{q}}, \tag{2.71}$$

for all $\delta\in[0,1]$.

Proof. Applying inequalities (2.32) and (2.33) on $\zeta_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ and summing over i from 0 to $n-1$ and using triangular inequality, we get (2.71). ■

Theorem 2.9 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping whose first derivative is absolutely continuous on $[a,b]$ and assume that the second derivative $f'' \in L_1(a,b)$, $p > 1$. Then, we have the following quadrature formula:*

$$\int_a^b f(t)\ dt = A(f, f', I_n, \zeta, \delta) + R_3(f, f', I_n, \zeta, \delta)$$

where

$$A(f, f', I_n, \zeta, \delta) = (1-2\delta)\sum_{i=0}^{n-1} h_i f(\zeta_i) + \delta \sum_{i=0}^{n-1} h_i \left(f(x_i) + f(x_{i+1})\right)$$
$$-(1-2\delta)\sum_{i=0}^{n-1} h_i \left(\zeta_i - \frac{x_i + x_{i+1}}{2}\right) f'(\zeta_i),$$

and the remainder satisfies the estimation:

$$\left|R_3(f, f', I_n, \zeta, \delta)\right|$$
$$\leq \frac{1}{2}\left\|f''\right\|_1 \begin{cases} \left[\left|\frac{1}{2} - \delta\right| v(h) + \sup_{i=0,...,n-1}\left|\zeta_i - \frac{x_i + x_{i+1}}{2}\right|\right]^2, & 0 \leq \delta < \frac{1}{2} \\ \frac{1}{2}\delta^2 v^2(h), & \frac{1}{2} \leq \delta < 1 \end{cases} \tag{2.72}$$

where $v(h) = \max\{h_i | i = 0, ..., n-1\}$.

Proof. Applying inequalities (2.32) and (2.34) on $\zeta_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ and summing over i from 0 to $n-1$ and using triangular inequality, we get (2.72). ■

Theorem 2.10 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping whose first derivative is absolutely continuous on $[a,b]$ and assume that the second derivative $f'' \in L_2(a,b)$. Then, we have the following quadrature formula:*

$$\int_a^b f(t)\ dt = A(f, f', I_n, \zeta, \delta) + R_4(f, f', I_n, \zeta, \delta),$$

where

$$A(f, f', I_n, \zeta, \delta) = (1-2\delta)\sum_{i=0}^{n-1} h_i f(\zeta_i) + \delta \sum_{i=0}^{n-1} h_i \left(f(x_i) + f(x_{i+1})\right)$$

$$- (1-2\delta)\sum_{i=0}^{n-1} h_i \left(\zeta_i - \frac{x_i + x_{i+1}}{2}\right) f^{'}(\zeta_i)$$

and the remainder satisfies the estimation:

$$\begin{aligned}
\left|R_4(f,f^{'},I_n,\zeta,\delta)\right| &\leq \frac{1}{2}\left\|f^{''}\right\|_2 \left[\frac{1}{80}\left(32\delta^5 + (1-2\delta)^5\right)\sum_{i=0}^{n-1} h_i^5 \right. \\
&+\frac{1}{2}(1-2\delta)^3 \sum_{i=0}^{n-1} h_i^3 \left(\zeta_i - \frac{x_i + x_{i+1}}{2}\right)^2 \\
&\left. +(1-2\delta)\sum_{i=0}^{n-1} h_i \left(\zeta_i - \frac{x_i + x_{i+1}}{2}\right)^4\right]^{\frac{1}{2}}, \qquad (2.73)
\end{aligned}$$

for all $\delta \in [0,1]$.

Proof. Applying inequality (2.52) on $\zeta_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ and summing over i from 0 to $n-1$ and using triangular inequality, we get (2.73). ■

2.3 A generalization of Ostrowski type inequality for (l, L) Lipschitzian mappings

In this section, we present an Ostrowski type inequality for n-times differentiable $(l, L)-$Lipschitzian functions. The presented inequality is a generalization of Ostrowski inequality for $L-$Lipschitzian and $(l, L)-$Lipschitzian functions and recaptures many previous results as special cases.

2.3.1 Introduction

In [30], S. S. Dragomir obtained Ostrowski's integral inequality for Lipschitzian mappings as follows:

Theorem 2.11 *Let* $u : [a,b] \to \mathbb{R}$ *be L-Lipschitzian mapping on* $[a,b]$ *i.e.,*

$$|u(x) - u(y)| \leq L|x-y|, \text{ for all } x \in [a,b].$$

Then, we have the inequality:

$$\left|\int_a^b u(t)\,dt - (b-a)\,u(x)\right| \leq L\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2, \qquad (2.74)$$

for all $x \in [a,b]$. *The constant* $\frac{1}{4}$ *is the best possible one.*

In [15], P. Cerone and S. S. Dragomir obtained three point inequalities of Ostowski and Grüss type in usual Lebesgue spaces and for Lipschitzian, monotonic and mappings of bounded total variation. They further generalized the inequalities obtained in [15] for n-differentiable function f, where $f^{(n)} \in L_p(a,b)$, $p > 1$ in [16]. Three-point inequality of Ostrowski type for L-Lipschitzian function obtained in [15] is stated below:

Theorem 2.12 *Let $f : [a,b] \to \mathbb{R}$ be L-Lipschitzian on $[a,b]$. Then, the following inequality holds:*

$$\left| \int_a^b f(t)\,dt - [(\alpha(x) - a) f(a) + (\beta(x) - \alpha(x)) f(x) + (b - \beta(x)) f(b)] \right|$$

$$\leq L \left\{ \left[\left(\frac{b-a}{2} \right)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] + \left(\alpha(x) - \frac{a+x}{2} \right)^2 + \left(\beta(x) - \frac{x+b}{2} \right)^2 \right\}, \tag{2.75}$$

where $\alpha : [a,x] \to \mathbb{R}$ and $\beta : (x,b] \to \mathbb{R}$.

Some special cases of Theorem 2.12 may also be considered for choice of $\alpha(x)$ and $\beta(x)$ as given in [15]. It may also be noted that (2.75) is a generalization of (2.74) for $\alpha(x) = a$ and $\beta(x) = b$.

In some recent papers [58, 57], Zheng Liu has obtained inequalities of Ostrowski and Grüss type for (l, L)-Lipschitzian mappings. Ostrowski type inequality obtained by Liu in [57] for (l, L)-Lipschitzian function is stated as follows:

Theorem 2.13 *Let $f : [a,b] \to \mathbb{R}$ be (l,L)-Lipschitzian on $[a,b]$. Then, for all $x \in [a,b]$, we have:*

$$\left| \frac{1}{2} [(x-a) f(a) + (b-a) f(x) + (b-x) f(b)] - \int_a^b f(t)\,dt \right|$$

$$\leq \frac{1}{4} \left[\left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{4} \right] (L - l), \tag{2.76}$$

$$\left| \frac{1}{2} [(x-a) f(a) + (b-a) f(x) + (b-x) f(b)] - \int_a^b f(t)\,dt \right|$$

$$\leq \frac{b-a}{2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right](S-l) \tag{2.77}$$

and

$$\left|\frac{1}{2}\left[(x-a)f(a)+(b-a)f(x)+(b-x)f(b)\right]-\int_a^b f(t)\,dt\right|$$
$$\leq \frac{b-a}{2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right](L-S), \tag{2.78}$$

where $S=\frac{f(b)-f(a)}{b-a}$.

In this section, we give a generalization of Theorem 2.11, Theorem 2.12 and Theorem 2.13 for $n-$times differentiable (l,L)-Lipschitzian functions.

For the sake of convenience, we re-state some definitions, lemmas and identities which are intended to be used to obtain our desired generalization.

Definition 2.1 *The function* $f:[a,b]\rightarrow\mathbb{R}$ *is said to be L-Lipschitzian on* $[a,b]$, *if for some* $L>0$ *and all* $x,y\in[a,b]$,

$$|f(x)-f(y)|\leq L|x-y|.$$

Definition 2.2 *The function* $f:[a,b]\rightarrow\mathbb{R}$ *is said to be* (l,L)*-Lipschitzian on* $[a,b]$ *if*

$$l(x_2-x_1)\leq f(x_2)-f(x_1)\leq L(x_2-x_1)\ \ \textit{for}\ a\leq x_1\leq x_2\leq b,$$

where $l,\ L\in\mathbb{R}$ *with* $l<L$.

Remark 2.18 *It may be noted that a* (l,L)*-Lipschitzian function is a L-Lipschitzian function for* $l=-L$.

The following known lemmas are useful in the sequel.

Lemma 2.1 *(see [15]) Let* $g,\upsilon:[a,b]\rightarrow\mathbb{R}$ *be such that* g *is Riemann integrable on* $[a,b]$ *and* υ *is L-Lipschitzian on* $[a,b]$. *Then,*

$$\left|\int_a^b g(t)\,d\upsilon(t)\right|\leq L\int_a^b |g(t)|\,dt,$$

where υ *is L-Lipschitzian if it satisfies*

$$|\upsilon(x)-\upsilon(y)|\leq L|x-y|,$$

for all $x,y\in[a,b]$.

Lemma 2.2 *(see [15]) Let $g, v : [a,b] \to \mathbb{R}$ be such that g is continuous and v is of bounded variation on $[a,b]$. Then, the Riemann-Stieltjes integral $\int_a^b g(t)\,dv(t)$ exists and is such that*

$$\left| \int_a^b g(t)\,dv(t) \right| \le \sup_{t\in[a,b]} |g(t)| \bigvee_a^b (v),$$

where $\bigvee_a^b (v)$ is the total variation of v on $[a,b]$.

Moreover, we will also use the following identity but from Riemann-Stieltjes point of view:

Theorem 2.14 *(see [16]) Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ with $\alpha : [a,b] \to [a,b]$ and $\beta : [a,b] \to [a,b]$, $\alpha(x) \le x \le \beta(x)$, then for all $x \in [a,b]$, the following identity holds:*

$$(-1)^n \int_a^b K_n(x,t) f^{(n)}(t)\,dt = \int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{1}{k!} \left[R_k(x) f^{(k-1)}(x) + S_k(x) \right], \tag{2.79}$$

where the kernel $K_n : [a,b]^2 \to \mathbb{R}$ is given by

$$K_n(x,t) = \begin{cases} \frac{(t-\alpha(x))^n}{n!}, & t \in [a,x] \\ \\ \frac{(t-\beta(x))^n}{n!}, & t \in (x,b]. \end{cases} \tag{2.80}$$

$$R_k(x) = (\beta(x) - x)^k + (-1)^{k-1} (x - \alpha(x))^k \tag{2.81}$$

and

$$S_k(x) = (\alpha(x) - a)^k f^{(k-1)}(a) + (-1)^{k-1} (b - \beta(x))^k f^{(k-1)}(b). \tag{2.82}$$

2.3.2 Main Results

Theorem 2.15 *Let $f : [a,b] \to \mathbb{R}$ be a n-times differentiable function and let $f^{(n-1)}$ be (l,L)-Lipschitzian function. Let $\alpha : [a,b] \to [a,b]$ and $\beta : [a,b] \to [a,b]$, $\alpha(x) \le x \le \beta(x)$. Then, for all $x \in [a,b]$, we have:*

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - \frac{(b^2-a^2)}{4}(L+l) - \sum_{k=1}^{n} \frac{1}{k!} \left[R_k(x) f^{(k-1)}(x) + S_{k,f}(x) \right.\right. \\ &\left.\left. -\frac{L+l}{2} (xR_k(x) + P_k(x)) \right] \right| \\ \le\ & \frac{L-l}{2} Q_n(x), \end{aligned} \tag{2.83}$$

$$\left| \int_a^b f(t)\,dt - \frac{(b^2-a^2)}{4}(L+l) - \sum_{k=1}^{n} \frac{1}{k!} \left[R_k(x) f^{(k-1)}(x) + S_{k,f}(x) \right. \right.$$
$$\left.\left. -\frac{L+l}{2}\left(xR_k(x)+P_k(x)\right)\right]\right|$$
$$\leq \frac{1}{n!}(b-a)\left(D^{n-1}-l\right)M^n(x) \tag{2.84}$$

and

$$\left| \int_a^b f(t)\,dt - \frac{(b^2-a^2)}{4}(L+l) - \sum_{k=1}^{n} \frac{1}{k!} \left[R_k(x) f^{(k-1)}(x) + S_{k,f}(x) \right. \right.$$
$$\left.\left. -\frac{L+l}{2}\left(xR_k(x)+P_k(x)\right)\right]\right|$$
$$\leq \frac{1}{n!}(b-a)\left(L-D^{n-1}\right)M^n(x), \tag{2.85}$$

with

$$\begin{aligned}
R_k(x) &= (\beta(x)-x)^k + (-1)^{k-1}(x-\alpha(x))^k, \\
P_k(x) &= (\alpha(x)-a)^k a + (-1)^{k-1}(b-\beta(x))^k b, \\
S_{k,f}(x) &= (\alpha(x)-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-\beta(x))^k f^{(k-1)}(b), \\
Q_n(x) &= \frac{1}{(n+1)!}\left[(\alpha(x)-a)^{n+1} + (x-\alpha(x))^{n+1}\right. \\
&\quad \left. + (\beta(x)-x)^{n+1} + (b-\beta(x))^{n+1}\right], \\
M(x) &= \max\{\alpha(x)-a, x-\alpha(x), x-\beta(x), b-\beta(x)\}
\end{aligned} \tag{2.86}$$

and

$$D^{n-1} = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}. \tag{2.87}$$

Proof. Let us consider kernel defined by (2.80)

$$K_n(x,t) = \begin{cases} \frac{(t-\alpha(x))^n}{n!}, & t \in [a,x] \\ \\ \frac{(t-\beta(x))^n}{n!}, & t \in (x,b]. \end{cases}$$

Let

$$g^{(n-1)}(t) = f^{(n-1)}(t) - \frac{L+l}{2}t. \tag{2.88}$$

It may be observed that the function $g^{(n-1)}(t)$ is M-Lipschitzian on $[a,b]$ with $M = \frac{L-l}{2}$. So, the Riemann-Stieltjes integral $\int_a^b K_n(x,t)\,dg^{(n-1)}(t)$ exists and we have by applying integration by parts formula for Riemann-Stieltjes integral:

$$(-1)^n \int_a^b K_n(x,t)\, dg^{(n-1)}(t) = \int_a^b g(t)\, dt - \sum_{k=1}^{n} \frac{1}{k!}\left[R_k(x)\, g^{(k-1)}(x) + S_{k,g}(x)\right], \tag{2.89}$$

where

$$R_k(x) = (\beta(x) - x)^k + (-1)^{k-1}(x - \alpha(x))^k$$

and

$$S_{k,g}(x) = (\alpha(x) - a)^k g^{(k-1)}(a) + (-1)^{k-1}(b - \beta(x))^k g^{(k-1)}(b). \tag{2.90}$$

From Lemma 2.1, we have

$$\left|\int_a^b K_n(x,t)\, dg^{(n-1)}(t)\right| \le \frac{L-l}{2} \int_a^b |K_n(x,t)|\, dt. \tag{2.91}$$

As calculated in [16],

$$\begin{aligned} \int_a^b |K_n(x,t)|\, dt &= \frac{1}{(n+1)!}\left[(\alpha(x) - a)^{n+1} + (x - \alpha(x))^{n+1}\right. \\ &\quad \left. + (\beta(x) - x)^{n+1} + (b - \beta(x))^{n+1}\right] \\ &= Q_n(x). \end{aligned} \tag{2.92}$$

So, from (2.89), (2.91) and (2.92), we obtain:

$$\begin{aligned} &\left|\int_a^b g(t)\, dt - \sum_{k=1}^{n} \frac{1}{k!}\left[R_k(x)\, g^{(k-1)}(x) + S_{k,g}(x)\right]\right| \\ &\le \frac{L-l}{2} Q_n(x). \end{aligned} \tag{2.93}$$

Consequently, substituting (2.88) in (2.93), we get the required inequality (2.83).

Next, let

$$\begin{aligned} g_1^{(n-1)}(t) &= f^{(n-1)}(t) - lt, \\ g_2^{(n-1)}(t) &= f^{(n-1)}(t) - Lt. \end{aligned} \tag{2.94}$$

It may be observed that $g_1^{(n-1)}(t)$ and $g_2^{(n-1)}(t)$ are functions of bounded variations on $[a,b]$ and

$$\begin{aligned} \bigvee_a^b \left(g_1^{(n-1)}\right) &= f^{(n-1)}(b) - f^{(n-1)}(a) - l(b-a), \\ \bigvee_a^b \left(g_2^{(n-1)}\right) &= L(b-a) - \left[f^{(n-1)}(b) - f^{(n-1)}(a)\right]. \end{aligned} \tag{2.95}$$

So, the Riemann-Stieltjes integrals $\int_a^b K_n(x,t)\,dg_1^{(n-1)}(t)$ and $\int_a^b K_n(x,t)\,dg_2^{(n-1)}(t)$ exist and we have by applying integration by parts formula for Riemann-Stieltjes integral:

$$(-1)^n \int_a^b K_n(x,t)\,dg_1^{(n-1)}(t) = \int_a^b g(t)\,dt - \sum_{k=1}^{n} \frac{1}{k!}\left[R_k(x)\,g_1^{(n-1)}(x) + S_{k,g_1}(x)\right] \tag{2.96}$$

and

$$(-1)^n \int_a^b K_n(x,t)\,dg_2^{(n-1)}(t) = \int_a^b g(t)\,dt - \sum_{k=1}^{n} \frac{1}{k!}\left[R_k(x)\,g_2^{(n-1)}(x) + S_{k,g_2}(x)\right], \tag{2.97}$$

where

$$R_k(x) = (\beta(x) - x)^k + (-1)^{k-1}(x - \alpha(x))^k,$$

and

$$\begin{aligned}
& S_{k,g_1}(x) \\
= \;& (\alpha(x) - a)^k g_1^{(n-1)}(a) + (-1)^{k-1}(b - \beta(x))^k g_1^{(n-1)}(b), \\
& S_{k,g_2}(x) \\
= \;& (\alpha(x) - a)^k g_2^{(n-1)}(a) + (-1)^{k-1}(b - \beta(x))^k g_2^{(n-1)}(b).
\end{aligned} \tag{2.98}$$

From Lemma 2.2, we have

$$\begin{aligned}
\left|\int_a^b K_n(x,t)\,dg_1^{(n-1)}(t)\right| &\le \max_{t\in[a,b]} |K_n(x,t)| \bigvee_a^b \left(g_1^{(n-1)}\right), \\
\left|\int_a^b K_n(x,t)\,dg_2^{(n-1)}(t)\right| &\le \max_{t\in[a,b]} |K_n(x,t)| \bigvee_a^b \left(g_2^{(n-1)}\right)
\end{aligned} \tag{2.99}$$

As calculated in [16], we have:

$$\begin{aligned}
\max_{t\in[a,b]} |K_n(x,t)| &= \frac{1}{n!}\left(\max\{\alpha(x) - a, x - \alpha(x), x - \beta(x), b - \beta(x)\}\right)^n \\
&= \frac{1}{n!} M^n(x).
\end{aligned} \tag{2.100}$$

So, from (2.95), (2.96), (2.97), (2.99) and (2.100), we obtain:

$$\begin{aligned}
& \left|\int_a^b g(t)\,dt - \sum_{k=1}^{n} \frac{1}{k!}\left[R_k(x)\,g_1^{(n-1)}(x) + S_{k,g_1}(x)\right]\right| \\
\le \;& \frac{1}{n!}(b-a)\left(D^{n-1} - l\right) M^n(x)
\end{aligned} \tag{2.101}$$

and

$$\left|\int_a^b g(t)\,dt - \sum_{k=1}^{n}\frac{1}{k!}\left[R_k(x)\,g_1^{(n-1)}(x) + S_{k,g_1}(x)\right]\right|$$
$$\leq \frac{1}{n!}(b-a)\left(L - D^{n-1}\right)M^n(x), \tag{2.102}$$

where

$$D^{n-1} = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}.$$

By substituting (2.95) in (2.101) and (2.102), we get the required inequalities (2.84) and (2.85). ■

Corollary 2.9 *Let f be as in Theorem 2.15. Then, for all $x \in [a,b]$, we have:*

$$\left|\int_a^b f(t)\,dt - \frac{(b^2-a^2)}{4}(L+l) - \sum_{k=1}^{n}\frac{1}{k!}\left[(1-h)^k r_k(x)\,f^{(k-1)}(x)\right.\right.$$
$$\left.\left.+h^k s_{k,f}(x) - \frac{L+l}{2}\left(x(1-h)^k r_k(x) + h^k p_k(x)\right)\right]\right|$$
$$\leq \frac{L-l}{2(n+1)!}H(h)\,G(x), \tag{2.103}$$

$$\left|\int_a^b f(t)\,dt - \frac{(b^2-a^2)}{4}(L+l) - \sum_{k=1}^{n}\frac{1}{k!}\left[(1-h)^k r_k(x)\,f^{(k-1)}(x)\right.\right.$$
$$\left.\left.+h^k s_{k,f}(x) - \frac{L+l}{2}\left(x(1-h)^k r_k(x) + h^k p_k(x)\right)\right]\right|$$
$$\leq \frac{(b-a)}{n!}\left(D^{n-1} - l\right)\upsilon^n(x) \tag{2.104}$$

and

$$\left|\int_a^b f(t)\,dt - \frac{(b^2-a^2)}{4}(L+l) - \sum_{k=1}^{n}\frac{1}{k!}\left[(1-h)^k r_k(x)\,f^{(k-1)}(x)\right.\right.$$
$$\left.\left.+h^k s_{k,f}(x) - \frac{L+l}{2}\left(x(1-h)^k r_k(x) + h^k p_k(x)\right)\right]\right|$$
$$\leq \frac{(b-a)}{n!}\left(L - D^{n-1}\right)\upsilon^n(x), \tag{2.105}$$

with

$$\begin{aligned}
r_k(x) &= (b-x)^k + (-1)^{k-1}(x-a)^k, \\
p_k(x) &= (x-a)^k a + (-1)^{k-1}(b-x)^k b, \\
s_{k,f}(x) &= (x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b), \\
G(x) &= (x-a)^{n+1} + (b-x)^{n+1}, \\
\upsilon(x) &= \left[\frac{1}{2} + \left|h - \frac{1}{2}\right|\right]\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right], \\
H(h) &= h^{n+1} + (1-h)^{n+1}, \qquad (2.106)
\end{aligned}$$

and D^{n-1} *is defined by (2.87).*

Proof. By choosing,

$$\begin{aligned}
\alpha(x) &= hx + (1-h)a, \\
\beta(x) &= hx + (1-h)b,
\end{aligned}$$

in (2.83), (2.84), (2.85) and (2.86), readily produces the required inequalities. ■

Remark 2.19 *It may be noted that for* $n = 1$, $\alpha(x) = a$, $\beta(x) = b$ *and* $l = -L$ *in (2.83), (2.74) is obtained.*

Remark 2.20 *It may be noted that for* $n = 1$ *and* $l = -L$ *in (2.83), (2.75) is obtained.*

Remark 2.21 *It may be noted that for* $n = 1$ *and* $h = \frac{1}{2}$ *in (2.103), (2.104), (2.105) and (2.106), the inequalities (2.76), (2.77) and (2.78) are recaptured.*

2.3.3 Applications in Numerical Integration

We may use Corollary 2.9 to get the estimates of composite quadrature rules with smaller error than that which may be obtained by the classical results.

Let $I_m : a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b$ be a division of the interval $[a, b]$ let $\zeta = (\zeta_0, ..., \zeta_{m-1})$ be a sequence of intermediate points where $\zeta_j \in [x_j, x_{j+1}]$ for $j = 0, 1, ..., m-1$, then the following theorem hold:

Theorem 2.16 *Let f be as in Theorem 2.15. Then, we have the following quadrature formula:*

$$\int_a^b f(t)\ dt = A_{m,n}(f, I_m, \zeta, \delta) + R_{m,n,1}(f, I_m, \zeta, \delta),$$

where

$$\begin{aligned} A_{m,n}(f, I_m, \zeta, \delta) &= \sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\frac{(-1)^k}{k!}\Big\{(1-\delta)^k r_k(\zeta_j) f^{(k-1)}(\zeta_j) \\ &\quad +\delta^k \left(A_j^k f^{(k-1)}(x_j) + (-1)^{k-1} B_j^k f^{(k-1)}(x_{j+1})\right) \end{aligned}$$

$$-\frac{L+l}{2}\left((1-\delta)^k r_k(\zeta_j)\zeta_j + \delta^k p_k(\zeta_j)\right)\Big\} - \frac{1}{4}(l+L)\sum_{j=0}^{m-1}\left(x_{j+1}^2 - x_j^2\right) \qquad (2.107)$$

and

$$\begin{aligned} A_j &= \zeta_j - x_j,\ B_j = x_{j+1} - \zeta_j,\ h_j = A_j + B_j = x_{j+1} - x_j, \\ r_k(\zeta_j) &= B_j^k + (-1)^{k-1} A_j^k,\ p_k(\zeta_j) = A_j^k x_j + (-1)^{k-1} B_j^k x_{j+1}, \end{aligned}$$

for $j = 0, ..., m-1$ and the remainder satisfies the estimation:

$$|R_{m,n,1}(f, I_m, \zeta, \delta)| \le \frac{L-l}{2(n+1)!} H(\delta) \sum_{j=0}^{m-1}\left(A_j^{n+1} + B_j^{n+1}\right), \qquad (2.108)$$

$$H(\delta) = \delta^{n+1} + (1-\delta)^{n+1},$$

for all $\delta \in [0, 1]$.

Proof. Applying inequality (2.103) on $\zeta_i \in [x_j, x_{j+1}]$ $(j = 0, ..., m-1)$ and summing over j from 0 to $m-1$ and using triangular inequality, we get (2.108). ■

Theorem 2.17 *Let f be as in Theorem 2.15. Then, we have the following quadrature formula:*

$$\int_a^b f(t)\ dt = A_{m,n}(f, I_m, \zeta, \delta) + R_{m,n,2}(f, I_m, \zeta, \delta),$$

where

$$\begin{aligned} A_{m,n}(f, I_m, \zeta, \delta) &= \sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\frac{(-1)^k}{k!}\Big\{(1-\delta)^k r_k(\zeta_j) f^{(k-1)}(\zeta_j) \\ &\quad +\delta^k \left(A_j^k f^{(k-1)}(x_j) + (-1)^{k-1} B_j^k f^{(k-1)}(x_{j+1})\right) \end{aligned}$$

$$-\frac{L+l}{2}\left((1-\delta)^k r_k\left(\zeta_j\right)\zeta_j+\delta^k p_k\left(\zeta_j\right)\right)\Bigg\}-\frac{1}{4}(l+L)\sum_{j=0}^{m-1}\left(x_{j+1}^2-x_j^2\right)$$

and

$$\begin{aligned} A_j &= \zeta_j - x_j,\ B_j = x_{j+1}-\zeta_j,\ h_j = A_j + B_j = x_{j+1}-x_j, \\ r_k\left(\zeta_j\right) &= B_j^k + (-1)^{k-1}A_j^k,\ p_k\left(\zeta_j\right) = A_j^k x_j + (-1)^{k-1}B_j^k x_{j+1}, \end{aligned}$$

for $j=0,...,m-1$ *and the remainder satisfies the estimation:*

$$\begin{aligned} |R_{m,n,2}(f,I_m,\zeta,\delta)| &\leq \frac{\left(\frac{1}{2}+\left|\delta-\frac{1}{2}\right|\right)^n}{n!}\left(f^{(n-1)}(b)-f^{(n-1)}(a)-l(b-a)\right) \\ &\quad\times\left(\frac{v(h)}{2}+\max_j\left|\zeta_j-\frac{x_j+x_{j+1}}{2}\right|\right)^n, \qquad (2.109) \\ &v(h) = \{h_j | j=0,...,m-1\}, \end{aligned}$$

for all $\delta\in[0,1]$.

Proof. Applying inequality (2.104) on $\zeta_i\in[x_j,x_{j+1}]$ $(j=0,...,m-1)$ and summing over j from 0 to $m-1$ and using triangular inequality, we get (2.109). ■

2.4 Conclusion

In this chapter, by the use of modified Peano kernels, some Ostrowski type inequalities depending on the second derivatives are highlighted. Ostrowski type inequalities for twice differentiable functions have been extensively addressed in the research papers [9] and [59]. We, in here, have presented some generalizations and improvements of the inequalities presented in [9] and [59].

In Section 2.1, we have presented a generalization (2.2) of the inequality (2.1) obtained in [38] (or see [9], Section 7) for twice differentiable functions whose first derivatives are absolutely continuous and second derivatives belong to $L_\infty(a,b)$ by introducing a parameter $h\in[0,1]$. From Remark 2.3, it is clear that (2.2) can present some better estimates for a specified range of h than (2.1). This generalization also results in obtaining a three-point inequality for a specific value of h as mentioned in Remark 2.4. The three-point inequality thus obtained has a better bound than the three-point inequalities presented in [9] and [59] for $\|.\|_\infty - norm$. Remark 2.4 also shows that the perturbed trapezoid inequality that can be obtained from (2.2) is better than the perturbed inequalities presented in [9] and [59] of perturbed trapezoid type for $\|.\|_\infty - norm$. The inequality is then applied for a partition of the interval $[a, b]$ to obtain some composite quadrature rules. The inequality is also applied to special means by properly choosing the function involved to get some direct relationships between different means.

In Section 2.2, some generalizations and refinements of the inequalities of Ostrowski type for twice differentiable functions are given in the sense of perturbations by introducing perturbed versions of inequalities of midpoint, trapezoid, Simpson's and averaged trapezoid-midpoint type which refines the results of [19, 20, 22, 38] and ([9], Theorem 20). The Remark 2.7-2.17 justify our claim. The corresponding composite quadrature rules are obtained in Section 2.2.3.

In Section 2.3, a generalization of Ostrowski type inequality is presented for (l, L)-Lipschitzian functions which not only extends some Ostrowski type inequalities L-Lipschitzian mappings to an higher space of (l, L)-Lipschitzian mappings but also generalizes some Ostrowski type inequalities for (l, L)-Lipschitzian mappings. Remark 2.19, 2.20 and 2.21 justify this fact. Applications for composite quadrature rules are also given in Section 2.3.3.

Chapter 3

Some Generalized Ostrowski-Grüss type inequalities

The integral inequality that measures the deviation of the integral of the product of two functions and the product of the integrals is known in the literature as the Grüss inequality. The inequality is stated in the form of Theorem 1.4.

In 1997, S. S. Dragomir and S. Wang [39], by the use of the Grüss inequality proved the following Ostrowski-Grüss type integral inequality:

Theorem 3.1 *Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I^0 of I, and let $a, b \in I^0$ with $a < b$. If $\gamma \leq f^{'}(x) \leq \Gamma$, $x \in [a, b]$ for some constants $\gamma, \Gamma \in \mathbb{R}$, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \leq \frac{1}{4}(b-a)(\Gamma - \gamma), \tag{3.1}$$

for all $x \in [a, b]$.

This inequality provides a relation between Ostrowski inequality [69] and the Grüss inequality [64].

We, in this chapter present some extensions of the inequality (3.1) for first and twice differentiable functions.

3.1 A generalization of Ostrowski-Grüss type inequality for first differentiable mappings

In this section, we improve and further generalize some Ostrowski-Grüss type inequalities involving first differentiable functions and apply them to probability density functions, generalized beta random variable and special means.

3.1.1 Introduction

In 2000, M. Matić, J. E. Pečarić and N. Ujević [61], by the use of pre-Grüss inequality improved the factor of the right membership of (3.1) with $\frac{1}{4\sqrt{3}}$ as follows:

Theorem 3.2 *Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I^0 of I, and let $a, b \in I^0$ with $a < b$. If $\gamma \leq f'(x) \leq \Gamma$, $x \in [a, b]$ for some constants $\gamma, \Gamma \in \mathbb{R}$, then*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \\
\leq\ & \frac{1}{4\sqrt{3}}(\Gamma - \gamma)(b-a), \qquad (3.2)
\end{aligned}$$

for all $x \in [a, b]$.

In 2000, N. S. Barnett et al.[12], by the use of Čebyšev functional, improved the Matić-Pecarić-Ujević result by providing first membership of the right side of (3.2) in terms of Euclidean norm as follows:

Theorem 3.3 *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function whose derivative $f' \in L_2[a, b]$. Then we have the inequality*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \\
\leq\ & \frac{(b-a)}{2\sqrt{3}}\left[\frac{1}{b-a}\left\|f'\right\|_2^2 - \left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}, \\
\leq\ & \frac{1}{4\sqrt{3}}(\Gamma - \gamma)(b-a), \\
& \text{if } \gamma \leq f'(t) \leq \Gamma \text{ for almost everywhere } t \text{ on } [a, b] \qquad (3.3)
\end{aligned}$$

for all $x \in [a, b]$.

Also in [12], we can find the pre-Grüss inequality as

$$T^2(f,g) \le T(f,f)\,T(g,g),$$

where $f,g \in L_2[a,b]$ and $T(f,g)$ is the Čebyšev functional as defined above.

In the following subsection, we give a generalization of (3.3) and then apply it to probability density functions, generalized beta random variable and special means.

3.1.2 Main Results

Theorem 3.4 *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function whose first derivative $f' \in L_2(a,b)$. Then, we have the inequality*

$$\begin{aligned}
&\left|(1-h)\left[f(x)-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right]+h\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\le\;& \left[\frac{(b-a)^2}{12}\left(3h^2-3h+1\right)+h(1-h)\left(x-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}} \times \\
& \left[\frac{1}{b-a}\left\|f'\right\|_2^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}, \\
\le\;& \frac{1}{2}(\Gamma-\gamma)\left[\frac{(b-a)^2}{12}\left(3h^2-3h+1\right)+h(1-h)\left(x-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}}, \\
& \text{if } \gamma \le f'(t) \le \Gamma \text{ almost everywhere } t \text{ on } [a,b]
\end{aligned} \tag{3.4}$$

for all $x\in\left[a+h\frac{b-a}{2},\,b-h\frac{b-a}{2}\right]$ and $h\in[0,1]$.

Proof. We consider the kernel as defined in [34] $p:[a,b]^2\to\mathbb{R}$

$$p(x,t)=\begin{cases} t-\left(a+h\frac{b-a}{2}\right), & \text{if } t\in[a,x] \\ t-\left(b-h\frac{b-a}{2}\right), & \text{if } t\in(x,b]. \end{cases}$$

Using Korkine's identity:

$$T(f,g):=\frac{1}{2(b-a)^2}\int_a^b\int_a^b (f(t)-f(s))(g(t)-g(s))\,dtds,$$

we obtain

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b p(x,t)f'(t)\,dt-\frac{1}{b-a}\int_a^b p(x,t)\,dt\frac{1}{b-a}\int_a^b f'(t)\,dt \\
=\;&\frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))\left(f'(t)-f'(s)\right)dtds.
\end{aligned} \tag{3.5}$$

Since,

$$\frac{1}{b-a}\int_a^b p(x,t) f'(t)\,dt = (1-h) f(x) + h\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt,$$

$$\frac{1}{b-a}\int_a^b p(x,t)\,dt = (1-h)\left(x-\frac{a+b}{2}\right)$$

and

$$\frac{1}{b-a}\int_a^b f'(t)\,dt = \frac{f(b)-f(a)}{b-a},$$

then by (3.5) we get the following identity:

$$(1-h)\left[f(x)-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right] + h\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt$$

$$= \frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))\left(f'(t)-f'(s)\right)dtds, \qquad (3.6)$$

for all $x \in \left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]$ and $h \in [0,1]$.

Using the Cauchy-Bunyakowski-Schwartz inequality for double integrals,we may write

$$\frac{1}{2(b-a)^2}\left|\int_a^b\int_a^b (p(x,t)-p(x,s))\left(f'(t)-f'(s)\right)dtds\right|$$

$$\leq \left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))^2\,dtds\right)^{\frac{1}{2}}$$

$$\times\left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(f'(t)-f'(s)\right)^2 dtds\right)^{\frac{1}{2}}. \qquad (3.7)$$

However,

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))^2\,dtds$$

$$\frac{1}{b-a}\int_a^b p^2(x,t)\,dt - \left(\frac{1}{b-a}\int_a^b p(x,t)\,dt\right)^2$$

$$= \frac{1}{b-a}\left[\frac{\left(x-\left(a+h\frac{b-a}{2}\right)\right)^3+\left(b-h\frac{b-a}{2}-x\right)^3}{3}+\frac{h^3(b-a)^3}{12}\right]$$

$$-(1-h)^2\left(x-\frac{a+b}{2}\right)^2. \qquad (3.8)$$

In addition, simple calculations show that

$$\begin{aligned}&\left(x-\left(a+h\frac{b-a}{2}\right)\right)^{3}+\left(b-h\frac{b-a}{2}-x\right)^{3}\\ =\ &(b-a)(1-h)\left[3\left(x-\frac{a+b}{2}\right)^{2}+\frac{(1-h)^{2}(b-a)^{2}}{4}\right] \qquad (3.9)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2(b-a)^{2}}\int_{a}^{b}\int_{a}^{b}\left(f^{'}(t)-f^{'}(s)\right)^{2}dtds\\ =\ &\frac{1}{b-a}\left\|f^{'}\right\|_{2}^{2}-\left(\frac{f(b)-f(a)}{b-a}\right)^{2}. \qquad (3.10)\end{aligned}$$

Using (3.6)-(3.10), we deduce the first inequality.

Moreover, if $\gamma\leq f^{'}(t)\leq\Gamma$ almost everywhere t on (a,b), then, by using Grüss inequality, we have

$$0\leq\frac{1}{b-a}\int_{a}^{b}\left(f^{'}(t)\right)^{2}dt-\left(\frac{1}{b-a}\int_{a}^{b}f^{'}(t)\,dt\right)^{2}\leq\frac{1}{4}(\Gamma-\gamma)^{2},$$

which proves the last inequality of (3.4). ■

Remark 3.1 *Since*

$$3h^{2}-3h+1\leq1,\ \forall\ h\in[0,1].$$

and is minimum for $h=\frac{1}{2}$.

Thus, (3.4) shows an overall improvement in the inequality obtained by Barnett et al. [12].

The following remark contains some special cases of (3.4):

Remark 3.2 *(i) For* $h=1$, *i.e.,* $x=\frac{a+b}{2}$, *(3.4) gives*

$$\begin{aligned}&\left|(b-a)\frac{f(a)+f(b)}{2}-\int_{a}^{b}f(t)\,dt\right|\\ \leq\ &\frac{(b-a)^{2}}{2\sqrt{3}}\left[\frac{1}{b-a}\left\|f^{'}\right\|_{2}^{2}-\left(\frac{f(b)-f(a)}{b-a}\right)^{2}\right]^{\frac{1}{2}},\\ \leq\ &\frac{1}{4\sqrt{3}}(\Gamma-\gamma)(b-a)^{2}\\ &\text{if }\gamma\leq f^{'}(t)\leq\Gamma\text{ almost everywhere }t\text{ on }[a,b], \qquad (3.11)\end{aligned}$$

which is trapezoid inequality.

(ii) For $h = 0$ and $x = \frac{a+b}{2}$, (3.4) gives

$$\begin{aligned}
&\left| (b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\,dt \right| \\
\leq\ & \frac{(b-a)^2}{2\sqrt{3}} \left[\frac{1}{b-a} \left\| f^{'} \right\|_2^2 - \left(\frac{f(b) - f(a)}{b-a} \right)^2 \right]^{\frac{1}{2}}, \\
\leq\ & \frac{1}{4\sqrt{3}} (\Gamma - \gamma)(b-a)^2 \\
& if\ \gamma \leq f^{'}(t) \leq \Gamma \text{ almost everywhere } t \text{ on } [a,b],
\end{aligned} \tag{3.12}$$

which is mid-point inequality.

(iii) For $h = \frac{1}{2}$ and $x = \frac{a+b}{2}$, (3.4) gives

$$\begin{aligned}
&\left| \frac{f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)}{4} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \\
\leq\ & \frac{(b-a)^2}{4\sqrt{3}} \left[\frac{1}{b-a} \left\| f^{'} \right\|_2^2 - \left(\frac{f(b) - f(a)}{b-a} \right)^2 \right]^{\frac{1}{2}}, \\
\leq\ & \frac{1}{8\sqrt{3}} (\Gamma - \gamma)(b-a)^2 \\
& if\ \gamma \leq f^{'}(t) \leq \Gamma \text{ almost everywhere } t \text{ on } [a,b],
\end{aligned} \tag{3.13}$$

which is an averaged mid-point and trapezoid inequality.

(iv) For $h = \frac{1}{3}$ and $x = \frac{a+b}{2}$, (3.4) gives

$$\begin{aligned}
&\left| (b-a) \frac{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)}{6} - \int_a^b f(t)\,dt \right| \\
\leq\ & \frac{(b-a)^2}{6} \left[\frac{1}{b-a} \left\| f^{'} \right\|_2^2 - \left(\frac{f(b) - f(a)}{b-a} \right)^2 \right]^{\frac{1}{2}}, \\
\leq\ & \frac{1}{12} (\Gamma - \gamma)(b-a)^2, \\
& if\ \gamma \leq f^{'}(t) \leq \Gamma \text{ almost everywhere } t \text{ on } (a,b),
\end{aligned} \tag{3.14}$$

which is a variant of Simpson's inequality for first differentiable function f, $f^{'}$ is integrable and there exist constants $\gamma, \Gamma \in R$ such that $\gamma \leq f^{'}(t) \leq \Gamma$, $t \in (a,b)$.

3.1.3 Application for Probability Density Functions

Let X be a continuous random variable having the probability density function $f:[a,b]\to\mathbb{R}_+$ and the cumulative distribution function $F:[a,b]\to[0,1]$, i.e.,

$$F(x)=\int_a^x f(t)\,dt,\ x\in\left[a+h\frac{b-a}{2},b-h\frac{b-a}{2}\right]\subset[a,b],$$

and

$$E(X)=\int_a^b t\ f(t)\,dt,$$

is the expectation of the random variable X on the interval $[a,b]$. Then, we may have the following.

Theorem 3.5 *Under the above assumptions and if the probability density function belongs to* $L_2[a,b]$, *then we have the inequality*

$$\begin{aligned}&\left|(1-h)\left[F(x)-\frac{1}{b-a}\left(x-\frac{a+b}{2}\right)\right]+\frac{h}{2}-\frac{b-E(X)}{b-a}\right|\\ \le\ &\frac{1}{b-a}\left[\frac{1}{12}\left(3h^2-3h+1\right)+h(1-h)\left(x-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}}\times\\ &\left[(b-a)\|f\|_2^2-1\right]^{\frac{1}{2}},\\ \le\ &\frac{(M-m)}{2(b-a)}\left[\frac{1}{12}\left(3h^2-3h+1\right)+h(1-h)\left(x-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}},\\ &\textit{if } m\le f\le M \textit{ almost everywhere on } [a,b], \qquad (3.15)\end{aligned}$$

for all $x\in\left[a+h\frac{b-a}{2},b-h\frac{b-a}{2}\right]$.

Proof. Put in (3.4), $f=F$ to get (3.15). ∎

Corollary 3.1 *Under the above assumptions, we have*

$$\begin{aligned}&\left|(1-h)\Pr\left(X\le\frac{a+b}{2}\right)+\frac{h}{2}-\frac{b-E(X)}{b-a}\right|\\ \le\ &\frac{1}{2\sqrt{3}}\left(3h^2-3h+1\right)^{\frac{1}{2}}\left[(b-a)\|f\|_2^2-1\right]^{\frac{1}{2}},\\ \le\ &\frac{1}{4\sqrt{3}}\left(3h^2-3h+1\right)^{\frac{1}{2}}(M-m),\ m\le f\le M \textit{ as above.} \qquad (3.16)\end{aligned}$$

3.1.4 Applications for generalized beta random variable

If X is a beta random variable with parameters $\beta_3 > -1$, $\beta_4 > -1$ and for $\beta_2 > 0$ and any β_1, the generalized beta random variable

$$Y = \beta_1 + \beta_2 X,$$

is said to have a generalized beta distribution [51] and the probability density function of the generalized beta distribution of beta random variable is given as

$$f(x) = \begin{cases} \frac{(x-\beta_1)^{\beta_3}(\beta_1+\beta_2-x)^{\beta_4}}{\beta(\beta_3+1,\beta_4+1)\beta_2^{(\beta_3+\beta_4+1)}}, & \text{for } \beta_1 < x < \beta_1+\beta_2 \\ 0, & \text{otherwise,} \end{cases}$$

where $\beta(l, m)$ is the beta function with $l,\ m > 0$ and is defined as

$$\beta(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx.$$

For p, $q > 0$ and $h \in [0, 1)$, we choose,

$$\begin{aligned} \beta_1 &= \frac{h}{2}, \\ \beta_2 &= (1-h), \\ \beta_3 &= p-1, \\ \beta_4 &= q-1. \end{aligned}$$

Then, the probability density function associated with generalized beta random variable

$$Y = \frac{h}{2} + (1-h)X,$$

takes the form

$$f(x) = \begin{cases} \frac{\left(x-\frac{h}{2}\right)^{p-1}\left(1-\frac{h}{2}-x\right)^{q-1}}{\beta(p,q)(1-h)^{p+q-1}}, & \frac{h}{2} < x < 1-\frac{h}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Now,

$$\begin{aligned} E(Y) &= \int_{\frac{h}{2}}^{1-\frac{h}{2}} x f(x)\,dx \\ &= (1-h)\frac{p}{p+q} + \frac{h}{2}. \end{aligned} \tag{3.17}$$

and

$$\|f(.;p,q)\|_2^2 = \frac{1}{(1-h)\beta^2(p,q)}\beta(2p-1,2q-1). \tag{3.18}$$

Then, by Theorem 3.5, we may state the following:

Proposition 3.1 *Let X be a beta random variable with parameters (p,q). Then for generalized beta random variable*

$$Y = \frac{h}{2} + (1-h)X,$$

we have the inequality

$$\begin{aligned}
&\left|\left[\Pr(Y \le x) - x + \frac{1}{2}\right] - \frac{q}{p+q}\right| \\
\le\ & \left[\frac{1}{12}\left(3h^2-3h+1\right) + h(1-h)\left(x-\frac{1}{2}\right)^2\right]^{\frac{1}{2}} \times \\
& \frac{\left[\beta(2p-1,2q-1)-(1-h)\beta^2(p,q)\right]^{\frac{1}{2}}}{(1-h)^{\frac{3}{2}}\beta(p,q)},
\end{aligned} \tag{3.19}$$

for all $x \in \left[\frac{h}{2}, 1-\frac{h}{2}\right]$.

