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All about L^1-convergence of modified trigonometric sums

– Monograph –

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”To my parents and my family”

Preface

Jean-Baptiste Joseph Fourier is best known for initiating the investigation of Fourier series and their applications to problems of heat transfer and vibrations. These series are special cases of trigonometric series and are named as Fourier series in honor of him. In the sequel, many questions pertaining to Fourier series has been raised. One of the main question about them is: When a trigonometric series is a Fourier series? Another interesting question related to the above question is: If a trigonometric series converges in L^1-norm to a function $f \in L^1$, then is it a Fourier series of the function f? The answer of this question has been interesting and challenging for almost one century and still receives considerable attentions. Among others, in the accessible literature we encounter the following answer: If a trigonometric series converges in L^1-norm to a function $f \in L^1$, then it is a Fourier series, however the converse of this is not true in general i.e., exist many Fourier series which are not convergent in L^1-metric. For example, F. Riesz, in 1932, gave the following counter example (the details of this example the interested reader can find in the well-known book of N. Bari [79]): The series

$$\sum_{m=2}^{\infty} \frac{\cos mx}{\log m}$$

is a Fourier series, however it does not converge in L^1-metric.

As a result of above observations, it seems that lots of researchers "curiously asked": How to make possible that this converse statement to be true? The answer has been obtained by modification of partial sums of a trigonometric series and by imposing several conditions on their coefficients. As initial work regarding to L^1-convergence of Fourier series, by imposing several conditions on their coefficients, belongs to W. H. Young in 1913 and A. N. Kolmogorov in 1923 who employed the class of convex sequences and the class of quasi-convex sequences, respectively. On the other side, regarding to modification of partial sums of a trigonometric series, were J. W. Garret and C. V. Stanojevic the first, in 1976, who introduced the modified trigonometric cosine sums as follows

$$f_n(x) := \frac{1}{2}\sum_{k=0}^{n}(\Delta a_k) + \sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_j)\cos kx.$$

Later on, were S. Kumari and B. Ram, during 1988-1989, who introduced next new modified trigonometric cosine and sine sums as follows

$$g_n^c(x) := \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right)k\cos kx$$

and

$$g_n^s(x) := \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right)k\sin kx.$$

These modified trigonometric sums and many others have been subject of studies for more than fifty years and still goes on. In this, monograph we have collected all results related to their L^1-convergence which have been published till now in several journals. These results are written in a chronological order as much as possible and are included in seven main sections.

The objective of the first section is to inform the reader about some real and, separately, to some complex classes of sequences which are used throughout this book and more. Moreover, in it are included some basic facts on trigonometric series and necessary auxiliary lemmas.

In the second section we deal with L^1-convergence of modified trigonometric sums $f_n(x)$ with coefficients which are quasi-convex, which belong to the class $\mathbf{C}$, are of bounded variation, belong to the class $\mathbf{S}$, belong to the class $\mathbf{S'}$, belong to the class $\mathbf{K}$, are semi-convex, belong to the class $\mathbf{S_r}$, are semi-convex of fractional order, and belong to the class $\mathbf{T}$. Also, $L^p(0 < p < 1)$-convergence of modified trigonometric sums $f_n(x)$ has been treated in this section. We have finalized this section with L^1-convergence of modified trigonometric sums $w_n^c(x)$ and $w_n^s(x)$ with coefficients of bounded variation as well as L^1-convergence of modified trigonometric sums $z_n^c(x)$ and $z_n^s(x)$ with generalized semi-convex coefficients.

In third section we deal with L^1-convergence of modified trigonometric sums $g_n^c(x)$ and $g_n^s(x)$ with coefficients from the class $\mathbf{S}$, from the class $\mathbf{R}$, from the class $\mathbf{S}(\delta)$, from the class $\mathbf{S}(\delta)$ without additional condition, from the class $\mathbf{S}$ without additional condition, from the class $\mathbf{F}_p$, from the class $\mathbf{BV} \cap \mathbf{C}$, and finally from the class $\mathbf{S}^{**}$. We close this section with L^1-convergence of modified trigonometric sums $g_{n,m}^c(x)$ and $g_{n,m}^s(x)$, which are generalizations of the sums $g_n^c(x)$ and $g_n^s(x)$ respectively, with coefficients from the class $\mathbf{R}$.

In the fourth section we deal with L^1-convergence of other modified trigonometric sums which are generalization of those previously introduced, those that are new, and with coefficients belonging to above generalized classes or to "old" and "new" ones.

The fifth section mainly deals with L^1-convergence of r-th derivative modified trigonometric sums whose coefficients belong to several above generalized classes of sequences.

The sixth section deals with L^1-convergence of complex modified trigonometric sums and their r-th derivative whose coefficients belong to several classes of complex sequences.

The seventh section, and the last one, deals L^1-convergence of double modified trigonometric sums with some special type of coefficients. It contains just few results on double trigonometric series, and we hope to complete it in the future with new results.

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Contents

1

Introduction

In this section we give some classes of real sequences, some classes of complex sequences, some basic facts of trigonometric series, and lots of auxiliary lemmas which are used throughout this monograph.

1.1 Some classes of real sequences

By an infinite real sequence we mean a mapping whose domain is the set of natural numbers and its range may be an arbitrary set of real numbers.

Let $u_0, u_1, \ldots, u_n, \ldots$ and $v_0, v_1, \ldots, v_n, \ldots$ be two real sequences. Then the following transformation holds true.

Lemma 1.1 (Abel's transformation–Discrete summation by parts). *The equality holds true*

$$\sum_{k=m}^{n} u_k v_k = \sum_{k=m}^{n-1} U_k(v_k - v_{k+1}) + U_n v_n - U_{m-1} v_m, \quad m \geq 0,$$

where $U_k = u_0 + u_1 + \ldots + u_k$ *and* $U_{-1} := 0$.

Proof. Since $u_k = U_k - U_{k-1}$, $k = m, m+1, \ldots, n$ we have

$$\begin{aligned}
\sum_{k=m}^{n} u_k v_k &= u_m v_m + u_{m+1} v_{m+1} + \cdots + u_n v_n \\
&= (U_m - U_{m-1})\, v_m + (U_{m+1} - U_m)\, v_{m+1} + \cdots + (U_n - U_{n-1})\, v_n \\
&= -U_{m-1} v_m + U_m(v_m - v_{m+1}) + \cdots + U_{n-1}\,(v_{n-1} - v_n) + U_n v_n \\
&= -U_{m-1} v_m + \sum_{k=m}^{n-1} U_k(v_k - v_{k+1}) + U_n v_n.
\end{aligned}$$

The proof is completed.

For $m = 0$ it is clear that

$$\sum_{k=0}^{n} u_k v_k = \sum_{k=0}^{n-1} U_k (v_k - v_{k+1}) + U_n v_n. \tag{1.1}$$

For any sequence $\{u_n\}$ we define the differences $\Delta^0 u_n := u_n, \Delta^1 u_n := \Delta u_n := u_n - u_{n+1}, \Delta^2 u_n := \Delta(\Delta^1 u_n) = u_n - 2u_{n+1} + u_{n+2}$, and in general

$$\Delta^m u_n := \Delta(\Delta^{m-1} u_n) = \Delta^{m-1} u_n - \Delta^{m-1} u_{n+1}, \quad m \in \{1, 2, \ldots\}.$$

In this context (1.1) can be written as follows

$$\sum_{k=0}^{n} u_k v_k = \sum_{k=0}^{n-1} \Delta v_k U_k + U_n v_n. \tag{1.2}$$

Using these notation we have,

Definition 1.2. *The sequence $\{u_n\}$ is monotone decreasing (it is written $u_n \downarrow$) if $\Delta u_n \geq 0$ for all n.*

Example 1.3. For example, since $\Delta \left(\frac{1}{n+1} \right) = \frac{1}{(n+1)(n+2)} \geq 0$ for all $n \geq 0$, then the sequence $\{(n+1)^{-1}\}$ is monotone decreasing one.

As an important application of (1.2) is the following.

Corollary 1.4 (Abel's Lemma). *If there exists a real number $M > 0$ such that $|U_k| \leq M$, for $0 \leq k \leq n$, and $v_0 \geq v_1 \geq \cdots \geq v_n \geq 0$, then*

$$\left| \sum_{k=0}^{n} u_k v_k \right| \leq M u_0.$$

Proof. Based on (1.2) and our assumptions we have

$$\begin{aligned}
\left| \sum_{k=0}^{n} u_k v_k \right| &\leq \sum_{k=0}^{n-1} |\Delta v_k| \, |U_k| + |U_n| \, |v_n| \\
&\leq \sum_{k=0}^{n-1} (\Delta v_k) \cdot M + M v_n \\
&= M \left[(v_0 - v_1) + (v_1 - v_2) + \cdots + (v_{n-1} - v_n) + v_n \right] = M v_0.
\end{aligned}$$

The proof is completed.

Using the second difference we can recall,

Definition 1.5. *The sequence $\{u_n\}$ is convex if $\Delta^2 u_n \geq 0$ for all n.*

Example 1.6. Defining $\Delta d_j = \frac{1}{j+1}$, $j \in \{0, 1, \ldots\}$, then we have $\Delta^2 d_j = \frac{1}{(j+1)(j+2)} > 0$ for $j \in \{0, 1, \ldots\}$, which means that the sequence $\{d_j\}_{j=0}^{\infty}$ is convex.

Next we give another example.

Example 1.7. We consider the sequence $d_j = \frac{1}{\ln j}$, $j \in \{2, 3, \ldots\}$, then we have

$$\Delta d_j = \frac{1}{\ln j} - \frac{1}{\ln(j+1)} = \int_j^{j+1} \frac{dx}{x \ln^2 x}$$

for $j \in \{2, 3, \ldots\}$, and thus

$$\Delta^2 d_j = \int_j^{j+1} \frac{dx}{x \ln^2 x} - \int_{j+1}^{j+2} \frac{dx}{x \ln^2 x}.$$

Since the function

$$h(x) = \frac{1}{x \ln^2 x}$$

is decreasing, then $\Delta^2 d_j \geq 0$ for all $j \in \{2, 3, \ldots\}$, which means that the sequence $\{d_j\}_{j=2}^{\infty}$ is convex.

Throughout this monograph for two sequence $\{u_n\}$ and $\{v_n\}$ we write $u_n = o(v_n)$ if $\frac{u_n}{v_n} \to 0$ as $n \to \infty$, and $u_n = \mathcal{O}(v_n)$ if $\left|\frac{u_n}{v_n}\right| \leq M$ for a real number $M > 0$ and for all n.

Some properties of convex sequences are included in next statement.

Lemma 1.8. *The following properties holds true.*

(i) If the sequence $\{u_n\}$ is convex and bounded from above, then $u_n \downarrow$.
(ii) If the sequence $\{u_n\}$ is convex and $u_n \to 0$, then $u_n \downarrow 0$.
(iii) If the sequence $\{u_n\}$ is convex and bounded, then

$$n\Delta u_n \to 0 \quad \text{and} \quad \sum_{n=1}^{\infty} (n+1)|\Delta^2 u_n| < \infty.$$

Proof. (i) Assuming that the opposite conclusion is true, then there would exists a value m such that $\Delta u_m < 0$. Since $\{u_n\}$ is convex at any $j \geq m$ we would have $\Delta u_j < 0$ and of course $|\Delta u_j| \geq |\Delta u_m|$. Therefore,

$$\begin{aligned} u_n - u_m &= (u_n - u_{n-1}) + (u_{n-1} - u_{n-2}) + \cdots + (u_{m+1} - u_m) \\ &= -\sum_{j=m}^{n-1} \Delta u_j = \sum_{j=m}^{n-1} |\Delta u_j| \geq (n-m)|\Delta u_m| \to \infty \end{aligned}$$

as $n \to \infty$, which is a contradiction as the sequence $\{u_n\}$ is bounded from above.

(ii) Condition $u_n \to 0$ implies that the sequence $\{u_n\}$ is bounded from above, therefore using the property (i) we obtain $u_n \downarrow$. So, $u_n \downarrow$ with $u_n \to 0$ imply $u_n \downarrow 0$.

(iii) Based on assumption we immediately get $u_n \downarrow$. The sequence $\{u_n\}$ has a finite limit since it is bonded from below. Let $\lim_{n\to} u_n = u$, then the series on the equality

$$u_0 - u = \sum_{j=0}^{\infty} \Delta u_j$$

has monotone decreasing terms and therefore it converges. This means that $n\Delta u_n \to 0$.

Moreover, using Lemma 1.1, we have

$$\sum_{m=0}^{n} \Delta u_m = \sum_{m=0}^{n-1} (m+1)\Delta^2 u_m + (n+1)\Delta u_n.$$

So, $(n+1)\Delta u_n \to 0$ and

$$\sum_{m=0}^{n} \Delta u_m = u_0 - u_n \to u_0 - u \quad as \quad n \to \infty$$

This means that

$$\sum_{m=0}^{n-1} (m+1)\Delta^2 u_m \to u_0 - u \quad or \quad \sum_{m=0}^{\infty} (m+1)\Delta^2 u_m < \infty.$$

The proof is completed.

Remark 1.9. If the sequence $\{u_n\}$ is convex and $u_n \to 0$, then

$$n\Delta u_n \to 0 \quad and \quad \sum_{n=1}^{\infty} (n+1)|\Delta^2 u_n| < \infty$$

as well.

The notion of a bounded sequence is given in next definition.

Definition 1.10. *The sequence $\{u_n\}$ is said to be bounded if there exists a positive real number M such that $|u_n| \le M$ for all n.*

Another class of sequences is the class of quasi-convex sequences.

Definition 1.11. *The sequence $\{u_n\}$ is said to be quasi-convex if*

$$\sum_{n=1}^{\infty} (n+1)|\Delta^2 u_n| < \infty.$$

It is a well-known fact that every bounded convex sequence is quasi-convex but the converse statement is not always true (see Lemma 1.8, (iii)).

Moreover, every quasi-convex zero sequence satisfies conditions: (i) $u_k = o(1)$ as $k \to \infty$ (ii) $S_1 := \sum_{k=0}^{\infty} |\Delta u_k| < \infty$, and

$$S_2 := \sum_{m=2}^{\infty} \left| \sum_{k=1}^{\left[\frac{m}{2}\right]} \frac{\Delta u_{m-k} - \Delta u_{m+k}}{k} \right| < \infty.$$

Namely, since $u_k = o(1)$ as $k \to \infty$, then

$$S_1 = \sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} \Delta^2 u_j \right| \le \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \left| \Delta^2 u_j \right| \le \sum_{k=0}^{\infty} k |\Delta^2 u_k|.$$

Also, since

$$\Delta u_{m-k} - \Delta u_{m+k} = \sum_{i=m-k}^{m+k-1} \Delta^2 u_i,$$

then we have

$$\begin{aligned} S_2 &\le \sum_{m=2}^{\infty} \sum_{k=1}^{\left[\frac{m}{2}\right]} \frac{|\Delta u_{m-k} - \Delta u_{m+k}|}{k} \le \sum_{m=2}^{\infty} \sum_{k=1}^{\left[\frac{m}{2}\right]} \frac{1}{k} \sum_{i=m-k}^{m+k-1} |\Delta^2 u_i| \\ &= \sum_{k=1}^{\infty} \sum_{m=2k}^{\infty} \frac{1}{k} \sum_{i=m-k}^{m+k-1} |\Delta^2 u_i| \le \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \frac{1}{k} \sum_{m=i-k+1}^{i+k} |\Delta^2 u_i| \\ &= 2 \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} |\Delta^2 u_i| = 2 \sum_{i=1}^{\infty} \sum_{k=1}^{i} |\Delta^2 u_i| = 2 \sum_{i=1}^{\infty} i |\Delta^2 u_i| < \infty. \end{aligned}$$

Now we are going to prove a lemma which deals with conditions (i) and (ii).

Lemma 1.12. *Let a_k, $k = 0, 1, 2, \ldots$, be a sequence of real numbers satisfying $S_2 < \infty$ and let n be a positive integer. If*

$$b_k = 0 \;\; for \;\; k \le n \quad and \quad b_k = a_k \;\; for \;\; k > n, \tag{1.3}$$

then, uniformly in respect to $n \ge 2$,

$$S_2 = \sum_{m=n}^{\infty} \sum_{k=1}^{\left[\frac{m}{2}\right]} \frac{|\Delta a_{m-k} - \Delta a_{m+k}|}{k} + \mathcal{O}\left(\max_{\frac{n}{2} \le k < \frac{3n}{2}} |a_k| \log n \right).$$

Proof. Let us estimate the sum

$$\sum_{m=2}^{n-1} \left| \sum_{k=1}^{\left[\frac{m}{2}\right]} \frac{\Delta a_{m-k} - \Delta a_{m+k}}{k} \right|.$$

By the definition of the numbers b_k, $\Delta a_{m-k} = 0$ for all m and k in this sum. Therefore

$$\sum_{m=2}^{n-1} \left| \sum_{k=1}^{\left[\frac{m}{2}\right]} \frac{\Delta a_{m+k}}{k} \right|.$$

Since

$$\sum_{k=1}^{\left[\frac{m}{2}\right]} \frac{\Delta a_{m+k}}{k} = b_{m+1} + \sum_{k=2}^{\left[\frac{m}{2}\right]} \left(\frac{1}{k} - \frac{1}{k+1} \right) b_{m+k} - \frac{1}{\left[\frac{m}{2}\right]} b_{m+\left[\frac{m}{2}\right]+1},$$

and $b_{m+k} = 0$ for $m + k \le n$, $b_{m+k} = a_{m+k}$ for $m + k > n$, then

$$\begin{aligned} &\sum_{m=2}^{n-1} \left| \sum_{k=1}^{\left[\frac{m}{2}\right]} \frac{\Delta a_{m+k}}{k} \right| \\ &\quad \le \sum_{m=2}^{n-1} \left[|b_{m+1}| + \sum_{k=2}^{\left[\frac{m}{2}\right]} \left(\frac{1}{k} - \frac{1}{k+1} \right) |b_{m+k}| + \frac{1}{\left[\frac{m}{2}\right]} \left| b_{m+\left[\frac{m}{2}\right]+1} \right| \right] \\ &\quad \le \max_{\frac{n}{2} \le k < \frac{3n}{2}} |a_k| \sum_{m=2}^{n-1} \frac{1}{n-m} \le \max_{\frac{n}{2} \le k < \frac{3n}{2}} |a_k| (1 + \log n). \end{aligned}$$

Now we estimate the difference

$$\sum_{m=n}^{\infty} \left| \sum_{k=1}^{\left[\frac{m}{2}\right]} \frac{\Delta b_{m-k} - \Delta b_{m+k}}{k} \right| - \sum_{m=n}^{\infty} \left| \sum_{k=1}^{\left[\frac{m}{2}\right]} \frac{\Delta a_{m-k} - \Delta a_{m+k}}{k} \right|.$$

Note that $\Delta b_{m+k} = \Delta a_{m+k}$ for all $m \ge n$. If $m - k > n$, then $\Delta b_{m-k} = \Delta a_{m-k}$ as well. But, $m - \left[\frac{m}{2}\right] > n$ for all $m > 2n$ and, therefore, the following estimate is valid

$$\begin{aligned} &\sum_{m=n}^{\infty} \left| \left| \sum_{k=1}^{\left[\frac{m}{2}\right]} \frac{\Delta b_{m-k} - \Delta b_{m+k}}{k} \right| - \left| \sum_{k=1}^{\left[\frac{m}{2}\right]} \frac{\Delta a_{m-k} - \Delta a_{m+k}}{k} \right| \right| \\ &\quad \le \sum_{m=n}^{2n} \left| \sum_{k=1}^{\left[\frac{m}{2}\right]} \frac{\Delta b_{m-k} - \Delta a_{m-k}}{k} \right|. \end{aligned}$$

Since

$$\sum_{k=1}^{\left[\frac{m}{2}\right]} \frac{\Delta b_{m-k} - \Delta a_{m-k}}{k} = -(b_m - a_m)$$
$$+ \sum_{k=1}^{\left[\frac{m}{2}\right]-1} \left(\frac{1}{k} - \frac{1}{k+1}\right)(b_{m-k} - a_{m-k}) + \frac{1}{\left[\frac{m}{2}\right]}\left(b_{m-\left[\frac{m}{2}\right]} - a_{m-\left[\frac{m}{2}\right]}\right)$$

by setting $d_{m,k} = -1$ if $k = 0$, $d_{m,k} = \frac{1}{k} - \frac{1}{k+1}$ if $k = 1, 2, \ldots, \left[\frac{m}{2}\right] - 1$, and $d_{m,k} = \frac{1}{\left[\frac{m}{2}\right]}$ if $k = \left[\frac{m}{2}\right]$, we can write

$$\sum_{k=1}^{\left[\frac{m}{2}\right]} \frac{\Delta b_{m-k} - \Delta a_{m-k}}{k} = \sum_{k=0}^{\left[\frac{m}{2}\right]} (b_{m-k} - a_{m-k}) d_{m,k}.$$

But $b_{m-k} = 0$ when $k \geq m - n$, $b_{m-k} = a_{m-k}$ when $k < m - n$, and therefore

$$\begin{aligned}
\sum_{m=n}^{2n} \left| \sum_{k=m-n}^{\left[\frac{m}{2}\right]} \frac{\Delta b_{m-k} - \Delta a_{m-k}}{k} \right| &= \sum_{m=n}^{2n} \left| \sum_{k=m-n}^{\left[\frac{m}{2}\right]} a_{m-k} d_{m,k} \right| \\
&\leq \max_{\frac{n}{2} \leq k < \frac{3n}{2}} |a_k| \sum_{m=n}^{2n} \sum_{k=m-n}^{\left[\frac{m}{2}\right]} |d_{m,k}| \\
&= \max_{\frac{n}{2} \leq k < \frac{3n}{2}} |a_k| \left(2 + \sum_{m=n+1}^{2n} \frac{1}{m-n}\right) \\
&\leq \max_{\frac{n}{2} \leq k < \frac{3n}{2}} |a_k| \left(3 + \log n\right).
\end{aligned}$$

The proof is completed.

Definition 1.13. *The sequence* $\{u_n\}$ *is said to be of bounded variation if the series*

$$\sum_{n=1}^{\infty} |\Delta u_n| \tag{1.4}$$

converges.

The class of zero-sequences of bounded variation usually is denoted by **BV**. It is obvious that if $u_n \downarrow 0$, then series (1.4) converges.

A very useful subclass of the class of bounded variation sequences is the so-called Sidon-Telyakovskii class denoted by **S**, seemingly in honor of Sidon, who was the first to introduce this class. We present here its equivalent form expressed by Telyakovskii.

Definition 1.14. *The zero-sequence* $\{u_n\}$ *is said to be in the class* $\mathbf{S}$ *if there exists a sequence* $\{A_n\}$ *such that*

(i) $A_n \downarrow 0$ *as* $n \to \infty$*,*
(ii) $\sum_{n=0}^{\infty} A_n < \infty$*, and*
(iii) $|\Delta u_n| \leq A_n$*, for all* n*.*

It is clear that the inclusion $\mathbf{S} \subset \mathbf{B}V$ holds true. Also it was shown that every quasi-convex zero-sequence satisfies the conditions of the class $\mathbf{S}$. Namely, taking

$$A_n = \sum_{k=n}^{\infty} |\Delta^2 u_k|$$

we have

$$A_n - A_{n+1} = |\Delta^2 u_n| \geq 0,$$

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{k=n}^{\infty} |\Delta^2 u_k| = 0,$$

$$\sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} |\Delta^2 u_k| = \sum_{k=0}^{\infty} |\Delta^2 u_k| \sum_{n=0}^{k} 1 = \sum_{k=0}^{\infty} (k+1)|\Delta^2 u_k| < \infty,$$

and

$$|\Delta u_n| = \left| \sum_{k=n}^{\infty} \Delta^2 u_k \right| \leq \sum_{k=n}^{\infty} \left|\Delta^2 u_k\right| = A_n.$$

The converse of this statement, in general, is not true.

A new criteria, in order that $\{u_n\} \in \mathbf{S}$, has been introduced latter by the following definition.

Definition 1.15. *A zero-sequence* $\{u_n\}_{n=1}^{\infty}$ *is said to belongs to the class* $\mathbf{F}_p$ *for some* $1 < p \leq 2$ *if*

$$\sum_{n=1}^{\infty} \left(\frac{|\Delta u_n|^p + |\Delta u_{n+1}|^p + \cdots}{n} \right)^{\frac{1}{p}} < \infty.$$

Before recalling an important class of sequences we need another definition.

Definition 1.16. *A sequence* $\{u_n\}_{n=1}^{\infty}$ *of non-negative numbers is said to be quasi-monotone if* $\frac{u_n}{n^\beta} \downarrow 0$ *for some* $\beta > 0$ *and for all* n *or equivalently* $u_{n+1} \leq u_n(1 + \alpha/n)$ *for some* $\alpha > 0$ *and for all* $n > n_0(\alpha)$*.*

Related to this definition is next lemma.

Lemma 1.17. *Let* $\{u_n\}_{n=1}^{\infty}$ *be a quasi-monotone sequence of real numbers. If* $\sum_{n=1}^{\infty} u_n$ *converges, then* $nu_n = o(1)$ *as* $n \to \infty$*.*

Proof. We have

$$u_{n-1} \geq (1 + \alpha/(n-1))^{-1} u_n, u_{n-2} \geq (1 + \alpha/(n-2))^{-2} u_n, \ldots$$

and therefore for $k < n$

$$\sum_{\nu=1}^{k} u_{n-\nu} \geq k(1 + \alpha/(n-k))^{-k} u_n$$

or

$$k u_n \leq (1 + \alpha/(n-k))^k \sum_{\nu=1}^{k} u_{n-\nu} < e^{k\alpha/(n-k)} \sum_{\nu=1}^{k} u_{n-\nu}.$$

Putting $k = \left[\frac{n}{2}\right]$ into last inequality and taking into account that $\sum_{n=1}^{\infty} u_n$ converges, we obtain $n u_n = o(1)$ as $n \to \infty$.

The proof is completed.

Lemma 1.18. *Let $\{u_n\}_{n=1}^{\infty}$ be a quasi-monotone sequence of constants. If $\sum_{n=1}^{\infty} u_n$ converges, then $\sum_{n=1}^{\infty}(n+1)|\Delta u_n|$ converges.*

Proof. Under assumption of this Lemma, Lemma 1.17, and Lemma 1.1 we have

$$\begin{aligned}\infty > \sum_{n=1}^{\infty} u_n &= \lim_{m\to\infty} \sum_{n=1}^{m} u_n \\ &= \lim_{m\to\infty} \left(\sum_{n=1}^{m-1} (n+1)\Delta u_n + (m+1) u_m \right) = \sum_{n=1}^{\infty} (n+1)\Delta u_n.\end{aligned}$$

The proof is completed.

Years later, a "new" class of sequences $\mathbf{S}'$ was introduced.

Definition 1.19. *The zero-sequence $\{u_n\}$ is said to be in the class $\mathbf{S}'$ if there exists a sequence $\{A_n\}$ such that*

(i) $\{A_n\}$ *is quasi-monotone,*
(ii) $\sum_{n=0}^{\infty} A_n < \infty$, *and*
(iii) $|\Delta u_n| \leq A_n$, *for all* n.

More general class of sequences than $\mathbf{S}'$ class is the class $\mathbf{S}^{**}$.

Definition 1.20. *The zero-sequence $\{u_n\}$ is said to be in the class $\mathbf{S}^{**}$ if*

$$n\Delta u_n = \mathcal{O}(1) \quad as \quad n \to \infty.$$

The latest definition is illustrated by next example.

Example 1.21. The sequence

$$u_n = \frac{(-1)^{n+1}}{n\log(n+1)}, \qquad n \in \{1, 2, \ldots\}$$

does not satisfy the conditions of the class $\mathbf{S}'$ since

$$|\Delta u_n| \geq n\log(n+1)^{-1} \quad \textit{and} \quad \sum_{n=0}^{\infty} |\Delta u_n| = \infty.$$

On the other hand,

$$n|\Delta u_n| \leq \frac{1}{\log(n+1)} = \mathcal{O}(1).$$

This class of sequences was generalized further to the class $\mathbf{S}_r^{**}$, $r \in \{0, 1, 2, \ldots\}$. Namely,

Definition 1.22. *The zero-sequence $\{u_n\}$ is said to be in the class $\mathbf{S}_r^{**}$, $r \in \{0, 1, 2, \ldots\}$, if*

$$n^{r+1}\Delta u_n = \mathcal{O}(1) \quad as \quad n \to \infty.$$

It is clear that for $r = 0$, as a special case, we have $\mathbf{S}_0^{**} \equiv \mathbf{S}^{**}$.

More than two decades later has been proved that the classes $\mathbf{S}$ and $\mathbf{S}'$ are equivalent. Now we recall $\mathbf{S}^2$ class of sequences, which indeed expresses an another equivalent form of the class $\mathbf{S}$.

Definition 1.23. *The zero-sequence $\{u_n\}$ is said to be in the class $\mathbf{S}^2$ if there exists a zero-sequence $\{A_n\}$ of non-negative numbers such that*

(i) $\sum_{n=1}^{\infty} n\,|\Delta A_n| < \infty$, and
(ii) $|\Delta u_n| \leq A_n$, for all n.

The definition of δ-quasi-monotone sequences $\mathbf{S}(\delta)$ is the following.

Definition 1.24. *A sequence $\{u_n\}$ is said to be δ-quasi-monotone, if $u_n \to 0$, $u_n > 0$ ultimately and $\Delta u_n \geq -\delta_n$, where $\{\delta_n\}$ is a sequence of positive numbers.*

Along with what we said above, even in this case, the classes $\mathbf{S}$ and $\mathbf{S}(\delta)$ are identical as well. Exactly, L. Leindler in 2000 has proved the following embedding relations

$$\mathbf{S} \subset \mathbf{S}' \subset \mathbf{S}(\delta) \subset \mathbf{S},$$

which factitive mean that the classes $\mathbf{S}$, $\mathbf{S}'$, and $\mathbf{S}(\delta)$ are indeed equivalent.

To show this firstly we prove two lemmas.

Lemma 1.25. *Let $\{u_n\}$ be a δ-quasi-monotone sequence with $\sum_{n=1}^{\infty} n\delta_n < \infty$. If $\sum_{n=1}^{\infty} u_n < \infty$, then $nu_n = o(1)$ as $n \to \infty$.*

Proof. Let $0 < m < n$. Adding the inequalities

$$n\Delta u_{n-1} \geq -n\delta_{n-1},$$
$$(n-1)\Delta u_{n-2} \geq -(n-1)\delta_{n-2},$$
$$\vdots$$
$$(m+1)\Delta u_m \geq -(m+1)\delta_m,$$

we obtain

$$-u_n n + \sum_{k=m+1}^{n-1} [(k+1)-k]u_k + (m+1)u_m \geq -\sum_{k=m}^{n-1} (k+1)\delta_k.$$

By assumptions, the right-hand side is $o(1)$ as $m, n \to \infty$. The sum on the left side is $o(1)$ since $\sum_{n=1}^{\infty} u_n < \infty$. Whence,

$$(m+1)u_m - nu_n \geq o(1), \quad (m, n \to \infty).$$

Since, $u_m \to 0$, then

$$mu_m - nu_n \geq o(1), \quad (m, n \to \infty).$$

We can not have $\liminf nu_n > 0$, since otherwise $\sum_{n=1}^{\infty} u_n$ could not converge. Hence, in particular, there is for each positive ε an infinite sequence of indices m for which $mu_m < \varepsilon$.

Now suppose that $\liminf nu_n > 0$. Then there is an infinite sequence of indices n such that $nu_n > 2\varepsilon > 0$. For each m satisfying $mu_m < \varepsilon$ we take larger n satisfying $nu_n > 2\varepsilon > 0$. So, we have a contradiction of

$$mu_m - nu_n \geq o(1), \quad (m, n \to \infty).$$

Thus $\liminf nu_n = 0$, as we required.

The proof is completed.

Lemma 1.26. *Let $\{u_n\}$ be a δ-quasi-monotone sequence with $\sum_{n=1}^{\infty} n\delta_n < \infty$. If $\sum_{n=1}^{\infty} u_n < \infty$, then $\sum_{n=1}^{\infty} n|\Delta u_n| < \infty$.*

Proof. We have

$$\sum_{n=1}^{\infty} n|\Delta u_n| = \sum_{n=1}^{\infty} n(\Delta u_n) + 2\sum_{n=1}^{\infty} n(\Delta u_n)^-,$$

where $(\Delta u_n)^-$ is $-\Delta u_n$ if $\Delta u_n < 0$, and 0 otherwise. Since $0 \leq (\Delta u_n)^- \leq \delta_n$, we have $\sum_{n=1}^{\infty} n(\Delta u_n)^-$ convergent, and it is therefore enough to show that $\sum_{n=1}^{\infty} n(\Delta u_n)$ is convergent. We may consider the series $\sum_{n=1}^{\infty}(n+1)(\Delta u_n)$ instead, since the difference between these series converges with $\sum_{n=1}^{\infty} |\Delta u_n|$.

Now $\sum_{n=1}^{\infty}(n+1)(\Delta u_n)$ is not necessarily a series of positive terms, but it differs from such series, namely

$$\sum_{n=1}^{\infty}[(n+1)(\Delta u_n)+(n+1)\delta_n],$$

by a convergent series. Hence it is enough to show that $\sum_{n=1}^{\infty}(n+1)(\Delta u_n)$ has bounded partial sums. By Lemma 1.1 we obtain

$$\sum_{k=1}^{n} u_k = \sum_{k=2}^{n-1}(k+1)\Delta u_k + (n+1)u_n + \mathcal{O}(1).$$

The series on the left converges, and therefore by Lemma 1.25, $(n+1)u_n \to 0$ as $n \to \infty$. So, the partial sums on the right are bounded.

The proof is completed.

Now, the following lemma holds true.

Lemma 1.27. *The classes* $\mathbf{S}$, $\mathbf{S}'$, $\mathbf{S}(\delta)$, *and* $\mathbf{S}^2$ *are all equivalent.*

Proof. Firstly we verify that classes $\mathbf{S}(\delta)$ and $\mathbf{S}^2$ are equivalent. Let $\{u_n\} \in \mathbf{S}(\delta)$. By Lemma 1.26 we have $\sum_{n=1}^{\infty} n|\Delta A_n| < \infty$, which means that $\{u_n\} \in \mathbf{S}^2$.

If $\{u_n\} \in \mathbf{S}^2$, then

$$nA_n = n\left|\sum_{k=n}^{\infty} \Delta A_k\right| \le \sum_{k=n}^{\infty} k\,|\Delta A_k| = o(1), \quad n \to \infty.$$

Using Lemma 1.1 we also have the inequality

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n-1} k\Delta A_k + nA_n \le \sum_{k=1}^{n-1} k|\Delta A_k| + nA_n$$

which implies that $\sum_{k=1}^{n} A_k < \infty$ i.e. $\{u_n\} \in \mathbf{S}(\delta)$.

Now we are going to prove that $\mathbf{S}$ and $\mathbf{S}(\delta)$ are equivalent. It is clear that $\mathbf{S} \subset \mathbf{S}(\delta)$. If $\{u_n\} \in \mathbf{S}(\delta)$, we define

$$B_k = A_k + \sum_{m=k}^{\infty} \delta_m.$$

Then $B_k - B_{k+1} = \Delta A_k + \delta_k \ge 0$, i.e. $B_k \downarrow 0$ as $k \to \infty$. On the other hand,

$$\begin{aligned}\sum_{k=1}^{\infty} B_k &= \sum_{k=1}^{\infty} A_k + \sum_{k=1}^{\infty}\sum_{m=k}^{\infty} \delta_m \\ &= \sum_{k=1}^{\infty} A_k + \sum_{m=1}^{\infty}\sum_{k=1}^{m} \delta_m = \sum_{k=1}^{\infty} A_k + \sum_{m=1}^{\infty} m\delta_m < \infty,\end{aligned}$$

and $|\Delta u_n| \le A_n < B_n$ for all n, which means that $\{u_n\} \in \mathbf{S}$. So, we have

$$\mathbf{S} \subset \mathbf{S}' \subset \mathbf{S}(\delta) \subset \mathbf{S}.$$

Subsequently, $\mathbf{S} \equiv \mathbf{S}' \equiv \mathbf{S}(\delta) \equiv \mathbf{S}^2$.

The proof is completed.

Even if the efforts to generalize the class $\mathbf{S}$ were in some way only virtual, such efforts were not without any success as shows next definition.

Definition 1.28. *The zero-sequence $\{u_n\}$ is said to be in the class $\mathbf{S}_r$, $(r = 1, 2, \ldots)$, if there exists a sequence $\{A_n\}$ such that*

(i) $A_n \downarrow 0$ *as* $n \to \infty$,
(ii) $\sum_{n=0}^{\infty} n^r A_n < \infty$, *and*
(iii) $|\Delta u_n| \le A_n$, *for all* n.

Here we write $\mathbf{S}_0 \equiv \mathbf{S}$. It should be noted that from $A_n \downarrow 0$ and $\sum_{n=1}^{\infty} n^r A_n < \infty$ we get $n^{r+1} A_n = o(1)$ as $n \to \infty$. Also, the inclusion $\mathbf{S}_{r+1} \subset \mathbf{S}_r$ for all $r = 1, 2, \ldots$, is trivial.

Now let $\{u_n\} \in \mathbf{S}_1$. For arbitrary real number u_0 we prove that $\{u_n\}_{n=0}^{\infty} \in \mathbf{S}_0$. Define $A_0 = \max\{|\Delta u_0|, A_1\}$. Then $|\Delta u_0| \le A_0$ i. e. $|\Delta u_n| \le A_n$ for all $n = 0, 1, 2, \ldots$, and $\{A_n\}_{n=0}^{\infty} \downarrow$.

On the other hand

$$\sum_{n=0}^{\infty} A_n \le A_0 + \sum_{n=1}^{\infty} n^r A_n < \infty.$$

Consequently, $\{u_n\}_{n=0}^{\infty} \in \mathbf{S}_0$ which indeed shows that $\mathbf{S}_{r+1} \subset \mathbf{S}_r$ for all $(r = 0, 1, 2, \ldots)$.

The next example proves that the implication

$$\{u_n\} \in \mathbf{S}_{r+1} \subset \{u_n\} \in \mathbf{S}_r, \quad (r = 0, 1, 2, \ldots)$$

is not always reversible.

Example 1.29. For $n = 0, 1, 2, \ldots$, we define $u_n = \sum_{k=n+1}^{\infty} \frac{1}{k^2}$. Then $u_n \to 0$ as $n \to \infty$ and for $n = 0, 1, 2, \ldots$, $\Delta u_n = \frac{1}{(n+1)^2}$. Let us show first that $\{u_n\}_{n=1}^{\infty} \notin \mathbf{S}_1$.

Let $\{A_n\}_{n=1}^{\infty}$ is an arbitrary positive sequence such that $A_n \downarrow 0$ and $\Delta u_n \le |\Delta u_n| \le A_n$, for all n. However $\sum_{n=1}^{\infty} n A_n \ge \sum_{n=1}^{\infty} \frac{n}{(n+1)^2}$ is divergent, i.e. $\{u_n\}_{n=0}^{\infty} \notin \mathbf{S}_1$.

Now, for all $n = 0, 1, 2, \ldots$, let $A_n = \frac{1}{(n+1)^2}$. Then $A_n \downarrow 0$, $|\Delta u_n| \le A_n$, and $\sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} < \infty$, i.e. $\{u_n\}_{n=0}^{\infty} \in \mathbf{S}_0$.

An example, which we are going to consider, will show that there exists a sequence such that $\{u_n\} \in \mathbf{S}_r$, but $\{u_n\} \notin \mathbf{S}_{r+1}$, $(r = 1, 2, \ldots)$.

Example 1.30. Indeed, for $n = 1, 2, \ldots$, let $u_n = \sum_{k=n+1}^{\infty} \frac{1}{k^{r+2}}$. Then $u_n \to 0$ as $n \to \infty$ and for $n = 1, 2, \ldots$, $\Delta u_n = \frac{1}{n^{r+2}}$. Let $\{A_n\}_{n=1}^{\infty}$ is an arbitrary positive sequence such that $A_n \downarrow 0$ and $\Delta u_n \leq |\Delta u_n| \leq A_n$, for all n. Even thought

$$\sum_{n=1}^{\infty} n^{r+1} A_n \geq \sum_{n=1}^{\infty} n^{r+1} \frac{1}{n^{r+2}} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

i.e. $\{u_n\} \notin \mathbf{S}_{r+1}$, $(r = 1, 2, \ldots)$. On the other hand, for all $n = 1, 2, \ldots$, let $A_n = \frac{1}{n^{r+2}}$. Then $A_n \downarrow 0$, $|\Delta u_n| \leq A_n$, and $\sum_{n=1}^{\infty} n^r A_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, i.e. $\{u_n\}_{n=1}^{\infty} \in \mathbf{S}_r$.

Now we give the definition of the class $\mathbf{C}$. It has an important place, among others, since as we will see below $\mathbf{S} \subset \mathbf{C} \cap \mathbf{BV}$.

Definition 1.31. *The zero-sequence $\{u_n\}$ is said to be in the class $\mathbf{C}$ if for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$, independent of n, and so that for all $n \geq 0$,*

$$\int_0^{\delta} \left| \sum_{k=n+1}^{\infty} \Delta u_k \frac{\sin\left(k + \frac{1}{2}\right) x}{2 \sin \frac{x}{2}} \right| dx < \varepsilon.$$

The class $\mathbf{C}$ was not remained without generalization. Namely,

Definition 1.32. *The zero-sequence $\{u_n\}$ is said to be in the class $\mathbf{C}_r$, $r = 0, 1, \ldots$, if for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$, independent of n, and so that for all $n \geq 0$,*

$$\int_0^{\delta} \left| \sum_{k=n+1}^{\infty} \Delta u_k \left[\frac{\sin\left(k + \frac{1}{2}\right) x}{2 \sin \frac{x}{2}} \right]^{(r)} \right| dx < \varepsilon.$$

One should note here that for $r = 0$ we obtain $\mathbf{C}_0 \equiv \mathbf{C}$.

In the following we need to prove an inequality known as *Cauchy–Schwarz* inequality for integrals.

Lemma 1.33. *Let f and g be real functions which are continuous on the closed interval $[a, b]$. Then*

$$\left\{ \int_a^b [f(x) \cdot g(x)] dx \right\}^2 \leq \int_a^b [f(x)]^2 dx \cdot \int_a^b [g(x)]^2 dx.$$

Proof. For any real number x we have

$$\begin{aligned} & 0 \leq [x f(t) + g(t)]^2 \\ \Longrightarrow\ & 0 \leq \int_a^b [x f(t) + g(t)]^2 dt \\ & = x^2 \int_a^b [f(t)]^2 dt + 2x \int_a^b [f(t) g(t)] dt + \int_a^b [g(t)]^2 dt \\ & = M x^2 + 2 N x + P, \end{aligned}$$

where

$$M = \int_a^b [f(t)]^2 dt, \quad N = \int_a^b [f(t)g(t)] dt, \quad \textit{and} \quad P = \int_a^b [g(t)]^2 dt.$$

So, the quadratic expression $Mx^2 + 2Nx + P$ is non-negative. Therefore its discriminant $(2N)^2 - 4MP$ must be non-positive

$$N^2 \leq MP.$$

Putting M, N, ad P into last inequality we obtain the proof of the requested inequality.

Now, as we mentioned above, we are in able to prove a lemma which shows that $\mathbf{S} \subset \mathbf{C} \cap \mathbf{BV}$. Truly, to show this we prove the following lemma which is known as Sidon-Fomin lemma. We give here its proof presented by Telyakovskii who gave an elementary proof.

Lemma 1.34. *Let the real numbers α_i, $i = 1, 2, \ldots, k$, satisfy conditions $|\alpha_i| \leq 1$. Then the following estimations hold true*

$$\int_0^\pi \left| \sum_{i=0}^k \alpha_i \frac{\sin\left(i+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} \right| dx \leq C(k+1),$$

and

$$\int_{\frac{\pi}{k+1}}^\pi \left| \sum_{i=0}^k \alpha_i \frac{\cos\left(i+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} \right| dx \leq C(k+1),$$

where C is a positive constant.

Proof. First we have

$$\int_0^{\frac{\pi}{k+1}} \left| \sum_{i=0}^k \alpha_i \frac{\sin\left(i+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} \right| dx \leq \sum_{i=0}^k |\alpha_i| \left(i + \frac{1}{2}\right) \frac{\pi}{k+1} \leq C(k+1).$$

On the other hand using Lemma 1.33 and the Jordan's inequality $\sin\beta \geq \frac{2}{\pi}\beta$, $\beta \in \left[0, \frac{\pi}{2}\right]$, see Lemma 1.79 page 37, we also have

$$\begin{aligned}
&\int_{\frac{\pi}{k+1}}^\pi \left| \sum_{i=0}^k \alpha_i \frac{\sin\left(i+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} \right| dx \\
&\quad \leq \left(\int_{\frac{\pi}{k+1}}^\pi \frac{dx}{4\sin^2\frac{x}{2}} \right)^{\frac{1}{2}} \cdot \left(\int_{\frac{\pi}{k+1}}^\pi \left[\sum_{i=0}^k \alpha_i \sin\left(i+\frac{1}{2}\right)x \right]^2 dx \right)^{\frac{1}{2}} \\
&\quad \leq C(k+1)^{\frac{1}{2}} \cdot \left(\int_0^\pi \left[\sum_{i=0}^k \alpha_i \sin\left(i+\frac{1}{2}\right)x \right]^2 dx \right)^{\frac{1}{2}} \leq C(k+1).
\end{aligned}$$

The second inequality of this lemma can be proved in a very similar way. The proof is completed.

Lemma 1.35. *The inclusion* $\mathbf{S} \subset \mathbf{C} \cap \mathbf{BV}$ *holds true.*

Proof. Since, according to the Lemma 1.27, the class $\mathbf{S}$ is equivalent to $\mathbf{S}^2$, then we prove that $\mathbf{S} \subset \mathbf{C} \cap \mathbf{BV}$ by proving $\mathbf{S}^2 \subset \mathbf{C} \cap \mathbf{BV}$.

From $\{u_n\} \in \mathbf{S}^2$ it implies that $\{u_n\} \in \mathbf{BV}$. Indeed,

$$\sum_{k=1}^{\infty} |\Delta A_k| \leq \sum_{k=1}^{\infty} k|\Delta A_k| < \infty.$$

On the other hand from $A_k = o(1)$ as $k \to \infty$ we have

$$nA_n = n\left|\sum_{k=n}^{\infty} \Delta A_k\right| \leq \sum_{k=n}^{\infty} k\,|\Delta A_k| = o(1), \ \textit{as } n \to \infty.$$

Applying Lemma 1.1 we obtain

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n-1} k\Delta A_k + nA_n, \tag{1.5}$$

which along with (1.5) we get that $\sum_{k=1}^{\infty} A_k < \infty$. Since $\{u_n\} \in \mathbf{S}^2$ it follows that $|\Delta u_k| \leq A_k$ for all k, and therefore

$$\sum_{k=1}^{\infty} |\Delta u_k| \leq \sum_{k=1}^{\infty} A_k < \infty, \ \textit{i.e.}\ \{u_n\} \in \mathbf{BV}.$$

Now we are going to prove the implication $\{u_n\} \in \mathbf{S}^2 \implies \{u_n\} \in \mathbf{C}$. Indeed, applying Lemma 1.34 we obtain

$$\begin{aligned}\int_0^{\delta} \left|\sum_{k=n+1}^{\infty} \Delta u_k \frac{\sin\left(k+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\right| dx &\leq \int_0^{\pi} \left|\sum_{k=n+1}^{\infty} \frac{\Delta u_k}{A_k} A_k \frac{\sin\left(k+\frac{1}{2}\right)}{2\sin\frac{t}{2}}\right| dx \\ &\leq K \lim_{m\to\infty} \left[\sum_{k=n+1}^{m-1} (k+1)|\Delta A_k| + (m+1)A_m + (n+1)A_{n+1}\right],\end{aligned}$$

where K is an absolute constant.

Since $mA_m \to 0$ as $m \to \infty$, because of $\sum_{k=1}^{\infty} A_k < \infty$, by Olivier's theorem, we find that

$$\begin{aligned}&\int_0^{\delta} \left|\sum_{k=n+1}^{\infty} \Delta u_k \frac{\sin\left(k+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\right| dx \\ &\qquad \leq K \left[\sum_{k=n+1}^{\infty} (k+1)|\Delta A_k| + (n+1)A_{n+1}\right].\end{aligned} \tag{1.6}$$

Both terms of the right-hand side (1.6) are $o(1)$ as $n \to \infty$. Subsequently, we can choose n big enough so that for every $\varepsilon > 0$ we have

$$\int_0^\delta \left| \sum_{k=n+1}^{\infty} \Delta u_k \frac{\sin\left(k+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} \right| dx < \varepsilon,$$

which means that $\{u_n\} \in \mathbf{C}$.

So, we have proved implications $\{u_n\} \in \mathbf{S} \Longrightarrow \{u_n\} \in \mathbf{BV}$ and $\{u_n\} \in \mathbf{S} \Longrightarrow \{u_n\} \in \mathbf{C}$ or $\{u_n\} \in \mathbf{BV} \cap \mathbf{C}$.

The proof is completed.

A wider class of sequences than the class of bounded variation is the class of bounded variation of higher order.

Definition 1.36. *The zero-sequence $\{u_n\}$ is said to be in the class $(\mathbf{BV})^m$ (the class of bounded variation of higher order) if*

$$\sum_{n=1}^{\infty} |\Delta^m u_n| < \infty,$$

where $\Delta^m u_n := \Delta(\Delta^{m-1} u_n) = \Delta^{m-1} u_n - \Delta^{m-1} u_{n+1}, \quad m \in \{1, 2, \ldots\}$.

Note that for $m = 1$, the class $(\mathbf{BV})^1$ is the class $\mathbf{BV}$. Also it is clear that

$$\{u_n\} \in (\mathbf{BV})^m \Longrightarrow \{u_n\} \in (\mathbf{BV})^{m+1},$$

but the converse inclusion is not true.

To show this fact we give an example.

Example 1.37. For $k = 1, 2, \ldots$, and $-k \le n < k$, let us define

$$u_{k^2+n} = \frac{k - |n|}{k^2}.$$

The sequence $\{u_i\}$ is well defined, for $k^2 + k = (k+1)^2 - (k+1)$. Since

$$|\Delta u_{k^2+n}| = \frac{1}{k^2},$$

we have

$$\sum_{i=0}^{\infty} |\Delta u_i| = \sum_{k=1}^{\infty} \sum_{n=-k}^{k-1} |\Delta u_{k^2+n}| = \sum_{k=1}^{\infty} \sum_{n=-k}^{k-1} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k} = +\infty,$$

which means that $\{u_i\}$ is not of bounded variation.

However,

$$\Delta^2 u_{k^2+n} = 0$$

for $-k \le n < -1$ or $0 \le n < k-2$, and

$$\Delta^2 u_{k^2-1} = -\frac{2}{k^2}, \quad \Delta^2 u_{k^2+k-1} = \frac{1}{k^2} + \frac{1}{(k+1)^2}.$$

Whence,

$$\sum_{i=0}^{\infty}|\Delta^2 u_i| = \sum_{k=1}^{\infty}\sum_{n=-k}^{k-1}|\Delta u_{k^2+n}| = \sum_{k=1}^{\infty}\left[\frac{1}{k^2}+\frac{1}{(k+1)^2}\right] < +\infty,$$

and therefore the sequence $\{u_i\}$ is of bounded variation of order two, but is not of bounded variation.

The class $\mathbf{R}$ of sequences also appears to be useful in literature.

Definition 1.38. *The zero-sequence* $\{u_n\}$ *is said to be in the class* $\mathbf{R}$ *if*

$$\sum_{n=1}^{\infty} n^2 \left|\Delta^2\left(\frac{u_n}{n}\right)\right| < \infty.$$

This class has been generalized to the class $\mathbf{R}_r$ as follows.

Definition 1.39. *The zero-sequence* $\{u_n\}$ *is said to be in the class* $\mathbf{R}_r$*,* $r \in \{0, 1, 2, \ldots\}$*, if*

$$\sum_{n=1}^{\infty} n^{r+2} \left|\Delta^2\left(\frac{u_n}{n}\right)\right| < \infty.$$

This class has been defined in the complex domain as we will see it in the next section. One should note here that for $r = 0$ we have $\mathbf{R}_0 \equiv \mathbf{R}$.

Also we introduce a new class of null sequences of real numbers related to the class $\mathbf{R}$.

Definition 1.40. *If* $u_k \to 0$ *as* $k \to \infty$ *and*

$$\sum_{k=1}^{\infty} k^2 \log k \left|\Delta^2\left(\frac{u_k}{k}\right)\right| < \infty$$

then we say that $\{u_k\}$ *belongs to the class* $\mathbf{R}^{\log}$*.*

Because of the inequality $1 < \ln n$ for $n \geq 3$, it is clear that $\mathbf{R}^{\log} \subset \mathbf{R}$. Moreover the following example shows that the class $\mathbf{R}^{\log}$ is not an empty subclass of the class $\mathbf{R}$.

Example 1.41. Let $u_k = \frac{1}{k^2 \ln^3 k}$, $k \geq 2$. Then it is obvious that $u_k \to 0$ and

$$\left|\Delta^2\left(\frac{u_k}{k}\right)\right| < \frac{4}{k^3 \ln^3 k} \Longrightarrow \sum_{k=2}^{\infty} k^2 \log k \left|\Delta^2\left(\frac{u_k}{k}\right)\right| < \sum_{k=2}^{\infty}\frac{1}{k \ln^2 k} < \infty.$$

Next definition introduces the notion of the semi-convexity.

Definition 1.42. *The zero-sequence* $\{u_n\}$ *is said to be semi-convex if*

$$\sum_{n=1}^{\infty} n \left|\Delta^2 u_{n-1} + \Delta^2 u_n\right| < \infty, \quad (u_0 = 0).$$

The class of semi-convex sequences has been generalized to the class of semi-convex sequences of order r, $r \in \{0, 1, 2, \ldots\}$.

Definition 1.43. *A sequence* (u_n) *is said to be semi-convex of order* r, $r \in \{0, 1, \ldots\}$, *or* $(a_n) \in (\mathbf{SC})^r$, *if* $u_n \to 0$ *as* $n \to \infty$, *and*

$$\sum_{n=1}^{\infty} n^{r+1} |\Delta^2 u_{n-1} + \Delta^2 u_n| < \infty, \quad (u_0 = 0).$$

We note that the class $(\mathbf{SC})^0$ is the same with the class of semi-convex sequences.

The semi-convexity of a sequence is generalized to hyper semi-convexity of non-negative integer and to non-integer positive order.

Definition 1.44. *The zero-sequence* $\{u_n\}$ *is said to be hyper semi-convex if*

$$\sum_{n=1}^{\infty} n^{\alpha+1} \left|\Delta^{\alpha+2} u_{n-1} + \Delta^{\alpha+2} u_n\right| < \infty, \quad (u_0 = 0),$$

where $\alpha \in \{0, 1, 2, \ldots\}$.

Definition 1.45. *The zero-sequence* $\{u_n\}$ *is said to be hyper semi-convex of non-integer positive order if*

$$\sum_{n=1}^{\infty} n^{\alpha+1} \left|\Delta^{\alpha+2} u_{n-1} + \Delta^{\alpha+2} u_n\right| < \infty, \quad (u_0 = 0),$$

where $\alpha > 0$ *any real number.*

Lately, the following class has been introduced.

Definition 1.46. *The sequence* $\{u_n\}$ *is said to belongs to the class* $\mathbf{K}$ *if* $u_n \to 0$ *as* $n \to \infty$ *and*

$$\sum_{n=1}^{\infty} n \left|\Delta^2 u_{n-1} - \Delta^2 u_{n+1}\right| < \infty, \quad (u_0 = 0).$$

The class $\mathbf{K}$ was generalized to the class $\mathbf{K}^\alpha$.

Definition 1.47. *The sequence* $\{u_n\}$ *is said to belongs to the class* $\mathbf{K}^\alpha$ *if* $u_n \to 0$ *as* $n \to \infty$ *and*

$$\sum_{n=1}^{\infty} n^{\alpha} \left|\Delta^{\alpha+1} u_{n-1} - \Delta^{\alpha+1} u_{n+1}\right| < \infty, \quad (u_0 = 0).$$

It is obvious that for $\alpha = 1$ we obtain $\mathbf{K}^1 \equiv \mathbf{K}$.
In what follows it is related in some way to the class $\mathbf{K}^\alpha$.
For any real value of α the binomial coefficients A_ν^α are defined by

$$(1-x)^{-\alpha-1} = \sum_{\nu=0}^{\infty} A_\nu^\alpha x^\nu, \quad |x| < 1.$$

Then $A_0^\alpha = 1$ and for $\nu \geq 1$

$$A_\nu^\alpha = (-1)^\nu \binom{-\alpha-1}{\nu} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+\nu)}{\nu!} = \binom{\nu+\alpha}{\nu}.$$

When $\alpha \neq -q$, where q is a positive integer, we have

$$\frac{A_\nu^\alpha}{\nu^\alpha} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+\nu)}{\nu!\nu^\alpha} \to \frac{1}{\Gamma(\alpha+1)} \; for \; \nu \to \infty,$$

and it follows that

$$A_\nu^\alpha = \mathcal{O}(\nu^\alpha) \quad and \quad \nu^\alpha = \mathcal{O}(A_\nu^\alpha),$$

where

$$\Gamma(x) = \int_0^{+\infty} x^{z-1} e^{-x} dz$$

is the well-known Euler integral of the second kind and we recall here only its main property that is $\Gamma(\alpha) = (\alpha-1)!$.

This relation concerning the order of magnitude of the binomial coefficients A_ν^α justifies the notation . Also by definition it follows that

$$\frac{A_\nu^{-q}}{\nu^\alpha} = 0 \; for \; \nu \geq q.$$

Concerning the sign and variation of the binomial coefficients we mention the following result, which are all immediate consequences of the definition of A_ν^α.

When $\alpha > -1$ and all A_ν^α are positive and the sequence $\{A_\nu^\alpha\}$ is monotone, for α increasing to ∞ and for $-1 < \alpha < 0$ decreasing to 0. If $\alpha = 0$, then all the coefficients are equal to 1.

When $\alpha < -1$ the sequence $\{A_\nu^\alpha\}$ contains both positive and negative terms. For small values of ν the terms have alternating sign, but from a certain value of ν all terms have same sign or vanish. In particular we note that in the sequence

$$A_0^{-\delta-1}, A_1^{-\delta-1}, A_2^{-\delta-1}, \ldots, A_\nu^{-\delta-1}, \ldots \;\; (0 < \delta < 1).$$

where $-2 < \alpha < -1$, all terms are negative except the firs one.

It should also be noticed that the series $\sum_{\nu=0}^{\infty} A_\nu^\alpha$ is absolutely convergent for $\alpha \leq 1$. Since for $\alpha < -1$

$$\sum_{\nu=0}^{\infty} A_\nu^\alpha x^\nu = (1-x)^{-\alpha-1} \to 0, \quad for \quad x \to 1,$$

we have

$$\sum_{\nu=0}^{\infty} A_\nu^\alpha = 0, \quad for \quad \alpha < -1.$$

By multiplication of two binomial series we obtain for $|x| < 1$

$$\sum_{\nu=0}^{\infty} A_\nu^\alpha x^\nu \sum_{\nu=0}^{\infty} A_\nu^\beta x^\nu = (1-x)^{-\alpha-1}(1-x)^{-\beta-1} = (1-x)^{-\alpha-\beta-2} = \sum_{\nu=0}^{\infty} A_\nu^{\alpha+\beta+1} x^\nu.$$

It follows that

$$\sum_{\nu=0}^{n} A_\nu^\alpha A_{n-\nu}^\beta = A_n^{\alpha+\beta+1}, \quad (n = 0, 1, 2, \dots)$$

for all values of α and β. For $\beta = 0$ this formula reduces to

$$\sum_{\nu=0}^{n} A_\nu^\alpha = A_n^{\alpha+1} \quad or \quad A_n^{\alpha+1} - A_{n-1}^{\alpha+1} = A_n^\alpha, \quad (n \geq 1),$$

which deals with addition and subtraction of binomial coefficients of the same order.