In particular, for $x = \frac{1}{2}$ in (3.19), we have:

$$\begin{aligned}
&\left|\Pr\left(Y \le \frac{1}{2}\right) - \frac{q}{p+q}\right| \\
\le\ & \frac{1}{2\sqrt{3}}\left(3h^2-3h+1\right)^{\frac{1}{2}} \frac{\left[\beta(2p-1,2q-1)-(1-h)\beta^2(p,q)\right]^{\frac{1}{2}}}{(1-h)^{\frac{3}{2}}\beta(p,q)}.
\end{aligned}$$

3.1.5 Applications for Special Means

Example 4 *Consider the mapping $f(x) = x^p$, $p \in \mathbb{R} \setminus \{-1, 0\}$. Then*

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(t)\,dt &= L_p^p(a,b), \\
\frac{f(b)-f(a)}{b-a} &= pL_{p-1}^{p-1}, \\
\frac{f(a)+f(b)}{2} &= \frac{a^p+b^p}{2} = A(a^p,b^p)
\end{aligned}$$

and

$$\frac{1}{b-a}\left\|f'\right\|_2^2 = \frac{1}{b-a}\int_a^b \left|f'(t)\right|^2 dt = p^2 L_{2(p-1)}^{2(p-1)}.$$

Therefore, (3.4) takes the form

$$\begin{aligned}
&\left|(1-h)\left[x^{p}-pL_{p-1}^{p-1}(x-A(a,b))\right]+hA(a^{p},b^{p})-L_{p}^{p}\right| \\
\leq\ & |p|\left[\frac{(b-a)^{2}}{12}\left(3h^{2}-3h+1\right)+h(1-h)(x-A)^{2}\right]^{\frac{1}{2}}\times \\
&\left[L_{2(p-1)}^{2(p-1)}-L_{p-1}^{2(p-1)}\right]^{\frac{1}{2}} \qquad (3.20)
\end{aligned}$$

Choose $x=A$ in (3.20), to get

$$\begin{aligned}
&\left|(1-h)A^{p}(a,b)+hA(a^{p},b^{p})-L_{p}^{p}\right| \\
\leq\ & \frac{(b-a)}{2\sqrt{3}}\left(3h^{2}-3h+1\right)^{\frac{1}{2}}|p|\left[L_{2(p-1)}^{2(p-1)}-L_{p-1}^{2(p-1)}\right]^{\frac{1}{2}},
\end{aligned}$$

which is minimum for $h=\frac{1}{2}$. Moreover for $h=1$,

$$\begin{aligned}
&\left|A(a^{p},b^{p})-L_{p}^{p}\right| \\
\leq\ & \frac{(b-a)}{2\sqrt{3}}|p|\left[L_{2(p-1)}^{2(p-1)}-L_{p-1}^{2(p-1)}\right]^{\frac{1}{2}}.
\end{aligned}$$

Example 5 *Consider the mapping $f(x)=\frac{1}{x}$, $\left(x\in\left[a+h\frac{b-a}{2},b-h\frac{b-a}{2}\right]\subset(0,\infty)\right)$.*

Then,

$$\begin{aligned}
\frac{1}{b-a}\int_{a}^{b}f(t)\,dt &= \frac{1}{L}, \\
\frac{f(b)-f(a)}{b-a} &= -\frac{1}{G^{2}}, \\
\frac{f(a)+f(b)}{2} &= \frac{A}{G^{2}}, \\
\frac{1}{b-a}\int_{a}^{b}\left|f^{'}(t)\right|^{2}dt &= \frac{a^{2}+ab+b^{2}}{3a^{3}b^{3}}
\end{aligned}$$

and

$$\frac{1}{b-a}\int_{a}^{b}\left|f^{'}(t)\right|^{2}dt-\left(\frac{f(b)-f(a)}{b-a}\right)^{2}=\frac{(b-a)^{2}}{3a^{3}b^{3}}.$$

Therefore, (3.4) becomes

$$\begin{aligned}
&\left|(1-h)\left[\frac{1}{x}+\frac{1}{G^{2}}(x-A)\right]+h\frac{A}{G^{2}}-\frac{1}{L}\right| \\
\leq\ & \left[\frac{(b-a)^{2}}{12}\left(3h^{2}-3h+1\right)+h(1-h)(x-A)^{2}\right]^{\frac{1}{2}}\times \\
&\frac{(b-a)}{\sqrt{3}G^{3}}. \qquad (3.21)
\end{aligned}$$

Choosing $x = A$ in (3.21),

$$\begin{aligned}&\left|(1-h)\frac{1}{A}+h\frac{A}{G^2}-\frac{1}{L}\right|\\ \leq\ &\frac{(b-a)^2}{6G^3}\left(3h^2-3h+1\right)^{\frac{1}{2}}.\end{aligned}$$

If we choose $x = L$ in (3.21), we get

$$\begin{aligned}&\left|(1-h)\frac{L}{G^2}+(2h-1)\frac{A}{G^2}-h\frac{1}{L}\right|\\ \leq\ &\left[\frac{(b-a)^2}{12}\left(3h^2-3h+1\right)+h(1-h)(L-A)^2\right]^{\frac{1}{2}}\times\\ &\frac{(b-a)}{\sqrt{3}G^3}.\end{aligned}$$

Example 6 *Consider the mapping $f(x) = \ln x$, $\left(x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right] \subset (0,\infty)\right)$. Then*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(t)\,dt &= \ln I,\\ \frac{f(b)-f(a)}{b-a} &= \frac{1}{L},\\ \frac{f(a)+f(b)}{2} &= \ln G,\\ \frac{1}{b-a}\int_a^b \left|f'(t)\right|^2 dt &= \frac{1}{G^2}\end{aligned}$$

and

$$\frac{1}{b-a}\int_a^b \left|f'(t)\right|^2 dt - \left(\frac{f(b)-f(a)}{b-a}\right)^2 = \frac{L^2-G^2}{L^2G^2}.$$

Thus, (3.4) takes the form

$$\begin{aligned}&\left|\ln\frac{x^{(1-h)}G^h}{I}-(1-h)\frac{x-A}{L}\right|\\ \leq\ &\left[\frac{(b-a)^2}{12}\left(3h^2-3h+1\right)+h(1-h)(x-A)^2\right]^{\frac{1}{2}}\times\\ &\frac{\left(L^2-G^2\right)^{\frac{1}{2}}}{LG}.\end{aligned} \tag{3.22}$$

For $x = A$,

$$\begin{aligned}&\left|\ln\frac{A^{(1-h)}G^h}{I}\right|\\ \leq\ &\frac{(b-a)\left(3h^2-3h+1\right)^{\frac{1}{2}}}{2\sqrt{3}}\frac{\left(L^2-G^2\right)^{\frac{1}{2}}}{LG}.\end{aligned}$$

which for $h = 1$, takes the form

$$\left| \ln \frac{G}{I} \right| \leq \frac{(b-a)\left(L^2 - G^2\right)^{\frac{1}{2}}}{2\sqrt{3}LG}.$$

Also, choosing $x = I$, we get

$$\left| \ln \frac{G^h}{I^h} - (1-h)\frac{I-A}{L} \right| \leq \left[\frac{(b-a)^2}{12}\left(3h^2 - 3h + 1\right) + h(1-h)(I-A)^2 \right]^{\frac{1}{2}} \times \frac{\left(L^2 - G^2\right)^{\frac{1}{2}}}{LG}.$$

3.2 A generalized Ostrowski-Grüss type inequality for twice differentiable bounded mappings and applications

In this section, a generalized Ostrowski-Grüss type inequality for twice differentiable mappings in terms of the upper and lower bounds of the second derivative is established. The inequality is applied to numerical integration.

3.2.1 Introduction

In [39], S. S. Dragomir and S. Wang proved the following Ostrowski type inequality in terms of lower and upper bounds of the first derivative which is known as Ostrowski-Grüss type inequality. In [9], S. S. Dragomir and N. S. Barnett, proved the following inequality:

Theorem 3.6 *Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous on $[a,b]$ and twice differentiable on (a,b) and where the second derivative $f'' : (a,b) \longrightarrow \mathbb{R}$ satisfies the condition,*

$$\varphi \leq f''(x) \leq \Phi, \; for \; all \; x \in (a,b),$$

then,

$$\left| f(x) + \left[\frac{(b-a)^2}{24} + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 \right] \frac{f'(b) - f'(a)}{b-a} - \left(x - \frac{a+b}{2}\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)dt \right|$$

$$\leq \frac{1}{8}(\Phi - \varphi)\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^2, \tag{3.23}$$

for all $x \in [a, b]$.

In the following subsection, we establish a more general form of (3.23) and apply the result to numerical integration.

3.2.2 Main Results

Theorem 3.7 *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be a continuous mapping on* $[a, b]$, *and twice differentiable on* (a, b) *with second derivative* $f'' : (a, b) \longrightarrow \mathbb{R}$ *satisfying the condition:*

$$\varphi \leq f''(x) \leq \Phi, \text{ for all } x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right].$$

It follows that,

$$\Bigg|(1-h)\left[f(x) - \left(x - \frac{a+b}{2}\right)f'(x)\right] + h\frac{f(a)+f(b)}{2}$$

$$+\left[\frac{1}{2}(1-h)\left(x - \frac{a+b}{2}\right)^2 - \frac{(3h-1)(b-a)^2}{24}\right]\left(\frac{f'(b) - f'(a)}{b-a}\right)$$

$$-\frac{1}{b-a}\int_a^b f(t)dt\Bigg|$$

$$\leq \frac{1}{8}(\Phi - \varphi)\left[\frac{1}{2}(b-a)(1-h) + \left|x - \frac{a+b}{2}\right|\right]^2, \tag{3.24}$$

for all $x \in [a + h\frac{b-a}{2}, b - h\frac{b-a}{2}]$ *and* $h \in [0, 1]$.

Proof. The proof uses the following identity:

$$\int_a^b f(t)dt = (b-a)(1-h)f(x) - (b-a)(1-h)\left(x - \frac{a+b}{2}\right)f'(x)$$
$$+h\frac{b-a}{2}(f(a)+f(b)) - \frac{h^2(b-a)^2}{8}\left(f'(b) - f'(a)\right)$$
$$+\int_a^b K(x,t) f''(t)\,dt, \tag{3.25}$$

for all $x \in [a + h\frac{b-a}{2}, b - h\frac{b-a}{2}]$, where the kernel $K : [a, b]^2 \to \mathbb{R}$ is defined by

$$K(x,t) = \begin{cases} \frac{1}{2}\left[t - \left(a + h\frac{b-a}{2}\right)\right]^2, & \text{if } t \in [a, x] \\ \frac{1}{2}\left[t - \left(b - h\frac{b-a}{2}\right)\right]^2, & \text{if } t \in (x, b]. \end{cases}$$

This is a particular form of the identity given in ([36], page 67, Theorem 28). Observe that the Kernel K satisfies the estimation

$$0 \le K(x,t) \le \begin{cases} \frac{1}{2}\left[\left(b - h\frac{b-a}{2}\right) - x\right]^2, & x \in \left[a + h\frac{b-a}{2}, \frac{a+b}{2}\right) \\ \frac{1}{2}\left[x - \left(a + h\frac{b-a}{2}\right)\right]^2, & x \in \left[\frac{a+b}{2}, b - h\frac{b-a}{2}\right]. \end{cases} \tag{3.26}$$

Applying Grüss inequality for the mappings $f''(.)$ and $K(x,.)$ we get,

$$\left|\frac{1}{b-a}\int_a^b K(x,t) f''(t)\,dt - \frac{1}{b-a}\int_a^b K(x,t)\,dt \frac{1}{b-a}\int_a^b f''(t)\,dt\right|$$

$$\le \frac{1}{4}(\Phi - \varphi) \times \begin{cases} \frac{1}{2}\left[\left(b - h\frac{b-a}{2}\right) - x\right]^2, & x \in \left[a + h\frac{b-a}{2}, \frac{a+b}{2}\right) \\ \frac{1}{2}\left[x - \left(a + h\frac{b-a}{2}\right)\right]^2, & x \in \left[\frac{a+b}{2}, b - h\frac{b-a}{2}\right]. \end{cases} \tag{3.27}$$

Observe that,

$$\int_a^b K(x,t)dt = \int_a^x \frac{\left[t - \left(a + h\frac{b-a}{2}\right)\right]^2}{2}dt + \int_x^b \frac{\left[t - \left(b - h\frac{b-a}{2}\right)\right]^2}{2}dt$$

$$= \frac{1}{6}\left[\left(x - \left(a + h\frac{b-a}{2}\right)\right)^3 + \left(\left(b - h\frac{b-a}{2}\right) - x\right)^3 + \frac{h^3(b-a)^3}{4}\right]$$

$$= (b-a)(1-h)\left[\frac{(b-a)^2(1-h)^2}{24} + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2\right] + \frac{h^3(b-a)^3}{24}. \tag{3.28}$$

Using (3.28) in (3.27), we get

$$\left|\frac{1}{b-a}\int_a^b K(x,t) f''(t)\,dt - \left[\frac{(b-a)^2(1-h)^3}{24} + \frac{1}{2}(1-h)\left(x - \frac{a+b}{2}\right)^2 + \frac{h^3(b-a)^2}{24}\right]\left(\frac{f'(b) - f'(a)}{b-a}\right)\right|$$

$$\le \frac{1}{4}(\Phi - \varphi) \times \begin{cases} \frac{1}{2}\left[\left(b - h\frac{b-a}{2}\right) - x\right]^2, & x \in \left[a + h\frac{b-a}{2}, \frac{a+b}{2}\right) \\ \frac{1}{2}\left[x - \left(a + h\frac{b-a}{2}\right)\right]^2, & x \in \left[\frac{a+b}{2}, b - h\frac{b-a}{2}\right]. \end{cases}$$

Also, by using identity (3.25), the above inequality reduces to,

$$\left|(1-h)\left[f(x)-\left(x-\frac{a+b}{2}\right)f'(x)\right]+h\frac{f(a)+f(b)}{2}\right.$$
$$+\left[\frac{1}{2}(1-h)\left(x-\frac{a+b}{2}\right)^2-\frac{(3h-1)(b-a)^2}{24}\right]\left(\frac{f'(b)-f'(a)}{b-a}\right)$$
$$\left.-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \frac{1}{4}(\Phi-\varphi)\times\begin{cases}\frac{1}{2}\left[\left(b-h\frac{b-a}{2}\right)-x\right]^2, & x\in\left[a+h\frac{b-a}{2},\frac{a+b}{2}\right)\\ \frac{1}{2}\left[x-\left(a+h\frac{b-a}{2}\right)\right]^2, & x\in\left[\frac{a+b}{2},b-h\frac{b-a}{2}\right].\end{cases}$$

Since,

$$\max\left\{\frac{\left[\left(b-h\frac{b-a}{2}\right)-x\right]^2}{2},\frac{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^2}{2}\right\}$$
$$=\begin{cases}\frac{1}{2}\left[\left(b-h\frac{b-a}{2}\right)-x\right]^2, & x\in\left[a+h\frac{b-a}{2},\frac{a+b}{2}\right)\\ \frac{1}{2}\left[x-\left(a+h\frac{b-a}{2}\right)\right]^2, & x\in\left[\frac{a+b}{2},b-h\frac{b-a}{2}\right],\end{cases}$$

but on the other hand,

$$\max\left\{\frac{\left[\left(b-h\frac{b-a}{2}\right)-x\right]^2}{2},\frac{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^2}{2}\right\}$$
$$=\frac{1}{2}\left[\frac{1}{2}(b-a)(1-h)+\left(x-\frac{a+b}{2}\right)\right]^2,$$

inequality (3.24) is proved. ■

Remark 3.3 *For $h=0$ in (3.24), we obtain (3.23).*

Corollary 3.2 *If f is as in Theorem 3.7, then we have the following perturbed midpoint inequality:*

$$\left|(1-h)f\left(\frac{a+b}{2}\right)+h\frac{f(a)+f(b)}{2}\right.$$
$$\left.-\frac{(3h-1)(b-a)}{24}\left(f'(b)-f'(a)\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \frac{1}{32}(\Phi-\varphi)(b-a)^2(1-h)^2, \tag{3.29}$$

giving,

$$\left|f\left(\frac{a+b}{2}\right)+\frac{(b-a)}{24}\left(f'(b)-f'(a)\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \frac{1}{32}(\Phi-\varphi)(b-a)^2, \tag{3.30}$$

for $h = 0$.

Remark 3.4 *The classical midpoint inequality states that*

$$\left| f(\frac{a+b}{2}) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{1}{24} (b-a)^2 \left\| f'' \right\|_\infty . \tag{3.31}$$

If $\Phi - \varphi \leq \frac{4}{3} \left\| f'' \right\|_\infty$, *then the estimation provided by (3.29) is better than estimation in the classical midpoint inequality (3.31). A sufficient condition for* $\Phi - \varphi \leq \frac{4}{3} \left\| f'' \right\|_\infty$ *to be true is* $0 \leq \varphi \leq \Phi$. *Indeed, if* $0 \leq \varphi \leq \Phi$, *then* $\Phi - \varphi \leq \left\| f'' \right\|_\infty < \frac{4}{3} \left\| f'' \right\|_\infty$.

Corollary 3.3 *Let* f *be as in Theorem 3.7, then,*

$$\left| \frac{f(a) + f(b)}{2} - \frac{(b-a)}{12} \left(f'(b) - f'(a) \right) - \frac{1}{b-a} \int_a^b f(t) dt \right|$$
$$\leq \frac{1}{32} (\Phi - \varphi) h^2 (b-a)^2 . \tag{3.32}$$

Proof. Put $x = a$ and $x = b$ in (3.24) and use the triangle inequality. ■

Corollary 3.4 *Let* f *be as in Theorem 3.7, then we have the following perturbed trapezoid inequality:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{(b-a)}{12} \left(f'(b) - f'(a) \right) - \frac{1}{b-a} \int_a^b f(t) dt \right|$$
$$\leq \frac{1}{32} (\Phi - \varphi) (b-a)^2 . \tag{3.33}$$

Proof. Put $h = 1$ in (3.32). ■

Remark 3.5 *The classical trapezoid inequality states that*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{1}{12} (b-a)^2 \left\| f'' \right\|_\infty . \tag{3.34}$$

If we assume that $\Phi - \varphi \leq \frac{2}{3} \left\| f'' \right\|_\infty$, *then the estimation provided by (3.32) is better than the estimation in the classical trapezoid inequality (3.34).*

3.2.3 Applications in Numerical Integration

Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $\xi_i \in [x_i + \delta\frac{h_i}{2}, x_{i+1} - \delta\frac{h_i}{2}], (i = 0,1,\cdots,n-1)$ a sequence of intermediate points and $h_i := x_{i+1} - x_i$, $(i = 0,1,\cdots,n-1)$. Then, we have the following composite quadrature rule:

Theorem 3.8 *Let f be as in Theorem 3.7, then we have the following quadrature formula:*

$$\int_a^b f(t)dt = A\left(f, f', I_n, \xi, \delta\right) + R\left(f, f', I_n, \xi, \delta\right), \tag{3.35}$$

where

$$\begin{aligned} A\left(f, f', I_n, \xi, \delta\right) &= (1-\delta)\sum_{i=0}^{n-1} h_i f(\xi_i) \\ &- (1-\delta)\sum_{i=0}^{n-1} h_i \left(\xi_i - \frac{x_i + x_{i-1}}{2}\right) f'(\xi_i) \\ &+ \delta\sum_{i=0}^{n-1} h_i \left(\frac{f(x_i) + f(x_{i+1})}{2}\right) \\ &+ \sum_{i=0}^{n-1}[\frac{1}{2}(1-\delta)\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2 \\ &- \frac{(3\delta - 1)h_i^2}{24}]\left(f'(x_{i+1}) - f'(x_i)\right) \end{aligned} \tag{3.36}$$

and the remainder $R\left(f, f', I_n, \xi, \delta\right)$ satisfies the estimation:

$$\begin{aligned} &\left|R\left(f, f', I_n, \xi, \delta\right)\right| \\ &\le \frac{1}{8}(\Phi - \varphi)\sum_{i=0}^{n-1} h_i \left[\frac{(1-\delta)}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]^2 \\ &\le \frac{1}{32}(\Phi - \varphi)(1-\delta)^2 \sum_{i=0}^{n-1} h_i^3, \end{aligned} \tag{3.37}$$

where $\delta \in [0,1]$ and $x_i + \delta\frac{h_i}{2} \le \xi_i \le x_{i+1} - \delta\frac{h_i}{2}$.

Proof. Applying Theorem 3.7 on the interval $[x_i, x_{i+1}]$ $(i = 0, \cdots, n-1)$ gives:

$$\Bigg|(1-\delta)\left[h_i f(\xi_i) - h_i\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) f'(\xi_i)\right] + \delta h_i\left(\frac{f(x_i) + f(x_{i+1})}{2}\right)$$
$$+\left[\frac{1}{2}(1-\delta)\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2 - \frac{(3\delta - 1)h_i^2}{24}\right]\left(f'(x_{i+1}) - f'(x_i)\right)$$
$$-\int_{x_i}^{x_{i+1}} f(t)\,dt\Bigg|$$
$$\leq \frac{1}{8}(\Phi - \varphi) h_i\left[\frac{1}{2}(1-\delta) h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]^2,$$
$$\leq \frac{1}{8}(\Phi - \varphi)(1-\delta)^2 h_i^3$$

as

$$\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \leq (1-\delta)\frac{h_i}{2} \text{ for all } i \in \{0, 1, ..., n-1\}$$

for any choice ξ_i of the intermediate points.

Summing the above inequalities over i from 0 to $n-1$, and using the generalized triangle inequality, we get the desired estimation (3.37). ■

Corollary 3.5 *The following perturbed midpoint rule holds:*

$$\int_a^b f(x)\,dx = M\left(f, f', I_n\right) + R_M\left(f, f', I_n\right),$$

where

$$M\left(f, f', I_n\right) = \sum_{i=0}^{n-1} h_i\, f\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{1}{24}\sum_{i=0}^{n-1} h_i^2\left(f'(x_{i+1}) - f'(x_i)\right) \qquad (3.38)$$

and the remainder term $R_M(f, f', I_n)$ satisfies the estimation:

$$\left|R_M(f, f', I_n)\right| \leq \frac{1}{32}(\Phi - \varphi)\sum_{i=0}^{n-1} h_i^3. \qquad (3.39)$$

Corollary 3.6 *The following perturbed trapezoid rule holds:*

$$\int_a^b f(x)\,dx = T\left(f, f', I_n\right) + R_T\left(f, f', I_n\right) \qquad (3.40)$$

where

$$T\left(f, f', I_n\right) = \sum_{i=0}^{n-1} h_i \frac{f(x_i) + f(x_{i+1})}{2} - \frac{1}{12}\sum_{i=0}^{n-1} h_i^2\left(f'(x_{i+1}) - f'(x_i)\right) \qquad (3.41)$$

and the remainder term $R_T\left(f, f', I_n\right)$ *satisfies the estimation:*

$$\left|R_T\left(f, f', I_n\right)\right| \leq \frac{1}{8}(\Phi-\varphi)\sum_{i=0}^{n-1} h_i^3. \tag{3.42}$$

Remark 3.6 *Note that the above mentioned perturbed midpoint formula (3.28) and perturbed trapezoid formula (3.41) can offer better approximations of the integral* $\int_a^b f(x)\,dx$ *for general classes of mappings as discussed in Remarks 3.4 and 3.5.*

3.3 A generalization of Ostrowski-Grüss type inequality for twice differentiable mappings in Euclidean norm

In this section, we improve and further generalize Ostrowski-Grüss type inequality involving twice differentiable functions. Some applications for probability density function and generalized beta random variable are also given.

3.3.1 Introduction

The Ostrowski-Grüss type inequality for first differentiable mappings has been extended by P. Cerone, S. S. Dragomir and J. Roumeliotis for twice differentiable mappings in [21] and the inequality is stated in the form of following theorem:

Theorem 3.9 *Let* $f: I \longrightarrow \mathbb{R}$, *where* $I \subseteq \mathbb{R}$ *is an interval. Suppose that* f *is twice differentiable in the interior* I^0 *of* I, *and let* $a, b \in I^0$ *with* $a < b$. *If*

$$\gamma \leq f''(x) \leq \Gamma,$$

for some constants $\gamma, \Gamma \in \mathbb{R}$, *then*

$$\begin{aligned} &\left| f(x) + \left(\frac{(b-a)^2}{24} + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2\right)\frac{f'(b) - f'(a)}{b-a} \right. \\ &\left. - \left(x - \frac{a+b}{2}\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ \leq\; & \frac{1}{8}(\Gamma - \gamma)\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^2, \end{aligned} \tag{3.43}$$

for all $x \in [a, b]$.

In 2000, M. Matić, J. Pečarić and N. Ujević [61], by the use of pre-Grüss inequality improved Theorem 3.9 as follows:

Theorem 3.10 *Let the assumptions of Theorem 3.9 hold, then for all* $x \in [a,b]$, *we have*

$$\begin{aligned} &\left| f(x) + \left(\frac{(b-a)^2}{24} + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 \right) \frac{f'(b) - f'(a)}{b-a} \right. \\ &\left. - \left(x - \frac{a+b}{2}\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ \leq\ & \frac{(\Gamma - \gamma)\, l}{6\sqrt{5}} \sqrt{l^2 + 15\xi^2}, \end{aligned} \tag{3.44}$$

where

$$l = \frac{b-a}{2} \text{ and } \xi = x - \frac{a+b}{2}.$$

This result has been further improved by X. L. Cheng in [23] as follows:

Theorem 3.11 *Let the assumptions of Theorem 3.9 hold. Then for all* $x \in [a,b]$, *we have*

$$\begin{aligned} &\left| f(x) + \left(\frac{(b-a)^2}{24} + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 \right) \frac{f'(b) - f'(a)}{b-a} \right. \\ &\left. - \left(x - \frac{a+b}{2}\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ \leq\ & (\Gamma_2 - \gamma_2)\, G(a,b,x), \end{aligned} \tag{3.45}$$

where

$$G(a,b,x) = \begin{cases} \frac{1}{3(b-a)} \left(\left| (x-a)\left(x - \frac{a+b}{2}\right)(b-x) \right| \right. \\ \left. + \left(\frac{1}{12}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2 \right)^{\frac{3}{2}} \right), & a \leq x \leq \frac{1}{3}(2a+b), \\ & \frac{1}{3}(a+2b) \leq x \leq b, \\ \frac{2}{3(b-a)} \left(\frac{1}{12}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2 \right)^{\frac{3}{2}}, & \frac{1}{3}(2a+b) \leq x \leq \frac{1}{3}(a+2b). \end{cases}$$

Further, in [61] we can find the special cases of (3.44) i.e., midpoint and trapezoid inequalities in the form of following corollary:

Corollary 3.7 *Let the assumptions of Theorem 3.9 hold. Then*

$$\left| f\left(\frac{a+b}{2}\right) + \frac{1}{24}(b-a)\left(f'(b) - f'(a)\right) - \frac{1}{b-a}\int_a^b f(t)\, dt \right|$$

$$\leq \frac{1}{24\sqrt{5}} (\Gamma - \gamma)(b-a)^2 . \tag{3.46}$$

Also,

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{12}(b-a)\left(f'(b) - f'(a)\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|$$
$$\leq \frac{1}{6\sqrt{5}} (\Gamma - \gamma)(b-a)^2 . \tag{3.47}$$

Moreover, in [101], a sharp Simpson's inequality for absolutely continuous functions with derivatives, which belong to $L_2(a,b)$ was given as follows:

Theorem 3.12 *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function, whose derivative $f' \in L_2(a,b)$. Then*

$$\left| \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \int_a^b f(t)\,dt \right|$$
$$\leq \frac{(b-a)^{\frac{3}{2}}}{6}\left[\left\| f' \right\|_2^2 - \frac{(f(b)-f(a))^2}{b-a} \right]^{\frac{1}{2}} . \tag{3.48}$$

The inequality is sharp in the sense that the constant $\frac{1}{6}$ cannot be replaced by a smaller one.

We know that for two mappings $f, g : [a,b] \to \mathbb{R}$, the Čebyšev functional is denoted by $T(f,g)$ and is defined as:

$$T(f,g) = \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \frac{1}{b-a}\int_a^b g(t)\,dt,$$

provided that f, g and fg are integrable on $[a,b]$.

Also in [61], we can find the pre-Grüss inequality as

$$T^2(f,g) \leq T(f,f)\,T(g,g),$$

where $f, g \in L_2[a,b]$ and $T(f,g)$ is the Čebyšev functional as defined above.

Moreover, we will use the Korkine's identity (see [52]) which is defined as

$$\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \frac{1}{b-a}\int_a^b g(t)\,dt$$
$$= \frac{1}{2(b-a)^2}\int_a^b\int_a^b (f(t)-f(s))(g(t)-g(s))\,dtds,$$

provided that $f, g : [a, b] \to \mathbb{R}$ are measurable and all the involved integrals exists.

In the following subsection, we improve and further generalize, by the use of Čebyšev functional, the M. Matić et al. [61] results by providing first membership of the right side of (3.44) in terms of Euclidean norm. The bound in (3.44) is given in terms of functions whose derivatives are bounded whereas the right membership of the new inequality is in terms of larger class of absolutely continuous functions whose second derivative $f'' \in L_2(a, b)$ which enlarges the applicability of the underlying quadrature rules. Some applications for probability density function and generalized beta random variable are also given.

3.3.2 Main Results

Theorem 3.13 *Let $f : [a, b] \to \mathbb{R}$ be a mapping whose first derivative is absolutely continuous and the second derivative $f'' \in L_2(a, b)$. Then we have the inequality*

$$\begin{aligned} & \Bigg| (1-h)\left[f(x) - \left(x - \frac{a+b}{2}\right) f'(x)\right] + h\frac{f(a)+f(b)}{2} \\ & - \left[\frac{1}{24}(3h-1)(b-a)^2 - \frac{1}{2}(1-h)\left(x - \frac{a+b}{2}\right)^2\right] \frac{f'(b)-f'(a)}{b-a} \\ & - \frac{1}{b-a}\int_a^b f(t)\,dt \Bigg| \\ \leq\ & (b-a)^2 \left[\frac{1}{2880}\left(4 - 15h + 15h^2\right) + \frac{1}{24}(2-3h)(1-h)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right. \\ & \left. + \frac{1}{4}h(1-h)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^4\right]^{\frac{1}{2}} \left[\frac{1}{b-a}\left\|f''\right\|_2^2 - \left(\frac{f'(b)-f'(a)}{b-a}\right)^2\right]^{\frac{1}{2}} \\ \leq\ & \frac{1}{2}(\Gamma - \gamma)(b-a)^2 \left[\frac{1}{2880}\left(4 - 15h + 15h^2\right)\right. \\ & \left. + \frac{1}{24}(2-3h)(1-h)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2 + \frac{1}{4}h(1-h)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^4\right]^{\frac{1}{2}}, \\ & \text{if } \gamma \leq f''(t) \leq \Gamma, \text{ almost everywhere } t \text{ on } [a, b], \end{aligned} \tag{3.49}$$

for all $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$ and $h \in [0, 1]$.

Proof. We defined in Section 3.2.1, the following kernel $K : [a,b]^2 \to \mathbb{R}$

$$K(x,t) = \begin{cases} \frac{1}{2}\left[t - \left(a + h\frac{b-a}{2}\right)\right]^2, & \text{if } t \in [a,x] \\ \frac{1}{2}\left[t - \left(b - h\frac{b-a}{2}\right)\right]^2, & \text{if } t \in (x,b]. \end{cases}$$

Using Korkine's identity for K and $f^{''}$, we obtain

$$\frac{1}{b-a}\int_a^b K(x,t) f^{''}(t)\,dt - \frac{1}{b-a}\int_a^b K(x,t)\,dt \frac{1}{b-a}\int_a^b f^{''}(t)\,dt$$

$$= \frac{1}{2(b-a)^2}\int_a^b\int_a^b (K(x,t) - K(x,s))\left(f^{''}(t) - f^{''}(s)\right) dtds, \tag{3.50}$$

for all $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$ and $h \in [0,1]$. Further in Section 3.2.1, we have developed the following identities:

$$\frac{1}{b-a}\int_a^b K(x,t) f^{''}(t)\,dt$$

$$= \frac{1}{b-a}\int_a^b f(t)dt - \frac{h}{2}(f(a) + f(b)) - (1-h)\left[f(x) - \left(x - \frac{a+b}{2}\right) f^{'}(x)\right]$$

$$+\frac{1}{8}h^2(b-a)\left(f^{'}(b) - f^{'}(a)\right),$$

$$\frac{1}{b-a}\int_a^b K(x,t)dt = \frac{1}{24}\left(3h^2 - 3h + 1\right)(b-a)^2 + \frac{1}{2}(1-h)\left(x - \frac{a+b}{2}\right)^2,$$

and

$$\frac{1}{b-a}\int_a^b f^{''}(t)\,dt = \frac{f^{'}(b) - f^{'}(a)}{b-a}.$$

Then, by (3.50), we get

$$\frac{1}{b-a}\int_a^b f(t)dt - (1-h)\left[f(x) - \left(x - \frac{a+b}{2}\right) f^{'}(x)\right] - \frac{h}{2}(f(a) + f(b))$$

$$+\left[\frac{1}{24}(3h-1)(b-a)^2 - \frac{1}{2}(1-h)\left(x - \frac{a+b}{2}\right)^2\right]\frac{f^{'}(b) - f^{'}(a)}{b-a}$$

$$= \frac{1}{2(b-a)^2}\int_a^b\int_a^b (K(x,t) - K(x,s))\left(f^{''}(t) - f^{''}(s)\right) dtds, \tag{3.51}$$

for all $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$ and $h \in [0,1]$.

Using the Cauchy-Bunyakowski-Schwartz inequality for double integrals, we may write

$$\begin{aligned}
&\left|\frac{1}{2(b-a)^2}\int_a^b\int_a^b (K(x,t)-K(x,s))\left(f''(t)-f''(s)\right)dtds\right| \\
\leq\ &\left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b (K(x,t)-K(x,s))^2\,dtds\right)^{\frac{1}{2}} \\
&\times\left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(f''(t)-f''(s)\right)^2 dtds\right)^{\frac{1}{2}}. \qquad (3.52)
\end{aligned}$$

However,

$$\begin{aligned}
&\frac{1}{2(b-a)^2}\int_a^b\int_a^b (K(x,t)-K(x,s))^2\,dtds \\
=\ &\frac{1}{b-a}\int_a^b K^2(x,t)\,dt-\left(\frac{1}{b-a}\int_a^b K(x,t)\,dt\right)^2, \qquad (3.53)
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{1}{b-a}\int_a^b K(x,t)\,dt\right)^2 \\
=\ &(b-a)^4\left[\frac{1}{576}\left(1-6h+15h^2-18h^3+9h^4\right)\right. \\
&\left.+\frac{1}{24}\left(1-4h+6h^2-3h^3\right)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2+\frac{1}{4}(1-h)^2\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^4\right] \\
&\qquad (3.54)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b K^2(x,t)\,dt \\
=\ &\frac{1}{20(b-a)}\left[\left(x-\left(a+h\frac{b-a}{2}\right)\right)^5+\left(b-h\frac{b-a}{2}-x\right)^5\right. \\
&\left.+\frac{1}{16}h^5(b-a)^5\right].
\end{aligned}$$

Taking $t=x-\frac{a+b}{2}$, we have

$$\begin{aligned}
x-\left(a+h\frac{b-a}{2}\right) &= t+\frac{1}{2}(1-h)(b-a), \\
b-h\frac{b-a}{2}-x &= \frac{1}{2}(1-h)(b-a)-t.
\end{aligned}$$

Thus,

$$\begin{aligned} & \left(x-\left(a+h\frac{b-a}{2}\right)\right)^5+\left(b-h\frac{b-a}{2}-x\right)^5 \\ = & \left(t+\frac{1}{2}(1-h)(b-a)\right)^5+\left(\frac{1}{2}(1-h)(b-a)-t\right)^5. \end{aligned}$$

For real numbers A and B, we have

$$A^5+B^5=(A+B)\left[\left(A^2+B^2\right)^2-(AB)^2-AB\left(A^2+B^2\right)\right].$$

Now, if $A=t+\frac{1}{2}(1-h)(b-a)$, $B=\frac{1}{2}(1-h)(b-a)-t$, then

$$\begin{aligned} A^2+B^2 &= \left(t+\frac{1}{2}(1-h)(b-a)\right)^2+\left(\frac{1}{2}(1-h)(b-a)-t\right)^2 \\ &= 2t^2+\frac{(1-h)^2(b-a)^2}{2}, \\ AB &= \frac{1}{4}(1-h)^2(b-a)^2-t^2, \\ A+B &= (1-h)(b-a). \end{aligned}$$

Thus,

$$\begin{aligned} & \left(x-\left(a+h\frac{b-a}{2}\right)\right)^5+\left(b-h\frac{b-a}{2}-x\right)^5 \\ = & 5(1-h)(b-a)^5\left[\frac{1}{80}(1-h)^4+\frac{1}{2}(1-h)^2\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^4\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{1}{b-a}\int_a^b K^2(x,t)\,dt \\ = & \frac{1}{4}(b-a)^4\left[\frac{1}{80}\left(1-5h+10h^2-10h^3+5h^4\right)\right. \\ & \left.+\frac{1}{2}(1-h)^3\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2+(1-h)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^4\right]. \end{aligned} \tag{3.55}$$

Using (3.54) and (3.55) in (3.53), we get

$$\begin{aligned} & \frac{1}{2(b-a)^2}\int_a^b\int_a^b (K(x,t)-K(x,s))^2\,dtds \\ = & (b-a)^4\left[\frac{1}{2880}\left(4-15h+15h^2\right)+\frac{1}{24}\left(2-5h+3h^2\right)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right. \\ & \left.+\frac{1}{4}h(1-h)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^4\right]. \end{aligned} \tag{3.56}$$

Moreover,

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(f''(t)-f''(s)\right)^2 dtds$$
$$= \frac{1}{b-a}\left\|f''\right\|_2^2-\left(\frac{f'(b)-f'(a)}{b-a}\right)^2. \tag{3.57}$$

Using (3.51)−(3.53), (3.56)−(3.57), we deduce the first inequality.

Moreover, if $\gamma \leq f''(t) \leq \Gamma$ almost everywhere t on (a,b), then, by using Grüss inequality, we have

$$0 \leq \frac{1}{b-a}\int_a^b f''^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b f''(t)\,dt\right)^2 \leq \frac{1}{4}(\Gamma-\gamma)^2,$$

which proves the last inequality of (3.49). ■

Remark 3.7 *(i) We can get the best estimation from (3.49), only when $x=\frac{a+b}{2}$ i.e.,*

$$\begin{aligned}
&\left|(1-h)f\left(\frac{a+b}{2}\right)+h\frac{f(a)+f(b)}{2}-\frac{1}{24}(3h-1)(b-a)^2\frac{f'(b)-f'(a)}{b-a}\right.\\
&\left.-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\
\leq\;& \frac{1}{24\sqrt{5}}(b-a)^2\left(4-15h+15h^2\right)^{\frac{1}{2}}\left[\frac{1}{b-a}\left\|f''\right\|_2^2-\left(\frac{f'(b)-f'(a)}{b-a}\right)^2\right]^{\frac{1}{2}}\\
\leq\;& \frac{1}{48\sqrt{5}}(\Gamma-\gamma)(b-a)^2\left(4-15h+15h^2\right)^{\frac{1}{2}},\\
&if\ \gamma\leq f''(t)\leq\Gamma,\ \textit{almost everywhere } t \textit{ on } [a,b].
\end{aligned} \tag{3.58}$$

As

$$4-15h+15h^2 \leq 4,\ \forall\ h\in[0,1].$$

and is minimum for $h=\frac{1}{2}$, implies

$$\frac{1}{48\sqrt{5}}\left(4-15h+15h^2\right)^{\frac{1}{2}} \leq \frac{1}{24\sqrt{5}}.$$

Thus (3.49) shows an overall improvement of the inequality obtained by M. Matić et al. [61].

(ii) *For* $h = 1$, *i.e.,* $x = \frac{a+b}{2}$, *(3.49) gives*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{12}(b-a)\left(f'(b)-f'(a)\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|$$
$$\leq \frac{1}{12\sqrt{5}}(b-a)^2\left[\frac{1}{b-a}\left\|f''\right\|_2^2 - \left(\frac{f'(b)-f'(a)}{b-a}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{24\sqrt{5}}(\Gamma-\gamma)(b-a)^2,$$
$$if\ \gamma \leq f''(t) \leq \Gamma,\ \text{almost everywhere } t \text{ on } [a,b], \tag{3.59}$$

which is perturbed trapezoid inequality (corrected trapezoid rule)and it is not difficult to see that it is better than the simple trapezoid inequality.

(iii) *For* $h = 0$ *and* $x = \frac{a+b}{2}$, *(3.49) gives*

$$\left| f\left(\frac{a+b}{2}\right) + \frac{1}{24}(b-a)\left(f'(b)-f'(a)\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|$$
$$\leq \frac{1}{12\sqrt{5}}(b-a)^2\left[\frac{1}{b-a}\left\|f''\right\|_2^2 - \left(\frac{f'(b)-f'(a)}{b-a}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{24\sqrt{5}}(\Gamma-\gamma)(b-a)^2,$$
$$if\ \gamma \leq f''(t) \leq \Gamma,\ \text{almost everywhere } t \text{ on } [a,b], \tag{3.60}$$

which is perturbed mid-point inequality.

(iv) *For* $h = \frac{1}{2}$ *and* $x = \frac{a+b}{2}$, *(3.49) gives*

$$\left| \frac{f(a)+2f\left(\frac{a+b}{2}\right)+f(b)}{4} - \frac{1}{48}(b-a)\left(f'(b)-f'(a)\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|$$
$$\leq \frac{1}{48\sqrt{5}}(b-a)^2\left[\frac{1}{b-a}\left\|f''\right\|_2^2 - \left(\frac{f'(b)-f'(a)}{b-a}\right)^2\right]^{\frac{1}{2}},$$
$$\leq \frac{1}{96\sqrt{5}}(\Gamma-\gamma)(b-a)^2,$$
$$if\ \gamma \leq f''(t) \leq \Gamma,\ \text{almost everywhere } t \text{ on } [a,b], \tag{3.61}$$

which is a linear combination of Trapezoid and Mid-point rule.

(v) *For* $h=\frac{1}{3}$ *and* $x=\frac{a+b}{2}$, *(3.49) gives*

$$\begin{aligned}
&\left|\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{1}{24\sqrt{30}}(b-a)^2\left[\frac{1}{b-a}\left\|f''\right\|_2^2-\left(\frac{f'(b)-f'(a)}{b-a}\right)^2\right]^{\frac{1}{2}}, \\
&\leq \frac{1}{48\sqrt{30}}(\Gamma-\gamma)(b-a)^2, \\
&\quad if\ \gamma\leq f''(t)\leq\Gamma,\ \textit{almost everywhere } t \textit{ on } [a,b], \qquad (3.62)
\end{aligned}$$

which is a variant of Simpson's inequality for twice differentiable function f, f'' *is integrable and there exist constants* $\gamma,\Gamma\in\mathbb{R}$ *such that* $\gamma\leq f''(t)\leq\Gamma$, $t\in(a,b)$.