For any real value of α the sums of order α belonging to a given sequence $\{a_\nu\}$ will be defined by

$$S_n^\alpha(a_\nu) = \sum_{\nu=0}^{n} A_{n-\nu}^\alpha a_\nu.$$

In particular

$$S_n^0(a_\nu) = a_0 + a_1 + \cdots + a_n \quad and \quad S_n^{-1}(a_\nu) = a_n.$$

As a consequence of the formula $\sum_{\nu=0}^{n} A_\nu^\alpha A_{n-\nu}^\beta = A_n^{\alpha+\beta+1}$, the formula

$$S_n^\beta(S_n^\alpha(a_\nu)) = S_n^{\alpha+\beta+1}(a_\nu),$$

holds for all values of α and β. For $\beta = 0$ this formula reduces to

$$\sum_{\nu=0}^{n} S_\nu^\alpha a_\nu = S_n^{\alpha+1} a_\nu \quad or \quad S_n^{\alpha+1}(a_\nu) - S_{n-1}^{\alpha+1}(a_\nu) = S_n^\alpha(a_\nu), \; n \geq 1,$$

which deals with addition and subtraction of sums of the same order.

For any real value of $\alpha > -1$ the Cesàro means of order α belonging to the series $\sum_{\nu=0}^{\infty} a_\nu$ are defined by

$$C_n^\alpha(u_\nu) = \frac{S_n^\alpha(a_\nu)}{A_n^\alpha}.$$

As we have seen earlier for any positive integer α the differences of order α belonging to a given sequence a_ν are defined by the equations

$$\Delta^1 a_\nu = a_\nu - a_{\nu+1}, \quad \Delta = \Delta^1(\Delta^{\alpha-1}a_\nu), \quad (\nu = 0, 1, 2, \ldots).$$

For these differences we have the formulas

$$\Delta^\alpha a_\nu = \sum_{\mu=0}^{\alpha} A_\mu^{-\alpha-1} a_{\nu+\mu}$$

and

$$\Delta^\alpha(a_\nu b_\nu) = \sum_{q=0}^{\alpha} \binom{\alpha}{q} \Delta^q(a_\nu)\Delta^{\alpha-q}(a_{\nu+q}),$$

both of which can be verified by mathematical induction.

Since $A_\mu^{-\alpha-1} = 0$ for $\mu \geq \alpha + 1$ we may replace the sum in the above equality by the infinite series

$$\Delta^\alpha a_\nu = \sum_{\mu=0}^{\infty} A_\mu^{-\alpha-1} a_{\nu+\mu}, \quad (\nu = 0, 1, 2, \ldots).$$

If these series are convergent for some α which is not a positive integer, we define the differences Δa_ν by the above equations. It should be noticed here that the convergence of this series for some value ν implies the convergence for all values of ν. If the series are not convergent these differences will not be defined. Thus, in any case where the differences Δa_ν exist, they can be determined by above equations.

After this extensions of the definition we have

$$\Delta^0 a_\nu = a_\nu.$$

We further notice that the difference $\Delta^\alpha a_\nu$ exists when $a_\nu = \mathcal{O}(\nu^{\alpha-\delta})$ for some positive value of δ. In particular, if the sequence $\{a_\nu\}$ is bounded, the differences of any positive order will exists.

Whether the series $\sum_{\mu=0}^{\infty} A_\mu^{-\alpha-1} a_{\nu+\mu}$ is convergent or not, we will call the partial sums

$$\Delta_n^\alpha a_\nu = A_0^{-\alpha-1} a_\nu + A_1^{-\alpha-1} a_{\nu+1} + \cdots + A_{n-\nu}^{-\alpha-1} a_n = \sum_{\mu=0}^{n-\nu} A_\mu^{-\alpha-1} a_{\nu+\mu},$$

where $n \geq \nu$, the broken differences of order α belonging to the sequence a_ν. The broken differences exist for any value of α. When α is a positive integer we have

$$\Delta_n^\alpha a_\nu = \Delta^\alpha a_\nu \quad for \quad n-\nu \geq \alpha,$$

since $A_\mu^{-\alpha-1} = 0$ for $\mu > \alpha$.

By means of the broken differences we can condense the Abel's formula for partial summation,

$$\sum_{\nu=0}^{n} a_\nu b_\nu = \sum_{\nu=0}^{n-1} S_\nu^0(a_\nu)\Delta^1 b_\nu + S_n^0(a_\nu)b_n,$$

into

$$\sum_{\nu=0}^{n} a_\nu b_\nu = \sum_{\nu=0}^{n-1} S_\nu^0(a_\nu)\Delta_n^1 b_\nu,$$

and generalize it, by introducing an unrestricted parameter α, to the transformation

$$\sum_{\nu=0}^{n} a_\nu b_\nu = \sum_{\nu=0}^{n} S_\nu^{\alpha-1}(a_\nu)\Delta_n^\alpha b_\nu, \tag{1.7}$$

which replaces the factors a_ν by the sums $S_\nu^{\alpha-1}(a_\nu)$ and the factors b_ν by the broken differences $\Delta^\alpha b_\nu$. The last generalized Abel's transformation is said to be of the order α. The formula for partial summation is of order 1 (Lemma 1.1).

If α is a positive integer we have

$$\Delta_n^\alpha b_\nu = \Delta^\alpha b_\nu \quad for \quad n-\nu \geq \alpha,$$

and the last transformation may be written

$$\sum_{\nu=0}^{n} a_\nu b_\nu = \sum_{\nu=0}^{n-\alpha} S_\nu^{\alpha-1}(a_\nu)\Delta^\alpha b_\nu + \sum_{\nu=n-\alpha+1}^{n} S_\nu^{\alpha-1}(a_\nu)\Delta_n^\alpha b_\nu, \tag{1.8}$$

which for $\alpha = 1$ reduces to the original form of the formula for partial summation. This transformation carries the product-sum $\sum_{\nu=0}^{n} a_\nu b_\nu$ into two different kinds of sum of which the first one will be called the main term of the transformation and the second one the reminder. In the main term we find only complete differences, in the remainder only broken differences of the sequence $\{b_\nu\}$.

Just as the usual formula for partial summation is of the utmost importance not only for the theory of convergence of numerical infinite series, but also for the convergence of Fourier series. The generalized Abel's transformation is a very valuable tool for investigation within the so-called L^1-convergence of Fourier series. In this short book it will be frequently applied in both of its forms i.e. the original and the generalized Abel's transformation.

In accordance with the generalized Abel's transformation we shall call the series

$$\sum_{\nu=0}^{\infty} S_\nu^{\alpha-1}(a_\nu)\Delta^\alpha b_\nu$$

an Abel transform (of order α) of the series $\sum_{\nu=0}^{n} a_\nu b_\nu$.

Lemma 1.48. *If $\alpha \geq 0$, $p \geq 0$,*

(a) $\varepsilon_n = \mathcal{O}(n^{-p})$, *and*
(b) $\sum_{n=0}^{\infty} A_n^{\alpha+p} |\Delta^{\alpha+1} \varepsilon_n| < \infty$, *then*

(i) $\sum_{n=0}^{\infty} A_n^{\lambda+p} |\Delta^{\lambda+1} \varepsilon_n| < \infty$, *for* $-1 \leq \lambda \leq \alpha$,
(ii) $A_n^{\lambda+p} \Delta^{\lambda} \varepsilon_n$ *is of bounded variation for* $0 \leq \lambda \leq \alpha$, *and tends to zero as* $n \to \infty$, *except when* $p = 0$ *and* $\lambda = 0$.

Proof. If $0 < \delta \leq 1$ and $0 \leq \sigma \leq \alpha$, we have (except when $p = 0$, $\sigma = 0$, $\delta = 1$),

$$\begin{aligned} \sum_{n=0}^{\infty} A_n^{\sigma-\delta+p} |\Delta^{\sigma-\delta+1} \varepsilon_n| &\leq \sum_{n=0}^{\infty} A_n^{\sigma-\delta+p} \sum_{v=n}^{\infty} A_{v-n}^{\delta-1} |\Delta^{\sigma+1} \varepsilon_n| \\ &= \sum_{v=0}^{\infty} |\Delta^{\sigma+1} \varepsilon_v| \sum_{n=0}^{v} A_{v-n}^{\delta-1} A_n^{\sigma-\delta+p} \\ &= \sum_{v=0}^{\infty} A_v^{\sigma+p} |\Delta^{\sigma+1} \varepsilon_v|, \end{aligned}$$

and (i) follows by induction, the case $p = 0$, $\lambda = -1$ being trivial.

Again, if $0 \leq \lambda \leq \alpha$,

$$\sum_{n=0}^{\infty} |\Delta(A_n^{\lambda+p} \Delta^{\lambda} \varepsilon_n)| \leq \sum_{n=0}^{\infty} A_n^{\lambda+p} |\Delta^{\lambda+1} \varepsilon_n| + \sum_{n=0}^{\infty} A_{n+1}^{\lambda+p-1} |\Delta^{\lambda} \varepsilon_{n+1}| < \infty.$$

Whence $A_n^{\lambda+p} \Delta^{\lambda} \varepsilon_n$ is of bounded variation, and tends to a limit, which can only be zero, except when $p = 0$ and $\lambda = 0$.

The proof is completed.

Lemma 1.49. *Let $r \geq 0$ be a real number and let s denote the integral part and δ the fractional part of r $(0 \leq \delta < 1)$. If the sequence ε_ν satisfies the conditions*

(i) $\varepsilon_\nu = \mathcal{O}(1)$ *and*
(ii) $\sum_{\nu=0}^{\infty} \nu^r |\Delta^{r+1} \varepsilon_\nu| < \infty$,

then we have

$$\Delta^{s+1} \varepsilon_\nu = \sum_{\mu=0}^{\infty} A_\mu^{\delta-1} \Delta^{r+1} \varepsilon_{\nu+\mu}.$$

Proof. We introduce the sequence

$$\varepsilon_\nu = \sum_{\mu=0}^{\infty} A_\mu^r \Delta^{r+1} \varepsilon_{\nu+\mu}.$$

Since $A_\mu^r \leq A_{\mu+1}^r$ for all values of μ, we have

$$|\varepsilon_\nu| \le \sum_{\mu=\nu}^{\infty} A_\mu^r |\Delta^{r+1}\varepsilon_\mu|, \quad i.e. \quad \varepsilon_\nu = o(1).$$

The differences of the order one being determined by

$$\Delta^1\varepsilon_\nu = A_0^r \Delta^{r+1}\varepsilon_\nu + \sum_{\mu=1}^{\infty}(A_\mu^r - A_{\mu-1}^r)\Delta^{r+1}\varepsilon_{\nu+\mu} = \sum_{\mu=0}^{\infty} A_\mu^{r-1}\Delta^{r+1}\varepsilon_{\nu+\mu},$$

we obtain, by mathematical induction, the difference formula

$$\Delta^q\varepsilon_\nu = \sum_{\mu=0}^{\infty} A_\mu^{r-q}\Delta^{r+1}\varepsilon_{\nu+\mu},$$

valid for any positive integer q. In particular,

$$\Delta^{s+1}\varepsilon_\nu = \sum_{\mu=0}^{\infty} A_\mu^{\delta-1}\Delta^{r+1}\varepsilon_{\nu+\mu}.$$

The proof is completed.

Lemma 1.50. *If $0 \le \delta \le 1$ and $0 \le m < n$, then*

$$\left|\sum_{p=0}^{m} A_{n-p}^{\delta-1} S_p\right| \le \max_{0\le\mu\le m} |S_m^\delta|.$$

Proof. We suppose that $0 < \delta < 1$ since the result is trivial for $\delta = 0$ or $\delta = 1$. By repeated application of Abel's Lemma 1.1 and since

$$\frac{A_{n-p}^{\delta-1}}{A_{m-p}^{\delta-1}} = \frac{\delta-1+n-p}{n-p}\cdot\frac{m-p}{\delta-1+m-p}\cdot\frac{A_{n-p-1}^{\delta-1}}{A_{m-p-1}^{\delta-1}} > \frac{A_{n-p-1}^{\delta-1}}{A_{m-p-1}^{\delta-1}}$$

for $0 \le p < m < n$, there exist integers m_p such that $m \ge m_1 \ge m_2 \ge \cdots \ge 0$, and

$$\begin{aligned}
\left|\sum_{p=0}^{m} A_{n-p}^{\delta-1} S_p\right| &= \left|\sum_{p=0}^{m} \frac{A_{n-p}^{\delta-1}}{A_{m-p}^{\delta-1}} A_{m-p}^{\delta-1} S_p\right| \\
&\le \frac{A_n^{\delta-1}}{A_m^{\delta-1}} \left|\sum_{p=0}^{m_1} A_{m-p}^{\delta-1} S_p\right| \\
&\le \frac{A_n^{\delta-1}}{A_m^{\delta-1}}\frac{A_m^{\delta-1}}{A_{m_1}^{\delta-1}} \left|\sum_{p=0}^{m_2} A_{m_1-p}^{\delta-1} S_p\right| \\
&\;\vdots \\
&\le \frac{A_n^{\delta-1}}{A_{m_k}^{\delta-1}} \left|\sum_{p=0}^{m_{k+1}} A_{m_k-p}^{\delta-1} S_p\right|.
\end{aligned}$$

Now since $m_1, m_2, \ldots$ is a non-decreasing sequence of non-negative integers, there is an integer ρ such that $m_\rho = m_{\rho+1}$. Therefore, since

$$\left|\sum_{p=0}^{m} A_{n-p}^{\delta-1} S_p\right| \leq \frac{A_n^{\delta-1}}{A_{m_\rho}^{\delta-1}} \left|\sum_{p=0}^{m_\rho} A_{m_\rho-p}^{\delta-1} S_p\right| \leq |S_{m_\rho}^{\delta}| \leq \max_{0\leq\mu\leq m} |S_m^{\delta}|.$$

Identically we note that for $0 < \delta < 1$, then $\frac{A_n^{\delta-1}}{A_{m_\rho}^{\delta-1}} < 1$, and whence there is strict inequality in this lemma unless $S_\mu^\delta = 0$ for $0 \leq \mu \leq m$, i.e. $S_0 = S_1 = \cdots = S_m = 0$.

The proof is completed.

Five other classes of sequences of the "semi" and "hyper" type also came across in the literature.

Definition 1.51. *The sequence $\{u_n\}$ is said to be twice quasi semi-convex if $u_n \to 0$ as $n \to \infty$ and*

$$\sum_{n=1}^{\infty} n \left|\Delta^4 u_{n-1} - \Delta^4 u_n\right| < \infty, \quad (u_0 = u_{-1} = 0).$$

Definition 1.52. *The sequence $\{u_n\}$ is said to be quasi semi-convex if $u_n \to 0$ as $n \to \infty$ and*

$$\sum_{n=1}^{\infty} n \left|\Delta^2 u_n - \Delta^2 u_{n+1}\right| < \infty.$$

Definition 1.53. *The sequence $\{u_n\}$ is said to be $r-$quasi convex, $r \geq 0$, if $u_n \to 0$ as $n \to \infty$ and*

$$\sum_{n=1}^{\infty} n^{r+1} \left|\Delta u_{n-1} - \Delta u_n\right| < \infty, \quad (u_0 = 0).$$

Definition 1.54. *The sequence $\{u_n\}$ is said to be third quasi hyper-convex if $u_n \to 0$ as $n \to \infty$ and*

$$\sum_{n=1}^{\infty} n^{3\alpha} \left|\Delta^{3\alpha-1} u_{n-1} - \Delta^{3\alpha-1} u_n\right| < \infty, \quad (u_0 = u_{-1} = 0).$$

The newly class of sequences **SJ** is given i next definition.

Definition 1.55. *The zero-sequence $\{u_n\}$ of positive numbers is said to be in the class* **SJ**, *if there exists a sequence $\{A_n\}$ such that*

(i) $A_n \downarrow 0$ *as* $n \to \infty$,
(ii) $\sum_{n=1}^{\infty} A_n < \infty$, *and*
(iii) $\left|\Delta\left(\frac{u_n}{n}\right)\right| \leq \frac{A_n}{n}$, *for all* n.

Since

$$\left|\Delta\left(\frac{u_n}{n}\right)\right| \le \frac{A_n}{n} \Longrightarrow |\Delta(u_n)| \le A_n,\ \forall n,$$

then it is clear that $\mathbf{SJ} \subset \mathbf{S}$.

The above inclusion is a proper one. Next example shows this fact.

Example 1.56. For $n \in \mathbf{Z}\setminus\{0,1,2\}$, where $\mathbf{Z}$ is the set of non-negative integers, define $u_n = \frac{1}{n^3}$. Then there exists the sequence $A_n = \frac{1}{n^2}$ such that u_n satisfies all conditions of the class S but not those of $\mathbf{SJ}$. But, for $n \in \{1,2,\ldots\}$ the sequence $v_n = \frac{1}{n^3}$ satisfies the conditions of the class S as well as conditions of the class $\mathbf{SJ}$. Therefore, the class $\mathbf{SJ}$ is indeed a proper subclass of the class S.

The class of sequences $\mathbf{SJ}_r$, $r \in \{0,1,2,\ldots\}$, is a natural extension of the class $\mathbf{SJ}$. It is defined as follows.

Definition 1.57. *The zero-sequence $\{u_n\}$ of positive numbers is said to be in the class $\mathbf{SJ}_r$, $r \in \{0,1,2,\ldots\}$, if there exists a sequence $\{A_n\}$ such that*

(i) $A_n \downarrow 0$ *as* $n \to \infty$,
(ii) $\sum_{n=1}^{\infty} n^r A_n < \infty$, *and*
(iii) $\left|\Delta\left(\frac{u_n}{n}\right)\right| \le \frac{A_n}{n}$, *for all* n.

It is also clear that for $r=0$ we have $\mathbf{SJ}_0 \equiv \mathbf{SJ}$. Moreover, the inclusion $\mathbf{SJ}_{r+1} \subset \mathbf{SJ}_r$ holds true for all $r \in \{0,1,2,\ldots\}$, but the converse statement does not hold as shows next example.

Example 1.58. For $n = 1,2,\ldots$, define $u_n = \frac{1}{n^{r+2}}$, $r \in \{0,1,2,\ldots\}$. Let us show first that $\{u_n\} \notin \mathbf{SJ}_{r+1}$. Namely, $u_n = \frac{1}{n^{r+2}} \to 0$ as $n \to \infty$. If we take $A_n = \frac{1}{n^{r+2}}$, $r \in \{0,1,2,\ldots\}$, then we have

$$\sum_{n=1}^{\infty} n^{r+1} A_n = \sum_{n=1}^{\infty} n^{r+1}\frac{1}{n^{r+2}} = \sum_{n=1}^{\infty}\frac{1}{n} = \infty,$$

which means that $\{u_n\} \notin \mathbf{SJ}_{r+1}$.

However, $A_n \downarrow 0$ as $n \to \infty$, and

$$\sum_{n=1}^{\infty} n^{r} A_n = \sum_{n=1}^{\infty} n^{r}\frac{1}{n^{r+2}} = \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty.$$

Finally,

$$\left|\Delta\left(\frac{u_n}{n}\right)\right| = \left|\Delta\left(\frac{1}{n^{r+3}}\right)\right| \le \frac{A_n}{n},\ \forall n.$$

So, we have verified that $\{u_n\} \in \mathbf{SJ}_r$.

The $\mathbf{BV}^{\log}$ class of sequences has been defined as follows.

Definition 1.59. *If* $u_j \to 0$ *as* $j \to \infty$ *and*

$$\sum_{j=1}^{\infty} \log^2(j+1) \left| \Delta \left(\frac{u_j}{\log(j+1)} \right) \right| < \infty$$

then we say that $\{u_j\}$ *belongs to the class* $\mathbf{BV}^{\log}$*.*

Also earlier has been introduced the so-called weakly even null-sequences and this class was denoted by $\mathbf{W}$.

Definition 1.60. *If* $u_j \to 0$ *as* $j \to \infty$ *and*

$$\sum_{j=1}^{\infty} \log(j+1) \, |\Delta u_j| < \infty$$

then we say that $\{u_j\}$ *is weakly even, briefly denoted by* $\{u_j\} \in \mathbf{W}$*.*

Moreover, in general context, next lemma shows that the class $\mathbf{BV}^{\log}$ is a wider class of sequences, and more useful in applications than the class $\mathbf{W}$.

Lemma 1.61. *The implication* $\{u_j\} \in \mathbf{W} \Longrightarrow \{u_j\} \in \mathbf{BV}^{\log}$ *holds true, i.e.* $\mathbf{W} \subseteq \mathbf{BV}^{\log}$*.*

Proof. Let $\{u_j\} \in \mathbf{W}$. After some elementary calculations we have

$$\Delta \left(\frac{u_j}{\log(j+1)} \right) = \frac{u_j - u_{j+1}}{\log(j+1)} + \frac{u_{j+1} \log \left(1 + \frac{1}{j+1} \right)}{\log(j+1) \log(j+2)}.$$

Hence,

$$\begin{aligned} \log^2(j+1) \left| \Delta \left(\frac{u_j}{\log(j+1)} \right) \right| &\leq \log(j+1)|\Delta u_j| + \frac{|u_{j+1}| \log{(j+1)}}{(j+1)\log(j+2)} \\ &\leq \log(j+1)|\Delta u_j| + 2\frac{|u_{j+1}|}{j+1}. \end{aligned}$$

Thus,

$$\begin{aligned} &\sum_{j=1}^{\infty} \log^2(j+1) \left| \Delta \left(\frac{u_j}{\log(j+1)} \right) \right| \leq \sum_{j=1}^{\infty} \log(j+1)|\Delta u_j| + 2\sum_{j=1}^{\infty} \frac{|u_{j+1}|}{j+1} \\ &\leq \sum_{j=1}^{\infty} \log(j+1)|\Delta u_j| + 2\sum_{j=1}^{\infty} \frac{1}{j+1} \sum_{i=j+1}^{\infty} |\Delta u_i| \leq \sum_{j=1}^{\infty} \log(j+1)|\Delta u_j| \\ &+2\sum_{i=1}^{\infty} |\Delta u_i| \sum_{j=1}^{i+1} \frac{1}{j} \leq \mathcal{O}\left(\sum_{j=1}^{\infty} \log(j+1)|\Delta u_j| \right) < +\infty, \end{aligned}$$

which clearly implies $\{u_j\} \in \mathbf{BV}^{\log}$.

The proof is completed.

1.2 Some classes of complex sequences

In this section we shall give the definitions of some classes of complex sequences. We assume that $\{c_j\}$ is a zero-sequence, i.e.

$$\lim_{|j|\to\infty} c_j = 0.$$

Definition 1.62. *A zero sequence $\{c_j\}$ of complex numbers satisfying*

$$\sum_{j=1}^{\infty} |\Delta (c_j - c_{-j})| \log j < \infty$$

is called weakly even.

It is clear that every even sequence is weakly even too.

The class $\mathbf{W}$ of weakly even sequences has been generalized in the following manner.

Definition 1.63. *A zero sequence $\{c_j\}$ of complex numbers belongs to the class $\mathbf{W}_r$, $r \in \{0, 1, 2, \ldots\}$ if*

$$\sum_{j=1}^{\infty} |\Delta (c_j - c_{-j})| \, j^r \log j < \infty.$$

For $r = 0$ we obviously have $\mathbf{W}_0 \equiv \mathbf{W}$.

Definition 1.64. *A zero sequence $\{c_j\}$ of complex numbers belongs to the class $\mathbf{R}^\star$ if*

$$\sum_{j=1}^{\infty} \left| \Delta \left(\frac{c_j - c_{-j}}{j} \right) \right| j \log j < \infty$$

and

$$\sum_{j=1}^{\infty} j^2 \left| \Delta^2 \left(\frac{c_j}{j} \right) \right| < \infty.$$

The class $\mathbf{R}^\star$ is generalized by the following definition.

Definition 1.65. *A zero sequence $\{c_j\}$ of complex numbers belongs to the class $\mathbf{R}^\star(r)$, $r \in \{0, 1, 2, \ldots\}$, if*

$$\sum_{j=1}^{\infty} \left| \Delta \left(\frac{c_j - c_{-j}}{j} \right) \right| j^{r+1} \log j < \infty$$

and

$$\sum_{j=1}^{\infty} j^{r+2} \left| \Delta^2 \left(\frac{c_j}{j} \right) \right| < \infty.$$

If we take $r = 0$ in this definition, then we obtain $\mathbf{R}^\star(0) \equiv \mathbf{R}^\star$.

The class $\mathbf{S}_p^\star$ of complex sequences is defined as follows.

Definition 1.66. *A weakly even zero sequence $\{c_j\}$ of complex numbers belongs to the class $\mathbf{S}_p^\star$ if for some $1 < p \leq 2$ and some monotone sequence $\{A_j\}$ such that $\sum_{j=1}^{\infty} A_j < \infty$ the condition*

$$\frac{1}{n} \sum_{j=1}^{n} \frac{|\Delta(c_j)|^p}{A_j^p} = \mathcal{O}(1)$$

holds.

An extension of the class $\mathbf{S}_p^\star$ is the class $\mathbf{S}_{p\alpha}^\star(\delta)$, where $1 < p \leq 2$ and $\alpha \geq 0$.

Definition 1.67. *A weakly even zero sequence $\{c_j\}$ of complex numbers belongs to the class $\mathbf{S}_{p\alpha}^\star(\delta)$ if for some $1 < p \leq 2$ and some δ-quasi-monotone sequence $\{A_j\}$ such that $\sum_{j=1}^{\infty} j^{\alpha+1}\delta_j < \infty$ and $\sum_{j=1}^{\infty} j^{\alpha} A_j < \infty$ the condition*

$$\frac{1}{n^{p\alpha+1}} \sum_{j=1}^{n} \frac{|\Delta(c_j)|^p}{A_j^p} = \mathcal{O}(1)$$

holds.

A new class of complex sequences also has been introduced e.i., the class $\mathbf{K}^\star$.

Definition 1.68. *A zero sequence $\{c_j\}$ of complex numbers belongs to the class $\mathbf{K}^\star$ if for some $1 < p \leq 2$ the conditions*

$$\frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} \left(\frac{c_j - c_{-j}}{k} \right) \log k = o(1),$$

$$\lim_{\lambda \downarrow 1} \overline{\lim}_{n\to\infty} \sum_{j=1}^{[\lambda n]} \left(\frac{c_j - c_{-j}}{k} \right) k \log k = 0,$$

and

$$\lim_{\lambda \downarrow 1} \overline{\lim}_{n\to\infty} \sum_{j=n}^{[\lambda n]} j^{p-1} \left| \Delta \left(\frac{c_j}{j} \right) \right|^p = 0,$$

hold, where $[\lambda]$ denotes the integer part of λ.

Definition 1.69. *A null sequence (c_j) of complex numbers belongs to class $\mathbf{J}^*$ if there exists a sequence (A_j) such that*

$$A_j \downarrow 0, \quad as \quad j \to \infty,$$

$$\sum_{j=1}^{\infty} jA_j < \infty,$$

and

$$\left|\Delta\left(\frac{c_j - c_{-j}}{j}\right)\right| \leq \frac{A_j}{j}, \quad \forall j.$$

Now we are going to introduce the following class of sequences of complex numbers.

Definition 1.70. *A null sequence (c_j) of complex numbers belongs to class $\mathcal{K}$ if there exists a sequence (A_j) such that*

$$A_j \downarrow 0, \quad as \quad j \to \infty,$$

$$\sum_{j=1}^{\infty} jA_j < \infty,$$

and

$$\max\left\{\left|\Delta\left(\frac{c_j}{j}\right)\right|, \left|\Delta\left(\frac{c_{-j}}{j}\right)\right|\right\} \leq \frac{A_j}{j}, \quad j \in \{1, 2, \ldots\}.$$

The class $\mathcal{K}$ has been extended to the class $\mathcal{K}^2$ given in the next definition.

Definition 1.71. *A null sequence (c_j) of complex numbers belongs to class $\mathcal{K}^2$ if there exists a sequence (A_j) such that*

$$A_j \downarrow 0, \quad as \quad j \to \infty,$$

$$\sum_{j=1}^{\infty} j^2 A_j < \infty,$$

and

$$\max\left\{\left|\Delta^2\left(\frac{c_j}{j}\right)\right|, \left|\Delta^2\left(\frac{c_{-j}}{j}\right)\right|\right\} \leq \frac{A_j}{j^2}, \quad j \in \{1, 2, \ldots\}.$$

Next example shows that the there exist sequences that belong or not belong to the class $\mathcal{K}^2$.

Example 1.72. Let (c_j) be a sequence defined by its general term $c_j := \frac{1}{j^2}$, $j \in \{1, 2, \ldots\}$. Then, $\left|\Delta^2\left(\frac{c_{\pm j}}{j}\right)\right| \leq \frac{4}{j^3} = \frac{A_j}{j^2}$, $A_j = \frac{4}{j} \downarrow 0$, and $\sum_{j=1}^{\infty} j^2 A_j = +\infty$, which means that $(c_j) \notin \mathcal{K}^2$.

On the other hand, let $(\overline{c}_j)$ be a sequence defined by its general term $\overline{c}_j = \frac{1}{j^5}$, $j \in \{1, 2, \ldots\}$. Then, $\left|\Delta^2\left(\frac{c_{\pm j}}{j}\right)\right| \leq \frac{4}{j^6} = \frac{A_j}{j^2}$, $A_j = \frac{4}{j^4} \downarrow 0$, and $\sum_{j=1}^{\infty} j^2 A_j < +\infty$, which means that $(\overline{c}_j) \in \mathcal{K}^2$.

The class $\mathcal{K}^2$ was generalized also to the class $\mathcal{K}^2_r$, $r \in \{1, 2, \ldots\}$.

Definition 1.73. *A null sequence* (c_j) *of complex numbers belongs to class* $\mathcal{K}^2_r$ *if there exists a sequence* (A_j) *such that*

$$A_j \downarrow 0, \quad as \quad j \to \infty,$$

$$\sum_{j=1}^{\infty} j^{r+1} A_j < \infty,$$

and

$$\max \left\{ \left| \Delta^2 \left(\frac{c_j}{j^r} \right) \right|, \left| \Delta^2 \left(\frac{c_{-j}}{j^r} \right) \right| \right\} \le \frac{A_j}{j^{r+1}}, \quad j, r \in \{1, 2, \ldots\}.$$

Note that for $r = 1$ we clearly have $\mathcal{K}^2_1 \equiv \mathcal{K}^2$.

Even in this case, next example shows that the there exist sequences that belong or not belong to the class $\mathcal{K}^2_r$ for some r.

Example 1.74. Let (c_j) be a sequence defined by its general term $c_j := \frac{1}{j^2}$, $j \in \{1, 2, \ldots\}$ and $r = 2$. Then, $\left| \Delta^2 \left(\frac{c_{\pm j}}{j^2} \right) \right| \le \frac{4}{j^4} = \frac{A_j}{j^3}$, $A_j = \frac{4}{j} \downarrow 0$, and $\sum_{j=1}^{\infty} j^3 A_j = +\infty$, which means that $(c_j) \notin \mathcal{K}^2_2$.

On the other hand, let $(\overline{c}_j)$ be a sequence defined by its general term $\overline{c}_j = \frac{1}{j^6}$, $j \in \{1, 2, \ldots\}$. Then, $\left| \Delta^2 \left(\frac{c_{\pm j}}{j^2} \right) \right| \le \frac{4}{j^8} = \frac{A_j}{j^3}$, $A_j = \frac{4}{j^5} \downarrow 0$, and $\sum_{j=1}^{\infty} j^3 A_j < +\infty$, which means that $(\overline{c}_j) \in \mathcal{K}^2_2$.

Some of the classes given in this section are extended to the two-dimensional case. Indeed, the notion of bounded variation of double sequences is given by next definition.

Definition 1.75. *We say that* $u_{j,k}$ *belongs to the class* $\mathbf{BV_2}$ *if*

$$u_{j,k} \to 0 \quad as \quad j + k \to \infty,$$

and

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\Delta_{11} u_{j,k}| < \infty,$$

where

$$\Delta_{11} u_{j,k} := u_{j,k} - u_{j+1,k} - u_{j,k+1} + u_{j+1,k+1}.$$

Definition 1.76. *The zero-sequence* $\{u_{j,k}\}$ *is said to be in the class* $\mathcal{C}_2$ *if for every* $\varepsilon > 0$*, there exists a* $\delta(\varepsilon) > 0$*, independent of* m, n*, and so that for all* $m \ge 0$ *and* $n \ge 0$*, we have*

$$\int \int_{D_\delta} \left| \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} \Delta u_{j,k} \frac{\sin\left(j + \frac{1}{2}\right) x \sin\left(k + \frac{1}{2}\right) y}{4 \sin \frac{x}{2} \sin \frac{y}{2}} \right| dxdy < \varepsilon,$$

where

$$D_\delta := \{(x, y) : 0 \le x, y \le \pi \ \& \ \min(x, y) \le \delta\}.$$

Definition 1.77. *The double sequence* $\{u_{j,k}\}$ *is said to be quasi-convex if*

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(j+1)(k+1)\left|\Delta_{22}u_{j,k}\right| < \infty,$$

where $\Delta_{22}u_{j,k} = \Delta_{11}\left(u_{j,k} - u_{j+1,k} - u_{j,k+1} + u_{j+1,k+1}\right).$

Definition 1.78. *A double null sequence* $\{a_{j,k}\}$ *of positive numbers is said to belong to the class* $\mathbf{J_d}$ *if there exists a double sequence* $\{A_{j,k}\}$ *such that*

$$A_{j,k} \downarrow 0, \quad j+k \to \infty,$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} jkA_{j,k} < \infty,$$

and

$$\left|\Delta_{p,q}\left(\frac{a_{j,k}}{jk}\right)\right| \le \frac{A_{j,k}}{jk}, \quad 1 \le p+q \le 2$$

for any non-negative integers p, q *and* $j, k \in \{1, 2, 3, \ldots\}$.

1.3 Basic facts on trigonometric series

A trigonometric series is the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{1.9}$$

where a_n and b_n are real numbers ($n = 0, 1, 2, \ldots$), known as the coefficients of the series.

If a trigonometric series converges for all $x \in (-\infty, +\infty)$, then it represents a function which has the period 2π.

Trigonometric series play an important role not only in mathematics itself but also in many of its applications. We will not discuss their entire role here but only one of them which we are going to reveal it at the and of this section.

Let us express the series (1.9) in a different form. Using the well-known Euler's identity

$$e^{ix} = \cos x + i \sin x \quad \textit{and} \quad e^{-ix} = \cos x - i \sin x$$

it follows that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \textit{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

Putting last identities into (1.9) we obtain

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{e^{ix} + e^{-ix}}{2} + ib_n \frac{e^{-ix} - e^{ix}}{2} \right).$$

Hence, taking

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}, \quad \textit{and} \quad c_{-n} = \frac{a_n + ib_n}{2},$$

then the series (1.9) takes its form

$$\sum_{n=-\infty}^{+\infty} c_n e^{inx}. \tag{1.10}$$

The series (1.10) is called complex form of the series (1.9).

The partial sum of the series (1.9) is

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx),$$

while its complex form is

$$S_n(x) = \sum_{k=-n}^{k=+n} c_k e^{ikx}, \tag{1.11}$$

in which case the convergence of the series (1.10) must be understood as the limit of sums of the form (1.11).

Let us assume that the function $f(x)$ is not only the sum of a trigonometric series but also that this series converges uniformly in $[-\pi, \pi]$. These conditions allow us easily to determine its coefficients. This is implied by multiplying

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

by $\cos kx$ or by $\sin kx$, then integrating it from $-\pi$ to π and taking into account that

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = 0, \quad m \neq n,$$
$$\int_{-\pi}^{\pi} \sin mx \cos nx dx = 0, \quad m \neq n,$$
$$\int_{-\pi}^{\pi} \cos mx \sin nx dx = 0, \quad m \neq n \quad \textit{and} \quad m = n,$$
$$\int_{-\pi}^{\pi} \cos^2 nx dx = \int_{-\pi}^{\pi} \sin^2 nx dx = \pi, \quad m, n \in N.$$

These equalities imply that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \; n \in \{0, 1, \ldots\}. \quad (1.12)$$

Formulae (1.12) are called Fourier formulae, the numbers a_n and b_n are Fourier coefficients and the series whose coefficients are determined by Fourier formulae derived from the function $f(x)$ is named the Fourier series of the function $f(x)$.

Regarding to the complex form of Fourier coefficients of the series (1.10) we write as in the following. Indeed, assuming that

$$f(x) = \sum_{k=-\infty}^{+\infty} c_k e^{ikx}, \quad (1.13)$$

(where the convergence is uniform), multiplying both sides of (1.13) by e^{-inx} and integrating term by term, we have

$$\int_{-\pi}^{\pi} f(x)dx = \sum_{k=-\infty}^{+\infty} \int_{-\pi}^{\pi} c_k e^{i(k-n)x}dx.$$

However,

$$\int_{-\pi}^{\pi} e^{i(k-n)x}dx = 2\pi \quad if \quad k = n \quad and \quad = 0 \quad if \quad k \neq n,$$

and whence

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \quad (n = 0, 1, 2, \ldots).$$

The numbers c_n are called the complex Fourier coefficients of the function $f(x)$.

Using formulae (1.12) we find that

$$\begin{aligned} S_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \sum_{k=1}^{n}\left[\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ktdt\right)\cos kx \right. \\ &\quad \left. + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ktdt\right)\sin kx\right] \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n}(\cos kt\cos kx + \sin kt\sin kx)\right]dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n}\cos k(t-x)\right]dt. \end{aligned}$$

Replacing $t - x = u$ into last equality and using 2π-periodicity of $f(x)$ we obtain

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\left[\frac{1}{2} + \sum_{k=1}^{n}\cos k(u)\right]du = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)D_n(u)du,$$

where

$$D_n(u) := \frac{1}{2} + \sum_{k=1}^{n}\cos ku.$$

The expression $D_n(u)$ is called the Dirichlet kernel which we are going to express it in a simpler form. Namely, using the elementary trigonometric identity

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$

we have

$$\begin{aligned}
D_n(u) &= \frac{1}{2} + \frac{1}{2\sin\frac{u}{2}}\sum_{k=1}^{n} 2\sin\frac{u}{2}\cos ku \\
&= \frac{1}{2} + \frac{1}{2\sin\frac{u}{2}}\sum_{k=1}^{n}\left[\sin\left(k+\frac{1}{2}\right)u - \sin\left(k-\frac{1}{2}\right)u\right] \\
&= \frac{1}{2} + \frac{1}{2\sin\frac{u}{2}}\left[\left(\sin\frac{3u}{2} - \sin\frac{u}{2}\right) + \left(\sin\frac{5u}{2} - \sin\frac{3u}{2}\right)\right. \\
&\quad + \cdots + \left(\sin\left(n-\frac{1}{2}\right)u - \sin\left(n-\frac{3}{2}\right)u\right) \\
&\quad \left. + \left(\sin\left(n+\frac{1}{2}\right)u - \sin\left(n-\frac{1}{2}\right)u\right)\right] \\
&= \frac{1}{2} + \frac{1}{2\sin\frac{u}{2}}\left[-\sin\frac{u}{2} + \sin\left(n+\frac{1}{2}\right)u\right] = \frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}
\end{aligned}$$

(This is why in the trigonometric series we take $\frac{a_0}{2}$ and not only a_0).
The series

$$\sum_{n=1}^{\infty}(-b_n\cos nx + a_n\sin nx)$$

usually is called the conjugate series of the the trigonometric series (1.9).
For its partial sum $\widetilde{S}_n(x)$ we similarly find that

$$\begin{aligned}
\widetilde{S}_n(x) &= \sum_{k=1}^{n}(-b_k\cos kx + a_k\sin kx) \\
&= \sum_{k=1}^{n}\left[\left(-\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt dt\right)\cos kx\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left(-\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ktdt\right)\sin kx\Bigg] \\
&=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\sum_{k=1}^{n}(-\sin kt\cos kx+\cos kt\sin kx)\right]dt \\
&=-\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\sum_{k=1}^{n}\sin k(t-x)\right]dt \\
&=-\frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\sum_{k=1}^{n}\sin kudu=-\frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\widetilde{D}_n(u)du,
\end{aligned}
$$

where

$$\widetilde{D}_n(u):=\sum_{k=1}^{n}\sin ku$$

is called the conjugate Dirichlet kernel.

The conjugate Dirichlet kernel $\widetilde{D}_n(x)$ has the following simplified form

$$
\begin{aligned}
\widetilde{D}_n(x)&=\frac{1}{2\sin\frac{u}{2}}\sum_{k=1}^{n}2\sin\frac{u}{2}\sin ku \\
&=\frac{1}{2\sin\frac{u}{2}}\sum_{k=1}^{n}\left[\cos\left(k-\frac{1}{2}\right)u-\cos\left(k+\frac{1}{2}\right)u\right] \\
&=\frac{1}{2\sin\frac{u}{2}}\left[\left(\cos\frac{u}{2}-\cos\frac{3u}{2}\right)+\left(\cos\frac{3u}{2}-\cos\frac{5u}{2}\right)+\cdots+\right. \\
&\quad\left.+\left(\cos\left(n-\frac{1}{2}\right)u-\cos\left(n+\frac{1}{2}\right)u\right)\right] \\
&=\frac{1}{2\sin\frac{u}{2}}\left[\cos\frac{u}{2}-\cos\left(n+\frac{1}{2}\right)u\right]=\frac{\cos\frac{u}{2}-\cos\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}.
\end{aligned}
$$

In order to estimate the kernels $D_n(x)$ and $\widetilde{D}_n(x)$ we prove a lemma known as Jordan's inequality.

Lemma 1.79. *If* $x\in\left[0,\frac{\pi}{2}\right]$, *then* $\sin x\geq\frac{2}{\pi}x$.

Proof. It is clear that for $x=0$ the inequality holds always true. Let $x\in\left(0,\frac{\pi}{2}\right]$. Since the function $\frac{\sin x}{x}$ is a decreasing one we have

$$\frac{\sin x}{x}\geq\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}}=\frac{2}{\pi}\quad i.e.\quad \sin x\geq\frac{2}{\pi}x.$$

The proof is completed.

Using Lemma 1.79 we have

$$|D_n(u)| \le \frac{1}{2\frac{2}{\pi}\frac{u}{2}} = \frac{\pi}{2u} = \mathcal{O}\left(\frac{1}{u}\right), \quad for \quad 0 < u \le \pi,$$

and also

$$\left|\widetilde{D}_n(u)\right| \le \frac{1}{2\frac{2}{\pi}\frac{u}{2}} = \frac{\pi}{2u} = \mathcal{O}\left(\frac{1}{u}\right), \quad for \quad 0 < u \le \pi.$$

Last estimates of the Dirichlet and the conjugate Dirichlet kernels have very important role in studying many questions related to the trigonometric series as we will see later in this book.

The Cesàro mean of the first order of the Dirichlet and of the conjugate Dirichlet kernels

$$K_n(u) := \frac{1}{n+1}\sum_{k=0}^{n} D_k(u)$$

and

$$\widetilde{K}_n(u) := \frac{1}{n+1}\sum_{k=1}^{n} \widetilde{D}_k(u)$$

respectively, are called the Fejér and the conjugate Fejér kernels.

Another form of the Fejér kernel can be derived as well. Namely, since

$$D_k(u) = \frac{2\sin\left(k+\frac{1}{2}\right)u\sin\frac{u}{2}}{4\sin^2\frac{u}{2}} = \frac{\cos ku - \cos(k+1)u}{4\sin^2\frac{u}{2}}$$

we have

$$\begin{aligned} K_n(u) &= \frac{1}{n+1}\sum_{k=0}^{n}\frac{\cos ku - \cos(k+1)u}{4\sin^2\frac{u}{2}} = \frac{1}{n+1}\frac{1-\cos(n+1)u}{4\sin^2\frac{u}{2}} \\ &= \frac{1}{n+1}\frac{2\sin^2\frac{(n+1)u}{2}}{4\sin^2\frac{u}{2}} = \frac{1}{2(n+1)}\left(\frac{\sin\frac{(n+1)u}{2}}{\sin\frac{u}{2}}\right)^2. \end{aligned}$$

This form implies that $K_n(u) \ge 0$ and using Lemma 1.79 for we get

$$K_n(u) \le \frac{1}{2(n+1)\sin^2\frac{u}{2}} \le \frac{1}{2(n+1)\left(\frac{2}{\pi}\frac{u}{2}\right)^2} = \mathcal{O}\left(\frac{1}{nu^2}\right),$$

for all $u \in (0,\pi]$.

Moreover, since

$$\int_{-\pi}^{\pi} D_k(u)du = \pi, \quad (k = 0,1,2,\ldots),$$

we also obtain

$$\int_{-\pi}^{\pi} K_n(u)du = \frac{1}{n+1}\sum_{k=0}^{n}\int_{-\pi}^{\pi} D_k(u)du = \pi, \quad (n = 0, 1, 2, \dots).$$

For the conjugate Fejér kernel we also have

$$\widetilde{K}_n(u) > 0 \quad for \quad 0 < u < \pi,$$

and for $n \in \{1, 2, \dots\}$,

$$|\widetilde{K}_n(u)| \le \frac{1}{n+1}\sum_{k=1}^{n} |\widetilde{D}_k(u)| \le \frac{1}{n+1}\sum_{k=1}^{n} k = \frac{n}{2}.$$

An interesting relation between $D_n(u)$, the derivative of $\widetilde{D}_n(u)$, and $K_n(u)$ is given in next lemma.

Lemma 1.80. *The equality*

$$\widetilde{D}'_n(u) = (n+1)D_n(u) - (n+1)K_n(u)$$

holds true.

Proof. Since

$$\begin{aligned}
&(n+1)D_n(u) - (n+1)K_n(u)\\
&\qquad = (n+1)\frac{\cos nu - \cos(n+1)u}{4\sin^2\frac{u}{2}} - \frac{1-\cos(n+1)u}{4\sin^2\frac{u}{2}}\\
&\qquad = \frac{(n+1)\cos nu - n\cos(n+1)u - 1}{4\sin^2\frac{u}{2}}\\
&\qquad = \frac{n[\cos nu - \cos(n+1)u] + \cos nu - 1}{4\sin^2\frac{u}{2}}\\
&\qquad = \frac{2n\sin\frac{u}{2}\sin\left(n+\frac{1}{2}\right)u + \cos nu - 1}{4\sin^2\frac{u}{2}}
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{D}'_n(u) &= \frac{-\sin^2\frac{u}{2} + (2n+1)\sin\frac{u}{2}\sin\left(n+\frac{1}{2}\right)u - \cos^2\frac{u}{2} + \cos\frac{u}{2}\cos\left(n+\frac{1}{2}\right)u}{4\sin^2\frac{u}{2}}\\
&= \frac{2n\sin\frac{u}{2}\sin\left(n+\frac{1}{2}\right)u + \cos nu - 1}{4\sin^2\frac{u}{2}}
\end{aligned}$$

we obtain

$$\widetilde{D}'_n(u) = (n+1)D_n(u) - (n+1)K_n(u).$$

The proof is completed.

Lemma 1.81. *For n big enough the estimation*

$$\int_{-\pi}^{\pi} \left| \frac{D_n(u)}{2 \sin u} \right| du = \mathcal{O}(n^2).$$

holds true.

Proof. Since

$$|D_n(u)| \le \frac{1}{2} + \sum_{k=1}^{n} |\cos kx| = \frac{1}{2} + n$$

for all $n \in \{1, 2, \ldots\}$, we have

$$\begin{aligned}
\int_{-\pi}^{\pi} \left| \frac{D_n(u)}{2 \sin u} \right| du &\le \left(\frac{1}{2} + n \right) \int_{-\pi}^{\pi} \frac{du}{2|\sin u|} = 4 \left(\frac{1}{2} + n \right) \int_{0}^{\frac{\pi}{2}} \frac{du}{2|\sin u|} \\
&\le \pi \left(\frac{1}{2} + n \right) \int_{0}^{\frac{\pi}{2}} \frac{du}{u} \le 8n \sum_{k=1}^{n-2} \int_{\frac{\pi}{k+2}}^{\frac{\pi}{k+1}} \frac{du}{u} \\
&\le 8n \sum_{k=1}^{n-2} \frac{1}{k+1} = \mathcal{O}(n \log n) = \mathcal{O}(n^2).
\end{aligned}$$

The proof is completed.

Lemma 1.82. *Let r be a non-negative integer, and $x \in [\pi/n, \pi]$, where $n \ge 1$. Then*

$$\begin{aligned}
D_n^{(r)}(x) = &\sum_{k=0}^{r-1} \frac{(n+1/2)^k \sin[(n+1/2)x + k\pi/2]}{[\sin(x/2)]^{r+1-k}} \varphi \\
&+ \frac{(n+1/2)^r \sin[(n+1/2)x + r\pi/2]}{\sin(x/2)},
\end{aligned}$$

where the same φ denotes various analytical functions of x independent of n, and $D_n(x)$ is the Dirichlet kernel.

Proof. For $r = 0$ the proof follows immediately. Supposing that this equality holds true and deriving its both sides we have

$$\begin{aligned}
D_n^{(r+1)}(x) = &\sum_{k=0}^{r-1} \Bigg\{ (n+1/2)^{k+1} \sin[(n+1/2)x + (k+1)\pi/2][\sin(x/2)]^{k-r-1} \varphi \\
&+ (n+1/2)^k \sin[(n+1/2)x + k\pi/2][\sin(x/2)]^{k-r-2} \varphi \\
&+ \frac{(n+1/2)^{r+1} \sin[(n+1/2)x + (r+1)\pi/2]}{2 \sin(x/2)} \\
&+ \frac{(n+1/2)^r \sin[(n+1/2)x + r\pi/2]}{\sin^2(x/2)} \varphi \Bigg\}
\end{aligned}$$

$$= \sum_{k=0}^{r}(n+1/2)^k \sin[(n+1/2)x + k\pi/2][\sin(x/2)]^{k-r-2}\varphi$$
$$+(n+1/2)^{r+1}\sin[(n+1/2)x+(r+1)\pi/2]2[\sin(x/2)]^{-1}.$$

By mathematical induction the proof is completed.

Lemma 1.83. *Let r be a non-negative integer, and $x \in [\varepsilon, \pi]$, where $n \geq 1$. Then*

$$|D_n^{(r)}(x)| \leq \frac{C_\varepsilon n^r}{x},$$

where C_ε is a positive constant depending on ε and $0 < \varepsilon < \pi$.

Proof. Using Lemmas 1.82 and 1.79 we have

$$|D_n^{(r)}(x)| = \mathcal{O}\left(\sum_{k=0}^{r-1}\frac{(n+1/2)^k}{x^{r+1-k}} + \frac{(n+1/2)^r}{x}\right) = \mathcal{O}_\varepsilon\left(\frac{n^r}{x}\right).$$

The proof is completed.

Lemma 1.84. *Let $r \in \{0, 1, 2, \ldots\}$. Then*

$$\int_{-\pi}^{\pi}|D_n^{(r)}(x)|dx = \mathcal{O}_\varepsilon\left(n^r \log n + n^r\right).$$

where $\mathcal{O}_\varepsilon$ contains a positive constant depending on ε and $0 < \varepsilon < \pi$.

Proof. Since

$$D_n^{(r)}(x) = \sum_{k=1}^{n} k^r \cos(kx + r\pi/2),$$

then we have

$$|D_n^{(r)}(x)| \leq \sum_{k=1}^{n} k^r \leq n^{r+1}.$$

Whence, using Lemmas 1.82 and 1.79 we obtain

$$\int_{-\pi}^{\pi}|D_n^{(r)}(x)|dx = 2\left(\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right)|D_n^{(r)}(x)|dx$$
$$= 2n^r\int_0^{\pi/n}\frac{|\sin(nx+r\pi/2)|}{x}dx$$
$$+\sum_{k=0}^{r-1}\int_{\pi/n}^{\pi} n^k x^{k-1-r}dx + \mathcal{O}(n^r) = \mathcal{O}_\varepsilon\left(n^r\log n + n^r\right).$$

The proof is completed.

Lemma 1.85. *Let $r \in \{0, 1, 2, \ldots\}$. Then*

$$\int_{-\pi}^{\pi} |\widetilde{D}_n^{(r)}(x)| dx = \mathcal{O}\left(n^r \log n\right).$$

Proof. Using the Bernstein inequality, for trigonometric polynomials, in the L space, we have

$$\int_0^{\pi} |\widetilde{D}_n^{(r)}(x)| dx \leq n^r \int_0^{\pi} |\widetilde{D}_n(x)| dx.$$

However,

$$\begin{aligned} \int_0^{\pi} |\widetilde{D}_n(x)| dx &\leq \int_0^{\pi} \frac{\sin^2(nx/2)}{x} dx + \mathcal{O}(1) \\ &= \log(1 + n\pi) + \mathcal{O}(1) = \mathcal{O}(\log n). \end{aligned}$$

Last two estimates complete the proof of this lemma.

Lemma 1.86. *Let the real numbers α_i, $i = 1, 2, \ldots, n$, satisfy conditions $|\alpha_i| \leq 1$. Then the following estimation holds true*

$$\int_0^{\pi} \left| \sum_{i=0}^{n} \alpha_i \widetilde{D}_i^{(r)}(x) \right| dx \leq C(n+1)^r,$$

where C is a positive constant.

Proof. Using the Bernstein inequality, for trigonometric polynomials, in the L space and Lemma 1.34, we have

$$\int_0^{\pi} \left| \sum_{i=0}^{n} \alpha_i \widetilde{D}_i^{(r)}(x) \right| dx \leq (n+1)^r \int_0^{\pi} \left| \sum_{i=0}^{n} \alpha_i \widetilde{D}_i(x) \right| dx \leq C(n+1)^{r+1},$$

where C is a positive constant.

The proof is completed.

Lemma 1.87. *The following estimates*

$$|\widetilde{D}_n^{\log}(x)| = \mathcal{O}\left(\frac{\log(n+1)}{x}\right), \quad 0 < x \leq \pi$$

and

$$|D_n^{\log}(x)| = \mathcal{O}\left(\left(\frac{\pi}{x} + \frac{1}{2}\right) \log(n+1)\right), \quad 0 < x \leq \pi,$$

hold true, where

$$D_n^{\log}(x) = \sum_{j=1}^{n} \log(j+1) \cos(jx)$$

and

$$\widetilde{D}_n^{\log}(x) = \sum_{j=1}^{n} \log(j+1) \sin(jx).$$

Proof. Applying Abel's transformation we obtain

$$\begin{aligned}\widetilde{D}_n^{\log}(x) &= \sum_{j=1}^{n} \log(j+1)\sin(jx)\\ &= \sum_{j=1}^{n-1} \Delta(\log(j+1)) \sum_{s=1}^{j} \sin(sx) + \log(n+1)\sum_{s=1}^{n}\sin(sx)\\ &= -\sum_{j=1}^{n-1} \log\left(1+\frac{1}{j+1}\right)\frac{\cos\frac{x}{2}-\cos\left(j+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\\ &\quad +\log(n+1)\frac{\cos\frac{x}{2}-\cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}, \ \textit{for}\ 0<x\le\pi.\end{aligned}$$

Thus, using Lemma 1.79, we have

$$|\widetilde{D}_n^{\log}(x)| \le \sum_{j=1}^{n-1}\frac{1}{j+1}\frac{\pi}{x} + \log(n+1)\frac{\pi}{x} = \mathcal{O}\left(\frac{\log(n+1)}{x}\right), \quad 0<x\le\pi.$$

Similarly, we have

$$\begin{aligned}D_n^{\log}(x) &= \sum_{j=1}^{n} \log(j+1)\cos(jx)\\ &= \sum_{j=1}^{n-1} \Delta(\log(j+1)) \sum_{s=1}^{j} \cos(sx) + \log(n+1)\sum_{s=1}^{n}\cos(sx)\\ &= -\sum_{j=1}^{n-1} \log\left(1+\frac{1}{j+1}\right)\left[\frac{\sin\left(j+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}-\frac{1}{2}\right]\\ &\quad +\log(n+1)\left[\frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}-\frac{1}{2}\right], \ \textit{for}\ 0<x\le\pi.\end{aligned}$$

Thus, using Lemma 1.79, we have

$$\begin{aligned}|D_n^{\log}(x)| &\le \sum_{j=1}^{n-1}\frac{1}{j+1}\left(\frac{\pi}{x}+\frac{1}{2}\right) + \log(n+1)\left(\frac{\pi}{x}+\frac{1}{2}\right)\\ &= \mathcal{O}\left(\left(\frac{\pi}{x}+\frac{1}{2}\right)\log(n+1)\right).\end{aligned}$$

The proof is completed.

Lemma 1.88. *If* $x \in [\epsilon, \pi-\epsilon], \epsilon > 0$ *and* $m \in N$*, then the following estimate holds*

$$\left|\left(\frac{\widetilde{D}_m(x)}{2\sin x}\right)^{(r)}\right| = \mathcal{O}_{r,\epsilon}\left(m^{r+1}\right), \quad (r = 0, 1, 2\ldots)$$

where $O_{r,\epsilon}$ depends only on r and ϵ.

Proof. By Leibniz formula we have

$$\begin{aligned}\left(\frac{\widetilde{D}_m(x)}{2\sin x}\right)^{(r)} &= \sum_{i=0}^{r}\binom{r}{i}\left(\frac{1}{2\sin x}\right)^{(r-i)}\left(\widetilde{D}_m(x)\right)^{(i)} \\ &= \sum_{i=0}^{r}\binom{r}{i}\left(\frac{1}{2\sin x}\right)^{(r-i)}\sum_{j=1}^{m} j^i \sin\left(jx + \frac{i\pi}{2}\right) \\ &= \mathcal{O}(1)m^{r+1}\sum_{i=0}^{r}\binom{r}{i}\left(\frac{1}{2\sin x}\right)^{(r-i)}. \end{aligned} \tag{1.14}$$

We shall prove by mathematical induction the equality

$$\left(\frac{1}{2\sin x}\right)^{(\tau)} = \frac{P_\tau(\cos x)}{\sin^{\tau+1} x},$$

where P_τ is a cosine polynomial of degree τ.

Namely, we have

$$\left(\frac{1}{2\sin x}\right)' = \frac{(-1/2)\cos x}{\sin^2 x} = \frac{P_1(\cos x)}{\sin^2 x},$$

so for $\tau = 1$ the above equality is true.

Assume that the equality

$$F(x) := \left(\frac{1}{2\sin x}\right)^{(\tau)} = \frac{P_\tau(\cos x)}{\sin^{\tau+1} x}$$

holds. For the $(\tau+1)-th$ derivative of $\frac{1}{2\sin x}$ we get

$$\begin{aligned} &F'(x) = \\ &= \frac{P_\tau'(\cos x)\left(-\sin^{\tau+2} x\right) - P_\tau(\cos x)(r+1)\sin^\tau x\cos x}{\sin^{2\tau+2} x} \\ &= \frac{(-1/2)H_{\tau-1}(\cos x) + (1/2)H_{\tau-1}(\cos x)\cos 2x - (r+1)P_\tau(\cos x)\cos x}{\sin^{\tau+2} x} \\ &= \frac{Q_{\tau+1}(\cos x) - (r+1)R_{\tau+1}(\cos x)}{\sin^{\tau+2} x} = \frac{T_{\tau+1}(\cos x)}{\sin^{\tau+2} x}, \end{aligned} \tag{1.15}$$

where $H_{\tau-1}$, $Q_{\tau+1}$, $R_{\tau+1}$, $T_{\tau+1}$ are cosine polynomials of degree $\tau-1$ and $\tau+1$ respectively.

Therefore for $x \in [\varepsilon, \pi - \varepsilon]$, $\varepsilon > 0$, from (1.14) dhe (1.15) we obtain

$$\left| \left(\frac{\widetilde{D}_m(x)}{2\sin x} \right)^{(r)} \right| = \mathcal{O}(1) m^{r+1} \sum_{i=0}^{r} \binom{r}{i} \frac{|P_{r-i}(\cos x)|}{\sin^{r-i+1} x} = \mathcal{O}_{r,\epsilon}\left(m^{r+1} \right).$$

The proof is completed.