The estimations (3.58), (3.59), (3.60), (3.61) and (3.62) are expressed in terms of second derivative of the integrand which are useful when the higher derivatives of f *do not exist or are very large at some points in the domain. Moreover, the three-point quadrature rule (3.61) which is a linear combination of Trapezoid and Mid-point rule, offers better estimations than the simple three-point Simpson's rule (3.62).*

Remark 3.8 *In [61], the result corresponding to (3.59) was given, but with* $\frac{1}{6\sqrt{5}}$ *in place of our factor* $\frac{1}{24\sqrt{5}}$ *showing an improvement of factor* $\frac{1}{4}$ *as it can be seen from (3.47). Also in [23], (3.59) was given with a factor of* $\frac{1}{18\sqrt{3}}$ *which shows that (3.59) also offers better estimation than as given in [23]. Moreover, we have also been able to present bounds for three-point quadrature rules as given in and (3.61) where (3.61) is a extension of (3.49) for twice differentiable mappings.*

3.3.3 Application in Numerical integration

Let $I_n: a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ be a division of the interval $[a,b]$ and $h_i=x_{i+1}-x_i=h=\frac{(b-a)}{n}$, $i=0,\cdots,n-1$, then we have the following quadrature formula:

Theorem 3.14 *Let* I_n *be the subdivision of the interval* $[a,b]$ *and let the assump-*

tions of Theorem 3.13 hold. Then,

$$\left|\int_a^b f(t)\,dt-(1-\delta)h\sum_{i=0}^{n-1}\left[f(\xi_i)-\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)f'(\xi_i)\right]\right.$$
$$+\sum_{i=0}^{n-1}\left[\frac{1}{24}h^2(3\delta-1)-\frac{1}{2}(1-\delta)\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right]\Big(f'(x_{i+1})-f'(x_i)\Big)$$
$$\left.-\delta\frac{h}{2}\sum_{i=0}^{n-1}[f(x_i)+f(x_{i+1})]\right|$$
$$\leq \left(\frac{b-a}{n}\right)^2\left[\frac{1}{2880}\left(4-15\delta+15\delta^2\right)+\frac{1}{24}(2-3\delta)(1-\delta)\sum_{i=0}^{n-1}\left(\frac{\xi_i-\frac{x_i+x_{i+1}}{2}}{h}\right)^2\right.$$
$$\left.+\frac{1}{4}\delta(1-\delta)\sum_{i=0}^{n-1}\left(\frac{\xi_i-\frac{x_i+x_{i+1}}{2}}{h}\right)^4\right]^{\frac{1}{2}}\left[\frac{b-a}{n}\left\|f''\right\|_2^2-\sum_{i=0}^{n-1}\Big(f'(x_{i+1})-f'(x_i)\Big)^2\right]^{\frac{1}{2}}.$$

Proof. Apply inequality (3.49) on the interval $[x_i, x_{i+1}]$, $i=0,\cdots,n-1$ to get,

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt-(1-\delta)h\left[f(\xi_i)-\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)f'(\xi_i)\right]\right.$$
$$+\left[\frac{1}{24}h^2(3\delta-1)-\frac{1}{2}(1-\delta)\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right]\Big(f'(x_{i+1})-f'(x_i)\Big)$$
$$\left.-\delta\frac{h}{2}[f(x_i)+f(x_{i+1})]\right|$$
$$\leq h^{\frac{5}{2}}\left[\frac{1}{2880}\left(4-15\delta+15\delta^2\right)+\frac{1}{24}(2-3\delta)(1-\delta)\left(\frac{\xi_i-\frac{x_i+x_{i+1}}{2}}{h}\right)^2\right.$$
$$\left.+\frac{1}{4}\delta(1-\delta)\left(\frac{\xi_i-\frac{x_i+x_{i+1}}{2}}{h}\right)^4\right]^{\frac{1}{2}}\left[\int_{x_i}^{x_{i+1}}\Big(f''(t)\Big)^2dt-\frac{\Big(f'(x_{i+1})-f'(x_i)\Big)^2}{h}\right]^{\frac{1}{2}},$$

for all $i=0,\cdots,n-1$.

Summing over i from 0 to $n-1$, using triangular inequality and Cauchy-Schwartz discrete inequality, we get,

$$\left|R'(f,f',I_n,\xi,\delta)\right|$$
$$\leq \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(t)\,dt-(1-\delta)h\left[f(\xi_i)-\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)f'(\xi_i)\right]\right.$$
$$+\left[\frac{1}{24}h^2(3\delta-1)-\frac{1}{2}(1-\delta)\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right]\Big(f'(x_{i+1})-f'(x_i)\Big)$$
$$\left.-\delta\frac{h}{2}[f(x_i)+f(x_{i+1})]\right|$$

$$\leq h^{\frac{5}{2}}\left(\sum_{i=0}^{n-1}\left(\left[\frac{1}{2880}\left(4-15\delta+15\delta^2\right)+\frac{1}{24}\left(2-3\delta\right)\left(1-\delta\right)\left(\frac{\xi_i-\frac{x_i+x_{i+1}}{2}}{h}\right)^2\right.\right.\right.$$

$$\left.\left.\left.+\frac{1}{4}\delta\left(1-\delta\right)\left(\frac{\xi_i-\frac{x_i+x_{i+1}}{2}}{h}\right)^4\right]^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}$$

$$\times\left(\sum_{i=0}^{n-1}\left(\left[\int_{x_i}^{x_{i+1}}\left(f''(t)\right)^2dt-\frac{\left(f'(x_{i+1})-f'(x_i)\right)^2}{h}\right]^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}$$

$$\leq h^2\left[\frac{1}{2880}\left(4-15\delta+15\delta^2\right)+\frac{1}{24}\left(2-3\delta\right)\left(1-\delta\right)\sum_{i=0}^{n-1}\left(\frac{\xi_i-\frac{x_i+x_{i+1}}{2}}{h}\right)^2\right.$$

$$\left.+\frac{1}{4}\delta\left(1-\delta\right)\sum_{i=0}^{n-1}\left(\frac{\xi_i-\frac{x_i+x_{i+1}}{2}}{h}\right)^4\right]^{\frac{1}{2}}\left[h\left\|f''\right\|_2^2-\sum_{i=0}^{n-1}\left(f'(x_{i+1})-f'(x_i)\right)^2\right]^{\frac{1}{2}}.$$

Thus, we get the required result.

Remark 3.9 *Note that if we choose* $\delta=\frac{1}{2}$, $\xi_i=\frac{x_i+x_{i+1}}{2}$, *then we get the quadrature rule which is a linear combination of midpoint rule and trapezoid rule and it offers the best estimate.*

3.3.4 Application for Probability Density Functions

Let X be a random variable having the probability density function $f:[a,b]\to\mathbb{R}_+$ and the cumulative distribution function $F:[a,b]\to[0,1]$, i.e.,

$$F(x)=\int_a^x f(t)\,dt,\ x\in\left[a+h\frac{b-a}{2},b-h\frac{b-a}{2}\right]\subseteq[a,b].$$

Then, we may have the following:

Theorem 3.15 *Under the above assumptions and if the probability density function*

f belongs to $L_2[a,b]$, then we have the inequality

$$\begin{aligned}
&\left|(1-h)\left[F(x)-\left(x-\frac{a+b}{2}\right)f(x)\right]+\frac{h}{2}-\frac{b-E(X)}{b-a}\right.\\
&\left.-\left[\frac{1}{24}(3h-1)(b-a)^2-\frac{1}{2}(1-h)\left(x-\frac{a+b}{2}\right)^2\right]\frac{f(b)-f(a)}{b-a}\right|\\
\leq\;& (b-a)^2\left[\frac{1}{2880}\left(4-15h+15h^2\right)+\frac{1}{24}(2-3h)(1-h)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right.\\
&\left.+\frac{1}{4}h(1-h)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^4\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\left\|f'\right\|_2^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}\\
\leq\;& \frac{(b-a)^2(M-m)}{2}\left[\frac{1}{2880}\left(4-15h+15h^2\right)+\frac{1}{24}\left(2-5h+3h^2\right)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right.\\
&\left.+\frac{1}{4}h(1-h)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^4\right]^{\frac{1}{2}},\\
&if\;\; m\leq f'\leq M,\;\; almost\ everywhere\ on\ [a,b], \qquad (3.63)
\end{aligned}$$

for all $x\in\left[a+h\frac{b-a}{2},b-h\frac{b-a}{2}\right]$ and $h\in[0,1]$.

Proof. Put in (3.49), $f=F$ to get (3.63). ∎

Corollary 3.8 *Under the above assumptions, we have*

$$\begin{aligned}
&\left|(1-h)\Pr\left(X\leq\frac{a+b}{2}\right)+\frac{h}{2}-\frac{b-E(X)}{b-a}-\frac{(3h-1)}{24}(b-a)(f(b)-f(a))\right|\\
\leq\;& \frac{1}{24\sqrt{5}}\left(4-15h+15h^2\right)^{\frac{1}{2}}(b-a)^2\left[\frac{1}{b-a}\left\|f'\right\|_2^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}\\
\leq\;& \frac{1}{48\sqrt{5}}(M-m)\left(4-15h+15h^2\right)^{\frac{1}{2}}(b-a)^2,\\
&if\;\; m\leq f'\leq M,\;\; almost\ everywhere\ on\ [a,b]. \qquad (3.64)
\end{aligned}$$

3.3.5 Application for generalized beta random variable

If X is a beta random variable with parameters $\beta_3>-1$, $\beta_4>-1$ and for $\beta_2>0$ and any β_1, the generalized beta random variable

$$Y=\beta_1+\beta_2X,$$

is said to have a generalized beta distribution [51] and the probability density function of the generalized beta distribution of beta random variable is,

$$f(x) = \begin{cases} \frac{(x-\beta_1)^{\beta_3}(\beta_1+\beta_2-x)^{\beta_4}}{\beta(\beta_3+1,\beta_4+1)\beta_2^{(\beta_3+\beta_4+1)}}, & \text{for } \beta_1 < x < \beta_1+\beta_2 \\ 0, & \text{otherwise,} \end{cases}$$

where $\beta(l, m)$ is the beta function with $l,\ m > 0$ and is defined as

$$\beta(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx.$$

For p, $q > 0$ and $h \in [0, 1)$, we choose,

$$\begin{aligned} \beta_1 &= \frac{h}{2}, \\ \beta_2 &= (1-h), \\ \beta_3 &= p-1, \\ \beta_4 &= q-1. \end{aligned}$$

Then, the probability density function associated with generalized beta random variable

$$Y = \frac{h}{2} + (1-h)X,$$

takes the form

$$f(x) = \begin{cases} \frac{\left(x-\frac{h}{2}\right)^{p-1}\left(1-\frac{h}{2}-x\right)^{q-1}}{\beta(p,q)(1-h)^{p+q-1}}, & \frac{h}{2} < x < 1-\frac{h}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Now,

$$\begin{aligned} E(Y) &= \int_{\frac{h}{2}}^{1-\frac{h}{2}} x f(x)\,dx \\ &= (1-h)\frac{p}{p+q} + \frac{h}{2}. \end{aligned}$$

$$\begin{aligned} \frac{df(x;p,q)}{dx} &= \frac{\left(x-\frac{h}{2}\right)^{p-2}\left(1-\frac{h}{2}-x\right)^{q-2}}{(1-h)^{p+q-1}\beta(p,q)} \times \\ &\quad \left[(p-1) - (p-q)\frac{h}{2} - (p+q-2)x\right], \end{aligned}$$

and

$$\left\|f'(.;p,q)\right\|_2^2 = \frac{1}{(1-h)^3\beta^2(p,q)}\left[(p-1)^2\beta(2p-3,2q-1)\right.$$
$$+(q-1)^2\beta(2p-1,2q-3)$$
$$\left.-2(p-1)(q-1)\beta(2p-2,2q-2)\right].$$

Then, by Theorem 3.14, we may state the following:

Proposition 3.2 *Let X be a beta random variable with parameters (p,q). Then, for generalized beta random variable*

$$Y=\frac{h}{2}+(1-h)X,$$

we have the inequality

$$\left|(1-h)\left[\Pr(Y\le x)-\left(x-\frac{1}{2}\right)f(x)-\frac{q}{p+q}\right]\right.$$
$$\left.-\left[\frac{1}{24}(3h-1)-\frac{1}{2}(1-h)\left(x-\frac{1}{2}\right)^2\right](f(1)-f(0))\right|$$
$$\le \frac{1}{(1-h)^{\frac{3}{2}}\beta(p,q)}\left[\frac{1}{2880}\left(4-15h+15h^2\right)+\right.$$
$$\left.\frac{1}{24}(2-3h)(1-h)\left(x-\frac{1}{2}\right)^2+\frac{1}{4}h(1-h)\left(x-\frac{1}{2}\right)^4\right]^{\frac{1}{2}}\times$$
$$\left[(p-1)^2\beta(2p-3,2q-1)+(q-1)^2\beta(2p-1,2q-3)\right.$$
$$-2(p-1)(q-1)\beta(2p-2,2q-2)$$
$$\left.-(1-h)^3\beta^2(p,q)(f(1)-f(0))^2\right]^{\frac{1}{2}}, \tag{3.65}$$

for all $x\in\left[\frac{h}{2},1-\frac{h}{2}\right]$.

3.4 New Estimates for first the inequality of Ostrowski-Grüss type and applications in numerical integration

In this section, some error bounds for the first inequalities of Ostrowski-Grüss type are obtained. These bounds provide some new and better estimates. Applications in numerical integration are also given.

3.4.1 Introduction

In [97], N. Ujević gave the following estimation of first inequality of Osrowski-Grüss type derived by S. S. Dragomir and S. Wang in [39].

Theorem 3.16 *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a differentiable mapping in the interior $Int\ I$ of I, and let $a, b \in Int\ I, a < b$. If there exists constants $\gamma, \Gamma \in \mathbb{R}$ such that $\gamma \le f'(t) \le \Gamma, \forall\ t \in [a, b]$ and $f' \in L_1(a, b)$, then we have*

$$\left| f(x) - \left(x - \frac{a+b}{2} \right) \frac{f(b) - f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \frac{b-a}{2} (S - \gamma), \quad (3.66)$$

and

$$\left| f(x) - \left(x - \frac{a+b}{2} \right) \frac{f(b) - f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \frac{b-a}{2} (\Gamma - S), \quad (3.67)$$

where $S = \frac{f(b)-f(a)}{b-a}$.

The main aim of this section is to point out better estimations of (3.66), (3.67) and to apply them in numerical integration. Some mid-point inequalities and corrected trapezoid inequalities are also given.

3.4.2 Main Results

We prove the following result:

Theorem 3.17 *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be mapping differentiable in the interior $Int\ I$ of I, and let $a, b \in Int\ I$, $a < b$. If there exists some constants $\gamma, \Gamma \in \mathbb{R}$, such that $\gamma \le f'(t) \le \Gamma$, $\forall\ t \in [a, b]$ and $f' \in L_1(a, b)$, then we have*

$$\left| (1-h) \left[f(x) - \left(x - \frac{a+b}{2} \right) f'(x) \right] + h \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right|$$
$$\le \frac{1}{2} \left(1 - h^2 \right) (b-a)(S - \gamma) \quad (3.68)$$

and

$$\left| (1-h) \left[f(x) - \left(x - \frac{a+b}{2} \right) f'(x) \right] + h \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right|$$

$$\leq \frac{1}{2}\left(1-h^2\right)(b-a)(\Gamma - S), \tag{3.69}$$

where $S = \frac{f(b)-f(a)}{b-a}$, $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$ *and* $h \in [0,1]$.

Proof. *Let us consider the mapping* $p(.,.) : [a,b]^2 \longrightarrow \mathbb{R}$ given by

$$p(x,t) = \begin{cases} t - \left(a + h\frac{b-a}{2}\right), & if\ t \in [a,x] \\ t - \left(b - h\frac{b-a}{2}\right), & if\ t \in (x,b], \end{cases} \tag{3.70}$$

where $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$ and $h \in [0,1]$.

Integrating by parts, we successively have

$$\frac{1}{b-a}\int_a^b p(x,t)f'(t)dt = (1-h)\,f(x) + h\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt. \tag{3.71}$$

Moreover,

$$\frac{1}{(b-a)}\int_a^b p(x,t)dt = (1-h)\left(x - \frac{a+b}{2}\right) \tag{3.72}$$

and

$$\frac{1}{(b-a)}\int_a^b f'(t)dt = f'(x). \tag{3.73}$$

From (3.71), (3.72) and (3.73), we have

$$\begin{aligned} &(1-h)\left[f(x) - \left(x - \frac{a+b}{2}\right)f'(x)\right] + h\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \\ = \;& \frac{1}{(b-a)}\int_a^b p(x,t)f'(t)dt - \frac{1}{(b-a)^2}\int_a^b p(x,t)dt\int_a^b f'(t)dt. \end{aligned} \tag{3.74}$$

We denote

$$R_n(x) = \frac{1}{(b-a)}\int_a^b f'(t)\left(p(x,t) - \frac{1}{(b-a)}\int_a^b p(x,s)ds\right)dt. \tag{3.75}$$

If $C \in \mathbb{R}$ is an arbitrary constant, then we also have

$$R_n(x) = \frac{1}{(b-a)}\int_a^b \left(f'(t) - C\right)\left(p(x,t) - \frac{1}{(b-a)}\int_a^b p(x,s)ds\right)dt. \tag{3.76}$$

Indeed

$$\int_a^b \left(p(x,t) - \frac{1}{(b-a)}\int_a^b p(x,s)ds\right)dt = 0. \tag{3.77}$$

If we choose $C = \gamma$ in (3.76)

$$R_n(x) = \frac{1}{(b-a)} \int_a^b \left(f'(t) - \gamma\right) \left(p(x,t) - \frac{1}{(b-a)} \int_a^b p(x,s)ds\right) dt,$$

implies

$$\begin{aligned} |R_n(x)| \;&\le\; \frac{1}{(b-a)} \sup_{t\in[a,b]} \left| p(x,t) - \frac{1}{(b-a)} \int_a^b p(x,s)ds \right| \\ &\quad \times \int_a^b \left| f'(t) - \gamma \right| dt. \end{aligned} \qquad (3.78)$$

Since

$$\sup_{t\in[a,b]} \left| p(x,t) - \frac{1}{(b-a)} \int_a^b p(x,s)ds \right| = \frac{(1-h^2)(b-a)}{2} \qquad (3.79)$$

and as

$$\int_a^b \left| f'(t) - \gamma \right| dt = (S - \gamma)(b-a). \qquad (3.80)$$

From (3.78), (3.79) and (3.80), we have

$$|R_n(x)| \le \frac{(1-h^2)(b-a)}{2} (S - \gamma). \qquad (3.81)$$

Next, we choose $C = \Gamma$ in (3.76)

$$R_n(x) = \frac{1}{(b-a)} \int_a^b \left(f'(t) - \Gamma\right) \left(p(x,t) - \frac{1}{(b-a)} \int_a^b p(x,s)ds\right) dt,$$

implies

$$\begin{aligned} |R_n(x)| \;&\le\; \frac{1}{(b-a)} \sup_{t\in[a,b]} \left| p(x,t) - \frac{1}{(b-a)} \int_a^b p(x,s)ds \right| \qquad (3.82) \\ &\quad \times \int_a^b \left| \left(f'(t) - \Gamma\right) \right| dt \end{aligned}$$

and as

$$\int_a^b \left| \left(f'(t) - \Gamma\right) \right| dt = (\Gamma - S)(b-a), \qquad (3.83)$$

from (3.79), (3.82) and (3.83), we have

$$|R_n(x)| \le \frac{(1-h^2)(b-a)}{2} (\Gamma - S).$$

This completes the proof. ■

Remark 3.10 *For $h = 0$ in Theorem 3.17, we recapture the results of Theorem 3.16.*

Corollary 3.9 *Let f be as in Theorem 3.17, then we have the inequalities:*

$$\begin{aligned}
& |(1-h)\frac{f(a+h\frac{b-a}{2})+f(b-h\frac{b-a}{2})}{2}+h\frac{f(a)+f(b)}{2} \\
& -\frac{(1-h)(b-a)}{4}\left(f'(b-h\frac{b-a}{2})-f'(a+h\frac{b-a}{2})\right) \\
& -\frac{1}{b-a}\int_a^b f(t)dt| \\
\leq\ & \frac{(1-h^2)(b-a)}{2}(S-\gamma),
\end{aligned} \tag{3.84}$$

and

$$\begin{aligned}
& |(1-h)\frac{f(a+h\frac{b-a}{2})+f(b-h\frac{b-a}{2})}{2}+h\frac{f(a)+f(b)}{2} \\
& -\frac{(1-h)(b-a)}{4}\left(f'(b-h\frac{b-a}{2})-f'(a+h\frac{b-a}{2})\right) \\
& -\frac{1}{b-a}\int_a^b f(t)dt| \\
\leq\ & \frac{(1-h^2)(b-a)}{2}(\Gamma-S),
\end{aligned} \tag{3.85}$$

where $h \in [0,1]$.

Proof. Putting in (3.68) $x = a + h\frac{b-a}{2}$ and $x = b - h\frac{b-a}{2}$ and then using the triangular inequality on the summoned of the two inequalities, we get the required inequality (3.84), and by the same substitution in (3.69), we can get (3.85). ∎

Remark 3.11 *If we choose in (3.84) and (3.85), $h = 0$, then we have the following perturbed trapezoid inequalities which are better than as we can have from (3.66) and (3.67).*

$$\begin{aligned}
& \left|\frac{f(a)+f(b)}{2}-\frac{(b-a)}{4}\left(f'(b)-f'(a)\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
\leq\ & \frac{(b-a)}{2}(S-\gamma),
\end{aligned} \tag{3.86}$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{(b-a)}{4}\left(f^{'}(b)-f^{'}(a)\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \frac{(b-a)}{2}(\Gamma-S). \tag{3.87}$$

Remark 3.12 *If we choose in (3.76)* $C=\frac{\Gamma+\gamma}{2}$, $h=0$ *and* $x=a$, *then* $x=b$ *and the summoned of the two inequalities is divided by* 2, *we get:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{(b-a)}{4}\left(f^{'}(b)-f^{'}(a)\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \tag{3.88}$$
$$\leq \frac{(b-a)}{8}(\Gamma-\gamma).$$

Remark 3.13 *If we put* $h=0$, $x=\frac{a+b}{2}$ *in (3.68) and (3.69) and add the results we have the midpoint inequality as:*

$$\left|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{(b-a)}{4}(\Gamma-\gamma). \tag{3.89}$$

Remark 3.14 *If we put* $h=\frac{1}{3}$, $x=\frac{a+b}{2}$ *in (3.68) and (3.69) and add the results we have the Simpson's inequality as:*

$$\left|\frac{1}{6}\left[f\left(a\right)+4f(\frac{a+b}{2})+f\left(b\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{2}{9}(b-a)(\Gamma-\gamma). \tag{3.90}$$

Remark 3.15 *If we put* $h=\frac{1}{2}$, $x=\frac{a+b}{2}$ *in (3.68) and (3.69) and add the results we have the averaged midpoint trapezoid inequality as:*

$$\left|\frac{1}{2}\left[f(\frac{a+b}{2})+\frac{f\left(a\right)+f\left(b\right)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{3}{16}(b-a)(\Gamma-\gamma). \tag{3.91}$$

We now present a result of $L_2\left(a,b\right)$.

Theorem 3.18 *Let* $f:I\to\mathbb{R}$, *where* $I\subset\mathbb{R}$ *is an interval, be continuously twice differentiable mapping in the interior Int* I *of* I, *with* $f^{''}\in L_2(a,b)$, *and let* $a,b\in$ *Int* I, $a<b$, *then we have*

$$\left|(1-h)\left[f(x)-\left(x-\frac{a+b}{2}\right)f^{'}(x)\right]+h\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \frac{(b-a)^{1/2}}{\pi}$$
$$\times\left[h\left(1-h\right)\left(x-\frac{a+b}{2}\right)+\frac{\left(b-a\right)^2}{12}\left(h^3+\left(1-h\right)^3\right)\right]^{1/2}\left\|f^{''}\right\|_2, \tag{3.92}$$

for all $x \in [a,b]$.

Proof. From (3.74) and (3.75), we get

$$R_n(x) = (1-h)\left[f(x) - \left(x - \frac{a+b}{2}\right) f'(x)\right] + h\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt. \tag{3.93}$$

If we choose $C = f'(\frac{a+b}{2})$ in (3.76) and use the Cauchy-Schwarz inequality, then we get

$$\begin{aligned} & |R_n(x)| \\ \leq\ & \frac{1}{(b-a)}\int_a^b \left|f'(t) - f'(\frac{a+b}{2})\right| \left|p(x,t) - \frac{1}{(b-a)}\int_a^b p(x,s)ds\right| dt \\ \leq\ & \frac{1}{(b-a)}\left[\int_a^b \left(f'(t) - f'\left(\frac{a+b}{2}\right)\right)^2 dt\right]^{1/2} \\ \times\ & \left[\int_a^b \left(p(x,t) - \frac{1}{(b-a)}\int_a^b p(x,s)ds\right)^2 dt\right]^{1/2}. \end{aligned} \tag{3.94}$$

By using the Diaz-Metcalf inequality (see Theorem 1.10) for $t_1 = t_2 = \frac{a+b}{2}$, we get

$$\int_a^b \left(f'(t) - f'\left(\frac{a+b}{2}\right)\right)^2 dt \leq \frac{(b-a)^2}{\pi^2}\left\|f''\right\|_2^2. \tag{3.95}$$

We have

$$\begin{aligned} & \int_a^b \left(p(x,t)dt - \frac{1}{(b-a)}\int_a^b p(x,s)ds\right)^2 dt \\ =\ & \int_a^b p^2(x,t)dt - \frac{1}{(b-a)}\left(\int_a^b p(x,s)ds\right)^2 dt, \end{aligned}$$

where

$$\begin{aligned} \int_a^b p^2(x,t)dt &= \frac{(b-a)^3}{12}\left[h^3 + (1-h)^3\right] \\ &\quad + (1-h)(b-a)\left(x - \frac{a+b}{2}\right)^2 \end{aligned}$$

and

$$\frac{1}{(b-a)}\left(\int_a^b p(x,s)ds\right)^2 = (b-a)(1-h)^2 (x - \frac{a+b}{2})^2.$$

Also

$$\int_a^b \left(p(x,t)dt - \frac{1}{(b-a)} \int_a^b p(x,s)ds \right)^2 dt$$

$$= \frac{(b-a)^3}{12} \left[h^3 + (1-h)^3 \right] + h(1-h)(b-a) \left(x - \frac{a+b}{2} \right)^2 . \quad (3.96)$$

With the help of (3.93), (3.94), (3.95) and (3.96) we get the required inequality. ■

3.4.3 Applications in Numerical Integration

Let $I_n : a = x_0 < x_1 < x_2 < .... < x_{n-1} < x_n = b$ be the division of the interval $[a,b]$, $\xi_i \in [x_i + \delta \frac{h_i}{2}, x_{i+1} - \delta \frac{h_i}{2}]$ $(i = 1, 2, 3....n-1)$. We have the following quadrature formula:

Theorem 3.19 *Let f be as in Theorem 3.17, then for every partition I_n of $[a,b]$ and for every intermediate point vector $\xi = (\xi_0, \xi_1, ..., \xi_{n-1})$, satisfying $\xi_i \in \left[x_i + \delta \frac{h_i}{2}, x_{i+1} - \delta \frac{h_i}{2}\right]$ $(i = 0, 1, ..., n-1)$, $\delta \in [0,1]$, then we have the following*

$$\left| \int_a^b f(t)dt - A'(f, f', \delta, \xi, I_n) \right| \leq \frac{1}{2} \left(1 - \delta^2\right) \sum_{i=0}^{n-1} h_i^2 (S_i - \gamma) \quad (3.97)$$

and

$$\left| \int_a^b f(t)dt - A'(f, f', \delta, \xi, I_n) \right| \leq \frac{1}{2} \left(1 - \delta^2\right) \sum_{i=0}^{n-1} h_i^2 (\Gamma - S_i) \quad (3.98)$$

where

$$A'(f, f', \delta, \xi, I_n) = (1-\delta) \left(\sum_{i=0}^{n-1} h_i f(\xi_i) - \sum_{i=0}^{n-1} h_i \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) f'(\xi_i) \right) - \delta \sum_{i=0}^{n-1} h_i \frac{f(x_i) + f(x_{i+1})}{2}, \quad (3.99)$$

for all $\xi_i \in [x_i + \delta \frac{h_i}{2}, x_{i+1} - \delta \frac{h_i}{2}]$, $\delta \in [0,1]$ and $S_i = \frac{f(x_{i+1}) - f(x_i)}{2}$, $h_i := x_{i+1} - x_i$, $(i = 0, ..., n-1)$.

Proof. Apply Theorem 3.17 on the interval $[x_i, x_{i+1}]$, $\xi_i \in [x_i + \delta \frac{h_i}{2}, x_{i+1} - \delta \frac{h_i}{2}]$ where $h_i := x_{i+1} - x_i$, $(i = 0, ..., n-1)$ to get

$$\left| \int_{x_i}^{x_{i+1}} f(t)dt - (1-\delta) \left[h_i f(\xi_i) - h_i \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) f'(\xi_i) \right] - \delta\, h_i \frac{f(x_i) + f(x_{i+1})}{2} \right|$$

$$\leq \frac{(1-\delta^2)}{2} h_i^2 (S_i - \gamma)$$

and

$$
\begin{aligned}
&|\int_{x_i}^{x_{i+1}} f(t)dt-(1-\delta)\left[h_i f(\xi_i)-h_i\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)f'(\xi_i)\right] \\
&-\delta\, h_i\frac{f(x_i)+f(x_{i+1})}{2}| \\
\leq\ &\frac{\left(1-\delta^2\right)}{2}{h_i}^2(\Gamma-S_i).
\end{aligned}
$$

Summing over i from 0 to $n-1$, we get and using the generalization triangle inequality we have deduced the desired estimations (3.97) and (3.98). ■

3.5 Conclusion

In this chapter, we have presented some generalizations of Ostrowski-Grüss type inequality for first and twice differentiable functions in Euclidean norm and for bounded functions. Since the bound in Ostrowski-Grüss inequality can be applied for absolutely continuous mappings whose first derivative is bounded, the new inequalities can also be applied for the larger classes of absolutely continuous mappings whose first or second derivatives are in $L_2(a,b)$.

In Section 3.1, the generalized Ostrowski-Grüss type inequality (3.4) has an advantage on (3.3) obtained in [12] in a way that it not only recaptures the special cases associated with (3.3) but can also present three-point inequalities of averaged trapezoid and Simpson's type. It also has applications in special means, for probability density functions, expectation of a random variable X and generalized beta random variable.

In Section 3.2, we have presented a generalization of Ostrowski-Grüss type inequality for twice differentiable with second derivative bounded obtained in ([9], Section 5). Remark 3.4 and 3.5 show that the estimates of perturbed mid-point and trapezoid inequalities presented in here are better than the classical estimates for these inequalities.

Section 3.3 is concerned with a new generalization Of Ostrowski's integral inequality that can be developed from Pre-Grüss and Grüss inequality. We have improved the Matić-Pečarić-Ujevic [61] result by providing a better bound for the first membership of Ostrowski-Grüss type inequality for twice differentiable functions. As special cases tighter bounds for mid-point,trapezoid, averaged trapezoid and Simpson's quadrature rules are also obtained and are shown to be better than the these quadrature rules presented in [61] and [23].

In Section 3.4, we have presented a generalization of Ostrowski-Grüss type inequalities obtained in [97] for bounded first derivatives. The inequalities (3.68) and (3.69) are more applicable than the first inequality of Osrowski-Grüss type derived by S. S. Dragomir and S. Wang in [39] and the inequalities (3.66) and (3.67) obtained in [97] because they can be applied for functions whose first derivative is either bounded above or bounded below. Moreover, as special cases we can also get the estimated for three-point inequalities in our case. A generalized version is also obtained for higher class of functions with $f'' \in L_2(a,b)$ by using the Diaz-Metcalf inequality.

Chapter 4

Some product inequalities of Ostrowski, Čebyšev and Grüss type

For two measurable functions $f,\ g : [a,b] \to \mathbb{R}$, define the functional,

$$T(f,g;a,b) := \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right),$$

which in literature is called the Čebyšev functional, provided the involved integrals exists.

Moreover, in 1882 P. L. Čebyšev (see [64], p. 297) proved that, if $f^{'}, g^{'} \in L_\infty[a,b]$, then

$$|T(f,g;a,b)| \le \frac{1}{12}(b-a)^2 \left\|f^{'}\right\|_\infty \left\|g^{'}\right\|_\infty .$$

In the recent past, Čebyšev functional has remained an area of special interest for many researchers and has yielded many variants and generalizations in the field of inequalities. It has also played a key role in obtaining some new inequalities of Ostrowski type, for example, Ostrowski-Grüss type, Ostrowski-Čebyšev type, etc. The research papers [97, 76] cover a comprehensive literature on the generalizations of Čebyšev functional and its associated bounds.

We, in this chapter, present some extensions of product Čebyšev type inequalities for first and twice differentiable functions.

4.1 A note on the generalization of some new Čebyšev type inequalities

In this section, we present a generalized Čebyšev type inequality for absolutely continuous functions whose derivatives belong to $L_p(a,b)$, $p>1$. Applications for probability density functions are also given.

4.1.1 Introduction

In [80], B. G. Pachpatte presented the following Čebyšev type inequality for p-norm:

Theorem 4.1 *Let $f,g:[a,b]\to\mathbb{R}$ be absolutely continuous functions whose derivatives $f^{'}$, $g^{'}\in L_p(a,b)$, $p>1$ then*

$$|P(C,D,f,g)|\leq\frac{1}{(b-a)^2}M^{\frac{2}{q}}\left\|f^{'}\right\|_p\left\|g^{'}\right\|_p, \tag{4.1}$$

where

$$\begin{aligned}C&=\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right],\\ D&=\frac{1}{3}\left[\frac{g(a)+g(b)}{2}+2g\left(\frac{a+b}{2}\right)\right],\end{aligned}$$

$$M=\frac{(2^{q+1}+1)(b-a)^{q+1}}{3(q+1)6^q},$$

with $\frac{1}{p}+\frac{1}{q}=1$ and

$$\|f\|_p=\left(\int_a^b|f(t)|^p\,dt\right)^{\frac{1}{p}}<\infty.$$

$$\begin{aligned}P(\alpha,\beta,f,g)&=\alpha\beta-\frac{1}{b-a}\left(\alpha\int_a^b g(t)\,dt+\beta\int_a^b f(t)\,dt\right)\\ &\quad+\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right),\end{aligned} \tag{4.2}$$

α and β are real constants.

Recently, in [56], Zheng Liu presented the following generalization of (4.1):

Theorem 4.2 *Let the assumptions of Theorem 4.1 hold, then for any* $\theta \in [0,1]$,

$$\left|P\left(\Gamma_{\theta},\Delta_{\theta},f,g\right)\right| \leq \frac{1}{(b-a)^{2}} M_{\theta}^{\frac{2}{q}} \left\|f^{'}\right\|_{p} \left\|g^{'}\right\|_{p}, \tag{4.3}$$

where

$$M_{\theta}=\frac{\theta^{q+1}+(1-\theta)^{q+1}}{(q+1)\,2^{q}}(b-a)^{q+1}$$

and

$$\begin{aligned}\Gamma_{\theta} &= \frac{\theta}{2}\left[f(a)+f(b)\right]+(1-\theta)\,f\left(\frac{a+b}{2}\right),\\ \Delta_{\theta} &= \frac{\theta}{2}\left[g(a)+g(b)\right]+(1-\theta)\,g\left(\frac{a+b}{2}\right).\end{aligned}$$

In the following subsection, we obtain a generalization of the inequalities (4.1), (4.3) and apply them to probability density functions.

4.1.2 Main Results

For suitable functions $f,g:[a,b]\to\mathbb{R}$ and $h\in[0,1]$, we present the following notations:

$$\Gamma_{h,x}=(1-h)\,f(x)+h\left(\frac{(x-a)\,f(a)+(b-x)\,f(b)}{b-a}\right),$$

$$\Delta_{h,x}=(1-h)\,g(x)+h\left(\frac{(x-a)\,g(a)+(b-x)\,g(b)}{b-a}\right). \tag{4.4}$$

and $P(\alpha,\beta,f,g)$ is as defined above in (4.2).

The following result holds:

Theorem 4.3 *Let the assumptions of Theorem 4.1 hold, then for any* $h\in[0,1]$ *and* $x\in[a,b]$, *we have:*

$$\begin{aligned}&\left|P\left(\Gamma_{h,x},\Delta_{h,x},f,g\right)\right|\\ &\leq \frac{1}{(b-a)^{2}} M_{h,x}^{\frac{2}{q}} \left\|f^{'}\right\|_{p}\left\|g^{'}\right\|_{p}\end{aligned} \tag{4.5}$$

where $\Gamma_{h,x}$ *and* $\Delta_{h,x}$ *are as defined by (4.4) and*

$$M_{h,x}=\frac{1}{q+1}\left[h^{q+1}+(1-h)^{q+1}\right]\left[(x-a)^{q+1}+(b-x)^{q+1}\right]. \tag{4.6}$$

Proof. We define the function

$$k(x,t;h)=\begin{cases} t-(1-h)a-hx, & t\in[a,x],\\ t-hx-(1-h)b, & t\in(x,b].\end{cases}$$

Then, we obtain the following identities:

$$\Gamma_{h,x}-\frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b k(x,t;h)\,f'(t)\,dt, \tag{4.7}$$

$$\Delta_{h,x}-\frac{1}{b-a}\int_a^b g(t)\,dt = \frac{1}{b-a}\int_a^b k(x,t;h)\,g'(t)\,dt. \tag{4.8}$$

Multiplying the left and right hand side of (4.7) and (4.8), we get,

$$P(\Gamma_{h,x},\Delta_{h,x},f,g)=\frac{1}{(b-a)^2}\left(\int_a^b k(x,t;h)\,f'(t)\,dt\right)\left(\int_a^b k(x,t;h)\,g'(t)\,dt\right),$$

implies

$$|P(\Gamma_{h,x},\Delta_{h,x},f,g)|\le\frac{1}{(b-a)^2}\left(\int_a^b |k(x,t;h)|\left|f'(t)\right|dt\right)\left(\int_a^b |k(x,t;h)|\left|g'(t)\right|dt\right). \tag{4.9}$$

Thus, by using the Hölder's integral inequality:

$$\begin{aligned}
&|P(\Gamma_{h,x},\Delta_{h,x},f,g)|\\
\le\ & \frac{1}{(b-a)^2}\left[\left(\int_a^b |k(x,t;h)|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^b \left|f'(t)\right|^p dt\right)^{\frac{1}{p}}\right]\\
&\times\left[\left(\int_a^b |k(x,t;h)|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^b \left|g'(t)\right|^p dt\right)^{\frac{1}{p}}\right]\\
=\ & \frac{1}{(b-a)^2}\left(\int_a^b |k(x,t;h)|^q\,dt\right)^{\frac{2}{q}}\left\|f'\right\|_p\left\|g'\right\|_p.
\end{aligned} \tag{4.10}$$

From the definition of $k(x,t;h)$, it follows that

$$\int_a^b |k(x,t;h)|^q\,dt=\frac{1}{(q+1)}\left[h^{q+1}+(1-h)^{q+1}\right]\left[(x-a)^{q+1}+(b-x)^{q+1}\right]. \tag{4.11}$$

By using (4.10)-(4.11), (4.5) follows. ■

Remark 4.1 *For* $x = \frac{a+b}{2}$, $h = \frac{1}{3}$ *in (4.5), (4.1) is recaptured.*

Remark 4.2 *For* $x = \frac{a+b}{2}$ *in (4.5), (4.3) is recaptured.*

We, now, state a special case of Theorem 4.3 in the form of the following corollary:

Corollary 4.1 *Let the assumptions of Theorem 4.1 hold, then*

$$\left|P\left(\Gamma_{1,\frac{a+b}{2}}, \Delta_{1,\frac{a+b}{2}}, f, g\right)\right| \leq \frac{1}{(b-a)^2} M_{1,\frac{a+b}{2}}^{\frac{2}{q}} \left\|f'\right\|_p \left\|g'\right\|_p \tag{4.12}$$

where

$$M_{1,\frac{a+b}{2}} = \frac{1}{2^q (q+1)} (b-a)^{q+1}, \tag{4.13}$$

and

$$\Gamma_{1,\frac{a+b}{2}} = \frac{f(a) + f(b)}{2},$$
$$\Delta_{1,\frac{a+b}{2}} = \frac{g(a) + g(b)}{2}. \tag{4.14}$$

We, now apply (4.12) to probability density functions as follows:

4.1.3 Applications for Probability Density Functions

Let X be a continuous random variable with the probability density function $f : [a,b] \to \mathbb{R}_+$ and the expectation of X is given by

$$E(X) = \int_a^b t\, f(t)\, dt. \tag{4.15}$$

The cumulative distribution function F is given as:

$$F(x) = \int_a^x f(t)\, dt, \tag{4.16}$$

for $x \in [a,b]$.

Moreover, let Y be another continuous variable with the probability density function $h : [a,b] \to \mathbb{R}_+$ and the expectation of Y is given by

$$E(Y) = \int_a^b t\, h(t)\, dt. \tag{4.17}$$

The cumulative distribution function H is given as:

$$H(y) = \int_a^y h(t)\,dt, \tag{4.18}$$

for $y \in [a,b]$. Then,

$$\begin{aligned} \int_a^b F(x)\,dx &= b - E(X), \\ F(a) &= 0, \; F(b) = 1, \\ \frac{F(a)+F(b)}{2} &= \frac{1}{2} \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} \int_a^b H(y)\,dy &= b - E(Y), \\ H(a) &= 0, \; H(b) = 1, \\ \frac{H(a)+H(b)}{2} &= \frac{1}{2}. \end{aligned} \tag{4.20}$$

Proposition 4.1 *Let X, Y, F and H be defined as above. Then, the following holds:*

$$\left| \frac{1}{4}\left(1 - \left(\frac{E(Y)-E(X)}{b-a}\right)\right) - \frac{1}{b-a}\left(b - \frac{E(X)+E(Y)}{2}\right)\left(1 - \frac{b - \frac{E(X)+E(Y)}{2}}{b-a}\right) \right|$$

$$\leq \frac{1}{4}\left(\frac{b-a}{q+1}\right)^{\frac{2}{q}} \|f\|_p \|h\|_p. \tag{4.21}$$

Proof. By choosing $f = F$ and $g = H$ in (4.12)-(4.14) and simplifying with the help of (4.15)-(4.20), we get the required inequality. ■

Remark 4.3 *If in (4.21), we choose $F = H$, then we have:*

$$\begin{aligned} &\left| \frac{1}{2} - \frac{1}{b-a}(b - E(X)) \right| \\ \leq\;& \frac{1}{2}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \|h\|_p, \end{aligned} \tag{4.22}$$

which is known in literature as "trapezoid inequality" for cumulative distribution functions (see [36], p. 34 for $f = H$).

4.2 Some new Čebyšev type inequalities

In this section, some new Čebyšev type inequalities have been developed by working with functions whose first derivatives are absolutely continuous and the second derivatives belong to the usual Lebesgue space $L_\infty(a,b)$. A unified treatment of the special cases is also given.

4.2.1 Introduction

In [77], B. G. Pachpatte presented the following Čebyšev type inequality via trapezoid like rules:

Theorem 4.4 *Let $f,g:[a,b]\to\mathbb{R}$ be differentiable functions so that f', g' are absolutely continuous on $[a,b]$, then*

$$\left|P\left(\bar{F},\bar{G},f,g\right)\right|\le\frac{(b-a)^4}{144}\left\|f''-\left[f';a,b\right]\right\|_\infty\left\|g''-\left[g';a,b\right]\right\|_\infty, \tag{4.23}$$

where

$$\begin{aligned}\bar{F}&=\frac{f(a)+f(b)}{2}-\frac{(b-a)^2}{12}\left[f';a,b\right],\\ \bar{G}&=\frac{g(a)+g(b)}{2}-\frac{(b-a)^2}{12}\left[g';a,b\right],\end{aligned}$$

and

$$\begin{aligned}P(\alpha,\beta,f,g)&=\alpha\beta-\frac{1}{b-a}\left(\alpha\int_a^b g(t)\,dt+\beta\int_a^b f(t)\,dt\right)\\ &\quad+\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right),\\ \left[f';a,b\right]&=\frac{f(b)-f(a)}{b-a}.\end{aligned}$$

Recently, in [56] Zheng Liu has presented the following generalization of (4.23):

Theorem 4.5 *Let the assumptions of Theorem 4.4 hold, then for any $\theta\in[0,1]$,*

$$\left|P\left(\bar{\Gamma}_\theta,\bar{\Delta}_\theta,f,g\right)\right|\le(b-a)^4I^2(\theta)\left\|f''-\left[f';a,b\right]\right\|_\infty\left\|g''-\left[g';a,b\right]\right\|_\infty, \tag{4.24}$$

where

$$I(\theta)=\begin{cases}\frac{\theta^3}{3}-\frac{\theta}{8}+\frac{1}{24}, & 0\le\theta\le\frac{1}{2},\\ \frac{1}{8}\left(\theta-\frac{1}{3}\right), & \frac{1}{2}\le\theta\le1,\end{cases}$$

and

$$\begin{aligned}
\Gamma_\theta &= \frac{\theta}{2}\left[f(a)+f(b)\right]+(1-\theta)f\left(\frac{a+b}{2}\right), \\
\Delta_\theta &= \frac{\theta}{2}\left[g(a)+g(b)\right]+(1-\theta)g\left(\frac{a+b}{2}\right) \\
\bar{\Gamma}_\theta &= \Gamma_\theta+\frac{1}{24}(1-3\theta)(b-a)^2\left[f';a,b\right], \\
\bar{\Delta}_\theta &= \Delta_\theta+\frac{1}{24}(1-3\theta)(b-a)^2\left[g';a,b\right].
\end{aligned}$$

In the following subsection, by following an approach similar to that of [56] and [77], we present some new Čebyšev type inequalities.

4.2.2 Main Results

For suitable functions $f,g:[a,b]\to\mathbb{R}$ and $h\in[0,1]$, we present the following notations:

$$\bar{T}_{h,x}=\frac{1}{2}(2-h)f(x)-(1-h)\left(x-\frac{a+b}{2}\right)f'(x)$$
$$+\frac{h}{2}\left(\frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right),$$

$$\bar{S}_{h,x}=\frac{1}{2}(2-h)g(x)-(1-h)\left(x-\frac{a+b}{2}\right)g'(x)$$
$$+\frac{h}{2}\left(\frac{(x-a)g(a)+(b-x)g(b)}{b-a}\right),$$

$$\bar{H}_{h,x}=(1-h)f(x)+h\left(\frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right)$$
$$-(1-h)^2\left(x-\frac{a+b}{2}\right)f'(x)-\frac{h^2}{2}\left(\frac{(b-x)^2f'(b)-(x-a)^2f'(a)}{b-a}\right),$$

$$\bar{L}_{h,x}=(1-h)g(x)+h\left(\frac{(x-a)g(a)+(b-x)g(b)}{b-a}\right)$$
$$-(1-h)^2\left(x-\frac{a+b}{2}\right)g'(x)-\frac{h^2}{2}\left(\frac{(b-x)^2g'(b)-(x-a)^2g'(a)}{b-a}\right),$$

$$T_{h,x}=\bar{T}_{h,x}+\frac{1}{4}(2-3h)\left[f';a,b\right](b-a)^2\Delta(x),$$

$$S_{h,x}=\bar{S}_{h,x}+\frac{1}{4}(2-3h)\left[g';a,b\right](b-a)^2\Delta(x),$$

$$H_{h,x}=\bar{H}_{h,x}+\frac{1}{2}\left(3h^2-3h+1\right)\left[f';a,b\right](b-a)^2\Delta(x)$$

and

$$L_{h,x}=\bar{L}_{h,x}+\frac{1}{2}\left(3h^2-3h+1\right)\left[g^{'};a,b\right](b-a)^2\Delta(x),$$

where

$$\Delta(x)=\frac{1}{12}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2 \tag{4.25}$$

and $\left[f^{'};a,b\right]$ is defined as above.

Theorem 4.6 *Let the assumptions of Theorem 4.4 hold, then for any* $h\in[0,1]$,

$$\begin{aligned}&|P(T_{h,x},S_{h,x},f,g)|\\ &\le\frac{1}{16}\omega^2(h)(b-a)^4\Delta^2(x)\left\|f^{''}-\left[f^{'};a,b\right]\right\|_{\infty}\left\|g^{''}-\left[g^{'};a,b\right]\right\|_{\infty},\end{aligned} \tag{4.26}$$

where $\Delta(x)$, $T_{h,x}$ *and* $S_{h,x}$ *are defined as above and*

$$\omega(h)=2h^3-3h+2. \tag{4.27}$$

Proof. We define the kernel

$$K^{'}(x,t;h)=\begin{cases}\frac{1}{2}(t-a)(t-(1-h)a-hx),\ t\in[a,x],\\ \frac{1}{2}(t-b)(t-hx-(1-h)b),\ t\in(x,b].\end{cases}$$

Through simple calculations it can be shown that

$$\frac{1}{b-a}\int_a^b f(t)\,dt-T_{h,x} = I\left(f^{'},f^{''};a,b\right), \tag{4.28}$$

$$\frac{1}{b-a}\int_a^b g(t)\,dt-S_{h,x} = I\left(g^{'},g^{''};a,b\right), \tag{4.29}$$

where

$$I\left(f^{'},f^{''};a,b\right)=\frac{1}{b-a}\int_a^b K^{'}(x,t;h)\left\{f^{''}(t)-\left[f^{'};a,b\right]\right\}dt.$$

Multiplying the left and right hand side of (4.28) and (4.29), we get:

$$P(T_{h,x},S_{h,x},f,g)=I\left(f^{'},f^{''};a,b\right)I\left(g^{'},g^{''};a,b\right),$$

implies

$$|P(T_{h,x},S_{h,x},f,g)|=\left|I\left(f^{'},f^{''};a,b\right)\right|\left|I\left(g^{'},g^{''};a,b\right)\right|. \tag{4.30}$$

Following an approach similar to [77], we calculate

$$\begin{aligned}\left|I\left(f^{'},f^{''};a,b\right)\right| &\leq \frac{1}{b-a}\int_a^b \left|K^{'}(x,t;h)\right|\left|f^{''}(t)-\left[f^{'};a,b\right]\right|dt \\ &\leq \frac{1}{b-a}\left\|f^{''}(t)-\left[f^{'};a,b\right]\right\|_\infty \int_a^b \left|K^{'}(x,t;h)\right|dt. \quad (4.31)\end{aligned}$$

In a similar manner,

$$\left|I\left(g^{'},g^{''};a,b\right)\right| \leq \frac{1}{b-a}\left\|g^{''}(t)-\left[g^{'};a,b\right]\right\|_\infty \int_a^b \left|K^{'}(x,t;h)\right|dt. \quad (4.32)$$

From the definition of $K^{'}(x,t;h)$, it follows that

$$\frac{1}{b-a}\int_a^b \left|K^{'}(x,t;h)\right|dt = \frac{1}{4}\omega(h)(b-a)^2\Delta(x), \quad (4.33)$$

where $\Delta(x)$ and $\omega(h)$ are defined by (4.25) and (4.27).

By using (4.30)-(4.31) and (4.26) follows. ■

The following corollary of Theorem 4.6 holds:

Corollary 4.2 *Let the assumptions of Theorem 4.4 hold, then for any* $h \in [0,1]$,

$$\begin{aligned}&\left|P\left(T_{h,\frac{a+b}{2}},S_{h,\frac{a+b}{2}},f,g\right)\right| \\ &\leq \frac{1}{2304}\omega^2(h)(b-a)^4\left\|f^{''}-\left[f^{'};a,b\right]\right\|_\infty\left\|g^{''}-\left[g^{'};a,b\right]\right\|_\infty, \quad (4.34)\end{aligned}$$

where

$$\begin{aligned}T_{h,\frac{a+b}{2}} &= \frac{1}{2}(2-h)f\left(\frac{a+b}{2}\right)+\frac{h}{4}(f(a)+f(b)) \\ &+\frac{1}{48}(2-3h)\left[f^{'};a,b\right](b-a)^2\end{aligned}$$

and

$$\begin{aligned}S_{h,\frac{a+b}{2}} &= \frac{1}{2}(2-h)g\left(\frac{a+b}{2}\right)+\frac{h}{4}(g(a)+g(b)) \\ &+\frac{1}{48}(2-3h)\left[g^{'};a,b\right](b-a)^2,\end{aligned}$$

$\omega(h)$ *is defined by (4.27).*

Remark 4.4 *It may observed that for $x=\frac{a+b}{2}$, the kernel defined in Theorem 4.6 takes the following form:*

$$K^{'}\left(\frac{a+b}{2},t;h\right)=\begin{cases}\frac{1}{2}(t-a)\left(t-\left(a+h\frac{b-a}{2}\right)\right), & t\in\left[a,\frac{a+b}{2}\right],\\ \frac{1}{2}(t-b)\left(t-\left(b-h\frac{b-a}{2}\right)\right), & t\in(\frac{a+b}{2},b].\end{cases}$$

We will now consider the following special cases of the above corollary:

The following special cases of Corollary 4.2 hold:

Remark 4.5 *(i) For $h=0$, (4.34) takes the form:*

$$\left|P\left(T_{o,\frac{a+b}{2}},S_{o,\frac{a+b}{2}},f,g\right)\right|$$
$$\leq\frac{1}{576}(b-a)^4\left\|f^{''}-\left[f^{'};a,b\right]\right\|_{\infty}\left\|g^{''}-\left[g^{'};a,b\right]\right\|_{\infty},\qquad(4.35)$$

with

$$T_{o,\frac{a+b}{2}}=f\left(\frac{a+b}{2}\right)+\frac{1}{24}\left[f^{'};a,b\right](b-a)^2$$

and

$$S_{o,\frac{a+b}{2}}=g\left(\frac{a+b}{2}\right)+\frac{1}{24}\left[g^{'};a,b\right](b-a)^2.$$

(ii) For $h=1$, (4.34) takes the form:

$$\left|P\left(T_{1,\frac{a+b}{2}},S_{1,\frac{a+b}{2}},f,g\right)\right|$$
$$\leq\frac{1}{2304}(b-a)^4\left\|f^{''}-\left[f^{'};a,b\right]\right\|_{\infty}\left\|g^{''}-\left[g^{'};a,b\right]\right\|_{\infty},\qquad(4.36)$$

where

$$T_{1,\frac{a+b}{2}}=\frac{1}{4}\left(f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{1}{48}\left[f^{'};a,b\right](b-a)^2$$

and

$$S_{1,\frac{a+b}{2}}=\frac{1}{4}\left(g(a)+2g\left(\frac{a+b}{2}\right)+g(b)\right)-\frac{1}{48}\left[g^{'};a,b\right](b-a)^2.$$

(iii) For $h=\frac{2}{3}$, (4.34) takes the form:

$$\left|P\left(T_{\frac{2}{3},\frac{a+b}{2}},S_{\frac{2}{3},\frac{a+b}{2}},f,g\right)\right|$$
$$\leq\frac{1}{6561}(b-a)^4\left\|f^{''}-\left[f^{'};a,b\right]\right\|_{\infty}\left\|g^{''}-\left[g^{'};a,b\right]\right\|_{\infty},\qquad(4.37)$$

where

$$T_{\frac{2}{3},\frac{a+b}{2}} = \frac{1}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

and

$$S_{\frac{2}{3},\frac{a+b}{2}} = \frac{1}{6}\left(g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)\right).$$

It may also be noted that $\omega(h)$ is minimum for $h = \frac{1}{\sqrt{2}}$.

Theorem 4.7 *Let the assumptions of Theorem 4.4 hold, then for any $h \in [0,1]$,*

$$|P(H_{h,x}, L_{h,x}, f, g)|$$

$$\leq \frac{1}{4}\eta^2(h)(b-a)^4 \Delta^2(x) \left\| f^{''} - \left[f^{'}; a, b\right] \right\|_\infty \left\| g^{''} - \left[g^{'}; a, b\right] \right\|_\infty, \tag{4.38}$$

where $\Delta(x)$, $H_{h,x}$ and $L_{h,x}$ are as defined above and

$$\eta(h) = 3h^2 - 3h + 1. \tag{4.39}$$

Proof. We define the kernel

$$K_1(x,t;h) = \begin{cases} \frac{1}{2}(t-(1-h)a-hx)^2, & t \in [a,x], \\ \frac{1}{2}(t-hx-(1-h)b)^2, & t \in (x,b]. \end{cases}$$

Through simple calculations, it can be shown that

$$\frac{1}{b-a}\int_a^b f(t)\,dt - H_{h,x} = J\left(f^{'}, f^{''}; a, b\right), \tag{4.40}$$

$$\frac{1}{b-a}\int_a^b g(t)\,dt - L_{h,x} = J\left(g^{'}, g^{''}; a, b\right), \tag{4.41}$$

where

$$J\left(f^{'}, f^{''}; a, b\right) = \frac{1}{b-a}\int_a^b K_1(x,t;h)\left\{f^{''}(t) - \left[f^{'}; a, b\right]\right\} dt.$$

Multiplying the left and right hand side of (4.40) and (4.41), we get:

$$P(H_{h,x}, L_{h,x}, f, g) = J\left(f^{'}, f^{''}; a, b\right) J\left(g^{'}, g^{''}; a, b\right).$$

This implies

$$|P(H_{h,x}, L_{h,x}, f, g)| = \left|J\left(f^{'}, f^{''}; a, b\right)\right| \left|J\left(g^{'}, g^{''}; a, b\right)\right|. \tag{4.42}$$

By following an approach similar to [77], we calculate

$$\begin{aligned}\left|J\left(f',f'';a,b\right)\right| &\le \frac{1}{b-a}\int_a^b |K_1(x,t;h)|\left|f''(t)-\left[f';a,b\right]\right|dt \\ &\le \frac{1}{b-a}\left\|f''(t)-\left[f';a,b\right]\right\|_\infty \int_a^b |K_1(x,t;h)|\,dt. \quad (4.43)\end{aligned}$$

In a similar manner,

$$\left|J\left(g',g'';a,b\right)\right| \le \frac{1}{b-a}\left\|g''(t)-\left[g';a,b\right]\right\|_\infty \int_a^b |K_1(x,t;h)|\,dt. \quad (4.44)$$

From the definition of $K_1(x,t;h)$, it follows that

$$\frac{1}{b-a}\int_a^b |K_1(x,t;h)|\,dt = \frac{1}{2}\eta(h)(b-a)^2\Delta(x), \quad (4.45)$$

where $\Delta(x)$ and $\eta(h)$ are defined by (4.25) and (4.39).

Therefore (4.38) follows directly from (4.42)-(4.45). ■

The following corollary of Theorem 4.7 holds:

Corollary 4.3 *Let the assumptions of Theorem 4.5 hold, then for any* $h \in [0,1]$,

$$\begin{aligned}&\left|P\left(H_{h,\frac{a+b}{2}},L_{h,\frac{a+b}{2}},f,g\right)\right| \\ &\le \frac{1}{576}\eta^2(h)(b-a)^4\left\|f''-\left[f';a,b\right]\right\|_\infty\left\|g''-\left[g';a,b\right]\right\|_\infty, \quad (4.46)\end{aligned}$$

where

$$H_{h,\frac{a+b}{2}} = (1-h)f\left(\frac{a+b}{2}\right)+\frac{h}{2}(f(a)+f(b))+\frac{1}{24}(1-3h)(b-a)^2\left[f';a,b\right],$$

and

$$L_{h,\frac{a+b}{2}} = (1-h)g\left(\frac{a+b}{2}\right)+\frac{h}{2}(g(a)+g(b))+\frac{1}{24}(1-3h)(b-a)^2\left[g';a,b\right],$$

$\eta(h)$ *is defined by (4.39).*

Remark 4.6 *It may observed that for* $x=\frac{a+b}{2}$, *the kernel defined in Theorem 4.7 takes the following form:*

$$K_1\left(\frac{a+b}{2},t;h\right) = \begin{cases} \frac{1}{2}\left(t-\left(a+h\frac{b-a}{2}\right)\right)^2, & t\in\left[a,\frac{a+b}{2}\right], \\ \frac{1}{2}\left(t-\left(b-h\frac{b-a}{2}\right)\right)^2, & t\in\left(\frac{a+b}{2},b\right]. \end{cases}$$

The following special cases of Corollary 4.3 hold:

Remark 4.7 *(i) For $h = 0$, (4.46) takes the form:*

$$\left|P\left(H_{o,\frac{a+b}{2}}, L_{o,\frac{a+b}{2}}, f, g\right)\right|$$
$$\leq \frac{1}{576}(b-a)^4 \left\|f^{''} - \left[f^{'}; a, b\right]\right\|_{\infty} \left\|g^{''} - \left[g^{'}; a, b\right]\right\|_{\infty}, \tag{4.47}$$

with

$$H_{o,\frac{a+b}{2}} = f\left(\frac{a+b}{2}\right) + \frac{1}{24}\left[f^{'}; a, b\right](b-a)^2$$

and

$$L_{o,\frac{a+b}{2}} = g\left(\frac{a+b}{2}\right) + \frac{1}{24}\left[g^{'}; a, b\right](b-a)^2.$$

(ii) For $h = 1$, (4.46) takes the form:

$$\left|P\left(H_{1,\frac{a+b}{2}}, L_{1,\frac{a+b}{2}}, f, g\right)\right|$$
$$\leq \frac{1}{576}(b-a)^4 \left\|f^{''} - \left[f^{'}; a, b\right]\right\|_{\infty} \left\|g^{''} - \left[g^{'}; a, b\right]\right\|_{\infty}, \tag{4.48}$$

where

$$H_{1,\frac{a+b}{2}} = \frac{1}{2}(f(a) + f(b)) - \frac{1}{12}\left[f^{'}; a, b\right](b-a)^2$$

and

$$L_{1,\frac{a+b}{2}} = \frac{1}{2}(g(a) + g(b)) - \frac{1}{12}\left[g^{'}; a, b\right](b-a)^2.$$

(iii) For $h = \frac{1}{2}$, (4.46) takes the form,

$$\left|P\left(H_{\frac{1}{2},\frac{a+b}{2}}, L_{\frac{1}{2},\frac{a+b}{2}}, f, g\right)\right|$$
$$\leq \frac{1}{9216}(b-a)^4 \left\|f^{''} - \left[f^{'}; a, b\right]\right\|_{\infty} \left\|g^{''} - \left[g^{'}; a, b\right]\right\|_{\infty}, \tag{4.49}$$

where

$$H_{\frac{1}{2},\frac{a+b}{2}} = \frac{1}{4}\left(f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right) - \frac{1}{48}\left[f^{'}; a, b\right](b-a)^2$$

and

$$L_{\frac{1}{2},\frac{a+b}{2}} = \frac{1}{4}\left(g(a) + 2g\left(\frac{a+b}{2}\right) + g(b)\right) - \frac{1}{48}\left[g^{'}; a, b\right](b-a)^2.$$

It may also be noted that $\eta(h)$ is minimum for $h = \frac{1}{2}$.

4.3 An integral inequality involving product of monotonic functions and applications

In this section, an integral inequality involving the product of two monotonic non-decreasing functions have been developed. Moreover, using the underlying assumptions, we have also presented some special cases.

4.3.1 Introduction

Ostrowski type inequalities have been developed for different types of functions, namely absolutely continuous function, function of bounded variation and monotonic function, etc. In [29], S. S. Dragomir established the following Ostrowski's inequality for monotonic mappings:

Theorem 4.8 *Let $f : [a, b] \to \mathbb{R}$ be a monotonic non-decreasing mapping on $[a, b]$. Then for all $x \in [a, b]$, we have the following inequality:*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\
\leq\ & \frac{1}{b-a} \left\{ [2x - (a+b)] f(x) + \int_a^b sgn(t-x) f(t)\, dt \right\}, \\
\leq\ & \frac{1}{b-a} \left[(x-a)(f(x) - f(a)) + (b-x)(f(b) - f(x)) \right], \\
\leq\ & \left[\frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|}{b-a} \right] (f(b) - f(a)).
\end{aligned}$$

The constant $\frac{1}{2}$ is the best possible one.

In [37], S. S. Dragomir et al. generalized the above theorem as follows:

Theorem 4.9 *Let $f : [a, b] \to \mathbb{R}$ be a monotonic non-decreasing mapping on $[a, b]$*

and $t_1, t_2, t_3 \in [a,b]$ *be such that* $t_1 \le t_2 \le t_3$. *Then*

$$\begin{aligned}
&\left| \int_a^b f(x)\,dx - [(t_1 - a) f(a) + (t_3 - t_1) f(t_2) + (b - t_3) f(b)] \right| \\
\le\ & (b - t_3) f(b) + (2t_2 - t_1 - t_3) f(t_2) - (t_1 - a) f(a) + \int_a^b T(x) f(x)\,dx \\
\le\ & (b - t_3)(f(b) - f(t_3)) + (t_3 - t_2)(f(t_3) - f(t_2)) \\
& + (t_2 - t_1)(f(t_2) - f(t_1)) + (t_1 - a)(f(t_1) - f(a)) \\
\le\ & \max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)),
\end{aligned}$$

where

$$T(x) = \begin{cases} sgn(t_1 - x) & \text{for } x \in [a, t_2], \\ sgn(t_3 - x) & \text{for } x \in [t_2, b]. \end{cases}$$

The following known lemmas is useful in the sequel.

Lemma 4.1 *(see [15]) Let* $p,\ v \in [a,b] \to \mathbb{R}$ *be such that* p *is Riemann integrable on* $[a,b]$ *and* v *is monotonic non-decreasing on* $[a,b]$. *Then*

$$\left| \int_a^b p(t)\,dv(t) \right| \le \int_a^b |p(t)|\,dv(t).$$

In the following subsection, we present an integral inequality involving product of two monotonic non-decreasing functions, thus providing a new estimation for these type of inequalities in terms of the functional values of monotonic mappings. The analysis is based on the inequality presented in [37].

4.3.2 Main Results

For the monotonic non-decreasing functions $f, g : [a,b] \to \mathbb{R}$, the following notations are presented:

$$\begin{aligned}
\overline{F} &= \frac{1}{b-a}[(t_1 - a) f(a) + (t_3 - t_1) f(t_2) + (b - t_3) f(b)], \\
\overline{G} &= \frac{1}{b-a}[(t_1 - a) g(a) + (t_3 - t_1) g(t_2) + (b - t_3) g(b)]. \qquad (4.50)
\end{aligned}$$

We define the functional

$$\overline{S}(f,g) = \bar{F}\bar{G} - \frac{1}{b-a}\left[\bar{F}\int_a^b g(t)\,dt + \bar{G}\int_a^b f(t)\,dt\right] + \left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right), \tag{4.51}$$

Next, we consider the functional

$$\overline{H}(f,g) = f(x)g(x) - \frac{1}{b-a}\left[f(x)\int_a^b g(t)\,dt + g(x)\int_a^b f(t)\,dt\right] + \left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right). \tag{4.52}$$

We shall start with the following:

Theorem 4.10 *Let $f,g : [a,b] \to \mathbb{R}$ be a monotonic non-decreasing functions on $[a,b]$ and $t_1,t_2,t_3 \in [a,b]$ be such that $t_1 \le t_2 \le t_3$. Then*

$$\left|\bar{S}(f,g)\right| \le \frac{1}{(b-a)^2}\left(\max\{t_1-a, t_2-t_1, t_3-t_2, b-t_3\}\right)^2 \times (f(b)-f(a))(g(b)-g(a)). \tag{4.53}$$

Proof. Using the identities developed in [37] by S. S. Dragomir et al.,

$$\bar{F} - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b s(t)\,df(t), \tag{4.54}$$

$$\bar{G} - \frac{1}{b-a}\int_a^b g(t)\,dt = \frac{1}{b-a}\int_a^b s(t)\,dg(t), \tag{4.55}$$

where

$$s(t) = \begin{cases} t-t_1 & \text{for } t \in [a,t_2], \\ t-t_3 & \text{for } t \in (t_2,b]. \end{cases}$$

Multiplying (4.54) and (4.55), we have

$$\bar{S}(f,g) = \frac{1}{(b-a)^2}\left(\int_a^b s(t)\,df(t)\right)\left(\int_a^b s(t)\,dg(t)\right). \tag{4.56}$$

By using the properties of modulus and applying Lemma 4.1 for $p(t) = s(t)$ and $v(t)$ for $f(t)$ and $g(t)$ respectively, we get

$$\left|\bar{S}(f,g)\right| \le \frac{1}{(b-a)^2}\left(\int_a^b |s(t)|\,df(t)\right)\left(\int_a^b |s(t)|\,dg(t)\right). \tag{4.57}$$

As in [37],

$$
\begin{aligned}
& \int_a^b |s(t)|\, df(t) \\
= \; & -(t_1 - a) f(a) + (t_2 - t_1) f(t_2) - (t_3 - t_2) f(t_2) \\
& + (b - t_3) f(b) + \int_a^b T(t) f(t)\, dt,
\end{aligned}
$$

where

$$
T(t) = \begin{cases} sgn(t_1 - t) & for\ t \in [a, t_2], \\ sgn(t_3 - t) & for\ t \in (t_2, b]. \end{cases}
$$

Since $f : [a, b] \to \mathbb{R}$ is monotonic non-decreasing in $[a, b]$, following the same direction as explained in [37], we have

$$
\int_a^b |s(t)|\, df(t) \le \max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)).
$$

In a similar manner,

$$
\int_a^b |s(t)|\, dg(t) \le \max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(g(b) - g(a)).
$$

Consequently (4.57) takes the form,

$$
\begin{aligned}
\left|\bar{S}(f, g)\right| \le \; & \frac{1}{(b-a)^2} (\max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\})^2 \times \\
& (f(b) - f(a))(g(b) - g(a)),
\end{aligned}
$$

which is the required inequality. ■

The following corollaries hold:

Corollary 4.4 *Let f, g be defined as above in Theorem 4.10. Then, for all $x \in [a, b]$, we have the following:*

$$
\begin{aligned}
& \left|\overline{H}(f, g)\right| \\
\le \; & \frac{1}{(b-a)^2} \left[\frac{1}{2}(b - a) + \left|x - \frac{a+b}{2}\right|\right]^2 \times \\
& (f(b) - f(a))(g(b) - g(a)),
\end{aligned}
$$

where $\overline{H}(f, g)$ is defined by (4.52).

Proof. Set in Theorem 4.10, $t_1 = a$, $t_2 = x$, $t_3 = b$. ∎

Corollary 4.5 *Let f, g be defined as above in Theorem 4.10. Then, for all $x \in [a, b]$, we have the following:*

$$\begin{aligned}
&\left| F^{'}G^{'} - \frac{1}{b-a}\left[F^{'}\int_a^b g(t)\,dt + G^{'}\int_a^b f(t)\,dt\right] \right. \\
&\left. + \left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)\right| \\
\leq\ & \frac{1}{(b-a)^2}\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^2 \times (f(b) - f(a))(g(b) - g(a)),
\end{aligned}$$

where

$$\begin{aligned}
F^{'} &= \frac{1}{b-a}\left[(x-a) f(a) + (b-x) f(b)\right], \\
G^{'} &= \frac{1}{b-a}\left[(x-a) g(a) + (b-x) g(b)\right].
\end{aligned}$$

Proof. Set in Theorem 4.10, $t_1 = t_2 = t_3 = x$. ∎

Corollary 4.6 *Let f, g be defined as above in Theorem 4.10. Then, for all $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$ and $h \in [0, 1]$, we have the following:*

$$\begin{aligned}
&\left| F_1 G_1 - \frac{1}{b-a}\left[F_1\int_a^b g(t)\,dt + G_1\int_a^b f(t)\,dt\right] \right. \\
&\left. + \left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)\right| \\
\leq\ & \frac{1}{(b-a)^2}\left(\max\left\{h\frac{b-a}{2}, \frac{1}{2}(1-h)(b-a) + \left|x - \frac{a+b}{2}\right|\right\}\right)^2 \times \\
& (f(b) - f(a))(g(b) - g(a)), \qquad (4.58)
\end{aligned}$$

where

$$\begin{aligned}
F_1 &= \frac{h}{2}(f(a) + f(b)) + (1-h) f(x), \\
G_1 &= \frac{h}{2}(g(a) + g(b)) + (1-h) g(x). \qquad (4.59)
\end{aligned}$$

Proof. Choose in Theorem 4.10, $t_1 = a + h\frac{b-a}{2}$, $t_2 = x$, $t_3 = b - h\frac{b-a}{2}$. ∎

Remark 4.8 *Consider the following result as a special cases of Corollary 4.6.*

(i) For $h = 1$, we obtain:

$$\begin{aligned} F_2 &= \frac{f(a) + f(b)}{2}, \\ G_2 &= \frac{g(a) + g(b)}{2}, \end{aligned}$$

$$\begin{aligned} \tilde{S}(f,g) &= F_2 G_2 - \frac{1}{b-a}\left[F_2 \int_a^b g(t)\,dt + G_2 \int_a^b f(t)\,dt\right] \\ &+ \left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right). \end{aligned}$$

Therefore, (4.58), becomes

$$\left|\tilde{S}(f,g)\right| \leq \frac{1}{4}(f(b) - f(a))(g(b) - g(a)). \tag{4.60}$$

(ii) For $h = \frac{1}{2}$ and $x = \frac{a+b}{2}$, $s(t)$ is defined as

$$s(t) = \begin{cases} t - \frac{3a+b}{4} \ if \ t \in [a, \frac{a+b}{2}], \\ t - \frac{a+3b}{4} \ if \ t \in (\frac{a+b}{2}, b]. \end{cases}$$

Moreover, for the monotonic non-decreasing functions $f, g : [a,b] \to \mathbb{R}$, with

$$\begin{aligned} F_3 &= \frac{1}{2}\left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right)\right], \\ G_3 &= \frac{1}{2}\left[\frac{g(a) + g(b)}{2} + g\left(\frac{a+b}{2}\right)\right], \end{aligned}$$

the functional defined by (4.51) takes the form

$$\begin{aligned} \tilde{S}(f,g) &= F_3 G_3 - \frac{1}{b-a}\left[F_3 \int_a^b g(t)\,dt + G_3 \int_a^b f(t)\,dt\right] \\ &+ \left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right). \end{aligned}$$

Therefore, (4.58), becomes

$$\left|\tilde{S}(f,g)\right| \leq \frac{1}{16}(f(b) - f(a))(g(b) - g(a)). \tag{4.61}$$

Similarly, we can obtain some more product inequalities as special cases of (4.58) for different values of h.

Remark 4.9 *It is important to mention that the estimates of product inequalities for monotonic non-decreasing functions are merely addressed in the literature of integral inequalities except a lower bound of the Čebyšev functional for monotonic non-decreasing functions was obtained by P. Cerone and S. S. Dragomir in [18]. It may also be noted that some variants of Čebyšev and Grüss type inequalities for monotonic non-decreasing functions can also be obtained from (4.54) and (4.55), by using the approach of ([78], Remark 4.1).*

We, now apply (4.60) to probability density functions as follows:

4.3.3 Applications for probability density functions

Let X, Y, G and H be as in Section 4.1.3. Then the following proposition holds:

Proposition 4.2 *Let X, Y, F and H be defined as above. Then, the following holds:*

$$\left|\frac{1}{4}\left(1-\left(\frac{E(Y)-E(X)}{b-a}\right)\right)-\frac{1}{b-a}\left(b-\frac{E(X)+E(Y)}{2}\right)\left(1-\frac{b-\frac{E(X)+E(Y)}{2}}{b-a}\right)\right| \leq \frac{1}{4}. \tag{4.62}$$

Proof. By choosing $f=F$ and $g=H$ in (4.60) and simplifying with the help of (4.15)-(4.20), we get the required inequality. ■

4.4 On Čebyšev-Grüss type inequalities for spherical shells and balls in $L_p[a,b]$, $p>1$

4.4.1 Introduction

Recently, in [3], Anastassiou has presented Čebyšev-Grüss type inequalities on $\mathbb{R}^N$ over spherical shells and balls based on the results of B. G. Pachpatte [79]. The main motivation of this work is to give Čebyšev-Grüss inequalities of Pachpatte type for $L_p[a,b]$, $p>1$ and then to extend these results on n-dimensional Euclidean space over spherical shells and balls by using the tools of [3].

Now we would restate the geometries defined in [3].

Definition 4.1 *(n-Ball) Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Then, a hyperball or n-ball of radius $R > 0$, centered at p denoted by B^n, is defined as:*

$$B^n(p, R) = \{x \in \mathbb{R}^n : \|x - p\| < R\},$$

where $\|.\|$ is the Euclidean norm.

Definition 4.2 *($n-1$-Sphere or Hypersphere) In mathematics, an $n-1$-Sphere $\mathbb{S}^{n-1}$ is a generalization of an ordinary sphere to arbitrary dimension. For any natural number n, an $n-1$-sphere of radius R is defined as the set of points in n-dimensional Euclidean space which are at distance R from a central point, where the radius R may be any positive real number. The $n-1$-sphere of unit radius centered at the origin is denoted by S^{n-1} and is defined as:*

$$S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\},$$

where $\|.\|$ is the Euclidean norm.

Definition 4.3 *(Spherical Shell) A spherical shell is a generalization of an annulus to three dimensions. A spherical shell is therefore the region between two concentric balls of differing radii. Let $0 < R_1 < R_2$, then a spherical shell $A \subseteq \mathbb{R}^n$, $n \geq 1$ is defined as:*

$$A = B^n(0, R_2) - \overline{B^n(0, R_1)}.$$

Definition 4.4 *(Radial functions) Let $F, G \in X\left(\overline{A}\right)$. Then, F and G are radial if $F(x) = f(r)$ and $G(x) = g(r)$ for $f, g \in X([R_1, R_2])$ and $\|x\| = r$, for $R_1 \leq r \leq R_2$.*

Definition 4.5 *(Hyperspherical Volume) The hyperdimensional volume of the space which a $(n-1)$-sphere encloses (the n-ball with radius R) is defined as:*

$$\begin{aligned} V_n &= \int_{\sum\limits_{i=0}^{n} x_i^2 \leq R} dx = \int_{S^{n-1}} d\Omega \int_0^R r^{n-1} dr \\ &= \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \frac{R^n}{n} = \frac{\pi^{\frac{n}{2}} R^n}{\Gamma\left(\frac{n}{2}+1\right)}, \end{aligned}$$

where Γ is the gamma function.

Definition 4.6 *(Surface area of unit $n-1-$Sphere) The surface area of unit $n-1$-sphere S^{n-1} is given by:*

$$\omega_n = \int_{S^{n-1}} d\omega = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

4.4.2 Čebyšev-Grüss type inequality for $L_p - spaces,\ p > 1$

Let $w : [a, b] \to [0, \infty)$ be a probability density function, that is, an integrable function satisfying $\int_a^b w(t)\,dt = 1$ and W be the corresponding cumulative distribution function. Then, $W(t) = \int_a^t w(x)\,dx$, for $t \in [a, b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$. Then, the Pecaric's weighted extension of Montgomery identity [82] is given as:

$$f(x) = \int_a^b w(t) f(t)\,dt + \int_a^b P_w(x,t) f'(t)\,dt,$$

where

$$P_w(x,t) = \begin{cases} W(t), & a \le t \le x \\ W(t) - 1, & x < t \le b. \end{cases}$$

The following results hold by using the weighted Montgomery identity and Hölder's integral inequality:

Theorem 4.11 *Let $f, g : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ and $f', g' : [a, b] \to \mathbb{R}$ be such that $f', g' \in L_p(a, b)$. Let $w : [a, b] \to [0, \infty)$ be an integrable function with $\int_a^b w(t)\,dt = 1$. Then*

$$|T(w,f,g)| \le \left\|f'\right\|_p \left\|g'\right\|_p \int_a^b w(x) \left(\int_a^b |P_w(x,t)|^q\,dt \right)^{\frac{2}{q}} dx, \tag{4.63}$$

for $x \in [a, b]$, where

$$T(w,f,g) = \int_a^b w(x) f(x) g(x)\,dx - \left(\int_a^b w(x) f(x)\,dx \right) \left(\int_a^b w(x) g(x)\,dx \right)$$

and $P_w(x,t)$ and $W(t)$ are defined as above.

Theorem 4.12 *Let f, g and w be as in Theorem 4.11. Then, the following inequality holds:*

$$|T(w,f,g)|$$

$$\le \frac{1}{2} \int_a^b w(x) \left[|g(x)| \left\|f'\right\|_p + |f(x)| \left\|g'\right\|_p \right] \left(\int_a^b |P_w(x,t)|^q\,dt \right)^{\frac{1}{q}} dx, \tag{4.64}$$

where $T(w,f,g)$ and $P_w(x,t)$ are defined as above.

We, now, extend the inequalities (4.63) and (4.64) for spherical shells and balls as follows:

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$. Then, for $x \in \mathbb{R}^n - \{0\}$, $x = r\omega$, where $r > 0$, $\omega \in S^{n-1}$. Thus, $\|x\| = r$. Let

$$w(s) = \frac{ns^{n-1}}{R_2^n - R_1^n}, \tag{4.65}$$

for $0 < R_1 < R_2$, $s \in [R_1, R_2]$ be the probability density function and

$$W(s) = \int_{R_1}^{s} w(\tau)\, d\tau = \frac{s^n - R_1^n}{R_2^n - R_1^n}, \tag{4.66}$$

be the corresponding cumulative distribution function. Let A be a spherical shell, region between two concentric balls of radii R_1 and R_2. Let F, G be differentiable on $\overline{A}$ and f, g be differentiable on $[R_1, R_2]$. Then, the weighted Peano kernel and the weighted Montgomery identity for spherical shells and balls can be written as:

$$P_w(r,s) = \begin{cases} W(s), & R_1 \le s \le r \\ W(t) - 1, & r < s \le R_2, \end{cases} \tag{4.67}$$

$$f(r) = \int_{R_1}^{R_2} \left(\frac{ns^{n-1}}{R_2^n - R_1^n}\right) f(s)\, ds + \int_{R_1}^{R_2} P_w(r,s)\, f'(s)\, ds, \tag{4.68}$$

and the Čebyšev functional is given as:

$$\begin{aligned} \tilde{T}(F,G) \quad : &= \frac{\int_A F(x) G(x)\, dx}{Vol(A)} - \frac{1}{(Vol(A))^2}\left(\int_A F(x)\, dx\right)\left(\int_A G(x)\, dx\right) \\ &= \frac{n}{\omega_n (R_2^n - R_1^n)} \int_A F(x) G(x)\, dx \\ &\quad - \left(\frac{n}{\omega_n (R_2^n - R_1^n)}\right)^2 \left(\int_A F(x)\, dx\right)\left(\int_A G(x)\, dx\right), \end{aligned} \tag{4.69}$$

where $Vol(A)$ is the volume of spherical shell region A and by using the concept that

$$\begin{aligned} \frac{1}{\omega_n}\int_A F(x) G(x)\, dx &= \frac{1}{\omega_n}\int_{S^{n-1}} \left(\int_{R_1}^{R_2} f(r)\, r^{n-1} dr\right) d\omega \\ &= \int_{R_1}^{R_2} f(r)\, r^{n-1} dr, \end{aligned}$$

we can write:

$$\begin{aligned} \tilde{T}(F,G) &= \frac{n}{R_2^n - R_1^n} \int_{R_1}^{R_2} f(r)\, g(r)\, r^{n-1} dr \\ &\quad - \left(\frac{n}{R_2^n - R_1^n}\right)^2 \left(\int_{R_1}^{R_2} f(r)\, dr\right)\left(\int_{R_1}^{R_2} g(r)\, dr\right) \\ &= T(w, f, g), \end{aligned} \tag{4.70}$$

with w defined by (4.65).

Now, the following inequalities hold:

Theorem 4.13 *Let f, g be differentiable on $[R_1, R_2]$ and $f', g' \in L_p([R_1, R_2])$, then the following inequality holds:*

$$\left|\tilde{T}(F,G)\right| \leq \left\|f'\right\|_p \left\|g'\right\|_p \frac{n}{(R_2^n - R_1^n)^3} \int_{R_1}^{R_2} r^{n-1} I_q^2(r)\, dr, \tag{4.71}$$

where

$$I_q(r) = \left(\int_{R_1}^{r} (s^n - R_1^n)^q\, ds + \int_{r}^{R_2} (R_2^n - s^n)^q\, ds \right)^{\frac{1}{q}} \tag{4.72}$$

and $\tilde{T}(F,G)$ is defined by (4.70).

Proof. Applying (4.65)-(4.70) on Theorem 4.11, we get the required result. ■

Theorem 4.14 *Let f, g be as in Theorem 4.13, then the following inequality holds:*

$$\left|\tilde{T}(F,G)\right|$$

$$\leq \frac{n}{2(R_2^n - R_1^n)^2} \int_{R_1}^{R_2} r^{n-1} \left[|g(r)| \left\|f'\right\|_p + |f(r)| \left\|g'\right\|_p \right] I_q(r)\, dr, \tag{4.73}$$

where $I_q(r)$ is defined by (4.72) and $\tilde{T}(F,G)$ is defined by (4.70).

Proof. Applying (4.65)-(4.70) on Theorem 4.12, we get the required result. ■

Theorem 4.15 *Let F, G be differentiable on $\overline{A}$ and $F', G' \in L_p(\overline{A})$. Let F, G be the radial functions, then from Theorem 4.13, we have:*

$$\left|\tilde{T}(F,G)\right| \leq \left\|\frac{\partial F}{\partial r}\right\|_p \left\|\frac{\partial G}{\partial r}\right\|_p \frac{n}{(R_2^n - R_1^n)^3} \int_{R_1}^{R_2} r^{n-1} I_q^2(r)\, dr,$$

or

$$\left|\tilde{T}(F,G)\right| \leq \left\|\frac{\partial F}{\partial r}\right\|_p \left\|\frac{\partial G}{\partial r}\right\|_p \frac{1}{Vol(A)} \int_{A} H_q^2(\|x\|)\, dx, \tag{4.74}$$

where $I_q(r)$ is defined by (4.72) and

$$H_q(\|x\|) = \frac{1}{R_2^n - R_1^n} \left(\int_{R_1}^{r} (s^n - R_1^n)^q\, ds + \int_{r}^{R_2} (R_2^n - s^n)^q\, ds \right)^{\frac{1}{q}} \tag{4.75}$$

and $\tilde{T}(F,G)$ is defined by (4.69).

Theorem 4.16 *Let F, G be as in Theorem 4.15, then from Theorem 4.14, we have:*

$$\left|\tilde{T}(F,G)\right|$$

$$\leq \frac{1}{2Vol(A)}\int_A \left[|G(x)|\left\|\frac{\partial F}{\partial r}\right\|_p + |F(x)|\left\|\frac{\partial G}{\partial r}\right\|_p\right] H_q(\|x\|)\,dx, \tag{4.76}$$

where $H_q(\|x\|)$ is defined by (4.75) and $\tilde{T}(F,G)$ is defined by (4.69).

The inequalities (4.74) and (4.76) defined over spherical shell can be transferred over a ball $B^n(0,R)$ by taking $R := R_2$ and $R > R_1 \to 0$ and the results are stated in the form of the following theorems:

Theorem 4.17 *Let F, G be differentiable on $\overline{B^n(0,R)}$ and $F', G' \in L_p\left(\overline{B^n(0,R)}\right)$. Let F, G be the radial functions, then from Theorem 4.15, we have:*

$$\left|\frac{1}{Vol(B^n(0,R))}\int_{B^n(0,R)} F(x)G(x)\,dx - \frac{1}{(Vol(B^n(0,R)))^2}\int_{B^n(0,R)} F(x)\,dx \int_{B^n(0,R)} G(x)\,dx\right|$$

$$\leq \left\|\frac{\partial F}{\partial r}\right\|_p \left\|\frac{\partial G}{\partial r}\right\|_p \frac{n}{R^n}\int_0^R r^{n-1}J_q^2(r)\,dr, \tag{4.77}$$

where

$$J_q(r) = \frac{1}{R^n}\left(\frac{r^{nq+1}}{(nq+1)} + \int_r^R (R^n - s^n)^q\,ds\right)^{\frac{1}{q}}, \tag{4.78}$$

or

$$\left|\int_{B^n(0,R)} F(x)G(x)\,dx - \frac{1}{Vol(B^n(0,R))}\int_{B^n(0,R)} F(x)\,dx \int_{B^n(0,R)} G(x)\,dx\right|$$

$$\leq \left\|\frac{\partial F}{\partial r}\right\|_p \left\|\frac{\partial G}{\partial r}\right\|_p \int_{B^n(0,R)} J_q^2(\|x\|)\,dx. \tag{4.79}$$

Theorem 4.18 *Let F, G be as in Theorem 4.17, then from Theorem 4.16, we have:*

$$\left|\int_{B^n(0,R)} F(x)G(x)\,dx - \frac{1}{Vol(B^n(0,R))}\int_{B^n(0,R)} F(x)\,dx \int_{B^n(0,R)} G(x)\,dx\right|$$

$$\leq \frac{1}{2}\int_{B^n(0,R)} \left[|G(x)|\left\|\frac{\partial F}{\partial r}\right\|_p + |F(x)|\left\|\frac{\partial G}{\partial r}\right\|_p\right] J_q(\|x\|)\,dx, \tag{4.80}$$

where $J_q(\|x\|)$ is defined by (4.78).

We, now, give the inequalities in spherical shell when F, G are not radial functions which are given in the form of the following theorems:

Theorem 4.19 *Let F, G be differentiable on $\overline{A}$ and $F', G' \in L_p(\overline{A})$. Then, from Theorem 4.11, we have:*

$$\frac{1}{Vol(A)}\left|\int_A F(x)G(x)\,dx - \frac{n}{(R_2^n - R_1^n)}\int_{S^{n-1}}\left(\int_{R_1}^{R_2} F(rw)\,dr\right)\left(\int_{R_1}^{R_2} G(rw)\,dr\right)dw\right|$$

$$\leq \left\|\frac{\partial F}{\partial r}\right\|_p \left\|\frac{\partial G}{\partial r}\right\|_p \frac{n}{(R_2^n - R_1^n)^3}\int_{R_1}^{R_2} r^{n-1} I_q^2(r)\,dr,$$

or

$$\left|\int_A F(x)G(x)\,dx - \frac{n}{(R_2^n - R_1^n)}\int_{S^{n-1}}\left(\int_{R_1}^{R_2} F(rw)\,dr\right)\left(\int_{R_1}^{R_2} G(rw)\,dr\right)dw\right|$$

$$\leq \left\|\frac{\partial F}{\partial r}\right\|_p \left\|\frac{\partial G}{\partial r}\right\|_p \int_A H_q^2(\|x\|)\,dx, \tag{4.81}$$

where $I_q(r)$ is defined by (4.72) and $H_q(\|x\|)$ is defined by (4.75).

Theorem 4.20 *Let F, G be differentiable on $\overline{A}$ and $F', G' \in L_p(\overline{A})$. Then, from Theorem 4.12, we have:*

$$\left|\int_A F(x)G(x)\,dx - \frac{n}{(R_2^n - R_1^n)}\int_{S^{n-1}}\left(\int_{R_1}^{R_2} F(rw)\,dr\right)\left(\int_{R_1}^{R_2} G(rw)\,dr\right)dw\right|$$

$$\leq \frac{1}{2}\int_A \left[|G(x)|\left\|\frac{\partial F}{\partial r}\right\|_p + |F(x)|\left\|\frac{\partial G}{\partial r}\right\|_p\right] H_q(\|x\|)\,dx, \tag{4.82}$$

where $I_q(r)$ is defined by (4.72) and $H_q(\|x\|)$ is defined by (4.75).

The inequalities (4.81) and (4.82) defined over spherical shell can be transferred over a ball $B^n(0,R)$ by taking $R := R_2$ and $R > R_1 \to 0$ and the results are stated in the form of the following theorems:

Theorem 4.21 *Let F, G be differentiable on $\overline{B^n(0,R)}$ and $F', G' \in L_p\left(\overline{B^n(0,R)}\right)$. Then from Theorem 4.19, we have:*

$$\left|\int_{B^n(0,R)} F(x)G(x)\,dx - \frac{n}{R^n}\int_{S^{n-1}}\left(\int_0^R F(rw)\,dr\right)\left(\int_0^R G(rw)\,dr\right)dw\right|$$

$$\leq \left\| \frac{\partial F}{\partial r} \right\|_p \left\| \frac{\partial G}{\partial r} \right\|_p \int_{B^n(0,R)} J_q^2 (\|x\|) \, dx, \tag{4.83}$$

where $J_q(\|x\|)$ is defined by (4.78).

Theorem 4.22 *Let F, G be differentiable on $\overline{B^n(0,R)}$ and $F', G' \in L_p\left(\overline{B^n(0,R)}\right)$. Then from Theorem 4.20, we have:*

$$\left| \int_{B^n(0,R)} F(x) G(x) \, dx - \frac{n}{R^n} \int_{S^{n-1}} \left(\int_0^R F(rw) \, dr \right) \left(\int_0^R G(rw) \, dr \right) dw \right|$$

$$\leq \frac{1}{2} \int_{B^n(0,R)} \left[|G(x)| \left\| \frac{\partial F}{\partial r} \right\|_p + |F(x)| \left\| \frac{\partial G}{\partial r} \right\|_p \right] J_q(\|x\|) \, dx, \tag{4.84}$$

where $J_q(\|x\|)$ is defined by (4.78).