Lemma 1.89. *If the set of real numbers $a_0, a_1, \ldots, a_n, \ldots$ satisfy conditions*

$$\lim_{n\to\infty} a_n = 0,$$

and

$$\sum_{n=1}^{\infty} n^k |\Delta^{k+1} a_n| < \infty,$$

then the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1.16}$$

will converge in the open interval $0 < x \leq \pi$, and will represent there an L-integrable function whose Fourier cosine development is given by (1.16). Further,

$$f(x) = \sum_{n=0}^{\infty} \Delta^{k+1} a_n S_n^k(x),$$

where $S_n^k(x)$ denotes the Cèsaro sum of order k of the series

$$\frac{1}{2} + \sum_{n=1}^{\infty} \cos nx. \tag{1.17}$$

Proof. First the Cesàro sum $\sigma_n^{(k)}(x)$ of order k of the series (1.17) is bounded in the mean in the interval $(0, \pi)$. Because of the conditions on $\{a_n\}$ and the boundedness of $S_n^{(k)}(x)$ for the series (1.17) in the interval $0 < \delta \leq x \leq \pi$, we may infer the convergence of the series (1.16) in that interval to a function

$$f(x) = \sum_{n=0}^{\infty} \Delta^{k+1} a_n S_n^{(k)}(x), \tag{1.18}$$

The series on the right hand side of (1.18) converges absolutely and uniformly in the interval $0 < \delta \leq x \leq \pi$ and can therefore be integrated term by term in that interval. Subsequently, we have

$$\int_{\delta}^{\pi} |f(x)| dx \leq \sum_{n=0}^{\infty} \left(\int_{\delta}^{\pi} \left| \frac{S_n^{(k)}(x)}{A_n^{(k)}} \right| dx \right) \frac{A_n^{(k)}}{n^k} n^k |\Delta^{k+1} a_n|. \tag{1.19}$$

As δ approaches to zero. the right hand side of (1.19) approaches to a definite limit in view of conditions of this lemma and the boundedness in the mean of $\sigma_n^{(k)}(x)$ in the interval $(0, \pi)$. Thus the left hand side does also, and the existence of the resulting integral shows that the function $f(x)$ is an L-integrable function whose Fourier cosine development is given by (1.16).

The proof is completed.

Lemma 1.90. *Let $S_n(x)$ and $T_n^{(k)}(x)$ be the n-th partial sum and Cesàro mean of order $k > 0$, respectively, of the infinite series*

$$\frac{1}{2} + \sum_{n=1}^{\infty} \cos nx.$$

Then

(i) $\int_0^\pi |S_n(x)| dx \sim \log n$,
(ii) $\int_0^\pi |T_n^{(k)}(x)| dx$ *remains bounded for all* n.

Proof. This Lemma intently is left without its proof. The interested reader can find it in [62].

Lemma 1.91. *Let r be a non-negative integer and $0 < \varepsilon < \pi$. Then there exists $M_{r\varepsilon} > 0$ such that for all $\varepsilon \le |x| \le \pi$ and all $n \ge 1$,*

(i) $|E_n^{(r)}(x)| \le \frac{M_{r\varepsilon} n^r}{|x|}$,
(ii) $|E_{-n}^{(r)}(x)| \le \frac{M_{r\varepsilon} n^r}{|x|}$,
(iii) $|D_n^{(r)}(x)| \le \frac{2M_{r\varepsilon} n^r}{|x|}$,
(iv) $|\widetilde{D}_n^{(r)}(x)| \le \frac{2M_{r\varepsilon} n^r}{|x|}$,

where

$$E_n(x) = \sum_{k=1}^{n} e^{ikx}, \quad E_{-n}(x) = \sum_{k=1}^{n} e^{-ikx}.$$

Proof. The case $r = 0$ is trivial. For $r \ge 1$, we have

$$-i^r E_n^{(r)}(x) = \sum_{k=1}^{n} k^r e^{-ikx} = \sum_{k=1}^{n} \Delta(k^r) E_k(x) + (n+1)^r E_n(x),$$

and so

$$|E_n^{(r)}(x)| \le \frac{M_{r\varepsilon}}{|x|} \left(\sum_{k=1}^{n} |\Delta(k^r)| + (n+1)^r \le \frac{M_{r\varepsilon} n^r}{|x|}, \right.$$

for some positive constant $M_{r\varepsilon}$.

Since

$$E_{-n}^{(r)}(x) = (-1)^r E_n^{(r)}(-x),$$

we also obtain

$$|E_{-n}^{(r)}(x)| \le \frac{M_{r\varepsilon} n^r}{|x|},$$

for some positive constant $M_{r\varepsilon}$.

Moreover, using the equalities

$$D_n^{(r)}(x) = E_n^{(r)}(x) + E_{-n}^{(r)}(x)$$

and

$$i\widetilde{D}_n^{(r)}(x) = E_n^{(r)}(x) - E_{-n}^{(r)}(x),$$

we obtain

$$|D_n^{(r)}(x)| \le \frac{M_{r\varepsilon} n^r}{|x|} + \frac{M_{r\varepsilon} n^r}{|x|} = \frac{2M_{r\varepsilon} n^r}{|x|}$$

and

$$|\widetilde{D}_n^{(r)}(x)| \le \frac{M_{r\varepsilon} n^r}{|x|} + \frac{M_{r\varepsilon} n^r}{|x|} = \frac{2M_{r\varepsilon} n^r}{|x|}.$$

The proof is completed.

Lemma 1.92. *Let r be a non-negative integer and $0 < \varepsilon < \pi$. Then there exists $M_{r\varepsilon} > 0$ such that for all $\varepsilon \le |x| \le \pi$ and all $n \ge 1$,*

(i) $|\overline{E}_n'(x)| \le \frac{M_{r\varepsilon} n^2}{|x|}$,
(ii) $|\overline{E}_{-n}'(x)| \le \frac{M_{r\varepsilon} n^2}{|x|}$,

where $\overline{E}_n(x) = \sum_{m=1}^n E_m(x)$.

Proof. (i) Under conditions of this Lemma and Lemma 1.91 we have

$$\begin{aligned} |\overline{E}_n'(x)| &\le \sum_{m=1}^n |E_m'(x)| \le \frac{M_{r\varepsilon}}{|x|} \sum_{m=1}^n m \\ &= \frac{M_{r\varepsilon}}{|x|} \cdot \frac{n(n+1)}{2} \le \frac{M_{r\varepsilon} n^2}{|x|}, \end{aligned}$$

for $0 < \varepsilon \le |x| \le \pi$.

(ii) Similarly we have obtained

$$|\overline{E}_{-n}'(x)| \le \sum_{m=1}^n |E_{-m}'(x)| \le \frac{M_{r\varepsilon}}{|x|} \sum_{m=1}^n m \le \frac{M_{r\varepsilon} n^2}{|x|},$$

for $0 < \varepsilon \le |x| \le \pi$.

The proof is completed.

Lemma 1.92 has been generalized to the next statement.

Lemma 1.93. *Let r be a non-negative integer and $0 < \varepsilon < \pi$. Then there exists $M_{r\varepsilon} > 0$ such that for all $\varepsilon \le |x| \le \pi$ and all $n \ge 1$,* *(i)* $|\overline{E}_n^{(r)}(x)| \le \frac{M_{r\varepsilon} n^{r+1}}{|x|}$,

(ii) $|\overline{E}_{-n}^{(r)}(x)| \le \frac{M_{r\varepsilon} n^{r+1}}{|x|}$,

where $\overline{E}_n(x) = \sum_{m=1}^{n} E_m(x)$.

Proof. (i) Under conditions of the Lemma and Lemma 1.91 we have

$$|\overline{E}_n^{(r)}(x)| \le \sum_{m=1}^{n} |E_m^{(r)}(x)| \le \frac{M_{r\varepsilon}}{|x|} \sum_{m=1}^{n} m^r$$
$$\le \frac{M_{r\varepsilon}}{|x|} \cdot \frac{n^{r-1} n(n+1)}{2} \le \frac{M_{r\varepsilon} n^{r+1}}{|x|},$$

for $0 < \varepsilon \le |x| \le \pi$.

(ii) Similarly we have obtained

$$|\overline{E}_{-n}^{(r)}(x)| \le \sum_{m=1}^{n} |E_{-m}^{(r)}(x)| \le \frac{M_{r\varepsilon}}{|x|} \sum_{m=1}^{n} m^r \le \frac{M_{r\varepsilon} n^{r+1}}{|x|},$$

for $0 < \varepsilon \le |x| \le \pi$.

The proof is completed.

Lemma 1.94. *The following statements hold true.*

(i) There exists a positive constants α *and* β *such that*

$$\alpha(\log n)| \le \|\widetilde{K}_n(x)\| \le \beta(\log n),$$

(ii) $|\widetilde{K}'_n(x)| = o(n)$, *where* $\widetilde{K}_n(x) = \sum_{m=1}^{n} \widetilde{D}_m(x)$.

Proof. (i) The existence of the constant β follows from the fact that $\widetilde{D}_n(x) = \mathcal{O}(\log n)$. Further, we have

$$\begin{aligned} 2\pi\|\widetilde{K}_n(x)\| &\ge \int_0^{\pi} \widetilde{K}_n(x)dx \\ &= \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) \int_0^{\pi} \sin kx dx \\ &= \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) \frac{1 - \cos k\pi}{k} \\ &= \frac{1}{n+1} \sum_{k=0}^{n} \left(\sum_{j=0}^{k} \frac{1 - \cos j\pi}{j} \right) \\ &\ge M \frac{\log(n!)}{n+1}, \end{aligned}$$

for some constant M and the last step being the implication of the relation $\sum_{v=1}^{n} \log v = \log(n!)$. Using Sterling's asymptotic formula $n! \sim \sqrt{2\pi n} \cdot n^n \cdot e^{-n}$, then we have

$$\|\widetilde{K}_n(x)\| \geq \alpha(\log n).$$

(ii) Firstly, we have

$$\widetilde{D}_n'(x) = \left| \sum_{k=0}^{n} k \cos kx \right| \leq \frac{n(n+1)}{2}$$

and so

$$\widetilde{K}_n'(x) \leq \frac{1}{n+1} \left| \sum_{k=0}^{n} \widetilde{D}_n'(x) \right| = o(n^2).$$

Subsequently,

$$\int_{|x| \leq \frac{\pi}{n}} |\widetilde{K}_n'(x)| dx = o(n).$$

Differentiating $\widetilde{K}_n(x)$ we get

$$\widetilde{K}_n'(x) := \Sigma_{1n}(x) - \Sigma_{2n}(x) + \Sigma_{3n}(x),$$

where

$$\Sigma_{1n}(x) = \frac{\cos x - \cos(n+1)x}{4 \sin^2 \frac{x}{2}}, \quad \Sigma_{2n}(x) = \frac{2 \sin^2 x}{\left(2 \sin \frac{x}{2}\right)^2},$$

and

$$\Sigma_{3n}(x) = \frac{2 \sin x \sin(n+1)x}{(n+1) \left(2 \sin \frac{x}{2}\right)^2}.$$

Clearly, $|\Sigma_{1n}(x)| = o(|x|^{-2})$ for $j = 1, 2$, and $(n+1)|\Sigma_{3n}(x)| = o(|x|^{-3})$. Using these estimates, we obtain

$$\int_{\frac{\pi}{n} \leq |x| \leq \pi} |\widetilde{K}_n'(x)| dx = o\left(\int_{\frac{\pi}{n} \leq |x| \leq \pi} \frac{dx}{x^2} \right) + o\left(\frac{1}{n+1} \int_{\frac{\pi}{n} \leq |x| \leq \pi} \frac{dx}{x^3} \right).$$

Combining the above estimates, we infer that $|\widetilde{K}_n'(x)| = o(n)$.

The proof is completed.

Lemma 1.95. *For each non-negative integer n, there holds*

$$\lim_{n \to \infty} \left\| c_n E_n^{(r)}(x) - c_{-n} E_{-n}^{(r)}(x) \right\| = 0$$

if and only if

$$\lim_{|n| \to \infty} n^r c_n \log |n| = 0,$$

where $\{c_n\}$ is a complex sequence.

Proof. Assuming $r \geq 1$ and denoting $J_n := \left\| c_n E_n^{(r)}(x) + c_{-n} E_{-n}^{(r)}(x) \right\|$, from Lemma 1.84 we have

$$\begin{aligned} J_n &= \int_0^\pi \left\{ \left| c_n E_n^{(r)}(x) + c_{-n} E_{-n}^{(r)}(x) \right| + \left| c_n E_{-n}^{(r)}(x) + c_{-n} E_n^{(r)}(x) \right| \right\} dx \\ &\geq |c_n + c_{-n}| \int_0^\pi \left| E_n^{(r)}(x) + E_{-n}^{(r)}(x) \right| dx \\ &= 2|c_n + c_{-n}| \int_0^\pi \left| D_n^{(r)}(x) \right| dx \geq \frac{4}{\pi} |c_n + c_{-n}| n^r \log n + \mathcal{O}(1). \end{aligned}$$

On the other hand, using

$$\begin{aligned} J_n &= \int_0^\pi \left| [c_n + c_{-n}] E_n^{(r)}(x) + c_{-n} [E_{-n}^{(r)}(x) - E_n^{(r)}(x)] \right| dx \\ &\leq |c_n + c_{-n}| \int_{-\pi}^\pi \left| E_n^{(r)}(x) \right| dx + |c_{-n}| \int_{-\pi}^\pi \left| E_{-n}^{(r)}(x) - E_n^{(r)}(x) \right| dx \end{aligned}$$

with Lemma 1.84 and Lemma 1.85 we have

$$J_n \leq \mathcal{O}\left(|c_n + c_{-n}| n^r \log n \right) + \mathcal{O}\left(|c_{-n}| n^r \log n \right) = \mathcal{O}\left(|c_n + c_{-n}| n^r \log n \right).$$

So, based on these estimates the results follow.
The proof is completed.

Lemma 1.96. *For $n \geq 1$, we have*

(i) $\left\| \frac{E_n(x)}{2 \sin x} \right\| = o(n),\ n \to \infty,$

(ii) $\left\| \frac{E_{-n}(x)}{2 \sin x} \right\| = o(n),\ n \to \infty,$

(iii) $\left\| \frac{e^{inx}}{2 \sin x} \right\| = o(\log n),\ n \to \infty.$

Proof. (i) For $x \neq 0$, $\sin x \geq \frac{2}{\pi} x$ for $x \in (0, \pi/2)$, and Lemma 1.93 we have

$$\begin{aligned} \left\| \frac{E_n(x)}{2 \sin x} \right\| &\leq \int_0^\pi \left| \frac{E_n(x)}{2 \sin x} \right| dx \leq \int_0^\pi \frac{M_\varepsilon}{2\,|\sin x|} dx \\ &\leq \int_0^{\frac{\pi}{2}} \frac{2M_\varepsilon}{2\,|x \sin x|} dx \leq \int_0^{\frac{\pi}{2}} \frac{M_\varepsilon dx}{x^2} = \lim_{n \to \infty} \left(-\frac{M_\varepsilon}{x} \right) \Bigg|_{\frac{\pi}{n}}^{\frac{\pi}{2}} = o(n), \end{aligned}$$

as $n \to \infty$.

(ii) In a similar way, we can prove that

$$\left\| \frac{E_{-n}(x)}{2 \sin x} \right\| = o(n), \quad n \to \infty.$$

(iii)

$$\begin{aligned}\left\|\frac{e^{inx}}{2\sin x}\right\| &\le \int_0^\pi \left|\frac{e^{inx}}{2\sin x}\right| dx \le \int_0^\pi \frac{1}{2\,|\sin x|}dx \\ &\le \int_0^{\frac{\pi}{2}} \frac{2}{2\,|x\sin x|}dx \le k\int_0^{\frac{\pi}{2}} \frac{dx}{x} = \lim_{n\to\infty} (\log x)\Big|_{\frac{\pi}{n}}^{\frac{\pi}{2}} = o(\log n),\end{aligned}$$

as $n \to \infty$.

The proof is completed.

2

L^1-convergence of modified sums $f_n(x)$

In this section we are going to present all collected results regrading to L^1-convergence of modified trigonometric sums $f_n(x)$ whose coefficients belong to several classes of real sequences.

2.1 L^1-convergence of modified trigonometric sums $f_n(x)$ with quasi-convex coefficients

We know that the trigonometric sums

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx$$

are called modified trigonometric cosine sums or simply modified cosine sums. Regarding to these sums we have next statement.

Theorem 2.1. *Let*

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

$\lim_{k\to\infty} a_k = 0$*, and* $\{a_k\}$ *a quasi-convex sequence. Then* $f_n(x)$ *converges to* $f(x)$ *in* L^1*-norm.*

Proof. Using Lemma 1.1 (transformation (1.1)) twice, we have

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \\ &= \lim_{n\to\infty}\left(\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx\right) \end{aligned}$$

$$= \lim_{n\to\infty} \left[\sum_{k=0}^{n-1} \Delta a_k D_k(x) + a_n D_n(x) \right]$$
$$= \lim_{n\to\infty} \left[\sum_{k=0}^{n-2} (k+1)\Delta^2 a_k F_k(x) + n\Delta a_{n-1} F_{n-1}(x) + a_n D_n(x) \right].$$

We know that $|D_n(x)| = \mathcal{O}\left(\frac{1}{x}\right)$ for $x \in (0,\pi]$, and by assumptions we obtain

$$\lim_{n\to\infty} a_n D_n(x) = 0.$$

Moreover, using the estimate

$$|K_n(x)| = \mathcal{O}\left(\frac{1}{nx^2}\right), \quad x \in (0,\pi],$$

we also have

$$\lim_{n\to\infty} n\Delta a_{n-1} F_{n-1}(x) = 0.$$

Whence,

$$f(x) = \sum_{k=0}^{\infty} (k+1)\Delta^2 a_k F_k(x), \tag{2.1}$$

where $F_k(x) := \frac{1}{k+1}\sum_{j=0}^{k} D_j(x)$.

On the other hand, using Lemma 1.1 we have

$$\begin{aligned} f_n(x) &= \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{2} - a_{n+1}\sum_{k=1}^{n} \cos kx \\ &= \sum_{k=0}^{n-1} \Delta a_k D_k(x) + a_n D_n(x) - a_{n+1} D_n(x) \\ &= \sum_{k=0}^{n} \Delta a_k D_k(x) = \sum_{k=0}^{n-1} (k+1)\Delta^2 a_k F_k(x) + (n+1)\Delta a_n F_n(x). \end{aligned} \tag{2.2}$$

So, (2.1) and (2.2) imply

$$\begin{aligned} \int_0^{\pi} |f(x) - f_n(x)| dx &= \int_0^{\pi} \left| \sum_{k=n}^{\infty} (k+1)\Delta^2 a_k F_k(x) - (n+1)\Delta a_n F_n(x) \right| dx \\ &\le \sum_{k=n}^{\infty} (k+1)|\Delta^2 a_k| \int_0^{\pi} |F_k(x)|\, dx \\ &\quad + (n+1)|\Delta a_n| \int_0^{\pi} |F_n(x)|\, dx \end{aligned}$$

$$= \frac{\pi}{2}\sum_{k=n}^{\infty}(k+1)|\Delta^2 a_k| + \frac{\pi}{2}(n+1)|\Delta a_n|.$$

Since

$$\lim_{n\to\infty}|\Delta a_n| = 0,$$

then we have

$$\begin{aligned}(n+1)|\Delta a_n| &= (n+1)\left|\sum_{k=n}^{\infty}\Delta^2 a_k\right| \\ &\leq (n+1)\sum_{k=n}^{\infty}|\Delta^2 a_k| \\ &\leq \sum_{k=n}^{\infty}(k+1)|\Delta^2 a_k|.\end{aligned}$$

Subsequently,

$$\int_0^{\pi}|f(x)-f_n(x)|dx \leq \pi\sum_{k=n}^{\infty}(k+1)|\Delta^2 a_k| = o(1) \text{ as } n\to\infty.$$

The proof is completed.

Corollary 2.2. *Let*

$$\lim_{n\to\infty}S_n(x) = \lim_{n\to\infty}\left(\frac{a_0}{2}+\sum_{k=1}^{n}a_k\cos kx\right) = f(x),$$

$\lim_{k\to\infty}a_k = 0$, *and* $\{a_k\}$ *a quasi-convex sequence. Then* $S_n(x)$ *converges to* $f(x)$ *in* L^1*-norm if and only if* $|a_{n+1}|\log n = o(1)$ *as* $n\to\infty$.

Proof. Using Theorem 2.1 and some parts of its proof we have

$$\begin{aligned}\int_0^{\pi}|f(x)-S_n(x)|dx &\leq \int_0^{\pi}|f(x)-f_n(x)|dx + \int_0^{\pi}|f_n(x)-S_n(x)|dx \\ &= o(1) + |a_{n+1}|\int_0^{\pi}|D_n(x)|dx \\ &= o(1) + \mathcal{O}(|a_{n+1}|\log n).\end{aligned}$$

Also, we have

$$\begin{aligned}\mathcal{O}(|a_{n+1}|\log n) &= \int_0^{\pi}|a_{n+1}D_n(x)|dx \\ &= \int_0^{\pi}|f_n(x)-S_n(x)|dx \\ &\leq \int_0^{\pi}|f_n(x)-f(x)|dx + \int_0^{\pi}|f(x)-S_n(x)|dx \\ &= o(1) + \int_0^{\pi}|f(x)-S_n(x)|dx.\end{aligned}$$

So,

$$\|f - S_n\|_L = o(1) \ as \ n \to \infty$$

if and only if

$$|a_{n+1}| \log n = o(1) \ as \ n \to \infty.$$

The proof is completed.

2.2 L^1-convergence of modified trigonometric sums $f_n(x)$ with coefficients from the class C

We consider the cosine series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

for which

$$f(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \left(\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx \right),$$

$$\lim_{k\to\infty} a_k = 0,$$

and

$$\sum_{k=1}^{\infty} |\Delta a_k| < \infty,$$

i.e. the sequence $\{a_k\}$ is a zero-sequence of bounded variation.

First we prove the following lemma.

Lemma 2.3. *Let*

$$f_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx.$$

Then

$$\lim_{n\to\infty} f_n(x) = f(x), \quad \forall x \in (0, \pi].$$

Proof. Since $|D_n(x)| = \mathcal{O}\left(\frac{1}{x}\right)$ for $x \in (0, \pi]$, $\lim_{n\to\infty} a_n D_n(x) = 0$, and

$$\lim_{n\to\infty} S_n(x) = f(x),$$

we have

$$\begin{aligned}
\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty}\left[\frac{1}{2}\sum_{k=0}^{n}\Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta a_j \cos kx\right]\\
&= \lim_{n\to\infty}\left[\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{2} - a_{n+1}D_n(x)\right]\\
&= \lim_{n\to\infty}[S_n(x) - a_{n+1}D_n(x)]\\
&= f(x) - 0 = f(x).
\end{aligned}$$

The proof is completed.

Theorem 2.4. *The sequence $\{f_n(x)\}$ converges to $f(x)$ in the L^1-metric if and only if given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that*

$$\int_0^{\delta}\left|\sum_{k=n+1}^{\infty}\Delta a_k D_k(x)\right| dx < \varepsilon, \quad \forall n \geq 0.$$

Proof. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\int_0^{\delta}\left|\sum_{k=n+1}^{\infty}\Delta a_k D_k(x)\right| dx < \frac{\varepsilon}{2}, \quad \forall n \geq 0.$$

Then by Lemma 2.3 we have

$$\begin{aligned}
\int_0^{\pi}|f(x) - f_n(x)|dx &= \int_0^{\pi}\left|\sum_{k=n+1}^{\infty}\Delta a_k D_k(x)\right| dx\\
&= \int_0^{\delta}\left|\sum_{k=n+1}^{\infty}\Delta a_k D_k(x)\right| dx + \int_{\delta}^{\pi}\left|\sum_{k=n+1}^{\infty}\Delta a_k D_k(x)\right| dx\\
&< \frac{\varepsilon}{2} + \sum_{k=n+1}^{\infty}|\Delta a_k|\int_{\delta}^{\pi}|D_k(x)|\, dx\\
&\leq \frac{\varepsilon}{2} + \sum_{k=n+1}^{\infty}|\Delta a_k|\int_{\delta}^{\pi}\csc\frac{x}{2}dx\\
&= \frac{\varepsilon}{2} + \sum_{k=n+1}^{\infty}|\Delta a_k|\left[-2\log\left|\tan\frac{\delta}{2}\right|\right] = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

since for $n \to \infty$,

$$\sum_{k=1}^{\infty}|\Delta a_k| < \infty \Longrightarrow \sum_{k=n+1}^{\infty}|\Delta a_k| = o(1).$$

In contrary, let $\varepsilon > 0$. Then there exists an integer M such that

$$\int_0^{\pi} |f(x) - f_n(x)| dx < \frac{\varepsilon}{2}, \quad n \geq M.$$

That is,

$$\int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx < \frac{\varepsilon}{2}, \quad n \geq M.$$

Now if $\sum_{k=0}^{M} |\Delta a_k| = 0$, then for $n > M$,

$$\int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx < \frac{\varepsilon}{2} < \varepsilon,$$

and for $0 \leq n \leq M$,

$$\int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx = \int_0^{\pi} \left| \sum_{k=M+1}^{\infty} \Delta a_k D_k(x) \right| dx < \frac{\varepsilon}{2} < \varepsilon.$$

If $\sum_{k=0}^{M} |\Delta a_k| \neq 0$, let

$$\delta = \frac{\varepsilon}{2} \sum_{k=0}^{M} |\Delta a_k|.$$

For $n \geq M$,

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx \leq \int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx < \frac{\varepsilon}{2} < \varepsilon.$$

For $0 \leq n < M$,

$$\begin{aligned}
\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx &\leq \int_0^{\delta} \left| \sum_{k=n}^{M-1} \Delta a_k D_k(x) \right| dx + \int_0^{\delta} \left| \sum_{k=M}^{\infty} \Delta a_k D_k(x) \right| dx \\
&\leq \int_0^{\delta} \sum_{k=n}^{M-1} k |\Delta a_k| dx + \int_0^{\pi} \left| \sum_{k=M}^{\infty} \Delta a_k D_k(x) \right| dx \\
&< \delta \sum_{k=n}^{M-1} k |\Delta a_k| dx + \frac{\varepsilon}{2} \\
&\leq \delta M \sum_{k=n}^{M-1} |\Delta a_k| dx + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

So given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx < \varepsilon$$

for all $n \geq 0$.

If

$$\lim_{n\to\infty} \int_0^\pi |f(x) - f_n(x)|dx = 0,$$

then

$$\int_0^\pi |f(x)|dx \leq \int_0^\pi |f(x) - f_n(x)|dx + \int_0^\pi |f_n(x)|dx < \infty,$$

that is $f \in L^1[0, \pi]$, since $f_n(x)$ is a trigonometric polynomial.

The proof is completed.

Corollary 2.5. *If for $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that*

$$\int_0^\delta \left| \sum_{k=n}^\infty \Delta a_k D_k(x) \right| dx < \varepsilon, \quad \forall n \geq 0,$$

then $\{S_n\}$ converges to f in the L^1 metric if and only if

$$\lim_{n\to\infty} a_n \log n = 0.$$

Proof. Using f_n as in the Lemma 2.3, we get

$$\begin{aligned}\int_0^\pi |f(x) - S_n(x)|dx &\leq \int_0^\pi |f(x) - f_n(x) + f_n(x) - S_n(x)|dx \\ &\leq \int_0^\pi |f(x) - f_n(x)|dx + \int_0^\pi |f_n(x) - S_n(x)|dx \\ &= \int_0^\pi |f(x) - f_n(x)|dx + \int_0^\pi |a_{n+1} D_n(x)|dx.\end{aligned}$$

Also,

$$\begin{aligned}\int_0^\pi |a_{n+1} D_n(x)|dx &= \int_0^\pi |f_n(x) - S_n(x)|dx \\ &\leq \int_0^\pi |f_n(x) - f(x)|dx + \int_0^\pi |f(x) - S_n(x)|dx.\end{aligned}$$

Since

$$\int_0^\pi |a_{n+1} D_n(x)|dx$$

behaves like $a_{n+1} \log n$ for large values of n, and

$$\lim_{n\to\infty} \int_0^\pi |f(x) - S_n(x)|dx = 0,$$

the corollary is proved.

2.3 L^1-convergence of modified trigonometric sums $f_n(x)$ with coefficients of generalized bounded variation

The following theorem regarding to L^1-convergence of modified trigonometric sums $f_n(x)$ holds true.

Theorem 2.6. *Let $k > 0$ be a real number. If*

$$\lim_{n\to\infty} a_n = 0, \tag{2.3}$$

and

$$\sum_{n=1}^{\infty} n^k |\Delta^{k+1} a_n| < \infty, \tag{2.4}$$

then $f_n(x)$ converges to $f(x)$ in the L^1-metric.

Proof. First using Lemma 1.1 we have

$$f_n(x) = \frac{1}{2}\sum_{i=0}^{n} \Delta a_k + \sum_{i=1}^{n}\sum_{j=i}^{n} \Delta a_j \cos ix = \sum_{i=0}^{n} \Delta a_i S_i(x).$$

Part 1. Let k be integral. Applying Abel's transformation of order k to $f_n(x)$ we get

$$f_n(x) = \sum_{i=0}^{n-k} \Delta^{k+1} a_i S_i^k(x) + \sum_{i=1}^{k} \Delta^i a_{n-i+1} S_{n-i+1}^i(x). \tag{2.5}$$

Now by Lemma 1.88,

$$f(x) = \sum_{n=0}^{\infty} \Delta^{k+1} a_n S_n^k(x). \tag{2.6}$$

So by (4.16) and (2.6),

$$\begin{aligned}
&\int_0^\pi |f(x) - f_n(x)| dx \\
&= \int_0^\pi \left| \sum_{i=n-k+1}^{\infty} \Delta^{k+1} a_i S_i^k(x) - \sum_{i=1}^{k} \Delta^i a_{n-i+1} S_{n-i+1}^i(x) \right| dx \\
&\le \int_0^\pi \left| \sum_{i=n-k+1}^{\infty} \Delta^{k+1} a_i S_i^k(x) \right| dx + \int_0^\pi \left| \sum_{i=1}^{k} \Delta^i a_{n-i+1} S_{n-i+1}^i(x) \right| dx \\
&\le \sum_{i=n-k+1}^{\infty} |\Delta^{k+1} a_i| \int_0^\pi \left| S_i^k(x) \right| dx + \sum_{i=1}^{k} |\Delta^i a_{n-i+1}| \int_0^\pi \left| S_{n-i+1}^i(x) \right| dx
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=n-k+1}^{\infty} A_i^k |\Delta^{k+1} a_i| \int_0^{\pi} \left| T_i^{(k)}(x) \right| dx \\
&\quad + \sum_{i=1}^{k} A_{n-i+1}^i |\Delta^i a_{n-i+1}| \int_0^{\pi} \left| T_{n-i+1}^{(i)}(x) \right| dx \\
&\leq C \sum_{i=n-k+1}^{\infty} A_i^k |\Delta^{k+1} a_i| + C \sum_{i=1}^{k} A_{n-i+1}^i |\Delta^i a_{n-i+1}| \\
&= o(1) + o(1) = o(1),
\end{aligned}
$$

by Lemmas 1.48, 1.90, and the assumptions of the theorem, where C is a positive real number.

Whence, we have

$$
\lim_{n \to \infty} \int_0^{\pi} |f(x) - f_n(x)| dx = 0. \tag{2.7}
$$

Part II. Let k be non-integral. Let $k = r + \delta$, r is the integral part of k, and δ is its fractional part i.e. $0 < \delta < 1$.

Case (i). Let $r = 0$. Applying Abel transformation of order $-\delta$ we have

$$
\sum_{i=0}^{n} \Delta^{\delta+1} a_i S_i^{\delta}(x) = \sum_{i=0}^{n} \sum_{m=0}^{n-i} \Delta^{\delta-1} a_{i+m} S_i(x)
$$

by (1.7).

Again by the result of Lemma 1.49 this formula can be transformed into

$$
\sum_{i=0}^{n} \Delta^{\delta+1} a_i S_i^{\delta}(x) = \sum_{i=0}^{n} \Delta a_i S_i(x) - R_n(x),
$$

where

$$
R_n(x) = \sum_{i=0}^{n} S_i(x) \left(A_{n-i+1}^{\delta-1} \Delta^{\delta+1} a_{n+1} + A_{n-i+2}^{\delta-1} \Delta^{\delta+1} a_{n+2} + \cdots \right).
$$

This implies that

$$
\sum_{i=0}^{n} \Delta a_i S_i(x) = \sum_{i=0}^{n} \Delta^{\delta+1} a_i S_i^{\delta}(x) + R_n(x),
$$

and thus,

$$
f_n(x) = \sum_{i=0}^{n} \Delta^{\delta+1} a_i S_i^{\delta}(x) + R_n(x).
$$

When $r = 0$, it clear that $k = \delta$ and

$$f(x) = \sum_{i=0}^{n} \Delta^{\delta+1} a_i S_i^{\delta}(x).$$

Whence, by Lemmas 1.48 and 1.90 we have

$$\begin{aligned}
\int_0^{\pi} &|f(x) - f_n(x)| dx \\
&= \int_0^{\pi} \left| \sum_{i=n+1}^{\infty} \Delta^{\delta+1} a_i S_i^{\delta}(x) - R_n(x) \right| dx \\
&\leq \sum_{i=n+1}^{\infty} |\Delta^{\delta+1} a_i| \int_0^{\pi} \left| S_i^{\delta}(x) \right| dx + \int_0^{\pi} |R_n(x)|\, dx \\
&= \sum_{i=n+1}^{\infty} A_i^{\delta} |\Delta^{\delta+1} a_i| \int_0^{\pi} \left| T_i^{\delta}(x) \right| dx + \int_0^{\pi} |R_n(x)|\, dx \\
&\leq C \sum_{i=n+1}^{\infty} A_i^{\delta} |\Delta^{\delta+1} a_i| + \int_0^{\pi} |R_n(x)|\, dx \\
&\leq o(1) + \int_0^{\pi} |R_n(x)|\, dx. \qquad (2.8)
\end{aligned}$$

Now we estimate $\int_0^{\pi} |R_n(x)|\, dx$. Namely, by Lemmas 1.48, 1.50, and 1.90 we obtain

$$\begin{aligned}
\int_0^{\pi} |R_n(x)|\, dx = \int_0^{\pi} \Bigg| &\left(\sum_{i=0}^{n} A_{n-i+1}^{\delta-1} S_i(x) \right) \Delta^{\delta+1} a_{n+1} \\
&+ \left(\sum_{i=0}^{n} A_{n-i+2}^{\delta-1} S_i(x) \right) \Delta^{\delta+1} a_{n+2} + \cdots \Bigg| dx \\
&\leq |\Delta^{\delta+1} a_{n+1}| \int_0^{\pi} \left| \left(\sum_{i=0}^{n} A_{n-i+1}^{\delta-1} S_i(x) \right) \right| dx \\
&\quad + |\Delta^{\delta+1} a_{n+2}| \int_0^{\pi} \left| \left(\sum_{i=0}^{n} A_{n-i+2}^{\delta-1} S_i(x) \right) \right| dx + \cdots \\
&\leq |\Delta^{\delta+1} a_{n+1}| \int_0^{\pi} \max_{0 \leq p \leq n+1} \left| S_p^{\delta}(x) \right| dx \\
&\quad + |\Delta^{\delta+1} a_{n+2}| \int_0^{\pi} \max_{0 \leq p \leq n+2} \left| S_p^{\delta}(x) \right| dx + \cdots \\
&= |\Delta^{\delta+1} a_{n+1} A_{n+1}^{\delta}| \int_0^{\pi} \max_{0 \leq p \leq n+1} \left| T_p^{\delta}(x) \right| dx \\
&\quad + |\Delta^{\delta+1} a_{n+2}| A_{n+2}^{\delta} \int_0^{\pi} \max_{0 \leq p \leq n+2} \left| T_p^{\delta}(x) \right| dx + \cdots \\
&\leq C \left[|\Delta^{\delta+1} a_{n+1} A_{n+1}^{\delta}| + |\Delta^{\delta+1} a_{n+2}| A_{n+2}^{\delta} + \cdots \right]
\end{aligned}$$

$$= C\,[o(1) + o(1) + \cdots] = o(1).$$

Consequently, by (2.8) we have

$$\int_0^{\pi} |f(x) - f_n(x)|dx = o(1), \ \ as\ n \to \infty. \tag{2.9}$$

Case (ii). Let $r \geq 1$. Applying Abel transformation of order r we have

$$\begin{aligned} f_n(x) &= \sum_{i=0}^{n} \Delta a_i S_i(x) \\ &= \sum_{i=0}^{n-r} \Delta^{r+1} a_i S_i^r(x) + \sum_{i=1}^{r} \Delta^i a_{n-i+1} S_{n-i+1}^i(x). \end{aligned} \tag{2.10}$$

Again, applying Abel transformation of order $-\delta$ we have

$$\sum_{i=0}^{n} \Delta^{k+1} a_i S_i^k(x) = \sum_{i=0}^{n} \sum_{p=0}^{n-i} A_p^{\delta-1} \Delta^{k+1} a_{i+p} S_i^r(x).$$

By the result of Lemma 1.49 this formula can be transformed into

$$\sum_{i=0}^{n} \Delta^{k+1} a_i S_i^k(x) = \sum_{i=0}^{n} \Delta^{r+1} a_i S_i^r(x) - R_n(x), \tag{2.11}$$

where

$$\begin{aligned} R_n(x) &= \sum_{i=0}^{n} S_i^r(x) \left(A_{n-i+1}^{\delta-1} \Delta^{\delta+1} a_{n+1} + A_{n-i+2}^{\delta-1} \Delta^{\delta+1} a_{n+2} + \cdots\right) \\ &= \left(\sum_{i=0}^{n} S_i^r(x) A_{n-i+1}^{\delta-1}\right) \Delta^{\delta+1} a_{n+1} \\ &\quad + \left(\sum_{i=0}^{n} S_i^r(x) A_{n-i+2}^{\delta-1}\right) \Delta^{\delta+1} a_{n+2} + \cdots. \end{aligned}$$

Replacing n by $n-r$ in (2.11) we have

$$\sum_{i=0}^{n-r} \Delta^{k+1} a_i S_i^k(x) = \sum_{i=0}^{n-r} \Delta^{r+1} a_i S_i^r(x) - R_{n-r}(x). \tag{2.12}$$

Now by (2.10) and (2.12) we get

$$f_n(x) = \sum_{i=0}^{n-r} \Delta^{k+1} a_i S_i^k(x) + R_{n-r}(x) + \sum_{i=0}^{r} \Delta^i a_{n-i+1} S_{n-i+1}^i(x). \tag{2.13}$$

Therefore by Lemmas 1.48, 1.49, 1.88, and the assumptions of the theorem, we obtain

$$\begin{aligned}
&\int_0^\pi |f(x) - f_n(x)| dx \\
&= \int_0^\pi \left| \sum_{i=n-k+1}^{\infty} \Delta^{k+1} a_i S_i^k(x) - R_{n-r}(x) - \sum_{i=1}^{r} \Delta^i a_{n-i+1} S_{n-i+1}^r(x) \right| dx \\
&\leq \int_0^\pi \left| \sum_{i=n-k+1}^{\infty} \Delta^{k+1} a_i S_i^k(x) \right| dx \\
&\quad + \int_0^\pi |R_{n-r}(x)|\, dx + \int_0^\pi \left| \sum_{i=1}^{r} \Delta^i a_{n-i+1} S_{n-i+1}^r(x) \right| dx \\
&= \sum_{i=n-k+1}^{\infty} A_i^k |\Delta^{k+1} a_i| \int_0^\pi \left| T_i^k(x) \right| dx \\
&\quad + \int_0^\pi |R_{n-r}(x)|\, dx + \sum_{i=1}^{r} A_{n-i+1}^i |\Delta^i a_{n-i+1}| \int_0^\pi \left| T_{n-i+1}^i(x) \right| dx \\
&\leq C \sum_{i=n-k+1}^{\infty} A_i^k |\Delta^{k+1} a_i| + C \sum_{i=1}^{r} A_{n-i+1}^i |\Delta^i a_{n-i+1}| + \int_0^\pi |R_{n-r}(x)|\, dx \\
&= o(1) + \int_0^\pi |R_{n-r}(x)|\, dx. \qquad (2.14)
\end{aligned}$$

By Lemma 1.49 we estimate $\int_0^\pi |R_{n-r}(x)|\, dx$. Indeed, we have

$$\begin{aligned}
\int_0^\pi |R_{n-r}(x)|\, dx &= \int_0^\pi \Bigg| \left(\sum_{i=0}^{n-r} A_{n-r-i+1}^{\delta-1} S_i^r(x) \right) \Delta^{k+1} a_{n-r+1} \\
&\quad + \left(\sum_{i=0}^{n-r} A_{n-r-i+2}^{\delta-1} S_i^r(x) \right) \Delta^{k+1} a_{n-r+2} + \cdots \Bigg| dx \\
&\leq |\Delta^{k+1} a_{n-r+1}| \int_0^\pi \left| \left(\sum_{i=0}^{n-r} A_{n-r-i+1}^{\delta-1} S_i^r(x) \right) \right| dx \\
&\quad + |\Delta^{k+1} a_{n-r+2}| \int_0^\pi \left| \left(\sum_{i=0}^{n-r} A_{n-r-i+2}^{\delta-1} S_i^r(x) \right) \right| dx + \cdots \\
&\leq |\Delta^{k+1} a_{n-r+1}| \sum_{i=0}^{n-r} A_{n-r-i+1}^{\delta-1} A_i^r \int_0^\pi |T_i^r(x)|\, dx \\
&\quad + |\Delta^{k+1} a_{n-r+2}| \sum_{i=0}^{n-r} A_{n-r-i+2}^{\delta-1} A_i^r \int_0^\pi |T_i^r(x)|\, dx + \cdots
\end{aligned}$$

$$\begin{aligned}
&\leq C|\Delta^{k+1}a_{n-r+1}|\sum_{i=0}^{n-r}A_{n-r-i+1}^{\delta-1}A_i^r \\
&\quad +C|\Delta^{k+1}a_{n-r+2}|\sum_{i=0}^{n-r}A_{n-r-i+2}^{\delta-1}A_i^r+\cdots \\
&\leq C|\Delta^{k+1}a_{n-r+1}|\sum_{i=0}^{n+1-r}A_{n-r-i+1}^{\delta-1}A_i^r \\
&\quad +C|\Delta^{k+1}a_{n-r+2}|\sum_{i=0}^{n+2-r}A_{n-r-i+2}^{\delta-1}A_i^r+\cdots \\
&=C|\Delta^{k+1}a_{n-r+1}|A_{n-r+1}^{r+\delta}+C|\Delta^{k+1}a_{n-r+2}|A_{n-r+2}^{r+\delta}+\cdots \\
&=C|\Delta^{k+1}a_{n-r+1}|A_{n-r+1}^{k}+C|\Delta^{k+1}a_{n-r+2}|A_{n-r+2}^{k}+\cdots \\
&=o(1)+o(1)+\cdots=o(1).
\end{aligned}$$

Whence,

$$\int_0^\pi |R_{n-r}(x)|\,dx=o(1)\quad as\quad n\to\infty.$$

Using (2.14) this implies that

$$\int_0^\pi |f(x)-f_n(x)|dx=o(1)\quad as\quad n\to\infty. \tag{2.15}$$

Subsequently by (2.9) and (2.15)

$$\int_0^\pi |f(x)-f_n(x)|dx=o(1)\quad as\quad n\to\infty, \tag{2.16}$$

where k is non-integral number.

Whence, in view of (2.7) and (2.16), we have

$$\lim_{n\to\infty}\int_0^\pi |f(x)-f_n(x)|dx=0,$$

for any $k>0$, which implies that

$$f_n(x)\to f(x)$$

in the L^1 metric.

The proof is completed.

2.4 L^1-convergence of modified trigonometric sums $f_n(x)$ with coefficients from the class S

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

and

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx.$$

Theorem 2.7. *Let* $\{a_k\} \in \mathbf{S}$, *then* $f_n(x)$ *converges to* $f(x)$ *in* L^1*-norm.*

Proof. Abel's transformation implies

$$\begin{aligned} f(x) &= \lim_{n\to\infty}\left(\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx\right) \\ &= \lim_{n\to\infty}\left[\sum_{k=1}^{n-1} \Delta a_k D_k(x) + a_n D_n(x)\right] \\ &= \lim_{n\to\infty}\left[\sum_{k=0}^{n-1} \Delta a_k D_k(x) + a_n D_n(x)\right] \\ &= \sum_{k=0}^{\infty} \Delta a_k D_k(x), \end{aligned}$$

since

$$\lim_{n\to\infty} a_n D_n(x) = 0$$

if $x \neq 0$, where

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx.$$

Also, the use of Abel's transformation yields

$$\begin{aligned} f_n(x) &= \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx \\ &= \sum_{k=0}^{n} \Delta a_k D_k(x). \end{aligned}$$

Now, using Lemma 1.34 we have

$$\begin{aligned}
\int_0^\pi |f(x) - f_n(x)|dx &= \int_0^\pi \left| \sum_{k=n+1}^\infty \Delta a_k D_k(x) \right| dx \\
&= \int_0^\pi \left| \sum_{k=n+1}^\infty A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx \\
&= \int_0^\pi \left| \sum_{k=n+1}^\infty \Delta A_k \sum_{\mu=1}^k \frac{\Delta a_\mu}{A_\mu} D_\mu(x) \right| dx \\
&\le \sum_{k=n+1}^\infty \Delta A_k \int_0^\pi \left| \sum_{\mu=1}^k \frac{\Delta a_\mu}{A_\mu} D_\mu(x) \right| dx \\
&\le C \sum_{k=n+1}^\infty (k+1)\Delta A_k \\
&= C(n+1)A_{n+1} + C \sum_{k=n+1}^\infty A_k = o(1) \quad as \quad n \to \infty.
\end{aligned}$$

taking into account that $\{a_k\} \in \mathbf{S}$.

So, we have obtained

$$\lim_{n\to\infty} \int_0^\pi |f(x) - f_n(x)|dx = 0.$$

The proof is completed.

Corollary 2.8. *Let* $\{a_k\} \in \mathbf{S}$. *The series*

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^\infty a_k \cos kx$$

converges in L^1*-norm if and only if*

$$\lim_{n\to\infty} a_n \log n = 0.$$

Proof. We notice that

$$\begin{aligned}
\int_0^\pi |f(x) - S_n(x)|dx &\le \int_0^\pi |f(x) - f_n(x)|dx + \int_0^\pi |f_n(x) - S_n(x)|dx \\
&= \int_0^\pi |f(x) - f_n(x)|dx + |a_{n+1}| \int_0^\pi |D_n(x)|dx,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\pi |a_{n+1} D_n(x)|dx &= \int_0^\pi |f_n(x) - S_n(x)|dx \\
&\le \int_0^\pi |f_n(x) - f(x)|dx + \int_0^\pi |f(x) - S_n(x)|dx.
\end{aligned}$$

So,

$$\|f - S_n\|_{L^1} = o(1) \ as \ n \to \infty$$

if and only if

$$|a_{n+1}| \log n = o(1) \ as \ n \to \infty,$$

since $\int_0^\pi |a_{n+1} D_n(x)| dx$ behaves as $|a_{n+1}| \log n$ for large values n.

The proof is completed.

2.5 L^1-convergence of modified trigonometric sums $f_n(x)$ with coefficients from the class S′

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

and

$$f_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx.$$

Theorem 2.9. *Let $\{a_k\} \in \mathbf{S}'$, then $f_n(x)$ converges to $f(x)$ in L^1-norm.*

Proof. By Abel's transformation we have

$$\begin{aligned} f(x) &= \lim_{n\to\infty} \left(\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx \right) \\ &= \lim_{n\to\infty} \left(\frac{a_0}{2} + \sum_{k=1}^{n-1} \Delta a_k D_k(x) + a_n D_n(x) - \frac{a_0}{2} \right) \\ &= \lim_{n\to\infty} \left(\sum_{k=0}^{n-1} \Delta a_k D_k(x) + a_n D_n(x) \right) \\ &= \sum_{k=0}^{\infty} \Delta a_k D_k(x), \end{aligned}$$

since

$$\lim_{n\to\infty} a_n D_n(x) = 0$$

if $x \neq 0$, where

$$D_n(x) = \frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx.$$

The use of Abel's transformation yields

$$
\begin{aligned}
f_n(x) &= \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx \\
&= \sum_{k=0}^{n} \Delta a_k D_k(x).
\end{aligned}
$$

Now, since $|\Delta a_k / A_k| \le 1$ by assumption, then applying Lemma 1.34 we have

$$
\begin{aligned}
&\int_0^{\pi} |f(x) - f_n(x)| dx = \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx \\
&= \lim_{N\to\infty} \int_0^{\pi} \left| \sum_{k=n+1}^{N} \Delta a_k D_k(x) \right| dx \\
&= \lim_{N\to\infty} \int_0^{\pi} \left| \sum_{k=n+1}^{N} \frac{\Delta a_k}{A_k} A_k D_k(x) \right| dx \\
&= \lim_{N\to\infty} \int_0^{\pi} \left| \sum_{k=n+1}^{N-1} \Delta A_k \sum_{\mu=1}^{k} \frac{\Delta a_\mu}{A_\mu} D_\mu(x) \right. \\
&\qquad \left. + A_N \sum_{\mu=1}^{N} \frac{\Delta a_k}{A_k} D_k(x) - A_{n+1} \sum_{k=1}^{n} \frac{\Delta a_k}{A_k} D_k(x) \right| dx \\
&\le \lim_{N\to\infty} \left(\sum_{k=n+1}^{N-1} |\Delta A_k| \int_0^{\pi} \left| \sum_{\mu=1}^{k} \frac{\Delta a_\mu}{A_\mu} D_\mu(x) \right| dx \right. \\
&\qquad \left. + |A_N| \int_0^{\pi} \left| \sum_{\mu=1}^{N} \frac{\Delta a_k}{A_k} D_k(x) \right| dx + |A_{n+1}| \int_0^{\pi} \left| \sum_{k=1}^{n} \frac{\Delta a_k}{A_k} D_k(x) \right| dx \right) \\
&\le C \lim_{N\to\infty} \left(\sum_{k=n+1}^{N-1} (k+1)|\Delta A_k| + (N+1)|A_N| + (n+1)|A_{n+1}| \right) \\
&\le C \left(\sum_{k=n+1}^{\infty} (k+1)|\Delta A_k| + (N+1)A_N + (n+1)|A_{n+1}| \right). \qquad (2.17)
\end{aligned}
$$

Taking into account Lemmas 1.17 and 1.18 we have

$$
(N+1)|A_N \to 0 \quad as \quad N \to \infty,
$$

and

$$
\sum_{k=n+1}^{\infty} (k+1)|\Delta A_k| = o(1) \quad as \quad n \to \infty.
$$

Finally, based on (2.17) we obtain

$$\lim_{n\to\infty}\int_0^\pi |f(x)-f_n(x)|dx = 0.$$

The proof is completed.

2.6 L^1-convergence of modified trigonometric sums $f_n(x)$ with coefficients from the class K

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

and

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx.$$

Theorem 2.10. *Let the sequence $\{a_n\}$ belong to the class* $\mathbf{K}$, *then $f_n(x)$ converges to $f(x)$ in the L^1-norm.*

Proof. We have

$$\begin{aligned}
f_n(x) &= \frac{a_0}{2} + \sum_{m=1}^{n} a_m \cos mx - a_{n+1}D_n(x)\\
&= \frac{1}{2\sin x}\sum_{m=1}^{n} 2a_m \sin x \cos mx - a_{n+1}D_n(x), \qquad (a_0 = 0)\\
&= \frac{1}{2\sin x}\sum_{m=1}^{n} a_m\big[\sin(m+1)x - \sin(m-1)x\big] - a_{n+1}D_n(x)\\
&= \frac{1}{2\sin x}\sum_{m=1}^{n} (a_{m-1}-a_{m+1})\sin mx\\
&\quad + a_n\frac{\sin(n+1)x}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x} - a_{n+1}D_n(x)\\
&= \frac{1}{2\sin x}\sum_{m=1}^{n} (a_{m-1}-a_{m+1})\sin mx\\
&\quad + (a_n - a_{n+1})\,\frac{\sin(n+1)x}{2\sin x}.
\end{aligned} \tag{2.18}$$

Applying the Abel's transformation in (2.18) we get

$$\begin{aligned}
f_n(x) &= \frac{1}{2\sin x}\sum_{m=1}^{n} (\Delta a_{m-1} - \Delta a_{m+1})\,\widetilde{D}_m(x)\\
&\quad + (a_n - a_{2n})\,\frac{\widetilde{D}_n(x)}{2\sin x} + (a_n - a_{n+1})\,\frac{\sin(n+1)x}{2\sin x},
\end{aligned}$$

and passing on limit when $n \to \infty$ we obtain

$$\lim_{n\to\infty} f_n(x) = \frac{1}{2\sin x} \sum_{m=1}^{\infty} (\varDelta a_{m-1} - \varDelta a_{m+1})\, \widetilde{D}_m(x). \tag{2.19}$$

In a similar fashion we can show that

$$S_n(x) = \frac{1}{2\sin x} \sum_{m=1}^{n} (\varDelta a_{m-1} - \varDelta a_{m+1})\, \widetilde{D}_m(x) + (a_n - a_{2n})\, \frac{\widetilde{D}_n(x)}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x} + a_n \frac{\sin(n+1)x}{2\sin x},$$

$$f(x) = \lim_{n\to\infty} S_n(x) = \frac{1}{2\sin x} \sum_{m=1}^{\infty} (\varDelta a_{m-1} - \varDelta a_{m+1})\, \widetilde{D}_m(x), \tag{2.20}$$

and the series

$$\frac{1}{2\sin x} \sum_{m=1}^{\infty} (\varDelta a_{m-1} - \varDelta a_{m+1})\, \widetilde{D}_m(x)$$

converges.

Therefore $\lim_{n\to\infty} f_n(x) = f(x)$ exists, and from (2.19) and (2.20) the following equality

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} S_n(x) = f(x)$$

holds.

Hence

$$f(x) - f_n(x) = \frac{1}{2\sin x} \sum_{m=n+1}^{\infty} (\varDelta a_{m-1} - \varDelta a_{m+1})\, \widetilde{D}_m(x) - (a_n - a_{2n})\, \frac{\widetilde{D}_n(x)}{2\sin x} - (a_n - a_{n+1})\, \frac{\sin(n+1)x}{2\sin x}.$$

Denoting with $\widetilde{F}_m(x) = \frac{1}{m+1} \sum_{i=0}^{m} \widetilde{D}_i(x)$ the conjugate Fejér kernel, then the use of Abel's transformation gives

$$f(x) - f_n(x) = \frac{1}{2\sin x} \lim_{\ell\to\infty} \Bigg[\sum_{m=n+1}^{\ell-1} (m+1)\left(\varDelta^2 a_{m-1} - \varDelta^2 a_{m+1}\right) \widetilde{F}_m(x) + (\ell+1)\,(\varDelta a_{\ell-1} - \varDelta a_{\ell+1})\, \widetilde{F}_\ell(x) - (n+1)\,(\varDelta a_n - \varDelta a_{n+2})\, \widetilde{F}_n(x) \Bigg] - (a_n - a_{2n})\, \frac{\widetilde{D}_n(x)}{2\sin x} - (a_n - a_{n+1})\, \frac{\sin(n+1)x}{2\sin x}$$

$$= \frac{1}{2\sin x}\left[\sum_{m=n+1}^{\infty}(m+1)\left(\Delta^2 a_{m-1} - \Delta^2 a_{m+1}\right)\widetilde{F}_m(x)\right.$$
$$\left. -(n+1)\left(\Delta a_n - \Delta a_{n+2}\right)\widetilde{F}_n(x)\right] - \left(a_n - a_{2n}\right)\frac{\widetilde{D}_n(x)}{2\sin x}$$
$$- \left(a_n - a_{n+1}\right)\frac{\sin(n+1)x}{2\sin x}.$$

Thus

$$\|f - f_n\| = O\left(\sum_{m=n+1}^{\infty}(m+1)\left|\Delta^2 a_{m-1} - \Delta^2 a_{m+1}\right|\int_{-\pi}^{\pi}\left|\widetilde{F}_m(x)\right|dx\right)$$
$$+(n+1)\left|\Delta a_n - \Delta a_{n+2}\right|\int_{-\pi}^{\pi}\left|\widetilde{F}_n(x)\right|dx$$
$$+\left|a_n - a_{2n}\right|\int_{-\pi}^{\pi}\left|\frac{\widetilde{D}_n(x)}{2\sin x}\right|dx + \left|a_n - a_{n+1}\right|\int_{-\pi}^{\pi}\left|\frac{\sin(n+1)x}{2\sin x}\right|dx.$$

The first and fourth terms tend to zero as $n \to \infty$ based on facts that $\int_{-\pi}^{\pi}\left|\widetilde{F}_m(x)\right|dx = \pi$ and $\{a_m\}$ belongs the class $\mathbf{K}$.

Further, for the second term, denoted by $\Lambda(n)$, for large enough n we obtain

$$\Lambda(n) = O\Big((n+1)\left|\Delta a_n - \Delta a_{n+2}\right|\Big)$$
$$= O\left((n+1)\left|\sum_{m=n}^{\infty}\left(\Delta^2 a_m - \Delta^2 a_{m+2}\right)\right|\right)$$
$$= O\left((n+1)\sum_{m=n+1}^{\infty}\left|\Delta^2 a_{m-1} - \Delta^2 a_{m+1}\right|\right)$$
$$= O\left(\sum_{m=n+1}^{\infty} m\left|\Delta^2 a_{m-1} - \Delta^2 a_{m+1}\right|\right) = o(1).$$

Since $\int_{-\pi}^{\pi}\left|\frac{\widetilde{D}_n(x)}{2\sin x}\right|dx = O(n)$ then the third term tends to zero, as well. Indeed, we have

$$\left|a_n - a_{2n}\right|\int_{-\pi}^{\pi}\left|\frac{\widetilde{D}_n(x)}{2\sin x}\right|dx = O\left(n\left|\sum_{m=n}^{\infty}\left(\Delta a_m - \Delta a_{m+2}\right)\right|\right)$$
$$= O\left((n+1)\sum_{m=n+1}^{\infty}\left|\Delta^2 a_{m-1} - \Delta^2 a_{m+1}\right|\right)$$
$$= O\left(\sum_{m=n+1}^{\infty} m\left|\Delta^2 a_{m-1} - \Delta^2 a_{m+1}\right|\right) = o(1).$$

The proof is completed.

Corollary 2.11. *If $\{a_n\} \in \mathbf{K}$, then the necessary and sufficient condition for the L^1-convergence of the cosine series*

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \quad is \quad \lim_{n\to\infty} a_n \log n = 0.$$

Proof. Sufficiency. We can write

$$\begin{aligned}
\|f - S_n\| &\le \|f - f_n\| + \|S_n - f_n\| \\
&= \|f - f_n\| + \left\| a_{n+1} \left(\frac{\sin nx}{2\sin x} + \frac{\sin(n+1)x}{2\sin x} \right) \right\| \\
&= \|f - f_n\| + |a_{n+1}| \int_{-\pi}^{\pi} |D_n(x)|\, dx.
\end{aligned}$$

From the well-known relation $\int_{-\pi}^{\pi} |D_n(x)|\, dx \sim \log n$ and our assumption that $a_n \log n = o(1)$, we obtain $|a_{n+1}| \int_{-\pi}^{\pi} |D_n(x)|\, dx = o(1)$ as $n \to \infty$.

Also, according to the Theorem 2.10

$$\|f - f_n\| = o(1) \quad as \quad n \to \infty.$$

This completes the sufficient condition.