4.5 Conclusion

In this chapter, we present some generalizations of product and Čebyšev type inequalities by working with absolutely continuous functions whose derivatives belong to usual Lebesgue spaces.

In Section 4.1 and 4.2 we have confined ourselves to obtain product inequalities of Čebyšev type in L_∞ and L_p-spaces for $p > 1$ by the use of generalized functionals. In Section 4.1, the product inequalities for absolutely continuous functions whose first derivatives belong to L_p-space are developed while in Section 4.2, Čebyšev type inequalities are obtained for twice differentiable functions whose first derivatives are absolutely continuous and second derivatives belong to L_∞-space are taken into account. Applications for expectation of a continuous random variable are also given in Section 4.1.

In Section 4.3, we have presented an integral inequality involving product of two monotonic non-decreasing functions, thus providing a new estimation for these types of inequalities in terms of the functional values of monotonic mappings. The inequality is then applied to the probability density functions.

In Section 4.4, we have obtained an inequality of Čebyšev-Grüss type for spherical shells and balls by working in $n-$dimensional Euclidean space $\mathbb{R}^n$, hence, inequalities of multivariate type in spherical coordinate system are established.

Chapter 5

Some Ostrowski type inequalities for Newton-Cotes formulae

5.1 Some generalized error inequalities and applications

In this section, we present a family of four-point Ostrowski type inequality which is a generalization of Gauss-two point, Simpson's $\frac{3}{8}$ and Lobatto four-point quadrature rule for twice differentiable mapping. Moreover, it is shown that the corresponding optimal quadrature formula presents better estimate in the context of four-point quadrature formulae of closed type. A unified treatment of error inequalities for different classes of function is also given.

5.1.1 Introduction

We define

$$I(f) = \int_a^b f(x)\,dx. \tag{5.1}$$

The problem of approximating $I(f)$ is usually referred to as numerical integration or quadrature (see [4]). Most numerical integration formulae are based on defining the approximation by using polynomial or piecewise polynomial interpolation. Formulae using such interpolation with evenly spaced nodes are referred to as Newton-Cotes formulae. The Gaussian quadrature formulae, which are optimal and converge rapidly by selecting the node points carefully that need not be equally spaced, are investigated in [94].

In [28, 34, 81], the quadrature problem, in particular, the investigation of error bounds of Newton-Cotes formulae namely the mid-point, trapezoid and Simpson's rule have been carried out by the use of Peano kernel approach in terms of variety of norms, from an inequality point of view.

The deduction of the optimal quadrature formulae in the sense of minimal error bounds, has not received the right attention as long as the papers of N. Ujević [99, 95, 96] and ([24], pp.153-166) have not appeared who used a new approach for obtaining optimal two-point and three-point quadrature formulae of open as well as closed type. Further, some error inequalities have also been presented by N. Ujević to ensure the applications of these optimal quadrature formulae for different classes of functions.

In this section, we present an approach similar to that of Ujevic's [99] to present some improvements and generalizations in this context.

Let us first formulate the main problem.

Consider

$$K(x,y,t) = \begin{cases} \frac{1}{2}(t-\alpha)^2 + \alpha_1, \ t \in [a,x], \\ \frac{1}{2}(t-\beta)^2 + \beta_1, \ t \in (x,y), \\ \frac{1}{2}(t-\gamma)^2 + \gamma_1, \ t \in [y,b], \end{cases} \tag{5.2}$$

as defined in [99], where $x, y \in [a + h(b-a), b - h(b-a)]$, $h \in \left[0, \frac{1}{2}\right]$, $x < y$ and $\alpha, \ \alpha_1, \ \beta, \ \beta_1, \ \gamma, \ \gamma_1 \in \mathbb{R}$ are parameters which are required to be determined.

We know that the exact value of the remainder term of the integral $\int_a^b K(x,y,t) f''(t)\,dt$ may not be found, thus, we may proceed as

$$\left| \int_a^b K(x,y,t) f''(t)\,dt \right| \le \max_{t\in[a,b]} \left| f''(t) \right| \int_a^b |K(x,y,t)|\,dt. \tag{5.3}$$

The main aim of this section is to present a minimal estimation of the error bound (5.3) by appropriately choosing the variables and parameters involved. Moreover, it is worth-mentioning that the family of quadrature formulae thus obtained hereafter is a generalization of that presented in [99].

5.1.2 A generalized optimal quadrature formula

Consider the above stated error inequality problem for $a = -1$, $b = 1$, so that $x, y \in [-1 + 2h, 1 - 2h]$. We will try to find out an optimal quadrature formula of the form:

$$\int_{-1}^{1} f(t)\,dt - [hf(-1) + (1-h)f(x) + (1-h)f(y) + hf(1)]$$
$$= \int_{-1}^{1} K(x,y,t) f''(t)\,dt, \tag{5.4}$$

where $K(x,y,t)$ is defined by (5.2) with $a=-1$, $b=1$ and $x,y \in [-1+2h, 1-2h]$ with $x<y$, $h \in \left[0, \frac{1}{2}\right]$.

The parameters $\alpha, a_1, \beta, \beta_1, \gamma, \gamma_1 \in \mathbb{R}$ involved in $K(x,y,t)$ are required to be determined in a way such that the representation (5.4) is obtained.

Integrating by parts right hand side of (5.4), we have:

$$\begin{aligned}
&\int_{-1}^{1} K(x,y,t) f''(t)\,dt \\
= \; & -\left[\frac{1}{2}(1+\alpha)^2 + \alpha_1\right] f'(-1) + \left[\frac{1}{2}(1-\gamma)^2 + \gamma_1\right] f'(1) \\
&+ \left[\frac{1}{2}\left\{(x-\alpha)^2 - (x-\beta)^2\right\} + \alpha_1 - \beta_1\right] f'(x) \\
&+ \left[\frac{1}{2}\left\{(y-\beta)^2 - (y-\gamma)^2\right\} + \beta_1 - \gamma_1\right] f'(y) \\
&- (1+\alpha) f(-1) - (1-\gamma) f(1) + (\alpha-\beta) f(x) + (\beta-\gamma) f(y) \\
&+ \int_{-1}^{1} f(t)\,dt.
\end{aligned} \tag{5.5}$$

For the representation (5.4), we require from (5.5),

$$\begin{aligned}
\frac{1}{2}(x-\alpha)^2 + \alpha_1 - \frac{1}{2}(x-\beta)^2 - \beta_1 &= 0, \\
\frac{1}{2}(y-\beta)^2 + \beta_1 - \frac{1}{2}(y-\gamma)^2 - \gamma_1 &= 0, \\
\frac{1}{2}(1+\alpha)^2 + \alpha_1 = 0, \; \frac{1}{2}(1-\gamma)^2 + \gamma_1 &= 0, \\
\beta - \gamma = -(1-h), \alpha - \beta &= -(1-h), \\
1+\alpha = h, 1-\gamma &= h.
\end{aligned}$$

This gives through simple calculations:

$$\alpha = -(1-h), \; \gamma = (1-h), \; \beta = 0,$$

$$\begin{aligned}\gamma_1 &= -\frac{1}{2}h^2,\ \alpha_1 = -\frac{1}{2}h^2,\\ \beta_1 &= \frac{1}{2} - h + (1-h)\,x\\ &= \frac{1}{2} - h - (1-h)\,y.\end{aligned}$$

Henceforth,

$$y = -x.$$

So, we have:

$$K(x,t) = \begin{cases} \frac{1}{2}(t + (1-h))^2 - \frac{1}{2}h^2,\ t \in [-1, x],\\ \frac{1}{2}t^2 + (1-h)\,x - h + \frac{1}{2},\ t \in (x, y),\\ \frac{1}{2}(t - (1-h))^2 - \frac{1}{2}h^2,\ \ t \in [y, 1]. \end{cases} \tag{5.6}$$

We further see that

$$\left|\int_{-1}^{1} K(x,t)\, f''(t)\, dt\right| \le \left\|f''\right\|_\infty \int_{-1}^{1} |K(x,t)|\, dt. \tag{5.7}$$

We are now required to find an x that minimizes $\int_{-1}^{1} |K(x,t)|\, dt$.

We next define

$$\begin{aligned} G(x) &= \int_{-1}^{1} |K(x,t)|\, dt\\ &= \frac{1}{2}\int_{-1}^{x} \left|(t + (1-h))^2 - h^2\right| dt + \int_{x}^{y} \left|\frac{1}{2}t^2 + (1-h)\,x - h + \frac{1}{2}\right| dt\\ &\quad + \frac{1}{2}\int_{y}^{1} \left|(t - (1-h))^2 - h^2\right| dt. \end{aligned}$$

and consider the problem

$$\mathit{minimize}\ G(x),\ x \in [-1+2h, 1-2h]\ \text{and}\ h \in \left[0, \frac{1}{2}\right].$$

Hence, we should like to find a global minimizer of G. Recall that a global minimizer is a point x^* that satisfies

$$G(x^*) \le G(x),\ \mathit{for\ all}\ x \in [-1+2h, 1-2h]\ \text{and}\ h \in \left[0, \frac{1}{2}\right].$$

We now consider the following cases:

(i) Let $x \in \left[-(1-2h), \frac{h-\frac{1}{2}}{1-h}\right]$. Then by symmetry, we may consider

$$\begin{aligned} G_1(x) &= -\frac{1}{2}\int_{-1}^{-1+2h}\left((t+(1-h))^2-h^2\right)dt \\ &+\frac{1}{2}\int_{-1+2h}^{x}\left((t+(1-h))^2-h^2\right)dt \\ &+\frac{1}{2}\int_{x}^{-\sqrt{2h-1-2(1-h)x}}\left(t^2+2(1-h)x-2h+1\right)dt \\ &-\frac{1}{2}\int_{-\sqrt{2h-1-2(1-h)x}}^{0}\left(t^2+2(1-h)x-2h+1\right)dt \\ &= \frac{1}{6}-\frac{1}{2}(1-h)x^2-\frac{4}{3}(1-h)\sqrt{2h-1-2(1-h)x}\;x \\ &+\frac{4}{3}\left(h-\frac{1}{2}\right)\sqrt{2h-1-2(1-h)x}+\frac{4}{3}h^3-\frac{h}{2}. \end{aligned} \tag{5.8}$$

We may note that

$$G(x) = 2G_1(x). \tag{5.9}$$

Combining (5.8) and (5.9) with (5.4) and (5.7), we get:

$$\begin{aligned} &\left|\int_{-1}^{1} f(t)\,dt - [hf(-1)+(1-h)f(x)+(1-h)f(-x)+hf(1)]\right| \\ \leq &\left[\frac{1}{3}-(1-h)x^2-\frac{8}{3}(1-h)\sqrt{2h-1-2(1-h)x}\;x\right. \\ &\left.+\frac{8}{3}\left(h-\frac{1}{2}\right)\sqrt{2h-1-2(1-h)x}+\frac{8}{3}h^3-h\right]\left\|f''\right\|_\infty. \end{aligned} \tag{5.10}$$

Moreover, simple calculations show that $G'(x) = 0$ for

$$x_{1,2} = -4+4h \pm 2\sqrt{3-6h+4h^2}.$$

It is not difficult to find that

$$G''(x_1) > 0 \text{ and } G''(x_2) < 0.$$

Thus, x_1 is the local minimizer of $G(x)$ for $x \in \left[-(1-2h), \frac{h-\frac{1}{2}}{1-h}\right]$. We have:

$$G_1(x_1) = \frac{52}{3}h^3 - 44h^2 + \frac{83}{2}h - \frac{83}{6} + 8(1-h)^2\sqrt{4h^2-6h+3}$$

$$+\frac{2}{3}\left(8h^2-14h+7\right)\sqrt{8h^2-14h+7-4(1-h)\sqrt{4h^2-6h+3}}$$

$$-\frac{8}{3}(1-h)\sqrt{8h^2-14h+7-4(1-h)\sqrt{4h^2-6h+3}}\sqrt{4h^2-6h+3},$$

such that

$$G(x_1)=2G_1(x_1).$$

(*ii*) Next, we check for the point $x_3=\frac{h-\frac{1}{2}}{1-h}$. We find that $\min_{h\in\left[0,\frac{1}{2}\right]} G_1(x_1) < \min_{h\in\left[0,\frac{1}{2}\right]} G_1(x_3)$.

Thus, from the above considerations, we find that $x^*=-4+4h+2\sqrt{3-6h+4h^2}$ is the global minima of G. Therefore, we get the following conclusion:

Theorem 5.1 *Let $I\subset\mathbb{R}$ be an open interval such that $[-1,1]\subset I$, and let $f:I\to\mathbb{R}$ be a twice differentiable function such that f'' is bounded and integrable. Then, we have:*

$$\begin{aligned}\int_{-1}^{1} f(t)\,dt &= \Big[hf(-1)+(1-h)f\left(-4+4h+2\sqrt{3-6h+4h^2}\right)\\ &\quad +(1-h)f\left(4-4h-2\sqrt{3-6h+4h^2}\right)+hf(1)\Big]\\ &\quad +R(f),\end{aligned} \tag{5.11}$$

where

$$|R(f)|\le 2\triangle(h)\left\|f''\right\|_\infty, \tag{5.12}$$

$h\in\left[0,\frac{1}{2}\right]$ and $\triangle(h)$ is defined as

$$\begin{aligned}\triangle(h) &= \frac{52}{3}h^3-44h^2+\frac{83}{2}h-\frac{83}{6}+8(1-h)^2\sqrt{4h^2-6h+3}\\ &\quad +\frac{2}{3}\left(8h^2-14h+7\right)\sqrt{8h^2-14h+7-4(1-h)\sqrt{4h^2-6h+3}}\\ &\quad -\frac{8}{3}(1-h)\sqrt{8h^2-14h+7-4(1-h)\sqrt{4h^2-6h+3}}\sqrt{4h^2-6h+3}.\end{aligned} \tag{5.13}$$

Proof. From the above discussion, we find that (5.11) holds with

$$R(f)=\int_{-1}^{1} K\left(-4+4h+2\sqrt{3-6h+4h^2},t\right)f''(t)\,dt,$$

and $K(x,t)$ is given by (5.6) with $y=-x$. We further have

$$\begin{aligned}|R(f)| &\le \left\|f''\right\|_\infty\int_{-1}^{1}\left|K\left(-4+4h+2\sqrt{3-6h+4h^2},t\right)\right|dt\\ &= G\left(-4+4h+2\sqrt{3-6h+4h^2}\right)\left\|f''\right\|_\infty,\end{aligned}$$

Since $G\left(-4+4h+2\sqrt{3-6h+4h^2}\right)=2G_1\left(-4+4h+2\sqrt{3-6h+4h^2}\right)$, thus (5.12) holds. ■

We would now like to mention here some special cases of (5.10):

Remark 5.1 *As it has been mentioned in [99], we recapture the Gauss two-point quadrature formula for* $h=0$ *and* $x=-\frac{\sqrt{3}}{3}$.

Remark 5.2 *It may be noted that for* $h=\frac{1}{6}$ *and* $x=-\frac{\sqrt{5}}{5}$, *we get Lobbato four-point quadrature rule as follows:*

$$\begin{aligned}\int_{-1}^{1} f(t)\,dt &= \frac{1}{6}\left[f(-1)+5f\left(-\frac{\sqrt{5}}{5}\right)+5f\left(\frac{\sqrt{5}}{5}\right)+f(1)\right] \\ &\quad +R_1(f),\end{aligned} \tag{5.14}$$

where

$$|R_1(f)|\le C_1\left\|f''\right\|_\infty$$

and $C_1=\frac{1}{81}+\frac{4}{27}\left(\sqrt{-6+3\sqrt{5}}\right)\left(\sqrt{5}-2\right)\approx 0.0418.$

Remark 5.3 *For* $h=\frac{1}{4}$ *and and* $x=-\frac{1}{3}$, *we get* $\frac{3}{8}$ *Simpson's rule as follows:*

$$\begin{aligned}\int_{-1}^{1} f(t)\,dt &= \frac{1}{4}\left[f(-1)+3f\left(-\frac{1}{3}\right)+3f\left(\frac{1}{3}\right)+f(1)\right] \\ &\quad +R_2(f),\end{aligned} \tag{5.15}$$

where

$$|R_2(f)|\le C_2\left\|f''\right\|_\infty$$

and $C_2=\frac{1}{24}\approx 0.0417.$

Remark 5.4 *Keeping in view the above special cases, (5.10) may be considered as a generalization of Gauss two-point, Simpson' s* $\frac{3}{8}$ *and Lobatto four-point quadrature rule for twice differentiable mappings.*

Remark 5.5 *For* $h=\frac{1}{5}$, $\triangle(h)$ *attains its minimum value.*

Corollary 5.1 *Let the assumptions of Theorem 5.1 hold. Then, we have the following optimal quadrature rule:*

$$\begin{aligned}\int_{-1}^{1} f(t)\,dt &= \frac{1}{5}\left[f(-1)+4f\left(-\frac{2}{5}\right)+4f\left(\frac{2}{5}\right)+f(1)\right] \\ &\quad +R_3(f),\end{aligned} \tag{5.16}$$

$$|R_3(f)| \le C_3 \left\| f'' \right\|_\infty ,$$

where $C_3 = \frac{14}{375} \approx 0.0373.$

Remark 5.6 *The comparison of (5.14), (5.15) and (5.16) shows that the later presents a much better estimate in the context of four-point quadrature rules of closed type.*

By considering the problem on the interval $[a,b]$, the following theorem is obvious:

Theorem 5.2 *Let* $I \subset \mathbb{R}$ *be an open interval such that* $[a,b] \subset I$ *and let* $f : I \to \mathbb{R}$ *be a twice differentiable function such that* f'' *is bounded and integrable. Then, we have*

$$\int_a^b f(t)\,dt = \frac{1}{2}(b-a)\left[hf(a) + (1-h) f(x_1) + (1-h) f(x_2) + hf(b)\right] + R(f), \tag{5.17}$$

where

$$x_1 = \frac{b-a}{2}x^* + \frac{a+b}{2}, \; x_2 = -\frac{b-a}{2}x^* + \frac{a+b}{2}, \tag{5.18}$$

with

$$x^* = -4 + 4h + 2\sqrt{3 - 6h + 4h^2}$$

and

$$|R(f)| \le \frac{1}{4}\triangle(h)(b-a)^3 \left\| f'' \right\|_\infty , \tag{5.19}$$

$h \in \left[0, \frac{1}{2}\right]$ *and* $\triangle(h)$ *is as defined above.*

5.1.3 Generalized error inequalities

From the basic properties of the $L_p(a,b)$ spaces, for $p = 1, 2, \infty$, we know that $L_2(a,b)$ is a Hilbert space with the inner product defined as:

$$\langle f, g \rangle_2 = \int_a^b f(t)\,g(t)\,dt.$$

We now define $X = (L_2(a,b), \langle .,. \rangle_2)$. In the space X, the norm $\|.\|_2$ is defined in the usual manner as:

$$\|f\|_2 = \left(\int_a^b f^2(t)\,dt \right)^{\frac{1}{2}}.$$

Let us also consider $Y = (L_2(a,b), \langle .,. \rangle)$, where the inner product $\langle .,. \rangle$ is defined by

$$\langle f,g \rangle = \frac{1}{b-a} \int_a^b f(t) g(t) \, dt,$$

with the corresponding norm $\|.\|$ defined by

$$\|f\| = \sqrt{\langle f,f \rangle}.$$

We know that the Čebyšev functional is defined as

$$T(f,g) = \langle f,g \rangle - \langle f,e \rangle \langle g,e \rangle, \tag{5.20}$$

where $f,g \in L_2(a,b)$ and $e = 1$ which satisfies the pre-Grüss inequality,

$$T^2(f,g) \leq T(f,f) T(g,g). \tag{5.21}$$

Let us denote

$$\sigma(f) = \sigma(f;a,b) = \sqrt{(b-a) T(f,f)}, \tag{5.22}$$

as defined in [99]. Moreover, the space $L_1(a,b)$ is a Banach space with the norm

$$\|f\|_1 = \int_a^b |f(t)| \, dt,$$

and the space $L_\infty(a,b)$ is also a Banach space with the norm

$$\|f\|_\infty = ess \sup_{t \in [a,b]} |f(t)|.$$

So, if $f \in L_1(a,b)$ and $g \in L_\infty(a,b)$, then we have

$$|\langle f,g \rangle_2| \leq \|f\|_1 \|g\|_\infty. \tag{5.23}$$

Finally, we define

$$\begin{aligned} J(f) &= J(f;a,b;h) \\ &= \int_a^b f(t) \, dt - \frac{1}{2}(b-a)[hf(a) + (1-h) f(x_1) \\ &\quad + (1-h) f(x_2) + hf(b)], \end{aligned} \tag{5.24}$$

where x_1 and x_2 are given by (5.18).

We would also like to mention the following Lemma [98].

Lemma 5.1 *Let*

$$f(t)=\begin{cases} f_1(t)\,, & t\in[a,x_1]\,,\\ f_2(t)\,, & t\in[x_1,x_2]\,,\\ f_3(t)\,, & t\in[x_2,b]\,,\end{cases} \tag{5.25}$$

where $a<x_1<x_2<b$, $f_1\in C^1(a,x_1)$, $f_2\in C^1(x_1,x_2)$, $f_3\in C^1(x_2,b)$ *and* $f_1(x_1)=f_2(x_1)$, $f_2(x_2)=f_3(x_2)$. *If*

$$\begin{aligned}\sup_{t\in(a,x_1)}|f^{'}(t)| &< \infty,\\ \sup_{t\in(x_1,x_2)}|f^{'}(t)| &< \infty,\\ \sup_{t\in(x_2,b)}|f^{'}(t)| &< \infty.\end{aligned}$$

Then, the function f *is an absolutely continuous function.*

Theorem 5.3 *Let* $f:[-1,1]\to\mathbb{R}$ *be a function such that* $f^{'}\in L_1(-1,1)$. *If there exists a real number* γ_1, *such that* $\gamma_1\le f^{'}(t)\,,\, t\in[-1,1]\,,$ *then*

$$|J(f;-1,1;h)|\le 2\triangle_0(h)(S-\gamma_1)\,, \tag{5.26}$$

and if there exists a real number Γ_1, *such that* $f^{'}(t)\le\Gamma_1,\, t\in[-1,1]\,,$ *then*

$$|J(f;-1,1;h)|\le 2\triangle_0(h)(\Gamma_1-S)\,, \tag{5.27}$$

where $J(f;-1,1;h)$ *is defined by (5.24),* $S=\frac{f(1)-f(-1)}{2}$ *and* $h\in[0,\frac{1}{2}]$. *If there exist real numbers* $\gamma_1,\ \Gamma_1$, *such that* $\gamma_1\le f^{'}(t)\le\Gamma_1,\, t\in[-1,1]\,,$ *then*

$$|J(f;-1,1;h)|\le\frac{1}{2}\triangle_1(h)(\Gamma_1-\gamma_1)\,. \tag{5.28}$$

$\triangle_0(h)$ *and* $\triangle_1(h)$ *are defined as:*

$$\triangle_0(h)=2\sqrt{4h^2-6h+3}-3(1-h)\,,$$

$$\triangle_1(h)=58h^2-98h+49-28(1-h)\sqrt{4h^2-6h+3}. \tag{5.29}$$

Proof. In order to prove (5.28), let us define

$$p_1(t)=\begin{cases} t+1-h, & t\in[-1,x]\,,\\ t, & t\in(x,y),\\ t-(1-h)\,, & t\in[y,1],\end{cases}$$

where $x=-4+4h+2\sqrt{3-6h+4h^2}$ and $y=-x$. Note that since $\langle p_1, e\rangle_2 = 0$, thus

$$\begin{aligned}\left\langle p_1, f^{'}\right\rangle_2 &= -J\left(f;-1,1;h\right), \\ \left\langle f^{'} - \frac{\Gamma_1+\gamma_1}{2}, p_1\right\rangle_2 &= \left\langle f^{'}, p_1\right\rangle_2 .\end{aligned} \tag{5.30}$$

From (5.23),

$$\begin{aligned}\left|\left\langle f^{'} - \frac{\Gamma_1+\gamma_1}{2}, p_1\right\rangle_2\right| &\le \left\| f^{'} - \frac{\Gamma_1+\gamma_1}{2}\right\|_\infty \|p_1\|_1 \\ &\le \frac{1}{2}\triangle_1(h)\left(\Gamma_1-\gamma_1\right),\end{aligned} \tag{5.31}$$

as

$$\left\| f^{'} - \frac{\Gamma_1+\gamma_1}{2}\right\|_\infty \le \frac{\Gamma_1-\gamma_1}{2}$$

and

$$\|p_1\|_1 = 58h^2 - 98h + 49 - 28\left(1-h\right)\sqrt{4h^2-6h+3}.$$

From (5.30) and (5.31), it may be observed that (5.28) holds. Further, it can be seen that

$$\begin{aligned}\left|\left\langle f^{'} - \gamma_1, p_1\right\rangle_2\right| &\le \|p_1\|_\infty \left\| f^{'} - \gamma_1\right\|_1 \\ &= 2\triangle_0(h)\left(S-\gamma_1\right),\end{aligned}$$

since

$$\|p_1\|_\infty = 2\sqrt{4h^2-6h+3} - 3\left(1-h\right)$$

and

$$\begin{aligned}\left\| f^{'} - \gamma_1\right\|_1 &= \int_{-1}^{1}\left(f^{'}(t) - \gamma_1\right) dt \\ &= f(1) - f(-1) - 2\gamma_1 \\ &= 2\left(S-\gamma_1\right).\end{aligned}$$

Hence, (5.26) holds. In the similar manner, we can prove (5.27). ■

Remark 5.7 *It may be noted that* $\triangle_0(h)$ *has its minimum value* 0.396 *at* $h = 0.259$. *In a similar way, it may be observed that* $\frac{1}{2}\triangle_1(h)$ *attains its minimum value* 0.1698 *at* $h = 0.296$.

Theorem 5.4 *Let $f:[a,b]\to\mathbb{R}$ be a function, such that $f^{'}\in L_1(a,b)$. If there exists a real number γ_1, such that $\gamma_1\le f^{'}(t)$, $t\in[a,b]$, then*

$$|J(f;a,b;h)|\le\frac{1}{2}\triangle_0(h)(S-\gamma_1)(b-a)^2, \tag{5.32}$$

and if there exists a real number Γ_1, such that $f^{'}(t)\le\Gamma_1$, $t\in[a,b]$, then

$$|J(f;a,b;h)|\le\frac{1}{2}\triangle_0(h)(\Gamma_1-S)(b-a)^2, \tag{5.33}$$

where $J(f;a,b;h)$ is defined by (5.24) and $S=\frac{f(a)-f(b)}{(b-a)}$ and $h\in[0,\frac{1}{2}]$. If there exist real numbers γ_1, Γ_1, such that $\gamma_1\le f^{'}(t)\le\Gamma_1$, $t\in[a,b]$, then

$$|J(f;a,b;h)|\le\frac{1}{8}\triangle_1(h)(\Gamma_1-\gamma_1)(b-a)^2. \tag{5.34}$$

$\triangle_0(h)$ and $\triangle_1(h)$ are as defined in (5.29).

Theorem 5.5 *Let $f:[-1,1]\to\mathbb{R}$ be an absolutely continuous function, such that $f^{'}\in L_2(-1,1)$. Then*

$$|J(f;-1,1;h)|\le\sqrt{\triangle_2(h)}\sigma\left(f^{'};-1,1\right), \tag{5.35}$$

where $\sigma\left(f^{'};-1,1\right)$ is defined by (5.22) and

$$\triangle_2(h)=-56h^3+154h^2-146h+\frac{146}{3}-28(1-h)^2\sqrt{4h^2-6h+3}, \tag{5.36}$$

for $h\in[0,\frac{1}{2}]$.

Proof. Let p_1 be the same as defined above. We have

$$\left\langle p_1,f^{'}\right\rangle_2=-J(f;-1,1;h), \tag{5.37}$$

since $\langle p_1,e\rangle_2=0$, if $[a,b]=[-1,1]$. Moreover, $\langle f,g\rangle=\frac{1}{2}\langle f,g\rangle_2$ and

$$\left\langle p_1,f^{'}\right\rangle=T\left(f^{'},p_1\right). \tag{5.38}$$

From (5.21), it follows that

$$\begin{aligned}T\left(f^{'},p_1\right) &\le \sqrt{T(p_1,p_1)}\sqrt{T(f^{'},f^{'})}\\ &= \frac{1}{2}\|p_1\|_2\,\sigma\left(f^{'};-1,1\right),\\ &= \frac{1}{2}\sqrt{\triangle_2(h)}\sigma\left(f^{'};-1,1\right)\end{aligned} \tag{5.39}$$

as

$$\|p_1\|_2^2=-56h^3+154h^2-146h+\frac{146}{3}-28(1-h)^2\sqrt{4h^2-6h+3}. \tag{5.40}$$

Using (5.37), (5.38), (5.39) and (5.40), inequality (5.35) is proved. ■

Remark 5.8 $\sqrt{\triangle_2(h)}$ *attains its minimum value* 0.2799 *at* $h = 0.2957$.

Theorem 5.6 *Let* $f : [a,b] \to \mathbb{R}$ *be an absolutely continuous function, such that* $f^{'} \in L_2(a,b)$. *Then*

$$|J(f;a,b;h)| \leq \frac{1}{2\sqrt{2}}\sqrt{\triangle_2(h)}\sigma\left(f^{'};a,b\right)(b-a)^{\frac{3}{2}}, \tag{5.41}$$

where $\sigma\left(f^{'};a,b\right)$ *is defined by (5.22) and* $\triangle_2(h)$ *is as defined above.*

5.1.4 Applications in Numerical Integration

Let $\pi = \{x_0 = a < x_1 < ... < x_n = b\}$ be a subdivision of the interval $[a,b]$, such that $h_i = x_{i+1} - x_i = h = \frac{(b-a)}{n}$. From (5.24), we have:

$$\begin{aligned} J(f) &= J(f;x_i,x_{i+1};\delta) = \int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{h}{2}\left[\delta f(x_i) + (1-\delta) f(x_{1i})\right. \\ &\quad \left. + (1-\delta) f(x_{2i}) + \delta f(x_{i+1})\right], \end{aligned}$$

where

$$x_{1i} = \frac{h}{2}x^* + \frac{x_i + x_{i+1}}{2}, \; x_2 = -\frac{h}{2}x^* + \frac{x_i + x_{i+1}}{2},$$

and

$$x^* = -4 + 4\delta + 2\sqrt{3 - 6\delta + 4\delta^2}, \; \delta \in \left[0, \frac{1}{2}\right].$$

Summing up the above relation from 1 to $n-1$, we get:

$$\begin{aligned} \sum_{i=0}^{n-1} J(f;x_i,x_{i+1};\delta) &= \int_a^b f(t)\,dt - \frac{h}{2}\sum_{i=0}^{n-1}\left[\delta f(x_i) + (1-\delta) f(x_{1i})\right. \\ &\quad \left. + (1-\delta) f(x_{2i}) + \delta f(x_{i+1})\right], \end{aligned}$$

Let us denote

$$S(f;a,b;\delta) = \sum_{i=0}^{n-1} J(f;x_i,x_{i+1};\delta). \tag{5.42}$$

Theorem 5.7 *Let the assumptions of Theorem 5.2 hold, then we have:*

$$|S(f;a,b;\delta)| \leq \frac{1}{4n^2}\triangle(\delta)\left\|f^{''}\right\|_\infty (b-a)^3,$$

where $S(f;a,b;\delta)$ *is defined by (5.42),* $\delta \in \left[0, \frac{1}{2}\right]$ *and* $\triangle(\delta)$ *is defined by (5.9).* π *is the uniform subdivision of* $[a,b]$.

Theorem 5.8 *Let the assumptions of Theorem 5.4 hold, then it follows*

$$|S(f;a,b;\delta)| \leq \frac{1}{8}\triangle_1(\delta)\frac{\Gamma_1-\gamma_1}{n}(b-a)^2,$$

$$|S(f;a,b;\delta)| \leq \frac{1}{2n}\triangle_0(\delta)(S-\gamma_1)(b-a)^2,$$

and if there exists a real number Γ_1, *such that* $f'(t) \leq \Gamma_1$, $t \in [a,b]$, *then*

$$|S(f;a,b;\delta)| \leq \frac{1}{2n}\triangle_0(\delta)(\Gamma_1-S)(b-a)^2,$$

where $S(f;a,b;\delta)$ *is defined by (5.42),* $\triangle_0(\delta)$, $\triangle_1(\delta)$ *are defined by (5.29) and* $S = \frac{f(a)-f(b)}{(b-a)}$. π *is the uniform subdivision of* $[a,b]$.

Theorem 5.9 *Let the assumptions of Theorem 5.6 hold, then it follows that*

$$|S(f;a,b;\delta)| \leq \frac{(b-a)^{\frac{3}{2}}}{2\sqrt{2}n}\sqrt{\triangle_2(\delta)}\sigma\left(f'\right),$$

where $S(f;a,b;\delta)$ *is defined by (5.42),* $\sigma\left(f'\right)$ *is defined by (5.22) and* $\triangle_2(\delta)$ *is as defined by (5.36).* π *is the uniform subdivision of* $[a,b]$.

Proof. Applying Theorem 5.6 on the interval $[x_i, x_{i+1}]$,

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{h}{2}\left[\delta f(x_i) + (1-\delta)f(x_{1i}) + (1-\delta)f(x_{2i}) + \delta f(x_{i+1})\right]\right|$$

$$\leq \frac{1}{2\sqrt{2}}\sqrt{\triangle_2(\delta)}h^{\frac{3}{2}}\left[\int_{x_i}^{x_{i+1}}\left(f'(t)\right)^2 dt - \frac{1}{h}\left(f(x_{i+1})-f(x_i)\right)^2\right]^{\frac{1}{2}}.$$

Summing over i from 0 to $n-1$,

$$|S(f;a,b;\delta)| \leq \frac{1}{2\sqrt{2}}\sqrt{\triangle_2(\delta)}h^{\frac{3}{2}}\sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+1}}\left(f'(t)\right)^2 dt - \frac{1}{h}\left(f(x_{i+1})-f(x_i)\right)^2\right]^{\frac{1}{2}}.$$

Using Cauchy-Schwartz inequality and the relation $h = (b-a)/n$, we obtain the required inequality:

$$|S(f;a,b;\delta)|$$

$$\leq \frac{1}{2\sqrt{2}}\sqrt{\triangle_2(\delta)}\frac{(b-a)^{\frac{3}{2}}}{n^{\frac{3}{2}}}n^{\frac{1}{2}}\left[\left\|f'\right\|_2^2 - \frac{n}{b-a}\sum_{i=0}^{n-1}\left(f(x_{i+1})-f(x_i)\right)^2\right]^{\frac{1}{2}}$$

$$\leq \frac{1}{2\sqrt{2}}\sqrt{\triangle_2(\delta)}\frac{(b-a)^{\frac{3}{2}}}{n}\left[\left\|f'\right\|_2^2 - \frac{\left(f(b)-f(a)\right)^2}{b-a}\right]^{\frac{1}{2}}.$$

■

5.2 A generalized integral inequality generating Newton-Cotes formulae of open and closed type

5.2.1 Introduction

In [50], Ming-How Hung et al. presented the following two-point open Newton-Cotes quadrature formula of open type for mappings of bounded variation. The result is given as follows:

Theorem 5.10 *Let $f^{(n-1)} : [a, b] \to \mathbb{R}$ be a mapping of bounded variation on $[a, b]$ and $n \in \{1, 2\}$. Then, we have the inequality:*

$$\left| \int_a^b f(x)\, dx - \frac{b-a}{2} \left[f(\alpha a + (1-\alpha) b) + f((1-\alpha) a + \alpha b) \right] \right|$$
$$\leq K_n (b-a)^n \bigvee_a^b \left(f^{(n-1)} \right), \qquad (5.43)$$

where

$$K_1 = \max \left\{ 1 - \alpha, \alpha - \frac{1}{2} \right\}, K_2 = \frac{1}{2} (1-\alpha)^2$$

and $\bigvee_a^b \left(f^{(n-1)} \right)$ denote the total variation of $f^{(n-1)}$ on the interval $[a, b]$ and $\frac{1}{2} \leq \alpha < 1$.

Recently, in [55] Wenjun Liu presented some error inequalities for a quadrature formula involving a parameter.

We, in the following subsection, present a two point Ostrowski type inequality, a generalization of the results of [50] and [55]. Moreover, it can also generate Newton-Cotes formulae of open as well as closed type for mappings of bounded variation. Furthermore, we also present the estimates of the generalized integral inequality for other classes of the function involved.

5.2.2 Main Results

We shall start with the following result:

Theorem 5.11 *Let $f^{(n-1)} : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $n \in \{1,2\}$. Then, we have the inequality:*

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{2}\left[f(\alpha_1) + f(\alpha_2)\right] \right|$$

$$\leq M_n \bigvee_a^b \left(f^{(n-1)}\right), \tag{5.44}$$

where

$$M_1 = \max\left\{\alpha_1 - a, \frac{a+b}{2} - \alpha_1, \alpha_2 - \frac{a+b}{2}, b - \alpha_2\right\},$$

$$\begin{aligned} M_2 &= \frac{1}{2}\max\left\{(\alpha_1 - a)^2, \left(\alpha_1 - \frac{a+b}{2}\right)^2 + (b-a)\left(\alpha_1 - \frac{3a+b}{4}\right),\right. \\ &\quad \left.(b-a)\left(\frac{3a+b}{4} - \alpha_1\right)\right\} \end{aligned} \tag{5.45}$$

and $\bigvee_a^b \left(f^{(n-1)}\right)$ denote the total variation of $f^{(n-1)}$ on the interval $[a,b]$ and $a \leq \alpha_1 \leq \alpha_2 \leq b$.

Proof. Let us define:

$$K(\alpha_1, \alpha_2, t) = \begin{cases} t - a, & t \in [a, \alpha_1) \\ t - \frac{a+b}{2}, & t \in [\alpha_1, \alpha_2) \\ t - b, & t \in [\alpha_2, b], \end{cases}$$

where α_1 and α_2 are to be taken in such a way that $a \leq \alpha_1 \leq \alpha_2 \leq b$.

Consider the Reimann-Stieltjes integral

$$\int_a^b K(\alpha_1, \alpha_2, t)\,df(t) = \int_a^{\alpha_1} (t-a)\,df(t) + \int_{\alpha_1}^{\alpha_2} \left(t - \frac{a+b}{2}\right) df(t) + \int_{\alpha_2}^b (t-b)\,df(t).$$

Integrating by parts, we obtain:

$$\int_a^b K(\alpha_1, \alpha_2, t)\,df(t) = \frac{b-a}{2}\left(f(\alpha_1) + f(\alpha_2)\right) - \int_a^b f(t)\,dt. \tag{5.46}$$

Similarly, if we define:

$$K'(\alpha_1, \alpha_2, t) = \begin{cases} \frac{1}{2}(t-a)^2, & t \in [a, \alpha_1) \\ \frac{1}{2}\left(t - \frac{a+b}{2}\right)^2 + \frac{1}{2}(b-a)\left(\alpha_1 - \frac{3a+b}{4}\right), & t \in [\alpha_1, \alpha_2) \\ \frac{1}{2}(t-b)^2, & t \in [\alpha_2, b]. \end{cases}$$

Then, from the definition of $K^{'}(\alpha_1,\alpha_2,t)$ and integration by parts of the Riemann-Stieltjes integral it follows that:

$$\int_a^b K^{'}(\alpha_1,\alpha_2,t)\,df^{'}(t) = -\int_a^b K(\alpha_1,\alpha_2,t)\,df(t).$$

Now, by using Lemma 2.2 for $p(x) = K(\alpha_1,\alpha_2,t)$ and $\upsilon(x) = f(x)$, we get:

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}\left[f(\alpha_1)+f(\alpha_2)\right]\right| \leq \sup_{t\in[a,b]}\left|K(\alpha_1,\alpha_2,t)\right| \bigvee_a^b(f). \tag{5.47}$$

It can be easily calculated that

$$\sup_{t\in[a,b]}\left|K(\alpha_1,\alpha_2,t)\right| = \max\left\{\alpha_1-a, \frac{a+b}{2}-\alpha_1, \alpha_2-\frac{a+b}{2}, b-\alpha_2\right\}. \tag{5.48}$$

Also, by using Lemma 2.2 for $p(x) = K^{'}(\alpha_1,\alpha_2,t)$ and $\upsilon(x) = f^{'}(x)$, we obtain:

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}\left[f(\alpha_1)+f(\alpha_2)\right]\right| \leq \sup_{t\in[a,b]}\left|K^{'}(\alpha_1,\alpha_2,t)\right| \bigvee_a^b\left(f^{'}\right). \tag{5.49}$$

Calculating,

$$\begin{aligned}&\sup_{t\in[a,b]}\left|K^{'}(\alpha_1,\alpha_2,t)\right| \\ = \;& \frac{1}{2}\max\left\{(\alpha_1-a)^2, (b-a)\left(\alpha_1-\frac{3a+b}{4}\right)\right. \\ &\left. +\left(\alpha_1-\frac{a+b}{2}\right)^2, (b-a)\left(\frac{3a+b}{4}-\alpha_1\right)\right\}\end{aligned} \tag{5.50}$$

Using (5.47)-(5.50), we obtain the required inequalities. ■

Corollary 5.2 *Let $f^{(n-1)}:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $n\in\{1,2\}$. Then, we have the inequality:*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}\left[f(\alpha_1)+f(\alpha_2)\right]\right| \leq \min_{n\in\{1,2\}}\left\{M_n\bigvee_a^b\left(f^{(n-1)}\right)\right\}, \tag{5.51}$$

where M_1 and M_2 be defined by (5.45).

Remark 5.9 *If we choose in (5.44)* $\alpha_1 = \alpha a + (1-\alpha) b$ *and* $\alpha_2 = (1-\alpha) a + \alpha b$*, then we get (5.43). It shows that (5.44) generalizes (5.43).*

Remark 5.10 *It may be noted that (5.51) can generate some Newton-Cotes formulae as special cases which are given as follows:*

(i) *For* $\alpha_1 = a$ *and* $\alpha_2 = b$ *in (5.51), we obtain:*

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{2}\left[f(a)+f(b)\right] \right|$$

$$\leq \min_{n\in\{1,2\}} \left\{ M_n \bigvee_a^b \left(f^{(n-1)}\right) \right\}, \tag{5.52}$$

where

$$M_1 = \frac{(b-a)}{2}, \; M_2 = \frac{(b-a)^2}{8}.$$

(ii) *For* $\alpha_1 = \alpha_2 = \frac{a+b}{2}$ *in (5.51), we obtain:*

$$\left| \int_a^b f(t)\,dt - (b-a) f\left(\frac{a+b}{2}\right) \right|$$

$$\leq \min_{n\in\{1,2\}} \left\{ M_n \bigvee_a^b \left(f^{(n-1)}\right) \right\}, \tag{5.53}$$

where

$$M_1 = \frac{(b-a)}{2}, \; M_2 = \frac{(b-a)^2}{8}.$$

(iii) *For* $\alpha_1 = \frac{2a+b}{3}$ *and* $\alpha_2 = \frac{a+2b}{3}$ *in (5.51), we obtain:*

$$\left| \int_a^b f(t)\,dt - \frac{(b-a)}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right] \right|$$

$$\leq \min_{n\in\{1,2\}} \left\{ M_n \bigvee_a^b \left(f^{(n-1)}\right) \right\}, \tag{5.54}$$

where

$$M_1 = \frac{(b-a)}{3}, \; M_2 = \frac{(b-a)^2}{18}.$$

(iv) *For $\alpha_1 = \frac{3a+b}{4}$ and $\alpha_2 = \frac{a+3b}{4}$ in (5.51), we obtain:*

$$\left| \int_a^b f(t)\,dt - \frac{(b-a)}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right]\right|$$

$$\leq \min_{n\in\{1,2\}} \left\{ M_n \bigvee_a^b \left(f^{(n-1)}\right)\right\}, \tag{5.55}$$

where

$$M_1 = \frac{(b-a)}{4},\ M_2 = \frac{(b-a)^2}{32}.$$

Corollary 5.3 *Let $f^{(n)}$ exists and is integrable on $[a,b]$ and for $n \in \{1,2\}$*

$$\left\| f^{(n)} \right\|_1 := \int_a^b \left| f^{(n)}(t)\right| dt < \infty.$$

Then, we have the inequality:

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{2}\left[f(\alpha_1) + f(\alpha_2)\right]\right|$$

$$\leq \min_{n\in\{1,2\}} \left\{ M_n \left\| f^{(n)}\right\|_1 \right\}, \tag{5.56}$$

where M_1 and M_2 be defined by (5.45).

Theorem 5.12 *Let $f^{(n-1)}$be L_n-Lipschitzian functions for $n \in \{1,2\}$. Then, we have the inequality:*

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{2}\left[f(\alpha_1) + f(\alpha_2)\right]\right|$$

$$\leq B_n L_n, \tag{5.57}$$

where

$$B_1 = \frac{(b-a)^2}{8} + \left(\alpha_1 - \frac{3a+b}{4}\right)^2 + \left(\alpha_2 - \frac{a+3b}{4}\right)^2, \tag{5.58}$$

and

$$B_2 = \begin{cases} \frac{1}{3}(\alpha_1-a)^3 + \frac{2}{3}(b-a)^{\frac{3}{2}}\left(\frac{3a+b}{4}-\alpha_1\right)^{\frac{3}{2}}, & \alpha_1 < \frac{3a+b}{4} \\ \frac{1}{3}(\alpha_1-a)^3 & \\ +\frac{2}{3}\left(\frac{a+b}{2}-\alpha_1\right)\left(\frac{1}{2}(\alpha_1-a)^2 + (b-a)\left(\alpha_1 - \frac{3a+b}{4}\right)\right), & \alpha_1 \geq \frac{3a+b}{4}, \end{cases} \tag{5.59}$$

provided that

$$\alpha_2 = (a+b) - \alpha_1.$$

Proof. Applying modulus and then by using Lemma 2.1 for $g(t) = K(\alpha_1, \alpha_2, t)$ and $\upsilon(t) = f(t)$ on (5.46), we obtain:

$$\left|\frac{b-a}{2}\left(f(\alpha_1) + f(\alpha_2)\right) - \int_a^b f(t)\,dt\right| \le L_1 \int_a^b \left|K(\alpha_1, \alpha_2, t)\right| dt. \tag{5.60}$$

Simple calculation shows that:

$$\begin{aligned} &\int_a^b \left|K(\alpha_1, \alpha_2, t)\right| dt \\ = \ &\frac{(b-a)^2}{8} + \left(\alpha_1 - \frac{3a+b}{4}\right)^2 + \left(\alpha_2 - \frac{a+3b}{4}\right)^2. \end{aligned} \tag{5.61}$$

In the similar way, applying Lemma 2.1 for $g(t) = K'(\alpha_1, \alpha_2, t)$ and $\upsilon(t) = f'(t)$, we have:

$$\left|\frac{b-a}{2}\left(f(\alpha_1) + f(\alpha_2)\right) - \int_a^b f(t)\,dt\right| \le L_2 \int_a^b \left|K'(\alpha_1, \alpha_2, t)\right| dt. \tag{5.62}$$

Now, by the definition of $K'(\alpha_1, \alpha_2, t)$ it follows that:

$$\begin{aligned} &\int_a^b \left|K'(\alpha_1, \alpha_2, t)\right| dt \\ = \ &\frac{1}{2}\int_a^{\alpha_1} (t-a)^2\,dt + \frac{1}{2}\int_{\alpha_2}^b (t-b)^2\,dt \\ &+\frac{1}{2}\int_{\alpha_1}^{\alpha_2} \left|\left(t - \frac{a+b}{2}\right)^2 + (b-a)\left(\alpha_1 - \frac{3a+b}{4}\right)\right| dt. \end{aligned}$$

Consider

$$I = \frac{1}{2}\int_{\alpha_1}^{\alpha_2} \left|\left(t - \frac{a+b}{2}\right)^2 + (b-a)\left(\alpha_1 - \frac{3a+b}{4}\right)\right| dt.$$

Here two cases arise:

Case 1. When $\alpha_1 > \frac{3a+b}{4}$. Then,

$$I = \frac{1}{2}\int_{\alpha_1}^{\alpha_2} \left(\left(t - \frac{a+b}{2}\right)^2 + (b-a)\left(\alpha_1 - \frac{3a+b}{4}\right)\right) dt.$$

Hence,

$$\begin{aligned}
&\int_a^b \left|K'(\alpha_1,\alpha_2,t)\right| dt \\
&= \frac{1}{3}(\alpha_1-a)^3 \\
&+\frac{2}{3}\left(\frac{a+b}{2}-\alpha_1\right)\left(\frac{1}{2}(\alpha_1-a)^2+(b-a)\left(\alpha_1-\frac{3a+b}{4}\right)\right). \quad (5.63)
\end{aligned}$$

Case 2. When $\alpha_1 < \frac{3a+b}{4}$. Then,

$$\begin{aligned}
I &= -\frac{1}{2}\int_{\alpha_1}^{\frac{a+b+\sqrt{(b-a)(3a+b-4\alpha_1)}}{2}} \left(\left(t-\frac{a+b}{2}\right)^2+(b-a)\left(\alpha_1-\frac{3a+b}{4}\right)\right)dt \\
&+\frac{1}{2}\int_{\frac{a+b+\sqrt{(b-a)(3a+b-4\alpha_1)}}{2}}^{\alpha_2} \left(\left(t-\frac{a+b}{2}\right)^2+(b-a)\left(\alpha_1-\frac{3a+b}{4}\right)\right)dt
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_a^b \left|K'(\alpha_1,\alpha_2,t)\right| dt \\
&= \frac{1}{3}(\alpha_1-a)^3+\frac{1}{12}(b-a)^{\frac{3}{2}}(3a+b-4\alpha_1)^{\frac{3}{2}}. \quad (5.64)
\end{aligned}$$

Therefore, from (5.60)-(5.64), we get the required inequalities. ■

Corollary 5.4 *Let $f^{(n-1)} \in C^n[a,b]$ for $n \in \{1,2\}$, then we have:*

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt-\frac{b-a}{2}\left[f(\alpha_1)+f(\alpha_2)\right]\right| \\
&\leq B_n\left\|f^{(n)}\right\|_\infty, \quad (5.65)
\end{aligned}$$

where B_1 and B_2 are defined by (5.59) and

$$\left\|f^{(n)}\right\|_\infty=\sup_{t\in[a,b]}|f(t)|<\infty.$$

Remark 5.11 *For $\alpha_1=\frac{3a+b}{4}$ and $\alpha_2=\frac{a+3b}{4}$, $n=2$ in (5.57) and (5.59), then we get:*

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt-\frac{b-a}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\right| \\
&\leq \frac{1}{96}L_2(b-a)^3, \quad (5.66)
\end{aligned}$$

which is a more generalized form of the two-point quadrature formula presented in ([55], Corollary 6).

5.3 Conclusion

By the use of Ostrowski type inequalities, we, in this chapter, have presented estimates for Newton-Cotes formulae.

In Section 5.1, a four-point generalized optimal quadrature rule is obtained which gives better error bound than the Simpson's $\frac{3}{8}$ and Lobatto type quadrature rules. The function involved is twice differentiable with bounded second derivative.

In Section 5.2, we have taken into account construction of one-point and two-point Newton-Cotes formulae of open and closed type for functions of bounded variations and for Lipschitzian functions. The inequalities are obtained for first and twice differentiable mappings. These inequalities generalize the results obtained in [50] and [55]. Remark 5.9 and 5.11 reveal this fact.

Later, in chapter 8, it has also been shown that such specially derived quadrature rules in the sense of can be applied to obtain iterative algorithms for solving non-linear equations.

Chapter 6

Weighted Ostrowski inequality for a continuous random variable

In this chapter, motivated and inspired by the results of ([36], Chapter 7), [10] and [33] (see also[8], Chapter 1), we have obtained some weighted Ostrowski type inequalities for a continuous random variable.

6.1 Weighted Ostrowski type inequality for a random variable whose probability density function belongs to $L_\infty[a,b]$

6.1.1 Introduction

The main aim of this section is to develop weighted Ostrowski type inequality for continuous random variables whose probability density functions are in $L_\infty[a,b]$. An application for a beta random variable is also given.

6.1.2 Main Results

Let the weight $\omega : [a,b] \to [0,\infty)$ be non-negative, integrable and

$$\int_a^b \omega(t)\,dt < \infty.$$

The domain of ω is finite and ω may vanish at boundary points. We denote the zero moment as

$$m(a,b) = \int_a^b \omega(t)dt.$$

We also know that expectation of any function $\phi(X)$ of the random variable X is given by

$$E[\phi(X)] = \int_a^b \phi(t)\, dF(t). \tag{6.1}$$

Taking $\phi(X) = \int \omega(X)\, dX$ as taken in [85], then from (6.1) and integration by parts, we have:

$$\begin{aligned} E_W &= E\left[\int \omega(X)\, dX\right] \\ &= \int_a^b \left(\int \omega(t)\, dt\right) dF(t) \\ &= W(b) - \int_a^b \omega(t) F(t)\, dt, \end{aligned} \tag{6.2}$$

where

$$W(b) = \left[\int \omega(t)\right]_{t=b}.$$

Also, we define

$$\begin{aligned} M_{a,x} &= \int_a^x m(a,t)\, dt,\ M_{x,b} = \int_x^b m(a,t)\, dt, \\ M'_{a,x} &= \int_a^x m(t,b)\, dt,\ M'_{x,b} = \int_x^b m(t,b)\, dt, \\ M_{a,b} &= M_{a,x} + M_{x,b} = \int_a^b m(a,t)\, dt, \\ M'_{a,b} &= M'_{a,x} + M'_{x,b} = \int_a^b m(t,b)\, dt, \end{aligned}$$

$$M_{a,b} + M'_{a,b} = \int_a^b m(a,b)\, dt. \tag{6.3}$$

Then the following theorem holds:

Theorem 6.1 *Let X be a continuous random variable with probability density function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}_+$ and with the cumulative distribution function $F(x) = \Pr(X \le x)$. If $f \in L_\infty[a,b]$ and $\|f\|_\infty := \sup_{t\in(a,b)} |f(t)| < \infty$, then we have the* *inequality:*

$$\left| \Pr(X \le x) - \frac{W(b) - E_W}{m(a,b)} \right| \le \frac{\|f\|_\infty}{m(a,b)} \left(M_{a,x} + M'_{x,b} \right), \tag{6.4}$$

or equivalently,

$$\left| 1 - \Pr(X \ge x) - \frac{W(b) - E_W}{m(a,b)} \right| \le \frac{\|f\|_\infty}{m(a,b)} \left(M_{a,x} + M'_{x,b} \right),$$

where

$$\begin{aligned} M_{a,x} &= \int_a^x m(a,t)\,dt, \\ M'_{x,b} &= \int_x^b m(t,b)\,dt, \end{aligned}$$

as defined above for all $x \in [a,b]$.

Proof. Consider the kernel $p_w : [a,b]^2 \to \mathbb{R}$ (see [36], Chapter 7) given by:

$$p_w(x,t) = \begin{cases} \int_a^t \omega(u)\,du, & if\ t \in [a,x] \\ \int_b^t \omega(u)\,du, & if\ t \in (x,b]. \end{cases}$$

Then, the Riemann-Stieltjes integral $\int_a^b p_w(x,t)\,dF(t)$ exists for any $x \in [a,b]$ and the following identity holds:

$$\int_a^b p_w(x,t)\,dF(t) = m(a,b)\,F(x) - \int_a^b \omega(t)\,F(t)\,dt. \tag{6.5}$$

Using (6.2) and (6.5), we get,

$$m(a,b)\,F(x) + E_W - W(b) = \int_a^b p_w(x,t)\,dF(t). \tag{6.6}$$

As shown in [10], if $p : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ and $\upsilon : [a,b] \to \mathbb{R}$ is L-Lipschitzian (with Lipschitz constant L), then

$$\left| \int_a^b p(x)\,d\upsilon(x) \right| \le L \int_a^b |p(x)|\,dx. \tag{6.7}$$

Since, for any $x, y \in [a,b]$

$$|F(x) - F(y)| \le \left| \int_x^y f(t)\,dt \right| \le \|f\|_\infty |x-y|.$$

Then, by using (6.7), we obtain:

$$\begin{aligned}
&\left|\int_a^b p_w(x,t)\,dF(t)\right| \\
\leq\; & \|f\|_\infty \int_a^b |p_w(x,t)|\,dt \\
=\; & \|f\|_\infty \left(\int_a^x \left(\int_a^t w(u)\,du\right) dt + \int_x^b \left(\int_t^b w(u)\,du\right) dt\right) \\
=\; & \|f\|_\infty \left(\int_a^x m(a,t)\,dt + \int_x^b m(t,b)\,dt\right),
\end{aligned}$$

implies

$$\left|\int_a^b p_w(x,t)\,dF(t)\right| \leq \|f\|_\infty \left(M_{a,x} + M'_{x,b}\right), \tag{6.8}$$

where

$$\begin{aligned}
m(a,t) &= \int_a^t w(u)\,du,\; M_{a,x} = \int_a^x m(a,t)\,dt, \\
m(t,b) &= \int_t^b w(u)\,du,\; M'_{x,b} = \int_a^b m(t,b)\,dt.
\end{aligned}$$

From (6.6) and (6.8), we have (6.4) and the second inequality follows directly from (6.4) by using

$$\Pr(X \leq x) = 1 - \Pr(X \geq x).$$

■

Remark 6.1 *Choosing $\omega(t) = 1$ in (6.4), we have the classical Ostrowski inequality for random variables whose probability density function belongs to $L_\infty[a,b]$. In this case, we have*

$$m(a,b) = \int_a^b \omega(t)\,dt = \int_a^b dt = (b-a),$$

and

$$\begin{aligned}
M_{a,x} + M'_{x,b} &= \int_a^x \left(\int_a^t du\right) dt + \int_x^b \left(\int_t^b du\right) dt \\
&= (b-a)^2 \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right].
\end{aligned}$$

Thus, (6.4) reduces to classical Ostrowski inequality

$$\left|\Pr(X \leq x) - \frac{b - E(X)}{b - a}\right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left\|f\right\|_\infty,$$

for all $x \in [a, b]$.

Corollary 6.1 *Under the assumptions of Theorem 6.1, we have the double inequality:*

$$W(b) - M'_{a,b}\left\|f\right\|_\infty \leq E_W \leq W(b) - m(a,b) + M_{a,b}\left\|f\right\|_\infty. \tag{6.9}$$

Proof. Choosing $x = a$ in (6.4), we obtain:

$$\left|W(b) - E_W\right| \leq M'_{a,b}\left\|f\right\|_\infty.$$

This implies

$$W(b) - E_W \leq M'_{a,b}\left\|f\right\|_\infty,$$

or

$$W(b) - M'_{a,b}\left\|f\right\|_\infty \leq E_W,$$

which proves left side of the inequality in (6.9).

Similarly, choosing $x = b$ in (6.4), we have:

$$\left|1 - \frac{W(b) - E_W}{m(a,b)}\right| \leq \frac{\left\|f\right\|_\infty}{m(a,b)} M_{a,b},$$

which gives

$$m(a,b) - W(b) + E_W \leq M_{a,b}\left\|f\right\|_\infty,$$

or

$$E_W \leq W(b) - m(a,b) + M_{a,b}\left\|f\right\|_\infty,$$

which proves right side of the inequality (6.9). ■

Remark 6.2 *Choosing* $\omega(t) = 1$ *in (6.9) gives us the inequality*

$$b - \frac{1}{2}(b-a)^2\left\|f\right\|_\infty \leq E(X) \leq a + \frac{1}{2}(b-a)^2\left\|f\right\|_\infty,$$

which was proved in [10] as Corollary 2.2.

Remark 6.3 *Let us define*

$$\begin{aligned} F_{\omega}(x) &= \int_{a}^{x} \omega(t) f(t)\, dt, \\ F_{\omega}(y) &= \int_{a}^{y} \omega(t) f(t)\, dt. \end{aligned}$$

It can be easily seen that

$$F_{\omega}(b) - F_{\omega}(a) = \int_{a}^{b} \omega(t) f(t)\, dt,$$

implies

$$\int_{x}^{y} F_{\omega}'(t)\, dt = \int_{x}^{y} \omega(t) f(t)\, dt,$$

which further implies

$$\int_{a}^{b} F_{\omega}'(t)\, dt = \int_{a}^{b} \omega(t) f(t)\, dt.$$

Now since $F_{\omega}(a) = 0$. We, therefore have:

$$\begin{aligned} F_{\omega}(b) &= \int_{a}^{b} \omega(t) f(t)\, dt \\ &\leq \sup_{t \in [a,b]} |f(t)| \int_{a}^{b} \omega(t)\, dt \\ &= \|f\|_{\infty} m(a,b), \end{aligned}$$

implies

$$\|f\|_{\infty} \geq \frac{F_{\omega}(b)}{m(a,b)}.$$

We assume that $\|f\|_{\infty}$ is not so large, say

$$\|f\|_{\infty} \leq \frac{2F_{\omega}(b)}{m(a,b)}, \tag{6.10}$$

then

$$\begin{aligned} E_W &\geq W(b) - M_{a,b}' \|f\|_{\infty} \\ &\geq W(b) - \frac{2F_{\omega}(b)}{m(a,b)} M_{a,b}', \end{aligned}$$

and

$$\begin{aligned} E_W &\leq W(b) - m(a,b) + M_{a,b}\|f\|_\infty \\ &\leq W(b) - m(a,b) + \frac{2F_\omega(b)}{m(a,b)}M_{a,b}. \end{aligned}$$

Thus

$$W(b) - \frac{2F_\omega(b)}{m(a,b)}M'_{a,b} \leq E_W \leq W(b) - m(a,b) + \frac{2F_\omega(b)}{m(a,b)}M_{a,b}. \tag{6.11}$$

We observe that the inequality

$$W(b) - M'_{a,b}\|f\|_\infty \leq E_W \leq W(b) - m(a,b) + M_{a,b}\|f\|_\infty,$$

is sharper than the inequality (6.11), when (6.10) holds.

Remark 6.4 *Choosing $\omega(t) = 1$ in (6.11) gives us the inequalities (2.8) and (2.9) in [10].*

Corollary 6.2 *With the above assumptions, we have:*

$$\begin{aligned} -M'_{a,b}\left(\|f\|_\infty - \frac{W(b)-W(a)}{2M'_{a,b}}\right) &\leq E_W - \frac{W(a)+W(b)}{2} \\ &\leq M_{a,b}\left(\|f\|_\infty - \frac{W(b)-W(a)}{2M_{a,b}}\right). \end{aligned} \tag{6.12}$$

Also

$$\left|E_W - \frac{W(a)+W(b)}{2}\right| \leq M_{a,b}\left(\|f\|_\infty - \frac{W(b)-W(a)}{2M_{a,b}}\right), \tag{6.13}$$

when

$$M'(a,b) = M(a,b). \tag{6.14}$$

Proof. From the inequality (6.9), we have

$$\begin{aligned} \frac{W(b)-W(a)}{2} - M'_{a,b}\|f\|_\infty &\leq E_W - \frac{W(a)+W(b)}{2} \\ &\leq \frac{W(b)-W(a)}{2} - m(a,b) + M_{a,b}\|f\|_\infty, \end{aligned}$$

implies

$$\begin{aligned} \frac{W(b)-W(a)}{2} - M'_{a,b}\|f\|_\infty &\leq E_W - \frac{W(a)+W(b)}{2} \\ &\leq \frac{W(b)-W(a)}{2} - m(a,b) + M_{a,b}\|f\|_\infty \\ &\leq -\frac{W(b)-W(a)}{2} + M_{a,b}\|f\|_\infty, \end{aligned}$$

as $m(a,b)=\int\limits_a^b \omega(u)\,du = W(b)-W(a)$.

This gives (6.12). Moreover, if (6.14) holds then

$$\left|E_W - \frac{W(a)+W(b)}{2}\right| \le M_{a,b}\left(\|f\|_\infty - \frac{W(b)-W(a)}{2M_{a,b}}\right).$$

This corollary helps in finding a sufficient condition in terms of $\|f\|_\infty$, for the expectation E_W to be close to the point $\frac{W(a)+W(b)}{2}$. ■

Corollary 6.3 *With the above assumptions, we have:*

$$-M'_{a,b}\left(\|f\|_\infty - \frac{W(b)-\frac{a+b}{2}}{M'_{a,b}}\right) \le E_W - \frac{a+b}{2}$$
$$\le M_{a,b}\left(\|f\|_\infty - \frac{W(b)-\frac{a+b}{2}}{M_{a,b}}\right), \tag{6.15}$$

if

$$m(a,b) = 2W(b)-(a+b).$$

Also, if (6.14) holds then

$$\left|E_W - \frac{a+b}{2}\right| \le M_{a,b}\left(\|f\|_\infty - \frac{W(b)-\frac{a+b}{2}}{M_{a,b}}\right). \tag{6.16}$$

Proof. From the inequality (6.9), we have

$$W(b)-\frac{a+b}{2} - M'_{a,b}\|f\|_\infty \le E_W - \frac{a+b}{2}$$
$$\le W(b)-\frac{a+b}{2} - m(a,b) + M_{a,b}\|f\|_\infty,$$

implies

$$\begin{aligned}
-M'_{a,b}\left(\|f\|_\infty - \frac{W(b)-\frac{a+b}{2}}{M'_{a,b}}\right) &\le E_W - \frac{a+b}{2} \\
&\le M_{a,b}\left(\|f\|_\infty + \frac{W(b)-\frac{a+b}{2}-m(a,b)}{M_{a,b}}\right) \\
&\le M_{a,b}\left(\|f\|_\infty - \frac{W(b)-\frac{a+b}{2}}{M_{a,b}}\right),
\end{aligned}$$

if $m(a,b)=2W(b)-(a+b)$.

This further gives

$$\left|E_W - \frac{a+b}{2}\right| \le M_{a,b}\left(\|f\|_\infty - \frac{W(b)-\frac{a+b}{2}}{M_{a,b}}\right),$$

when

$$M_{a,b} = M'_{a,b}.$$

This corollary helps in finding a condition in terms of $\|f\|_\infty$, for the expectation E_W to be close to the midpoint $\frac{a+b}{2}$ of the interval $[a,b]$. ■

Remark 6.5 *If we choose $\omega(t) = 1$ in (6.13) or (6.16), then we get the inequality (2.10) in [10].*

Remark 6.6 *It may be observed that for $\varepsilon > 0$, if*

$$\|f\|_\infty \leq \frac{\varepsilon}{M_{a,b}} - \frac{W(b) - \frac{a+b}{2} - m(a,b)}{M_{a,b}},$$

then

$$E_W - \frac{a+b}{2} \leq \varepsilon.$$

Moreover, for $\varepsilon > 0$

$$E_W - \frac{a+b}{2} \geq -\varepsilon,$$

if

$$\|f\|_\infty \leq \frac{\varepsilon}{M'_{a,b}} + \frac{W(b) - \frac{a+b}{2}}{M'_{a,b}}.$$

Obviously the two definitions of $\|f\|_\infty$ coincides when

$$\begin{aligned} M'_{a,b} &= M_{a,b}, \\ m(a,b) &= 2W(b) - (a+b), \end{aligned}$$

and therefore with these conditions

$$\left| E_W - \frac{a+b}{2} \right| \leq \varepsilon.$$

Corollary 6.4 *Let X and f be defined as above, then*

$$\left| \Pr\left(X \leq \frac{a+b}{2} \right) - \frac{W(b) - W(a)}{2m(a,b)} \right|$$

$$\begin{aligned} &\leq \frac{\|f\|_\infty}{m(a,b)} \left(M_{a,\frac{a+b}{2}} + M'_{\frac{a+b}{2},b} \right) + \frac{1}{m(a,b)} \left| E_W - \frac{W(a)+W(b)}{2} \right| \\ &\leq \frac{\|f\|_\infty}{m(a,b)} \left(M_{a,b} + M_{a,\frac{a+b}{2}} + M'_{\frac{a+b}{2},b} \right) - \frac{W(b) - W(a)}{2m(a,b)}, \qquad (6.17) \end{aligned}$$

if (6.14) holds.

Proof. If we choose $x = \frac{a+b}{2}$ in (2.4), then we have

$$\left|\Pr\left(X \le \frac{a+b}{2}\right) - \frac{W(b) - E_W}{m(a,b)}\right| \le \frac{\|f\|_\infty}{m(a,b)}\left(M_{a,\frac{a+b}{2}} + M'_{\frac{a+b}{2},b}\right).$$

Thus, the cited inequality can be written as:

$$\left|\Pr(X \le \frac{a+b}{2}) - \frac{W(b) - W(a)}{2m(a,b)} + \frac{1}{m(a,b)}\left(E_W - \frac{W(a) + W(b)}{2}\right)\right|$$
$$\le \frac{\|f\|_\infty}{m(a,b)}\left(M_{a,\frac{a+b}{2}} + M'_{\frac{a+b}{2},b}\right). \tag{6.18}$$

Using triangular inequality,

$$\left|\Pr(X \le \frac{a+b}{2}) - \frac{W(b) - W(a)}{2m(a,b)}\right|$$
$$= \left|\Pr(X \le \frac{a+b}{2}) - \frac{W(b) - W(a)}{2m(a,b)} + \frac{1}{m(a,b)}(E_W - \frac{W(a) + W(b)}{2})\right.$$
$$\left. - \frac{1}{m(a,b)}(E_W - \frac{W(a) + W(b)}{2})\right|$$
$$\le \left|\Pr(X \le \frac{a+b}{2}) - \frac{W(b) - W(a)}{2m(a,b)} + \frac{1}{m(a,b)}(E_W - \frac{W(a) + W(b)}{2})\right|$$
$$+ \left|\frac{1}{m(a,b)}(E_W - \frac{W(a) + W(b)}{2})\right|$$

Using (6.18) in the above inequality, we get the required inequality:

$$\left|\Pr(X \le \frac{a+b}{2}) - \frac{W(b) - W(a)}{2m(a,b)}\right|$$
$$\le \frac{\|f\|_\infty}{m(a,b)}\left(M_{a,\frac{a+b}{2}} + M'_{\frac{a+b}{2},b}\right) + \frac{1}{m(a,b)}\left|E_W - \frac{W(a) + W(b)}{2}\right|$$
$$\le \frac{\|f\|_\infty}{m(a,b)}\left(M_{a,b} + M_{a,\frac{a+b}{2}} + M'_{\frac{a+b}{2},b}\right) - \frac{W(b) - W(a)}{2m(a,b)},$$

if (6.14) holds. ■

Remark 6.7 *A similar result holds for* $\Pr(X \ge \frac{a+b}{2})$.

Remark 6.8 *If we assume that* f *is continuous on* (a,b), *then* F *is differentiable on* (a,b) *and we get in view of (6.4)-(6.6), a weighted Ostowski inequality (see [36], Chapter 7):*

$$\left|F(x) - \frac{1}{m(a,b)}\int_a^b \omega(t)F(t)dt\right|$$
$$\le \frac{\|f\|_\infty}{m(a,b)}\left(M_{a,x} + M_{x,b}\right),$$

for all $x \in [a,b]$.

6.1.3 Applications for a Beta Random Variable

A beta random variable X with parameter (p,q) has the probability density function

$$f(x;p,q) = \frac{x^{p-1}(1-x)^{q-1}}{\beta(p,q)};\ 0 < x < 1, \tag{6.19}$$

where

$$\Omega = \{(p,q) : p,q \geq 1\} \text{ and } \beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt,$$

and

$$E_{WB} = \frac{1}{\beta(p,q)}\int_0^1 \left(\int \omega(t)dt\right) t^{p-1}(1-t)^{q-1}dt. \tag{6.20}$$

We observe that

$$\|f(x;p,q)\|_\infty = \sup_{0<x<1}\left[\frac{x^{p-1}(1-x)^{q-1}}{\beta(p,q)}\right].$$

Assume that $p,q \geq 1$,then we find that

$$\frac{df(x;p,q)}{dx} = \frac{x^{p-2}(1-x)^{q-2}}{\beta(p,q)}\left[-(p+q-2)x + (p-1)\right].$$

We further observe that for $p,q > 1$, $\frac{df}{dx} = 0$ if and only if $x = x_0 = \frac{p-1}{p+q-2}$. We, therefore, have $\frac{df}{dx} > 0$ on $(0,x_0)$ and $\frac{df}{dx} < 0$ on $(x_0,1)$. Thus, we have:

$$\begin{aligned}\|f(x;p,q)\|_\infty &= \|f(x_0;p,q)\|_\infty \\ &= \frac{1}{\beta(p,q)}\left[\frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}\right].\end{aligned}$$

Proposition 6.1 *Let ω and F be as in Theorem 6.1 and X be a random variable with parameters $(p,q), p,q \geq 1$. Then, we have the inequalities:*

$$\begin{aligned}&\left|\Pr(X \leq x) - \frac{W(1) - E_{WB}}{m(0,1)}\right| \\ \leq\ &\frac{1}{\beta(p,q)m(0,1)}\frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}\left[M_{0,x} + M'_{x,1}\right],\end{aligned}$$

or equivalently,

$$\begin{aligned}&\left|1 - \Pr(X \geq x) - \frac{W(1) - E_{WB}}{m(0,1)}\right| \\ \leq\ &\frac{1}{\beta(p,q)m(0,1)}\frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}\left[M_{0,x} + M'_{x,1}\right],\end{aligned}$$

for all $x \in [0,1]$*, where* E_{WB} *is given by (6.20) and*

$$\begin{aligned} M_{0,x} &= \int_0^x \left(\int_0^t \omega(u)\, du \right) dt, \\ M'_{x,1} &= \int_x^1 \left(\int_t^1 \omega(u)\, du \right) dt, \end{aligned}$$

and

$$W(1) = \int \omega(t) dt \mid_{t=1} .$$

In particular, we have:

$$\begin{aligned} & \left| \Pr(X \le \frac{1}{2}) - \frac{W(1) - E_{WB}}{m(0,1)} \right| \\ \le & \frac{1}{\beta(p,q) m(0,1)} \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}} \left[M_{0,\frac{1}{2}} + M'_{\frac{1}{2},1} \right], \end{aligned}$$

or equivalently,

$$\begin{aligned} & \left| 1 - \Pr(X \ge \frac{1}{2}) - \frac{W(1) - E_{WB}}{m(0,1)} \right| \\ \le & \frac{1}{\beta(p,q) m(0,1)} \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}} \left[M_{0,\frac{1}{2}} + M'_{\frac{1}{2},1} \right]. \end{aligned}$$

6.2 Weighted Ostrowski type inequality for a random variable whose probability density function belongs to $L_p[a,b]$, $p > 1$

6.2.1 Introduction

The main aim of this section is to develop weighted Ostrowski type inequality for random variables whose probability density functions are in $L_p[a,b], p > 1$. An application for a beta random variable is also given.

6.2.2 Main Results

Let X be a continuous random variable with probability density function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}_+$ and with cumulative distribution function $F(x) = \Pr(X \le x)$. Also, let $f \in L_p[a,b]$. The weighted norm in $L_p(a,b)$ is defined as $\|f\|_p = \left(\int_a^b |f(t)|^p\, dt \right)^{\frac{1}{p}}$.

Let the weight $\omega : [a,b] \to [0,\infty)$ be as defined in Section 6.1.2 and let us define

$$m_q(a,b) = \left(\int_a^b |\omega(t)|^q \, dt \right)^{\frac{1}{q}}. \tag{6.21}$$

The expectation of any function $\phi(X)$ of the random variable X is defined by (6.1).

Also, we define

$$\begin{aligned} Q_{a,x}(x) &= \int_a^x m_q(t,x)\, dt, \; Q_{x,b}(x) = \int_x^b m_q(x,t)\, dt, \\ Q_{a,b}(x) &= Q_{a,x}(x) + Q_{x,b}(x) = \int_a^b |m_q(t,x)|\, dt. \end{aligned} \tag{6.22}$$

Then, the following inequality for random variable holds:

Theorem 6.2 *Let X, ω, f and F be as defined above. Then, if $f \in L_p[a,b]$, $p > 1$, then we have the inequality:*

$$\begin{aligned} &\left| \Pr(X \leq x) - \frac{W(b) - E_W}{m(a,b)} \right| \\ \leq \; &\frac{1}{m(a,b)} \max_{t \in [a,b]} \omega(t) \|f\|_p \left[Q_{a,x}(x) + Q_{x,b}(x) \right], \end{aligned} \tag{6.23}$$

where

$$\begin{aligned} Q_{a,x}(x) &= \int_a^x m_q(t,x)\, dt, \\ Q_{x,b}(x) &= \int_x^b m_q(x,t)\, dt, \end{aligned}$$

for all $x \in [a,b]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Now

$$F(x) = \int_a^x \omega(u) f(u) du, F(t) = \int_a^t \omega(u) f(u) du.$$

This gives

$$\begin{aligned} F(x) - F(t) &= \int_a^x \omega(u) f(u) du - \int_a^t \omega(u) f(u) du \\ &= -\left(\int_x^a \omega(u) f(u) du + \int_a^t \omega(u) f(u) du \right) \\ &= -\int_x^t \omega(u) f(u) du, \end{aligned}$$

implies

$$\begin{aligned} |F(x)-F(t)| &= \left|\int_x^t \omega(u)f(u)du\right| \\ &\leq \left|\int_x^t |\omega(u)|^q\,du\right|^{\frac{1}{q}} \left|\int_x^t |f(u)|^p\,du\right|^{\frac{1}{p}} \\ &\leq \left|\int_x^t |\omega(u)|^q\,du\right|^{\frac{1}{q}} \|f\|_p\,, \end{aligned} \tag{6.24}$$

for all $x \in [a,b], p>1, \frac{1}{p}+\frac{1}{q}=1$.

Now using (6.24), we obtain:

$$|F(x)-F(t)|\,\omega(t) \leq \omega(t)\left|\int_x^t |\omega(u)|^q\,du\right|^{\frac{1}{q}} \|f\|_p\,.$$

Integrating with respect to t over $[a,b]$, we have:

$$\begin{aligned} &\int_a^b |F(x)-F(t)|\,\omega(t)dt \\ &\leq \|f\|_p \int_a^b \omega(t)\left|\int_x^t |\omega(u)|^q\,du\right|^{\frac{1}{q}} dt \\ &= \|f\|_p \left[\int_a^x \omega(t)\left(\int_t^x |\omega(u)|^q\,du\right)^{\frac{1}{q}} dt + \int_x^b \omega(t)\left(\int_x^t |\omega(u)|^q\,du\right)^{\frac{1}{q}} dt\right] \end{aligned}$$

$$\begin{aligned} &= \|f\|_p \left[\int_a^x \omega(t)\left(\int_t^x |\omega(u)|^q\,du\right)^{\frac{1}{q}} dt + \int_x^b \omega(t)\left(\int_x^t |\omega(u)|^q\,du\right)^{\frac{1}{q}} dt\right] \\ &\leq \|f\|_p \left[\sup_{t\in[a,x]} \omega(t)\int_a^x \left(\int_t^x |\omega(u)|^q\,du\right)^{\frac{1}{q}} dt + \sup_{t\in(x,b]} \omega(t)\int_x^b \left(\int_x^t |\omega(u)|^q\,du\right)^{\frac{1}{q}} dt\right] \end{aligned}$$

$$\begin{aligned}
&\leq \max\left(\sup_{t\in[a,x]}\omega(t), \sup_{t\in(x,b]}\omega(t)\right)\|f\|_p \\
&\quad\times\left[\int_a^x\left(\int_t^x|\omega(u)|^q\,du\right)^{\frac{1}{q}}dt+\int_x^b\left(\int_x^t|\omega(u)|^q\,du\right)^{\frac{1}{q}}dt\right] \\
&= \max_{t\in[a,b]}\omega(t)\|f\|_p\left[\int_a^x m_q(t,x)\,dt+\int_x^b m_q(x,t)\,dt\right] \\
&= \max_{t\in[a,b]}\omega(t)\|f\|_p\left[Q_{a,x}(x)+Q_{x,b}(x)\right], \qquad (6.25)
\end{aligned}$$

where $Q_{a,x}(x)$ and $Q_{x,b}(x)$ are defined by (6.22).

Consider

$$\begin{aligned}
&\int_a^b(F(x)-F(t))\,\omega(t)dt \\
&= F(x)\int_a^b\omega(t)dt-\int_a^b\omega(t)F(t)dt \\
&= m(a,b)F(x)-\int_a^b\omega(t)F(t)dt.
\end{aligned}$$

This implies

$$\frac{1}{m(a,b)}\int_a^b(F(x)-F(t))\,\omega(t)dt=F(x)-\frac{1}{m(a,b)}\int_a^b\omega(t)F(t)dt,$$

or

$$\left|F(x)-\frac{1}{m(a,b)}\int_a^b\omega(t)F(t)dt\right|\leq\frac{1}{m(a,b)}\int_a^b|F(x)-F(t)|\,\omega(t)dt. \qquad (6.26)$$

Using (6.2) and (6.25) in (6.26), we get:

$$\begin{aligned}
&\left|\Pr(X\leq x)-\frac{W(b)-E_W}{m(a,b)}\right| \\
&\leq \frac{1}{m(a,b)}\max_{t\in[a,b]}\omega(t)\|f\|_p\left[Q_{a,x}(x)+Q_{x,b}(x)\right],
\end{aligned}$$

and the theorem is completely proved. ■

Remark 6.9 *A similar inequality can be deduced for* $\Pr(X\geq x)$.

Corollary 6.5 *Under the above assumptions, we have the double inequality:*

$$W(b)-\max_{t\in[a,b]}\omega(t)\|f\|_p\,Q_{a,b}(a)\leq E_W$$

$$\leq W(b) - m(a,b) + \max_{t\in[a,b]} \omega(t) \|f\|_p Q_{a,b}(b), \tag{6.27}$$

where

$$Q_{a,b}(a) = \int_a^b m_q(a,t)\,dt,$$

$$Q_{a,b}(b) = \int_a^b m_q(t,b)\,dt.$$

Proof. Choose $x = a$ in (6.23), to get:

$$\left| -\frac{W(b) - E_W}{m(a,b)} \right| \leq \frac{\max_{t\in[a,b]} \omega(t)}{m(a,b)} \|f\|_p Q_{a,b}(a).$$

This gives

$$W(b) - \max_{t\in[a,b]} \omega(t) \|f\|_p Q_{a,b}(a) \leq E_W,$$

which is equivalent to the first inequality in (2.9).

Also, by choosing $x = b$ in (6.23), we get:

$$\left| 1 - \frac{W(b) - E_W}{m(a,b)} \right| \leq \frac{\max\limits_{t\in[a,b]} \omega(t)}{m(a,b)} \|f\|_p Q_{a,b}(b),$$

or

$$\left| \frac{m(a,b) - W(b) + E_W}{m(a,b)} \right| \leq \frac{\max\limits_{t\in[a,b]} \omega(t)}{m(a,b)} \|f\|_p Q_{a,b}(b).$$

This gives

$$E_W \leq W(b) - m(a,b) + \max_{t\in[a,b]} \omega(t) \|f\|_p Q_{a,b}(b),$$

which is the right hand side of the inequality (6.27). ■

Remark 6.10 *Choosing $\omega(t) = 1$, in (6.27) gives us the inequality which was proved in [33]:*

$$b - \frac{q}{q+1} \|f\|_p (b-a)^{\frac{1}{q}+1} \leq E(X) \leq a + \frac{q}{q+1} \|f\|_p (b-a)^{\frac{1}{q}+1}.$$

Remark 6.11 *Let us define*

$$F_\omega(x) = \int_a^x \omega(t) f(t)dt, \;\; F_\omega(y) = \int_a^y \omega(t) f(t)dt.$$

It can be easily seen that

$$F_\omega(b) = \int_a^b \omega(t) f(t)dt \leq \left(\int_a^b |\omega(t)|^q \,dt \right)^{\frac{1}{q}} \|f\|_p,$$

which gives

$$\|f\|_p \geq \frac{F_\omega(b)}{m_q(a,b)}.$$

If we assume that $\|f\|_p$ is not too large, i.e., say

$$\|f\|_p \leq \frac{q+1}{q}\frac{F_\omega(b)}{m_q(a,b)}, \tag{6.28}$$

then

$$\begin{aligned} E_W &\geq W(b) - \max_{t\in[a,b]} \omega(t) \|f\|_p Q_{a,b}(a) \\ &\geq W(b) - \frac{q+1}{q}\frac{\left(\max\limits_{t\in[a,b]} \omega(t)\right) F_\omega(b)}{m_q(a,b)} Q_{a,b}(a), \end{aligned}$$

and

$$\begin{aligned} E_W &\leq W(b) - m(a,b) + \max_{t\in[a,b]} \omega(t) \|f\|_p Q_{a,b}(b) \\ &\leq W(b) - m(a,b) + \frac{q+1}{q}\frac{\left(\max\limits_{t\in[a,b]} \omega(t)\right) F_\omega(b)}{m_q(a,b)} Q_{a,b}(b). \end{aligned}$$

Thus,

$$W(b) - \frac{q+1}{q}\frac{\left(\max\limits_{t\in[a,b]} \omega(t)\right) F_\omega(b)}{m_q(a,b)} Q_{a,b}(a) \leq E_W$$

$$\leq W(b) - m(a,b) + \frac{q+1}{q}\frac{\left(\max\limits_{t\in[a,b]} \omega(t)\right) F_\omega(b)}{m_q(a,b)} Q_{a,b}(b). \tag{6.29}$$

We observe that the inequality

$$\begin{aligned} & W(b) - \max_{t\in[a,b]} \omega(t) \|f\|_p Q_{a,b}(a) \\ \leq\ & E_W \\ \leq\ & W(b) - m(a,b) + \max_{t\in[a,b]} \omega(t) \|f\|_p Q_{a,b}(b), \end{aligned}$$

is sharper than the inequality (6.29), when (6.28) holds.

Remark 6.12 *Choosing $\omega(t) = 1$, in (6.28) gives us the condition mentioned in [33].*

Corollary 6.6 *Under the above assumptions, we have the inequality:*

$$\begin{aligned}
& -\left(\max_{t\in[a,b]}\omega(t)\right)Q_{a,b}(a)\left(\|f\|_p-\frac{W(b)-W(a)}{2\left(\max_{t\in[a,b]}\omega(t)\right)Q_{a,b}(a)}\right)\\
\leq\ & E_W-\frac{W(a)+W(b)}{2}\\
\leq\ & \left(\max_{t\in[a,b]}\omega(t)\right)Q_{a,b}(b)\left(\|f\|_p-\frac{W(b)-W(a)}{2\left(\max_{t\in[a,b]}\omega(t)\right)Q_{a,b}(a)}\right). \qquad (6.30)
\end{aligned}$$

Moreover,

$$\begin{aligned}
& \left|E_W-\frac{W(a)+W(b)}{2}\right|\\
\leq\ & \max_{t\in[a,b]}\omega(t)\,Q_{a,b}(a)\left[\|f\|_p-\frac{W(b)-W(a)}{2\left(\max_{t\in[a,b]}\omega(t)\right)Q_{a,b}(a)}\right], \qquad (6.31)
\end{aligned}$$

provided that

$$Q_{a,b}(a)=Q_{a,b}(b). \qquad (6.32)$$

Proof. From inequality (6.27), we have:

$$\begin{aligned}
& \frac{W(b)-W(a)}{2}-\max_{t\in[a,b]}\omega(t)\,\|f\|_p\,Q_{a,b}(a)\\
\leq\ & E_W-\frac{W(a)+W(b)}{2}\\
\leq\ & \frac{W(b)-W(a)}{2}-m(a,b)+\max_{t\in[a,b]}\omega(t)\,\|f\|_p\,Q_{a,b}(b),
\end{aligned}$$

implies

$$\begin{aligned}
& -\left(\max_{t\in[a,b]}\omega(t)\right)Q_{a,b}(a)\left(\|f\|_p-\frac{W(b)-W(a)}{2\left(\max_{t\in[a,b]}\omega(t)\right)Q_{a,b}(a)}\right)\\
\leq\ & E_W-\frac{W(a)+W(b)}{2}\\
\leq\ & \left(\max_{t\in[a,b]}\omega(t)\right)Q_{a,b}(b)\left(\|f\|_p-\frac{W(b)-W(a)}{2\left(\max_{t\in[a,b]}\omega(t)\right)Q_{a,b}(b)}\right),
\end{aligned}$$

since,

$$m(a,b) = W(b) - W(a).$$

Also,

$$\left| E_W - \frac{W(a) + W(b)}{2} \right|$$
$$\leq \left(\max_{t\in[a,b]} \omega(t) \right) Q_{a,b}(a) \left(\|f\|_p - \frac{W(b) - W(a)}{2\left(\max_{t\in[a,b]} \omega(t) \right) Q_{a,b}(a)} \right),$$

if (6.32) holds. ■

This corollary helps in finding a condition in terms of $\|f\|_p$, for the expectation $E_W(X)$ to be close to the point $\frac{W(a)+W(b)}{2}$ and this completes the proof.