Necessity. The following holds

$$\begin{aligned}
|a_{n+1}| \log n &\sim |a_{n+1}| \int_{-\pi}^{\pi} |D_n(x)|\, dx = \|a_{n+1} D_n(x)\| \\
&= \|S_n - f_n\| \le \|S_n - f\| + \|f - f_n\| = o(1) \quad as \quad n \to \infty,
\end{aligned}$$

by our assumption and Theorem 2.10.

The proof is completed.

2.7 L^1-convergence of modified trigonometric sums $f_n(x)$ with generalized semi-convex coefficients

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

and

$$f_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx.$$

For $0 < x \le \pi$, let

$$\widetilde{D}_0(x) = -\frac{1}{2}\cot\frac{x}{2},$$
$$\widetilde{S}_n(x) = \widetilde{D}_0(x) + \widetilde{D}_n(x),$$
$$\widetilde{S}_n^1(x) = \widetilde{S}_0(x) + \widetilde{S}_1(x) + \widetilde{S}_2(x) + \cdots + \widetilde{S}_n(x),$$
$$\widetilde{S}_n^2(x) = \widetilde{S}_0^1(x) + \widetilde{S}_1^1(x) + \widetilde{S}_2^1(x) + \cdots + \widetilde{S}_n^1(x),$$
$$\vdots$$
$$\widetilde{S}_n^k(x) = \widetilde{S}_0^{k-1}(x) + \widetilde{S}_1^{k-1}(x) + \widetilde{S}_2^{k-1}(x) + \cdots + \widetilde{S}_n^{k-1}(x).$$

The conjugate Cèsaro means $\widetilde{T}_k^\alpha(x)$ of order α is denoted by

$$\widetilde{T}_k^\alpha(x) = \frac{\widetilde{S}_k^\alpha(x)}{\widetilde{A}_k^\alpha}.$$

The following result holds true.

Theorem 2.12. *If $\{a_n\}$ is a generalized semi-convex null sequence, then $f_n(x)$ converges to $f(x)$ in the L^1-norm if and only if*

$$\lim_{n\to\infty} \Delta a_n \log n = 0.$$

Proof. We have

$$\begin{aligned} f_n(x) &= \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x) \\ &= \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x), \quad (a_0 = 0) \\ &= \sum_{k=1}^{n} (a_{k-1} - a_{k+1}) \frac{\sin kx}{2\sin x} + a_{n-1}\frac{\sin nx}{2\sin x} \\ &\quad + a_n \frac{\sin(n+1)x}{2\sin x} - a_{n+1} D_n(x), \end{aligned}$$

where

$$D_n(x) = \frac{\sin nx + \sin(n+1)x}{2\sin x}.$$

Applying Lemma 1.1 we obtain

$$f_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n} (\Delta a_{k-1} + \Delta a_k)\sin kx + \Delta a_n \frac{\sin(n+1)x}{2\sin x}.$$

Applying Lemma 1.1 again we have

$$\begin{aligned}
f_n(x) &= \frac{1}{2\sin x}\sum_{k=1}^{n-1}(\Delta^2 a_{k-1}+\Delta^2 a_k)\widetilde{D}_k(x) \\
&\quad +(\Delta a_{n-1}+\Delta a_n)\widetilde{D}_n(x)+\Delta a_n\frac{\sin(n+1)x}{2\sin x} \\
&= \frac{1}{2\sin x}\Bigg[\sum_{k=1}^{n-1}(\Delta^2 a_{k-1}+\Delta^2 a_k)(\widetilde{S}_k^0(x)-\widetilde{S}_0(x)) \\
&\quad +(\Delta a_{n-1}+\Delta a_n)(\widetilde{S}_n^0(x)-\widetilde{S}_0(x))\Bigg]+\Delta a_n\frac{\sin(n+1)x}{2\sin x} \\
&= \frac{1}{2\sin x}\Bigg[\sum_{k=1}^{n-1}(\Delta^2 a_{k-1}+\Delta^2 a_k)\widetilde{S}_k^0(x)-\sum_{k=1}^{n-1}(\Delta^2 a_{k-1}+\Delta^2 a_k)\widetilde{S}_0(x) \\
&\quad +(\Delta a_{n-1}+\Delta a_n)\widetilde{S}_n^0(x)-(\Delta a_{n-1}+\Delta a_n)\widetilde{S}_0(x)\Bigg]+\Delta a_n\frac{\sin(n+1)x}{2\sin x} \\
&= \frac{1}{2\sin x}\Bigg[\sum_{k=1}^{n-1}(\Delta^2 a_{k-1}+\Delta^2 a_k)\widetilde{S}_k^0(x)-(\Delta a_{n-1}+\Delta a_n)\widetilde{S}_n^0(x) \\
&\quad +a_2\widetilde{S}_0(x)\Bigg]+\Delta a_n\frac{\sin(n+1)x}{2\sin x}.
\end{aligned}$$

Similarly, if we continue to apply Lemma 1.1 α times, we obtain

$$\begin{aligned}
f_n(x) = \frac{1}{2\sin x}\Bigg[&\sum_{k=1}^{n-\alpha}(\Delta^{\alpha+1} a_{k-1}+\Delta^{\alpha+1} a_k)\widetilde{S}_k^{\alpha-1}(x)+\sum_{k=1}^{\alpha}\Delta^k a_{n-k}\widetilde{S}_{n-k+1}^{k-1}(x) \\
&+\sum_{k=1}^{\alpha}\Delta^k a_{n-k+1}\widetilde{S}_{n-k+1}^{k-1}(x)+a_2\widetilde{S}_0(x)\Bigg]+\Delta a_n\frac{\sin(n+1)x}{2\sin x}.
\end{aligned}$$

Since $\widetilde{S}_k^\alpha(x)$ and $\widetilde{T}_k^\alpha(x)$ are bounded on every segment $[\varepsilon, \pi-\varepsilon]$, $\varepsilon > 0$, have

$$\begin{aligned}
f(x) &= \lim_{n\to\infty} f_n(x) \\
&= \frac{1}{2\sin x}\Bigg[\sum_{k=1}^{\infty}(\Delta^{\alpha+1} a_{k-1}+\Delta^{\alpha+1} a_k)\widetilde{S}_k^{\alpha-1}(x)+a_2\widetilde{S}_0(x)\Bigg].
\end{aligned}$$

Consequently,

$$f(x)-f_n(x) = \frac{1}{2\sin x}\Bigg[\sum_{k=n-\alpha+1}^{\infty}(\Delta^{\alpha+1} a_{k-1}+\Delta^{\alpha+1} a_k)\widetilde{S}_k^{\alpha-1}(x)$$

$$-\sum_{k=1}^{\alpha}\Delta^k a_{n-k}\widetilde{S}_{n-k+1}^{k-1}(x)-\sum_{k=1}^{\alpha}\Delta^k a_{n-k+1}\widetilde{S}_{n-k+1}^{k-1}(x)\Bigg]$$
$$-\Delta a_n\frac{\sin(n+1)x}{2\sin x}.$$

Whence,

$$\begin{aligned}\|f(x)-f_n(x)\| &\le C\Bigg[\int_0^{\pi}\Bigg|\sum_{k=n-\alpha+1}^{\infty}(\Delta^{\alpha+1}a_{k-1}+\Delta^{\alpha+1}a_k)\widetilde{S}_k^{\alpha-1}(x)\Bigg|dx\\
&\quad+\int_0^{\pi}\Bigg|\sum_{k=1}^{\alpha}\Delta^k a_{n-k}\widetilde{S}_{n-k+1}^{k-1}(x)\Bigg|dx\\
&\quad+\int_0^{\pi}\Bigg|\sum_{k=1}^{\alpha}\Delta^k a_{n-k+1}\widetilde{S}_{n-k+1}^{k-1}(x)\Bigg|dx\Bigg]\\
&\quad+\int_0^{\pi}\Bigg|\Delta a_n\frac{\sin(n+1)x}{2\sin x}\Bigg|dx\\
&\le C\Bigg[\sum_{k=n-\alpha+1}^{\infty}|\Delta^{\alpha+1}a_{k-1}+\Delta^{\alpha+1}a_k|\int_0^{\pi}\big|\widetilde{S}_k^{\alpha-1}(x)\big|dx\\
&\quad+\sum_{k=1}^{\alpha}|\Delta^k a_{n-k}|\int_0^{\pi}\big|\widetilde{S}_{n-k+1}^{k-1}(x)\big|dx\\
&\quad+\sum_{k=1}^{\alpha}|\Delta^k a_{n-k+1}|\int_0^{\pi}\big|\widetilde{S}_{n-k+1}^{k-1}(x)\big|dx\Bigg]\\
&\quad+\int_0^{\pi}\Bigg|\Delta a_n\frac{\sin(n+1)x}{2\sin x}\Bigg|dx\\
&\le C\Bigg[\sum_{k=n-\alpha+1}^{\infty}A_k^{\alpha}|\Delta^{\alpha+1}a_{k-1}+\Delta^{\alpha+1}a_k|\int_0^{\pi}\big|\widetilde{T}_k^{\alpha}(x)\big|dx\\
&\quad+\sum_{k=1}^{\alpha}A_{n-k+1}^{k}|\Delta^k a_{n-k}|\int_0^{\pi}\big|\widetilde{T}_{n-k+1}^{k}(x)\big|dx\\
&\quad+\sum_{k=1}^{\alpha}A_{n-k+1}^{k}|\Delta^k a_{n-k+1}|\int_0^{\pi}\big|\widetilde{T}_{n-k+1}^{k-1}(x)\big|dx\Bigg]\\
&\quad+\int_0^{\pi}\Bigg|\Delta a_n\frac{\sin(n+1)x}{2\sin x}\Bigg|dx. \qquad (2.21)\end{aligned}$$

Based on Lemma 1.48 and assumptions of the theorem, first three terms of (2.21) are of order $o(1)$ as $n\to\infty$.

Moreover, since

$$\int_0^{\pi}\left|\frac{\sin(n+1)x}{2\sin x}\right|dx\le C\log n,\quad n\ge 2,$$

then

$$\int_0^{\pi} \left| \Delta a_n \frac{\sin(n+1)x}{2\sin x} \right| dx \sim \Delta a_n \log n.$$

So, it follows that

$$\lim_{n\to\infty} \|f(x) - f_n(x)\| = 0,$$

if and only if

$$\lim_{n\to\infty} \Delta a_n \log n = 0.$$

The proof is completed.

2.8 L^1-convergence of modified trigonometric sums $f_n(x)$ with coefficients from the class $\mathbf{S_r}$

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

and

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx.$$

Theorem 2.13. *Let $\{a_k\} \in \mathbf{S_r}$, $r \in \{1, 2, \ldots\}$, then $f_n(x)$ converges to $f(x)$ in L^1-norm.*

Proof. Firstly, we have

$$\begin{aligned} f_n(x) &= \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x). \end{aligned}$$

Consequently,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} S_n(x) = f(x),$$

since $D_n(x)$ is bounded in $(0, \pi]$ and $\{a_k\} \in \mathbf{S_r}$, $r \in \{1, 2, \ldots\}$, where

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx.$$

So, we can write

$$f(x) - f_n(x) = \sum_{k=n+1}^{\infty} a_k \cos kx + a_{n+1} D_n(x).$$

Now, using Lemmas 1.1, 1.34, and $\{a_k\} \in \mathbf{S_r}$, $r \in \{1,2,\ldots\}$, we have

$$\begin{aligned}
\int_0^\pi |f(x) - f_n(x)|dx &= \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx \\
&= \int_0^\pi \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx \\
&= \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta A_k \sum_{\mu=1}^{k} \frac{\Delta a_\mu}{A_\mu} D_\mu(x) \right| dx \\
&\le \sum_{k=n+1}^{\infty} \Delta A_k \int_0^\pi \left| \sum_{\mu=1}^{k} \frac{\Delta a_\mu}{A_\mu} D_\mu(x) \right| dx \\
&= \sum_{k=n+1}^{\infty} \frac{k^{r-1}}{k^{r-1}} |\Delta A_k| \int_0^\pi \left| \sum_{\mu=1}^{k} \frac{\Delta a_\mu}{A_\mu} D_\mu(x) \right| dx \\
&\le \frac{C}{(n+1)^{r-1}} \sum_{k=n+1}^{\infty} (k+1)^r |\Delta A_k| \\
&\le 2^r C \sum_{k=n+1}^{\infty} k^r A_k + C \sum_{k=n+1}^{\infty} (k+1)^r A_{k+1} \\
&= (2^r+1) C \sum_{k=n}^{\infty} k^r A_k = o(1) \quad as \quad n \to \infty,
\end{aligned}$$

taking into account that $\{a_k\} \in \mathbf{S}$.

Thus, we have obtained

$$\lim_{n\to\infty} \int_0^\pi |f(x) - f_n(x)|dx = 0.$$

The proof is completed.

Corollary 2.14. *Let* $\{a_k\} \in \mathbf{S_r}$, $r \in \{1,2,\ldots\}$. *The series*

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

converges in L^1*-norm if and only if*

$$\lim_{n\to\infty} a_{n+1} \log n = 0.$$

Proof. We note that

$$\int_0^\pi |f(x) - S_n(x)|dx = \int_0^\pi |f(x) - f_n(x) + f_n(x) - S_n(x)|dx$$

$$\leq \int_0^\pi |f(x) - f_n(x)|dx + \int_0^\pi |f_n(x) - S_n(x)|dx$$
$$= \int_0^\pi |f(x) - f_n(x)|dx + |a_{n+1}| \int_0^\pi |D_n(x)|dx,$$

and

$$\int_0^\pi |a_{n+1} D_n(x)|dx = \int_0^\pi |f_n(x) - S_n(x)|dx$$
$$\leq \int_0^\pi |f_n(x) - f(x)|dx + \int_0^\pi |f(x) - S_n(x)|dx.$$

Subsequently,

$$\|f - S_n\|_{L^1} = o(1) \ as \ n \to \infty$$

if and only if

$$|a_{n+1}| \log n = o(1) \ as \ n \to \infty,$$

since $\int_0^\pi |a_{n+1} D_n(x)|dx$ behaves as $|a_{n+1}| \log n$ for large values n.

The proof is completed.

2.9 L^1-convergence of modified sums $f_n(x)$ with generalized semi-convex coefficients of fractional order

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

and

$$f_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx.$$

For $0 < x \leq \pi$, let

$$\begin{aligned}
\widetilde{D}_0(x) &= -\frac{1}{2} \cot \frac{x}{2}, \\
\widetilde{S}_n(x) &= \widetilde{D}_0(x) + \widetilde{D}_n(x), \\
\widetilde{S}_n^1(x) &= \widetilde{S}_0(x) + \widetilde{S}_1(x) + \widetilde{S}_2(x) + \cdots + \widetilde{S}_n(x), \\
\widetilde{S}_n^2(x) &= \widetilde{S}_0^1(x) + \widetilde{S}_1^1(x) + \widetilde{S}_2^1(x) + \cdots + \widetilde{S}_n^1(x), \\
&\vdots \\
\widetilde{S}_n^k(x) &= \widetilde{S}_0^{k-1}(x) + \widetilde{S}_1^{k-1}(x) + \widetilde{S}_2^{k-1}(x) + \cdots + \widetilde{S}_n^{k-1}(x).
\end{aligned}$$

The conjugate Cèsaro means $\widetilde{T}_k^\alpha(x)$ of order α is denoted by

$$\widetilde{T}_k^\alpha(x) = \frac{\widetilde{S}_k^\alpha(x)}{\widetilde{A}_k^\alpha}.$$

Next result holds true.

Theorem 2.15. *If $\{a_n\}$ is a generalized semi-convex null sequence of fractional order, then $f_n(x)$ converges to $f(x)$ in the L^1-norm if and only if*

$$\lim_{n\to\infty} \Delta a_n \log n = 0.$$

Proof. We have

$$\begin{aligned}
f_n(x) &= \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x) \\
&= \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x), \quad (a_0 = 0) \\
&= \sum_{k=1}^{n} (a_{k-1} - a_{k+1}) \frac{\sin kx}{2\sin x} + a_{n-1}\frac{\sin nx}{2\sin x} \\
&\quad + a_n \frac{\sin(n+1)x}{2\sin x} - a_{n+1} D_n(x),
\end{aligned}$$

where

$$D_n(x) = \frac{\sin nx + \sin(n+1)x}{2\sin x}.$$

Applying Lemma 1.1 we obtain

$$f_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n} (\Delta a_{k-1} + \Delta a_k)\sin kx + \Delta a_n \frac{\sin(n+1)x}{2\sin x}.$$

Applying Lemma 1.1 again, we have

$$\begin{aligned}
f_n(x) &= \frac{1}{2\sin x}\sum_{k=1}^{n-1} (\Delta^2 a_{k-1} + \Delta^2 a_k)\widetilde{D}_k(x) \\
&\quad + (\Delta a_{n-1} + \Delta a_n)\widetilde{D}_n(x) + \Delta a_n \frac{\sin(n+1)x}{2\sin x} \\
&= \frac{1}{2\sin x}\left[\sum_{k=1}^{n-1} (\Delta^2 a_{k-1} + \Delta^2 a_k)(\widetilde{S}_k^0(x) - \widetilde{S}_0(x)) \right. \\
&\quad \left. + (\Delta a_{n-1} + \Delta a_n)(\widetilde{S}_n^0(x) - \widetilde{S}_0(x))\right] + \Delta a_n \frac{\sin(n+1)x}{2\sin x}
\end{aligned}$$

$$= \frac{1}{2\sin x}\Bigg[\sum_{k=1}^{n-1}(\Delta^2 a_{k-1}+\Delta^2 a_k)\widetilde{S}_k^0(x) - \sum_{k=1}^{n-1}(\Delta^2 a_{k-1}+\Delta^2 a_k)\widetilde{S}_0(x)$$
$$+(\Delta a_{n-1}+\Delta a_n)\widetilde{S}_n^0(x) - (\Delta a_{n-1}+\Delta a_n)\widetilde{S}_0(x)\Bigg] + \Delta a_n\frac{\sin(n+1)x}{2\sin x}$$
$$= \frac{1}{2\sin x}\Bigg[\sum_{k=1}^{n-1}(\Delta^2 a_{k-1}+\Delta^2 a_k)\widetilde{S}_k^0(x) - (\Delta a_{n-1}+\Delta a_n)\widetilde{S}_n^0(x)$$
$$+a_2\widetilde{S}_0(x)\Bigg] + \Delta a_n\frac{\sin(n+1)x}{2\sin x}. \tag{2.22}$$

Since $\alpha > 0$ is non-integral, then let $\alpha = r+\delta$, where r is integral part of α, δ is its fractional part, and $0<\delta<1$.

Case (i). Let $r=0$. Applying Abel's transformation of order $-\delta+1$, we have by (1.7)

$$\sum_{k=1}^{n-1}\widetilde{S}_k^{\delta-1}(x)(\Delta^{\delta+1}a_{k-1}+\Delta^{\delta+1}a_k)$$
$$= \sum_{k=1}^{n-1}\widetilde{S}_k(x)\sum_{m=1}^{n-(k+1)}A_m^{\delta-2}(\Delta^{\delta+1}a_{m+k-1}+\Delta^{\delta+1}a_{m+k}).$$

Moreover, applying Lemma 1.49, we have

$$\sum_{k=1}^{n-1}\widetilde{S}_k^{\delta-1}(x)\left(\Delta^{\delta+1}a_{k-1}+\Delta^{\delta+1}a_k\right)$$
$$= \sum_{k=1}^{n-1}\widetilde{S}_k(x)\left\{\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right) - \sum_{m=n-k}^{\infty}A_m^{\delta-2}\left(\Delta^{\delta+1}a_{m+k-1}+\Delta^{\delta+1}a_{m+k}\right)\right\}.$$
$$= \sum_{k=1}^{n-1}\widetilde{S}_k(x)\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right) - R_n(x),$$

where

$$R_n(x) = \sum_{k=1}^{n-1}\widetilde{S}_k(x)\Big\{A_{n-k}^{\delta-2}\left(\Delta^{\delta+1}a_{n-1}+\Delta^{\delta+1}a_n\right)$$
$$+A_{n-k+1}^{\delta-2}\left(\Delta^{\delta+1}a_n+\Delta^{\delta+1}a_{n+1}\right)+\cdots\Big\}.$$

Therefore,

$$\frac{1}{2\sin x}\sum_{k=1}^{n-1}\widetilde{S}_k(x)\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)$$

$$= \frac{1}{2\sin x}\left\{\sum_{k=1}^{n-1} \widetilde{S}_k^{\delta-1}(x)\left(\Delta^{\delta+1}a_{k-1} + \Delta^{\delta+1}a_k\right) + R_n(x)\right\},$$

and thus by (2.22) we get

$$f_n(x) = \frac{1}{2\sin x}\left\{\sum_{k=1}^{n-1} \widetilde{S}_k^{\delta-1}(x)\left(\Delta^{\delta+1}a_{k-1} + \Delta^{\delta+1}a_k\right)\right.$$
$$\left. +R_n(x) + (\Delta a_{n-1} + \Delta a_n)\widetilde{S}_n^0(x) + a_2\widetilde{S}_0(x)\right\} + \Delta a_n \frac{\sin(n+1)x}{2\sin x}.$$

When $r = 0$, then $\alpha = \delta$ and

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f_n(x) \\ &= \frac{1}{2\sin x}\left\{\sum_{k=1}^{\infty} \widetilde{S}_k^{\delta-1}(x)\left(\Delta^{\delta+1}a_{k-1} + (\Delta^{\delta+1}a_k\right) + a_2\widetilde{S}_0(x)\right\}. \end{aligned}$$

So, by Lemmas 1.48 and 1.90 we obtain

$$\begin{aligned} \int_0^\pi |f(x) - f_n(x)|dx &\le \int_0^\pi \left|\frac{1}{2\sin x}\left\{\sum_{k=n}^{\infty} \widetilde{S}_k^{\delta-1}(x)\left(\Delta^{\delta+1}a_{k-1} + \Delta^{\delta+1}a_k\right)\right.\right. \\ &\quad \left.\left. -R_n(x) - (\Delta a_{n-1} + \Delta a_n)\widetilde{S}_n^0(x)\right\}\right|dx \\ &\quad + \int_0^\pi \left|\Delta a_n \frac{\sin(n+1)x}{2\sin x}\right|dx \\ &\le C\left\{\sum_{k=n}^{\infty} \left|\Delta^{\delta+1}a_{k-1} + \Delta^{\delta+1}a_k\right| \int_0^\pi \left|\widetilde{S}_k^{\delta-1}(x)\right|dx\right. \\ &\quad \left. + \int_0^\pi \left|R_n(x)\right|dx + |\Delta a_{n-1} + \Delta a_n| \int_0^\pi \left|\widetilde{S}_n^0(x)\right|dx\right\} \\ &\quad + \int_0^\pi \left|\Delta a_n \frac{\sin(n+1)x}{2\sin x}\right|dx \\ &= C\left\{\sum_{k=n}^{\infty} A_k^{\delta-1}\left|\Delta^{\delta+1}a_{k-1} + \Delta^{\delta+1}a_k\right| \int_0^\pi \left|\widetilde{T}_k^{\delta-1}(x)\right|dx\right. \\ &\quad \left. + \int_0^\pi \left|R_n(x)\right|dx + |\Delta a_{n-1} + \Delta a_n| \int_0^\pi \left|\widetilde{S}_n^0(x)\right|dx\right\} \\ &\quad + \int_0^\pi \left|\Delta a_n \frac{\sin(n+1)x}{2\sin x}\right|dx \\ &\le C_1\left\{\sum_{k=n}^{\infty} A_k^{\delta-1}\left|\Delta^{\delta+1}a_{k-1} + \Delta^{\delta+1}a_k\right| + \int_0^\pi \left|R_n(x)\right|dx\right. \end{aligned}$$

$$
\begin{aligned}
&+|\varDelta a_{n-1}+\varDelta a_n|\Big\} + \int_0^\pi \left|\varDelta a_n \frac{\sin(n+1)x}{2\sin x}\right| dx \\
&= o(1) + C\int_0^\pi \big|R_n(x)\big| dx \\
&\quad + C_1|a_{n-1}-a_{n+1}| + \int_0^\pi \left|\varDelta a_n \frac{\sin(n+1)x}{2\sin x}\right| dx. \quad (2.23)
\end{aligned}
$$

In order to estimate $\int_0^\pi \big|R_n(x)\big| dx$ we use Lemmas 1.48, 1.50, and 1.90:

$$
\begin{aligned}
&\int_0^\pi \big|R_n(x)\big| dx \\
&\quad = \int_0^\pi \Bigg|\sum_{k=1}^{n-1} \widetilde{S}_k(x)\Big\{A_{n-k}^{\delta-2}\left(\varDelta^{\delta+1}a_{n-1}+\varDelta^{\delta+1}a_n\right) \\
&\qquad\quad + A_{n-k+1}^{\delta-2}\left(\varDelta^{\delta+1}a_n+\varDelta^{\delta+1}a_{n+1}\right)+\cdots\Big\}\Bigg| dx \\
&\quad \le \big|\varDelta^{\delta+1}a_{n-1}+\varDelta^{\delta+1}a_n\big| \int_0^\pi \Bigg|\sum_{k=1}^{n-1} A_{n-k}^{\delta-2}\widetilde{S}_k(x)\Bigg| dx \\
&\qquad\quad + \big|\varDelta^{\delta+1}a_n+\varDelta^{\delta+1}a_{n+1}\big| \int_0^\pi \Bigg|\sum_{k=1}^{n-1} A_{n-k+1}^{\delta-2}\widetilde{S}_k(x)\Bigg| dx + \cdots \\
&\quad \le \big|\varDelta^{\delta+1}a_{n-1}+\varDelta^{\delta+1}a_n\big| \int_0^\pi \max_{1\le p\le n-1}\left|\widetilde{S}_p^{\delta-1}(x)\right| dx \\
&\qquad\quad + \big|\varDelta^{\delta+1}a_n+\varDelta^{\delta+1}a_{n+1}\big| \int_0^\pi \max_{1\le p\le n}\left|\widetilde{S}_p^{\delta-1}(x)\right| dx + \cdots \\
&\quad = A_n^{\delta-1}\big|\varDelta^{\delta+1}a_{n-1}+\varDelta^{\delta+1}a_n\big| \int_0^\pi \max_{1\le p\le n-1}\left|\widetilde{T}_p^{\delta-1}(x)\right| dx \\
&\qquad\quad + A_{n+1}^{\delta-1}\big|\varDelta^{\delta+1}a_n+\varDelta^{\delta+1}a_{n+1}\big| \int_0^\pi \max_{1\le p\le n}\left|\widetilde{T}_p^{\delta-1}(x)\right| dx + \cdots \\
&\quad \le CA_n^{\delta-1}\big|\varDelta^{\delta+1}a_{n-1}+\varDelta^{\delta+1}a_n\big| \\
&\qquad\quad + CA_{n+1}^{\delta-1}\big|\varDelta^{\delta+1}a_n+\varDelta^{\delta+1}a_{n+1}\big| + \cdots \\
&\quad \le CA_n^{\delta}\big|\varDelta^{\delta+1}a_{n-1}+\varDelta^{\delta+1}a_n\big| \\
&\qquad\quad + CA_{n+1}^{\delta}\big|\varDelta^{\delta+1}a_n+\varDelta^{\delta+1}a_{n+1}\big| + \cdots \\
&\quad = o(1)+o(1)+\cdots = o(1).
\end{aligned}
$$

Moreover, since

$$
\int_0^\pi \left|\frac{\sin(n+1)x}{2\sin x}\right| dx \le C\log n, \quad n\ge 2,
$$

then

$$\int_0^\pi \left| \varDelta a_n \frac{\sin(n+1)x}{2\sin x} \right| dx \sim \varDelta a_n \log n.$$

Thus, it follows that

$$\lim_{n\to\infty} \|f(x) - f_n(x)\| = 0,$$

if and only if

$$\lim_{n\to\infty} \varDelta a_n \log n = 0.$$

Case (ii). Let $r \geq 1$. Applying Abel's transformation r times to equality

$$f_n(x) = \frac{1}{2\sin x} \sum_{k=1}^{n} (\varDelta a_{k-1} + \varDelta a_k) \sin kx + \varDelta a_n \frac{\sin(n+1)x}{2\sin x},$$

we obtain

$$\begin{aligned} f_n(x) = \frac{1}{2\sin x} \Bigg\{ & \sum_{k=1}^{n-r} (\varDelta^{r+1} a_{k-1} + \varDelta^{r+1} a_k) \widetilde{S}_k^{r-1}(x) \\ & + \sum_{k=1}^{r} (\varDelta^k a_{n-k} + \varDelta^k a_{n-k+1}) \widetilde{S}_{n-k-1}^{k-1}(x) + a_2 \widetilde{S}_0(x) \Bigg\} \\ & + \varDelta a_n \frac{\sin(n+1)x}{2\sin x}. \end{aligned} \tag{2.24}$$

Applying Abel's transformation of order $-\delta$ again, we get

$$\begin{aligned} & \sum_{k=1}^{n-1} \widetilde{S}_k^{\alpha-1}(x) (\varDelta^{\alpha+1} a_{k-1} + \varDelta^{\alpha+1} a_k) \\ & \qquad = \sum_{k=1}^{n-1} \widetilde{S}_k^{r-1}(x) \sum_{m=0}^{n-(k+1)} A_m^{\delta-1} (\varDelta^{\alpha+1} a_{m+k-1} + \varDelta^{\alpha+1} a_{m+k}). \end{aligned}$$

By Lemma 1.49 we have

$$\begin{aligned} & \frac{1}{2\sin x} \sum_{k=1}^{n-1} \widetilde{S}_k^{\alpha-1}(x) (\varDelta^{\alpha+1} a_{k-1} + \varDelta^{\alpha+1} a_k) \\ & = \frac{1}{2\sin x} \left\{ \sum_{k=1}^{n-1} \widetilde{S}_k^{r-1}(x) (\varDelta^{r+1} a_{k-1} + \varDelta^{r+1} a_k) - R_n(x) \right\}, \end{aligned} \tag{2.25}$$

where

$$R_n(x) = \sum_{k=1}^{n-1} \widetilde{S}_k^{r-1}(x) \Big\{ A_{n-k}^{\delta-1} \left(\varDelta^{\alpha+1} a_{n-1} + \varDelta^{\alpha+1} a_n \right)$$

$$
\begin{aligned}
&+A_{n-k+1}^{\delta-1}\left(\Delta^{\alpha+1}a_n+\Delta^{\alpha+1}a_{n+1}\right)+\cdots\Big\}\\
&=\left(\Delta^{\alpha+1}a_{n-1}+\Delta^{\alpha+1}a_n\right)\sum_{k=1}^{n-1}A_{n-k}^{\delta-1}\widetilde{S}_k^{r-1}(x)\\
&+\left(\Delta^{\alpha+1}a_n+\Delta^{\alpha+1}a_{n+1}\right)\sum_{k=1}^{n-1}A_{n-k+1}^{\delta-1}\widetilde{S}_k^{r-1}(x)+\cdots
\end{aligned}
$$

Replacing n by $n-r+1$ in (2.25), we obtain

$$
\begin{aligned}
&\frac{1}{2\sin x}\sum_{k=1}^{n-r}\widetilde{S}_k^{\alpha-1}(x)(\Delta^{\alpha+1}a_{k-1}+\Delta^{\alpha+1}a_k)\\
&=\frac{1}{2\sin x}\left\{\sum_{k=1}^{n-r}\widetilde{S}_k^{r-1}(x)(\Delta^{r+1}a_{k-1}+\Delta^{r+1}a_k)-R_{n-r+1}(x)\right\}. \quad (2.26)
\end{aligned}
$$

Now, based on (2.24) and (2.26), we get

$$
\begin{aligned}
f_n(x)=\frac{1}{2\sin x}\Bigg\{&\sum_{k=1}^{n-r}\widetilde{S}_k^{\alpha-1}(x)(\Delta^{\alpha+1}a_{k-1}+\Delta^{\alpha+1}a_k)-R_{n-r+1}(x)\\
&+\sum_{k=1}^{r}(\Delta^k a_{n-k}+\Delta^k a_{n-k+1})\widetilde{S}_{n-k-1}^{k-1}(x)+a_2\widetilde{S}_0(x)\Bigg\}\\
&+\Delta a_n\frac{\sin(n+1)x}{2\sin x}. \qquad (2.27)
\end{aligned}
$$

Whence, under assumptions of theorem and Lemma 1.48, we have

$$
\begin{aligned}
&\int_0^\pi|f(x)-f_n(x)|dx\\
&\le\int_0^\pi\left|\frac{1}{2\sin x}\left\{\sum_{k=n-r+1}^{\infty}\widetilde{S}_k^{\alpha-1}(x)(\Delta^{\alpha+1}a_{k-1}+\Delta^{\alpha+1}a_k)\right.\right.\\
&\left.\left.-R_{n-r+1}(x)+\sum_{k=1}^{r}(\Delta^k a_{n-k}+\Delta^k a_{n-k+1})\widetilde{S}_{n-k+1}^{k-1}(x)\right\}\right|dx\\
&+\int_0^\pi\left|\Delta a_n\frac{\sin(n+1)x}{2\sin x}\right|dx\\
&\le C\sum_{k=n-r+1}^{\infty}|\Delta^{\alpha+1}a_{k-1}+\Delta^{\alpha+1}a_k|\int_0^\pi\left|\widetilde{S}_k^{\alpha-1}(x)\right|dx\\
&+\int_0^\pi\left|R_{n-r+1}(x)\right|dx\\
&+\sum_{k=1}^{r}|\Delta^k a_{n-k}+\Delta^k a_{n-k+1}|\int_0^\pi\left|\widetilde{S}_{n-k+1}^{k-1}(x)\right|dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^\pi \left|\Delta a_n \frac{\sin(n+1)x}{2\sin x}\right| dx \\
&= C \sum_{k=n-r+1}^{\infty} A_k^{\alpha-1} |\Delta^{\alpha+1} a_{k-1} + \Delta^{\alpha+1} a_k| \int_0^\pi \left|\widetilde{T}_k^{\alpha-1}(x)\right| dx \\
&+\int_0^\pi \left|R_{n-r+1}(x)\right| dx \\
&+\sum_{k=1}^{r} A_{n-k+1}^{k-1} |\Delta^k a_{n-k} + \Delta^k a_{n-k+1}| \int_0^\pi \left|\widetilde{T}_{n-k+1}^{k-1}(x)\right| dx \\
&+\int_0^\pi \left|\Delta a_n \frac{\sin(n+1)x}{2\sin x}\right| dx \\
&\le C_1 \sum_{k=n-r+1}^{\infty} A_k^{\alpha-1} |\Delta^{\alpha+1} a_{k-1} + \Delta^{\alpha+1} a_k| + \int_0^\pi \left|R_{n-r+1}(x)\right| dx \\
&+C_1 \sum_{k=1}^{r} A_{n-k+1}^{k-1} |\Delta^k a_{n-k} + \Delta^k a_{n-k+1}| + \int_0^\pi \left|\Delta a_n \frac{\sin(n+1)x}{2\sin x}\right| dx \\
&= o(1) + \int_0^\pi \left|R_{n-r+1}(x)\right| dx + o(1) + \int_0^\pi \left|\Delta a_n \frac{\sin(n+1)x}{2\sin x}\right| dx. \qquad (2.28)
\end{aligned}
$$

However, by the assumptions of the theorem we obtain

$$
\begin{aligned}
&\int_0^\pi \left|R_{n-r+1}(x)\right| dx \\
&\le \int_0^\pi \left| \left(\sum_{k=1}^{n-r} A_{n-r-k+1}^{\delta-1} \widetilde{S}_k^{r-1}(x)\right) \left(\Delta^{\alpha+1} a_{n-r} + \Delta^{\alpha+1} a_{n-r+1}\right) \right| dx \\
&\quad + \int_0^\pi \left| \left(\sum_{k=1}^{n-r} A_{n-r-k+2}^{\delta-1} \widetilde{S}_k^{r-1}(x)\right) \left(\Delta^{\alpha+1} a_{n-r+1} + \Delta^{\alpha+1} a_{n-r+2}\right) \right| dx \\
&\quad + \int_0^\pi \left| \left(\sum_{k=1}^{n-r} A_{n-r-k+3}^{\delta-1} \widetilde{S}_k^{r-1}(x)\right) \left(\Delta^{\alpha+1} a_{n-r+2} + \Delta^{\alpha+1} a_{n-r+3}\right) \right| dx \\
&\quad + \cdots \\
&\le \sum_{k=1}^{n-r} A_{n-r-k+1}^{\delta-1} A_k^{r-1} |\Delta^{\alpha+1} a_{n-r} + \Delta^{\alpha+1} a_{n-r+1}| \int_0^\pi \left|\widetilde{T}_k^{r-1}(x)\right| dx \\
&\quad + \sum_{k=1}^{n-r} A_{n-r-k+2}^{\delta-1} A_k^{r-1} |\Delta^{\alpha+1} a_{n-r+1} + \Delta^{\alpha+1} a_{n-r+2}| \int_0^\pi \left|\widetilde{T}_k^{r-1}(x)\right| dx \\
&\quad + \sum_{k=1}^{n-r} A_{n-r-k+3}^{\delta-1} A_k^{r-1} |\Delta^{\alpha+1} a_{n-r+2} + \Delta^{\alpha+1} a_{n-r+3}| \int_0^\pi \left|\widetilde{T}_k^{r-1}(x)\right| dx \\
&\quad + \cdots
\end{aligned}
$$

$$\begin{aligned}
&\le C_1 \sum_{k=1}^{n-r} A_{n-r-k+1}^{\delta-1} A_k^{r-1} |\Delta^{\alpha+1} a_{n-r} + \Delta^{\alpha+1} a_{n-r+1}| \\
&\quad + C_1 \sum_{k=1}^{n-r} A_{n-r-k+2}^{\delta-1} A_k^{r-1} |\Delta^{\alpha+1} a_{n-r+1} + \Delta^{\alpha+1} a_{n-r+2}| \\
&\quad + C_1 \sum_{k=1}^{n-r} A_{n-r-k+3}^{\delta-1} A_k^{r-1} |\Delta^{\alpha+1} a_{n-r+2} + \Delta^{\alpha+1} a_{n-r+3}| \\
&\quad + \cdots \\
&\le C_1 \sum_{k=1}^{n+1-r} A_{n+1-r-k}^{\delta-1} A_k^{r-1} |\Delta^{\alpha+1} a_{n-r} + \Delta^{\alpha+1} a_{n-r+1}| \\
&\quad + C_1 \sum_{k=1}^{n+2-r} A_{n+2-r-k}^{\delta-1} A_k^{r-1} |\Delta^{\alpha+1} a_{n-r+1} + \Delta^{\alpha+1} a_{n-r+2}| \\
&\quad + C_1 \sum_{k=1}^{n+3-r} A_{n+3-r-k}^{\delta-1} A_k^{r-1} |\Delta^{\alpha+1} a_{n-r+2} + \Delta^{\alpha+1} a_{n-r+3}| \\
&\quad + \cdots \\
&\le C_1 A_{n+1-r}^{r+\delta-1} |\Delta^{\alpha+1} a_{n-r} + \Delta^{\alpha+1} a_{n-r+1}| \\
&\quad + C_1 A_{n+2-r}^{r+\delta-1} |\Delta^{\alpha+1} a_{n-r+1} + \Delta^{\alpha+1} a_{n-r+2}| \\
&\quad + C_1 A_{n+3-r}^{r+\delta-1} |\Delta^{\alpha+1} a_{n-r+2} + \Delta^{\alpha+1} a_{n-r+3}| \\
&\quad + \cdots \\
&\le C_1 A_{n+1-r}^{r+\delta} |\Delta^{\alpha+1} a_{n-r} + \Delta^{\alpha+1} a_{n-r+1}| \\
&\quad + C_1 A_{n+2-r}^{r+\delta} |\Delta^{\alpha+1} a_{n-r+1} + \Delta^{\alpha+1} a_{n-r+2}| \\
&\quad + C_1 A_{n+3-r}^{r+\delta} |\Delta^{\alpha+1} a_{n-r+2} + \Delta^{\alpha+1} a_{n-r+3}| \\
&\quad + \cdots \\
&= C_1 A_{n+1-r}^{\alpha} |\Delta^{\alpha+1} a_{n-r} + \Delta^{\alpha+1} a_{n-r+1}| \\
&\quad + C_1 A_{n+2-r}^{\alpha} |\Delta^{\alpha+1} a_{n-r+1} + \Delta^{\alpha+1} a_{n-r+2}| \\
&\quad + C_1 A_{n+3-r}^{\alpha} |\Delta^{\alpha+1} a_{n-r+2} + \Delta^{\alpha+1} a_{n-r+3}| \\
&\quad + \cdots \\
&= o(1) + o(1) + o(1) + \cdots = o(1) \quad as \quad n \to \infty.
\end{aligned}$$

So,

$$\lim_{n\to\infty} \|f(x) - f_n(x)\| = 0,$$

if and only if

$$\lim_{n\to\infty} \Delta a_n \log n = 0.$$

Finally, the cases (i) and (ii) imply

$$\lim_{n\to\infty} \|f(x) - f_n(x)\| = 0.$$

if and only if

$$\lim_{n\to\infty} \Delta a_n \log n = 0,$$

where α is non-integral number.

The proof is completed.

2.10 L^p-convergence of modified trigonometric sums $f_n(x)$ with coefficients from the class T

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

and

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{\infty} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx.$$

Firstly, there were run into in literature the following class of sequences.

Definition 2.16. *A sequence $\{a_n\}$ is said to be in the class $\mathbf{T}$, if:*

(i) $a_n \to 0$ as $n \to \infty$,
(ii) There exists a δ quasi-monotone sequence $\{A_n\}$, and the series $\sum_{n=1}^{\infty} n\delta_n$ and $\sum_{n=1}^{\infty} A_n$ converge, and
(iii) $|\Delta a_n| \le A_n$ ultimately.

Now we are going to prove first the following.

Theorem 2.17. *If $\{a_k\} \in \mathbf{T}$, then $S_n(x)$ converges point-wise to $f(x)$.*

Proof. Let $N' > N$. Then

$$|S_{N'}(x) - S_N(x)| = \left|\sum_{k=N}^{N'} a_k \cos kx\right|,$$

and by discrete summation by parts, we get

$$|S_{N'}(x) - S_N(x)| = \left|\sum_{k=N}^{N'-1} \Delta a_k D_k(x) - a_{N+1} D_N(x) + a_{N'} D_{N'}(x)\right|,$$

where $D_M(x) = \sum_{i=1}^{M} \sin ix$.

Using the inequality $|D_M(x)| = \mathcal{O}(x^{-1})$, for $\pi \ge x > 0$, then for arbitrary small $\varepsilon > 0$ and $N', N > N_0(\varepsilon)$, we have

$$|S_{N'}(x) - S_N(x)| \le \mathcal{O}\left[\sum_{k=N}^{N'-1} |\Delta a_k| + |a_{N+1}| + |a_{N'}|\right],$$

Since $\{a_k\} \in \mathbf{T}$, then it follows that

$$|S_{N'}(x) - S_N(x)| \le \varepsilon,$$

for arbitrary small $\varepsilon > 0$ and $N', N > N_0(\varepsilon)$.

Consequently,

$$f(x) = \lim_{N\to\infty} S_N(x)$$

exists for $(0, \pi]$.

The proof is completed.

Theorem 2.18. *Let* $\{a_k\} \in \mathbf{T}$. *Then* $f_n(x)$ *converges to* $f(x)$ *in* L^1*-norm.*

Proof. Applying Abel's transformation in the equality

$$f(x) = \lim_{N\to\infty} S_N(x),$$

we have

$$\begin{aligned} f(x) &= \lim_{N\to\infty} \left[\frac{a_0}{2} + \sum_{k=0}^{N-1} \Delta a_k D_k(x) + a_N D_N(x) - \frac{a_0}{2} \right] \\ &= \lim_{N\to\infty} \left[\sum_{k=0}^{N-1} \Delta a_k D_k(x) + a_N D_N(x) \right], \end{aligned}$$

since $D_0(x) = \frac{1}{2}$.

Hence, using $|D_N(x)| = \mathcal{O}(x^{-1})$ we obtain

$$f(x) = \sum_{k=0}^{\infty} \Delta a_k D_k(x).$$

The use of Abel's transformation also implies

$$f_n(x) = \sum_{k=0}^{n} \Delta a_k D_k(x),$$

and

$$f(x) - f_n(x) = \lim_{m\to\infty} \left[\sum_{k=n+1}^{m-1} \Delta A_k T_k(x) + A_n T_n(x) - A_{m+1} T_m(x), \right]$$

where $T_n(x) = \sum_{k=1}^{n} \frac{\Delta a_k}{A_k} D_k(x)$.

If we put $\alpha_k = \frac{\Delta a_k}{A_k}$, then for k big enough $|\alpha_k| \le 1$, and since $\{a_k\} \in \mathbf{T}$, we have

$$\int_0^\pi |f(x) - f_n(x)|dx \le \lim_{m\to\infty} \Bigg[\sum_{k=n+1}^{m-1} |\Delta A_k| \int_0^\pi |T_k(x)|dx + A_n \int_0^\pi |T_n(x)|dx + A_{m+1} \int_0^\pi |T_m(x)|dx. \Bigg]$$

So, applying Lemma 1.34, we get

$$\int_0^\pi |f(x) - f_n(x)|dx \le \lim_{m\to\infty} \Bigg[\sum_{k=n+1}^{m-1} (k+1)|\Delta A_k| + (n+1)A_n + (m+1)A_{m+1}. \Bigg]$$

Now, using Lemma 1.17, we obtain

$$\int_0^\pi |f(x) - f_n(x)|dx \le \sum_{k=n+1}^{\infty} (k+1)|\Delta A_k| + (n+1)A_n.$$

Subsequently, using Lemma 1.17 again and the hypothesis of the theorem we get

$$\lim_{n\to\infty} \int_0^\pi |f(x) - f_n(x)|dx = 0.$$

The proof is completed.

As a consequence of the above theorem we have next corollary.

Corollary 2.19. *Let* $\{a_k\} \in \mathbf{T}$. *Then* $S_n^c(x)$ *converges to* $f(x)$ *in* L^1*-norm if and only if* $\lim_{n\to\infty} a_n \log n = 0$.

$$\begin{aligned}\int_0^\pi |f(x) - S_n^c(x)|\,dx &= \int_0^\pi |f(x) - f_n(x) + f_n(x) - S_n^c(x)|\,dx \\ &\le \int_0^\pi |f(x) - f_n(x)|\,dx + \int_0^\pi |f_n(x) - S_n^c(x)|\,dx \\ &= \int_0^\pi |f(x) - f_n(x)|\,dx + \int_0^\pi |a_{n+1} D_n(x)|\,dx.\end{aligned}$$

Since by Theorem,

$$\int_0^\pi |f(x) - f_n(x)|\,dx = o(1) \text{ as } n \to \infty$$

and $\int_0^\pi |a_{n+1} D_n(x)|\,dx$ behaves like $|a_{n+1}| \log n$ for large values n, the conclusion the necessity part follows.

Conversely, assume that $S_n^c(x)$ converges to $f(x)$ in L^1-norm. Then,

$$\begin{aligned}|a_{n+1}|\log n &\sim \int_0^\pi |a_{n+1}D_n(x)|\,dx = \int_0^\pi |f_n(x) - S_n^c(x)|\,dx\\ &\le \int_0^\pi |f(x) - S_n^c(x)|\,dx + \int_0^\pi |f(x) - f_n(x)|\,dx = o(1),\end{aligned}$$

as $n \to \infty$.

The proof is completed.

Now we are going to prove some results on L^p-convergence.

Theorem 2.20. *Let $\{a_k\}$ be a sequence of bounded variation such that $a_n \to 0$, as $n \to \infty$. Then for $0 < p < 1$,*

$$\lim_{n\to\infty} \int_0^\pi |f(x) - f_n(x)|^p\,dx = 0.$$

Proof. In what we said in the proof of above theorem, we have

$$|f(x) - f_n(x)| = \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right|.$$

Based on $|D_N(x)| = \mathcal{O}(x^{-1})$ for $x \in (0, \pi]$ we get

$$|f(x) - f_n(x)| = \mathcal{O}\left(\frac{1}{x} \sum_{k=n+1}^{\infty} |\Delta a_k| \right).$$

Subsequently,

$$\int_0^\pi |f(x) - f_n(x)|^p dx = \mathcal{O}\left(\sum_{k=n+1}^{\infty} |\Delta a_k| \right)^p \int_0^\pi \frac{dx}{x^p} = 0,$$

since

$$\int_0^\pi \frac{dx}{x^p} < \infty,$$

for $0 < p < 1$ and $\{a_k\}$ is a sequence of bounded variation.

The proof is completed.

Now we consider the sums

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx.$$

Theorem 2.21. *Let $\{a_k\}$ be a sequence of bounded variation such that $a_n \to 0$, as $n \to \infty$. Then for $0 < p < \frac{1}{2}$ holds*

$$\lim_{n\to\infty} \int_0^\pi |g(x) - g_n(x)|^p\,dx = 0.$$

Proof. Discrete summation by parts gives

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} a_k + \sum_{k=1}^{n} a_k D_k(x) - \frac{1}{2}\sum_{k=0}^{n} a_k + a_n D_n(x)$$
$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k D_k(x),$$

where $D_k(x)$ is Dirichlet's kernel.

Applying the summation by parts once again, we get

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n-1}(k+1)\Delta a_k K_k(x) + (n+1)a_n K_n(x) - \frac{a_0}{2}$$
$$= \sum_{k=1}^{n-1}(k+1)\Delta a_k K_k(x) + (n+1)a_n K_n(x),$$

where $K_k(x)$ is Fejér's kernel.

Since,

$$K_k(x) = \mathcal{O}\left(\frac{1}{kx^2}\right),$$

for all $x \in (0,\pi]$, and $a_n \to \infty$, as $n \to \infty$, then we obtain

$$g(x) = \lim_{n\to\infty} g_n(x) = \sum_{k=1}^{\infty}(k+1)\Delta a_k K_k(x).$$

Whence,

$$g(x) - g_n(x) = \sum_{k=n}^{\infty}(k+1)\Delta a_k K_k(x) - (n+1)a_n K_n(x),$$

and

$$|g(x) - g_n(x)|dx \le \frac{C}{x^2}\left(\sum_{k=n}^{\infty}|\Delta a_k| + |a_n|\right).$$

Thus,

$$0 \le \lim_{n\to\infty}\int_0^{\pi}|g(x) - g_n(x)|^p dx \le C\int_0^{\pi}\frac{dx}{x^{2p}}\left(\sum_{k=n}^{\infty}|\Delta a_k| + |a_n|\right)^p \to 0,$$

as $n \to \infty$, since

$$\int_0^{\pi}\frac{dx}{x^{2p}} < \infty,$$

for $0 < p < \frac{1}{2}$ and $\{a_k\}$ is a sequence of bounded variation.

The proof is completed.

Corollary 2.22. *Let $\{a_k\}$ be a sequence of bounded variation such that $a_n \to 0$, as $n \to \infty$. Then $g \in L^p[0,\pi]$ for $0 < p < \frac{1}{2}$.*

Proof. We write

$$g(x) = g(x) - g_n(x) + g_n(x).$$

Using the well-known inequality

$$(a+b)^p \le 2^p(a^p + b^p),\ \ a \ge 0,\ \ b \ge 0,$$

we have

$$|g(x)|^p \le 2^p[|g(x) - g_n(x)|^p + |g_n(x)|^p].$$

Also, taking into account the equality

$$g_n(x) = \sum_{k=1}^{n-1}(k+1)\Delta a_k K_k(x) + (n+1)a_n K_n(x)$$

and

$$K_k(x) = \mathcal{O}\left(\frac{1}{kx^2}\right),$$

we have

$$|g_n(x)| \le \frac{C}{x^2}\sum_{k=n}^{\infty}|\Delta a_k| + \frac{C}{x^2}|a_n|.$$

Thus, we have

$$|g(x)|^p \le 2^p\left\{|g(x) - g_n(x)|^p + 2^p\left[\frac{C}{x^{2p}}\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^p + \frac{C}{x^{2p}}|a_n|^p\right]\right\}.$$

Hence,

$$\int_0^{\pi}|g(x)|^p dx \le 2^p\left\{\int_0^{\pi}|g(x) - g_n(x)|^p dx \right.$$
$$\left. +2^pC\left[\int_0^{\pi}\frac{dx}{x^{2p}}\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^p + \int_0^{\pi}\frac{dx}{x^{2p}}|a_n|^p\right]\right\}.$$

Based on the above theorem and assumptions of the corollary we obtain

$$\int_0^{\pi}|g(x)|^p dx \le C(p)\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^p < \infty,$$

where $C(p)$ is a positive constant depending only on p.

Subsequently, $g \in L^p[0,\pi]$ for $0 < p < \frac{1}{2}$.

The proof is completed.

2.11 L^p-convergence of modified trigonometric sums $w_n^c(x)$ and $w_n^s(x)$ with coefficients of bounded variation

We consider cosine series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

sine series

$$g(x) = \sum_{k=1}^{\infty} a_k \sin kx,$$

modified cosine sums

$$w_n^c(x) = \frac{1}{2}\left(a_1 + \sum_{k=0}^{n} \Delta^2 a_k\right) + \sum_{k=1}^{n}\left(a_{k+1} + \sum_{j=k}^{n} \Delta^2 a_j\right)\cos kx,$$

and modified sine sums

$$w_n^s(x) = \sum_{k=1}^{n}\left(a_{k+1} + \sum_{j=k}^{n} \Delta^2 a_j\right)\sin kx,$$

where $\Delta^2 a_i = a_i - 2a_{i+1} + a_{i+2}$.

We present here the following result.

Theorem 2.23. *Let $\{a_n\}$ be a sequence such that $a_n \to 0$, as $n \to \infty$ and $\sum_{n=1}^{\infty} |\Delta a_n| < \infty$. Then for any $p \in (0,1)$*

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |f(x) - w_n^c(x)|^p dx = 0,$$

and

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |g(x) - w_n^s(x)|^p dx = 0.$$

Proof. Firstly, we have

$$\begin{aligned} w_n^c(x) &= \frac{1}{2}\left(a_1 + \sum_{k=0}^{n} \Delta^2 a_k\right) + \sum_{k=1}^{n}\left(a_{k+1} + \sum_{j=k}^{n} \Delta^2 a_j\right)\cos kx \\ &= S_n^c(x) - \Delta a_{n+1} D_n(x), \end{aligned}$$

where $D_n(x) = \frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx$ represents the Dirichlet's kernel.

Using Abel's transformation, we get

$$\begin{aligned} w_n^c(x) &= \sum_{k=1}^{n-1} \Delta a_k D_k(x) + a_n D_n(x) - \Delta a_{n+1} D_n(x) \\ &= \sum_{k=1}^{n} \Delta a_k D_k(x) + a_{n+2} D_n(x). \end{aligned}$$

Therefore,

$$f(x) - w_n^c(x) = \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - a_{n+2} D_n(x), \quad x \neq 0.$$

Hence, taking into account that $D_n(x) = \mathcal{O}(x^{-1})$, we obtain

$$|f(x) - w_n^c(x)|^p = \mathcal{O}\left(\frac{1}{|x|}\right)^p \left(\sum_{k=n+1}^{\infty} |\Delta a_k| + |a_{n+2}|\right)^p,$$

and therefore

$$\int_{-\pi}^{\pi} |f(x) - w_n^c(x)|^p dx = \mathcal{O}_p\left(\int_{-\pi}^{\pi} \frac{dx}{|x|^p}\right) \left[\left(\sum_{k=n+1}^{\infty} |\Delta a_k|\right)^p + (|a_{n+2}|)^p\right] \to 0,$$

as $n \to \infty$, since the integral

$$\int_{-\pi}^{\pi} \frac{dx}{|x|^p}$$

is finite for $p \in (0,1)$.

So, we have proved that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |f(x) - w_n^c(x)|^p dx = 0.$$

Similar arguments can be used to prove the equality

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |g(x) - w_n^s(x)|^p dx = 0.$$

The proof is completed.

Corollary 2.24. *Let $\{a_n\}$ be a sequence such that $a_n \to 0$, as $n \to \infty$ and $\sum_{n=1}^{\infty} |\Delta a_n| < \infty$. Then for any $p \in (0,1)$*

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |f(x) - S_n^c(x)|^p dx = 0.$$

Proof. We can write

$$\begin{aligned}\int_{-\pi}^{\pi} &|f(x) - S_n^c(x)|^p dx \\ &\le \int_{-\pi}^{\pi} |f(x) - w_n^c(x)|^p dx + \lim_{n\to\infty} \int_{-\pi}^{\pi} |w_n^c(x) - S_n^c(x)|^p dx \\ &= \int_{-\pi}^{\pi} |f(x) - w_n^c(x)|^p dx + \int_{-\pi}^{\pi} |\Delta a_{n+1} D_n(x)|^p dx.\end{aligned}$$

The first term, on right hand side of last equality tends to zero as $n \to \infty$, by the above theorem, while the second term tend to zero as well, since

$$\begin{aligned}\int_{-\pi}^{\pi} |\Delta a_{n+1} D_n(x)|^p dx &\le \int_{-\pi}^{\pi} \left(\frac{2}{x}\right)^p |\Delta a_{n+1}|^p dx \\ &= 2^p |\Delta a_{n+1}|^p \int_{-\pi}^{\pi} x^{-p} dx \to 0,\end{aligned}$$

as $n \to \infty$, and $\int_{-\pi}^{\pi} x^{-p} dx$ is finite for $p \in (0,1)$.

The proof is completed.

Corollary 2.25. *Let $\{a_n\}$ be a sequence such that $a_n \to 0$, as $n \to \infty$ and $\sum_{n=1}^{\infty} |\Delta a_n| < \infty$. Then for any $p \in (0,1)$*

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |g(x) - S_n^s(x)|^p dx = 0.$$

Proof. The proof can be done using the same arguments.

Theorem 2.26. *Let $a_n \to 0$ as $n \to \infty$. Then (i) $f(x) = \lim_{n\to\infty} w_n^c(x)$ exists, and (ii) $f \in L^1(0, \pi]$.*

Proof. (i) We have

$$w_n^c(x) = \sum_{k=1}^{n} \Delta a_k D_k(x) + a_{n+2} D_n(x).$$

Applying the Abel's transformation, we obtain

$$w_n^c(x) = \sum_{k=1}^{n-1} (k+1) \Delta^2 a_k K_k(x) + (n+1) \Delta a_k K_n(x) + a_{n+2} D_n(x),$$

where $K_n(x)$ denotes the Fejér's kernel.

Since, $K_n(x) = \mathcal{O}(1/(nx^2))$ for $x \neq 0$, and $a_n \to 0$ as $n \to \infty$, the last two terms in the above equality tend to zero.

Also,

$$0 \le \sum_{k=1}^{n-1} (k+1) \Delta^2 a_k K_k(x) + (n+1) \Delta a_k K_n(x) \le (C/(nx^2))(a_0 - \Delta a_n),$$

so

$$\lim_{n\to\infty}\sum_{k=1}^{n-1}(k+1)\Delta^2 a_k K_k(x)$$

always exists for $x \neq 0$ and $a_n \to 0$ as $n \to \infty$, which proves the part (i).

(ii) We proved that

$$f(x) = \sum_{k=1}^{\infty}(k+1)\Delta^2 a_k K_k(x),\ \ x \neq 0.$$

Integrating term by term, we get

$$\int_{-\pi}^{\pi}|f(x)|dx = \frac{\pi}{2}\sum_{k=1}^{\infty}(k+1)\Delta^2 a_k = \frac{\pi}{2}a_0 < \infty.$$

The proof is completed.

2.12 L^1-convergence of modified trigonometric sums $z_n^c(x)$ and $z_n^s(x)$ with generalized semi-convex coefficients

We consider cosine series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

sine series

$$g(x) = \sum_{k=1}^{\infty} a_k \sin kx,$$

their parital sums

$$S_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx,$$

$$S_n^s(x) = \sum_{k=1}^{n} a_k \sin kx,$$

modified cosine sums

$$z_n^c(x) = \sum_{k=1}^{n}\left[\frac{a_{k+1}}{k+1} + \sum_{j=k}^{n}\Delta^2\left(\frac{a_j}{j}\right)\right]k\cos kx, \quad (a_0 = a_1 = a_2 = 0),$$

and modified sine sums

$$z_n^s(x) = \sum_{k=1}^{n}\left[\frac{b_{k+1}}{k+1} + \sum_{j=k}^{n}\Delta^2\left(\frac{b_j}{j}\right)\right]k\sin kx, \quad (b_1 = b_2 = 0),$$

where $\Delta^2 c_i := c_i - 2c_{i+1} + c_{i+2}$.

We prove the following result.

Theorem 2.27. *If* $\{a_k\}$ *is a generalized semi-convex sequence, then* $z_n^c(x)$ *converges to* $f(x)$ *in* L^1*-metric if and only if the condition*

$$\lim_{n\to\infty} a_n \log n = 0$$

holds true.

Proof. At first, we have

$$\begin{aligned} z_n^c(x) &= \sum_{k=1}^{n}\left[\frac{a_{k+1}}{k+1} + \sum_{j=k}^{n} \Delta^2\left(\frac{a_j}{j}\right)\right] k\cos kx \\ &= \sum_{k=1}^{n}\left(\frac{a_k}{k} - \frac{a_{n+1}}{n+1} + \frac{a_{n+2}}{n+2}\right) k\cos kx \\ &= \sum_{k=1}^{n} a_k \cos kx + \left(\frac{a_{n+2}}{n+2} - \frac{a_{n+1}}{n+1}\right)\widetilde{D}'_n(x), \end{aligned}$$

where $\widetilde{D}'_n(x)$ represents the first derivative of the conjugate Dirichlet kernel.

Applying the Abel's transformation we get

$$\begin{aligned} z_n^c(x) &= \sum_{k=1}^{n-1} \Delta a_k D_k(x) + a_n D_n(x) + \left(\frac{a_{n+2}}{n+2} - \frac{a_{n+1}}{n+1}\right)\widetilde{D}'_n(x) \\ &= \sum_{k=1}^{n-1}(a_{k-1} - a_{k+1})\frac{\sin kx}{2\sin x} + a_{n-1}\frac{\sin nx}{2\sin x} \\ &\quad + a_n\frac{\sin(n+1)x}{2\sin x} + \left(\frac{a_{n+2}}{n+2} - \frac{a_{n+1}}{n+1}\right)\widetilde{D}'_n(x), \end{aligned}$$

where $D_n(x) = \frac{\sin nx + \sin(n+1)x}{2\sin x}$.

Also, we have

$$\begin{aligned} z_n^c(x) &= \sum_{k=1}^{n}(\Delta a_{k-1} + \Delta a_k)\frac{\sin kx}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x} \\ &\quad + a_n\frac{\sin(n+1)x}{2\sin x} + \left(\frac{a_{n+2}}{n+2} - \frac{a_{n+1}}{n+1}\right)\widetilde{D}'_n(x), \end{aligned}$$

Using Abel's transformation again, we have

$$\begin{aligned} z_n^c(x) = &\frac{1}{2\sin x}\sum_{k=1}^{n-1}(\Delta^2 a_{k-1} + \Delta^2 a_k)\sum_{v=1}^{k}\sin vx \\ &+ \frac{1}{2\sin x}\left[(\Delta a_{n-1} + \Delta a_n)\sum_{v=1}^{n}\sin vx\right] \end{aligned}$$

$$
\begin{aligned}
&+a_{n+1}\frac{\sin nx}{2\sin x}+a_n\frac{\sin(n+1)x}{2\sin x}+\left(\frac{a_{n+2}}{n+2}-\frac{a_{n+1}}{n+1}\right)\widetilde{D}'_n(x)\\
=&\frac{1}{2\sin x}\sum_{k=1}^{n-1}(\Delta^2a_{k-1}+\Delta^2a_k)\left(\widetilde{S}_k^0(x)-\widetilde{S}_0(x)\right)\\
&+\frac{1}{2\sin x}\left[(\Delta a_{n-1}+\Delta a_n)\left(\widetilde{S}_n^0(x)-\widetilde{S}_0(x)\right)\right]\\
&+a_{n+1}\frac{\sin nx}{2\sin x}+a_n\frac{\sin(n+1)x}{2\sin x}+\left(\frac{a_{n+2}}{n+2}-\frac{a_{n+1}}{n+1}\right)\widetilde{D}'_n(x)\\
=&\frac{1}{2\sin x}\left[\sum_{k=1}^{n-1}(\Delta^2a_{k-1}+\Delta^2a_k)\widetilde{S}_k^0(x)-\sum_{k=1}^{n-1}(\Delta^2a_{k-1}+\Delta^2a_k)\widetilde{S}_0(x)\right]\\
&+\frac{1}{2\sin x}(\Delta a_{n-1}+\Delta a_n)\widetilde{S}_n^0(x)-\frac{1}{2\sin x}(\Delta a_{n-1}+\Delta a_n)\widetilde{S}_0(x)\\
&+a_{n+1}\frac{\sin nx}{2\sin x}+a_n\frac{\sin(n+1)x}{2\sin x}+\left(\frac{a_{n+2}}{n+2}-\frac{a_{n+1}}{n+1}\right)\widetilde{D}'_n(x)\\
=&\frac{1}{2\sin x}\left[\sum_{k=1}^{n-1}(\Delta^2a_{k-1}+\Delta^2a_k)\widetilde{S}_k^0(x)+(\Delta a_{n-1}+\Delta a_n)\widetilde{S}_n^0(x)\right]\\
&-\frac{1}{2\sin x}\sum_{k=1}^{n-1}(\Delta a_{k-1}-\Delta a_{k+1})\widetilde{S}_0(x)-\frac{1}{2\sin x}(a_{n-1}-a_{n+1})\widetilde{S}_0(x)\\
&+a_{n+1}\frac{\sin nx}{2\sin x}+a_n\frac{\sin(n+1)x}{2\sin x}+\left(\frac{a_{n+2}}{n+2}-\frac{a_{n+1}}{n+1}\right)\widetilde{D}'_n(x)\\
=&\frac{1}{2\sin x}\left[\sum_{k=1}^{n-1}(\Delta^2a_{k-1}+\Delta^2a_k)\widetilde{S}_k^0(x)+(\Delta a_{n-1}+\Delta a_n)\widetilde{S}_n^0(x)\right]\\
&+a_{n+1}\frac{\sin nx}{2\sin x}+a_n\frac{\sin(n+1)x}{2\sin x}+\left(\frac{a_{n+2}}{n+2}-\frac{a_{n+1}}{n+1}\right)\widetilde{D}'_n(x).
\end{aligned}
$$

Moreover, applying Abel's transformation α-times, the latest equality becomes

$$
\begin{aligned}
z_n^c(x)=&\frac{1}{2\sin x}\left[\sum_{k=1}^{n-\alpha}(\Delta^{\alpha+1}a_{k-1}+\Delta^{\alpha+1}a_k)\widetilde{S}_k^{\alpha-1}(x)\right.\\
&\left.+\sum_{k=1}^{\alpha}\Delta^k a_{n-k}\widetilde{S}_{n-k+1}^{k-1}(x)\right]+a_{n+1}\frac{\sin nx}{2\sin x}+a_n\frac{\sin(n+1)x}{2\sin x}\\
&+\sum_{k=1}^{\alpha}\Delta^k a_{n-k+1}\widetilde{S}_{n-k+1}^{k-1}(x)-\Delta\left(\frac{a_{n+1}}{n+1}\right)\widetilde{D}'_n(x).
\end{aligned}
$$

Since, $\widetilde{S}_k(x)$ and $\widetilde{T}_k(x)$ are uniformly bounded on every segment $[\varepsilon,\pi-\varepsilon]$, $\varepsilon>0$, and $\frac{\sin nx}{2\sin x}$ is bounded in $(0,\pi)$, then

$$f(x) = \lim_{n\to\infty} z_n^c(x) = \frac{1}{2\sin x}\sum_{k=1}^{\infty}(\Delta^{\alpha+1}a_{k-1} + \Delta^{\alpha+1}a_k)\widetilde{S}_k^{\alpha-1}(x)$$

exists in $(0,\pi)$ by given hypothesis.

Next, we consider

$$\begin{aligned}
f(x) - z_n^c(x) = \frac{1}{2\sin x}\Bigg[&\sum_{k=n-\alpha+1}^{\infty}(\Delta^{\alpha+1}a_{k-1} + \Delta^{\alpha+1}a_k)\widetilde{S}_k^{\alpha-1}(x)\\
&-\sum_{k=1}^{\alpha}\Delta^k a_{n-k}\widetilde{S}_{n-k+1}^{k-1}(x)\Bigg] - a_{n+1}\frac{\sin nx}{2\sin x} - a_n\frac{\sin(n+1)x}{2\sin x}\\
&-\sum_{k=1}^{\alpha}\Delta^k a_{n-k+1}\widetilde{S}_{n-k+1}^{k-1}(x) + \Delta\left(\frac{a_{n+1}}{n+1}\right)\widetilde{D}_n'(x).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|f(x) - z_n^c(x)\| \le\ & C\int_0^{\pi}\left|\sum_{k=n-\alpha+1}^{\infty}(\Delta^{\alpha+1}a_{k-1} + \Delta^{\alpha+1}a_k)\widetilde{S}_k^{\alpha-1}(x)\right|dx\\
&+C\int_0^{\pi}\left|\sum_{k=1}^{\alpha}\Delta^k a_{n-k}\widetilde{S}_{n-k+1}^{k-1}(x)\right|dx\\
&+|a_{n+1}|\int_0^{\pi}\left|\frac{\sin nx}{2\sin x}\right|dx + |a_n|\int_0^{\pi}\left|\frac{\sin(n+1)x}{2\sin x}\right|dx\\
&+C\int_0^{\pi}\left|\sum_{k=1}^{\alpha}\Delta^k a_{n-k+1}\widetilde{S}_{n-k+1}^{k-1}(x)\right|dx\\
&+\left|\Delta\left(\frac{a_{n+1}}{n+1}\right)\right|\int_0^{\pi}\left|\widetilde{D}_n'(x)\right|dx\\
\le\ & C\sum_{k=n-\alpha+1}^{\infty}|\Delta^{\alpha+1}a_{k-1} + \Delta^{\alpha+1}a_k|\int_0^{\pi}\left|\widetilde{S}_k^{\alpha-1}(x)\right|dx\\
&+C\sum_{k=1}^{\alpha}|\Delta^k a_{n-k}|\int_0^{\pi}\left|\widetilde{S}_{n-k+1}^{k-1}(x)\right|dx\\
&+|a_{n+1}|\int_0^{\pi}\left|\frac{\sin nx}{2\sin x}\right|dx + |a_n|\int_0^{\pi}\left|\frac{\sin(n+1)x}{2\sin x}\right|dx\\
&+C\sum_{k=1}^{\alpha}|\Delta^k a_{n-k+1}|\int_0^{\pi}\left|\widetilde{S}_{n-k+1}^{k-1}(x)\right|dx\\
&+\left|\Delta\left(\frac{a_{n+1}}{n+1}\right)\right|\int_0^{\pi}\left|\widetilde{D}_n'(x)\right|dx\\
=\ & C\sum_{k=n-\alpha+1}^{\infty}A_k^{\alpha}|\Delta^{\alpha+1}a_{k-1} + \Delta^{\alpha+1}a_k|\int_0^{\pi}\left|\widetilde{T}_k^{\alpha-1}(x)\right|dx
\end{aligned}$$

$$+C\sum_{k=1}^{\alpha} A_{n-k+1}^{k}|\Delta^k a_{n-k}| \int_0^\pi \left|\widetilde{T}_{n-k+1}^{k-1}(x)\right| dx$$
$$+|a_{n+1}| \int_0^\pi \left|\frac{\sin nx}{2\sin x}\right| dx + |a_n| \int_0^\pi \left|\frac{\sin(n+1)x}{2\sin x}\right| dx$$
$$+C\sum_{k=1}^{\alpha} A_{n-k+1}^{k}|\Delta^k a_{n-k+1}| \int_0^\pi \left|\widetilde{T}_{n-k+1}^{k-1}(x)\right| dx$$
$$+\left|\Delta\left(\frac{a_{n+1}}{n+1}\right)\right| \int_0^\pi \left|\widetilde{D}_n'(x)\right| dx.$$

By Lemma 1.90 and given hypothesis, the first three terms of the above inequality are of $o(1)$ order as $n \to \infty$. Further, it is a well-known that

$$\int_0^\pi \left|\widetilde{D}_n'(x)\right| dx = o\left(n\log n\right) \quad \textit{and} \quad \int_0^\pi \left|\frac{\sin nx}{2\sin x}\right| dx = O\left(\log n\right), \ n > 2.$$

Thus, the conclusion of the statement holds true if and only if

$$\lim_{n\to\infty} a_n \log n = 0.$$

The proof is completed.

Corollary 2.28. *If $\{a_k\}$ is a generalized semi-convex sequence, then the necessary and sufficient condition for L^1-convergence of the cosine series is the condition*

$$\lim_{n\to\infty} a_n \log n = 0.$$

Proof. It is obvious that

$$\begin{aligned}
\|f(x) - S_n^c(x)\| &= \|f(x) - z_n^c(x) + z_n^c(x) - S_n^c(x)\| \\
&\le \int_0^\pi |f(x) - z_n^c(x)|dx + \int_0^\pi |z_n^c(x) - S_n^c(x)|dx \\
&= \int_0^\pi |f(x) - z_n^c(x)|dx + \int_0^\pi \left|\left(\frac{a_{n+2}}{n+2} - \frac{a_{n+1}}{n+1}\right)\widetilde{D}_n'(x)\right| dx \\
&\le \int_0^\pi |f(x) - z_n^c(x)|dx + \left(\left|\frac{a_{n+2}}{n+2}\right| + \left|\frac{a_{n+1}}{n+1}\right|\right)\left|\int_0^\pi \left|\widetilde{D}_n'(x)\right| dx.\right.
\end{aligned}$$

By Theorem 2.27

$$\lim_{n\to\infty} \int_0^\pi |f(x) - z_n^c(x)|dx = 0,$$

holds true, and while $\int_0^\pi \left|\widetilde{D}_n'(x)\right| dx$ behaves like $n\log n$ for large values of n, then by given hypothesis we conclude that

$$\lim_{n\to\infty} \|f(x) - S_n^c(x)\| = 0 \Longleftrightarrow \lim_{n\to\infty} a_n \log n = 0.$$

The proof is completed.

Pertaining to the sine series we prove the following.

Theorem 2.29. *If $\{a_k\}$ is a generalized semi-convex sequence, then $z_n^s(x)$ converges to $g(x)$ in L^1-metric if and only if the condition*

$$\lim_{n\to\infty} a_n \log n = 0$$

holds true.

Proof. Modified sine sums $z_n^s(x)$ can be written as follows

$$\begin{aligned} z_n^s(x) &= \sum_{k=1}^{n}\left[\frac{b_{k+1}}{k+1} + \sum_{j=k}^{n} \Delta^2\left(\frac{b_j}{j}\right)\right] k\sin kx \\ &= \sum_{k=1}^{n}\left(\frac{b_k}{k} - \frac{b_{n+1}}{n+1} + \frac{b_{n+2}}{n+2}\right) k\sin kx \\ &= \sum_{k=1}^{n} b_k \sin kx + \left(\frac{b_{n+2}}{n+2} - \frac{b_{n+1}}{n+1}\right) D_n'(x), \end{aligned}$$

where $D_n'(x)$ represents the first derivative of the Dirichlet kernel.

Abel's transformation gives

$$\begin{aligned} z_n^s(x) &= \sum_{k=1}^{n} \Delta b_k \overline{D}_k(x) + b_n \overline{D}_n(x) + \left(\frac{b_{n+2}}{n+2} - \frac{b_{n+1}}{n+1}\right) D_n'(x) \\ &= \sum_{k=1}^{n-1} \Delta b_k \left[\frac{1+\cos x - \cos kx - \cos(k+1)x}{2\sin x}\right] \\ &\quad + b_n \overline{D}_n(x) + \left(\frac{b_{n+1}}{n+1} - \frac{b_{n+2}}{n+2}\right) D_n'(x), \end{aligned}$$

where $\overline{D}_n(x) = \frac{1+\cos x - \cos nx - \cos(n+1)x}{2\sin x}$.

Last equality can be transformed as follows

$$\begin{aligned} z_n^s(x) &= \sum_{k=1}^{n-1} (b_{k-1} - b_{k+1}) \left[\frac{1+\cos x - \cos kx - \cos(k+1)x}{2\sin x}\right] \\ &\quad + b_n \overline{D}_n(x) + \left(\frac{b_{n+1}}{n+1} - \frac{b_{n+2}}{n+2}\right) D_n'(x) \\ &\quad + \sum_{k=1}^{n-1} (b_k - b_{k-1}) \left[\frac{1+\cos x - \cos kx - \cos(k+1)x}{2\sin x}\right] \\ &= \sum_{k=1}^{n-1} (b_{k-1} - b_{k+1}) \frac{1+\cos x}{2\sin x} - \sum_{k=1}^{n-1} (b_{k-1} - b_{k+1}) \frac{\cos kx}{2\sin x} \end{aligned}$$

$$
\begin{aligned}
&-\sum_{k=1}^{n-1}(b_{k-1}-b_{k+1})\frac{\cos(k+1)x}{2\sin x}+\sum_{k=1}^{n-1}(b_k-b_{k-1})\frac{1+\cos x}{2\sin x}\\
&-\sum_{k=1}^{n-1}(b_k-b_{k-1})\frac{\cos kx}{2\sin x}-\sum_{k=1}^{n-1}(b_k-b_{k-1})\frac{\cos(k+1)x}{2\sin x}\\
&+b_n\overline{D}_n(x)+\left(\frac{b_{n+1}}{n+1}-\frac{b_{n+2}}{n+2}\right)D_n'(x)
\end{aligned}
$$

or

$$
\begin{aligned}
z_n^s(x)=&-\sum_{k=1}^{n-1}(b_{k-1}-b_{k+1})\frac{\cos kx}{2\sin x}+(b_1-b_n)\frac{1+\cos x}{2\sin x}\\
&+b_n\frac{\cos nx}{2\sin x}-b_1\frac{\cos x}{2\sin x}+b_0\frac{\cos x}{2\sin x}-b_{n-1}\frac{\cos nx}{2\sin x}\\
&+b_n\frac{1+\cos x}{2\sin x}-b_n\frac{\cos nx}{2\sin x}-b_n\frac{\cos(n+1)x}{2\sin x}\\
&+b_n\overline{D}_n(x)+\left(\frac{b_{n+1}}{n+1}-\frac{b_{n+2}}{n+2}\right)D_n'(x)\\
=&-\sum_{k=1}^{n}(b_{k-1}-b_{k+1})\frac{\cos kx}{2\sin x}-b_{n+1}\frac{\cos nx}{2\sin x}\\
&-b_n\frac{\cos(n+1)x}{2\sin x}+\left(\frac{b_{n+1}}{n+1}-\frac{b_{n+2}}{n+2}\right)D_n'(x),\quad b_0=b_1=0\\
=&-\frac{1}{2\sin x}\sum_{k=1}^{n}(\Delta b_{k-1}+\Delta b_k)\cos kx-b_{n+1}\frac{\cos nx}{2\sin x}\\
&-b_n\frac{\cos(n+1)x}{2\sin x}+\left(\frac{b_{n+1}}{n+1}-\frac{b_{n+2}}{n+2}\right)D_n'(x).
\end{aligned}
$$

Using again Abel's transformation in last equality we have

$$
\begin{aligned}
z_n^s(x)=&-\frac{1}{2\sin x}\sum_{k=1}^{n-1}(\Delta^2 b_{k-1}+\Delta^2 b_k)\sum_{v=1}^{k}\cos vx\\
&-\frac{1}{2\sin x}(\Delta b_{n-1}+\Delta b_n)\sum_{v=1}^{n}\cos vx\\
&-b_{n+1}\frac{\cos nx}{2\sin x}-b_n\frac{\cos(n+1)x}{2\sin x}+\left(\frac{b_{n+1}}{n+1}-\frac{b_{n+2}}{n+2}\right)D_n'(x)\\
=&-\frac{1}{2\sin x}\sum_{k=1}^{n-1}(\Delta^2 b_{k-1}+\Delta^2 b_k)\left(S_k^0(x)-S_0(x)\right)\\
&-\frac{1}{2\sin x}(\Delta b_{n-1}+\Delta b_n)\left(S_n^0(x)-S_0(x)\right)\\
&-b_{n+1}\frac{\cos nx}{2\sin x}-b_n\frac{\cos(n+1)x}{2\sin x}+\left(\frac{b_{n+1}}{n+1}-\frac{b_{n+2}}{n+2}\right)D_n'(x)
\end{aligned}
$$

$$= -\frac{1}{2\sin x}\left[\sum_{k=1}^{n-1}(\Delta^2 b_{k-1} + \Delta^2 b_k)S_k^0(x) + (\Delta b_{n-1} + \Delta b_n)S_n^0(x)\right]$$
$$-b_{n+1}\frac{\cos nx}{2\sin x} - b_n\frac{\cos(n+1)x}{2\sin x} + \left(\frac{b_{n+1}}{n+1} - \frac{b_{n+2}}{n+2}\right)D'_n(x).$$

Repeating Abel's transformation α times to the last equality we obtain

$$z_n^s(x) = -\frac{1}{2\sin x}\left[\sum_{k=1}^{n-\alpha}(\Delta^{\alpha+1}b_{k-1} + \Delta^{\alpha+1}b_k)S_k^{\alpha-1}(x)\right.$$
$$\left.+\sum_{k=1}^{\alpha}\Delta^k b_{n-k}S_{n-k+1}^{k-1}(x) + \sum_{k=1}^{\alpha}\Delta^k b_{n-k+1}S_{n-k+1}^{k-1}(x)\right]$$
$$-b_{n+1}\frac{\cos nx}{2\sin x} - b_n\frac{\cos(n+1)x}{2\sin x} + \left(\frac{b_{n+1}}{n+1} - \frac{b_{n+2}}{n+2}\right)D'_n(x).$$

Since $S_n(x)$ and $T_n(x)$ are uniformly bounded on every segment $[\varepsilon, \pi-\varepsilon]$, $\varepsilon > 0$, then based on the given hypothesis we conclude that

$$g(x) = \lim_{n\to\infty} z_n^s(x) = -\frac{1}{2\sin x}\sum_{k=1}^{\infty}(\Delta^{\alpha+1}b_{k-1} + \Delta^{\alpha+1}b_k)S_k^{\alpha-1}(x)$$

exists in $[\varepsilon, \pi-\varepsilon]$, $\varepsilon > 0$.

Next, we have

$$g(x) - z_n^s(x) = -\frac{1}{2\sin x}\sum_{k=n-\alpha+1}^{\infty}(\Delta^{\alpha+1}b_{k-1} + \Delta^{\alpha+1}b_k)S_k^{\alpha-1}(x)$$
$$+\frac{1}{2\sin x}\sum_{k=1}^{\alpha}\Delta^k b_{n-k}S_{n-k+1}^{k-1}(x)$$
$$+\frac{1}{2\sin x}\sum_{k=1}^{\alpha}\Delta^k b_{n-k+1}S_{n-k+1}^{k-1}(x)$$
$$+b_{n+1}\frac{\cos nx}{2\sin x} + b_n\frac{\cos(n+1)x}{2\sin x} - \left(\frac{b_{n+1}}{n+1} - \frac{b_{n+2}}{n+2}\right)D'_n(x).$$

Subsequently, we get

$$\|g(x) - z_n^s(x)\| \le C\int_0^\pi \left|\sum_{k=n-\alpha+1}^{\infty}(\Delta^{\alpha+1}b_{k-1} + \Delta^{\alpha+1}b_k)S_k^{\alpha-1}(x)\right|dx$$
$$+C\int_0^\pi\left|\sum_{k=1}^{\alpha}\Delta^k b_{n-k}S_{n-k+1}^{k-1}(x)\right|dx$$
$$+C\int_0^\pi\left|\sum_{k=1}^{\alpha}\Delta^k b_{n-k+1}S_{n-k+1}^{k-1}(x)\right|dx$$

$$
\begin{aligned}
&+|b_{n+1}|\int_0^\pi \left|\frac{\cos nx}{2\sin x}\right| dx + |b_n|\int_0^\pi \left|\frac{\cos(n+1)x}{2\sin x}\right| dx \\
&+\left|\frac{b_{n+1}}{n+1}-\frac{b_{n+2}}{n+2}\right|\int_0^\pi |D_n'(x)|\, dx \\
\le\, & C\sum_{k=n-\alpha+1}^{\infty} |\Delta^{\alpha+1}b_{k-1}+\Delta^{\alpha+1}b_k| \int_0^\pi \left|S_k^{\alpha-1}(x)\right| dx \\
&+C\sum_{k=1}^{\alpha} |\Delta^k b_{n-k}| \int_0^\pi \left|S_{n-k+1}^{k-1}(x)\right| dx \\
&+C\sum_{k=1}^{\alpha} |\Delta^k b_{n-k+1}| \int_0^\pi \left|S_{n-k+1}^{k-1}(x)\right| dx \\
&+|b_{n+1}|\int_0^\pi \left|\frac{\cos nx}{2\sin x}\right| dx + |b_n|\int_0^\pi \left|\frac{\cos(n+1)x}{2\sin x}\right| dx \\
&+\left|\frac{b_{n+1}}{n+1}-\frac{b_{n+2}}{n+2}\right|\int_0^\pi |D_n'(x)|\, dx \\
=\, & C\sum_{k=n-\alpha+1}^{\infty} A_k^\alpha|\Delta^{\alpha+1}b_{k-1}+\Delta^{\alpha+1}b_k| \int_0^\pi \left|T_k^{\alpha-1}(x)\right| dx \\
&+C\sum_{k=1}^{\alpha} A_{n-k+1}^k|\Delta^k b_{n-k}| \int_0^\pi \left|T_{n-k+1}^{k-1}(x)\right| dx \\
&+C\sum_{k=1}^{\alpha} A_{n-k+1}^k|\Delta^k b_{n-k+1}| \int_0^\pi \left|T_{n-k+1}^{k-1}(x)\right| dx \\
&+|b_{n+1}|\int_0^\pi \left|\frac{\cos nx}{2\sin x}\right| dx + |b_n|\int_0^\pi \left|\frac{\cos(n+1)x}{2\sin x}\right| dx \\
&+\left|\frac{b_{n+1}}{n+1}-\frac{b_{n+2}}{n+2}\right|\int_0^\pi |D_n'(x)|\, dx.
\end{aligned}
$$

As we know $\int_0^\pi |D_n'(x)|\, dx = o(n\log n)$ and $\int_0^\pi \left|\frac{\cos nx}{2\sin x}\right| dx \sim O(\log n)$, $n > 2$. So by given hypothesis and Lemma 1.48, all terms that appear in the right hand side of the above inequality are of order $o(1)$ as $n \to \infty$.

Subsequently, it follows that

$$\lim_{n\to\infty} \|g(x) - z_n^s(x)\| = 0$$

if and only if the condition

$$\lim_{n\to\infty} a_n \log n = 0.$$

The proof is completed.

Corollary 2.30. *If $\{b_k\}$ is a generalized semi-convex sequence, then the necessary and sufficient condition for L^1-convergence of the sine series is the condition*

$$\lim_{n\to\infty} b_n \log n = 0.$$

Proof. We can see that

$$\begin{aligned}
\|g(x) - S_n^s(x)\| &= \|g(x) - z_n^s(x) + z_n^s(x) - S_n^s(x)\| \\
&\leq \int_0^\pi |g(x) - z_n^s(x)|dx + \int_0^\pi |z_n^s(x) - S_n^s(x)|dx \\
&= \int_0^\pi |g(x) - z_n^s(x)|dx + \int_0^\pi \left|\left(\frac{b_{n+2}}{n+2} - \frac{b_{n+1}}{n+1}\right) D_n'(x)\right| dx \\
&\leq \int_0^\pi |g(x) - z_n^s(x)|dx + \left(\left|\frac{b_{n+2}}{n+2}\right| + \left|\frac{b_{n+1}}{n+1}\right|\right)\bigg| \int_0^\pi |D_n'(x)|\, dx.
\end{aligned}$$

By Theorem 2.29

$$\lim_{n\to\infty} \int_0^\pi |g(x) - z_n^s(x)|dx = 0,$$

holds true, and while $\int_0^\pi |D_n'(x)|\, dx$ behaves like $n \log n$ for large values of n, then by given hypothesis we verify that

$$\lim_{n\to\infty} \|g(x) - S_n^c(x)\| = 0 \Longleftrightarrow \lim_{n\to\infty} b_n \log n = 0.$$

The proof is completed.

3

L^1-convergence of modified sums $g_n^c(x)$ and $g_n^s(x)$

In this section we are going to present all results regrading to L^1-convergence of modified trigonometric sums $g_n^c(x)$ and $g_n^s(x)$ whose coefficients belong to several classes of real sequences.

3.1 L^1-convergence of modified trigonometric sums $g_n^c(x)$ with coefficients from the class S

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

be a cosine series and

$$g_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left(\frac{a_j}{j} \right) k \cos kx$$

the modified trigonometric cosine sums.

In what follows we prove the following.

Theorem 3.1. *Let* $\{a_k\} \in \mathbf{S}$. *If*

$$\lim_{n\to\infty} |a_{n+1}| \log n = 0,$$

then $g_n^c(x)$ *converges to* $f(x)$ *in* L^1*-norm.*

Proof. We have

$$g_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left(\frac{a_j}{j} \right) k \cos kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k\cos kx \left[\Delta\left(\frac{a_k}{k}\right) + \Delta\left(\frac{a_{k+1}}{k+1}\right) + \cdots + \Delta\left(\frac{a_n}{n}\right)\right]$$
$$= \frac{a_0}{2} + \sum_{k=1}^{n} k\cos kx \left(\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right)$$
$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos kx - \frac{a_{n+1}}{n+1}\sum_{k=1}^{n} k\cos kx$$
$$= S_n(x) - \frac{a_{n+1}}{n+1}\widetilde{D}_n'(x),$$

where $\widetilde{D}_n'(x)$ denotes the first derivative of the conjugate Dirichlet kernel.

Since $\{a_k\}$ is a null sequence and $\widetilde{D}_n'(x) = \mathcal{O}(n)$ for $x \in (0,\pi]$, then

$$\lim_{n\to\infty} g_n^c(x) = \lim_{n\to\infty} S_n(x) = f(x), \quad \forall x \in (0,\pi].$$

Applying Abel's transformation and Lemma 1.34, we obtain

$$\int_0^\pi |f(x) - g_n^c(x)|dx$$
$$\leq \int_0^\pi \left|\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)\right| dx + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}_n'(x)\right| dx$$
$$= \int_0^\pi \left|\sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x)\right| dx + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}_n'(x)\right| dx$$
$$= \int_0^\pi \left|\sum_{k=n+1}^{\infty} \Delta A_k \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x)\right| dx + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}_n'(x)\right| dx$$
$$\leq \sum_{k=n+1}^{\infty} \Delta A_k \int_0^\pi \left|\sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x)\right| dx + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}_n'(x)\right| dx$$
$$\leq C \sum_{k=n+1}^{\infty} (k+1)\Delta A_k + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}_n'(x)\right| dx. \tag{3.1}$$

Under the assumed hypothesis, the series

$$\sum_{k=1}^{\infty} (k+1)\Delta A_k$$

converges and therefore the first term in (3.1) tends to zero as $n \to \infty$.

Moreover,

$$\int_0^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}_n'(x)\right| dx \leq \int_{-\pi}^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}_n'(x)\right| dx = \frac{|a_{n+1}|}{n+1}\int_{-\pi}^\pi \left|\widetilde{D}_n'(x)\right| dx$$
$$= C|a_{n+1}|\int_{-\pi}^\pi \left|\widetilde{D}_n'(x)\right| dx \sim |a_{n+1}|\log n, \tag{3.2}$$

since $\int_{-\pi}^{\pi}\left|\widetilde{D}_n'(x)\right|dx$ behaves like $\log n$ for large values of n.

Now the conclusion of the theorem follows from (3.1) and (3.2).

The proof is completed.

In the sequel we prove the following.

Corollary 3.2. *Let* $\{a_k\} \in \mathbf{S}$ *and*

$$\lim_{n\to\infty} a_{n+1}\log n = 0.$$

Then $\|f - S_n\|_{L^1} = o(1)$ *as* $n \to \infty$.

Proof. We note that

$$\begin{aligned}\int_{-\pi}^{\pi} |f(x) - S_n(x)|\,dx &= \int_{-\pi}^{\pi} |f(x) - g_n^c(x) + g_n^c(x) - S_n(x)|\,dx \\ &\le \int_{-\pi}^{\pi} |f(x) - g_n^c(x)|\,dx + \int_{-\pi}^{\pi} |g_n^c(x) - S_n(x)|\,dx \\ &= \int_{-\pi}^{\pi} |f(x) - g_n^c(x)|\,dx + \int_{-\pi}^{\pi} \left|\frac{a_{n+1}}{n+1}\widetilde{D}_n'\right|dx.\end{aligned}$$

Since by Theorem 3.1,

$$\int_{-\pi}^{\pi} |f(x) - g_n^c(x)|\,dx = o(1) \ \text{as} \ n \to \infty$$

and

$$\int_{-\pi}^{\pi} \left|\frac{a_{n+1}}{n+1}\widetilde{D}_n'\right|dx$$

behaves like $|a_{n+1}|\log n$ for large values n, the conclusion of the corollary follows.

The proof is completed.

3.2 L^1-convergence of modified trigonometric sums $g_n^c(x)$ and $g_n^s(x)$ with coefficients from the class R

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

and

$$g(x) = \sum_{k=1}^{\infty} a_k \sin kx$$

be cosine and sine series.

Also we consider the modified trigonometric cosine and sine sums

$$g_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k\cos kx$$

and

$$g_n^s(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k\sin kx.$$

In what follows we denote by $t(x)$ either $f(x)$ or $g(x)$, and $t_n(x)$ either $g_n^c(x)$ or $g_n^s(x)$.

Theorem 3.3. *Let* $\{a_k\} \in \mathbf{R}$. *Then*

$$\lim_{n\to\infty} t_n(x) = t(x), \quad \forall x \in (0,\pi],$$

and $t_n(x)$ *converges to* $t(x)$ *in* L^1*-norm.*

Proof. We consider only cosine sums $g_n^c(x)$ since the proof for the sine sums follows in the same line. We have

$$\begin{aligned} t_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} k\cos kx \left[\Delta\left(\frac{a_k}{k}\right) + \Delta\left(\frac{a_{k+1}}{k+1}\right) + \cdots + \Delta\left(\frac{a_n}{n}\right)\right] \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} k\cos kx \left(\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right) \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos kx - \frac{a_{n+1}}{n+1}\sum_{k=1}^{n} k\cos kx \\ &= S_n(x) - \frac{a_{n+1}}{n+1}\widetilde{D}_n'(x), \end{aligned} \tag{3.3}$$

where $\widetilde{D}_n'(x)$ denotes the first derivative of the conjugate Dirichlet kernel.

Since $\{a_k\}$ is a null sequence and $\widetilde{D}_n'(x) = \mathcal{O}(n)$ for $x \in (0,\pi]$, then

$$\lim_{n\to\infty} t_n(x) = \lim_{n\to\infty} S_n(x) = t(x), \quad \forall x \in (0,\pi].$$

The relation (3.3) gives

$$\begin{aligned} t(x) - t_n(x) &= \sum_{k=n+1}^{\infty} a_k\cos kx + \frac{a_{n+1}}{n+1}\widetilde{D}_n'(x) \\ &= \lim_{m\to\infty}\frac{d}{dx}\left(\sum_{k=n+1}^{m}\frac{a_k}{k}\sin kx\right) + \frac{a_{n+1}}{n+1}\widetilde{D}_n'(x). \end{aligned}$$

Applying Abel's transformation twice, we obtain

$$
\begin{aligned}
t(x) - t_n(x) &= \lim_{m\to\infty}\Bigg[\sum_{k=n+1}^{m-1} \Delta\left(\frac{a_k}{k}\right)\widetilde{D}'_k(x) + \frac{a_m}{m}\widetilde{D}'_m(x) \\
&\quad - \frac{a_{n+1}}{n+1}\widetilde{D}'_n(x)\Bigg] + \frac{a_{n+1}}{n+1}\widetilde{D}'_n(x) \\
&= \lim_{m\to\infty}\Bigg[\frac{a_m}{m}\widetilde{D}'_m(x) + \sum_{k=n+1}^{m-2}(k+1)\Delta^2\left(\frac{a_k}{k}\right)\widetilde{K}'_k(x) \\
&\quad + m\Delta\left(\frac{a_{m-1}}{m-1}\right)\widetilde{K}'_{m-1}(x) - (n+1)\Delta\left(\frac{a_{n+1}}{n+1}\right)\widetilde{K}'_n(x) \\
&= \sum_{k=n+1}^{\infty}(k+1)\Delta^2\left(\frac{a_k}{k}\right)\widetilde{K}'_k(x) - (n+1)\Delta\left(\frac{a_{n+1}}{n+1}\right)\widetilde{K}'_n(x),
\end{aligned}
$$

where $\widetilde{K}'_n(x)$ denotes the first derivative of the conjugate Fejér kernel.

Hence,

$$
\begin{aligned}
\int_{-\pi}^{\pi} |t(x) - t_n(x)| dx &\leq \sum_{k=n+1}^{\infty}(k+1)\left|\Delta^2\left(\frac{a_k}{k}\right)\right|\int_{-\pi}^{\pi}\left|\widetilde{K}'_k(x)\right| dx \\
&\quad + (n+1)\left|\Delta\left(\frac{a_{n+1}}{n+1}\right)\right|\int_{-\pi}^{\pi}\left|\widetilde{K}'_n(x)\right| dx
\end{aligned}
$$

However, using the inequality

$$
\int_{-\pi}^{\pi}\left|\widetilde{K}'_k(x)\right| dx = \mathcal{O}(k),
$$

we have

$$
\begin{aligned}
\left|\Delta\left(\frac{a_{n+1}}{n+1}\right)\right| &= \left|\sum_{k=n+1}^{\infty}\Delta^2\left(\frac{a_k}{k}\right)\right| \\
&\leq \sum_{k=n+1}^{\infty}\frac{k^2}{k^2}\left|\Delta^2\left(\frac{a_k}{k}\right)\right| \\
&\leq \frac{1}{(n+1)^2}\sum_{k=n+1}^{\infty}k^2\left|\Delta^2\left(\frac{a_k}{k}\right)\right| \\
&= o\left(\frac{1}{(n+1)^2}\right)
\end{aligned}
$$

by given hypothesis.

Thus, it follows that

$$
\int_{-\pi}^{\pi}|t(x) - t_n(x)|\, dx = \mathcal{O}\left(\sum_{k=n+1}^{\infty}(k+1)^2\left|\Delta^2\left(\frac{a_k}{k}\right)\right|\right) + o(1) = o(1),
$$

as $n \to \infty$.

The proof is completed.

Now we prove the following.

Corollary 3.4. *Let* $\{a_k\} \in \mathbf{R}$. *The series*

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

converges in L^1*-norm if and only if*

$$\lim_{n\to\infty} a_{n+1} \log n = 0.$$

Proof. We shall prove the corollary only for cosine series, since the proof for the sine series is very similar. Indeed, we have

$$\begin{aligned}
\int_{-\pi}^{\pi} |t(x) - S_n(x)|\, dx &= \int_{-\pi}^{\pi} |t(x) - t_n(x) + t_n(x) - S_n(x)|\, dx \\
&\le \int_{-\pi}^{\pi} |t(x) - t_n(x)|\, dx + \int_{-\pi}^{\pi} |t_n(x) - S_n(x)|\, dx \\
&= \int_{-\pi}^{\pi} |t(x) - t_n(x)|\, dx + \int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \widetilde{D}_n' \right| dx
\end{aligned}$$

and

$$\begin{aligned}
\int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \widetilde{D}_n'(x) \right| dx &= \int_{-\pi}^{\pi} |t_n(x) - S_n(x)|\, dx \\
&\le \int_{-\pi}^{\pi} |t(x) - S_n(x)|\, dx + \int_{-\pi}^{\pi} |t(x) - t_n(x)|\, dx.
\end{aligned}$$

Since by Theorem 3.3,

$$\int_{-\pi}^{\pi} |t(x) - t_n(x)|\, dx = o(1) \textit{ as } n \to \infty$$

and

$$\int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \widetilde{D}_n'(x) \right| dx$$

behaves like $|a_{n+1}| \log n$ for large values n, the conclusion of the statement follows.

The proof is completed.

3.3 L^1-convergence of modified trigonometric sums $g_n^c(x)$ with coefficients from the class $\mathrm{S}(\delta)$

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

be a cosine series and

$$g_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

the modified trigonometric cosine sums.

In what follows we prove the following.

Theorem 3.5. *Let* $\{a_k\} \in \mathbf{S}(\delta)$*. If*

$$\lim_{n\to\infty} |a_{n+1}| \log n = 0,$$

then $g_n^c(x)$ *converges to* $f(x)$ *in* L^1*-norm.*

Proof. We have

$$\begin{aligned}
g_n^c(x) &= \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left[\Delta\left(\frac{a_k}{k}\right) + \Delta\left(\frac{a_{k+1}}{k+1}\right) + \cdots + \Delta\left(\frac{a_n}{n}\right)\right] \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left(\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right) \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1}\sum_{k=1}^{n} k \cos kx \\
&= S_n(x) - \frac{a_{n+1}}{n+1}\widetilde{D}'_n(x),
\end{aligned}$$

where $\widetilde{D}'_n(x)$ denotes the first derivative of the conjugate Dirichlet kernel.

Since $\{a_k\}$ is a null sequence and $\widetilde{D}'_n(x) = \mathcal{O}(n)$ for $x \in (0,\pi]$, then

$$\lim_{n\to\infty} g_n^c(x) = \lim_{n\to\infty} S_n(x) = f(x), \quad \forall x \in (0,\pi].$$

Applying Abel's transformation and Lemma 1.34, we obtain

$$\begin{aligned}
&\int_0^\pi |f(x)-g_n^c(x)|dx \\
&\quad\le \int_0^\pi \left|\sum_{k=n+1}^\infty \Delta a_k D_k(x)\right| dx + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}'_n(x)\right| dx \\
&\quad= \int_0^\pi \left|\sum_{k=n+1}^\infty A_k \frac{\Delta a_k}{A_k} D_k(x)\right| dx + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}'_n(x)\right| dx \\
&\quad= \int_0^\pi \left|\sum_{k=n+1}^\infty \Delta A_k \sum_{i=0}^k \frac{\Delta a_i}{A_i} D_i(x)\right| dx + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}'_n(x)\right| dx \\
&\quad\le \sum_{k=n+1}^\infty \Delta A_k \int_0^\pi \left|\sum_{i=0}^k \frac{\Delta a_i}{A_i} D_i(x)\right| dx + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}'_n(x)\right| dx \\
&\quad\le C\sum_{k=n+1}^\infty (k+1)\Delta A_k + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}'_n(x)\right| dx. \qquad (3.4)
\end{aligned}$$

Based on Lemma 1.26 the series

$$\sum_{k=1}^\infty (k+1)\Delta A_k$$

converges and therefore the first term in (3.4) tends to zero as $n \to \infty$.

Moreover,

$$\begin{aligned}
\int_0^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}'_n(x)\right| dx &\le \int_{-\pi}^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}'_n(x)\right| dx = \frac{|a_{n+1}|}{n+1}\int_{-\pi}^\pi \left|\widetilde{D}'_n(x)\right| dx \\
&= C|a_{n+1}|\int_{-\pi}^\pi \left|\widetilde{D}'_n(x)\right| dx \sim |a_{n+1}|\log n, \qquad (3.5)
\end{aligned}$$

since

$$\int_{-\pi}^\pi \left|\widetilde{D}'_n(x)\right| dx$$

behaves like $\log n$ for large values of n.

Now the conclusion of the theorem follows from (3.4) and (3.5).

The proof is completed.

Now we prove the following.

Corollary 3.6. *Let* $\{a_k\} \in \mathbf{S}(\delta)$ *and*

$$\lim_{n\to\infty} a_{n+1}\log n = 0.$$

Then $\|f - S_n\|_{L^1} = o(1)$ *as* $n \to \infty$.

Proof. We note that

$$\begin{aligned}\int_{-\pi}^{\pi} |f(x) - S_n(x)|\, dx &= \int_{-\pi}^{\pi} |f(x) - g_n^c(x) + g_n^c(x) - S_n(x)|\, dx \\ &\leq \int_{-\pi}^{\pi} |f(x) - g_n^c(x)|\, dx + \int_{-\pi}^{\pi} |g_n^c(x) - S_n(x)|\, dx \\ &= \int_{-\pi}^{\pi} |f(x) - g_n^c(x)|\, dx + \int_{-\pi}^{\pi} \left|\frac{a_{n+1}}{n+1}\widetilde{D}_n'\right| dx.\end{aligned}$$

Since by Theorem 3.5,

$$\int_{-\pi}^{\pi} |f(x) - g_n^c(x)|\, dx = o(1) \text{ as } n \to \infty$$

and

$$\int_{-\pi}^{\pi} \left|\frac{a_{n+1}}{n+1}\widetilde{D}_n'\right| dx$$

behaves like $|a_{n+1}| \log n$ for large values n, the conclusion of the corollary follows.

The proof is completed.

3.4 L^1-convergence of modified sums $g_n^c(x)$ with coefficients from S(δ) without additional condition

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

be a cosine series and

$$g_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

the modified trigonometric cosine sums.

Here we prove the following.

Theorem 3.7. *If $\{a_k\} \in \mathbf{S}(\delta)$, then $g_n^c(x)$ converges to $f(x)$ in L^1-norm.*

Proof. After some calculations we have

$$\begin{aligned}g_n^c(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left(\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right) \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1} \sum_{k=1}^{n} k \cos kx \\ &= S_n(x) - \frac{a_{n+1}}{n+1}\widetilde{D}_n'(x),\end{aligned}$$

where $\widetilde{D}'_n(x)$ denotes the first derivative of the conjugate Dirichlet kernel.

Applying Abel's transformation and Lemma 1.80, we obtain

$$\begin{aligned} g_n^c(x) &= \sum_{k=0}^{n} \Delta a_k D_k(x) + a_{n+1} D_n(x) - \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \\ &= \sum_{k=0}^{n} \Delta a_k D_k(x) + a_{n+1} K_n(x). \end{aligned}$$

Applying Abel's transformation again and Lemma 1.34, we get

$$\begin{aligned} &\int_0^{\pi} |f(x) - g_n^c(x)| dx \\ &\le \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx + \int_0^{\pi} |a_{n+1} K_n(x)| \, dx \\ &= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx + |a_{n+1}| \int_0^{\pi} |K_n(x)| \, dx \\ &= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta A_k \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x) \right| dx + |a_{n+1}| \int_0^{\pi} |K_n(x)| \, dx \\ &\le \sum_{k=n+1}^{\infty} \Delta A_k \int_0^{\pi} \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x) \right| dx \\ &\quad + A_{n+1} \int_0^{\pi} \left| \sum_{i=0}^{n} \frac{\Delta a_i}{A_i} D_i(x) \right| dx + |a_{n+1}| \int_0^{\pi} |K_n(x)| \, dx \\ &\le C \sum_{k=n+1}^{\infty} (k+1) \Delta A_k + C(n+1) A_{n+1} + |a_{n+1}| \int_0^{\pi} |K_n(x)| \, dx. \end{aligned} \tag{3.6}$$

Based on Lemma 1.26 the series

$$\sum_{k=1}^{\infty} (k+1) \Delta A_k$$

converges and therefore the first term in (3.6) tends to zero as $n \to \infty$, and

$$\int_0^{\pi} |K_n(x)| \, dx \le \int_{-\pi}^{\pi} |K_n(x)| \, dx = \pi. \tag{3.7}$$

Thus, the conclusion of the theorem follows from (3.6) and (3.7).

The proof is completed.

As a consequence of Theorem 3.7 is the following.

Corollary 3.8. *Let* $\{a_k\} \in \mathbf{S}(\delta)$. *Then* $\|f - S_n\|_{L^1} = o(1)$ *as* $n \to \infty$ *if and only if* $\lim_{n\to\infty} |a_{n+1}| \log n = 0$.

Proof. It is clear that we can write

$$\int_{-\pi}^{\pi} |f(x) - S_n(x)|\, dx = \int_{-\pi}^{\pi} |f(x) - g_n^c(x) + g_n^c(x) - S_n(x)|\, dx$$
$$\leq \int_{-\pi}^{\pi} |f(x) - g_n^c(x)|\, dx + \int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx \tag{3.8}$$

Since by Theorem 3.7,

$$\int_{-\pi}^{\pi} |f(x) - g_n^c(x)|\, dx = o(1) \ as\ n \to \infty$$

and

$$\int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx \sim |a_{n+1}| \log n$$

as $\int_{-\pi}^{\pi} \left|\widetilde{D}'_n(x)\right| dx \sim n|\log n$, for large values n, from (3.7) we obtain

$$\|f - S_n\|_{L^1} = o(1) \quad as \quad n \to \infty.$$

Conversely, we have

$$\int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx = \|g_n^c - S_n\|_{L^1} \leq \|f - g_n^c\|_{L^1} + \|f - S_n\|_{L^1}.$$

Subsequently,

$$\|f - S_n\|_{L^1} = o(1) \quad as \quad n \to \infty$$

if and only if

$$\lim_{n\to\infty} |a_{n+1}| \log n = 0.$$

The proof is completed.

3.5 L^1-convergence of modified sums $g_n^c(x)$ with coefficients from S without additional condition

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

be a cosine series and

$$g_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

the modified trigonometric cosine sums.

Next statement holds true.

Theorem 3.9. *Let $\{a_k\} \in \mathbf{S}$. Then $g_n^c(x)$ converges to $f(x)$ in L^1-norm.*

Proof. We have

$$\begin{aligned} g_n^c(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left[\Delta\left(\frac{a_k}{k}\right) + \Delta\left(\frac{a_{k+1}}{k+1}\right) + \cdots + \Delta\left(\frac{a_n}{n}\right)\right] \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1} \sum_{k=1}^{n} k \cos kx \\ &= S_n(x) - \frac{a_{n+1}}{n+1} \widetilde{D}_n'(x), \end{aligned}$$

where $\widetilde{D}_n'(x)$ denotes the first derivative of the conjugate Dirichlet kernel.

Applying Abel's transformation and Lemma 1.80, we obtain

$$\begin{aligned} g_n^c(x) &= \sum_{k=0}^{n} \Delta a_k D_k(x) + a_{n+1} D_n(x) - \frac{a_{n+1}}{n+1} \widetilde{D}_n'(x) \\ &= \sum_{k=0}^{n} \Delta a_k D_k(x) + a_{n+1} D_n(x) - a_{n+1} D_n(x) + a_{n+1} K_n(x) \\ &= \sum_{k=0}^{n} \Delta a_k D_k(x) + a_{n+1} K_n(x). \end{aligned}$$

Applying Abel's transformation again and Lemma 1.34, we get

$$\begin{aligned} &\int_0^\pi |f(x) - g_n^c(x)| dx \\ &\quad \le \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx + \int_0^\pi |a_{n+1} K_n(x)|\, dx \\ &\quad = \int_0^\pi \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx + |a_{n+1}| \int_0^\pi |K_n(x)|\, dx \\ &\quad \le \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta A_k \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x) \right| dx + |a_{n+1}|\pi \\ &\quad \le \sum_{k=n+1}^{\infty} \Delta A_k \int_0^\pi \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x) \right| dx \\ &\qquad + A_{n+1} \int_0^\pi \left| \sum_{i=0}^{n} \frac{\Delta a_i}{A_i} D_i(x) \right| dx + |a_{n+1}|\pi \\ &\quad \le C \sum_{k=n+1}^{\infty} (k+1) \Delta A_k + (n+1) A_n + |a_{n+1}|\pi. \end{aligned} \tag{3.9}$$

Based on Lemma 1.26 the series

$$\sum_{k=1}^{\infty}(k+1)\Delta A_k$$

converges and therefore the first term in (3.9) tends to zero as $n \to \infty$, and

$$\int_0^{\pi} |K_n(x)|\, dx \le \int_{-\pi}^{\pi} |K_n(x)|\, dx = \pi. \tag{3.10}$$

Thus, the conclusion of the theorem follows from (3.9) and (3.10).
The proof is completed.

As a consequence of Theorem 3.9 is the following.

Corollary 3.10. *Let* $\{a_k\} \in \mathbf{S}$. *Then* $\|f - S_n\|_{L^1} = o(1)$ *as* $n \to \infty$ *if and only if*

$$\lim_{n\to\infty} |a_{n+1}| \log n = 0.$$

Proof. It is clear that

$$\begin{aligned}\int_{-\pi}^{\pi} |f(x) - S_n(x)|\, dx &\le \int_{-\pi}^{\pi} |f(x) - g_n^c(x)|\, dx + \int_{-\pi}^{\pi} |g_n^c(x) - S_n(x)|\, dx \\ &\le \int_{-\pi}^{\pi} |f(x) - g_n^c(x)|\, dx \\ &\qquad + \int_{-\pi}^{\pi} \left|\frac{a_{n+1}}{n+1}\widetilde{D}'_n(x)\right| dx. \end{aligned} \tag{3.11}$$

Since by Theorem 3.9,

$$\int_{-\pi}^{\pi} |f(x) - g_n^c(x)|\, dx = o(1) \text{ as } n \to \infty$$

and

$$\int_{-\pi}^{\pi} \left|\frac{a_{n+1}}{n+1}\widetilde{D}'_n(x)\right| dx \sim |a_{n+1}| \log n$$

as $\int_{-\pi}^{\pi} \left|\widetilde{D}'_n(x)\right| dx \sim n \log n$, for large values n, from (3.11) we obtain

$$\|f - S_n\|_{L^1} = o(1) \quad as \quad n \to \infty.$$

Conversely, we have

$$\int_{-\pi}^{\pi} \left|\frac{a_{n+1}}{n+1}\widetilde{D}'_n(x)\right| dx = \|g_n^c - S_n\|_{L^1} \le \|f - g_n^c\|_{L^1} + \|f - S_n\|_{L^1}.$$

Subsequently,

$$\|f - S_n\|_{L^1} = o(1) \quad as \quad n \to \infty$$

if and only if

$$\lim_{n\to\infty} |a_{n+1}| \log n = 0.$$

The proof is completed.

3.6 L^1-convergence of modified trigonometric sums $g_n^c(x)$ with coefficients from the class $\mathbf{F}_p$

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

be a cosine series and

$$g_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

the modified trigonometric cosine sums.

We have the following statement.

Theorem 3.11. *Let $\{a_k\} \in \mathbf{F}_p$, $1 < p \leq 2$. Then $g_n^c(x)$ converges to $f(x)$ in L^1-norm if and only if*

$$\lim_{n\to\infty} a_n \log n = 0.$$

Proof. We have

$$\begin{aligned} g_n^c(x) &= \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx \\ &= S_n(x) - \frac{a_{n+1}}{n+1}\widetilde{D}_n'(x) \\ &= S_n(x) - a_{n+1}D_n(x) + a_{n+1}K_n(x) \\ &= \sum_{k=0}^{n} \Delta a_k D_k(x) + a_{n+1}K_n(x), \end{aligned} \tag{3.12}$$

where $K_n(x)$ denotes the Fejér's kernel.

Since $|K_n(x)| \leq \mathcal{O}\left(x^{-2}\right)$, $x \in (0,\pi]$, and $a_n \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} g_n^c(x)$ exists for $x \in (0,\pi]$, and $\lim_{n\to\infty} g_n^c(x) = f(x)$.

Now,

$$\int_0^{\pi} |f(x) - g_n^c(x)|dx = \int_0^{\frac{1}{n}} |f(x) - g_n^c(x)|dx + \int_{\frac{1}{n}}^{\pi} |f(x) - g_n^c(x)|dx. \tag{3.13}$$

For the first integral of the right hand side of (3.13), we have

$$\int_0^{\frac{1}{n}} |f(x) - g_n^c(x)|dx \leq \int_0^{\frac{1}{n}} |\sigma_n(x) - f(x)|dx + \int_0^{\frac{1}{n}} |g_n^c(x) - \sigma_n(x)|dx,$$

where $\sigma_n(x)$ is the Fejér sum of $S_n(x)$, and

$$\int_0^{\frac{1}{n}} |\sigma_n(x) - f(x)|dx = \mathcal{O}\left(\|\sigma_n(x) - f(x)\|\right), \quad n \to \infty.$$

From (3.12), we get

$$g_n^c(x) - \sigma_n(x) = a_{n+1}K_n(x) + \frac{1}{n+1}\left(\sum_{k=1}^{n} k\Delta a_k D_k(x) - \sum_{k=1}^{n} a_k D_k(x)\right).$$

Therefore,

$$\int_0^{\frac{1}{n}} |g_n^c(x) - \sigma_n(x)|dx \le |a_{n+1}|\frac{\pi}{2} + \frac{1}{n+1}\sum_{k=1}^{n} k|\Delta a_k| \int_0^{\frac{1}{n}} |D_k(x)|dx$$
$$+\frac{1}{n+1}\sum_{k=1}^{n}|a_k|\int_0^{\frac{1}{n}} |D_k(x)|dx,$$

or

$$\int_0^{\frac{1}{n}} |g_n^c(x) - \sigma_n(x)|dx = \mathcal{O}\left(\frac{1}{n}\sum_{k=1}^{n} k|\Delta a_k|\right), \quad n \to \infty.$$

Thus, for the first integral in (3.13), we have

$$\int_0^{\frac{1}{n}} |g_n^c(x) - f(x)|dx = \mathcal{O}\left(\|\sigma_n(x) - f(x)\| + \frac{1}{n}\sum_{k=1}^{n} k|\Delta a_k|\right), \quad n \to \infty.$$

For the second integral of the right hand side of (3.13), we have

$$\begin{aligned}\int_{\frac{1}{n}}^{\pi} |g_n^c(x) - \sigma_n(x)|dx &\le \int_{\frac{1}{n}}^{\pi} |a_{n+1}F_n(x)|dx + \int_{\frac{1}{n}}^{\pi}\left|\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)\right|dx \\ &\le \int_0^{\pi} |a_{n+1}F_n(x)|dx + \int_{\frac{1}{n}}^{\pi}\left|\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)\right|dx \\ &= |a_{n+1}|\frac{\pi}{2} + \int_{\frac{1}{n}}^{\pi}\left|\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)\right|dx. \qquad (3.14)\end{aligned}$$

So, we have

$$\int_0^{\pi} |g_n^c(x) - f(x)|dx = \mathcal{O}\left(\|\sigma_n(x) - f(x)\|\right)$$
$$+\mathcal{O}\left(\sum_{k=1}^{n} k|\Delta a_k| + \sum_{k=n+1}^{\infty} k^{p-1}|\Delta a_k|^p\right) = o(1), \quad n \to \infty.$$

The proof is completed.

As a consequence of Theorem 3.9 is the following.

Corollary 3.12. *Let* $\{a_k\} \in \mathbf{F}_p$, $1 < p \leq 2$. *Then*

$$\lim_{n\to\infty} \|f - S_n\|_{L^1} = 0$$

if and only if

$$\lim_{n\to\infty} |a_{n+1}| \log n = 0.$$

Proof. We have

$$\begin{aligned}\int_0^\pi |f(x) - S_n(x)|\, dx &\leq \int_0^\pi |f(x) - g_n^c(x)|\, dx + \int_0^\pi |g_n^c(x) - S_n(x)|\, dx \\ &\leq \int_0^\pi |f(x) - g_n^c(x)|\, dx \\ &\quad + \int_0^\pi |a_{n+1} D_n(x)|\, dx + \int_{-\pi}^\pi |a_{n+1} K_n(x)|\, dx\end{aligned}$$

and

$$\begin{aligned}\int_0^\pi |a_{n+1} D_n(x)|\, dx &\leq \int_0^\pi |g_n^c(x) - S_n(x)|\, dx + \int_{-\pi}^\pi |a_{n+1} K_n(x)|\, dx \\ &\leq \int_0^\pi |f(x) - g_n^c(x)|\, dx + \int_{-\pi}^\pi |a_{n+1} K_n(x)|\, dx.\end{aligned}$$

Since $\int_{-\pi}^\pi D_n(x)|\, dx \sim n \log n$, for large values n, from Theorem 3.11, we obtain the conclusion of the corollary.