Corollary 6.7 *With the above assumptions, we have:*

$$-\left(\max_{t\in[a,b]} \omega(t) \right) Q_{a,b}(a) \left(\|f\|_p - \frac{W(b) - \frac{a+b}{2}}{\left(\max_{t\in[a,b]} \omega(t) \right) Q_{a,b}(a)} \right)$$
$$\leq E_W - \frac{a+b}{2}$$
$$\leq \left(\max_{t\in[a,b]} \omega(t) \right) Q_{a,b}(b) \left(\|f\|_p - \frac{W(b) - \frac{a+b}{2}}{\left(\max_{t\in[a,b]} \omega(t) \right) Q_{a,b}(b)} \right), \qquad (6.33)$$

if

$$m(a,b) = 2W(b) - (a+b).$$

Also, if (6.32) holds then

$$\left| E_W - \frac{a+b}{2} \right|$$
$$\leq \left(\max_{t\in[a,b]} \omega(t) \right) Q_{a,b}(a) \left(\|f\|_p - \frac{W(b) - \frac{a+b}{2}}{\left(\max_{t\in[a,b]} \omega(t) \right) Q_{a,b}(a)} \right). \qquad (6.34)$$

Proof. From the inequality (6.27), we have

$$W(b) - \frac{a+b}{2} - \max_{t\in[a,b]} \omega(t) \|f\|_p Q_{a,b}(a) \leq E_W - \frac{a+b}{2}$$
$$\leq W(b) - \frac{a+b}{2} - m(a,b) + \max_{t\in[a,b]} \omega(t) \|f\|_p Q_{a,b}(b),$$

implies

$$
\begin{aligned}
& -\max_{t\in[a,b]}\omega(t)\,Q_{a,b}(a)\left(\|f\|_p-\frac{W(b)-\frac{a+b}{2}}{\max\limits_{t\in[a,b]}\omega(t)\,Q_{a,b}(a)}\right) \\
\leq\ & E_W-\frac{a+b}{2} \\
\leq\ & \max_{t\in[a,b]}\omega(t)\,Q_{a,b}(b)\left(\|f\|_p+\frac{W(b)-\frac{a+b}{2}-m(a,b)}{\max\limits_{t\in[a,b]}\omega(t)\,Q_{a,b}(b)}\right) \\
\leq\ & \max_{t\in[a,b]}\omega(t)\,Q_{a,b}(b)\left(\|f\|_p-\frac{W(b)-\frac{a+b}{2}}{\max\limits_{t\in[a,b]}\omega(t)\,Q_{a,b}(b)}\right),
\end{aligned}
$$

if $m(a,b)=2W(b)-(a+b)$. This further gives

$$
\left|E_W-\frac{a+b}{2}\right|\leq\max_{t\in[a,b]}\omega(t)\,Q_{a,b}(a)\left(\|f\|_p-\frac{W(b)-\frac{a+b}{2}}{\max\limits_{t\in[a,b]}\omega(t)\,Q_{a,b}(a)}\right),
$$

when

$$
Q_{a,b}(a)=Q_{a,b}(b)\,.
$$

This corollary helps in finding a sufficient condition in terms of $\|f\|_p$, for the expectation E_W to be close to the midpoint $\frac{a+b}{2}$ of the interval $[a,b]$. ■

Remark 6.13 *If we choose $\omega(t)=1$ in (6.31) and (6.34), then we get the inequality proved in [33].*

Remark 6.14 *It may be observed that for $\varepsilon>0$, if*

$$
\|f\|_p\leq\frac{\varepsilon}{\max\limits_{t\in[a,b]}\omega(t)\,Q_{a,b}(b)}-\frac{W(b)-\frac{a+b}{2}-m(a,b)}{\max\limits_{t\in[a,b]}\omega(t)\,Q_{a,b}(b)},
$$

then

$$
E_W-\frac{a+b}{2}\leq\varepsilon.
$$

Moreover, for $\varepsilon>0$

$$
E_W-\frac{a+b}{2}\geq-\varepsilon,
$$

if

$$
\|f\|_p\leq\frac{\varepsilon}{\max\limits_{t\in[a,b]}\omega(t)\,Q_{a,b}(a)}+\frac{W(b)-\frac{a+b}{2}}{\max\limits_{t\in[a,b]}\omega(t)\,Q_{a,b}(a)}.
$$

Obviously the two definitions of $\|f\|_p$ coincides when

$$\begin{aligned} Q_{a,b}(a) &= Q_{a,b}(b), \\ m(a,b) &= 2W(b)-(a+b), \end{aligned}$$

and therefore with these conditions

$$\left|E_W - \frac{a+b}{2}\right| \leq \varepsilon.$$

Corollary 6.8 *Let X and f be defined as above, then*

$$\left|\Pr\left(X \leq \frac{a+b}{2}\right) - \frac{W(b)-W(a)}{2m(a,b)}\right|$$

$$\begin{aligned} &\leq \frac{\|f\|_p}{m(a,b)} \max_{t\in[a,b]} \omega(t) Q_{a,b}\left(\frac{a+b}{2}\right) + \frac{1}{m(a,b)}\left|E_W - \frac{W(a)+W(b}{2}\right| \\ &\leq \frac{\|f\|_p}{m(a,b)} \max_{t\in[a,b]} \omega(t)\left(Q_{a,b}\left(\frac{a+b}{2}\right) + Q_{a,b}(a)\right) - \frac{W(b)-W(a)}{2m(a,b)}, \quad (6.35) \end{aligned}$$

provided that (6.32) holds.

Proof. If we choose $x = \frac{a+b}{2}$ in (6.23), then we have

$$\begin{aligned} &\left|\Pr\left(X \leq \frac{a+b}{2}\right) - \frac{W(b)-E_W(X)}{m(a,b)}\right| \\ &\leq \frac{\|f\|_p}{m(a,b)} \max_{t\in[a,b]} \omega(t)\left(Q_{a,\frac{a+b}{2}}\left(\frac{a+b}{2}\right) + Q_{\frac{a+b}{2},b}\left(\frac{a+b}{2}\right)\right) \\ &= \frac{\|f\|_p}{m(a,b)} \max_{t\in[a,b]} \omega(t) Q_{a,b}\left(\frac{a+b}{2}\right). \end{aligned}$$

Thus, the cited inequality can be written as:

$$\begin{aligned} &\left|\Pr(X \leq \frac{a+b}{2}) - \frac{W(b)-W(a)}{2m(a,b)} + \frac{1}{m(a,b)}\left(E_W(X) - \frac{W(a)+W(b)}{2}\right)\right| \\ &\qquad \leq \frac{\|f\|_p}{m(a,b)} \max_{t\in[a,b]} \omega(t) Q_{a,b}\left(\frac{a+b}{2}\right). \quad (6.36) \end{aligned}$$

Using triangular inequality,

$$\left|\Pr(X \leq \frac{a+b}{2}) - \frac{W(b)-W(a)}{2m(a,b)}\right|$$

$$\begin{aligned} = &\left|\Pr(X \leq \frac{a+b}{2}) - \frac{W(b)-W(a)}{2m(a,b)} + \frac{1}{m(a,b)}(E_W(X) - \frac{W(a)+W(b}{2})\right. \\ &\left. - \frac{1}{m(a,b)}(E_W(X) - \frac{W(a)+W(b)}{2})\right| \end{aligned}$$

$$\leq \left|\Pr(X \leq \frac{a+b}{2}) - \frac{W(b)-W(a)}{2m(a,b)} + \frac{1}{m(a,b)}(E_W(X) - \frac{W(a)+W(b}{2})\right|$$
$$+\left|\frac{1}{m(a,b)}(E_W(X) - \frac{W(a)+W(b)}{2})\right|$$

Using (6.36) in the above inequality, we get the required inequality:

$$\begin{aligned} &\left|\Pr(X \leq \frac{a+b}{2}) - \frac{W(b)-W(a)}{2m(a,b)}\right| \\ \leq\ & \frac{\|f\|_p}{m(a,b)} \max_{t\in[a,b]} \omega(t)\, Q_{a,b}\left(\frac{a+b}{2}\right) + \frac{1}{m(a,b)}\left|E_W(X) - \frac{W(a)+W(b}{2}\right| \\ \leq\ & \frac{\|f\|_p}{m(a,b)} \max_{t\in[a,b]} \omega(t)\left(Q_{a,b}\left(\frac{a+b}{2}\right) + Q_{a,b}(a)\right) - \frac{W(b)-W(a)}{2m(a,b)}, \end{aligned}$$

if (6.32) holds. ■

Remark 6.15 *A similar result also holds for* $\Pr\left(X \geq \frac{a+b}{2}\right)$ *and the details are omitted.*

6.2.3 Application for a Beta Random Variable

A beta random variable X with parameter (s,t) has the probability density function

$$f(x;s,t) = \frac{x^{s-1}(1-x)^{t-1}}{\beta(s,t)}; 0 < x < 1,$$

where

$$\Omega = \{(s,t) : s,t \geq 1\} \text{ and } \beta(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1}dx.$$

We observe that for $p > 1$,

$$\begin{aligned} \|f(x;s,t)\|_p &= \frac{1}{\beta(s,t)}\left(\int_0^1 x^{p(s-1)}(1-x)^{p(t-1)}dx\right)^{\frac{1}{p}} \\ &= \frac{1}{\beta(s,t)}\left(\int_0^1 x^{p(s-1)+-1}(1-x)^{p(t-1)+1-1}dx\right)^{\frac{1}{p}} \\ &= \frac{1}{\beta(s,t)}\left[\beta\left(p(s-1)+1, p(t-1)+1\right)\right]^{\frac{1}{p}}, \end{aligned} \tag{6.37}$$

provided $p(s-1)+1,\ p(t-1)+1 > 0$, namely $s > 1-\frac{1}{p}$ and $t > 1-\frac{1}{p}$. Moreover,

$$E_{WB} = \frac{1}{\beta(s,t)}\int_0^1 \left(\int \omega(x)dx\right) x^{s-1}(1-x)^{t-1}dx. \tag{6.38}$$

The following proposition holds:

Proposition 6.2 *Let ω and F be as defined in Theorem 6.2 and X be a beta random variable with parameters (s,t), $s > 1-\frac{1}{p}, t > 1-\frac{1}{p}$. Then, we have the inequalities:*

$$\begin{aligned} & \left| \Pr(X \leq x) - \frac{W(1) - E_{WB}}{m(0,1)} \right| \\ \leq & \frac{1}{m(0,1)} \frac{\max\limits_{t\in[0,1]} \omega(t)}{\beta(s,t)} \\ & \times \left[\beta\left(p(s-1)+1, p(t-1)+1\right)\right]^{\frac{1}{p}} \left[Q_{0,x}(x) + Q_{x,1}(x)\right], \end{aligned}$$

for all $x \in [0,1]$, where E_{WB} is defined by (6.38) and

$$\begin{aligned} Q_{0,x}(x) &= \int_0^x m_q(t,x)\,dt, \\ Q_{x,1}(x) &= \int_x^1 m_q(x,t)\,dt, \end{aligned}$$

and

$$W(1) = \left[\int \omega(t)dt\right]_{t=1}$$

In particular, we have:

$$\begin{aligned} & \left| \Pr\left(X \leq \frac{1}{2}\right) - \frac{W(1) - E_{WB}}{m(0,1)} \right| \\ \leq & \frac{1}{m(0,1)} \frac{\max\limits_{t\in[0,1]} \omega(t)}{\beta(s,t)} \left[\beta\left(p(s-1)+1, p(t-1)+1\right)\right]^{\frac{1}{p}} \left[Q_{0,\frac{1}{2}}\left(\frac{1}{2}\right) + Q_{\frac{1}{2},1}\left(\frac{1}{2}\right)\right], \end{aligned}$$

where $W(1)$ and E_{WB} are defined as above and

$$\begin{aligned} Q_{0,\frac{1}{2}}\left(\frac{1}{2}\right) &= \int_0^{\frac{1}{2}} m_q\left(t,\frac{1}{2}\right)dt, \\ Q_{\frac{1}{2},1}\left(\frac{1}{2}\right) &= \int_{\frac{1}{2}}^1 m_q\left(\frac{1}{2},t\right)dt. \end{aligned}$$

6.3 Conclusion

In this chapter, some weighted Ostrowski type inequalities for a random variable have been obtained whose probability density functions belong to $\{L_p[a,b], p=\infty, p>1\}$. The inequalities obtained in this chapter recapture the inequalities of Ostrowski type for random variables given in [10] and [33]. Moreover, it may also be noted that these inequalities are also applicable to obtain expectations of random variables defined on infinite intervals in contrast to the previous results of this type. The inequalities for generalized beta random variables are also presented.

Chapter 7

Applications of Ostrowski type inequalities for probability density functions

7.1 A generalized Ostrowski type inequality for a random variable whose probability density function belongs to $L_\infty[a,b]$

We establish here an inequality of Ostrowski type for a continuous random variable whose probability density function belongs to $L_\infty[a,b]$, in terms of the cumulative distribution function and expectation. The inequality is then applied to generalized beta random variable.

7.1.1 Introduction

In [10], N. S. Barnett and S. S. Dragomir established the following version of Ostrowski type inequality for cumulative and probability distribution functions.

Theorem 7.1 *Let X be a continuous random variable with probability density function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}_+$ and with cumulative distribution function $F(x) = \Pr(X \leq x)$. If $f \in L_\infty [a,b]$ and$\|f\|_\infty := \sup_{t\in[a,b]} |f(t)| < \infty$, then we have the inequality:*

$$\begin{aligned} &\left|\Pr(X \leq x) - \frac{b - E(X)}{b-a}\right| \\ \leq\ & \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f\|_\infty, \end{aligned} \tag{7.1}$$

for all $x \in [a,b]$.

Equivalently,

$$\left|\Pr(X \geq x) - \frac{E(X)-a}{b-a}\right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f\|_\infty . \tag{7.2}$$

The constant $\frac{1}{4}$ *in (7.1) and (7.2) is sharp.*

In the following subsection, we establish a generalized Ostrowski type inequality for cumulative distribution function and expectation of a random variable. Applications for the generalized beta distribution are also given.

7.1.2 Main Results

The following theorem holds.

Theorem 7.2 *Let* X *and* F *be as defined above. Let* $f \in L_\infty[a,b]$ *and put* $\|f\|_\infty = \sup_{t\in[a,b]} f(t) < \infty$*. Then, we have the inequality:*

$$\left|(1-h)\Pr(X \leq x) + \frac{h}{2} - \frac{b-E(X)}{b-a}\right| \leq \left[\frac{1}{4}\left(h^2 + (1-h)^2\right) + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f\|_\infty , \tag{7.3}$$

or equivalently,

$$\left|(1-h)\Pr(X \geq x) + \frac{h}{2} - \frac{E(X)-a}{b-a}\right| \leq \left[\frac{1}{4}\left(h^2 + (1-h)^2\right) + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f\|_\infty , \tag{7.4}$$

for all $x \in [a + h\frac{b-a}{2}, b - h\frac{b-a}{2}]$ *and* $h \in [0,1]$.

Proof. As defined in [34], consider the kernel $p : [a,b]^2 \longrightarrow \mathbb{R}$ given by

$$p(x,t) = \begin{cases} t - \left(a + h\frac{b-a}{2}\right), & if\ t \in [a,x] \\ t - \left(b - h\frac{b-a}{2}\right), & if\ t \in (x,b]. \end{cases}$$

Then, the Riemann-Stieltjes integral $\int_a^b p(x,t)\,dF(t)$ exists for any $x \in [a+h\frac{b-a}{2}, b-h\frac{b-a}{2}]$ and the following identity holds:

$$\begin{aligned}
\int_a^b p(x,t)dF(t) &= \int_a^x \left[t-\left(a+h\frac{b-a}{2}\right)\right] dF(t) + \int_x^b \left[t-\left(b-h\frac{b-a}{2}\right)\right] dF(t) \\
&= \left[t-\left(a+h\frac{b-a}{2}\right)\right] F(t) \Big|_a^x - \int_a^x F(t)dt \\
&\quad + \left[t-\left(b-h\frac{b-a}{2}\right) F(t)\right] \Big|_x^b - \int_x^b F(t)dt \\
&= (b-a)\left[(1-h)F(x)+\frac{h}{2}\right] - \int_a^b F(t)dt. \qquad (7.5)
\end{aligned}$$

Further, we have

$$\begin{aligned}
E(X) &= \int_a^b t dF(t) = tF(t) \Big|_a^b - \int_a^b F(t)dt \\
&= b - \int_a^b F(t)dt,
\end{aligned}$$

implies

$$\int_a^b F(t)dt = b - E(X). \qquad (7.6)$$

Using (7.5) and (7.6), we get the identity

$$\int_a^b p(x,t)dF(t) = (b-a)\left[(1-h)F(x)+\frac{h}{2}\right] + E(X) - b. \qquad (7.7)$$

As shown in [10], if $p : [a,b]^2 \to \mathbb{R}$ is Riemann integrable on $[a,b]$ and $\upsilon : [a,b] \to R$ is L-Lipschitzian (Lipschitzian with the constant L), then we have

$$\left| \int_a^b p(x)\,d\upsilon(x) \right| \le L \int_a^b |p(x)|\,dx. \qquad (7.8)$$

Applying (7.8) for the mapping $p(x,.)$ and the function $F(.)$, we get

$$\begin{aligned}
&\left|\int_a^b p(x,t)\,dF(t)\right| \\
\leq\ & \|f\|_\infty \int_a^b |p(x,t)|\,dt \\
=\ & \|f\|_\infty \left[\int_a^x \left|t-\left(a+h\frac{b-a}{2}\right)\right| dt + \int_x^b \left|t-\left(b-h\frac{b-a}{2}\right)\right| dt\right] \\
=\ & \|f\|_\infty \left[\int_a^{a+h\frac{b-a}{2}} \left(a+h\frac{b-a}{2}-t\right) dt + \int_{a+h\frac{b-a}{2}}^{x} \left(t-\left(a+h\frac{b-a}{2}\right)\right) dt\right. \\
&\left. + \int_x^{b-h\frac{b-a}{2}} \left(b-h\frac{b-a}{2}-t\right) dt + \int_{b-h\frac{b-a}{2}}^{b} \left(t-\left(b-h\frac{b-a}{2}\right)\right) dt\right] \\
=\ & \|f\|_\infty (b-a)^2 \left[\frac{1}{4}\left(h^2+(1-h)^2\right) + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right],
\end{aligned}$$

for all $x \in \left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]$.

Finally, by the identity (7.7) we deduce for all $x \in \left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]$,

$$\begin{aligned}
&\left|(1-h)F(x)+\frac{h}{2}-\frac{b-E(X)}{b-a}\right| \\
\leq\ & \left[\frac{1}{4}\left(h^2+(1-h)^2\right) + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f\|_\infty,
\end{aligned}$$

which proves (7.3).

Also, since

$$\Pr(X \leq x) = 1 - \Pr(X \geq x),$$

the inequality (7.4) is obtained. ■

Remark 7.1 *For* $h=0$ *in (7.3) and (7.4), we recapture (7.1) and (7.2). Moreover, as*

$$h^2+(1-h)^2 \leq 1, \text{ for all } h \in [0,1],$$

therefore, (7.3) and (7.4) gives better estimates than (7.1) and (7.2).

We now give some corollaries of Theorem 7.2 for the expectations of the variable X.

Corollary 7.1 *Under the above assumptions, we have the double inequality:*

$$\begin{aligned} & b-\frac{h}{2}(b-a)-\frac{1}{2}\triangle(b-a)^2\|f\|_\infty \\ \leq\ & E(X) \\ \leq\ & a+\frac{h}{2}(b-a)+\frac{1}{2}\triangle(b-a)^2\|f\|_\infty, \end{aligned} \tag{7.9}$$

where

$$\triangle = h^2-h+1, \tag{7.10}$$

for $h\in[0,1]$.

Proof. It is known that

$$a\leq E(X)\leq b.$$

If $x=a$ in (7.3), we obtain

$$\left|\frac{h}{2}-\frac{b-E(X)}{b-a}\right|\leq\frac{1}{2}\triangle(b-a)\|f\|,$$

where $\triangle$ is as defined above and

$$\begin{aligned} & b-\frac{h}{2}(b-a)-\frac{1}{2}\triangle(b-a)^2\|f\| \\ & \leq E(X) \\ & \leq b-\frac{h}{2}(b-a)+\frac{1}{2}\triangle(b-a)^2\|f\|. \end{aligned} \tag{7.11}$$

The left hand estimate of the inequality (7.11) is equivalent to first inequality in (7.9).

Also, if $x=b$ in (7.3)

$$\left|\frac{E(X)-a}{b-a}-\frac{h}{2}\right|\leq\frac{1}{2}\triangle(b-a)\|f\|_\infty,$$

which reduces to

$$\begin{aligned} & a+\frac{h}{2}(b-a)-\frac{1}{2}\triangle(b-a)^2\|f\|_\infty \\ & \leq E(X) \\ & \leq a+\frac{h}{2}(b-a)+\frac{1}{2}\triangle(b-a)^2\|f\|_\infty. \end{aligned} \tag{7.12}$$

The right hand side of the inequality (7.12) proves the second inequality of (7.9).

■

Remark 7.2 *As for the probability density function f associated with random variable X*

$$1 = \int_a^b f(t)\,dt,$$

implies

$$\|f\|_\infty \geq \frac{1}{b-a}.$$

If we suppose that f is not too large and

$$\|f\|_\infty \leq \frac{2-h}{\triangle\,(b-a)}, \tag{7.13}$$

where $\triangle$ is defined by (7.10) and $h \in [0,1]$. Then from the double inequality (7.9) it can be verified that

$$a + \frac{h}{2}(b-a) + \frac{1}{2}\triangle\,(b-a)^2\|f\|_\infty \leq b,$$

and

$$b - \frac{h}{2}(b-a) - \frac{1}{2}\triangle\,(b-a)^2\|f\|_\infty \geq a,$$

when (7.13) holds. It shows that (7.9) gives a much tighter estimate of the expected value of the random variable X.

Corollary 7.2 *Under the above assumptions, we have:*

$$\left|E(X) - \frac{a+b}{2}\right| \leq \frac{1}{2}(b-a)^2\left[\triangle\,\|f\|_\infty - \frac{1-h}{b-a}\right]. \tag{7.14}$$

Proof. From the inequality (7.9),

$$\begin{aligned}
& -\frac{1}{2}(b-a)^2\left[\triangle\,\|f\|_\infty - \frac{1-h}{b-a}\right] \\
\leq\ & E(X) - \frac{a+b}{2} \\
\leq\ & \frac{1}{2}(b-a)^2\left[\triangle\,\|f\|_\infty - \frac{1-h}{b-a}\right],
\end{aligned}$$

which is exactly (7.14).

This corollary helps in finding a condition, in terms of $\|f\|_\infty$, for the expectation $E(X)$ to be close to the midpoint of the interval, $\frac{a+b}{2}$. ■

Corollary 7.3 *Let X and f be as above and $\varepsilon > 0$. If*

$$\|f\|_\infty \leq \frac{(1-h)}{\triangle\,(b-a)} + \frac{2\varepsilon}{\triangle\,(b-a)^2}, \tag{7.15}$$

then

$$\left|E(X)-\frac{a+b}{2}\right|\leq\varepsilon$$

The following corollary of Theorem 7.2 also holds.

Corollary 7.4 *Let X and F be as above, then*

$$\begin{aligned}
&\left|(1-h)\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{1}{2}(1-h)\right| \\
\leq\ &\frac{1}{4}\left[h^2+(1-h)^2\right](b-a)\|f\|_\infty+\frac{1}{b-a}\left|E(X)-\frac{a+b}{2}\right| \\
\leq\ &\left(\triangle-\frac{1}{4}\right)(b-a)\|f\|_\infty-\frac{1}{2}(1-h). \qquad (7.16)
\end{aligned}$$

Proof. If we choose $x=\frac{a+b}{2}$ in (7.3), then we get

$$\left|(1-h)\Pr\left(X\leq\frac{a+b}{2}\right)+\frac{h}{2}-\frac{b-E(X)}{b-a}\right|$$
$$\leq\frac{1}{2}\left(\triangle-\frac{1}{2}\right)(b-a)\|f\|_\infty,$$

which may be rewritten in the following form

$$\left|(1-h)\Pr\left(X\leq\frac{a+b}{2}\right)+\frac{h}{2}-\frac{1}{2}+\frac{1}{b-a}\left(E(X)-\frac{a+b}{2}\right)\right|$$
$$\leq\frac{1}{2}\left(\triangle-\frac{1}{2}\right)(b-a)\|f\|_\infty.$$

Using the triangular inequality, we get

$$\left|(1-h)\Pr\left(X\leq\frac{a+b}{2}\right)+\frac{h}{2}-\frac{1}{2}+\frac{1}{b-a}\left(E(X)-\frac{a+b}{2}\right)-\frac{1}{b-a}\left(E(X)-\frac{a+b}{2}\right)\right|$$

$$\begin{aligned}
\leq\ &\left|(1-h)\Pr\left(X\leq\frac{a+b}{2}\right)+\frac{h}{2}-\frac{1}{2}+\frac{1}{b-a}\left(E(X)-\frac{a+b}{2}\right)\right| \\
&+\frac{1}{b-a}\left|E(X)-\frac{a+b}{2}\right| \\
\leq\ &\frac{1}{2}\left(\triangle-\frac{1}{2}\right)(b-a)\|f\|_\infty+\frac{1}{b-a}\left|E(X)-\frac{a+b}{2}\right| \\
\leq\ &\left(\triangle-\frac{1}{4}\right)(b-a)\|f\|_\infty-\frac{1}{2}(1-h),
\end{aligned}$$

which gives the desired result.

A similar inequality holds for

$$\Pr\left(X\geq\frac{a+b}{2}\right).$$

■

Corollary 7.5 *Let X and F be as above, then*

$$\left|E(X)-\frac{a+b}{2}\right| \le \frac{1}{2}\left(\triangle-\frac{1}{2}\right)(b-a)^2\|f\|_\infty + (b-a)\left|\Pr\left(X\le\frac{a+b}{2}\right)-\frac{1}{2}(1-h)\right|. \quad (7.17)$$

Following the above corollary the proof is obvious and the details are omitted.

Remark 7.3 *If we assume that f is continuous on $[a,b]$, then F is differentiable on (a,b), and we get*

$$\left|(1-h)F(x)+\frac{h}{2}-\frac{1}{b-a}\int_a^b F(t)\,dt\right| \le \left[\frac{1}{4}\left(h^2+(1-h)^2\right)+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f\|_\infty. \quad (7.18)$$

Using the identity (7.6), we recapture (7.3) and (7.4) for random variables whose probability density function are continuous on $[a,b]$.

7.1.3 Applications for Beta Random Variable

If X is a beta random variable with parameters $\beta_3>-1$, $\beta_4>-1$ and for $\beta_2>0$ and any β_1, the generalized beta random variable

$$Y=\beta_1+\beta_2X,$$

is said to have a generalized beta distribution [51] and the probability density function of the generalized beta distribution of beta random variable is given as:

$$f(x)=\begin{cases}\frac{(x-\beta_1)^{\beta_3}(\beta_1+\beta_2-x)^{\beta_4}}{\beta(\beta_3+1,\beta_4+1)\beta_2^{(\beta_3+\beta_4+1)}}, & \text{for } \beta_1<x<\beta_1+\beta_2\\ 0, & \text{otherwise}\end{cases}$$

where $\beta(l,m)$ is the beta function with $l,\ m>0$ and is defined as

$$\beta(l,m)=\int_0^1 x^{l-1}(1-x)^{m-1}\,dx.$$

For p, $q > 0$ and $h \in [0, 1)$, we choose,

$$\begin{aligned}\beta_1 &= \frac{h}{2}, \\ \beta_2 &= (1-h), \\ \beta_3 &= p-1, \\ \beta_4 &= q-1.\end{aligned}$$

Then, the probability density function associated with generalized beta random variable

$$Y = \frac{h}{2} + (1-h)X,$$

takes the form

$$f(x) = \begin{cases} \frac{\left(x-\frac{h}{2}\right)^{p-1}\left(1-\frac{h}{2}-x\right)^{q-1}}{\beta(p,q)(1-h)^{p+q-1}}, & \frac{h}{2} < x < 1-\frac{h}{2} \\ 0, & \text{otherwise.} \end{cases} \tag{7.19}$$

Now,

$$\begin{aligned}E(Y) &= \int_{\frac{h}{2}}^{1-\frac{h}{2}} xf(x)\,dx \\ &= (1-h)\frac{p}{p+q} + \frac{h}{2}.\end{aligned} \tag{7.20}$$

We observe that for $p < 1$

$$\|f(x;p,q)\|_{\infty} = \sup_{\frac{h}{2}<x<1-\frac{h}{2}} \left[\frac{\left(x-\frac{h}{2}\right)^{p-1}\left(1-\frac{h}{2}-x\right)^{q-1}}{\beta(p,q)(1-h)^{p+q-1}}\right].$$

Assume that $p, q > 1$, then we find that

$$\begin{aligned}\frac{df(x;p,q)}{dx} &= \frac{\left(x-\frac{h}{2}\right)^{p-2}\left(1-\frac{h}{2}-x\right)^{q-2}}{(1-h)^{p+q-1}\beta(p,q)} \times \\ &\quad \left[(p-1) + \frac{h}{2}(q-p) - (p+q-2)x\right].\end{aligned}$$

We further observe that for $p, q > 1$, $\frac{df}{dx} = 0$ if and only if $x = x_0 = \frac{(p-1)+\frac{h}{2}(q-p)}{p+q-2}$. We therefore have $\frac{df}{dx} > 0$ on $(\frac{h}{2}, x_0)$ and $\frac{df}{dx} < 0$ on $(x_0, 1-\frac{h}{2})$. Consequently, we see that

$$\begin{aligned}\|f(x;p,q)\|_{\infty} &= \|f(x_0;p,q)\|_{\infty} \\ &= \frac{1}{(1-h)\beta(p,q)}\left[\frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}\right].\end{aligned} \tag{7.21}$$

Then, by Theorem 7.2, we may state the following.

Proposition 7.1 *Let X be a beta random variable with parameters (p, q). Then, for generalized beta random variable:*

$$Y = \frac{h}{2} + (1-h)\,X,$$

we have the inequality

$$\begin{aligned} &\left|\Pr(Y \le x) - \frac{q}{p+q}\right| \\ \le\ & \frac{1}{(1-h)^2\,\beta(p,q)}\left[\frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}\right]\times \\ & \left[\frac{1}{4}\left(h^2+(1-h)^2\right)+\left(x-\frac{1}{2}\right)^2\right], \end{aligned} \tag{7.22}$$

for all $x \in \left[\frac{h}{2}, 1-\frac{h}{2}\right]$.

In particular,

$$\begin{aligned} &\left|\Pr\left(Y \le \frac{1}{2}\right) - \frac{q}{p+q}\right| \\ \le\ & \frac{1}{4\,(1-h)^2\,\beta(p,q)}\left(h^2+(1-h)^2\right)\left[\frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}\right]. \end{aligned} \tag{7.23}$$

7.2 A generalized Ostrowski type inequality for a random variable whose probability density function belongs to $L_p[a,b]$, $p > 1$.

7.2.1 Introduction

In [33], S. S. Dragomir, N. S. Barnett and S. Wang developed Ostrowski's type inequality for a random variable whose probability density function belongs to $L_p[a,b]$ in terms of the cumulative distribution function and expectation. The inequality is given in the form of the following theorem:

Theorem 7.3 *Let X be a continuous random variable with the probability density function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}_+$ and with cumulative distribution function $F(x) =$*

$\Pr(X \leq x)$. *If* $f \in L_p[a,b]$, $p > 1$, *then, we have the inequality:*

$$\begin{aligned}
&\left|\Pr(X \leq x) - \frac{b - E(X)}{b-a}\right| \\
\leq\ & \frac{q}{q+1}\|f\|_p (b-a)^{\frac{1}{q}} \left[\left(\frac{x-a}{b-a}\right)^{\frac{1+q}{q}} + \left(\frac{b-x}{b-a}\right)^{\frac{1+q}{q}}\right] \\
\leq\ & \frac{q}{1+q}\|f\|_p (b-a)^{\frac{1}{q}},
\end{aligned} \tag{7.24}$$

for all $x \in [a,b]$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

In [106], we can find the following theorem:

Theorem 7.4 *Let* $f : [a,b] \to \mathbb{R}$ *be continuous, differentiable on* $[a,b]$ *and* $f' \in L_p(a,b)$ *for some* $p > 1$. *Then*

$$\begin{aligned}
&\left|(b-a)\left[(1-h)f(x) + h\frac{f(a)+f(b)}{2}\right] - \int_a^b f(t)\,dt\right| \\
\leq\ & \frac{1}{(q+1)^{\frac{1}{q}}}\left[2\left(\frac{h(b-a)}{2}\right)^{q+1} + \left(x-a-\frac{h(b-a)}{2}\right)^{q+1}\right. \\
&\left. + \left(b-x-\frac{h(b-a)}{2}\right)^{q+1}\right]^{\frac{1}{q}} \left\|f'\right\|_p,
\end{aligned} \tag{7.25}$$

where $q = \frac{p}{p-1}$, $h \in [0,1]$ *and* $a + h\frac{b-a}{2} \leq x < b - h\frac{b-a}{2}$.

The main aim of this section is to develop an Ostrowski type inequality for random variables whose probability density functions are in $L_p[a,b]$ based on (7.25). An application for a generalized beta random variable is also given.

7.2.2 Main Results

The following theorem holds:

Theorem 7.5 *Let* X *and* F *be as defined above. Then from Theorem 7.4, we have*

$$\begin{aligned}
&\left|(1-h)F(x) + \frac{h}{2} - \frac{1}{b-a}\int_a^b F(t)\,dt\right| \\
\leq\ & \frac{1}{(b-a)(q+1)^{\frac{1}{q}}}\left[2\left(\frac{h(b-a)}{2}\right)^{q+1} + \left(x-a-\frac{h(b-a)}{2}\right)^{q+1}\right. \\
&\left. + \left(b-x-\frac{h(b-a)}{2}\right)^{q+1}\right]^{\frac{1}{q}} \|f\|_p,
\end{aligned} \tag{7.26}$$

where f is the probability distribution function associated with the cumulative distribution function F.

Equivalently,

$$\begin{aligned}
&\left|(1-h)\Pr(X \leq x) + \frac{h}{2} - \frac{b-E(X)}{b-a}\right| \\
\leq\ & \frac{1}{(b-a)(q+1)^{\frac{1}{q}}}\left[2\left(\frac{h(b-a)}{2}\right)^{q+1} + \left(x-a-\frac{h(b-a)}{2}\right)^{q+1}\right. \\
&\left.+\left(b-x-\frac{h(b-a)}{2}\right)^{q+1}\right]^{\frac{1}{q}} \|f\|_p, \qquad (7.27)
\end{aligned}$$

for all $x \in [a+h\frac{b-a}{2}, b-h\frac{b-a}{2}]$ and $h \in [0,1]$.

Proof. Proof is obvious. Hence, the details are omitted. ∎

We now give some corollaries of the above theorem for the expectations of the variable X.

Corollary 7.6 *Under the above assumptions, we have the double inequality*

$$\begin{aligned}
& b - \frac{h}{2}(b-a) - \frac{1}{(q+1)^{\frac{1}{q}}} \triangle(q,h)(b-a)^{1+\frac{1}{q}} \|f\|_p \\
\leq\ & E(X) \\
\leq\ & a + \frac{h}{2}(b-a) + \frac{1}{(q+1)^{\frac{1}{q}}} \triangle(q,h)(b-a)^{1+\frac{1}{q}} \|f\|_p, \qquad (7.28)
\end{aligned}$$

for $h \in [0,1]$ and

$$\triangle(q,h) = \left(\left(\frac{h}{2}\right)^{q+1}(2-(-1)^q) + \left(1-\frac{h}{2}\right)^{q+1}\right)^{\frac{1}{q}}. \qquad (7.29)$$

Proof. It is known that

$$a \leq E(X) \leq b.$$

If $x = a$ in (7.27), we obtain

$$\left|\frac{h}{2} - \frac{b-E(X)}{b-a}\right| \leq \left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \triangle(q,h)\|f\|_p,$$

implies

$$\begin{aligned}
& b - \frac{h}{2}(b-a) - \frac{1}{(q+1)^{\frac{1}{q}}} \triangle(q,h)(b-a)^{1+\frac{1}{q}} \|f\|_p \\
& \leq E(X)
\end{aligned}$$

$$\leq b-\frac{h}{2}(b-a)+\frac{1}{(q+1)^{\frac{1}{q}}}\bigtriangleup(q,h)(b-a)^{1+\frac{1}{q}}\|f\|_p. \qquad (7.30)$$

The left hand estimate of the inequality (7.30) is equivalent to first inequality in (7.28).

Also, if $x=b$ in (7.27)

$$\left|\frac{E(X)-a}{b-a}-\frac{h}{2}\right|\leq\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\bigtriangleup(q,h)\|f\|_p,$$

which reduces to

$$a+\frac{h}{2}(b-a)-\frac{1}{(q+1)^{\frac{1}{q}}}\bigtriangleup(q,h)(b-a)^{1+\frac{1}{q}}\|f\|_p$$

$$\leq E(X)$$

$$\leq a+\frac{h}{2}(b-a)+\frac{1}{(q+1)^{\frac{1}{q}}}\bigtriangleup(q,h)(b-a)^{1+\frac{1}{q}}\|f\|_p. \qquad (7.31)$$

The right hand side of the inequality (7.31) proves the second inequality of (7.28).

■

Remark 7.4 *As for the probability density function f associated with the random variable X*

$$1=\int_a^b f(t)\,dt,$$

implies

$$\|f\|_p\geq\frac{1}{(b-a)^{\frac{1}{q}}}.$$

If we suppose that f is not too large and

$$\|f\|_p\leq\frac{(q+1)^{\frac{1}{q}}\left(1-\frac{h}{2}\right)}{(b-a)^{\frac{1}{q}}\bigtriangleup(q,h)}. \qquad (7.32)$$

Then from the double inequality (7.28) it can be verified that

$$a+\frac{h}{2}(b-a)+\frac{1}{(q+1)^{\frac{1}{q}}}\bigtriangleup(q,h)(b-a)^{1+\frac{1}{q}}\|f\|_p\leq b,$$

and

$$b-\frac{h}{2}(b-a)-\frac{1}{(q+1)^{\frac{1}{q}}}\bigtriangleup(q,h)(b-a)^{1+\frac{1}{q}}\|f\|_p\geq a,$$

when (7.32) holds. It shows that (7.28) gives a much tighter estimate of the expected value of the random variable X.

Corollary 7.7 *With the above assumptions, we have:*

$$\left|E(X)-\frac{a+b}{2}\right|\leq(b-a)\left[\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\triangle(q,h)\|f\|_p-\frac{1-h}{2}\right], \tag{7.33}$$

where $\triangle(q,h)$ *is defined by (7.29).*

Proof. From the inequality (7.28),

$$\begin{aligned}
&\frac{1}{2}(b-a)(1-h)-\frac{1}{(q+1)^{\frac{1}{q}}}\triangle(q,h)(b-a)^{1+\frac{1}{q}}\|f\|_p\\
\leq\ & E(X)-\frac{a+b}{2}\\
\leq\ & -\frac{1}{2}(b-a)(1-h)+\frac{1}{(q+1)^{\frac{1}{q}}}\triangle(q,h)(b-a)^{1+\frac{1}{q}}\|f\|_p
\end{aligned}$$

which is exactly (7.33). ■

This corollary helps in finding a condition, in terms of $\|f\|_p$, for the expectation $E(X)$ to be close to the midpoint of the interval, $\frac{a+b}{2}$.

Corollary 7.8 *Let* X *and* f *be as above and* $\varepsilon>0$. *If*

$$\|f\|_p\leq\frac{(1-h)(q+1)^{\frac{1}{q}}}{2\triangle(q,h)(b-a)^{\frac{1}{q}}}+\frac{(q+1)^{\frac{1}{q}}\varepsilon}{\triangle(q,h)(b-a)^{1+\frac{1}{q}}}, \tag{7.34}$$

then

$$\left|E(X)-\frac{a+b}{2}\right|\leq\varepsilon$$

The following corollary of Theorem 7.5 also holds.

Corollary 7.9 *Let* X *and* F *be as above, then*

$$\begin{aligned}
&\left|(1-h)\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{1}{2}(1-h)\right|\\
\leq\ & \frac{1}{2}\left(h^{q+1}+(1-h)^{q+1}\right)^{\frac{1}{q}}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\|f\|_p\\
&+\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\triangle(q,h)\|f\|_p-\frac{1}{2}(1-h).
\end{aligned} \tag{7.35}$$

Proof. If we choose $x=\frac{a+b}{2}$ in (7.27), then we get

$$\left|(1-h)\Pr\left(X\leq\frac{a+b}{2}\right)+\frac{h}{2}-\frac{b-E(X)}{b-a}\right|$$

$$\leq\frac{1}{2}\left(h^{q+1}+(1-h)^{q+1}\right)^{\frac{1}{q}}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\|f\|_p,$$

which may be rewritten in the following form

$$\left|(1-h)\Pr\left(X\le\frac{a+b}{2}\right)+\frac{h}{2}-\frac{1}{2}+\frac{1}{b-a}\left(E(X)-\frac{a+b}{2}\right)\right|$$
$$\le\frac{1}{2}\left(h^{q+1}+(1-h)^{q+1}\right)^{\frac{1}{q}}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\|f\|_p .$$

Using the triangular inequality, we get

$$\left|(1-h)\Pr\left(X\le\frac{a+b}{2}\right)+\frac{h}{2}-\frac{1}{2}+\frac{1}{b-a}\left(E(X)-\frac{a+b}{2}\right)-\frac{1}{b-a}\left(E(X)-\frac{a+b}{2}\right)\right|$$

$$\le \left|(1-h)\Pr\left(X\le\frac{a+b}{2}\right)+\frac{h}{2}-\frac{1}{2}+\frac{1}{b-a}\left(E(X)-\frac{a+b}{2}\right)\right| +\frac{1}{b-a}\left|E(X)-\frac{a+b}{2}\right|$$

gives the desired result.

A similar inequality holds for

$$\Pr\left(X\ge\frac{a+b}{2}\right).$$

■

Moreover, the following applications of Theorem 7.5 hold:

7.2.3 Applications for Generalized Beta Random Variable

If X be as in Section 7.1.3, then by using (7.19), we have:

$$\|f\|_p=\frac{1}{(1-h)^{1-\frac{1}{p}}\beta(s,t)}\beta^{\frac{1}{p}}\left(p(s-1)+1,p(t-1)+1\right), \tag{7.36}$$

provided

$$s > 1-\frac{1}{p},$$
$$t > 1-\frac{1}{p},$$

for $p>1$. Then, by Theorem 7.5, we may state the following.

Proposition 7.2 *Let X be a beta random variable with parameters (s,t). Then, for generalized beta random variable:*

$$Y=\frac{h}{2}+(1-h)X,$$

we have the inequality

$$\begin{aligned}
&\left|\Pr(Y \le x) - \frac{t}{s+t}\right| \\
\le\ & \frac{1}{(1-h)^{2-\frac{1}{p}}\beta(s,t)}\left(\frac{2\left(\frac{h}{2}\right)^{q+1} + \left(x-\frac{h}{2}\right)^{q+1} + \left(1-x-\frac{h}{2}\right)^{q+1}}{q+1}\right)^{\frac{1}{q}} \times \\
& \beta^{\frac{1}{p}}(p(s-1)+1, p(t-1)+1), \qquad (7.37)
\end{aligned}$$

for all $x \in \left[\frac{h}{2}, 1-\frac{h}{2}\right]$.

In particular,

$$\begin{aligned}
&\left|\Pr\left(Y \le \frac{1}{2}\right) - \frac{t}{s+t}\right| \\
\le\ & \frac{1}{2(1-h)^{2-\frac{1}{p}}\beta(s,t)}\left(\frac{h^{q+1} + (1-h)^{q+1}}{q+1}\right)^{\frac{1}{q}} \\
& \times\beta^{\frac{1}{p}}(p(s-1)+1, p(t-1)+1). \qquad (7.38)
\end{aligned}$$

Remark 7.5 *For* $h = 0$ *in (7.37), we have the inequality*

$$\begin{aligned}
&\left|\Pr(X \le x) - \frac{t}{s+t}\right| \\
\le\ & \left(\frac{x^{q+1} + (1-x)^{q+1}}{q+1}\right)^{\frac{1}{q}} \frac{\beta^{\frac{1}{p}}(p(s-1)+1, p(t-1)+1)}{\beta(s,t)}, \qquad (7.39)
\end{aligned}$$

for all $x \in [0, 1]$, *and particularly,*

$$\begin{aligned}
&\left|\Pr\left(X \le \frac{1}{2}\right) - \frac{t}{s+t}\right| \\
\le\ & \frac{1}{2(q+1)^{\frac{1}{q}}} \frac{\beta^{\frac{1}{p}}(p(s-1)+1, p(t-1)+1)}{\beta(s,t)}. \qquad (7.40)
\end{aligned}$$

It is interesting to compare these two inequalities with the results of Proposition 3.1 in [33]. Actually, we, in here, have sharpened and improved the previous results.

7.3 Conclusion

In this chapter, Ostrowski type inequalities are applied to obtain various tight bounds for the random variables defined on a finite interval whose probability density functions belong to $\{L_p[a,b] : p = \infty, p > 1\}$. Moreover, as it has been shown in Remark 7.2 and 7.4 that some tighter estimates of the expectation of a random variable have been obtained. The inequalities obtained are then applied to a generalized beta random variable to get some new and generalized estimates in this context. Moreover, Remark 7.5 also reflects that we have improved some previous inequalities of [33] for a beta random variable.

Chapter 8

Applications of Ostrowski type inequalities to iterative methods

Let us consider the equation

$$f(x) = 0, \tag{8.1}$$

where f is a real valued univariate non-linear function.

Locating zeros of such functions has been given much attention from several decades due to its importance in applied sciences. Newton's method is the most widely used quadratically convergent iterative method in solving such problems; yet in the recent past many other efficient iterative methods for solving non-linear equations have appeared in the literature by the use of Taylor's series, interpolating polynomials, decomposition techniques and quadrature formulae. The books and research papers [14, 6, 25] provide an extensive amount of literature in the context of Newton's method, its variants and modifications.

The connection of quadrature formulae and iterative methods has already been established by S. Weerakoon and T. G. I. Fernando in [105] by using the indefinite integral representation of Newton's method [26] to obtain quadrature based iterative methods. The trend continued with the publication of the papers by G. Nedzhibov [67], V. I. Hasanov et al. [48] and M. Frontini and E. Sormani [43, 44]. However, this domain is addressed only for classical quadrature rules e.g., trapezoid, mid-point, Simpson's, etc. N. Ujević in [102, 103], however, adopted a quite different approach by using specially derived quadrature rule, infact the equivalence of two quadrature rules to re-establish this connection and to obtain quadrature based iterative predictor-corrector type methods for solving non-linear equations.

The applications of mathematical inequalities, particularly inequalities of Ostrowski-Grüss and Čebyšev type have already been explored by S. S. Dragomir, N. S. Barnett, P. Cerone, Th. M. Rassias and S. Wang, etc., in Numerical integration, Special means and Probability theory, see e.g., [8, 36, 39, 40]. We, however, by using the approach of S. Weerakoon and T. G. I. Fernando [105] give some new applications of such inequalities to obtain iterative methods for solving non-linear equations. We, thus, establish the fact that the specially derived quadrature rules developed in the sense of inequalities may be applied to develop many other iterative methods.

8.1 A generalized family of quadrature based iterative methods

8.1.1 Introduction

In this section, we present a family of iterative methods for solving non-linear equations as an application of integral inequalities. Thus, we give a new application of such inequalities other than their natural applications in Numerical integration and Special means. Moreover, it is shown that the family of two-step iterative methods thus established has third-order convergence and it recaptures many previously presented quadrature based iterative methods.