The proof is completed.

3.7 L^1-convergence of modified sums $g_n^c(x)$ with coefficients from the intersection class $\mathbf{BV} \cap \mathbf{C}$

Let us consider the cosine series

$$\frac{a_0}{2} + \sum_{k=1}^\infty a_k \cos kx,$$

and modified cosine sums

$$g_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^n \sum_{j=k}^n \Delta\left(\frac{a_j}{j}\right) k \cos kx.$$

Theorem 3.13. *Let* $\{a_k\} \in \mathbf{BV} \cap \mathbf{C}$. *Then* $g_n^c(x)$ *converges to* $f(x)$ *in* L^1-*norm if and only if*

$$\lim_{n\to\infty} a_{n+1} \log n = 0.$$

Proof. Using Lemma 1.80 we have

$$\begin{aligned} g_n^c(x) &= S_n(x) - \frac{a_{n+1}}{n+1}\widetilde{D}'_n(x) \\ &= S_n(x) - a_{n+1}D_n(x) + a_{n+1}K_n(x) \\ &= \sum_{k=0}^{n-1} \Delta a_k D_k(x) + \Delta a_{n+1}D_n(x) + a_{n+1}K_n(x), \end{aligned} \tag{3.15}$$

where $K_n(x)$ denotes the Fejér's kernel.

Therefore,

$$\begin{aligned} &\int_0^\pi |f(x) - g_n^c(x)| dx \\ &\le \int_0^\pi \left| \sum_{k=n}^\infty \Delta a_k D_k(x) - \Delta a_{n+1} D_n(x) - a_{n+1}K_n(x) \right| dx \\ &\le \int_0^\pi \left| \sum_{k=n}^\infty \Delta a_k D_k(x) \right| dx + \int_0^\pi |\Delta a_{n+1} D_n(x)|\, dx + \int_0^\pi |a_{n+1}K_n(x)|\, dx \\ &= \int_0^\delta \left| \sum_{k=n}^\infty \Delta a_k D_k(x) \right| dx + \int_\delta^\pi \left| \sum_{k=n}^\infty \Delta a_k D_k(x) \right| dx \\ &\quad + \int_0^\pi |\Delta a_{n+1} D_n(x)|\, dx + |a_{n+1}| \frac{\pi}{2} \\ &< \frac{\varepsilon}{4} + \sum_{k=n}^\infty |\Delta a_k| \int_\delta^\pi \csc\frac{x}{2} dx + |\Delta a_{n+1}| \int_0^\pi |D_n(x)|\, dx + \frac{\varepsilon}{4} \\ &< \frac{\varepsilon}{4} + \sum_{k=n}^\infty |\Delta a_k| \left[-2\log\left| \csc\frac{\delta}{2} - \cot\frac{\delta}{2} \right| \right] + C(|a_{n+1}| + |a_{n+2}|)\log n + \frac{\varepsilon}{4} < \frac{\varepsilon}{4}, \end{aligned}$$

since $\int_0^\pi |D_n(x)|\, dx$ behaves like $\log n$ for large n and $\{a_k\} \in \mathbf{BV} \cap \mathbf{C}$.

Also, using (3.15) we have

$$\begin{aligned} \int_0^\pi |a_{n+1}D_n(x)|\, dx &\le \int_0^\pi |S_n(x) - g_n^c(x)| dx + |a_{n+1}| \int_0^\pi |a_{n+1}K_n(x)|\, dx \\ &\le \int_0^\pi |f(x) - g_n^c(x)| dx + |a_{n+1}| \frac{\pi}{2} = o(1) + o(1) = o(1) \end{aligned}$$

as $n \to \infty$.

The proof is completed.

Corollary 3.14. *Let* $\{a_k\} \in \mathbf{BV} \cap \mathbf{C}$. *Then* $S_n(x)$ *converges to* $f(x)$ *in* L^1-*norm if and only if*

$$\lim_{n\to\infty} |a_{n+1}| \log n = 0.$$

Proof. We have

$$\begin{aligned}\int_0^\pi |f(x)-S_n(x)|dx &\le \int_0^\pi |f(x)-g_n^c(x)+g_n^c(x)-S_n(x)|dx\\ &\le \int_0^\pi |f(x)-g_n^c(x)|dx+\int_0^\pi |g_n^c(x)-S_n(x)|dx\\ &= \int_0^\pi |f(x)-g_n^c(x)|dx+\int_0^\pi |a_{n+1}D_n(x)|dx+\int_0^\pi |a_{n+1}K_n(x)|dx.\end{aligned}$$

Also,

$$\begin{aligned}\int_0^\pi |a_{n+1}D_n(x)|dx &\le \int_0^\pi |g_n^c(x)-S_n(x)|dx+\int_0^\pi |a_{n+1}K_n(x)|dx\\ &\le \int_0^\pi |f(x)-S_n(x)|dx+|a_{n+1}|\frac{\pi}{2}.\end{aligned}$$

Since

$$\int_0^\pi |a_{n+1}D_n(x)|dx$$

behaves like $|a_{n+1}|\log n$ for large values of n, and by Theorem 2.13

$$\lim_{n\to\infty}\int_0^\pi |f(x)-S_n(x)|dx=0,$$

then the corollary is proved.

3.8 L^1-convergence of modified sums $g_n^c(x)$ with coefficients from the class $\mathbf{S}^{**}$

Let us consider the cosine series

$$\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k\cos kx,$$

and modified cosine sums

$$g_n^c(x)=\frac{a_0}{2}+\sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right)k\cos kx.$$

Theorem 3.15. *Let $\{a_k\}\in\mathbf{S}^{**}$. Then $g_n^c(x)$ converges to $f(x)$ in L^1-norm if and only if $|a_{n+1}|\log n=o(1)$ as $n\to\infty$.*

Proof. Firstly we have

$$g_n^c(x)=\frac{a_0}{2}+\sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right)k\cos kx=S_n(x)-\frac{a_{n+1}}{n+1}\widetilde{D}_n'(x).$$

Since, $\widetilde{D}'_n(x) = \mathcal{O}(n)$ in $(0, \pi]$ and $\{a_n\}$ is a null sequence, then

$$\lim_{n\to\infty} g_n^c(x) = \lim_{n\to\infty} S_n(x) = f(x) \quad for \quad x \in (0, \pi].$$

This implies the equality

$$f(x) - g_n^c(x) = \sum_{k=n+1}^{\infty} a_k \cos kx + \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x).$$

Applying Abel's transformation, we obtain

$$f(x) - g_n^c(x) = \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - a_{n+1} D_n(x) + \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x).$$

Whence,

$$\begin{aligned}\|f - g_n^c\| &= \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - a_{n+1} D_n(x) + \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx \\ &\le \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx + \int_0^\pi |a_{n+1} D_n(x)|\, dx \\ &\quad + \int_0^\pi \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx.\end{aligned}$$

Since $\int_0^\pi \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx \sim |a_{n+1}| \log n$, by Zygmund's theorem, $\{a_n\} \in S^{**}$ and Lemma 1.85, we get

$$\begin{aligned}\|f - g_n^c\| &= \mathcal{O}\left(\sum_{k=n+1}^{\infty} k \Delta a_k \right) + o\left(|a_{n+1}| \log n\right) + o\left(|a_{n+1}| \log n\right) \\ &= o(1) + o\left(|a_{n+1}| \log n\right).\end{aligned}$$

Subsequently, $\|f - g_n^c(x)\| = o(1)$ as $n \to \infty$ if and only if $|a_{n+1}| \log n = o(1)$ as $n \to \infty$.

The proof is completed.

Corollary 3.16. *Let $\{a_k\} \in \mathbf{S}^{**}$. Then $S_n(x)$ converges to $f(x)$ in L^1-norm if and only if $|a_{n+1}| \log n = o(1)$ as $n \to \infty$.*

Proof. We consider,

$$\begin{aligned}\|f - S_n\| &\le \|f - g_n^c + g_n^c - S_n\| \\ &\le \|f - g_n^c\| + \|g_n^c - S_n\| \\ &= \|f - g_n^c\| + \int_0^\pi \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx.\end{aligned}$$

Again, since

$$\int_0^\pi \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx$$

behaves like $|a_{n+1}| \log n$ for large values of n, and by Theorem 3.15 we get

$$\lim_{n\to\infty} \|f - S_n\| = 0$$

if and only if $|a_{n+1}| \log n = o(1)$ as $n \to \infty$.

The corollary is proved.

3.9 L^1-convergence of modified trigonometric sums $g_{n,m}^c(x)$ and $g_{n,m}^s(x)$ with coefficients from the class R

We consider cosine series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

sine series

$$g(x) = \sum_{k=1}^{\infty} a_k \sin kx,$$

their parital sums

$$S_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx,$$

$$S_n^s(x) = \sum_{k=1}^{n} a_k \sin kx,$$

modified cosine sums

$$g_{n,m}^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} k^m \Delta\left(\frac{a_j}{j^m}\right) \cos kx,$$

and modified sine sums

$$g_{n,m}^s(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} k^m \Delta\left(\frac{a_j}{j^m}\right) \sin kx,$$

where $m \in \{1, 2, \ldots\}$ is a fixed number.

We prove the following result.

Theorem 3.17. *Let the coefficients of the cosine or sine series belong to the class* **R**. *Then*

$$\lim_{n\to\infty} g_{n,m}(x) = r(x) \ exists for all \ x \in (0,\pi],$$

$$r \in L^1(0,\pi] \quad and \quad \lim_{n\to\infty} \|r - g_{n,m}\| = 0,$$

where $g_{n,m}(x)$ *denotes either* $g^c_{n,m}(x)$ *or* $g^s_{n,m}(x)$ *and* $m \in \{1,3,5,\ldots\}$.

Proof. We are going to prove these results only for the cosine series. The proof for the sine series is very similar. Indeed, we have

$$\begin{aligned} g^c_{n,m}(x) &= \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} k^m \Delta\left(\frac{a_j}{j^m}\right)\cos kx \\ &= \frac{a_0}{2} + \sum_{k=1}^{n}\left[\Delta\left(\frac{a_k}{k^m}\right) + \cdots + \Delta\left(\frac{a_n}{n^m}\right)\right] k^m \cos kx \\ &= \frac{a_0}{2} + \sum_{k=1}^{n}\left[\frac{a_k}{k^m} - \frac{a_{n+1}}{(n+1)^m}\right] k^m \cos kx \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{(n+1)^m}\sum_{k=1}^{n} k^m \cos kx \\ &= S^c_n(x) - \frac{a_{n+1}}{(n+1)^m}\left[\pm \widetilde{D}^{(m)}_n(x)\right], \end{aligned} \tag{3.16}$$

where $\widetilde{D}^{(m)}_n(x)$ denotes $m-$th derivative of conjugate Dirichlet kernel.

Since

$$|\widetilde{D}^{(m)}_n(x)| = \mathcal{O}(n^{m+1})$$

then for second term of the right side of (3.16) we have

$$\frac{|a_{n+1}|}{(n+1)^m}|\widetilde{D}^{(m)}_n(x)| = \mathcal{O}(n|a_{n+1}|).$$

Therefore, by (3.16) and our assumptions, we obtain that

$$\lim_{n\to\infty} g^c_{n,m}(x) = \lim_{n\to\infty} S^c_n(x) = f(x)$$

exists for all $x \in (0,\pi]$ and $m \in \{1,3,5,\ldots\}$.

Consequently, Kano's theorem implies that $f \in L^1(0,\pi]$.

Using (3.16) we can write

$$\begin{aligned} f(x) - g^c_{n,m}(x) &= \sum_{k=n+1}^{\infty} a_k \cos kx + \frac{a_{n+1}}{(n+1)^m}\widetilde{D}^{(m)}_n(x) \\ &= \lim_{s\to\infty}\frac{d}{dx}\left(\sum_{k=n+1}^{s}\frac{a_k}{k}\sin kx\right) + \frac{a_{n+1}}{(n+1)^m}\widetilde{D}^{(m)}_n(x). \end{aligned}$$

Applying summation by parts twice we obtain

$$\begin{aligned}
&f(x) - g_{n,m}^c(x)\\
&= \lim_{s\to\infty}\Bigg[\sum_{k=n+1}^{s-1} \Delta\left(\frac{a_k}{k}\right)\widetilde{D}'_k(x) + \frac{a_s}{s}\widetilde{D}'_s(x)\\
&\quad -\frac{a_{n+1}}{n+1}\widetilde{D}'_n(x)\Bigg] + \frac{a_{n+1}}{(n+1)^m}\widetilde{D}_n^{(m)}(x)\\
&= \lim_{s\to\infty}\Bigg[\sum_{k=n+1}^{s-2}(k+1)\Delta^2\left(\frac{a_k}{k}\right)\widetilde{K}'_k(x) + s\Delta\left(\frac{a_{s-1}}{s-1}\right)\widetilde{K}'_{s-1}(x)\\
&\quad -(n+1)\Delta\left(\frac{a_{n+1}}{n+1}\right)\widetilde{K}'_n(x) + \frac{a_s}{s}\widetilde{D}'_s(x)\\
&\quad -\frac{a_{n+1}}{n+1}\widetilde{D}'_n(x)\Bigg] + \frac{a_{n+1}}{(n+1)^m}\widetilde{D}_n^{(m)}(x)\\
&= \sum_{k=n+1}^{\infty}(k+1)\Delta^2\left(\frac{a_k}{k}\right)\widetilde{K}'_k(x) - (n+1)\Delta\left(\frac{a_{n+1}}{n+1}\right)\widetilde{K}'_n(x)\\
&\quad +a_{n+1}\left[\frac{1}{(n+1)^m}\widetilde{D}_n^{(m)}(x) - \frac{1}{n+1}\widetilde{D}'_n(x)\right],
\end{aligned}$$

where $\widetilde{K}_n(x)$ denotes the conjugate Fejér kernel.

Whence, we have

$$\begin{aligned}
\int_{-\pi}^{\pi}|f(x) - g_{n,m}^c(x)|dx \le &\sum_{k=n+1}^{\infty}(k+1)\left|\Delta^2\left(\frac{a_k}{k}\right)\right|\int_{-\pi}^{\pi}\left|\widetilde{K}'_k(x)\right|dx\\
&+(n+1)\left|\Delta\left(\frac{a_{n+1}}{n+1}\right)\right|\int_{-\pi}^{\pi}\left|\widetilde{K}'_n(x)\right|dx\\
&+|a_{n+1}|\Bigg[\frac{1}{(n+1)^m}\int_{-\pi}^{\pi}\left|\widetilde{D}_n^{(m)}(x)\right|dx\\
&+\frac{1}{n+1}\int_{-\pi}^{\pi}\left|\widetilde{D}'_n(x)\right|dx\Bigg] := \sum_{i=1}^{3}J_i.
\end{aligned}$$

Let us estimate alternately each J_i, $i = 1, 2, 3$. Indeed, using the estimation

$$\int_{-\pi}^{\pi}\left|\widetilde{K}'_k(x)\right|dx = \mathcal{O}(k)$$

we have

$$\begin{aligned}
J_1 &:= \sum_{k=n+1}^{\infty}(k+1)\left|\Delta^2\left(\frac{a_k}{k}\right)\right|\int_{-\pi}^{\pi}\left|\widetilde{K}'_k(x)\right|dx\\
&= \mathcal{O}\left(\sum_{k=n+1}^{\infty}k^2\left|\Delta^2\left(\frac{a_k}{k}\right)\right|\right) = o(1) \quad as \quad n\to\infty,
\end{aligned}$$

since $\{a_k\} \in \mathbf{R}$.

Also, we have

$$\begin{aligned} J_2 &:= (n+1)\left|\Delta\left(\frac{a_{n+1}}{n+1}\right)\right| \int_{-\pi}^{\pi} \left|\widetilde{K}'_n(x)\right| dx \\ &= \mathcal{O}\left((n+1)^2 \left|\Delta\left(\frac{a_{n+1}}{n+1}\right)\right|\right) \\ &= \mathcal{O}\left((n+1)^2 \sum_{k=n+1}^{\infty} \left|\Delta^2\left(\frac{a_k}{k}\right)\right|\right) \\ &= \mathcal{O}\left(\sum_{k=n+1}^{\infty} k^2 \left|\Delta^2\left(\frac{a_k}{k}\right)\right|\right) = o(1) \quad as \quad n \to \infty, \end{aligned}$$

since $\{a_k\} \in \mathbf{R}$.

Also, we have

$$\begin{aligned} J_3 &:= |a_{n+1}|\left[\frac{1}{(n+1)^m} \int_{-\pi}^{\pi} \left|\widetilde{D}_n^{(m)}(x)\right| dx + \frac{1}{n+1} \int_{-\pi}^{\pi} \left|\widetilde{D}'_n(x)\right| dx\right] \\ &= |a_{n+1}|\left[\frac{1}{(n+1)^m}\mathcal{O}\left((n+1)^{m+1}\right) + \frac{1}{n+1}\mathcal{O}\left((n+1)^2\right)\right] = \mathcal{O}\left((n+1)|a_{n+1}|\right). \end{aligned}$$

Consequently, by given hypothesis we obtain

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |f(x) - g^c_{n,m}(x)| dx = 0.$$

The proof is completed.

Corollary 3.18. *Let the coefficients of the cosine or sine series belong to the class* $\mathbf{R}$. *Then*

$$\lim_{n\to\infty} |a_{n+1}| \log n = 0 \Longrightarrow \lim_{n\to\infty} \|f - S^c_n\| = 0.$$

Proof. Let $\lim_{n\to\infty} |a_{n+1}| \log n = 0$ be satisfied. Then, based on (3.16) we have

$$\begin{aligned} \int_{-\pi}^{\pi} |f(x) - S^c_n(x)| dx &= \int_{-\pi}^{\pi} |f(x) - g^c_{n,m}(x) + g^c_{n,m}(x) - S^c_n(x)| dx \\ &\le \int_{-\pi}^{\pi} |f(x) - g^c_{n,m}(x)| dx + \int_{-\pi}^{\pi} |g^c_{n,m}(x) - S^c_n(x)| dx \\ &\le \int_{-\pi}^{\pi} |f(x) - g^c_{n,m}(x)| dx + \frac{|a_{n+1}|}{(n+1)^m} \int_{-\pi}^{\pi} \left|\widetilde{D}_n^{(m)}(x)\right| dx \\ &= o(1) + \mathcal{O}\left(|a_{n+1}| \log n\right) = o(1) + o(1) = o(1) \quad as \quad n \to \infty. \end{aligned}$$

The proof is completed.

3.10 Equivalent theorems pertaining L^1-convergence of modified trigonometric sums $f_n(x)$ and $g_n(x)$

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

be the cosine series with its partial sums

$$S_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx,$$

and the known modified cosine sums

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx$$

and

$$g_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx.$$

We prove here that all results which have been proved by considering $f_n(x)$ are also true for $g_n^c(x)$ as far as L^1-convergence of cosine series is concerned irrespective of the consideration of classes. When we say that $\mathcal{S}$ is any subclass of coefficients of the cosine series, we mean that the sequence $\{a_k\}$ is a null-sequence and convex or quasi-convex or belongs to the class $\mathbf{S}$ of Sidon or any class of coefficients for which $\|f - f_n(x)\| = o(1)$ or $\|f - g_n^c(x)\| = o(1)$ as $n \to \infty$.

First we prove next statement.

Theorem 3.19. *If $\{a_k\}$ belongs to the class $\mathbf{S}$, then $\|f - g_n^c\| = o(1)$ as $n \to \infty$.*

Proof. We have

$$g_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1}\widetilde{D}_n'(x).$$

Using summation by parts and Lemma 1.80 we get

$$g_n^c(x) = \sum_{k=1}^{n} \Delta a_k D_k(x) + a_{n+1} K_n(x).$$

Applying Abel's transformation again and Lemma 1.34, we get

$$\begin{aligned}
&\int_0^\pi |f(x) - g_n^c(x)|dx \\
&\quad\le \int_0^\pi \left| \sum_{k=n+1}^\infty \Delta a_k D_k(x) \right| dx + \int_0^\pi |a_{n+1} K_n(x)|\, dx \\
&\quad= \int_0^\pi \left| \sum_{k=n+1}^\infty A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx + |a_{n+1}| \int_0^\pi |K_n(x)|\, dx \\
&\quad\le \int_0^\pi \left| \sum_{k=n+1}^\infty \Delta A_k \sum_{i=0}^k \frac{\Delta a_i}{A_i} D_i(x) \right| dx + |a_{n+1}|\pi \\
&\quad\le \sum_{k=n+1}^\infty \Delta A_k \int_0^\pi \left| \sum_{i=0}^k \frac{\Delta a_i}{A_i} D_i(x) \right| dx \\
&\qquad + A_{n+1} \int_0^\pi \left| \sum_{i=0}^n \frac{\Delta a_i}{A_i} D_i(x) \right| dx + |a_{n+1}|\pi \\
&\quad\le C \sum_{k=n+1}^\infty (k+1)\Delta A_k + (n+1)A_n + |a_{n+1}|\pi. \qquad (3.17)
\end{aligned}$$

Based on Lemma 1.26 the series

$$\sum_{k=1}^\infty (k+1)\Delta A_k$$

converges and therefore the first term in (3.17) tends to zero as $n \to \infty$, and

$$\int_0^\pi |K_n(x)|\, dx \le \int_{-\pi}^\pi |K_n(x)|\, dx = \pi. \qquad (3.18)$$

Thus, the conclusion of the theorem follows from (3.17) and (3.18).
The proof is completed.

Theorem 3.20. *Let $\mathcal{S}$ be any subclass of coefficients of the cosine series and $\{a_k\}$ belongs to the class $\mathcal{S}$. Then $\|f - f_n\| = o(1) \iff \|f - g_n^c\| = o(1)$ as $n \to \infty$.*

Proof. Using the equalities

$$f_n(x) = S_n(x) - a_{n+1} D_n(x)$$

and

$$g_n^c(x) = S_n(x) - \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x),$$

we have

$$f_n(x) - g_n^c(x) = -a_{n+1} D_n(x) + \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x).$$

Whence, using Lemma 1.80 we get

$$f_n(x) - g_n^c(x) = -a_{n+1}K_n(x).$$

Assume that $\|f - f_n\| = o(1)$ as $n \to \infty$. Then,

$$\begin{aligned}\|f - g_n^c\| &= \|f - f_n - a_{n+1}K_n(x)\| \\ &\le \|f - f_n\| + |a_{n+1}|\|K_n(x)\| = o(1), \quad n \to \infty.\end{aligned}$$

Conversely,

$$\|f_n - g_n^c\| = |a_{n+1}|\|K_n(x)\| = o(1), \quad n \to \infty.$$

The proof is completed.

Theorem 3.21. *Let $\mathcal{S}$ be any subclass of coefficients of the cosine series and $\{a_k\}$ belongs to the class $\mathcal{S}$ and $r \in \{0, 1, 2, \ldots\}$. Then:*

(1) $\|f^{(r)} - f_n^{(r)}\| = \mathcal{O}(n^r) \Longrightarrow \|f^{(r)} - [g_n^c]^{(r)}\| = o(1)$ *as* $n^r a_n \to 0,\ n \to \infty.$
(2) $\|f^{(r)} - [g_n^c]^{(r)}\| = \mathcal{O}(n^r) \Longrightarrow \|f^{(r)} - f_n^{(r)}\| = o(1)$ *as* $n^r a_n \to 0,\ n \to \infty.$

Proof. Using the equalities

$$f_n^{(r)}(x) = S_n^{(r)}(x) - a_{n+1}D_n^{(r)}(x)$$

and

$$[g_n^c(x)]^{(r)} = S_n^{(r)}(x) - \frac{a_{n+1}}{n+1}\widetilde{D}_n^{(r+1)}(x),$$

we have

$$f_n^{(r)}(x) - [g_n^c(x)]^{(r)} = -a_{n+1}D_n^{(r)}(x) + \frac{a_{n+1}}{n+1}\widetilde{D}_n^{(r+1)}(x).$$

Whence, using Lemma 1.80 we get

$$f_n^{(r)}(x) - [g_n^c(x)]^{(r)} = -a_{n+1}K_n^{(r)}(x).$$

Now, the proof is very similar to the proof of Theorem 3.20. Therefore we omit it.

The proof is completed.

Remark 3.22. Similar results hold true when we consider sine series

$$g(x) = \sum_{k=1}^{\infty} a_k \sin kx,$$

and the modified sine sums

$$f_n^s(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \sin kx \quad \textit{or} \quad g_n^s(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \sin kx.$$

Other modified trigonometric sums similar to $g_n^c(x)$ and $g_n^s(x)$ have been introduced as follows

$$\overline{g}_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j \cos jx}{2^j}\right) 2^k$$

and

$$\overline{g}_n^s(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j \sin jx}{2^j}\right) 2^k.$$

The sums $\overline{g}_n^c(x)$ can be rewritten as follows

$$\overline{g}_n^c(x) = S_n^c(x) - a_{n+1}\cos(n+1)x + \frac{a_{n+1}\cos(n+1)x}{2^n}.$$

From last equality we obviously obtain

$$\lim_{n\to\infty} \overline{g}_n^c(x) = \lim_{n\to\infty} S_n^c(x) = f(x)$$

whenever $\{a_k\}$ is a null-sequence.

Using this fact we are in able to prove next statement.

Theorem 3.23. *Let $\{a_k\}$ be a null-sequence and convex or quasi-convex or belongs to the class $\underline{S}$. Then $\|f - \overline{g}_n^c\| = o(1)$ as $n \to \infty$ if and only if $a_n \log n = o(1)$ as $n \to \infty$.*

Proof. The proof can be done easily. This is why we have omitted it.

Remark 3.24. The modified sums $\overline{g}_n^c(x)$ and $\overline{g}_n^s(x)$ are generalized as follows

$$\overline{G}_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j \cos jx}{d^j}\right) d^k$$

and

$$\overline{G}_n^s(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j \sin jx}{d^j}\right) d^k,$$

where $d > 1$ is a real number.

Remark 3.25. We note that, in the specific case $d = 2$, it holds $\overline{G}_n^c(x) = \overline{g}_n^c(x)$ and $\overline{G}_n^s(x) = \overline{g}_n^s(x)$.

Remark 3.26. Using modified sums $\overline{G}_n^c(x)$ and $\overline{G}_n^s(x)$ we can obtain similar results as Theorem 3.23.

L^1-convergence of some other modified trigonometric sums

In this section we show all results regrading to L^1-convergence of of some other modified trigonometric sums with coefficients from some new classes of real sequences.

4.1 L^1-convergence of modified sums $j_n^c(x)$ and $j_n^s(x)$ with coefficients from the class SJ

Let us consider the cosine

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

sine series

$$g(x) = \sum_{k=1}^{\infty} a_k \sin kx,$$

or together

$$\phi(x) = \sum_{k=1}^{\infty} a_k \phi_k(x),$$

where $\phi_k(x)$ is $\cos kx$ or $\sin kx$ and $\phi_(x)$ is $f(x)$ or $g(x)$ respectively.

We also consider modified cosine and sine sums

$$j_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left(a_j \cos jx \right)$$

and

$$j_n^s(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left(a_j \sin jx \right).$$

Now we prove the following.

Theorem 4.1. *Let* $\{a_k\} \in \mathbf{SJ}$. *Then*

(i) $\lim_{n\to\infty} t_n(x) = t(x)$ *exists for all* $x \in (0,\pi]$, *where* $t_n(x)$ *is either* $j_n^c(x)$ *or* $j_n^s(x)$,
(ii) $t(x) \in L^1(0,\pi]$, *and*
(iii) $\|t - S_n(t)\| = o(1)$ *as* $n \to \infty$.

Proof. We are going to give the proof of this statement only for cosine sums since for the sine sums the proof can be done in the same manner. We note that

$$\begin{aligned} j_n^c(x) &= \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(a_j \cos jx\right) \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} \left[\Delta\left(a_k \cos kx\right) + \cdots + \Delta\left(a_n \cos nx\right)\right] \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \sum_{k=1}^{n} a_{n+1} \cos(n+1)x \\ &= S_n(x) - n a_{n+1} \cos(n+1)x. \end{aligned}$$

Since $A_k \downarrow 0$ as $k \to \infty$ and $\sum_{k=1}^{\infty} A_k < \infty$, then by Olivier's theorem we have $kA_k \to 0$ as $k \to \infty$ and therefore

$$n a_n = n^2 \sum_{k=n}^{\infty} \Delta\left(\frac{a_k}{k}\right) \le \sum_{k=n}^{\infty} k^2 \left(\frac{A_k}{k}\right) = o(1) \quad as \quad n \to \infty.$$

Also $\cos(n+1)x$ is finite in $(0,\pi]$ and therefore

$$\lim_{n\to\infty} t_n(x) = \lim_{n\to\infty} S_n(x) = t(x).$$

Moreover,

$$t(x) = \lim_{n\to\infty} t_n(x) = \lim_{n\to\infty} S_n(x) = \frac{a_0}{2} + \lim_{n\to\infty} \sum_{k=1}^{n} a_k \cos kx.$$

Abel's transformation implies that

$$\frac{a_0}{2} + \lim_{n\to\infty} \sum_{k=1}^{\infty} a_k \cos kx = \lim_{n\to\infty} \left[\sum_{k=1}^{n-1} \Delta\left(\frac{a_k}{k}\right) \widetilde{D}_k'(x) + \frac{a_n}{n} \widetilde{D}_n'(x)\right]$$

or

$$t(x) = \sum_{k=1}^{\infty} \Delta\left(\frac{a_k}{k}\right) \widetilde{D}_k'(x).$$

Based on our assumptions and Lemma 1.83 the series $\sum_{k=1}^{\infty} \Delta\left(\frac{a_k}{k}\right) \widetilde{D}_k'(x)$ converges, and hence the limit-function $t(x)$ exists for $x \in (0,\pi]$ and subsequently the statement (i) holds true.

For $x \neq 0$ we have

$$t(x) - t_n(x) = \sum_{k=n+1}^{\infty} a_k \cos kx + na_{n+1}\cos(n+1)x$$
$$= \lim_{m\to\infty}\left[\sum_{k=n+1}^{m}\left(\frac{a_k}{k}\right) k\cos kx\right] + na_{n+1}\cos(n+1)x.$$

Now applying Abel's transformation, Lemma 1.85 and Lemma 1.86 we get

$$\begin{aligned}
&\int_0^\pi |t(x) - t_n(x)|dx \\
&= \int_0^\pi \left|\sum_{k=n+1}^{\infty} \Delta\left(\frac{a_k}{k}\right)\widetilde{D}_k'(x) - \frac{a_{n+1}}{n+1}\widetilde{D}_n'(x) + na_{n+1}\cos(n+1)x\right| dx \\
&\le \int_0^\pi \left|\sum_{k=n+1}^{\infty}\left(\frac{A_k}{k}\right)\frac{\Delta\left(\frac{a_k}{k}\right)}{\frac{A_k}{k}}\widetilde{D}_k'(x)\right| dx \\
&\quad + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}_n'(x)\right| dx + \int_0^\pi |na_{n+1}\cos(n+1)x|\,dx \\
&\le \int_0^\pi \left|\sum_{k=n+1}^{\infty}\Delta\left(\frac{A_k}{k}\right)\sum_{j=1}^{k}\frac{\Delta\left(\frac{a_j}{j}\right)}{\frac{A_j}{j}}\widetilde{D}_j'(x)\right| dx \\
&\quad + \left|\frac{a_{n+1}}{n+1}\right|\int_0^\pi \left|\widetilde{D}_n'(x)\right| dx + |na_{n+1}|\int_0^\pi |\cos(n+1)x|\,dx \\
&\le \sum_{k=n+1}^{\infty}\Delta\left(\frac{A_k}{k}\right)\int_0^\pi\left|\sum_{j=1}^{k}\frac{\Delta\left(\frac{a_j}{j}\right)}{\frac{A_j}{j}}\widetilde{D}_j'(x)\right| dx \\
&\quad + \frac{a_{n+1}}{n+1}\mathcal{O}((n+1)\log n) + na_{n+1}\cdot\frac{2}{n+1} \\
&= \mathcal{O}\left(\sum_{k=n+1}^{\infty}(k+1)^2\Delta\left(\frac{A_k}{k}\right) + na_{n+1}\right).
\end{aligned}$$

However using the identity

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n-1}\frac{k(k+1)}{2}\Delta\left(\frac{A_k}{k}\right) + \frac{n(n+1)}{2}\cdot\frac{A_n}{n}$$

and $\{a_n\} \in \mathbf{SJ}$ we get

$$\frac{n(n+1)}{2}\cdot\frac{A_n}{n} = (n+1)A_n \to 0 \quad as \quad n\to\infty,$$

which means that the series

$$\sum_{k=1}^{\infty}(k+1)^2\Delta\left(\frac{A_k}{k}\right)$$

converges. Since we have already proved that $na_n = o(1) \quad as \quad n \to \infty$, it follows that $\|t - t_n\| = o(1)$ as $n \to \infty$. Finally, the fact that $t_n(x)$ is a trigonometric polynomial implies that $t \in L^1(0, \pi]$. This conclusion verifies completely statement (ii).

Now we consider,

$$\begin{aligned}\|t - S_n\| &\le \|t - t_n + t_n - S_n\| \\ &\le \|t - t_n\| + \|t_n - S_n\| \\ &= \|t - t_n\| + \int_0^{\pi} |na_{n+1}\cos(n+1)x|\,dx \\ &= o(1) + n|a_{n+1}| \cdot \frac{2}{n+1} = o(1) \quad as \quad n \to \infty.\end{aligned}$$

The proof of (iii) is completed.

The proof of theorem is completed.

Remark 4.2. This theorem holds true for a weaker class than the class **S**, but the results of Theorem 4.1 have been proved without any conditions like $a_n \log n \to 0$ as $n \to \infty$.

4.2 L^1-convergence of modified sums $K_n^s(x)$ with coefficients from the class K

In this unit we consider the cosine

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

series and the modified sine sums of the form

$$K_n^s(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_{j-1} - \Delta a_{j+1})\sin kx.$$

Then we prove the following statement.

Theorem 4.3. *If $\{a_k\} \in \mathbf{K}$, then $K_n^s(x)$ converges to $g(x)$ in L^1-norm.*

Proof. By definition of the class **K** we know that $a_0 = 0$ and making some elementary transformation we obtain

$$\begin{aligned}S_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx = \frac{1}{2\sin x}\sum_{k=1}^{n} 2a_k \sin x \cos kx \\ &= \frac{1}{2\sin x}\sum_{k=1}^{n} a_k[\sin(k+1)x - \sin(k-1)x] \\ &= \frac{1}{2\sin x}\sum_{k=1}^{n}(a_{k-1}-a_{k+1})\sin kx + a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x}.\end{aligned}$$

Applying Abel's transformation, we have

$$\begin{aligned}S_n(x) = &\frac{1}{2\sin x}\left(\sum_{k=1}^{n}(\Delta a_{k-1}-\Delta a_{k+1})\widetilde{D}_k(x) + (a_n - a_{n+2})\widetilde{D}_n(x)\right) \\ &+ a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x},\end{aligned}$$

where $\widetilde{D}_k(x)$ is the Dirichlet conjugate kernel. Thus

$$g(x) = \lim_{n\to\infty} S_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{\infty}(\Delta a_{k-1}-\Delta a_{k+1})\widetilde{D}_k(x)$$

if the series is convergent. Also

$$\begin{aligned}K_n^s(x) &= \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_{j-1}-\Delta a_{j+1})\sin kx \\ &= \frac{1}{2\sin x}\left(\sum_{k=1}^{n}(a_{k-1}-a_{k+1})\sin kx - (a_n - a_{n+2})\widetilde{D}_n(x)\right)\end{aligned}$$

in which after applying the Abel's transformation again we get

$$K_n^s(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}(\Delta a_{k-1}-\Delta a_{k+1})\widetilde{D}_k(x).$$

Before we go further we have to prove that the series

$$\frac{1}{2\sin x}\sum_{k=1}^{n}(\Delta a_{k-1}-\Delta a_{k+1})\widetilde{D}_k(x),$$

converges.

Indeed, since $\left|\frac{\widetilde{D}_k(x)}{2\sin x}\right| = \mathcal{O}(k)$ and

$$\sum_{k=1}^{\infty} k|\Delta^2 a_{k-1} - \Delta^2 a_{k+1}| < \infty,$$

then the series

$$\frac{1}{2\sin x}\sum_{k=1}^{n}(\Delta a_{k-1}-\Delta a_{k+1})\widetilde{D}_k(x)$$

is a convergent one. This means that $\lim_{n\to\infty} K_n^s(x)=g(x)$ exists.

Whence,

$$\begin{aligned} g(x)-K_n^s(x) &= \frac{1}{2\sin x}\sum_{k=n+1}^{\infty}(\Delta a_{k-1}-\Delta a_{k+1})\widetilde{D}_k(x)\\ &= \frac{1}{2\sin x}\lim_{m\to\infty}\left[\sum_{k=n+1}^{m}(\Delta a_{k-1}-\Delta a_{k+1})\widetilde{D}_k(x)\right]. \end{aligned}$$

Applying the Abel's transformation, even in this case, we obtain

$$\begin{aligned} g(x)-K_n^s(x) &= \frac{1}{2\sin x}\lim_{m\to\infty}\left[\sum_{k=n+1}^{m-1}(k+1)(\Delta^2 a_{k-1}-\Delta^2 a_{k+1})\widetilde{F}_k(x)\right.\\ &\quad +(m+1)(\Delta a_{m-1}-\Delta a_{m+1})\widetilde{F}_m(x)\\ &\quad \left.-(n+1)(\Delta a_n-\Delta a_{n+2})\widetilde{F}_n(x)\right]\\ &= \frac{1}{2\sin x}\left[\sum_{k=n+1}^{\infty}(k+1)(\Delta^2 a_{k-1}-\Delta^2 a_{k+1})\widetilde{F}_k(x)\right.\\ &\quad \left.-(n+1)(\Delta a_n-\Delta a_{n+2})\widetilde{F}_n(x)\right], \end{aligned}$$

where $\widetilde{F}_k(x)=\frac{1}{k+1}\sum_{j=0}^{k}\widetilde{D}_j(x)$ denotes the conjugate Fejér kernel.

Therefore we have

$$\begin{aligned} \int_{-\pi}^{\pi}|g(x)-K_n^s(x)|\,dx &= \int_{-\pi}^{\pi}\left|\frac{1}{2\sin x}\left[\sum_{k=n+1}^{\infty}(k+1)(\Delta^2 a_{k-1}-\Delta^2 a_{k+1})\widetilde{F}_k(x)\right.\right.\\ &\quad \left.\left.-(n+1)(\Delta a_n-\Delta a_{n+2})\widetilde{F}_n(x)\right]\right|dx\\ &\le C\left[\sum_{k=n+1}^{\infty}(k+1)|\Delta^2 a_{k-1}-\Delta^2 a_{k+1}|\int_{-\pi}^{\pi}\left|\widetilde{F}_k(x)\right|dx\right.\\ &\quad \left.+(n+1)|\Delta a_n-\Delta a_{n+2}|\int_{-\pi}^{\pi}\left|\widetilde{F}_n(x)\right|dx\right]. \end{aligned}$$

Taking into account that $\int_{-\pi}^{\pi}\left|\widetilde{F}_k(x)\right|dx=\pi$ and

$$(n+1)|\Delta a_n-\Delta a_{n+2}| = (n+1)\left|\sum_{k=n}^{\infty}(\Delta^2 a_k-\Delta^2 a_{k+2})\right|$$

$$= (n+1)\left|\sum_{k=n+1}^{\infty}\frac{k}{k}(\varDelta^2 a_{k-1}-\varDelta^2 a_{k+1})\right|$$
$$\leq \frac{n+1}{n+1}\sum_{k=n+1}^{\infty} k\left|\varDelta^2 a_{k-1}-\varDelta^2 a_{k+1}\right| = o(1)$$

as $n\to\infty$, we get

$$\int_{-\pi}^{\pi}|g(x)-K_n^s(x)|\,dx = \mathcal{O}\left(\sum_{k=n+1}^{\infty}(k+1)|\varDelta^2 a_{k-1}-\varDelta^2 a_{k+1}|\right)+o(1)=o(1)$$

as $n\to\infty$.

The proof is completed.

As an important consequence of this theorem is the following.

Corollary 4.4. *Let* $\{a_k\}\in\mathbf{K}$. *The necessary and sufficient condition for the* L^1*-convergence of the cosine series is* $\lim_{n\to\infty} a_n\log n = 0$.

Proof. Firstly we have

$$\|S_n(x)-g(x)\|\leq\|S_n(x)-K_n^s(x)\|+\|K_n^s(x)-g(x)\| = \|K_n^s(x)-g(x)\|$$
$$+\left\|(a_n-a_{n+2})\frac{\widetilde{D}_n(x)}{2\sin x}+a_{n+1}\frac{\sin nx}{2\sin x}+a_n\frac{\sin(n+1)x}{2\sin x}\right\|.$$

Also we have

$$\left\|(a_n-a_{n+2})\frac{\widetilde{D}_n(x)}{2\sin x}+a_{n+1}\frac{\sin nx}{2\sin x}+a_n\frac{\sin(n+1)x}{2\sin x}\right\|$$
$$=\|S_n(x)-K_n^s(x)\|\leq\|K_n^s(x)-g(x)\|+\|S_n(x)-g(x)\|,$$

and

$$n|a_n-a_{n+2}| = n\left|\sum_{k=n}^{\infty}(\varDelta a_k-\varDelta a_{k+2})\right|$$
$$= n\left|\sum_{k=n+1}^{\infty}\frac{k}{k}(\varDelta a_{k-1}-\varDelta a_{k+2}-\varDelta a_{k+1}+\varDelta a_{k+2})\right|$$
$$\leq\frac{n+1}{n+1}\sum_{k=n+1}^{\infty}k\left|\varDelta^2 a_{k-1}-\varDelta^2 a_{k+1}\right| = o(1),$$

as $n\to\infty$.

Hence, since $\int_{-\pi}^{\pi}\left|\frac{\widetilde{D}_n(x)}{2\sin x}\right|dx=\mathcal{O}(n)$ we obtain

$$|a_n-a_{n+2}|\int_{-\pi}^{\pi}\left|\frac{\widetilde{D}_n(x)}{2\sin x}\right|dx=\mathcal{O}(n|a_n-a_{n+2}|)=o(1)$$

as $n \to \infty$.

Moreover,

$$\int_{-\pi}^{\pi} \left| a_{n+1} \frac{\sin nx}{2 \sin x} + a_n \frac{\sin(n+1)x}{2 \sin x} \right| dx$$
$$\le \int_{-\pi}^{\pi} a_n \left| \frac{\sin nx}{2 \sin x} + \frac{\sin(n+1)x}{2 \sin x} \right| dx = a_n \int_{-\pi}^{\pi} |D_n(x)| dx \sim a_n \log n.$$

Since $\|K_n^s(x) - g(x)\| = o(1)$ as $n \to \infty$, by the above theorem, then

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} |S_n(x) - g(x)| = 0$$

if and only if

$$\lim_{n \to \infty} a_n \log n = 0.$$

The proof is completed.

4.3 L^1-convergence of modified sums $K_n^c(x)$ with coefficients from the class J

Here we are going to consider the sine

$$f(x) = \sum_{k=1}^{\infty} a_k \sin kx,$$

series and the modified cosine sums of the form

$$K_n^c(x) = \frac{1}{2 \sin x} \sum_{k=1}^{n} \sum_{j=k}^{n} (\Delta a_{j-1} - \Delta a_{j+1}) \cos kx,$$

where $a_0 := a_1 := 0$.

The main result is the following statement.

Theorem 4.5. *If $\{a_k\} \in \mathbf{J}$, then $K_n^c(x)$ converges to $f(x)$ in L^1-norm.*

Proof. We have

$$\begin{aligned} K_n^c(x) &= \frac{1}{2 \sin x} \sum_{k=1}^{n} \sum_{j=k}^{n} (\Delta a_{j-1} - \Delta a_{j+1}) \cos kx \\ &= \frac{1}{2 \sin x} \left[\sum_{k=1}^{n} (a_{k+1} - a_{k-1}) \cos kx + (a_n - a_{n+2}) \right] D_n(x) \\ &= \sum_{k=1}^{n} a_k \sin kx + \frac{1}{2 \sin x} [a_n \cos(n+1)x \\ &\quad + a_{n+1} \cos nx + (a_n - a_{n+2}) D_n(x)] \\ &= S_n(x) + \frac{1}{2 \sin x} [a_n \cos(n+1)x \\ &\quad + a_{n+1} \cos nx + (a_n - a_{n+2}) D_n(x)]. \end{aligned}$$

Since $\left|\frac{\cos nx}{2\sin x}\right|$, $\left|\frac{D_n(x)}{2\sin x}\right|$ are bounded in $(0,\pi)$, and $a_n \to 0$ as $n\to\infty$, then the last three terms of the last equality tend to zero as $n\to\infty$.

Thus

$$f(x) = \lim_{n\to\infty} K_n^c(x) = \lim_{n\to\infty} S_n(x).$$

Considering the partial sums

$$S_n(x) = \sum_{k=1}^{n} a_k \sin kx$$

of the sine series we apply the Abel's transformation:

$$S_n(x) = \sum_{k=1}^{n-1} \Delta a_k \widetilde{D}_k(x) + a_n \widetilde{D}_n(x).$$

Since $\left|\widetilde{D}_k(x)\right| = \mathcal{O}(k)$,

$$\left|\Delta\left(\frac{a_k}{k}\right)\right| \le \frac{A_k}{k} \Longrightarrow |\Delta a_k| \le A_k, \quad \forall k \in \{1,2,\ldots\},$$

then

$$\lim_{n\to\infty} S_n(x) = \mathcal{O}\left(\sum_{k=1}^{\infty} kA_k\right) + o(na_n),$$

and based on our assumptions it holds

$$na_n = n\sum_{k=n}^{\infty} \Delta a_k \le \sum_{k=n}^{\infty} kA_k = o(1), \quad n\to\infty.$$

So, $\lim_{n\to\infty} S_n(x) = f(x)$ exists.

Whence, applying Abel's transformation again we have

$$\begin{aligned}
\int_{-\pi}^{\pi} |f(x) - K_n^c(x)|\,dx &= \int_{-\pi}^{\pi} \Bigg| \sum_{k=n+1}^{\infty} a_k \sin kx - \frac{1}{2\sin x}[a_n \cos(n+1)x \\
&\quad + a_{n+1}\cos nx + (a_n - a_{n+2})D_n(x)]\Bigg|dx \\
&\le \int_{-\pi}^{\pi} \left|\sum_{k=n+1}^{\infty} \Delta a_k \widetilde{D}_k(x)\right|dx + \int_{-\pi}^{\pi}\left|a_{n+1}\widetilde{D}_n(x)\right|dx \\
&\quad + \int_{-\pi}^{\pi}\left|\frac{1}{2\sin x} a_n \cos(n+1)x\right|dx + \int_{-\pi}^{\pi}\left|\frac{1}{2\sin x} a_{n+1}\cos nx\right|dx \\
&\quad + \int_{-\pi}^{\pi}\left|(a_n - a_{n+2})D_n(x)\right|dx
\end{aligned}$$

or

$$\begin{aligned}\|f(x)-K_n^c(x)\| &= \mathcal{O}\left(\sum_{k=n+1}^{\infty} A_k \log k\right) + o(a_{n+1}\log n) \\ &\quad + o(a_{n+1}\log n) + o(a_n \log n) + o(n^2(a_n - a_{n+2})) \\ &= o(1) + o(1) + o(1) + o(1) + o(n^2(a_n - a_{n+2})) \\ &= o(1) + o(n^2(a_n - a_{n+2})).\end{aligned}$$

Since, by assumptions, $A_k \downarrow 0$ as $k \to \infty$ and $\sum_{k=1}^{\infty} kA_k < \infty$, then by Olivier's theorem we have $k^2 A_k \to 0$ as $k \to \infty$, which implies that

$$\begin{aligned}n^2|a_n - a_{n+2}| &= n^2|a_n - a_{n+1} + a_{n+1} - a_{n+2}| \\ &= n^2|\Delta a_n + \Delta a_{n+1}| \\ &\le n^2|\Delta a_n| + |\Delta a_{n+1}|) \\ &\le n^2 A_n + (n+1)^2 A_{n+1}) = o(1), \quad n \to \infty.\end{aligned}$$

Subsequently,

$$\|f(x) - K_n^c(x)\| = o(1), \quad n \to \infty.$$

The proof is completed.

As an important consequence of this theorem is the following.

Corollary 4.6. *Let* $\{a_k\} \in \mathbf{J}$. *Then* $\|f - S_n\| = o(1)$ *as* $n \to \infty$.

Proof. We have

$$\begin{aligned}\|S_n(x) - f(x)\| &= \|S_n(x) - K_n^c(x) + K_n^c(x) - f(x)\| \\ &\le \|S_n(x) - K_n^c(x)\| + \|K_n^s(x) - f(x)\| \\ &\le \|K_n^s(x) - f(x)\| \\ &\quad + \int_0^{\pi} \left|\frac{1}{2\sin x}[a_n \cos(n+1)x + a_{n+1}\cos nx]\right| dx \\ &\quad + \int_0^{\pi} |(a_n - a_{n+2})D_n(x)|\, dx.\end{aligned}$$

The conclusion of the corollary follows by making the same augmentations as in the proof of the above theorem.

The proof is completed.

4.4 L^1-convergence of modified sums $K_n^s(x)$ with semi-convex coefficients

Here we are going to show the L^1-convergence of the cosine

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

series with semi-convex coefficients using the modified sine sums of the form

$$K_n^s(x) = \frac{1}{2\sin x} \sum_{k=1}^{n} \sum_{j=k}^{n} (\Delta a_{j-1} - \Delta a_{j+1}) \sin kx.$$

Namely,

Theorem 4.7. *Let $\{a_k\}$ be a semi-convex null sequence. Then $K_n^s(x)$ converges to $g(x)$ in L^1-norm.*

Proof. We have

$$\begin{aligned} S_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx \\ &= \frac{1}{2\sin x} \sum_{k=1}^{n} a_k \cos kx 2 \sin x \\ &= \frac{1}{2\sin x} \sum_{k=1}^{n} a_k \left[\sin(k+1)x - \sin(k-1)x\right] \\ &= \frac{1}{2\sin x} \sum_{k=1}^{n} (a_{k-1} - a_{k+1}) \sin kx + a_{n+1} \frac{\sin nx}{2\sin x} + a_n \frac{\sin(n+1)x}{2\sin x} \\ &= \frac{1}{2\sin x} \sum_{k=1}^{n} (\Delta a_k + \Delta a_{k-1}) \sin kx + a_{n+1} \frac{\sin nx}{2\sin x} + a_n \frac{\sin(n+1)x}{2\sin x} \end{aligned}$$

Applying Abel's transformation, we have

$$\begin{aligned} S_n(x) = {} & \frac{1}{2\sin x} \left(\sum_{k=1}^{n} \left(\Delta^2 a_k + \Delta^2 a_{k+1}\right) \widetilde{D}_k(x) + (a_n - a_{n+2}) \widetilde{D}_n(x) \right) \\ & + a_{n+1} \frac{\sin nx}{2\sin x} + a_n \frac{\sin(n+1)x}{2\sin x}. \end{aligned}$$

Thus

$$g(x) = \lim_{n\to\infty} S_n(x) = \frac{1}{2\sin x} \sum_{k=1}^{\infty} \left(\Delta^2 a_k + \Delta^2 a_{k-1}\right) \widetilde{D}_k(x).$$

Also

$$\begin{aligned} g_n(x) &= \frac{1}{2\sin x} \sum_{k=1}^{n} \sum_{j=k}^{n} (\Delta a_{j-1} - \Delta a_{j+1}) \sin kx \\ &= \frac{1}{2\sin x} \left(\sum_{k=1}^{n} (a_{k-1} - a_{k+1}) \sin kx - (a_n - a_{n+2}) \widetilde{D}_n(x) \right). \end{aligned}$$

Applying Abel's transformation, we have

$$\begin{aligned} g_n(x) &= \frac{1}{2\sin x}\sum_{k=1}^{n}\left(\Delta a_{k-1}-\Delta a_{k+1}\right)\widetilde{D}_k(x) \\ &= \frac{1}{2\sin x}\sum_{k=1}^{n}\left(\Delta^2 a_k+\Delta^2 a_{k-1}\right)\widetilde{D}_k(x) \end{aligned}$$

and

$$\begin{aligned} g(x)-g_n(x) &= \frac{1}{2\sin x}\sum_{k=n+1}^{\infty}\left(\Delta^2 a_k+\Delta^2 a_{k-1}\right)\widetilde{D}_k(x) \\ &= \lim_{m\to\infty}\left(\frac{1}{2\sin x}\sum_{k=n+1}^{m}\left(\Delta^2 a_k+\Delta^2 a_{k-1}\right)\widetilde{D}_k(x)\right). \end{aligned}$$

Consequently, taking into account that $\{a_k\}$ is semi-convex, we have

$$\int_{-\pi}^{\pi}|g(x)-g_n(x)|\,dx=\mathcal{O}\left(\sum_{k=n+1}^{\infty}k\left|\left(\Delta^2 a_k+\Delta^2 a_{k-1}\right)\right|\right)=o(1),\ n\to\infty.$$

The proof is completed.

As a consequence of this theorem is the following.

Corollary 4.8. *If $\{a_n\}$ be the semi-convex null sequence, then the necessary and sufficient for L^1-convergence of the cosine series is* $\lim_{n\to\infty} a_n\log n=0$.

Proof. We have

$$\begin{aligned} \|S_n(x)-g(x)\| &\le \|S_n(x)-g_n(x)\|+\|g_n(x)-g(x)\| \\ &= \|g_n(x)-g(x)\| \\ &\quad+\left\|(a_n-a_{n+2})\frac{\widetilde{D}_n(x)}{2\sin x}+a_{n+1}\frac{\sin nx}{2\sin x}+a_n\frac{\sin(n+1)x}{2\sin x}\right\|. \end{aligned}$$

Also

$$\begin{aligned} &\left\|(a_n-a_{n+2})\frac{\widetilde{D}_n(x)}{2\sin x}+a_{n+1}\frac{\sin nx}{2\sin x}+a_n\frac{\sin(n+1)x}{2\sin x}\right\| \\ &\qquad= \|g_n(x)-S_n(x)\|\le\|g_n(x)-g(x)\|+\|S_n(x)-g(x)\|, \end{aligned}$$

and

$$\begin{aligned}|(a_n - a_{n+2})| &= \left|\sum_{k=n}^{\infty}(\Delta a_k - \Delta a_{k+2})\right| \\ &= \left|\sum_{k=n+1}^{\infty}\frac{k}{k}(\Delta a_{k-1} - \Delta a_{k+1})\right| \\ &\le \frac{1}{n}\left|\sum_{k=n+1}^{\infty} k\left(\Delta^2 a_k + \Delta^2 a_{k-1}\right)\right| = o\left(\frac{1}{n}\right).\end{aligned}$$

Since $\int\limits_{-\pi}^{\pi}\frac{\widetilde{D}_n(x)}{2\sin x}dx = \mathcal{O}(n)$, then

$$(a_n - a_{n+2})\int\limits_{-\pi}^{\pi}\frac{\widetilde{D}_n(x)}{2\sin x}dx = \mathcal{O}\left((a_n - a_{n+2})n\right) = o(1).$$

Moreover,

$$\begin{aligned}\int\limits_{-\pi}^{\pi}\left|a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x}\right| dx &\le \int\limits_{-\pi}^{\pi} a_n\left|\frac{\sin nx}{2\sin x} + \frac{\sin(n+1)x}{2\sin x}\right| dx \\ &= a_n\int\limits_{-\pi}^{\pi}|D_n(x)|\, dx \sim (a_n \log n).\end{aligned}$$

Since $\|g_n(x) - g(x)\| = o(1)$, as $n \to \infty$, by the already proved theorem, then it implies that

$$\lim_{n\to\infty}\int\limits_{-\pi}^{\pi}|g(x) - S_n(x)|\, dx = 0$$

if and only if $\lim\limits_{n\to\infty} a_n \log n = 0$.

The proof is completed.

4.5 L^1-convergence of modified sums $K_n^s(x)$ with coefficients from the class $\mathbf{K}^\alpha$

Let

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

and

$$K_n^s(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_{j-1} - \Delta a_{j+1})\sin kx.$$

We give the proof of next result.

Theorem 4.9. *Let* $\{a_n\} \in \mathbf{K}^\alpha$*, where* $\alpha > 0$ *is a real number. Then* $K_n^s(x)$ *converges to* $g(x)$ *in the* L^1*-norm.*

Proof. We have

$$K_n^s(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_{j-1} - \Delta a_{j+1})\sin kx$$
$$= \frac{1}{2\sin x}\sum_{k=1}^{n}(\Delta a_{k-1} - \Delta a_{k+1})\,\widetilde{D}_k(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}(\Delta a_{k-1} - \Delta a_{k+1})\,\widetilde{S}_k(x).$$

where

$$\widetilde{D}_k(x) := \widetilde{S}_k(x) := \sum_{j=1}^{k}\sin kx.$$

Part 1. Let $\alpha > 0$ be non-integral. Applying Lemma 1.1 α times we obtain

$$K_n^s(x) = \frac{1}{2\sin x}\Bigg\{\sum_{k=1}^{n-\alpha}(\Delta^{\alpha+1}a_{k-1} - \Delta^{\alpha+1}a_{k+1})\widetilde{S}_k^\alpha(x)$$
$$+\sum_{k=1}^{\alpha}(\Delta^k a_{n-k} - \Delta^k a_{n-k+2})\widetilde{S}_{n-k+1}^k(x)\Bigg\}.$$

Then

$$g(x) = \lim_{n\to\infty} K_n^s(x)$$
$$= \frac{1}{2\sin x}\sum_{k=1}^{\infty}(\Delta^{\alpha+1}a_{k-1} - \Delta^{\alpha+1}a_{k+1})\widetilde{S}_k^\alpha(x).$$

Thus we have

$$\int_0^\pi |g(x) - K_n^s(x)|dx$$
$$= \int_0^\pi \Bigg|\frac{1}{2\sin x}\sum_{k=n-\alpha+1}^{\infty}(\Delta^{\alpha+1}a_{k-1} - \Delta^{\alpha+1}a_{k+1})\widetilde{S}_k^\alpha(x)$$
$$-\sum_{k=1}^{\alpha}(\Delta^k a_{n-k} - \Delta^k a_{n-k+2})\widetilde{S}_{n-k+1}^k(x)\Bigg|dx$$
$$\le C\int_0^\pi \Bigg|\sum_{k=n-\alpha+1}^{\infty}(\Delta^{\alpha+1}a_{k-1} - \Delta^{\alpha+1}a_{k+1})\widetilde{S}_k^\alpha(x)\Bigg|dx$$
$$+C\int_0^\pi \Bigg|\sum_{k=1}^{\alpha}(\Delta^k a_{n-k} - \Delta^k a_{n-k+2})\widetilde{S}_{n-k+1}^k(x)\Bigg|dx$$

$$\le C \sum_{k=n-\alpha+1}^{\infty} |\Delta^{\alpha+1} a_{k-1} - \Delta^{\alpha+1} a_{k+1}| \int_0^\pi \left|\widetilde{S}_k^\alpha(x)\right| dx$$
$$+C \sum_{k=1}^{\alpha} |\Delta^k a_{n-k} - \Delta^k a_{n-k+2}| \int_0^\pi \left|\widetilde{S}_{n-k+1}^k(x)\right| dx$$
$$= C \sum_{k=n-\alpha+1}^{\infty} A_k^\alpha |\Delta^{\alpha+1} a_{k-1} - \Delta^{\alpha+1} a_{k+1}| \int_0^\pi \left|\widetilde{T}_k^\alpha(x)\right| dx$$
$$+C \sum_{k=1}^{\alpha} A_{n-k+1}^\alpha |\Delta^k a_{n-k} - \Delta^k a_{n-k+2}| \int_0^\pi \left|\widetilde{T}_{n-k+1}^k(x)\right| dx$$
$$\le C_1 \sum_{k=n-\alpha+1}^{\infty} A_k^\alpha |\Delta^{\alpha+1} a_{k-1} - \Delta^{\alpha+1} a_{k+1}|$$
$$+C_1 \sum_{k=1}^{\alpha} A_{n-k+1}^\alpha |\Delta^k a_{n-k} - \Delta^k a_{n-k+2}| = O(1) + o(1) = o(1),$$

by Lemmas 1.48, 1.90 and hypothesis of the theorem. So,

$$\lim_{n\to\infty} \int_0^\pi |g(x) - K_n^s(x)| dx = 0.$$

Part 2. Let $\alpha > 0$ be non-integral. Let $\alpha = r + \delta$, r is the integral part of α, and δ is the fractional part i.e. $0 < \delta < 1$.

Case (i). Let $r = 0$. Applying Abel's transformation of order $-\delta$, we have

$$\sum_{k=1}^{n} \widetilde{S}_k^\delta (\Delta^{\delta+1} a_{k-1} - \Delta^{\delta+1} a_{k+1})$$
$$= \sum_{k=1}^{n} \widetilde{S}_k \sum_{m=0}^{n-k} A_m^{\delta+1} (\Delta^{\delta+1} a_{m+k-1} - \Delta^{\delta+1} a_{m+k+1}).$$

Also applying Lemma 1.49 we have

$$\sum_{k=1}^{n} \widetilde{S}_k^\delta(x) (\Delta^{\delta+1} a_{k-1} - \Delta^{\delta+1} a_{k+1})$$
$$= \sum_{k=1}^{n} \widetilde{S}_k(x) \Bigg\{ (\Delta a_{k-1} - \Delta a_{k+1})$$
$$- \sum_{m=n-k+1}^{\infty} A_m^{\delta-1} (\Delta^{\delta+1} a_{m+k-1} - \Delta^{\delta+1} a_{m+k+1}) \Bigg\}$$
$$= \sum_{k=1}^{n} \widetilde{S}_k(x) (\Delta a_{k-1} - \Delta a_{k+1}) - R_n(x),$$

where

$$R_n(x) := \sum_{k=1}^{n} \widetilde{S}_k(x)\Big\{A_{n-k+1}^{\delta-1}(\Delta^{\delta+1}a_{n+2} - \Delta^{\delta+1}a_n) \\ + A_{n-k+2}^{\delta-1}(\Delta^{\delta+1}a_{n+3} - \Delta^{\delta+1}a_{n+1}) + \cdots\Big\}.$$

Therefore,

$$\frac{1}{2\sin x}\sum_{k=1}^{n} \widetilde{S}_k(x)\Bigg\{(\Delta a_{k-1} - \Delta a_{k+1}) \\ = \frac{1}{2\sin x}\Bigg\{\sum_{k=1}^{n} \widetilde{S}_k^{\delta}(x)(\Delta^{\delta+1}a_{k-1} - \Delta^{\delta+1}a_{k+1}) + R_n(x)\Bigg\},$$

and consequently

$$K_n^s(x) = \frac{1}{2\sin x}\Bigg\{\sum_{k=1}^{n} \widetilde{S}_k^{\delta}(x)(\Delta^{\delta+1}a_{k-1} - \Delta^{\delta+1}a_{k+1}) + R_n(x)\Bigg\}.$$

When $r = 0$, then $\alpha = \delta$ and

$$g(x) = \frac{1}{2\sin x}\sum_{k=1}^{\infty} \widetilde{S}_k^{\delta}(x)\left(\Delta^{\delta+1}a_{k-1} - \Delta^{\delta+1}a_{k+1}\right).$$

Therefore,

$$\begin{aligned}
&\int_0^{\pi} |g(x) - K_n^s(x)|dx \\
&= \int_0^{\pi} \left|\frac{1}{2\sin x}\sum_{k=n+1}^{\infty} \widetilde{S}_k^{\delta}(x)\left(\Delta^{\delta+1}a_{k-1} - \Delta^{\delta+1}a_{k+1}\right) - R_n(x)\right| \\
&\le C\Bigg\{\sum_{k=n+1}^{\infty}\int_0^{\pi}\left|\widetilde{S}_k^{\delta}(x)\right|dx\left|\Delta^{\delta+1}a_{k-1} - \Delta^{\delta+1}a_{k+1}\right| + \int_0^{\pi}|R_n(x)|dx\Bigg\} \\
&\le C\Bigg\{\sum_{k=n+1}^{\infty} A_k^{\delta}\int_0^{\pi}\left|\widetilde{T}_k^{\delta}(x)\right|dx\left|\Delta^{\delta+1}a_{k-1} - \Delta^{\delta+1}a_{k+1}\right| + \int_0^{\pi}|R_n(x)|dx\Bigg\} \\
&\le C_1\sum_{k=n+1}^{\infty} A_k^{\delta}\left|\Delta^{\delta+1}a_{k-1} - \Delta^{\delta+1}a_{k+1}\right| + C\int_0^{\pi}|R_n(x)|dx \\
&= o(1) + C\int_0^{\pi}|R_n(x)|dx,
\end{aligned}$$

by Lemma 1.48 and Lemma 1.90.

Now we estimate $\int_0^{\pi}\left|R_n(x)\right|dx$ using Lemmas 1.48, 1.50, and 1.90:

$$\begin{aligned}
&\int_0^\pi \left|R_n(x)\right| dx \\
&\quad = \int_0^\pi \left| \sum_{k=1}^n \widetilde{S}_k(x) \Big\{ A_{n-k+1}^{\delta-1} \left(\Delta^{\delta+1} a_{n+2} - \Delta^{\delta+1} a_n\right) \right. \\
&\qquad\quad \left. + A_{n-k+2}^{\delta-1} \left(\Delta^{\delta+1} a_{n+3} - \Delta^{\delta+1} a_{n+1}\right) + \cdots \Big\} \right| dx \\
&\quad \le \left|\Delta^{\delta+1} a_{n+2} - \Delta^{\delta+1} a_n\right| \int_0^\pi \left| \sum_{k=1}^n A_{n-k+1}^{\delta-1} \widetilde{S}_k(x) \right| dx \\
&\qquad\quad + \left|\Delta^{\delta+1} a_{n+3} - \Delta^{\delta+1} a_{n+1}\right| \int_0^\pi \left| \sum_{k=1}^n A_{n-k+2}^{\delta-1} \widetilde{S}_k(x) \right| dx + \cdots \\
&\quad \le \left|\Delta^{\delta+1} a_{n+2} - \Delta^{\delta+1} a_n\right| \int_0^\pi \max_{0 \le p \le n+1} \left| \widetilde{S}_p^\delta(x) \right| dx \\
&\qquad\quad + \left|\Delta^{\delta+1} a_{n+3} - \Delta^{\delta+1} a_{n+1}\right| \int_0^\pi \max_{0 \le p \le n+2} \left| \widetilde{S}_p^\delta(x) \right| dx + \cdots \\
&\quad = A_{n+1}^{\delta+1} \left|\Delta^{\delta+1} a_{n+2} - \Delta^{\delta+1} a_n\right| \int_0^\pi \max_{0 \le p \le n+1} \left| \widetilde{T}_p^{\delta-1}(x) \right| dx \\
&\qquad\quad + A_{n+2}^{\delta} \left|\Delta^{\delta+1} a_{n+3} - \Delta^{\delta+1} a_{n+1}\right| \int_0^\pi \max_{0 \le p \le n+2} \left| \widetilde{T}_p^{\delta}(x) \right| dx + \cdots \\
&\quad \le C A_{n+1}^{\delta} \left|\Delta^{\delta+1} a_{n+2} - \Delta^{\delta+1} a_n\right| \\
&\qquad\quad + C A_{n+2}^{\delta} \left|\Delta^{\delta+1} a_{n+3} - \Delta^{\delta+1} a_{n+1}\right| + \cdots \\
&\quad = o(1) + o(1) + \cdots = o(1).
\end{aligned}$$

Thus, it follows that

$$\lim_{n\to\infty} \|g(x) - K_n^s(x)\| = 0.$$

Case (ii). Let $r \ge 1$. Applying Abel's transformation of order r we have

$$\begin{aligned}
K_n^s(x) &= \frac{1}{2\sin x} \sum_{k=1}^n (\Delta a_{k-1} - \Delta a_{k+1}) \widetilde{S}_k(x) \\
&= \frac{1}{2\sin x} \sum_{k=1}^{n-r} (\Delta^{r+1} a_{k-1} - \Delta^{r+1} a_{k+1}) \widetilde{S}_k^r(x) \\
&\quad + \sum_{k=1}^r (\Delta^k a_{k-k} - \Delta^k a_{n-k+2}) \widetilde{S}_{n-k+1}^k(x).
\end{aligned}$$

Applying Abel's transformation of order $-\delta$ again, we get

$$\frac{1}{2\sin x}\sum_{k=1}^{n}\widetilde{S}_k^{\alpha-1}(x)(\varDelta^{\alpha+1}a_{k-1}-\varDelta^{\alpha+1}a_{k+1})$$
$$=\frac{1}{2\sin x}\sum_{k=1}^{n}\widetilde{S}_k^{r}(x)\sum_{m=0}^{n-k}A_m^{\delta-1}(\varDelta^{\alpha+1}a_{m+k-1}-\varDelta^{\alpha+1}a_{m+k+1}).$$

By Lemma 1.49 we have

$$\frac{1}{2\sin x}\sum_{k=1}^{n}\widetilde{S}_k^{\alpha}(x)(\varDelta^{\alpha+1}a_{k-1}-\varDelta^{\alpha+1}a_{k+1})$$
$$=\frac{1}{2\sin x}\left\{\sum_{k=1}^{n}\widetilde{S}_k^{r}(x)(\varDelta^{r+1}a_{k-1}-\varDelta^{r+1}a_{k+1})-R_n(x)\right\},$$

where

$$R_n(x)=\sum_{k=1}^{n}\widetilde{S}_k^{r}(x)\Big\{A_{n-k+1}^{\delta-1}\left(\varDelta^{\alpha+1}a_{n+2}-\varDelta^{\alpha+1}a_n\right)$$
$$+A_{n-k+2}^{\delta-1}\left(\varDelta^{\alpha+1}a_{n+3}-\varDelta^{\alpha+1}a_{n+1}\right)+\cdots\Big\}$$
$$=\left(\varDelta^{\alpha+1}a_{n+2}-\varDelta^{\alpha+1}a_n\right)\sum_{k=1}^{n}A_{n-k+1}^{\delta-1}\widetilde{S}_k^{r}(x)$$
$$+\left(\varDelta^{\alpha+1}a_{n+3}-\varDelta^{\alpha+1}a_{n+1}\right)\sum_{k=1}^{n}A_{n-k+2}^{\delta-1}\widetilde{S}_k^{r}(x)+\cdots$$

Replacing n by $n-r$ we obtain

$$\frac{1}{2\sin x}\sum_{k=1}^{n-r}\widetilde{S}_k^{\alpha}(x)(\varDelta^{\alpha+1}a_{k-1}-\varDelta^{\alpha+1}a_{k+1})$$
$$=\frac{1}{2\sin x}\left\{\sum_{k=1}^{n-r}\widetilde{S}_k^{r}(x)(\varDelta^{r+1}a_{k-1}-\varDelta^{r+1}a_{k+1})-R_{n-r}(x)\right\}.$$

Now, based on what have obtained above, we get

$$K_n^s(x)=\frac{1}{2\sin x}\Bigg\{\sum_{k=1}^{n-r}\widetilde{S}_k^{\alpha}(x)(\varDelta^{\alpha+1}a_{k-1}-\varDelta^{\alpha+1}a_{k+1})+R_{n-r}(x)$$
$$+\sum_{k=1}^{r}(\varDelta^{k}a_{n-k}-\varDelta^{k}a_{n-k+2})\widetilde{S}_{n-k+1}^{k}(x)\Bigg\}.$$

Hence, under assumptions of theorem and Lemma 1.48, we have

$$\begin{aligned}
&\int_0^\pi |g(x) - K_n^s(x)|dx \\
&\le \int_0^\pi \left| \frac{1}{2\sin x} \left\{ \sum_{k=n-r+1}^{\infty} \widetilde{S}_k^\alpha(x)(\Delta^{\alpha+1}a_{k-1} - \Delta^{\alpha+1}a_{k+1}) \right.\right. \\
&\quad \left.\left. -R_{n-r}(x) - \sum_{k=1}^{r}(\Delta^k a_{n-k} - \Delta^k a_{n-k+2})\widetilde{S}_{n-k+1}^k(x) \right\} \right| dx \\
&\le C \sum_{k=n-r+1}^{\infty} |\Delta^{\alpha+1}a_{k-1} - \Delta^{\alpha+1}a_{k+1}| \int_0^\pi \left|\widetilde{S}_k^\alpha(x)\right| dx \\
&\quad + \int_0^\pi \left|R_{n-r}(x)\right| dx + \sum_{k=1}^{r} |\Delta^k a_{n-k} - \Delta^k a_{n-k+2}| \int_0^\pi \left|\widetilde{S}_{n-k+1}^k(x)\right| dx \\
&= C \sum_{k=n-r+1}^{\infty} A_k^\alpha |\Delta^{\alpha+1}a_{k-1} - \Delta^{\alpha+1}a_{k+1}| \int_0^\pi \left|\widetilde{T}_k^\alpha(x)\right| dx \\
&\quad + \int_0^\pi \left|R_{n-r}(x)\right| dx + \sum_{k=1}^{r} A_{n-k+1}^k |\Delta^k a_{n-k+1} - \Delta^k a_{n-k+2}| \int_0^\pi \left|\widetilde{T}_{n-k+1}^k(x)\right| dx \\
&\le C_1 \sum_{k=n-r+1}^{\infty} A_k^\alpha |\Delta^{\alpha+1}a_{k-1} - \Delta^{\alpha+1}a_{k+1}| + C \int_0^\pi \left|R_{n-r}(x)\right| dx \\
&\quad + C_1 \sum_{k=1}^{r} A_{n-k+1}^k |\Delta^k a_{n-k} - \Delta^k a_{n-k+2}| \\
&= o(1) + \int_0^\pi \left|R_{n-r}(x)\right| dx + o(1) = o(1) + \int_0^\pi \left|R_{n-r}(x)\right| dx.
\end{aligned}$$

However, by the assumptions of the theorem we obtain

$$\begin{aligned}
&\int_0^\pi \left|R_{n-r}(x)\right| dx \\
&\le \int_0^\pi \left| \left(\sum_{k=1}^{n-r} A_{n-r-k+1}^{\delta-1} \widetilde{S}_k^r(x) \right) \left(\Delta^{\alpha+1}a_{n-r+2} - \Delta^{\alpha+1}a_{n-r}\right) \right| dx \\
&\qquad + \int_0^\pi \left| \left(\sum_{k=1}^{n-r} A_{n-r-k+2}^{\delta-1} \widetilde{S}_k^r(x) \right) \left(\Delta^{\alpha+1}a_{n-r+3} - \Delta^{\alpha+1}a_{n-r+1}\right) \right| dx \\
&\qquad + \int_0^\pi \left| \left(\sum_{k=1}^{n-r} A_{n-r-k+3}^{\delta-1} \widetilde{S}_k^r(x) \right) \left(\Delta^{\alpha+1}a_{n-r+5} - \Delta^{\alpha+1}a_{n-r+2}\right) \right| dx \\
&\qquad + \cdots \\
&\le \sum_{k=1}^{n-r} A_{n-r-k+1}^{\delta-1} A_k^r |\Delta^{\alpha+1}a_{n-r+2} - \Delta^{\alpha+1}a_{n-r}| \int_0^\pi \left|\widetilde{T}_k^r(x)\right| dx
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{n-r} A_{n-r-k+2}^{\delta-1} A_k^r |\Delta^{\alpha+1} a_{n-r+3} - \Delta^{\alpha+1} a_{n-r+1}| \int_0^\pi \left|\widetilde{T}_k^r(x)\right| dx \\
&+\sum_{k=1}^{n-r} A_{n-r-k+3}^{\delta-1} A_k^r |\Delta^{\alpha+1} a_{n-r+5} - \Delta^{\alpha+1} a_{n-r+2}| \int_0^\pi \left|\widetilde{T}_k^r(x)\right| dx \\
&+\cdots \\
\le\ & C_1 \sum_{k=1}^{n-r} A_{n-r-k+1}^{\delta-1} A_k^r |\Delta^{\alpha+1} a_{n-r+2} - \Delta^{\alpha+1} a_{n-r}| \\
&+C_1 \sum_{k=1}^{n-r} A_{n-r-k+2}^{\delta-1} A_k^r |\Delta^{\alpha+1} a_{n-r+3} - \Delta^{\alpha+1} a_{n-r+1}| \\
&+C_1 \sum_{k=1}^{n-r} A_{n-r-k+3}^{\delta-1} A_k^r |\Delta^{\alpha+1} a_{n-r+5} - \Delta^{\alpha+1} a_{n-r+2}| \\
&+\cdots \\
\le\ & C_1 \sum_{k=1}^{n+1-r} A_{n+1-r-k}^{\delta-1} A_k^r |\Delta^{\alpha+1} a_{n-r+2} - \Delta^{\alpha+1} a_{n-r}| \\
&+C_1 \sum_{k=1}^{n+2-r} A_{n+2-r-k}^{\delta-1} A_k^r |\Delta^{\alpha+1} a_{n-r+3} - \Delta^{\alpha+1} a_{n-r+1}| \\
&+C_1 \sum_{k=1}^{n+3-r} A_{n+3-r-k}^{\delta-1} A_k^r |\Delta^{\alpha+1} a_{n-r+5} - \Delta^{\alpha+1} a_{n-r+2}| \\
&+\cdots \\
\le\ & C_2 A_{n+1-r}^{r+\delta} |\Delta^{\alpha+1} a_{n-r+2} - \Delta^{\alpha+1} a_{n-r}| \\
&+C_2 A_{n+2-r}^{r+\delta} |\Delta^{\alpha+1} a_{n-r+3} - \Delta^{\alpha+1} a_{n-r+1}| \\
&+C_2 A_{n+3-r}^{r+\delta} |\Delta^{\alpha+1} a_{n-r+5} - \Delta^{\alpha+1} a_{n-r+2}| \\
&+\cdots \\
\le\ & C_2 A_{n+1-r}^{\alpha} |\Delta^{\alpha+1} a_{n-r+2} - \Delta^{\alpha+1} a_{n-r}| \\
&+C_2 A_{n+2-r}^{\alpha} |\Delta^{\alpha+1} a_{n-r+3} - \Delta^{\alpha+1} a_{n-r+1}| \\
&+C_2 A_{n+3-r}^{\alpha} |\Delta^{\alpha+1} a_{n-r+5} - \Delta^{\alpha+1} a_{n-r+2}| \\
&+\cdots \\
=\ & o(1) + o(1) + o(1) + \cdots = o(1) \quad as \quad n \to \infty.
\end{aligned}
$$

Therefore, we proved that

$$\lim_{n\to\infty} \|g(x) - K_n^s(x)\| = 0.$$

So, the cases (i) and (ii) imply

$$\lim_{n\to\infty} \|g(x) - K_n^s(x)\| = 0,$$

when α is non-integral.

Finally, we have deduced that

$$\lim_{n\to\infty} \|g(x) - K^s_n(x)\| = 0,$$

for any $\alpha > 0$, which means $K^s_n(x) \to g(x)$ in L^1-norm.

The proof is completed.

4.6 L^1-convergence of modified sums $K^s_{nr}(x)$ with coefficients from the class $\mathbf{K}^\alpha$

Let

$$g(x) = \sum_{k=1}^{\infty} a_k \cos kx$$

and

$$K^s_{nr}(x) = \frac{1}{2\sin x} \sum_{k=1}^{n} \left(\Delta^r a_{k-1} - \Delta^r a_{k+1}\right) \widetilde{S}^{r-1}_k(x),$$

where r is any real number greater or equal to 1.

We give here the proof of next result.

Theorem 4.10. *Let α be a positive real number. If $\{a_n\} \in \mathbf{K}^\alpha$, then for $\alpha \le r \le \alpha + 1$*

(i) $K^s_n(x)$ converges to $g(x)$ pointwise for $0 < \delta \le x \le \pi$, and
(ii)$K^s_n(x) \to g(x)$ in the L^1-norm.

Proof. We have

$$g(x) = \frac{1}{2\sin x} \sum_{k=1}^{\infty} \left(\Delta^r a_{k-1} - \Delta^r a_{k+1}\right) \widetilde{S}^{r-1}_k(x)$$

and

$$K^s_{nr}(x) = \frac{1}{2\sin x} \sum_{k=1}^{n} \left(\Delta^r a_{k-1} - \Delta^r a_{k+1}\right) \widetilde{S}^{r-1}_k(x).$$

Case 1. Let $r = \alpha + 1$. Then

$$K^s_{nr}(x) = \frac{1}{2\sin x} \sum_{k=1}^{n} \left(\Delta^{\alpha+1} a_{k-1} - \Delta^{\alpha+1} a_{k+1}\right) \widetilde{S}^{\alpha}_k(x).$$

So $K^s_{nr}(x)$ to $g(x)$ point-wise for $0 < \delta \le \pi$.

Now, by hypothesis of the theorem we obtain

$$\begin{aligned}
&\int_0^{\pi} |g(x) - K_{nr}^s(x)| dx \\
&\qquad = \int_0^{\pi} \left| \frac{1}{2\sin x} \sum_{k=n+1}^{\infty} \left(\Delta^r a_{k-1} - \Delta^r a_{k+1}\right) \widetilde{S}_k^{r-1}(x) \right| dx \\
&\qquad \le C \sum_{k=n+1}^{\infty} |\Delta^r a_{k-1} - \Delta^r a_{k+1}| \int_0^{\pi} \left| \widetilde{S}_k^{r-1}(x) \right| dx \\
&\qquad = C \sum_{k=n+1}^{\infty} A_k^{\alpha} |\Delta^r a_{k-1} - \Delta^r a_{k+1}| \int_0^{\pi} \left| \widetilde{T}_k^{r-1}(x) \right| dx \\
&\qquad \le C_1 \sum_{k=n+1}^{\infty} A_k^{\alpha} |\Delta^r a_{k-1} - \Delta^r a_{k+1}| = o(1), \; n \to \infty.
\end{aligned}$$

Therefore, $K_n^s(x)$ converges to $g(x)$, as $n \to \infty$ in the L^1-norm.
Case 2. Let $\alpha < r < \alpha + 1$. Take $r = \alpha + 1 - \delta$ and $0 < \delta < 1$. Then

$$\begin{aligned}
K_{nr}^s(x) &= \frac{1}{2\sin x} \sum_{k=1}^{n} \left(\Delta^r a_{k-1} - \Delta^r a_{k+1}\right) \widetilde{S}_k^{r-1}(x) \\
&= \frac{1}{2\sin x} \sum_{k=1}^{n} \left(\Delta^{\alpha+1-\delta} a_{k-1} - \Delta^{\alpha+1-\delta} a_{k+1}\right) \widetilde{S}_k^{\alpha-\delta}(x).
\end{aligned}$$

Applying Abel's transformation of order $-\delta$ again and using Lemma 1.49, we get

$$\begin{aligned}
&\frac{1}{2\sin x} \sum_{k=1}^{n} \left(\Delta^{\alpha+1} a_{k-1} - \Delta^{\alpha+1} a_{k+1}\right) \widetilde{S}_k^{\alpha}(x) \\
&\qquad = \frac{1}{2\sin x} \left\{ \sum_{k=1}^{n} \widetilde{S}_k^{\alpha-\delta}(x) \sum_{m=1}^{n-k} A_m^{\delta-1} \left(\Delta^{\alpha+1} a_{m+k-1} - \Delta^{\alpha+1} a_{m+k+1}\right) \right\} \\
&\qquad = \frac{1}{2\sin x} \left\{ \sum_{k=1}^{n} \widetilde{S}_k^{\alpha-\delta}(x) \Big[\left(\Delta^{\alpha-\delta+1} a_{k-1} - \Delta^{\alpha-\delta+1} a_{k+1}\right) \right. \\
&\qquad\qquad \left. - \sum_{m=n-k+1}^{\infty} A_m^{\delta-1} \left(\Delta^{\delta+1} a_{m+k-1} - \Delta^{\delta+1} a_{m+k+1}\right) \Big] \right\} \\
&\qquad = \frac{1}{2\sin x} \left[\sum_{k=1}^{n} \left(\Delta^{\alpha-\delta+1} a_{k-1} - \Delta^{\alpha-\delta+1} a_{k+1}\right) \widetilde{S}_k^{\alpha-\delta}(x) - R_n(x) \right],
\end{aligned}$$

where

$$R_n(x) := \sum_{k=1}^{n} \widetilde{S}_k^{\alpha-\delta}(x) \Big[A_{n-k+1}^{\delta-1} \left(\Delta^{\delta+1} a_n - \Delta^{\delta+1} a_{n+2}\right)$$

$$+A_{n-k+2}^{\delta-1}\left(\Delta^{\delta+1}a_{n+1}-\Delta^{\delta+1}a_{n+3}\right)+\cdots\Big]$$

$$=\sum_{k=1}^{n}\widetilde{S}_k^{r-\delta}(x)A_{n-k+1}^{\delta-1}\left(\Delta^{\delta+1}a_{n+2}-\Delta^{\delta+1}a_{n}\right)$$

$$+\sum_{k=1}^{n}\widetilde{S}_k^{r-\delta}(x)A_{n-k+2}^{\delta-1}\left(\Delta^{\delta+1}a_{n+1}-\Delta^{\delta+1}a_{n+3}\right)+\cdots.$$

This implies that

$$K_{nr}^s(x)=\frac{1}{2\sin x}\left[\sum_{k=1}^{n}\widetilde{S}_k^{\alpha}(x)\left(\Delta^{\alpha+1}a_{k-1}-\Delta^{\alpha+1}a_{k+1}\right)+R_n(x)\right].$$

Whence, by hypothesis of the theorem we have

$$\begin{aligned}
&\int_0^\pi |g(x)-K_{nr}^s(x)|dx\\
&=\int_0^\pi\left|\frac{1}{2\sin x}\left[\sum_{k=n+1}^{\infty}\widetilde{S}_k^{\alpha}(x)\left(\Delta^{\alpha+1}a_{k-1}-\Delta^{\alpha+1}a_{k+1}\right)-R_n(x)\right]\right|dx\\
&\le C\left[\sum_{k=n+1}^{\infty}\left|\Delta^{\alpha+1}a_{k-1}-\Delta^{\alpha+1}a_{k+1}\right|\int_0^\pi\left|\widetilde{S}_k^{\alpha}(x)\right|dx+\int_0^\pi|R_n(x)|\,dx\right]\\
&\le C\left[\sum_{k=n+1}^{\infty}A_k^\alpha\left|\Delta^{\alpha+1}a_{k-1}-\Delta^{\alpha+1}a_{k+1}\right|\int_0^\pi\left|\widetilde{T}_k^{\alpha}(x)\right|dx+\int_0^\pi|R_n(x)|\,dx\right]\\
&\le C_1\left[\sum_{k=n+1}^{\infty}A_k^\alpha\left|\Delta^{\alpha+1}a_{k-1}-\Delta^{\alpha+1}a_{k+1}\right|+\int_0^\pi|R_n(x)|\,dx\right]\\
&=o(1)+C_1\int_0^\pi|R_n(x)|\,dx.
\end{aligned}$$

Let us estimate now the quantity $\int_0^\pi|R_n(x)|\,dx$. Indeed, using Lemmas 1.48, 1.50, and 1.90 we obtain

$$\begin{aligned}
&\int_0^\pi\big|R_n(x)\big|dx\\
&=\int_0^\pi\Bigg|\left(\sum_{k=1}^{n}A_{n-k+1}^{\delta-1}\widetilde{S}_k^{\alpha-\delta}(x)\right)\left(\Delta^{\delta+1}a_{n}-\Delta^{\delta+1}a_{n+2}\right)\\
&\qquad+\left(\sum_{k=1}^{n}A_{n-k+2}^{\delta-1}\widetilde{S}_k^{\alpha-\delta}(x)\right)\left(\Delta^{\delta+1}a_{n+1}-\Delta^{\delta+1}a_{n+3}\right)+\cdots\Bigg|dx\\
&\le\left|\Delta^{\delta+1}a_n-\Delta^{\delta+1}a_{n+2}\right|\int_0^\pi\left|\sum_{k=1}^{n}A_{n-k+1}^{\delta-1}\widetilde{S}_k^{\alpha-\delta}(x)\right|dx
\end{aligned}$$

$$
\begin{aligned}
&\quad + \left|\Delta^{\delta+1} a_{n+1} - \Delta^{\delta+1} a_{n+3}\right| \int_0^\pi \left|\sum_{k=1}^n A_{n-k+2}^{\delta-1} \widetilde{S}_k^{\alpha-\delta}(x)\right| dx + \cdots \\
&\le \left|\Delta^{\delta+1} a_n - \Delta^{\delta+1} a_{n+2}\right| \int_0^\pi \max_{0\le p\le n+1} \left|\widetilde{S}_p^{\delta}(x)\right| dx \\
&\quad + \left|\Delta^{\delta+1} a_{n+1} - \Delta^{\delta+1} a_{n+3}\right| \int_0^\pi \max_{0\le p\le n+2} \left|\widetilde{S}_p^{\delta}(x)\right| dx + \cdots \\
&= C A_{n+1}^{\delta} \left|\Delta^{\delta+1} a_n - \Delta^{\delta+1} a_{n+2}\right| \int_0^\pi \max_{0\le p\le n+1} \left|\widetilde{T}_p^{\delta}(x)\right| dx \\
&\quad + C A_{n+2}^{\delta} \left|\Delta^{\delta+1} a_{n+1} - \Delta^{\delta+1} a_{n+3}\right| \int_0^\pi \max_{0\le p\le n+2} \left|\widetilde{T}_p^{\delta}(x)\right| dx + \cdots \\
&= C_1 A_{n+1}^{\delta} \left|\Delta^{\delta+1} a_n - \Delta^{\delta+1} a_{n+2}\right| + C_1 A_{n+2}^{\delta} \left|\Delta^{\delta+1} a_{n+1} - \Delta^{\delta+1} a_{n+3}\right| + \cdots \\
&= o(1) \quad as \quad n \to \infty.
\end{aligned}
$$

Using this fact we clearly have

$$\lim_{n\to\infty} \|g(x) - K_{nr}^s(x)\| = 0.$$

Case 3. Let $\alpha = r$. In this case

$$K_{nr}^s(x) = \frac{1}{2\sin x} \sum_{k=1}^n \left(\Delta^\alpha a_{k-1} - \Delta^\alpha a_{k+1}\right) \widetilde{S}_k^{\alpha-1}(x).$$

Applying Abel's transformation, we have

$$
\begin{aligned}
K_{nr}^s(x) = \frac{1}{2\sin x}\Big[&\sum_{k=1}^n \left(\Delta^{\alpha+1} a_{k-1} - \Delta^{\alpha+1} a_{k+1}\right) \widetilde{S}_k^{\alpha}(x) \\
&+ \left(\Delta^\alpha a_n - \Delta^\alpha a_{n+2}\right) \widetilde{S}_n^{\alpha-1}(x)\Big].
\end{aligned}
$$

Since, $\widetilde{S}_k^{\alpha}(x)$ are bounded for $0 < \delta \le x \le \pi$, then

$$K_{nr}^s(x) \to g(x) = \frac{1}{2\sin x} \sum_{k=1}^\infty \left(\Delta^{\alpha+1} a_{k-1} - \Delta^{\alpha+1} a_{k+1}\right) \widetilde{S}_k^{\alpha}(x)$$

pointwise for $0 < \delta \le x \le \pi$.

Subsequently, by hypothesis of the theorem and Lemma 1.48 we get

$$
\begin{aligned}
&\int_0^\pi |g(x) - K_{nr}^s(x)| dx \\
&\quad = \int_0^\pi \left| \frac{1}{2\sin x}\Big[\sum_{k=1}^\infty \left(\Delta^{\alpha+1} a_{k-1} - \Delta^{\alpha+1} a_{k+1}\right) \widetilde{S}_k^{\alpha}(x) \right. \\
&\qquad\qquad \left. - \left(\Delta^\alpha a_n - \Delta^\alpha a_{n+2}\right) \widetilde{S}_n^{\alpha}(x)\Big] \right| dx
\end{aligned}
$$

$$\leq C\left[\sum_{k=1}^{\infty}\left(\Delta^{\alpha+1}a_{k-1}-\Delta^{\alpha+1}a_{k+1}\right)\int_0^{\pi}\left|\widetilde{S}_k^{\alpha}(x)\right|dx\right.$$
$$\left.+\left|\Delta^{\alpha}a_n-\Delta^{\alpha}a_{n+2}\right|\int_0^{\pi}\left|\widetilde{S}_n^{\alpha}(x)\right|dx\right]$$
$$=C\left[\sum_{k=1}^{\infty}A_k^{\alpha}\left|\Delta^{\alpha+1}a_{k-1}-\Delta^{\alpha+1}a_{k+1}\right|\int_0^{\pi}\left|\widetilde{T}_k^{\alpha}(x)\right|dx\right.$$
$$\left.+A_n^{\alpha}\left|\Delta^{\alpha}a_n-\Delta^{\alpha}a_{n+2}\right|\int_0^{\pi}\left|\widetilde{T}_n^{\alpha}(x)\right|dx\right]$$
$$\leq C_1\sum_{k=1}^{\infty}A_k^{\alpha}\left|\Delta^{\alpha+1}a_{k-1}-\Delta^{\alpha+1}a_{k+1}\right|$$
$$+C_1A_n^{\alpha}\left|\Delta^{\alpha}a_n-\Delta^{\alpha}a_{n+2}\right|=o(1)+o(1)=o(1),\ \ as\ n\to\infty.$$

The proof is completed.

4.7 L^1-convergence of modified sums $k_n^c(x)$ and $k_n^s(x)$ with semi-convex coefficients

Let
$$\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos kx \quad and \quad \sum_{k=1}^{\infty}a_k\sin kx$$
be cosine and sine series, with their partial sums
$$S_n^c(x)=\frac{a_0}{2}+\sum_{k=1}^{n}a_k\cos kx \quad and \quad S_n^s(x)=\sum_{k=1}^{n}a_k\sin kx$$
respectively, and let
$$f(x)=\lim_{n\to\infty}S_n^c(x) \quad and \quad g(x)=\lim_{n\to\infty}S_n^s(x).$$

Also we recall the following modified trigonometric sums
$$k_n^c(x)=-\frac{1}{2\sin x}\sum_{k=0}^{n}\sum_{j=k}^{n}\Delta\left[(a_{j-1}-a_{j+1})\cos jx\right]$$
and
$$k_n^s(x)=\frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left[(a_{j-1}-a_{j+1})\sin jx\right].$$

Firstly, we prove the following.

Theorem 4.11. *Let $\{a_n\}$ be a semi-convex null sequence, then $k_n^c(x)$ converges to $g(x)$ in L^1-norm.*

Proof. We have

$$\begin{aligned} k_n^c(x) &= -\frac{1}{2\sin x}\sum_{k=0}^{n}\sum_{j=k}^{n}\Delta\left[(a_{j-1}-a_{j+1})\cos jx\right] \qquad (4.1)\\ &= -\frac{1}{2\sin x}\sum_{k=0}^{n}\left[(a_{k-1}-a_{k+1})\cos kx-(a_n-a_{n+2})\cos(n+1)x\right]\\ &= -\frac{1}{2\sin x}\sum_{k=0}^{n}(a_{k-1}-a_{k+1})\cos kx+(n+1)\left(a_n-a_{n+2}\right)\frac{\cos(n+1)x}{2\sin x}\\ &= -\frac{1}{2\sin x}\sum_{k=0}^{n}(\Delta a_{k-1}+\Delta a_k)\cos kx+(n+1)\left(a_n-a_{n+2}\right)\frac{\cos(n+1)x}{2\sin x}. \end{aligned}$$

Applying Abel's transformation in (4.1), we have

$$\begin{aligned} k_n^c(x) &= -\frac{1}{2\sin x}\left\{\sum_{k=0}^{n-1}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)\left(D_k(x)+\frac{1}{2}\right)\right. \qquad (4.2)\\ &\quad\left.+\left(\Delta a_{n-1}+\Delta a_n\right)\left(D_n(x)+\frac{1}{2}\right)\right\}+(n+1)\left(a_n-a_{n+2}\right)\frac{\cos(n+1)x}{2\sin x}\\ &= -\frac{1}{2\sin x}\left\{\sum_{k=0}^{n}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)D_k(x)-\left(\Delta a_{n-1}-\Delta a_{n+1}\right)D_n(x)\right.\\ &\quad\left.+\frac{1}{2}\sum_{k=0}^{n-1}\left(\Delta a_{k-1}-\Delta a_{k+1}\right)+\frac{\Delta a_{n-1}+\Delta a_n}{2}\right\}+(n+1)\left(a_n-a_{n+2}\right)\frac{\cos(n+1)x}{2\sin x}\\ &= -\frac{1}{2\sin x}\sum_{k=0}^{n}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)D_k(x)+(n+1)\left(a_n-a_{n+2}\right)\frac{\cos(n+1)x}{2\sin x}. \end{aligned}$$

On the other side we have

$$\begin{aligned} S_n^s(x) &= \frac{1}{\sin x}\sum_{k=1}^{n}a_k\sin kx\sin x \qquad (4.3)\\ &= -\frac{1}{2\sin x}\sum_{k=1}^{n}a_k\left[\cos(k+1)x-\cos(k-1)x\right]\\ &= -\frac{1}{2\sin x}\sum_{k=0}^{n}(a_{k-1}-a_{k+1})\cos kx-a_{n+1}\frac{\cos nx}{2\sin x}-a_n\frac{\cos(n+1)x}{2\sin x}\\ &= -\frac{1}{2\sin x}\sum_{k=0}^{n}(\Delta a_{k-1}+\Delta a_k)\cos kx-a_{n+1}\frac{\cos nx}{2\sin x}-a_n\frac{\cos(n+1)x}{2\sin x}. \end{aligned}$$

Applying Abel's transformation to (4.3) we get

$$\begin{aligned} S_n^s(x) = &-\frac{1}{2\sin x}\left\{\sum_{k=0}^{n-1}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)\left(D_k(x)+\frac{1}{2}\right)\right. \\ &\left.+\left(\Delta a_{n-1}+\Delta a_n\right)\left(D_n(x)+\frac{1}{2}\right)\right\} - a_{n+1}\frac{\cos nx}{2\sin x} - a_n\frac{\cos(n+1)x}{2\sin x} \\ = &-\frac{1}{2\sin x}\left\{\sum_{k=0}^{n}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)D_k(x)+\frac{1}{2}\sum_{k=0}^{n-1}\left(\Delta a_{k-1}-\Delta a_{k+1}\right)\right. \\ &\left.+\left(\Delta a_n+\Delta a_{n+1}\right)D_n(x)+\frac{\Delta a_{n-1}+\Delta a_n}{2}\right\} - a_{n+1}\frac{\cos nx}{2\sin x} - a_n\frac{\cos(n+1)x}{2\sin x} \\ = &-\frac{1}{2\sin x}\sum_{k=0}^{n}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)D_k(x) \\ &-\left(a_n-a_{n+2}\right)\frac{D_n(x)}{2\sin x} - a_{n+1}\frac{\cos nx}{2\sin x} - a_n\frac{\cos(n+1)x}{2\sin x}. \end{aligned} \tag{4.4}$$

Since $\{a_n\}$ is semi-convex sequence, then we have

$$\begin{aligned} \left|(n+1)\left(a_n-a_{n+2}\right)\right| &= (n+1)\left|\sum_{k=n}^{\infty}\left(\Delta a_k-\Delta a_{k+2}\right)\right| \\ &= (n+1)\left|\sum_{k=n+1}^{\infty}\left(\Delta a_{k-1}-\Delta a_{k+1}\right)\right| \\ &\le \sum_{k=n+1}^{\infty} k\left|\Delta^2 a_{k-1}+\Delta^2 a_k\right| = o(1),\ \ n\to\infty. \end{aligned} \tag{4.5}$$

Using (4.5) and passing on limit as $n\to\infty$ to (4.2) and (4.2) we get

$$\begin{aligned} g(x) &= \lim_{n\to\infty} S_n^s(x) = \lim_{n\to\infty} k_n^c(x) \\ &= -\frac{1}{2\sin x}\sum_{k=0}^{\infty}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)D_k(x). \end{aligned} \tag{4.6}$$

Applying well-known inequality $|D_k(x)|\le 1/2+k, k=1,2,\ldots,$ and relations (4.4), (4.5) and (4.6) we obtain

$$\begin{aligned} &\int_{-\pi}^{\pi}\left|g(x)-k_n^c(x)\right|dx \\ &= \int_{-\pi}^{\pi}\left|\frac{1}{2\sin x}\sum_{k=n+1}^{\infty}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)D_k(x)+(n+1)\left(a_n-a_{n+2}\right)\frac{\cos(n+1)x}{2\sin x}\right|dx \\ &= \mathcal{O}\left(\sum_{k=n+1}^{\infty} k\left|\Delta^2 a_{k-1}+\Delta^2 a_k\right|\right)+\mathcal{O}\left(\left|(n+1)\left(a_n-a_{n+2}\right)\right|\right) = o(1),\quad n\to\infty. \end{aligned}$$

The proof is completed.

Corollary 4.12. *Let* $\{a_n\}$ *be a semi-convex null sequence. Then the necessary and sufficient condition for* L^1*-convergence of the sine series is* $\lim_{n\to\infty} a_n \log n = 0$.

Proof. Let $\|S_n^s(x) - g(x)\| = o(1), n \to \infty$. We are going to show that $a_n \log n = o(1), n \to \infty$.

Indeed, we have

$$\begin{aligned}
&\left\|g(x) - k_n^c(x)\right\| + \left\|S_n^s(x) - g(x)\right\| + \left\|(n+1)\left(a_n - a_{n+2}\right)\frac{\cos(n+1)x}{2\sin x}\right\| \\
&+\left\|\left(a_n - a_{n+2}\right)\frac{D_n(x)}{2\sin x}\right\| \geq \left\|a_{n+1}\frac{\cos nx}{2\sin x} + a_n\frac{\cos(n+1)x}{2\sin x}\right\| \\
&\geq a_{n+1}\int_{-\pi}^{\pi}\left|\frac{\cos nx}{2\sin x} + \frac{\cos(n+1)x}{2\sin x}\right| dx = a_{n+1}\int_{-\pi}^{\pi}\left|\widetilde{D}_n(x) - \frac{1}{2}ctg\frac{x}{2}\right| dx \\
&= a_{n+1}\left(\int_{-\pi}^{\pi}\left|\widetilde{D}_n(x)\right| dx - \int_{-\pi}^{\pi}\left|\cot\frac{x}{2}\right| dx\right) = \mathcal{O}\left(a_{n+1}\log n\right). \qquad (4.7)
\end{aligned}$$

Since $\left\|g(x) - k_n^c(x)\right\| = o(1)$ by Theorem, $\left\|S_n^s(x) - g(x)\right\| = o(1)$ by assumption of corollary, and (4.5), the third and fourth term in the left side of relation (4.7) tends to 0. This means that $a_n \log n = o(1),\ n \to \infty$.

Conversely, let $a_n \log n = o(1), n \to \infty$. Then by (4.7)

$$\begin{aligned}
\left\|S_n^s(x) - g(x)\right\| &\leq \left\|k_n^c(x) - g(x)\right\| + \left\|k_n^c(x) - S_n^s(x)\right\| \\
&= \left\|k_n^c(x) - g(x)\right\| + \left\|(n+1)\left(a_n - a_{n+2}\right)\frac{\cos(n+1)x}{2\sin x}\right. \\
&\quad \left. + \left(a_n - a_{n+2}\right)\frac{D_n(x)}{2\sin x} + a_{n+1}\frac{\cos nx}{2\sin x} + a_n\frac{\cos(n+1)x}{2\sin x}\right\| \\
&= o(1) + \mathcal{O}\left((n+1)|a_n - a_{n+2}|\right) + \mathcal{O}\left(n|a_n - a_{n+2}|\right) \\
&\quad + \left\|a_{n+1}\frac{\cos nx}{2\sin x} + a_n\frac{\cos(n+1)x}{2\sin x}\right\| \\
&= o(1) + \left\|a_{n+1}\frac{\cos nx}{2\sin x} + a_n\frac{\cos(n+1)x}{2\sin x}\right\|. \qquad (4.8)
\end{aligned}$$

However,

$$\begin{aligned}
\int_{-\pi}^{\pi}\left|a_{n+1}\frac{\cos nx}{2\sin x} + a_n\frac{\cos(n+1)x}{2\sin x}\right| dx &\leq a_n\int_{-\pi}^{\pi}\left|\frac{\cos nx + \cos(n+1)x}{2\sin x}\right| dx \\
= a_n\int_{-\pi}^{\pi}\left|\frac{1}{2}\cot\frac{x}{2} - \widetilde{D}_n(x)\right| dx &= \mathcal{O}\left(a_n\log n\right) = o(1),\ n \to \infty,
\end{aligned}$$

and therefore from (4.8) we get $\left\|S_n^s(x) - g(x)\right\| = o(1),\ n \to \infty$.

The proof is completed.

Theorem 4.13. *Let $\{a_n\}$ be a semi-convex null sequence. Then $k_n^s(x)$ converges to $f(x)$ in L^1-norm.*

Proof. The proof is very similar to the proof of Theorem 4.11.

Corollary 4.14. *Let $\{a_n\}$ be a semi-convex null sequence. Then the necessary and sufficient condition for L^1-convergence of the cosine series is $\lim_{n\to\infty} a_n \log n = 0$.*

Proof. The proof is very similar to the proof of Corollary 4.12.

4.8 L^1-convergence of modified sums $\beta_n^{\sin}(x)$ and $\beta_n^{\cos}(x)$ with coefficients from the class $\mathbf{BV}^{\log}$

We consider

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

and

$$\sum_{k=1}^{\infty} a_k \sin kx$$

cosine and sine series, and the modified trigonometric sums

$$\beta_n^{\sin}(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left(\frac{a_j}{\log(j+1)} \right) \log(k+1) \sin kx$$

and

$$\beta_n^{\cos}(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left(\frac{a_j}{\log(j+1)} \right) \log(k+1) \cos kx.$$

Here and throughout this unit we will denote by $S_n(x)$ the partial sums of the cosine or sine series and $\lim_{n\to\infty} S_n(x) = \beta(x)$, where $\beta(x)$ is the sum-function of sine or cosine series.

We prove here the following.

Theorem 4.15. *Let $\{a_k\}$ be a sequence that belongs to the class $\mathbf{BV}^{\log}$, then*

(i) $\beta_n(x)$ converges point-wise to $\beta(x)$ for $\delta \le x \le \pi$, $\delta > 0$,
(ii) $\beta_n(x)$ converges to $\beta(x)$ in the L^1-norm, and
(iii) $\beta(x)$ is an integrabile function i.e. $\beta \in L^1$,

where $\beta_n(x)$ represents either $\beta_n^{\cos}(x)$ or $\beta_n^{\sin}(x)$.

Proof. (i) We consider only the case of the sine sums $\beta_n^{\sin}(x)$, since the case of cosine sums $\beta_n^{\cos}(x)$ can be treated in a similar way.

We have

$$\begin{aligned}\beta_n^{\sin}(x) &= \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{\log(j+1)}\right)\log(k+1)\sin kx \\ &= \sum_{k=1}^{n}\left(\Delta\left(\frac{a_k}{\log(k+1)}\right)+\cdots+\Delta\left(\frac{a_n}{\log(n+1)}\right)\right)\log(k+1)\sin kx \\ &= \sum_{k=1}^{n}\left(\frac{a_k}{\log(k+1)}-\frac{a_{n+1}}{\log(n+2)}\right)\log(k+1)\sin kx \\ &= \sum_{k=1}^{n} a_k \sin kx - \frac{a_{n+1}}{\log(n+2)}\widetilde{D}_n^{\log}(x). \end{aligned} \tag{4.9}$$

Since the sequence $\{a_k\}$ tends to zero then the second term in (4.9) tends to zero. Namely, using Lemma 1.87 for $0<\delta\le x\le\pi$, we find that

$$\begin{aligned}\left|\frac{a_{n+1}}{\log(n+2)}\widetilde{D}_n^{\log}(x)\right| &= \frac{a_{n+1}}{\log(n+2)}\left|\widetilde{D}_n^{\log}(x)\right| \\ &= \frac{a_{n+1}}{\log(n+2)}\mathcal{O}\left(\frac{\log(n+1)}{x}\right) = \frac{1}{\delta}\mathcal{O}\left(\frac{a_{n+1}\log(n+1)}{\log(n+2)}\right) \\ &= \frac{1}{\delta}\mathcal{O}\left(a_{n+1}\right) = o(1), \quad as \quad n\to\infty, \end{aligned}$$

in view of $\frac{\log(n+1)}{\log(n+2)} \le 1$ for all $n\in N$.

Therefore, we obtain that

$$\lim_{n\to\infty}\beta_n^{\sin}(x) = \lim_{n\to\infty} S_n(x) = \beta(x), \quad for \quad 0<x\le\pi.$$

(ii) Based on equality (4.10) we can write

$$\begin{aligned}\beta(x)-\beta_n^{\sin}(x) &= \sum_{k=n+1}^{\infty} a_k\sin kx + \frac{a_{n+1}}{\log(n+2)}\widetilde{D}_n^{\log}(x) \\ &= \lim_{p\to\infty}\sum_{k=n+1}^{p}\frac{a_k}{\log(k+1)}\log(k+1)\sin kx \\ &\quad + \frac{a_{n+1}}{\log(n+2)}\widetilde{D}_n^{\log}(x). \end{aligned}$$

Applying the summation by parts to the above equality we get

$$\begin{aligned}\beta(x)-\beta_n^{\sin}(x) = \lim_{p\to\infty}\Bigg[&\sum_{k=n+1}^{p-1}\Delta\left(\frac{a_k}{\log(k+1)}\right)\widetilde{D}_k^{\log}(x) \\ &+\frac{a_p}{\log(p+1)}\widetilde{D}_p^{\log}(x) - \frac{a_{n+1}}{\log(n+2)}\widetilde{D}_n^{\log}(x)\Bigg] + \frac{a_{n+1}}{\log(n+2)}\widetilde{D}_n^{\log}(x). \end{aligned}$$

In a similar way one can show that (based on discussion we made in the proof of the assertion (i)) the second term in brackets of the above equality tend to zero, and hence we obtain

$$\beta(x) - \beta_n^{\sin}(x) = \sum_{k=n+1}^{\infty} \Delta\left(\frac{a_k}{\log(k+1)}\right) \widetilde{D}_k^{\log}(x).$$

Since

$$\sum_{j=1}^{k} \log(j+1)\sin(jx) = \sum_{j=1}^{k-1} \Delta(\log(j+1)) \sum_{s=1}^{j} \sin(sx) + \log(k+1) \sum_{s=1}^{k} \sin(sx),$$

then we proceed as follows

$$\begin{aligned}\int_0^{\pi} |\widetilde{D}_k^{\log}(x)|dx &\le \sum_{j=1}^{k-1} |\Delta(\log(j+1))| \int_0^{\pi} \left|\sum_{s=1}^{j} \sin(sx)\right| dx \\ &\quad + \log(k+1) \int_0^{\pi} \left|\sum_{s=1}^{k} \sin(sx)\right| dx \\ &= \sum_{j=1}^{k-1} \log\left(1 + \frac{1}{j+1}\right) |\mathcal{O}(\log j) + \mathcal{O}(\log^2 k) \\ &= \mathcal{O}(\log k) \sum_{j=1}^{k} \frac{1}{j+1} + \mathcal{O}(\log^2(k+1)) = \mathcal{O}(\log^2(k+1)).\end{aligned}$$

Subsequently, since $\{a_k\} \in \mathbf{BV}^{\log}$ we get

$$\begin{aligned}\|\beta(x) - \beta_n^{\sin}(x)\| &\le \sum_{k=n+1}^{\infty} \left|\Delta\left(\frac{a_k}{\log(k+1)}\right)\right| \int_0^{\pi} |\widetilde{D}_k^{\log}(x)|dx \\ &= \mathcal{O}\left(\sum_{k=n+1}^{\infty} \log^2(k+1) \left|\Delta\left(\frac{a_k}{\log(k+1)}\right)\right|\right) = o(1),\end{aligned}$$

as $n \to \infty$, which obviously means that $\beta_n^{\sin}(x) \to \beta(x)$ in the L^1-norm.

(iii) Since $\beta_n^{\sin}(x)$ is a polynomial, then the obtained relation $\|\beta(x) - \beta_n^{\sin}(x)\| = o(1)$ as $n \to \infty$ in (ii), clearly implies $\beta \in L^1$.

The proof is completed.

Now we will deduce a sufficient condition for the L^1-convergence of the sine series. Let $g(x)$ be the sum function of the sine series and $S_n(g;x)$ its partial sums.

Corollary 4.16. *Let $\{a_k\} \in \mathbf{BV}^{\log}$, then $\|g - S_n(g)\| = o(1)$ as $n \to \infty$, i.e. the sine series is Fourier series of the function g.*

Proof. Using (4.9), the already proved Theorem (ii), and

$$\int_0^\pi |\widetilde{D}_n^{\log}(x)|dx = \mathcal{O}(\log^2(n+1))$$

we have

$$\begin{aligned}\int_0^\pi |g(x)-S_n(x)|dx &\le \int_0^\pi |f(x)-\beta_n^{\sin}(x)|dx + \int_0^\pi |\beta_n^{\sin}(x)-S_n(x)|dx \\ &= o(1) + \frac{|a_{n+1}|}{\log(n+1)}\int_0^\pi |\widetilde{D}_n^{\log}(x)|dx \\ &= o(1) + \mathcal{O}\left(|a_{n+1}|\log(n+1)\right).\end{aligned}$$

Since

$$0 \le |a_{n+1}|\log(n+1) \le \sum_{j=n}^{\infty} \log^2(j+1)\left|\frac{a_j}{\log(j+1)} - \frac{a_{j+1}}{\log(j+2)}\right| \to 0$$

as $n\to\infty$, then we have

$$\int_0^\pi |g(x)-S_n(x)|dx \to 0 \quad as \quad n\to\infty.$$

The proof is completed.

4.9 L^1-convergence of modified sums $\psi_n^c(x)$ and $\psi_n^s(x)$ with coefficients from the class $\mathrm{R}^{\log}$

We consider

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

and

$$\sum_{k=1}^{\infty} a_k \sin kx$$

cosine and sine series, and the modified trigonometric sums

$$\psi_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n}\left(\sum_{i=j}^{n} \Delta^2\left(\frac{a_i}{i}\right)\right)k\cos kx$$

and

$$\psi_n^s(x) = \sum_{k=1}^{n}\sum_{j=k}^{n}\left(\sum_{i=j}^{n} \Delta^2\left(\frac{a_i}{i}\right)\right)k\sin kx.$$

The following theorem presents the main result of this unit.

Theorem 4.17. *Let $\{a_k\}$ be a null sequence. If $\{a_k\} \in \mathbf{R}^{\log}$ then*

(i) $\psi_n(x)$ converges to $\psi(x)$ pointwise for $0 < \delta \le x \le \pi$,
(ii) $\psi \in L(0,\pi]$, and
(iii) $\psi_n(x) \to \psi(x)$ in the L^1-norm,

where $\psi_n(x)$ represents either $\psi_n^c(x)$ or $\psi_n^s(x)$.

Proof. (i) We consider only the case of the cosine sums $\psi_n^c(x)$, since for the case of sine sums $\psi_n^s(x)$ it can be treated in the same way.