8.1.2 A generalized family of two-step Iterative methods

Consider the following family of quadrature rules derived in the sense of inequalities in Section 3.4:

Theorem 8.1 *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be mapping differentiable in the interior $Int\ I$ of I, and let $a, b \in Int\ I$, $a < b$. If there exists some constants $\gamma, \Gamma \in \mathbb{R}$, such that $\gamma \leq f^{'}(t) \leq \Gamma$, $\forall\ t \in [a, b]$ and $f^{'} \in L_1(a, b)$, then we have:*

$$\left|(1-h)\left[f(x)-\left(x-\frac{a+b}{2}\right)f^{'}(x)\right]+h\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{1}{2}\left(1-h^2\right)(b-a)(S-\gamma) \tag{8.2}$$

and

$$\left|(1-h)\left[f(x)-\left(x-\frac{a+b}{2}\right)f^{'}(x)\right]+h\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{1}{2}\left(1-h^2\right)(b-a)(\Gamma-S) \tag{8.3}$$

where $S=\frac{f(b)-f(a)}{b-a}$, $x\in\left[a+h\frac{b-a}{2},b-h\frac{b-a}{2}\right]$ *and* $h\in[0,1]$.

Moreover, in Section 3.1, we have derived the following inequality:

Theorem 8.2 *Let* $f:[a,b]\to\mathbb{R}$ *be an absolutely continuous function whose first derivative* $f^{'}\in L_2(a,b)$. *Then, we have the inequality:*

$$\left|(1-h)\left[f(x)-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right]+h\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\leq \left[\frac{(b-a)^2}{12}\left(3h^2-3h+1\right)+h(1-h)\left(x-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}}\times$$

$$\left[\frac{1}{b-a}\left\|f^{'}\right\|_2^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}},$$

$$\leq \frac{1}{2}(\Gamma-\gamma)\left[\frac{(b-a)^2}{12}\left(3h^2-3h+1\right)+h(1-h)\left(x-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}},$$

$$if\ \gamma\leq f^{'}(t)\leq\Gamma\ almost\ everywhere\ t\ on\ [a,b], \tag{8.4}$$

for all $x\in\left[a+h\frac{b-a}{2},b-h\frac{b-a}{2}\right]$ *and* $h\in[0,1]$.

Remark 8.1 *It may be noted that for* $x=\frac{a+b}{2}$ *and for* $h\in[0,1]$ *the left hand sides of (2.1), (2.2) and (2.3) give the following family of quadrature rule:*

$$\int_a^b f(t)\,dt=(b-a)\left[(1-h)f\left(\frac{a+b}{2}\right)+h\frac{f(a)+f(b)}{2}\right]+R(f), \tag{8.5}$$

which is a combination of mid-point and trapezoid rule.

We proceed with the indefinite integral representation of Newton's method [26]:

$$f(x)=f(x_n)+\int_{x_n}^{x} f^{'}(t)\,dt. \tag{8.6}$$

Now approximating the integral in (8.6) with the quadrature rule (8.5), we obtain:

$$\int_{x_n}^{x} f'(t)\,dt = (x - x_n)\left[(1-h)\,f'\left(\frac{x_n + x}{2}\right) + h\frac{f'(x_n) + f'(x)}{2}\right]. \tag{8.7}$$

Using the approximation (8.7) in (8.6) implies

$$-2f(x_n) = (x - x_n)\left[2(1-h)\,f'\left(\frac{x_n + x}{2}\right) + h\left(f'(x_n) + f'(x)\right)\right]$$

which finally results into the following implicit method:

$$x = x_n - \frac{2f(x_n)}{2(1-h)\,f'\left(\frac{x_n+x}{2}\right) + h\left(f'(x_n) + f'(x)\right)}.$$

This implies

$$x_{n+1} = x_n - \frac{2f(x_n)}{2(1-h)\,f'\left(\frac{x_n+y_n}{2}\right) + h\left(f'(x_n) + f'(y_n)\right)}, \tag{8.8}$$

where y_n is some explicit method.

If we choose y_n as Newton's method in (8.8), then we have the following two-step method:

$$\begin{aligned} x_{n+1} &= x_n - \frac{2f(x_n)}{2(1-h)\,f'\left(\frac{x_n+y_n}{2}\right) + h\left(f'(x_n) + f'(y_n)\right)}, \\ y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{8.9}$$

or

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{8.10}$$

$$z_n = x_n - \frac{f(x_n)}{2f'(x_n)}, \tag{8.11}$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{2(1-h)\,f'(z_n) + h\left(f'(x_n) + f'(y_n)\right)}. \tag{8.12}$$

We, now, compute the order of convergence of algorithm (8.9) using Maple 7.0 and is given in the form of the following theorem:

Theorem 8.3 *Let $w \in I$ be a simple zero of sufficiently differentiable function $f : I \sqsubseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to w, then the algorithm (8.9) is cubically convergent for all $h \in [0,1]$.*

Proof. Let w be a simple zero of f and $x_n = w + e_n$ with an error e_n. By Taylor's expansion, we have:

$$f(x_n) = f^{'}(w)\left(e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6\right) + O(e_n^7) \tag{8.13}$$

$$f^{'}(x_n) = f^{'}(w)\left(1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5\right) + O(e_n^6), \tag{8.14}$$

where

$$c_k = \left(\frac{1}{k!}\right)\frac{f^{(k)}(w)}{f'(w)}, k = 2, 3, ...\text{and } e_n = x_n - w. \tag{8.15}$$

Using (8.14) and (8.13), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2\left(c_2^2 - c_3\right)e_n^3 + \left(7c_2c_3 - 3c_4 - 4c_2^3\right)e_n^4 + O(e_n^5). \tag{8.16}$$

Using (8.16) in (8.10), we obtain

$$\begin{aligned} y_n &= w + c_2 e_n^2 + \left(-2c_2^2 + 2c_3\right)e_n^3 - \left(7c_2c_3 - 4c_2^3 - 3c_4\right)e_n^4 \\ &\quad + O\left(e_n^5\right). \end{aligned} \tag{8.17}$$

Expanding $f(y_n)$ by Taylor's series about w,we have:

$$f(y_n) = f^{'}(w)\left(c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + (-7c_2c_3 + 3c_4 + 5c_2^3)e_n^4\right) + O\left(e_n^5\right). \tag{8.18}$$

By Taylor's series, we have

$$\begin{aligned} f^{'}(y_n) &= f^{'}(w)\,(1 + 2c_2^2 e_n^2 + \left(-4c_2^3 + 4c_2c_3\right)e_n^3 \\ &\quad +(-11c_3c_2^2 + 8c_2^4 + 6c_2c_4)e_n^4) + O(e_n^5). \end{aligned} \tag{8.19}$$

Using (8.16) in (8.11), we thus have

$$\begin{aligned} z_n &= w + \frac{1}{2}e_n + \frac{1}{2}c_2 e_n^2 + \left(-c_2^2 + c_3\right)e_n^3 \\ &\quad + \left(\frac{3}{2}c_4 - \frac{7}{2}c_2c_3 + 2c_2^3\right)e_n^4 + O\left(e_n^5\right). \end{aligned} \tag{8.20}$$

Expanding $f(z_n)$ by Taylor's series about w,we have:

$$\begin{aligned} f(z_n) &= f^{'}(w)\,(\frac{1}{2}e_n + \frac{3}{4}c_2 e_n^2 + \left(-\frac{1}{2}c_2^2 + \frac{9}{8}c_3\right)e_n^3 \\ &\quad + \left(\frac{5}{4}c_2^3 - \frac{17}{8}c_2c_3 + \frac{25}{16}c_4\right)e_n^4 \\ &\quad + \left(-3c_2^4 + \frac{57}{8}c_3c_2^2 - \frac{9}{4}c_3^2 - \frac{13}{4}c_2c_4 + \frac{65}{32}c_5\right)e_n^5) \\ &\quad + O(e_n^6). \end{aligned} \tag{8.21}$$

By Taylor's series, we have

$$\begin{aligned} f^{'}(z_n) &= f^{'}(w)(1 + c_2 e_n + \left(c_2^2 + \frac{3}{4}c_3\right) e_n^2 + (-2c_2^3 + \frac{7}{2}c_2c_3 + \frac{1}{2}c_4)e_n^3 \\ &+ \left(\frac{9}{2}c_2c_4 + c_2^4 - \frac{37}{4}c_2^2c_3 + 3c_3^2 + \frac{5}{16}c_5\right) e_n^4) \\ &+ O\left(e_n^5\right). \end{aligned} \tag{8.22}$$

Using (8.14), (8.19) and (8.22) in

$$\begin{aligned} &\frac{2f(x_n)}{2(1-h) f'(z_n) + h(f'(x_n) + f'(y_n))} \\ &= e_n + (\frac{1}{4}(1-3h)c_3 - c_2^2)e_n^3 \\ &+ (3c_2^3 + \frac{3}{4}(3h-5)c_2c_3 + \frac{1}{2}(1-3h)c_4)e_n^4 \\ &+ O\left(e_n^5\right). \end{aligned} \tag{8.23}$$

Therefore, by using (8.23) in (8.12), we have:

$$x_{n+1} = w + (c_2^2 - \frac{1}{4}(1-3h)c_3)e_n^3 + O\left(e_n^4\right).$$

Hence, we obtain:

$$e_{n+1} = (c_2^2 - \frac{1}{4}(1-3h)c_3)e_n^3 + O\left(e_n^4\right).$$

Thus, we observe that the method is cubically convergent for all $h \in [0, 1]$. ■

Remark 8.2 *It is clear from Theorem 8.3 that algorithm (8.9) is cubically convergent and*

1. *For* $h = 1$, *it recaptures the trapezoid Newton's method given by S. Weerakoon and T. G. I. Fernando in [105].*

2. *For* $h = 0$, *it recaptures the midpoint Newton's method given by A. Y. Özban in [72] and by Frontini et al. in [43].*

3. *For* $h = \frac{1}{3}$, *it recaptures the Simpson Newton's method given by V. I. Hasanov et. al. in [48].*

4. *For* $h = \frac{1}{2}$, *it recaptures the averaged trapezoid midpoint Newton's method given by G. Nedzhibov in [67].*

Remark 8.3 *The computational efficiency of the algorithm (8.9) is less than the computational efficiency of the Newton's method except for the cases for which* $h = 0$ *and* $h = 1$. *However, the implicit method (8.8) can be used in combination with the other known methods to increase the convergence order and computational efficiency.*

8.2 Applications of error inequalities to iterative methods

8.2.1 Introduction

In this section, we, by the use of quadrature rule developed in Section 5.1 in the sense of error inequalities present some two-step and three-step iterative algorithms for solving non-linear equations. The two-step algorithms and their derivation are given in Section 8.2.2 followed by their convergence analysis in Section 8.2.3. The three-step iterative algorithms are suggested in Section 8.2.4 with their convergence analysis in Section 8.2.5. It is proved that the new algorithms are of three, four, six and eighth order. In Section 8.2.6, several numerical examples are given to ensure that the new algorithms are comparable with the existing methods. The comparisons have been carried out with the respective known methods of cubic, fourth, sixth and seventh order.

8.2.2 Two-step Iterative Methods

Consider the following family of quadrature rules in the sense of error inequalities derived in Section 5.1:

Theorem 8.4 *Let* $I \subset \mathbb{R}$ *be an open interval such that* $[a, b] \subset I$ *and let* $f : I \to \mathbb{R}$ *be a twice differentiable function such that* f'' *is bounded and integrable. Then, we have:*

$$\int_a^b f(t)\,dt = \frac{1}{2}(b-a)\left[hf(a) + (1-h)f(x_1) + (1-h)f(x_2) + hf(b)\right] + R(f), \tag{8.24}$$

with

$$x_1 = \frac{b-a}{2}x^* + \frac{a+b}{2}, \; x_2 = -\frac{b-a}{2}x^* + \frac{a+b}{2},$$
$$x^* = -4 + 4h + 2\sqrt{3 - 6h + 4h^2}, \tag{8.25}$$

and

$$|R(f)| \leq \frac{1}{4}\triangle(h)(b-a)^3 \left\| f^{''} \right\|_\infty, \tag{8.26}$$

$h \in \left[0, \frac{1}{2}\right]$ *and* $\triangle(h)$ *is defined as:*

$$\triangle(h) = \frac{52}{3}h^3 - 44h^2 + \frac{83}{2}h + \frac{83}{6} + 8(1-h)^2\sqrt{4h^2 - 6h + 3}$$
$$+\frac{2}{3}\left(8h^2 - 14h + 7\right)\sqrt{8h^2 - 14h + 7 - 4(1-h)\sqrt{4h^2 - 6h + 3}}$$
$$-\frac{8}{3}(1-h)\sqrt{8h^2 - 14h + 7 - 4(1-h)\sqrt{4h^2 - 6h + 3}}\sqrt{4h^2 - 6h + 3}. \tag{8.27}$$

Remark 8.4 *For* $h = \frac{1}{5}$, $\triangle(h)$ *attains its minimum value and the corresponding quadrature rule is as follows:*

$$\int_a^b f(t)\,dt = \frac{1}{10}(b-a)\left[f(a) + 4f\left(\frac{7a+3b}{10}\right)\right.$$
$$\left. +4f\left(\frac{3a+7b}{10}\right) + f(b)\right] + R(f), \tag{8.28}$$

and

$$|R(f)| \leq C(b-a)^3 \left\| f^{''} \right\|_\infty,$$

where $C = \frac{7}{1500} \approx 0.00467$.

We proceed with the indefinite integral representation of Newton's method [26]:

$$f(x) = f(x_n) + \int_{x_n}^{x} f^{'}(t)\,dt. \tag{8.29}$$

Now approximating the integral in (8.29) with the quadrature rule (8.28), we obtain:

$$\int_{x_n}^{x} f^{'}(t)\,dt = \frac{1}{10}(x - x_n)\left[f^{'}(x_n) + 4f^{'}\left(\frac{7x_n + 3x}{10}\right)\right.$$
$$\left. +4f^{'}\left(\frac{3x_n + 7x}{10}\right) + f^{'}(x)\right]. \tag{8.30}$$

Using the approximation (8.30) in (8.29) implies

$$\begin{aligned} -10f(x_n) &= (x - x_n)\left[f'(x_n) + 4f'\left(\frac{7x_n + 3x}{10}\right)\right. \\ &\quad \left. +4f'\left(\frac{3x_n + 7x}{10}\right) + f'(x)\right], \end{aligned}$$

which finally results into the following implicit method:

$$x = x_n - \frac{10f(x_n)}{f'(x_n) + 4f'\left(\frac{7x_n+3x}{10}\right) + 4f'\left(\frac{3x_n+7x}{10}\right) + f'(x)}.$$

This implies

$$x_{n+1} = x_n - \frac{10f(x_n)}{f'(x_n) + 4f'\left(\frac{7x_n+3y_n}{10}\right) + 4f'\left(\frac{3x_n+7y_n}{10}\right) + f'(y_n)}, \tag{8.31}$$

where y_n is some explicit method.

If we choose y_n as Newton's method in (8.31), then we have the following two-step method:

$$\begin{aligned} x_{n+1} &= x_n - \frac{10f(x_n)}{f'(x_n) + 4f'\left(\frac{7x_n+3y_n}{10}\right) + 4f'\left(\frac{3x_n+7y_n}{10}\right) + f'(y_n)}, \\ y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \end{aligned}$$

or

$$\begin{aligned} x_{n+1} &= x_n - \frac{10f(x_n)}{f'(x_n) + 4f'(h_n) + 4f'(z_n) + f'(y_n)}, \\ z_n &= x_n - \frac{7f(x_n)}{10f'(x_n)}, \\ h_n &= x_n - \frac{3f(x_n)}{10f'(x_n)}, \\ y_n &= x_n - \frac{f(x_n)}{f'(x_n)}. \end{aligned} \tag{8.32}$$

Next, we, compute the order of convergence of algorithm (8.32) using Maple 7.0 and give it in the form of the following theorem:

Theorem 8.5 *Let $w \in I$ be a simple zero of sufficiently differentiable function $f : I \sqsubseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to w, then the algorithm (8.32) is cubically convergent and the error equation is given by*

$$e_{n+1} = \left(c_2^2 - \frac{1}{250}c_3\right)e_n^3 + O\left(e_n^4\right).$$

Remark 8.5 *It is clear from Theorem 8.5 that algorithm (8.32) is cubically convergent. Moreover, it may be observed that the computational efficiency of algorithm (8.32) is less than the Newton's method. Therefore, some reduction or decomposition techniques may further be applied on algorithm (8.32) to obtain some new computationally efficient two-step and three-step variants of algorithm (8.32).*

We, however, now suggest the following reductions of two step iterative algorithm (8.32) to increase the computational efficiency.

Algorithm 1. For a given initial guess x_0, find the approximate solution of (8.1) by:

$$h_n = x_n - \frac{3f(x_n)}{10f'(x_n)}, \tag{8.33}$$

$$x_{n+1} = x_n - \frac{10f(x_n)}{Af'(x_n) + Bf'(h_n)}. \tag{8.34}$$

Algorithm 2. For a given initial guess x_0, find the approximate solution of (8.1) by:

$$z_n = x_n - \frac{7f(x_n)}{10f'(x_n)}, \tag{8.35}$$

$$x_{n+1} = x_n - \frac{10f(x_n)}{Af'(x_n) + Bf'(z_n)}. \tag{8.36}$$

Algorithm 3. For a given initial guess x_0, find the approximate solution of (8.1) by:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{8.37}$$

$$h_n = x_n - \frac{3f(x_n)}{10f'(x_n)}, \tag{8.38}$$

$$x_{n+1} = y_n - \frac{10f(y_n)}{Af'(x_n) + Bf'(h_n)}. \tag{8.39}$$

Algorithm 4. For a given initial guess x_0, find the approximate solution of (8.1) by:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{8.40}$$

$$z_n = x_n - \frac{7f(x_n)}{10f'(x_n)}, \tag{8.41}$$

$$x_{n+1} = y_n - \frac{10f(y_n)}{Af'(x_n) + Bf'(z_n)}. \tag{8.42}$$

We now compute the convergence orders of the above suggested algorithms using Maple 7.0.

8.2.3 Convergence Analysis of Two-step Iterative methods

Theorem 8.6 *Let $w \in I$ be a simple zero of sufficiently differentiable function $f : I \sqsubseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to w, then the algorithm 1 is cubically convergent for $A = -\frac{20}{3}$ and $B = \frac{50}{3}$.*

Proof. Let w be a simple zero of f and $x_n = w + e_n$ with an error e_n. By Taylor's expansion, we have:

$$f(x_n) = f^{'}(w)\left(e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6\right) + O(e_n^7). \tag{8.43}$$

$$f^{'}(x_n) = f^{'}(w)\left(1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_{65} e_n^5\right) + O(e_n^6), \tag{8.44}$$

where

$$c_k = \left(\frac{1}{k!}\right)\frac{f^{(k)}(w)}{f^{'}(w)}, k = 2, 3, ..., \text{ and } e_n = x_n - w. \tag{8.45}$$

Using (8.43) and (8.44), we have

$$\frac{f(x_n)}{f^{'}(x_n)} = e_n - c_2 e_n^2 + 2\left(c_2^2 - c_3\right)e_n^3 + \left(7c_2c_3 - 3c_4 - 4c_2^3\right)e_n^4 + O(e_n^5). \tag{8.46}$$

Using (8.46) in (8.33), we thus have

$$\begin{aligned} h_n &= w + \frac{7}{10}e_n + \frac{3}{10}c_2 e_n^2 + \left(-\frac{3}{5}c_2^2 + \frac{3}{5}c_3\right)e_n^3 \\ &\quad + \left(\frac{9}{10}c_4 - \frac{21}{10}c_2c_3 + \frac{6}{5}c_2^3 -\right)e_n^4 + O\left(e_n^5\right) \end{aligned} \tag{8.47}$$

Expanding $f(h_n)$ by Taylor's series about w,we have:

$$\begin{aligned} f(h_n) &= f^{'}(w)(\frac{7}{10}e_n + \frac{79}{100}c_2 e_n^2 + \frac{1}{50}(\frac{943}{20}c_3 - 9c_2^2)e_n^3 \\ &\quad + \frac{1}{20}(-\frac{819}{50}c_2c_3 + \frac{11401}{500}c_4 + 9c_2^3)e_n^4) + O\left(e_n^5\right). \end{aligned} \tag{8.48}$$

By Taylor's series, we have

$$\begin{aligned} f^{'}(h_n) &= f^{'}(w)(1 + \frac{7}{5}c_2 e_n + \left(\frac{3}{5}c_2^2 + \frac{147}{100}c_3\right)e_n^2 \\ &\quad + \frac{1}{5}\left(-6c_2^3 + \frac{123}{10}c_2c_3 + \frac{343}{50}c_4\right)e_n^3 \\ &\quad + \frac{1}{5}\left(12c_2^4 - \frac{129}{4}c_3c_2^2 + \frac{63}{5}c_3^2 + \frac{891}{50}c_2c_4 + \frac{2401}{400}c_5\right)e_n^4) \\ &\quad + O(e_n^5). \end{aligned} \tag{8.49}$$

Using (8.43), (8.44) and (8.49), we have:

$$\begin{aligned}\frac{10f(x_n)}{Af'(x_n)+Bf'(h_n)} &= \frac{10}{A+B}e_n - \frac{2\left(5A+2B\right)}{\left(A+B\right)^2}c_2e_n^2 \\ &\quad +\frac{1}{10}(4\left(50A^2-40AB-B^2\right)c_2^2 \\ &\quad -(A+B)(200A+47B)c_3)e_n^3 + O\left(e_n^4\right). \qquad (8.50)\end{aligned}$$

Therefore, by using (8.50) in (8.34), we have

$$\begin{aligned}x_{n+1} &= w+\left(1-\frac{10}{A+B}\right)e_n + \frac{2\left(5A+2B\right)}{\left(A+B\right)^2}c_2e_n^2 \\ &\quad -\frac{1}{10}(4\left(50A^2-40AB-B^2\right)c_2^2 \\ &\quad -(A+B)(200A+47B)c_3)e_n^3 + O\left(e_n^4\right).\end{aligned}$$

For $A=-\frac{20}{3}$ and $B=\frac{50}{3}$, we obtain:

$$e_{n+1} = \left(c_2^2 - \frac{11}{20}c_3\right)e_n^3 + O\left(e_n^4\right).$$

Thus, we observe that the method is cubically convergent. ■

For $A=-\frac{20}{3}$ and $B=\frac{50}{3}$, algorithm 1 takes the following form:

Algorithm 1 (FM1). For a given initial guess x_0, find the approximate solution of (8.1) by:

$$\begin{aligned}h_n &= x_n - \frac{3f\left(x_n\right)}{10f'\left(x_n\right)}, \\ x_{n+1} &= x_n - \frac{3f\left(x_n\right)}{5f'\left(h_n\right)-2f'\left(x_n\right)}. \qquad (8.51)\end{aligned}$$

Similarly, we can compute the convergence orders of algorithm 2 to algorithm 4 using Maple 7.0 and are given in the form of the following Theorem 8.7 to Theorem 8.9:

Theorem 8.7 *Let $w\in I$ be a simple zero of sufficiently differentiable function $f: I \sqsubseteq \mathbb{R}\rightarrow\mathbb{R}$ for an open interval I. If x_0 is sufficiently close to w, then the algorithm 2 is cubically convergent for $A=\frac{20}{7}$ and $B=\frac{50}{7}$ and the error equation is given by*

$$e_{n+1} = \left(c_2^2 + \frac{1}{20}c_3\right)e_n^3 + O\left(e_n^4\right).$$

Thus, for $A=\frac{20}{7}$ and $B=\frac{50}{7}$, algorithm 2 takes the following form:

Algorithm 2 (FM2). For a given initial guess x_0, find the approximate solution of (8.1) by:

$$\begin{aligned} z_n &= x_n - \frac{7f(x_n)}{10f'(x_n)}, \\ x_{n+1} &= x_n - \frac{7f(x_n)}{5f'(z_n) + 2f'(x_n)}. \end{aligned} \tag{8.52}$$

Theorem 8.8 *Let $w \in I$ be a simple zero of sufficiently differentiable function $f : I \sqsubseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to w, then the algorithm 3 has fourth order convergence for $A = -\frac{70}{3}$ and $B = \frac{100}{3}$ and the error equation is given by*

$$e_{n+1} = \left(c_2^3 - \frac{21}{10}c_2c_3\right)e_n^4 + O\left(e_n^5\right).$$

Thus, for $A = -\frac{70}{3}$ and $B = \frac{100}{3}$, algorithm 3 takes the following form:

Algorithm 3 (FM3). For a given initial guess x_0, find the approximate solution of (8.1) by:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ h_n &= x_n - \frac{3f(x_n)}{10f'(x_n)}, \\ x_{n+1} &= y_n - \frac{3f(y_n)}{10f'(h_n) - 7f'(x_n)}. \end{aligned} \tag{8.53}$$

Theorem 8.9 *Let $w \in I$ be a simple zero of sufficiently differentiable function $f : I \sqsubseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to w, then the algorithm 4 has fourth order convergence for $A = -\frac{30}{7}$ and $B = \frac{100}{7}$ and the error equation is given by*

$$e_{n+1} = \left(c_2^3 - \frac{9}{10}c_2c_3\right)e_n^4 + O\left(e_n^5\right).$$

Thus, for $A = -\frac{30}{7}$ and $B = \frac{100}{7}$, algorithm 4 takes the following form:

Algorithm 4 (FM4). For a given initial guess x_0, find the approximate solution of (8.1) by:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{7f(x_n)}{10f'(x_n)}, \\ x_{n+1} &= y_n - \frac{7f(y_n)}{10f'(z_n) - 3f'(x_n)}. \end{aligned} \tag{8.54}$$

8.2.4 Three Step iterative methods

We now suggest some three-step iterative algorithms based on algorithm 1, 2, 3 and 4.

Algorithm 5 (FM5). For a given initial guess x_0, find the approximate solution of (8.1) by:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{8.55}$$

$$h_n = y_n - \frac{3f(y_n)}{10f'(y_n)}, \tag{8.56}$$

$$x_{n+1} = y_n - \frac{3f(y_n)}{5f'(h_n) - 2f'(y_n)}. \tag{8.57}$$

Algorithm 6 (FM6). For a given initial guess x_0, find the approximate solution of (8.1) by:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{8.58}$$

$$z_n = y_n - \frac{7f(y_n)}{10f'(y_n)}, \tag{8.59}$$

$$x_{n+1} = y_n - \frac{7f(y_n)}{5f'(z_n) + 2f'(y_n)}. \tag{8.60}$$

Algorithm 7 (FM7). For a given initial guess x_0, find the approximate solution of (8.1) by:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{8.61}$$

$$z_n = y_n - \frac{f(y_n)}{f'(y_n)}, \tag{8.62}$$

$$h_n = y_n - \frac{3f(y_n)}{10f'(y_n)}, \tag{8.63}$$

$$x_{n+1} = z_n - \frac{3f(z_n)}{10f'(h_n) - 7f'(y_n)}. \tag{8.64}$$

Algorithm 8 (FM8). For a given initial guess x_0, find the approximate solution of (8.1) by:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{8.65}$$

$$z_n = y_n - \frac{f(y_n)}{f'(y_n)}, \tag{8.66}$$

$$h_n = y_n - \frac{7f(y_n)}{10f'(y_n)}, \tag{8.67}$$

$$x_{n+1} = z_n - \frac{7f(z_n)}{10f'(k_n) - 3f'(y_n)}. \tag{8.68}$$

8.2.5 Convergence Analysis of Three-step Iterative Methods

Theorem 8.10 *Let $w \in I$ be a simple zero of sufficiently differentiable function $f : I \sqsubseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to w, then the algorithm 5 has sixth order convergence.*

Proof. Let w be a simple zero of f and $x_n = w + e_n$ with an error e_n. By Taylor's expansion, we have:

$$f(x_n) = f^{'}(w)\left(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6\right) + O(e_n^7), \tag{8.69}$$

$$f^{'}(x_n) = f^{'}(w)\left(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_{65}e_n^5\right) + O(e_n^6), \tag{8.70}$$

where

$$c_k = \left(\frac{1}{k!}\right)\frac{f^{(k)}(w)}{f'(w)}, k = 2, 3, ...\text{and } e_n = x_n - w. \tag{8.71}$$

Using (8.69) and (8.70), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2\left(c_2^2 - c_3\right)e_n^3 + \left(7c_2c_3 - 3c_4 - 4c_2^3\right)e_n^4 + O(e_n^5). \tag{8.72}$$

Using (8.72) in (8.55), we obtain

$$\begin{aligned} y_n &= w + c_2e_n^2 + \left(-2c_2^2 + 2c_3\right)e_n^3 - \left(7c_2c_3 - 4c_2^3 - 3c_4\right)e_n^4 \\ &\quad + O\left(e_n^5\right). \end{aligned} \tag{8.73}$$

Expanding $f(y_n)$ by Taylor's series about w,we have:

$$f(y_n) = f(w) + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (-7c_2c_3 + 3c_4 + 5c_2^3)e_n^4 + O\left(e_n^5\right). \tag{8.74}$$

By Taylor's series, we have

$$\begin{aligned} f^{'}(y_n) &= f^{'}(w)(1 + 2c_2^2e_n^2 + \left(-4c_2^3 + 4c_2c_3\right)e_n^3 \\ &\quad + (-11c_3c_2^2 + 8c_2^4 + 6c_2c_4)e_n^4) + O(e_n^5). \end{aligned} \tag{8.75}$$

Using (8.74) and (8.75), we have

$$\frac{f(y_n)}{f'(y_n)} = e_n - c_2e_n^2 + 2\left(c_2^2 - c_3\right)e_n^3 + \left(7c_2c_3 - 3c_4 - 4c_2^3\right)e_n^4 + O(e_n^5). \tag{8.76}$$

Using (8.76) in (8.56), we thus have

$$\begin{aligned} h_n &= w + \frac{7}{10}c_2e_n^2 + \left(-\frac{7}{5}c_2^2 + \frac{7}{5}c_3\right)e_n^3 \\ &\quad + \left(\frac{21}{10}c_4 - \frac{49}{10}c_2c_3 + \frac{31}{10}c_2^3\right)e_n^4 + O\left(e_n^5\right). \end{aligned} \tag{8.77}$$

Expanding $f(h_n)$ by Taylor's series about w,we have:

$$\begin{aligned} f(h_n) &= f^{'}(w)\,(\frac{7}{10}c_2e_n^2+\left(-\frac{7}{5}c_2^2+\frac{7}{5}c_3\right)e_n^3 \\ &+\frac{1}{10}\left(\frac{359}{10}c_2^3-49c_2c_3+21c_4\right)e_n^4 \\ &+\left(-\frac{219}{25}c_2^4+\frac{429}{25}c_3c_2^2-\frac{21}{5}c_3^2-7c_2c_4+\frac{14}{5}c_5\right)e_n^5) \\ &+O(e_n^6). \qquad (8.78) \end{aligned}$$

By Taylor's series, we have

$$\begin{aligned} f^{'}(h_n) &= f^{'}(w)(1+\frac{7}{5}c_2^2e_n^2-\frac{14}{5}(c_2^3-c_2c_3)e_n^3 \\ &+\frac{1}{5}\left(31c_2^4+21c_2c_4-\frac{833}{20}c_2^2c_3\right)e_n^4 \\ &+\left(-\frac{68}{5}c_2^5+\frac{613}{25}c_2^3c_3-14c_2^2c_4-\frac{63}{25}c_2c_3^2+\frac{28}{5}c_2c_5\right)e_n^5) \\ &+O\left(e_n^6\right). \qquad (8.79) \end{aligned}$$

Using (8.74), (8.75) and (8.79) in

$$\begin{aligned} \frac{3f(y_n)}{5f'(h_n)-2f'(y_n)} &= c_2e_n^2+(2c_3-2c_2^2)e_n^3+(4c_2^3+3c_4-7c_2c_3)e_n^4 \\ &+(-6c_3^2-8c_2^4+20c_3c_2^2-10c_2c_4+4c_5)e_n^5 \\ &+O\left(e_n^6\right). \qquad (8.80) \end{aligned}$$

Therefore, by using (8.80) in (8.57), we have

$$x_{n+1}=w+(c_2^5-\frac{11}{20}c_3c_2^3)e_n^6+O\left(e_n^7\right).$$

Hence, we obtain

$$e_{n+1}=(c_2^5-\frac{11}{20}c_3c_2^3)e_n^6+O\left(e_n^7\right).$$

Thus, we observe that the method has sixth order convergence. ■

Similarly, we can compute the convergence orders of algorithm 6 to algorithm 8 and are given in the form of the following Theorem 8.11 to Theorem 8.13:

Theorem 8.11 *Let $w \in I$ be a simple zero of sufficiently differentiable function $f : I \sqsubseteq \mathbb{R} \rightarrow \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to w, then the algorithm 6 has sixth order convergence and the error equation is given by*

$$e_{n+1}=(c_2^5+\frac{1}{20}c_3c_2^3)e_n^6+O\left(e_n^7\right).$$

Theorem 8.12 *Let $w \in I$ be a simple zero of sufficiently differentiable function $f : I \sqsubseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to w, then the algorithm 7 has eighth order convergence and the error equation is given by*

$$\begin{aligned} e_{n+1} &= (126c_2^7 - \frac{3671}{10}c_3c_2^5 + 174c_4c_2^4 + (199c_3^2 - 50c_5)c_2^3 + \\ &\quad (-105c_4c_3 + 6c_6)c_2^2 + (30c_3^3 + 2c_4^2 + 2c_7)c_2 \\ &\quad +9c_4c_5 + 9c_3c_6 - 33c_3^2c_4)e_n^8 + O\left(e_n^9\right). \end{aligned}$$

Theorem 8.13 *Let $w \in I$ be a simple zero of sufficiently differentiable function $f : I \sqsubseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to w, then the algorithm 8 has eighth order convergence and the error equation is given by*

$$\begin{aligned} e_{n+1} &= (126c_2^7 - \frac{3659}{10}c_3c_2^5 + 174c_4c_2^4 + (199c_3^2 - 50c_5)c_2^3 + \\ &\quad (-105c_4c_3 + 6c_6)c_2^2 + (30c_3^3 + 2c_4^2 + 2c_7)c_2 \\ &\quad +9c_4c_5 + 9c_3c_6 - 33c_3^2c_4)e_n^8 + O\left(e_n^9\right). \end{aligned}$$

8.2.6 Numerical Examples

In this subsection, we now consider some numerical examples to demonstrate the performance of the newly developed iterative methods. The methods being chosen for numerical comparison are some of the efficient methods developed in the recent past. All the computations for the above mentioned methods, are performed using Maple 7 with 128 digits precision and $\varepsilon = 10^{-15}$ is taken as tolerance. The following criteria is used for estimating the zeros:

(i) $\delta = |x_{n+1} - x_n| < \varepsilon$

(ii) $|f(x_n)| < \varepsilon$

Thus for convergence criteria, it is required that the distance between two consecutive iterates be less than 10^{-15}. x_0 represents the initial guess, and w, the exact zero of the non-linear function $f(x)$. In all the tables, the columns below each method represents the number of iterations required to find the approximate solution of the respective functions.

The following examples are used for comparison, most of which are taken from [54, 105, 68].

Examples	Exact Zero
$f_1(x) = x^2 - e^x - 3x + 2,$	$w = .257530285439860760455367304 9$
$f_2(x) = x^3 + 4x^2 - 15,$	$w = 1.631980805566063517522106445$
$f_3(x) = \ln(x),$	$w = 1$
$f_4(x) = \sin(x) - 10^{-x},$	$w = 3.140869666785039238259944749$
$f_5(x) = (x-1)^6 - 1,$	$w = 2$
$f_6(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5,$	$w = -1.207647827130918927009416 75$
$f_7(x) = e^{-x} + \cos(x),$	$w = 1.746139530408012417650703089$
$f_8(x) = x^3 - 10,$	$w = 2.154434690031883721759293566$
$f_9(x) = e^{x^2+7x-30-1},$	$w = 3$
$f_{10}(x) = cos(x) - x,$	$w = .7390851332151606416553120876$
$f_{11}(x) = \frac{1}{x} - 1,$	$w = 1$
$f_{12}(x) = x^2 - 10\cos(x),$	$w = -1.379364594222030825391587 9$
$f_{13}(x) = \sin(x) - \frac{x}{2},$	$w = 1.895494267033980947144035738$
$f_{14}(x) = \sin^2(x) - x^2 + 1,$	$w = 1.404491648215341226035086817$
$f_{15}(x) = 10xe^{-x^2} - 1,$	$w = 1.679630610428449940674920338$

Table 8.1: Numerical Examples

In Table 8.2, we compare the classical Newton's method (CN), the Weerakoon-Fernando method (WF) [105], the midpoint method (MM) [43], the method of H. H. H. Homeier (HM) [49], the method of J. Kou et al. (KLW) [53] and the newly developed cubically convergent two-step methods, FM1 and FM2. All these methods except Newton's method are cubically convergent. D stands for divergent.

$f(x)$	x_0	CN	WF	MM	HM	KLW	$FM1$	$FM2$
f_1	2.0	6	5	4	5	5	4	4
	3.0	7	5	5	5	6	5	5
f_2	0.8	7	5	5	4	6	5	5
	2.5	6	5	4	4	5	4	4
f_3	3.0	D	D	5	5	5	5	6
	5.5	D	D	7	7	D	5	D

$f(x)$	x_0	CN	WF	MM	HM	KLW	$FM1$	$FM2$
f_4	3.0	4	4	4	4	4	4	3
	3.5	4	4	4	4	4	4	3
f_5	2.5	8	6	6	5	6	5	6
	3.0	10	7	6	6	7	6	7
f_6	0.0	72	$Fails$	D	14	D	27	D
	−2.0	9	6	6	6	6	6	6
f_7	1.6	5	4	4	4	4	4	3
	2.0	5	4	4	4	4	4	4
f_8	2.5	6	4	4	4	4	4	4
	3.0	6	5	5	4	5	4	5
f_9	3.25	9	7	6	6	6	6	6
	3.5	9	9	9	8	8	9	9
f_{10}	3.0	7	10	5	6	6	5	5
	3.5	19	9	6	D	D	5	8
f_{11}	2.1	$Fails$	D	6	5	5	5	7
	3.0	$Fails$	$Fails$	D	$Fails$	7	7	7
f_{12}	0.3	8	5	5	5	6	5	4
	−1.0	5	4	4	4	4	4	4
f_{13}	1.5	6	5	5	5	6	5	4
	3.5	6	4	4	4	5	4	4
f_{14}	1.8	6	4	4	4	5	4	4
	1.1	6	5	4	4	5	4	4
f_{15}	1.3	6	4	4	4	5	4	4
	1.7	5	3	3	3	3	3	3

Table 8.2: Comparison of Cubically Convergent Iterative Methods

In Table 8.3, we compare the classical Newton's method (CN), the method of M. A. Noor et al. (NA) [68], the method of J. F. Traub (MT) [93], the Ostrowski's method (OM) [71] and the newly developed fourth order convergent two-step methods, FM3 and FM4. All these methods except Newton's method are fourth-order convergent.

$f(x)$	x_0	CN	NA	MT	OM	$FM3$	$FM4$
f_1	0.9	4	4	3	3	3	3
	2.0	6	4	4	4	5	4
f_2	0.8	7	5	4	4	4	4
	2.0	6	4	4	4	4	4
f_3	0.7	6	4	4	3	4	3
	1.2	5	4	3	3	3	3
f_4	3.0	4	3	3	3	3	3
	3.5	4	3	3	3	3	3
f_5	2.5	7	4	4	4	4	4
	3.5	8	5	5	5	4	5
f_6	−1.2	6	4	4	3	4	3
	−2.0	9	6	5	5	5	5
f_7	1.0	5	4	3	3	3	3
	2.0	5	3	3	3	3	3
f_8	0.5	11	7	6	6	5	6
	1.5	7	4	4	4	4	4
f_9	3.2	8	5	5	5	4	5
	3.5	13	8	7	6	5	7
f_{10}	0.5	5	3	3	3	3	3
	2.0	5	4	3	3	4	3
f_{11}	0.8	6	4	4	$Fails$	4	3
	1.2	6	4	4	$Fails$	4	3
f_{12}	0.3	8	5	5	4	5	4
	−1.0	5	4	3	4	4	4
f_{13}	2.0	5	3	3	3	3	3
	3.5	6	5	4	4	4	4
f_{14}	−1.0	7	4	4	4	4	4
	1.1	6	4	4	4	4	4
f_{15}	1.3	6	4	4	4	4	4
	1.7	5	3	3	3	3	3

Table 8.3: Comparison of Fourth-order Convergent Iterative Methods

In Table 8.4, we compare the method of M. Grau (MG) [45], a sixth order variant of Ostrowski's method, the seventh order convergent J. Kou method (KM) [54], the newly developed three-step sixth order convergent algorithms, FM5 and FM6 and the eighth order convergent algorithms, FM7 and FM8.

$f(x)$	x_0	MG	KM	$FM5$	$FM6$	$FM7$	$FM8$
f_1	0.9	3	3	3	3	2	2
	2.0	4	4	3	3	3	3
f_2	0.8	4	4	4	4	3	3
	2.0	3	3	3	3	3	3
f_3	0.7	3	3	3	3	3	3
	1.2	3	3	3	3	3	3
f_4	3.0	3	3	3	3	2	2
	3.5	3	3	3	3	3	3
f_5	2.5	3	3	3	3	3	3
	3.5	4	4	4	4	3	3
f_6	−1.2	3	2	3	3	2	2
	−1.0	3	3	3	3	3	3
f_7	1.7	3	3	3	3	2	2
	2.0	3	3	3	3	3	3
f_8	0.5	5	14	5	5	5	5
	1.5	3	3	3	3	3	3
f_9	3.2	4	4	4	4	4	4
	3.5	6	5	6	6	4	5
f_{10}	0.5	3	3	3	3	3	3
	1.0	3	3	3	3	3	3
f_{11}	0.8	3	$Fails$	3	3	3	3
	1.2	$Fails$	2	3	3	3	3

$f(x)$	x_0	MG	KM	$FM5$	$FM6$	$FM7$	$FM8$
f_{12}	0.3	4	5	4	4	4	4
	−1.4	3	2	3	2	2	2
f_{13}	1.0	$Fails$	7	6	5	4	6
	2.0	3	3	3	3	3	3
f_{14}	1.1	3	3	3	3	3	3
	1.6	3	3	3	3	3	3
f_{15}	1.0	3	3	3	3	3	3
	1.8	4	3	3	3	3	3

Table 8.4: Comparison of Higher Order Convergent Iterative Methods

New algorithms are tested for almost all types of non-linear functions, polynomials and transcendental functions. Table 8.2 shows that if the initial guess is far from the exact root then the newly developed cubically convergent methods specially FM1 converges while most of the existing methods diverge or fail to converge. Tables 8.3 and 8.4 show that the new fourth order, sixth order and eighth order convergent methods namely FM3, FM4, FM5, FM6, FM7 and FM8 are at least comparable with the existing methods of respective orders and in some cases perform better than the existing methods. It can be further noted that the algorithms FM3 and FM4 are free from second derivative in contrast to other recently developed fourth order convergent methods [92]. Moreover, the computational efficiency of all the methods derived in this section is either equal or greater than the computational efficiency of Newton's method.

8.3 Conclusion

We, in this chapter, have established the fact that the specially derived quadrature rules developed in the sense of inequalities may be applied to develop many other iterative methods. Moreover, the presented iterative methods are extendable to the system of non-linear equations. The iterative algorithms obtained in this chapter are of cubic, fourth, sixth and eighth order and are computationally efficient in comparison with other known algorithms of this type.

Chapter 9

Concluding Remarks

9.1 Critical Analysis

By analyzing the Ostrowski type inequalities obtained in this dissertation, the following concluding aspects of this dissertation are highlighted:

1. The insertion of an arbitrary parameter, hence modifying the Peano kernels in this manner can improve the bounds.

2. The bounds are presented for first and twice differentiable functions which are more applicable in the cases where higher derivatives do not exists.

3. The bounds are also obtained for functions of bounded variation, Lipschitzian functions and for Euclidean norm which enlarges the applicability of the results.

4. The results obtained in here can also give estimates for three-point inequalities in contrast to the existing inequalities of corresponding domains.

5. The bounds are obtained by using Grüss and Pre-Grüss inequalities which provide more accurate approximations, since the bounds are expressed in terms of the oscillation of a function rather than its sup norm that is usually not as tight.

6. The composite quadrature rules mainly involve an arbitrary point. The concept is useful in the sense when the data is given at discrete points and is not uniform.

7. The results are also obtained for shells, spheres and balls by moving in the n-dimensional Euclidean space $\mathbb{R}^n$.

8. More refined bounds are obtained for the expectation of random variables defined on finite as well as on infinite intervals.

9. The results obtained are applied to special means to show their applicability towards obtaining direct relationship of these means.

10. Estimates for the beta random variables are provided as applications of the inequalities presented.

11. Applicability of the obtained inequalities towards constructing some quadrature based iterative methods for solving non-linear equation is also shown.

9.2 Future Extensions

The outcomes of this dissertation may further be extended:

- To obtain Ostrowski type inequalities for n-differentiable functions.
- To present multivariate analogues of the inequalities, extending to inequalities involving double integrals or more than one independent variable.
- To obtain weighted versions of the inequalities.
- To obtain the inequalities in other environments such as for linear spaces.
- To obtain fractional Ostrowski type inequalities.
- The inequalities may also be extended to time scale domains.

9.3 Research Publications

The following research material has been published in some international journals from the thesis:

- [111] is based on the results obtained in Section 2.1.
- [107] is based on the results presented in Section 2.3

- The results presented in Section 3.1 have been published in [113].
- The results of Section 3.2 have been published in [87].
- The results from Section 3.3 have been published in [88].
- The results presented in Section 3.4 are published in [91].
- [109] is based on the results presented in Section 4.1.
- [114] covers the work of Section 4.2.
- The results presented in Section 4.3 have been publication in [112].
- The work of Section 5.1 has been published in [110].
- The work presented in Section 7.1 has been published in [89].
- Section 7.2 has been published in [90].
- [108] presents the results of Section 8.1.

The final versions of the research papers may slightly vary from the original version in terms of presentation and bibliography.

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