We have

$$\begin{aligned}
\psi_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n}\left(\sum_{i=j}^{n} \Delta^2\left(\frac{a_i}{i}\right)\right)k\cos kx \\
&= \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n}\left(\Delta^2\left(\frac{a_j}{j}\right) + \Delta^2\left(\frac{a_{j+1}}{j+1}\right) + \cdots + \Delta^2\left(\frac{a_n}{n}\right)\right)k\cos kx \\
&= \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n}\left(\Delta\left(\frac{a_j}{j}\right) - \Delta\left(\frac{a_{n+1}}{n+1}\right)\right)k\cos kx \\
&= \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right)k\cos kx - \Delta\left(\frac{a_{n+1}}{n+1}\right)\sum_{k=1}^{n}\sum_{j=k}^{n}k\cos kx \\
&= \frac{a_0}{2} + \sum_{k=1}^{n}\left(\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right)k\cos kx - \Delta\left(\frac{a_{n+1}}{n+1}\right)\left[\sum_{k=1}^{n}\sum_{j=k}^{n}\sin kx\right]' \\
&= \frac{a_0}{2} + \sum_{k=1}^{n}\left(\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right)k\cos kx - \Delta\left(\frac{a_{n+1}}{n+1}\right)\times \\
&\quad\times\left(\sum_{j=1}^{n}\sin kx + \sum_{j=2}^{n}\sin kx + \sum_{j=3}^{n}\sin kx + \cdots + \sum_{j=n}^{n}\sin kx\right)' \\
&= \frac{a_0}{2} + \sum_{k=1}^{n}\left(\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right)k\cos kx - \Delta\left(\frac{a_{n+1}}{n+1}\right)\times \\
&\quad\times\Big[\widetilde{D}_n(x) + \left(\widetilde{D}_n(x) - \widetilde{D}_1(x)\right) \\
&\qquad + \left(\widetilde{D}_n(x) - \widetilde{D}_2(x)\right) + \cdots + \left(\widetilde{D}_n(x) - \widetilde{D}_{n-1}(x)\right)\Big]' \\
&= \frac{a_0}{2} + \sum_{k=1}^{n}\left(\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right)k\cos kx - \Delta\left(\frac{a_{n+1}}{n+1}\right)\times \\
&\quad\times\left[n\widetilde{D}_n(x) - \left(\widetilde{D}_1(x) + \widetilde{D}_2(x) + \cdots + \widetilde{D}_{n-1}(x)\right)\right]' \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos kx - \frac{a_{n+1}}{n+1}\widetilde{D}'_n(x) - \Delta\left(\frac{a_{n+1}}{n+1}\right)\left[n\widetilde{D}_n(x) - \sum_{s=1}^{n-1}\widetilde{D}_s(x)\right]'
\end{aligned}$$

$$= S_n(x) - \frac{a_{n+1}}{n+1}\widetilde{D}'_n(x) - n\Delta\left(\frac{a_{n+1}}{n+1}\right)\left(\widetilde{D}'_n(x) - \widetilde{F}'_{n-1}(x)\right), \qquad (4.10)$$

where $\widetilde{D}_n(x)$ and $\widetilde{F}_n(x)$ denote the conjugate Dirichlet and Fejer's kernels respectively. It is not difficult to verify the estimations

$$|\widetilde{D}'_n(x)| \le \frac{2\pi^2 n}{\delta^2} \quad \textit{and} \quad |\widetilde{F}'_{n-1}(x)| \le \frac{\pi^2 n}{\delta^2}, \quad 0 < \delta \le x \le \pi.$$

Since the sequence $\{a_k\}$ tends to zero then the second term in (4.10) tends to zero as well, based on the above estimation. For the third term we also have

$$\begin{aligned}
\left|n\Delta\left(\frac{a_{n+1}}{n+1}\right)\left(\widetilde{D}'_n(x) - \widetilde{F}'_{n-1}(x)\right)\right| &\le \frac{2\pi^2}{\delta^2} n^2 \left|\Delta\left(\frac{a_{n+1}}{n+1}\right)\right| \\
&\le \frac{2\pi^2}{\delta^2} n^2 \sum_{m=n+1}^{\infty} \left|\Delta^2\left(\frac{a_m}{m}\right)\right| \\
&\le \frac{2\pi^2}{\delta^2} \sum_{m=n}^{\infty} m^2 \left|\Delta^2\left(\frac{a_m}{m}\right)\right| \\
&= o(1) \quad \textit{as} \quad n \to \infty, \qquad (4.11)
\end{aligned}$$

in view of $\{a_k\} \in \mathbf{R}^{\log}$.
Therefore, we obtain that

$$\lim_{n\to\infty} \psi_n(x) = \lim_{n\to\infty} S_n(x) = \psi(x), \quad \textit{for} \quad 0 < x \le \pi.$$

(ii) The statement $\psi \in L(0,\pi]$ is an immediate result of Theorem 4.17 since by our assumption $\{a_k\} \in \mathbf{R}^{\log}$.
(iii) Based on equality (4.10) we can write

$$\begin{aligned}
\psi(x) - \psi_n(x) &= \sum_{k=n+1}^{\infty} a_k \cos kx \\
&\quad + \frac{a_{n+1}}{n+1}\widetilde{D}'_n(x) + n\Delta\left(\frac{a_{n+1}}{n+1}\right)\left(\widetilde{D}'_n(x) + \widetilde{F}'_{n-1}(x)\right) \\
&= \lim_{p\to\infty}\left(\sum_{k=n+1}^{p} \frac{a_k}{k}\sin kx\right)' \\
&\quad + \frac{a_{n+1}}{n+1}\widetilde{D}'_n(x) + n\Delta\left(\frac{a_{n+1}}{n+1}\right)\left(\widetilde{D}'_n(x) + \widetilde{F}'_{n-1}(x)\right).
\end{aligned}$$

Applying the summation by parts to the above equality twice we get

$$\psi(x) - \psi_n(x) = \lim_{p\to\infty}\left(\sum_{k=n+1}^{p-1} \Delta\left(\frac{a_k}{k}\right)\widetilde{D}'_k(x) + \frac{a_p}{p}\widetilde{D}'_p(x) - \frac{a_{n+1}}{n+1}\widetilde{D}'_n(x)\right)$$

$$+\frac{a_{n+1}}{n+1}\widetilde{D}'_n(x)+n\Delta\left(\frac{a_{n+1}}{n+1}\right)\left(\widetilde{D}'_n(x)+\widetilde{F}'_{n-1}(x)\right)$$

$$=\lim_{p\to\infty}\left(\sum_{k=n+1}^{p-2}(k+1)\Delta^2\left(\frac{a_k}{k}\right)\widetilde{F}'_k(x)+p\Delta\left(\frac{a_{p-1}}{p-1}\right)\widetilde{F}'_{p-1}(x)\right.$$
$$\left.-(n+1)\Delta\left(\frac{a_{n+1}}{n+1}\right)\widetilde{F}'_n(x)+\frac{a_p}{p}\widetilde{D}'_p(x)\right)$$
$$+n\Delta\left(\frac{a_{n+1}}{n+1}\right)\left(\widetilde{D}'_n(x)+\widetilde{F}'_{n-1}(x)\right).$$

In a similar way one can show that (based on discussion we made in the proof of the assertion (i)) the second term and the fourth term in brackets of the above equality tend to zero, and hence we obtain

$$\psi(x)-\psi_n(x)=\sum_{k=n+1}^{\infty}(k+1)\Delta^2\left(\frac{a_k}{k}\right)\widetilde{F}'_k(x)$$
$$-(n+1)\Delta\left(\frac{a_{n+1}}{n+1}\right)\widetilde{F}'_n(x)+n\Delta\left(\frac{a_{n+1}}{n+1}\right)\left(\widetilde{D}'_n(x)+\widetilde{F}'_{n-1}(x)\right).$$

Subsequently, we get

$$\int_{-\pi}^{\pi}|\psi(x)-\psi_n(x)|dx\le\sum_{k=n+1}^{\infty}(k+1)\left|\Delta^2\left(\frac{a_k}{k}\right)\right|\int_{-\pi}^{\pi}\left|\widetilde{F}'_k(x)\right|dx$$
$$+(n+1)\left|\Delta\left(\frac{a_{n+1}}{n+1}\right)\right|\int_{-\pi}^{\pi}\left|\widetilde{F}'_n(x)\right|dx$$
$$+n\left|\Delta\left(\frac{a_{n+1}}{n+1}\right)\right|\left(\int_{-\pi}^{\pi}\left|\widetilde{D}'_n(x)\right|dx+\int_{-\pi}^{\pi}\left|\widetilde{F}'_{n-1}(x)\right|dx\right).$$

Since by Zygmund's theorem

$$\int_{-\pi}^{\pi}\left|\widetilde{F}'_k(x)\right|dx=\mathcal{O}(k),$$

we obtain

$$(n+1)\left|\Delta\left(\frac{a_{n+1}}{n+1}\right)\right|\int_{-\pi}^{\pi}\left|\widetilde{F}'_n(x)\right|dx=\mathcal{O}\left(\sum_{m=n+1}^{\infty}m^2\left|\Delta^2\left(\frac{a_m}{m}\right)\right|\right)=o(1),$$

and similarly using Lemma 1.85 (for $r=1$) we get

$$n\left|\Delta\left(\frac{a_{n+1}}{n+1}\right)\right|\int_{-\pi}^{\pi}\left|\widetilde{D}'_n(x)\right|dx=\mathcal{O}\left(\sum_{m=n+1}^{\infty}m^2\log m\left|\Delta^2\left(\frac{a_m}{m}\right)\right|\right)=o(1),\quad n\to\infty,$$

because of $\{a_k\}\in\mathbf{R}^{\log}$.

Thus, it follows that

$$\int_{-\pi}^{\pi} |\psi(x) - \psi_n(x)| dx = o(1) \quad as \quad n \to \infty,$$

which obviously means that

$$\psi_n(x) \to \psi(x) \textit{ in the } L^1\textit{−norm}.$$

The proof is completed.

Now we will deduce a necessary and sufficient condition for the L^1-convergence of a cosine series. Let $f(x)$ be the sum-function of the cosine series and $\mathcal{S}_n(f;x)$ its partial sums.

Theorem 4.18. *Let $\{a_k\} \in \mathbf{R}^{\log}$ be a null sequence. Then $\|f - \mathcal{S}_n(f)\| = o(1)$ if and only if $|a_{n+1}| \log n = o(1)$, $n \to \infty$.*

Proof. Let $|a_{n+1}| \log n = o(1)$, as $n \to \infty$. Using (4.10), Theorem 4.17 (iii), Zygmund's theorem and Lemma 1.85 we have

$$\begin{aligned}
\int_{-\pi}^{\pi} |f(x) - \mathcal{S}_n(x)| dx &\leq \int_{-\pi}^{\pi} |f(x) - \psi_n(x)| dx + \int_{-\pi}^{\pi} |\psi_n(x) - \mathcal{S}_n(x)| dx \\
&= o(1) + \frac{|a_{n+1}|}{n+1} \int_{-\pi}^{\pi} \left|\widetilde{D}'_n(x)\right| dx \\
&\quad + n \left| \Delta \left(\frac{a_{n+1}}{n+1} \right) \right| \left(\int_{-\pi}^{\pi} \left|\widetilde{D}'_n(x)\right| dx + \int_{-\pi}^{\pi} \left|\widetilde{F}'_{n-1}(x)\right| dx \right) \\
&= o(1) + \mathcal{O}\left(|a_{n+1}| \log n\right) + o(1) + o(1) = o(1), \; n \to \infty.
\end{aligned}$$

Conversely,

$$\begin{aligned}
\frac{|a_{n+1}|}{n+1} & \int_{-\pi}^{\pi} \left|\widetilde{D}'_n(x)\right| dx leq \int_{-\pi}^{\pi} |\psi_n(x) - \mathcal{S}_n(x)| dx \\
& + n \left| \Delta \left(\frac{a_{n+1}}{n+1} \right) \right| \left(\int_{-\pi}^{\pi} \left|\widetilde{D}'_n(x)\right| dx + \int_{-\pi}^{\pi} \left|\widetilde{F}'_{n-1}(x)\right| dx \right) \\
& \leq \int_{-\pi}^{\pi} |f(x) - \mathcal{S}_n(x)| dx + \int_{-\pi}^{\pi} |\psi_n(x) - f(x)| dx \\
& + \mathcal{O}\left(n \log n \left| \Delta \left(\frac{a_{n+1}}{n+1} \right) \right| + n^2 \left| \Delta \left(\frac{a_{n+1}}{n+1} \right) \right| \right) \\
& = o(1) + \mathcal{O}\left(\sum_{m=n+1}^{\infty} m \log m \left| \Delta^2 \left(\frac{a_m}{m} \right) \right| + \sum_{\ell=n+1}^{\infty} \ell^2 \left| \Delta^2 \left(\frac{a_\ell}{\ell} \right) \right| \right) \\
& = o(1), \; n \to \infty.
\end{aligned}$$

Since

$$\frac{|a_{n+1}|}{n+1} \int_{-\pi}^{\pi} \left|\widetilde{D}'_n(x)\right| dx$$

by Zygmund's theorem behaves as $|a_{n+1}| \log n$ for larg n.

The proof is completed.

4.10 L^1-convergence of modified sums $j_n^{(2,c)}(x)$ with coefficients from the class $\mathbf{S_2}$

We consider cosine series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

and modified cosine sums

$$j_n^{(2,c)}(x) = \frac{a_0}{2} + \sum_{k_1=1}^{n} \sum_{k_2=k_1}^{n} \sum_{k_3=k_2}^{n} \Delta^2 \left(a_{k_3} \cos k_3 x\right),$$

where $\Delta^2 a_k = \Delta\left(\Delta a_k\right) = \Delta\left(a_k - a_{k+1}\right) = a_k - 2a_{k+1} + a_{k+2}$.

The main result of this unit is the following.

Theorem 4.19. *Let $\{a_k\} \in S_2$, then $\|f - j_n^{(2,c)}\|_{L^1} = o(1)$, as $n \to \infty$.*

Proof. We have

$$\begin{aligned}
j_n^{(2,c)}(x) &= \frac{a_0}{2} + \sum_{k_1=1}^{n} \sum_{k_2=k_1}^{n} \sum_{k_3=k_2}^{n} \Delta^2 \left(a_{k_3} \cos k_3 x\right) \\
&= \frac{a_0}{2} + \sum_{k_1=1}^{n} \sum_{k_2=k_1}^{n} \Big[\Delta\left(a_{k_2} \cos k_2 x\right) - \Delta\left(a_{k_2+1} \cos (k_2+1) x\right) + \\
&\quad \cdots + \Delta\left(a_n \cos nx\right) - \Delta\left(a_{n+1} \cos\left(n+1\right)x\right)\Big] \\
&= \frac{a_0}{2} + \sum_{k_1=1}^{n} \sum_{k_2=k_1}^{n} \Big[\Delta\left(a_{k_2} \cos k_2 x\right) - \Delta\left(a_{n+1} \cos\left(n+1\right)x\right)\Big] \\
&= \frac{a_0}{2} + \sum_{k_1=1}^{n} \Big[a_{k_1} \cos k_1 x - a_{k_1+1} \cos (k_1+1) x + \cdots + a_n \cos nx \\
&\quad - a_{n+1} \cos\left(n+1\right)x\Big] - \Delta\left(a_{n+1} \cos\left(n+1\right)x\right) \sum_{k_1=1}^{n} (n - k_1 + 1) \\
&= S_n(x) - n a_{n+1} \cos\left(n+1\right)x - \frac{1}{2} n(n+1) \Delta\left(a_{n+1} \cos\left(n+1\right)x\right) \\
&= S_n(x) \\
&\qquad - \frac{1}{2} n(n+3) a_{n+1} \cos\left(n+1\right)x \\
&\qquad\qquad + \frac{1}{2} n(n+1) a_{n+2} \cos\left(n+2\right)x.
\end{aligned} \tag{4.12}$$

From $A_k \downarrow 0$ and $\sum_{k=1}^{\infty} k^2 A_k < \infty$ it follows $k^3 A_k = o(1), k \to \infty$, which gives $k^2 A_k = o(1), k \to \infty$. Therefore from

$$0 \leq n^2|a_n| = n^2\Big|\sum_{k=n}^{\infty} \Delta a_k\Big| \leq \Big|\sum_{k=n}^{\infty} k^2 \Delta a_k\Big| \leq \sum_{k=n}^{\infty} k^2 A_k = o(1), n \to \infty$$

follow

$$n^2 a_n = o(1), \quad n a_n = o(1), \quad n \to \infty. \tag{4.13}$$

Also, $\cos(n+1)x$ and $\cos(n+2)x$ are finite in $[0, \pi]$ therefore from (4.12) and (4.13) we get

$$\lim_{n\to\infty} j_n^{(2,c)}(x) = \lim_{n\to\infty} S_n(x) = f(x).$$

On the other side, using Abel's transformation we have

$$\begin{aligned}
f(x) - j_n^{(2,c)}(x) &= \lim_{m\to\infty}\left(\sum_{k=n+1}^{m-1} \Delta a_k D_k(x) + a_m D_m(x) - a_{n+1} D_n(x)\right) \\
&+\frac{1}{2}n(n+3)a_{n+1}\cos{(n+1)}\,x - \frac{1}{2}n(n+1)a_{n+2}\cos{(n+2)}\,x \\
&= \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - a_{n+1}D_n(x) + \frac{1}{2}n(n+3)a_{n+1}\cos{(n+1)}\,x \\
&-\frac{1}{2}n(n+1)a_{n+2}\cos{(n+2)}\,x.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^\pi |f(x) - j_n^{(2,c)}(x)|dx \;\leq\; &\int_0^\pi \Big|\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)\Big|dx + |a_{n+1}| \int_0^\pi \big|D_n(x)\big|dx \\
&+\frac{1}{2}n(n+3)|a_{n+1}| \int_0^\pi \big|\cos{(n+1)}\,x\big|dx \\
&+\frac{1}{2}n(n+1)|a_{n+2}| \int_0^\pi \big|\cos{(n+2)}\,x\big|dx \\
:=& \sum_{\nu=1}^{4} B_\nu(n).
\end{aligned} \tag{4.14}$$

Since $a_k \in S_2 \subset S_0 \equiv S$ then $\sum_{k=n+1}^{\infty}(k+1)\Delta A_k = o(1)$ as $n \to \infty$, therefore from this fact, Lemma 1.34, and using Abel's transformation we have

$$\begin{aligned}
B_1(n) = \int_0^\pi \Big|\sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x)\Big|dx &\leq \sum_{k=n+1}^{\infty} \Delta A_k \int_0^\pi \Big|\sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x)\Big|dx \\
= \mathcal{O}\left(\sum_{k=n+1}^{\infty}(k+1)\Delta A_k\right) &= o(1), n \to \infty.
\end{aligned} \tag{4.15}$$

By well-known Zygmund's theorem, for large enough n, the following relation holds

$$\int_0^\pi \big|D_n(x)\big|dx \sim \log n,$$

therefore from last relation and (4.13) we have

$$B_2(n) = |a_{n+1}|\log n \le n|a_{n+1}| = o(1), n \to \infty. \tag{4.16}$$

Moreover, from the fact that integrals $\int_0^\pi \big|\cos(n+1)x\big|dx$, $\int_0^\pi \big|\cos(n+2)x\big|dx$ are bounded, and from relation (4.13) we conclude that

$$B_3(n) = O\bigg(n(n+3)|a_{n+1}|\bigg) = o(1), n \to \infty \tag{4.17}$$

and similarly

$$B_4(n) = O\bigg(n(n+1)|a_{n+2}|\bigg) = o(1), n \to \infty. \tag{4.18}$$

Finally, from (4.13)-(4.18) it follows that

$$\|f - j_n^{(2,c)}(x)\|_{L^1} = o(1),\ n \to \infty.$$

The proof is completed.

Corollary 4.20. *Let $\{a_k\} \in S_2$, then $\|f - S_n\|_{L^1} = o(1)$ as $n \to \infty$.*

Proof. From Theorem 4.19, and relations (4.17), (4.18), we have

$$\begin{aligned}\|f - S_n\|_{L^1} &= \|f - j_n^{(2,c)}(x) + j_n^{(2,c)}(x) - S_n\|_{L^1}\\ &\le \|f - j_n^{(2,c)}(x)\|_{L^1} + \|j_n^{(2,c)}(x) - S_n\|_{L^1}\\ &\le \|f - j_n^{(2,c)}(x)\|_{L^1} + \frac{1}{2}n(n+3)|a_{n+1}|\int_0^\pi \big|\cos{(n+1)\,x}\big|dx\\ &+\frac{1}{2}n(n+1)|a_{n+2}|\int_0^\pi \big|\cos{(n+2)\,x}\big|dx = o(1)\end{aligned}$$

as $n \to \infty$, which completely proves the corollary.

Remark 4.21. We noticed during the proofs of Theorem 4.19 and Corollary 4.20 that condition $a_k \in S_2$ we can replace with conditions $a_k \in S$ and $n^2|a_n| = o(1)$. This enables us to formulate Theorem 4.19 and Corollary 4.20 as follows.

Theorem 4.22. *Let (a_k) belong to the class S and $n^2|a_n| = o(1)$, then*

$$\|f - j_n^{(2,c)}(x)\|_{L^1} = o(1) \quad as \quad n \to \infty.$$

Corollary 4.23. *Let (a_k) belong to the class S and $n^2|a_n| = o(1)$, then*

$$\|f - S_n\|_{L^1} = o(1) \quad as \quad n \to \infty.$$

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4.11 L^1-convergence of modified sums $w_n^c(x)$ with coefficients satisfying some special conditions

We consider cosine series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

and new modified cosine sums

$$w_n^c(x) = \frac{1}{2}\left(a_1 + \sum_{k=0}^{n} \Delta^2 a_k\right) + \sum_{k=1}^{n}\left(a_{k+1} + \sum_{j=k}^{n} \Delta^2 a_j\right)\cos kx,$$

where $\Delta^2 a_i = a_i - 2a_{i+1} + a_{i+2}$.

We present here the following result.

Theorem 4.24. *Let $\{a_k\}$ be a sequence of numbers such that the conditions: (i) $a_k = o(1)$ as $k \to \infty$ (ii) $S_1 := \sum_{k=0}^{\infty} |\Delta a_k| < \infty$, and*

$$S_2 := \sum_{m=2}^{\infty}\left|\sum_{k=1}^{\left[\frac{m}{2}\right]} \frac{\Delta a_{m-k} - \Delta a_{m+k}}{k}\right| < \infty$$

are satisfied.

If $\lim_{n\to\infty} a_n \log n = 0$, then $\lim_{n\to\infty}\|f - w_n^c(x)\| = 0$.

Proof. We have

$$\begin{aligned}
w_n^c(x) &= \frac{1}{2}\left(a_1 + \sum_{k=0}^{n} \Delta^2 a_k\right) + \sum_{k=1}^{n}\left(a_{k+1} + \sum_{j=k}^{n} \Delta^2 a_j\right)\cos kx \\
&= \frac{1}{2}\left(a_1 + \sum_{k=0}^{n}(a_k - 2a_{k+1} + a_{k+2})\right) \\
&\quad + \sum_{k=1}^{n}\left(a_{k+1} + \sum_{j=k}^{n}(a_j - 2a_{j+1} + a_{j+2})\right)\cos kx \\
&= \frac{1}{2}(a_0 - a_{n+1} + a_{n+2}) + \sum_{k=1}^{n}(a_k - a_{n+1} + a_{n+2})\cos kx \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{\Delta a_{n+1}}{2} - \Delta a_{n+1}\sum_{k=1}^{n}\cos kx \\
&= S_n^c(x) - \Delta a_{n+1} D_n(x).
\end{aligned}$$

Choosing numbers b_k as in Lemma 1.12, we can write

$$\frac{b_0}{2} + \sum_{k=1}^{\infty} b_k \cos kx = \sum_{k=n+1}^{\infty} a_k \cos kx.$$

Now using Lemma 1.12 and the inequality (see [70])

$$\int_0^{\pi} |f(x)| \le C(S_1 + S_2),$$

we obtain

$$\int_0^{\pi} |f(x) - w_n^c(x)| dx = \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} a_k \cos kx + \Delta a_{n+1} D_n(x) \right| dx$$

$$= \int_0^{\pi} \left| \frac{b_0}{2} + \sum_{k=1}^{\infty} b_k \cos kx + \Delta a_{n+1} D_n(x) \right| dx$$

$$\le \int_0^{\pi} \left| \frac{b_0}{2} + \sum_{k=1}^{\infty} b_k \cos kx \right| dx + \int_0^{\pi} |\Delta a_{n+1} D_n(x)| \, dx$$

$$\le C \left(\sum_{k=0}^{\infty} |\Delta b_k| + \sum_{m=2}^{\infty} \left| \sum_{k=1}^{\left[\frac{m}{2}\right]} \frac{\Delta b_{m-k} - \Delta b_{m+k}}{k} \right| \right)$$

$$+ \int_0^{\pi} |a_{n+1} D_n(x)| \, dx + \int_0^{\pi} |a_{n+2} D_n(x)| \, dx$$

$$\le C \Bigg[|a_{n+1}| + \sum_{k=n+1}^{\infty} |\Delta a_k| + \sum_{m=n}^{\infty} \left| \sum_{k=1}^{\left[\frac{m}{2}\right]} \frac{\Delta a_{m-k} - \Delta a_{m+k}}{k} \right|$$

$$+ \max_{\frac{n}{2} \le k < \frac{3n}{2}} |a_n| \log n \Bigg] + |a_{n+1}| \int_0^{\pi} |D_n(x)| \, dx + |a_{n+2}| \int_0^{\pi} |D_n(x)| \, dx.$$

Since $\int_0^{\pi} |D_n(x)| \, dx$ behaves like $\log n$ for large values of n, then the results follows.

The proof is completed.

Corollary 4.25. *Let $\{a_k\}$ be a sequence of numbers such that the conditions: (i) $a_k = o(1)$ as $k \to \infty$ (ii) $S_1 < \infty$, and $S_2 < \infty$ are satisfied. If $\lim_{n\to\infty} a_n \log n = 0$, then $\lim_{n\to\infty} \|f - S_n^c(x)\| = 0$.*

Proof. We have

$$\int_0^{\pi} |f(x) - S_n^c(x)| dx = \int_0^{\pi} |f(x) - w_n^c(x) + w_n^c(x) - S_n^c(x)| dx$$

$$\le \int_0^{\pi} |f(x) - w_n^c(x)| dx + \int_0^{\pi} |w_n^c(x) - S_n^c(x)| dx$$

$$= \int_0^\pi |f(x) - w_n^c(x)|dx + \int_0^\pi |\Delta a_{n+1} D_n(x)|dx$$
$$\leq \int_0^\pi |f(x) - w_n^c(x)|dx + |a_{n+1}| \int_0^\pi |D_n(x)|dx + |a_{n+2}| \int_0^\pi |D_n(x)|dx.$$

Since $\lim_{n\to\infty} \int_0^\pi |f(x)-w_n^c(x)|dx = 0$ by Theorem 4.24 and $\int_0^\pi |D_n(x)|\, dx$ behaves like $\log n$ for large values of n, then the results follows.

The proof is completed.

4.12 L^1-convergence of modified sums $w_n^c(x)$ with coefficients from the class S

In this unit we are going to consider cosine series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

modified cosine sums

$$w_n^c(x) = \frac{1}{2}\left(a_1 + \sum_{k=0}^{n} \Delta^2 a_k\right) + \sum_{k=1}^{n}\left(a_{k+1} + \sum_{j=k}^{n} \Delta^2 a_j\right) \cos kx,$$

where $\Delta^2 a_i = a_i - 2a_{i+1} + a_{i+2}$, and the class **S**.

We present the following result.

Theorem 4.26. *Let* $\{a_k\} \in \mathbf{S}$. *Then* $\lim_{n\to\infty} \|f - w_n^c(x)\| = 0$ *if and only if* $\lim_{n\to\infty} a_n \log n = 0$.

Proof. As in the previous unit we have

$$\begin{aligned} w_n^c(x) &= \frac{1}{2}\left(a_1 + \sum_{k=0}^{n} \Delta^2 a_k\right) + \sum_{k=1}^{n}\left(a_{k+1} + \sum_{j=k}^{n} \Delta^2 a_j\right) \cos kx \\ &= S_n^c(x) - \Delta a_{n+1} D_n(x). \end{aligned}$$

Using Abel's transformation, we get

$$\begin{aligned} w_n^c(x) &= \sum_{k=1}^{n-1} \Delta a_k D_k(x) + a_n D_n(x) - \Delta a_{n+1} D_n(x) \\ &= \sum_{k=1}^{n} \Delta a_k D_k(x) + a_{n+2} D_n(x). \end{aligned}$$

Hence,

$$f(x) - w_n^c(x) = \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - a_{n+2} D_n(x).$$

Abel's transformation with Lemma 1.34 yield,

$$\begin{aligned}
\int_0^{\pi} |f(x) - w_n^c(x)| dx &\le \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx + \int_0^{\pi} |a_{n+2} D_n(x)| \, dx \\
&= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx + |a_{n+2}| \int_0^{\pi} |D_n(x)| \, dx \\
&= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta A_k \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x) \right| dx + |a_{n+2}| \int_0^{\pi} |D_n(x)| \, dx \\
&\le \sum_{k=n+1}^{\infty} (k+1) \Delta A_k + |a_{n+2}| \int_0^{\pi} |D_n(x)| \, dx.
\end{aligned}$$

Now, $\int_0^{\pi} |a_{n+2} D_n(x)| \, dx$ behaves like $a_n \log n$ for large values of n, and under assumed hypothesis $a_n \log n \to 0$, $n \to \infty$, as well as $\sum_{k=1}^{\infty} (k+1) \Delta A_k < \infty$ and $\sum_{k=n+1}^{\infty} (k+1) \Delta A_k = o(1)$ as $n \to \infty$.

So, we have obtained

$$\lim_{n \to \infty} \|f - w_n^c(x)\| = 0.$$

On the other hand,

$$a_{n+2} D_n(x) = \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - [f(x) - w_n^c(x)],$$

and

$$\int_0^{\pi} |a_{n+2} D_n(x)| \, dx \le \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx + \int_0^{\pi} |f(x) - w_n^c(x)| \, dx.$$

Using the hypothesis of the theorem along with above estimates, the right hand side tends to zero as $n \to \infty$.

The proof is completed.

Corollary 4.27. *Let* $\{a_k\} \in \mathbf{S}$. *Then* $\lim_{n \to \infty} \|f - S_n^c(x)\| = 0$ *if and only if* $\lim_{n \to \infty} a_n \log n = 0$.

Proof. We have

$$\begin{aligned}\int_0^\pi |f(x)-S_n^c(x)|dx &= \int_0^\pi |f(x)-w_n^c(x)+w_n^c(x)-S_n^c(x)|dx\\ &\le \int_0^\pi |f(x)-w_n^c(x)|dx+\int_0^\pi |w_n^c(x)-S_n^c(x)|dx\\ &= \int_0^\pi |f(x)-w_n^c(x)|dx+\int_0^\pi |\Delta a_{n+1}D_n(x)|dx,\end{aligned}$$

whereas

$$\int_0^\pi |\Delta a_{n+1}D_n(x)|dx \le \int_0^\pi |f(x)-w_n^c(x)|dx+\int_0^\pi |f(x)-S_n^c(x)|dx.$$

Since $\int_0^\pi |D_n(x)|\,dx$ behaves like $\log n$ for large values of n, then by the hypothesis of our result the corollary follows.

The proof is completed.

Remark 4.28. This corollary is indeed a result proved earlier by S. A. Telyakovskii in 1973.

4.13 L^1-convergence of modified sums $l_n^c(x)$ and $l_n^s(x)$ with coefficients from the class S

We consider cosine series

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^\infty a_k\cos kx,$$

sine series

$$g(x)=\sum_{k=1}^\infty b_k\sin kx,$$

modified cosine

$$l_n^c(x)=\sum_{k=1}^n\left[\sum_{j=k}^n\left(\Delta a_{j+1}+\sum_{i=j}^n\Delta^3 a_i\right)\right]\cos kx,$$

and modified sine sums

$$l_n^s(x)=\sum_{k=1}^n\left[\sum_{j=k}^n\left(\Delta b_{j+1}+\sum_{i=j}^n\Delta^3 b_i\right)\right]\sin kx,$$

where $\Delta^3 c_i=\Delta^2 c_i-\Delta^2 c_{i+1}$, and the class **S**.

We prove first the following result.

Theorem 4.29. *Let* $\{a_k\} \in \mathbf{S}$ *and* $\lim_{n\to\infty} n^2 a_n = 0$. *Then*

$$\lim_{n\to\infty} \|f - l_n^c(x)\| = 0.$$

Proof. We have

$$\begin{aligned} l_n^c(x) &= \sum_{k=1}^{n} \left[\sum_{j=k}^{n} \left(\Delta a_{j+1} + \sum_{i=j}^{n} \Delta^3 a_i \right) \right] \cos kx \\ &= \sum_{k=1}^{n} \left[\sum_{j=k}^{n} \left(\Delta a_{j+1} + \Delta^2 a_j - \Delta^2 a_{n+1} \right) \right] \cos kx \\ &= \sum_{k=1}^{n} \left[a_k - a_{n+1} - (n-k+1)\Delta^2 a_{n+1} \right] \cos kx \\ &= \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x) \\ &\quad -(n+1)\Delta^2 a_{n+1} D_n(x) + \Delta^2 a_{n+1} \widetilde{D}_n'(x). \end{aligned}$$

We apply the Abel's transformation to get

$$l_n^c(x) = \sum_{k=1}^{n} \Delta a_k D_k(x) - (n+1)\Delta^2 a_{n+1} D_n(x) + \Delta^2 a_{n+1} \widetilde{D}_n'(x).$$

Since $D_n(x)$ is bounded and $|\Delta a_k| \le A_k$ for all $k \in \{1, 2, \ldots\}$, then by given hypothesis and Lemma 1.83 we conclude that $\lim_{n\to\infty} l_n^c(x) = f(x)$ exists in $(0, \pi)$.

Now, we consider the difference

$$\begin{aligned} f(x) - l_n^c(x) = &\sum_{k=n+1}^{\infty} a_k \cos kx + a_{n+1} D_n(x) \\ &+(n+1)\Delta^2 a_{n+1} D_n(x) - \Delta^2 a_{n+1} \widetilde{D}_n'(x). \end{aligned}$$

Applying Abel's transformation we get

$$\begin{aligned} f(x) - l_n^c(x) = &\sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \\ &+(n+1)\Delta^2 a_{n+1} D_n(x) - \Delta^2 a_{n+1} \widetilde{D}_n'(x). \end{aligned}$$

Thus,

$$\int_0^{\pi} |f(x) - l_n^c(x)|\, dx = \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right.$$

$$+(n+1)a_{n+1}D_n(x) - 2(n+1)a_{n+2}D_n(x) + (n+1)a_{n+3}D_n(x)$$
$$-a_{n+1}\widetilde{D}'_n(x) + 2a_{n+2}\widetilde{D}'_n(x) - a_{n+3}\widetilde{D}'_n(x)\Big| dx$$
$$\leq \int_0^\pi \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx$$
$$+(n+1)|a_{n+1}| \int_0^\pi \big|D_n(x)\big| dx + 2(n+1)|a_{n+2}| \int_0^\pi \big|D_n(x)\big| dx$$
$$+(n+1)|a_{n+3}| \int_0^\pi \big|D_n(x)\big| dx + |a_{n+1}| \int_0^\pi \big|\widetilde{D}'_n(x)\big| dx$$
$$+2|a_{n+2}| \int_0^\pi \big|\widetilde{D}'_n(x)\big| dx + |a_{n+3}| \int_0^\pi \big|\widetilde{D}'_n(x)\big| dx.$$

Using the fact that $\int_0^\pi \big|D_n(x)\big| dx \sim \log n$, for n big enough, Abel's transformation, Lemma 1.85 and Lemma 1.34, we have

$$\int_0^\pi |f(x) - l_n^c(x)|\, dx \leq \sum_{k=n+1}^{\infty} \Delta A_k \int_0^\pi \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x) \right| dx$$
$$+(n+1)|a_{n+1}| \log n + 2(n+1)|a_{n+2}| \log n$$
$$+(n+1)|a_{n+3}| \log n + |a_{n+1}| n \log n + 2|a_{n+2}| n \log n + |a_{n+3}| n \log n$$
$$\leq C \left[\sum_{k=n+1}^{\infty} \Delta (k+1) A_k + n^2|a_{n+1}| + n^2|a_{n+2}| \right.$$
$$\left. + n^2|a_{n+3}| + n^2|a_{n+1}| + n^2|a_{n+2}| + n^2|a_{n+3}| \right].$$

Since, $\{a_k\} \in \mathbf{S}$ and $\lim_{n\to\infty} n^2 a_n = 0$, then $\lim_{n\to\infty} \|f - l_n^c(x)\| = 0$. The proof is completed.

Corollary 4.30. *Let* $\{a_k\} \in \mathbf{S}$ *and* $\lim_{n\to\infty} n^2 a_n = 0$. *Then*

$$\lim_{n\to\infty} \|f - S_n^c(x)\| = 0 \Longleftrightarrow \lim_{n\to\infty} a_n \log n = 0.$$

Proof. Similarly, we notice that

$$\|f - S_n^c(x)\| \leq \|f - l_n^c(x)\| + \|l_n^c(x) - S_n^c(x)\|$$
$$\leq \|f - l_n^c(x)\| + \int_0^\pi |a_{n+1}D_n(x)$$
$$+(n+1)\Delta^2 a_{n+1} D_n(x) - \Delta^2 a_{n+1} \widetilde{D}'_n(x)| dx$$
$$\leq \|f - l_n^c(x)\| + n^2|a_{n+1}| + n^2|a_{n+2}|$$
$$+n^2|a_{n+3}| + n^2|a_{n+1}| + n^2|a_{n+2}| + n^2|a_{n+3}| \to 0,$$

as $n \to \infty$, based on Theorem 4.29, the fact that $\int_0^\pi \left|D_n(x)\right|dx \sim \log n$, for n big enough, Lemma 1.85, and given assumptions.

The proof is completed.

Pertaining to the sine series and the modified sine sums $l_n^s(x)$ we can prove, in a similar way, the following results. We have omitted their proofs.

Theorem 4.31. *Let* $\{a_k\} \in \mathbf{S}$ *and* $\lim_{n\to\infty} n^2 a_n = 0$. *Then*

$$\lim_{n\to\infty} \|f - l_n^s(x)\| = 0.$$

Corollary 4.32. *Let* $\{a_k\} \in \mathbf{S}$ *and* $\lim_{n\to\infty} n^2 a_n = 0$. *Then*

$$\lim_{n\to\infty} \|f - S_n^s(x)\| = 0 \Longleftrightarrow \lim_{n\to\infty} a_n \log n = 0.$$

4.14 L^1-convergence of $N_n(x)$ sums with quasi semi-convex coefficients

We consider cosine series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

with its partial sums

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx,$$

$$f(x) = \lim_{n\to\infty} S_n(x),$$

and modified cosine sums

$$N_n(x) = \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2} - \frac{1}{\left(2\sin\frac{x}{2}\right)^2} \sum_{k=1}^{n}\sum_{j=k}^{n} (\Delta^2 a_{j-1} - \Delta^2 a_j) \cos kx.$$

We prove the following result.

Theorem 4.33. *Let* $\{a_n\}$ *be a quasi semi-convex null sequence, then* $N_n(x)$ *converges to* $f(x)$ *in* L^1 *norm.*

Proof. We have

$$
\begin{aligned}
S_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx \\
&= \frac{1}{\left(2\sin\frac{x}{2}\right)^2} \sum_{k=1}^{n} a_k \cos kx \left(2\sin\frac{x}{2}\right)^2 \\
&= -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \sum_{k=1}^{n} a_k [\cos(k+1)x - 2\cos kx + \cos(k-1)x] \\
&= -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \sum_{k=1}^{n} (a_{k-1} - 2a_k + a_{k+1}) \cos kx \\
&\quad - \frac{a_0 \cos x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_n \cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2} - \frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2}
\end{aligned}
$$

or

$$
\begin{aligned}
S_n(x) &= -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \sum_{k=1}^{n} \Delta^2 a_{k-1} \cos kx \\
&\quad - \frac{a_0 \cos x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_n \cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2} - \frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2}.
\end{aligned}
$$

Applying Abel's transformation, we have

$$
\begin{aligned}
S_n(x) &= -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} - \Delta^2 a_k) \widetilde{D}_k(x) + \frac{\Delta^2 a_{n-1} \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2} \\
&\quad - \frac{a_0 \cos x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_n \cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2} - \frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2}.
\end{aligned}
$$

Since $\widetilde{D}_n(x)$ is uniformly bounded on every segment $[\epsilon, \pi - \epsilon]$ for every $\epsilon > 0$, then

$$
f(x) = \lim_{n\to\infty} S_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \sum_{k=1}^{\infty} (\Delta^2 a_{k-1} - \Delta^2 a_k) \widetilde{D}_k(x) + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}.
$$

Also,

$$
N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \sum_{k=1}^{n} \sum_{j=k}^{n} (\Delta^2 a_{j-1} - \Delta^2 a_j) \cos kx + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}
$$

can be rewritten as

$$
N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \sum_{k=1}^{n} \Delta^2 a_{k-1} \cos kx + \frac{\Delta^2 a_n \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}.
$$

Now applying Abel's transformation we get

$$N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n-1}(\Delta^2 a_{k-1} - \Delta^2 a_k)\widetilde{D}_k(x) + \frac{\Delta^2 a_{n-1}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2} + \frac{\Delta^2 a_n\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}.$$

From above relation we have

$$f(x) - N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=n+1}^{\infty}(\Delta^2 a_{k-1} - \Delta^2 a_k)\widetilde{D}_k(x) - \frac{\Delta^2 a_{n-1}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2} - \frac{\Delta^2 a_n\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}$$

or

$$f(x) - N_n(x) = -\lim_{m\to\infty}\left(\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=n+1}^{m}(\Delta^2 a_{k-1} - \Delta^2 a_k)\widetilde{D}_k(x)\right) - \frac{\Delta^2 a_{n-1}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2} - \frac{\Delta^2 a_n\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}.$$

Consequently, based on the assumption that $\{a_n\}$ is a quasi semi-convex null sequence, we obtain

$$\lim_{n\to\infty}\int_0^\pi |f(x) - N_n(x)|dx = 0.$$

The proof is completed.

Corollary 4.34. *Let $\{a_n\}$ be a quasi semi-convex null sequence, then $N_n(x)$ converges to $f(x)$ in L_1 norm.*

Proof. The proof follows directly from Theorem 4.33 and the fact that every quasi-convex null sequence is a quasi semi-convex sequence as well.

Corollary 4.35. *If $\{a_n\}$ is a quasi semi-convex null sequence, then the necessary and sufficient condition for L^1-convergence of the cosine series is*

$$\lim_{n\to\infty} a_n \log n = 0.$$

Proof. We have

$$\|S_n(x) - g(x)\| \le \|S_n(x) - N_n(x)\| + \|N_n(x) - g(x)\| = \|N_n(x) - g(x)\| + \left\|\frac{a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2} - \frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2} - \frac{\Delta^2 a_n\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}\right\|,$$

$$\begin{aligned}\left\| \frac{a_n \cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2} - \frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2} - \frac{\Delta^2 a_n \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2} \right\| \\ = \|N_n(x) - S_n(x)\| \le \|N_n(x) - f(x)\| + \|f(x) - S_n(x)\|,\end{aligned}$$

and

$$\begin{aligned}\Delta^2 a_n &= \sum_{k=n}^{\infty} (\Delta^2 a_k - \Delta^2 a_{k+1}) \\ &= \sum_{k=n}^{\infty} \frac{k}{k} (\Delta^2 a_k - \Delta^2 a_{k+1}) \le \frac{1}{n} \sum_{k=n}^{\infty} k(\Delta^2 a_k - \Delta^2 a_{k+1}) = o\left(\frac{1}{n}\right).\end{aligned}$$

Since

$$\int_0^{\pi} \frac{\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2} = \mathcal{O}(n),$$

then

$$\Delta^2 a_n \int_0^{\pi} \frac{\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2} = o(1).$$

Moreover,

$$\begin{aligned}\int_0^{\pi} \left| \frac{a_n \cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2} - \frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2} \right| dx \le \int_0^{\pi} a_n \left| \frac{\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2} - \frac{\cos nx}{\left(2\sin\frac{x}{2}\right)^2} \right| dx \\ = \int_0^{\pi} a_n \left| \widetilde{D}_n(x) - \frac{1}{2} \right| dx \sim (a_n \log n).\end{aligned}$$

From Theorem 4.33 it follows that

$$\|N_n(x) - f(x)\| = o(1), \ n \to \infty.$$

Finally we get

$$\lim_{n\to\infty} \int_0^{\pi} |f(x) - S_n(x)| dx = 0$$

if and only if

$$\lim_{n\to\infty} a_n \log n = 0.$$

The proof is completed.

The following consequence also holds true.

Corollary 4.36. *If $\{a_n\}$ is a quasi semi-convex null sequence, then the necessary and sufficient condition for L^1-convergence of the cosine series is*

$$\lim_{n\to\infty} a_n \log n = 0.$$

4.15 L^1-convergence of $N_n^{(1)}(x)$ sums with third quasi hyper convex coefficients

We consider cosine series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

with its partial sums

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx,$$

$$f(x) = \lim_{n\to\infty} S_n(x),$$

and modified cosine sums

$$\begin{aligned} N_n^{(1)}(x) = {} & \frac{a_2(6-\cos x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_1(15-6\cos x+\cos 2x)}{\left(2\sin\frac{x}{2}\right)^6} \\ & - \frac{1}{\left(2\sin\frac{x}{2}\right)^6} \sum_{k=1}^{n}\sum_{j=k}^{n} (\Delta^5 a_{j-3} - \Delta^5 a_{j-2}) \cos kx. \end{aligned}$$

We prove here the following result.

Theorem 4.37. *Let $\{a_n\}$ be a third quasi hyper convex zero sequence, then $N_n^{(1)}(x)$ converges to $f(x)$ in L^1 norm.*

Proof. Applying Abels transformation to the sums

$$\begin{aligned} N_n^{(1)}(x) = {} & \frac{a_2(6-\cos x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_1(15-6\cos x+\cos 2x)}{\left(2\sin\frac{x}{2}\right)^6} \\ & - \frac{1}{\left(2\sin\frac{x}{2}\right)^6} \sum_{k=1}^{n}\sum_{j=k}^{n} (\Delta^5 a_{j-3} - \Delta^5 a_{j-2}) \cos kx \end{aligned}$$

we have

$$\begin{aligned} N_n^{(1)}(x) = {} & -\frac{1}{\left(2\sin\frac{x}{2}\right)^6} \sum_{k=1}^{n-1} (\Delta^5 a_{j-3} - \Delta^5 a_{j-2}) \widetilde{D}_k(x) \\ & + \frac{\Delta^5 a_{n-3} \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6} + \frac{\Delta^5 a_{n-2} \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6} \\ & - \frac{a_1(15-6\cos x+\cos 2x)}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_2(6-\cos x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6} \end{aligned}$$

Repeating the Abel's transformation $3\alpha - 5$ times, we get

$$N_n^{(1)}(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=1}^{n-3\alpha+5}(\Delta^{3\alpha-1}a_{k-3} - \Delta^{3\alpha-1}a_{k-2})S_k^{3\alpha-6}(x)$$
$$-\sum_{k=1}^{3\alpha-6}\frac{(\Delta^{k+4}a_{n-k-3} - \Delta^{k+4}a_{n-k-2})S_{n-k}^k(x)}{\left(2\sin\frac{x}{2}\right)^6}$$
$$+\frac{\Delta^5 a_{n-3} + \Delta^5 a_{n-2}}{\left(2\sin\frac{x}{2}\right)^6}\widetilde{D}_n(x)$$
$$-\frac{a_1(15 - 6\cos x + \cos 2x)}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_2(6-\cos x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6}.$$

Since $S_n^k(x)$, $T_n(x)$, $\widetilde{D}_n(x)$ are uniformly bounded in any segment $[\epsilon, \pi-\epsilon]$, $\epsilon > 0$, and $T_n^k(x) = \frac{S_n^k(x)}{A_n^k}$, we obtain

$$f(x) = \lim_{n\to\infty} N_n^{(1)}(x)$$
$$= -\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=1}^{\infty}(\Delta^{3\alpha-1}a_{k-3} - \Delta^{3\alpha-1}a_{k-2})S_k^{3\alpha-6}(x)$$
$$-\frac{a_1(15 - 6\cos x + \cos 2x)}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_2(6-\cos x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6}.$$

From last two equalities we have

$$f(x) - N_n^{(1)}(x)$$
$$= -\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=n-(3\alpha-5)}^{\infty}(\Delta^{3\alpha-1}a_{k-3} - \Delta^{3\alpha-1}a_{k-2})S_k^{3\alpha-6}(x)$$
$$+\sum_{k=1}^{3\alpha-6}\frac{(\Delta^{k+4}a_{n-k-3} - \Delta^{k+4}a_{n-k-2})S_{n-k}^k(x)}{\left(2\sin\frac{x}{2}\right)^6}$$
$$-\frac{\Delta^5 a_{n-3} + \Delta^5 a_{n-2}}{\left(2\sin\frac{x}{2}\right)^6}\widetilde{D}_n(x).$$

Whence,

$$\|f(x) - N_n^{(1)}(x)\|$$
$$\le \left\|\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=n-(3\alpha-5)}^{\infty}(\Delta^{3\alpha-1}a_{k-3} - \Delta^{3\alpha-1}a_{k-2})S_k^{3\alpha-6}(x)\right\|$$
$$+\left\|\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=1}^{3\alpha-6}\Delta^{k+4}a_{n-k-3}S_{n-k}^k(x)\right\|$$

$$+\left\|\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=1}^{3\alpha-6}\Delta^{k+4}a_{n-k-2}S_{n-k}^k(x)\right\|$$
$$+\left\|\frac{\Delta^5 a_{n-3}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6}\right\|+\left\|\frac{\Delta^5 a_{n-2}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6}\right\|$$

Subsequently,

$$\begin{aligned}
&\|f(x)-N_n^{(1)}(x)\| \\
&\le C_1\int_0^\pi\left|\sum_{k=n-(3\alpha-5)}^{\infty}(\Delta^{3\alpha-1}a_{k-3}-\Delta^{3\alpha-1}a_{k-2})S_k^{3\alpha-6}(x)\right|dx \\
&\quad +C_1\int_0^\pi\left|\sum_{k=1}^{3\alpha-6}\Delta^{k+4}a_{n-k-3}S_{n-k}^k(x)\right|dx \\
&\quad +C_1\int_0^\pi\left|\sum_{k=1}^{3\alpha-6}\Delta^{k+4}a_{n-k-2}S_{n-k}^k(x)\right|dx \\
&\quad +C_1\int_0^\pi\left|\Delta^5 a_{n-3}\widetilde{D}_n(x)\right|dx+C_1\int_0^\pi\left|\Delta^5 a_{n-2}\widetilde{D}_n(x)\right|dx \\
&\le C_1\sum_{k=n-(3\alpha-5)}^{\infty}A_k^{3\alpha-6}\left|(\Delta^{3\alpha-5}a_{k-3}-\Delta^{3\alpha-5}a_{k-2})\right|\int_0^\pi\left|T_k^{\alpha-1}(x)\right|dx \\
&\quad +C_1\sum_{k=1}^{3\alpha-6}A_{n-k}^k\left|\Delta^{k+4}a_{n-k-3}\right|\int_0^\pi\left|T_{n-k}^k(x)\right|dx \\
&\quad +C_1\sum_{k=1}^{3\alpha-6}A_{n-k}^k\left|\Delta^{k+4}a_{n-k-2}\right|\int_0^\pi\left|T_{n-k}^k(x)\right|dx \\
&\quad +C_1A_n^0|\Delta^5 a_{n-3}|\int_0^\pi|T_n^0(x)|dx \\
&\quad +C_1A_n^0|\Delta^5 a_{n-2}|\int_0^\pi|T_n^0(x)|dx.
\end{aligned}$$

Next, we can write

$$\begin{aligned}
&\sum_{k=1}^{3\alpha-6}A_{n-k}^k\left|\Delta^{k+4}a_{n-k-2}\right|\int_0^\pi\left|T_{n-k}^k(x)\right|dx \\
&\quad=\sum_{k=1}^{3\alpha-6}A_{n-k}^k\big|\Delta^k a_{n-k-3}-4\Delta^k a_{n-k-2} \\
&\qquad +6\Delta^k a_{n-k-1}-4\Delta^k a_{n-k}+\Delta^k a_{n-k+1}\big|\int_0^\pi\left|T_{n-k}^k(x)\right|dx
\end{aligned}$$

Based on Lemma 1.48 the right hand side of last equality tends to zero. Also, based on the hypothesis that $\{a_n\}$ is a third quasi hyper convex sequence, then it holds

$$\sum_{k=1}^{\infty} k^{3\alpha} |(\Delta^{3\alpha-1} a_{k-1} - \Delta^{3\alpha-1} a_k)| < \infty.$$

So, finally we get

$$\|g(x) - N_n^{(1)}(x)\| \to 0, \quad n \to \infty.$$

The proof is completed.

4.16 L^1-convergence of $N_n^{(2)}(x)$ sums with twice quasi semi-convex coefficients

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

be cosine series with its partial sums

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx,$$

$$f(x) = \lim_{n\to\infty} S_n(x),$$

and modified cosine sums

$$\begin{aligned} N_n^{(2)}(x) = \frac{a_1(\cos x - 4)}{\left(2\sin\frac{x}{2}\right)^4} + \frac{a_2}{\left(2\sin\frac{x}{2}\right)^4} \\ + \frac{1}{\left(2\sin\frac{x}{2}\right)^4} \sum_{k=1}^{n} \sum_{j=k}^{n} (\Delta^4 a_{j-2} - \Delta^4 a_{j-1}) \cos kx. \end{aligned}$$

Now we are going to prove next result.

Theorem 4.38. *Let $\{a_n\}$ be a twice quasi semi-convex null sequence, then $N_n^{(2)}(x)$ converges to $f(x)$ in L^1 norm.*

Proof. We have

$$\begin{aligned} S_n(x) &= \frac{1}{\left(2\sin\frac{x}{2}\right)^4} \sum_{k=1}^{n} a_k \cos kx \left(2\sin\frac{x}{2}\right)^4 \\ &= \frac{1}{\left(2\sin\frac{x}{2}\right)^4} \sum_{k=1}^{n} a_k [\cos(k+2)x - 4\cos(k+1)x \\ &\quad + 6\cos kx - 4\cos(k-1)x + \cos(k-2)x] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\left(2\sin\frac{x}{2}\right)^4}\sum_{k=1}^{n}\left(a_{k-2}-4a_{k-1}+6a_k-4a_{k+1}+a_{k+2}\right)\cos kx \\
&\quad -\frac{a_{-1}\cos x}{\left(2\sin\frac{x}{2}\right)^4}-\frac{a_0\cos 2x}{\left(2\sin\frac{x}{2}\right)^4}+\frac{a_{n-1}\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^4}+\frac{a_n\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^4} \\
&\quad +\frac{4a_0\cos x}{\left(2\sin\frac{x}{2}\right)^4}-\frac{4a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^4}-\frac{4a_1}{\left(2\sin\frac{x}{2}\right)^4}+\frac{4a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^4} \\
&\quad +\frac{a_1\cos x}{\left(2\sin\frac{x}{2}\right)^4}+\frac{a_2}{\left(2\sin\frac{x}{2}\right)^4}-\frac{a_{n+1}\cos(n-1)x}{\left(2\sin\frac{x}{2}\right)^4}-\frac{a_{n+2}\cos nx}{\left(2\sin\frac{x}{2}\right)^4}
\end{aligned}
$$

or

$$
\begin{aligned}
S_n(x) &= \frac{1}{\left(2\sin\frac{x}{2}\right)^4}\sum_{k=1}^{n}\Delta^4 a_{k-2}\cos kx-\frac{a_{-1}\cos x}{\left(2\sin\frac{x}{2}\right)^4}-\frac{a_0\cos 2x}{\left(2\sin\frac{x}{2}\right)^4} \\
&\quad +\frac{a_{n-1}\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^4}+\frac{a_n\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^4}+\frac{4a_0\cos x}{\left(2\sin\frac{x}{2}\right)^4} \\
&\quad -\frac{4a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^4}-\frac{4a_1}{\left(2\sin\frac{x}{2}\right)^4}+\frac{4a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^4} \\
&\quad +\frac{a_1\cos x}{\left(2\sin\frac{x}{2}\right)^4}+\frac{a_2}{\left(2\sin\frac{x}{2}\right)^4}-\frac{a_{n+1}\cos(n-1)x}{\left(2\sin\frac{x}{2}\right)^4}-\frac{a_{n+2}\cos nx}{\left(2\sin\frac{x}{2}\right)^4}.
\end{aligned}
$$

Applying the Abel's transformation, we have

$$
\begin{aligned}
S_n(x) &= \frac{1}{\left(2\sin\frac{x}{2}\right)^4}\sum_{k=1}^{n-1}\left(\Delta^4 a_{k-2}-\Delta^4 a_{k-1}\right)\widetilde{D}_k(x) \\
&\quad -\frac{\left(\Delta^4 a_{n-2}-\Delta^4 a_{n-1}\right)\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4}+\frac{a_{n-1}\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^4} \\
&\quad +\frac{a_n\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^4}-\frac{4a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^4}-\frac{4a_1}{\left(2\sin\frac{x}{2}\right)^4}+\frac{4a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^4} \\
&\quad +\frac{a_1\cos x}{\left(2\sin\frac{x}{2}\right)^4}+\frac{a_2}{\left(2\sin\frac{x}{2}\right)^4}-\frac{a_{n-1}\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^4}-\frac{a_{n+2}\cos nx}{\left(2\sin\frac{x}{2}\right)^4}.
\end{aligned}
$$

Since $\widetilde{D}_n(x)$ is uniformly bounded on every segment $[\epsilon, \pi-\varepsilon]$, $\varepsilon > 0$, then

$$
\begin{aligned}
f(x) &= \lim_{n\to\infty} S_n(x) \\
&= \frac{1}{\left(2\sin\frac{x}{2}\right)^4}\sum_{k=1}^{\infty}\left(\Delta^4 a_{k-2}-\Delta^4 a_{k-1}\right)\widetilde{D}_k(x) \\
&\qquad +\frac{a_1(\cos x-4)}{\left(2\sin\frac{x}{2}\right)^4}+\frac{a_2}{\left(2\sin\frac{x}{2}\right)^4}.
\end{aligned}
$$

Moreover,

$$N_n^{(2)}(x) = \frac{a_1(\cos x - 4)}{\left(2\sin\frac{x}{2}\right)^4} + \frac{a_2}{\left(2\sin\frac{x}{2}\right)^4} + \frac{1}{\left(2\sin\frac{x}{2}\right)^4}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta^4 a_{j-2} - \Delta^4 a_{j-1})\cos kx.$$

can be rewritten as follows

$$N_n^{(2)}(x) = \frac{1}{\left(2\sin\frac{x}{2}\right)^4}\sum_{k=1}^{n}\Delta^4 a_{k-2}\cos kx - \frac{\Delta^4 a_{n-1}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4} + \frac{a_1(\cos x - 4)}{\left(2\sin\frac{x}{2}\right)^4} + \frac{a_2}{\left(2\sin\frac{x}{2}\right)^4}.$$

Again, applying the Abel's transformation we get the following

$$N_n^{(2)}(x) = \frac{1}{\left(2\sin\frac{x}{2}\right)^4}\sum_{k=1}^{n-1}(\Delta^4 a_{k-2} - \Delta^4 a_{k-1})\widetilde{D}_k(x) - \frac{\Delta^4 a_{n-2}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4} - \frac{\Delta^4 a_{n-1}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4} + \frac{a_1(\cos x - 4)}{\left(2\sin\frac{x}{2}\right)^4} + \frac{a_2}{\left(2\sin\frac{x}{2}\right)^4}.$$

Therefore, from above relations we obtain

$$f(x) - N_n^{(2)}(x) = \frac{1}{\left(2\sin\frac{x}{2}\right)^4}\sum_{k=n+1}^{\infty}(\Delta^4 a_{k-2} - \Delta^4 a_{k-1})\widetilde{D}_k(x) + \frac{\Delta^4 a_{n-2}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4} + \frac{\Delta^4 a_{n-1}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4}.$$

Consequently, based on our assumptions, we have

$$\lim_{n\to\infty}\int_0^{\pi}|f(x) - N_n^{(2)}(x)|dx = 0.$$

The proof is completed.

As a consequence of the above theorem is the following.

Corollary 4.39. *If $\{a_n\}$ is a twice quasi semi-convex null sequence, then the necessary and sufficient condition for L^1-convergence of the cosine series is*

$$\lim_{n\to\infty} a_n \log n = 0.$$

Proof. At first, we have

$$\begin{aligned}\|S_n(x)-g(x)\| &\le \|S_n(x)-N_n^{(2)}(x)\|+\|N_n^{(2)}(x)-f(x)\| \\ &\le \|N_n^{(2)}(x)-f(x)\|+\left\|\frac{2\Delta^4 a_{n-1}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4}\right\| \\ &\quad+\left\|\frac{a_n\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^4}-\frac{a_{n+2}\cos nx}{\left(2\sin\frac{x}{2}\right)^4}\right\| \\ &\quad+4\left\|\frac{a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^4}-\frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^4}\right\|.\end{aligned}$$

On the other hand

$$\begin{aligned}\Delta^4 a_{n-1} &= \sum_{k=n-1}^{\infty}(\Delta^4 a_k-\Delta^4 a_{k+1}) \\ &= \sum_{k=n-1}^{\infty}\frac{k}{k}(\Delta^4 a_k-\Delta^4 a_{k+1}) \\ &\le \frac{1}{n-1}\sum_{k=n-1}^{\infty}k(\Delta^4 a_k-\Delta^4 a_{k+1})=o\left(\frac{1}{n}\right).\end{aligned}$$

Taking into account that

$$\int_0^{\pi}\left|\frac{\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4}\right|dx=\mathcal{O}(n),$$

then

$$\Delta^4 a_{n-1}\int_0^{\pi}\left|\frac{\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4}\right|dx=o(1).$$

We also have

$$\begin{aligned}&\int_0^{\pi}\left|\frac{a_n\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^4}-\frac{a_{n+2}\cos nx}{\left(2\sin\frac{x}{2}\right)^4}\right|dx \\ &\qquad\le C_1\int_0^{\pi}a_n\left|\frac{\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^2}-\frac{\cos nx}{\left(2\sin\frac{x}{2}\right)^2}\right|dx \\ &\qquad\le C_2\int_0^{\pi}a_n\left|\widetilde{D}_n(x)-\frac{1}{2}\right|dx\sim C_3(a_n\log n).\end{aligned}$$

In a similar way we find that

$$\int_0^{\pi}\left|\frac{a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^4}-\frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^4}\right|dx\sim C_4(a_n\log n),$$

where C_1, C_2, C_3, and C_4 are positive constants. From Theorem 4.38 it follows that

$$\|N_n^{(2)}(x) - f(x)\| = o(1),\ n \to \infty.$$

Subsequently,

$$\lim_{n\to\infty} \int_0^\pi |f(x) - S_n(x)| dx = 0$$

if and only if

$$\lim_{n\to\infty} a_n \log n = 0.$$

The proof is completed.

4.17 L^1-convergence of the sine series whose coefficients belong to some generalized classes of sequences

We consider trigonometric sine series

$$g(x) = \sum_{k=1}^{\infty} a_k \sin kx,$$

with its partial sums

$$S_n^s(x) = \sum_{k=1}^{n} a_k \sin kx,$$

and

$$\lim_{n\to\infty} S_n^s(x) = g(x).$$

Here we have the following generalized modified sine sums

$$z_{n,m}^s(x) = \sum_{k=1}^{n} \left[\frac{a_{k+1}}{(k+1)^m} + \sum_{j=k}^{n} \Delta^2 \left(\frac{a_j}{j^m} \right) \right] k^m \sin kx,\ m \in \{1, 2, \ldots\},$$

where $\Delta^2 c_i := \Delta(\Delta c_i) := c_i - 2c_{i+1} + c_{i+2}$.

Remark 4.40. Note that $z_{n,1}^s(x) \equiv z_n^s(x)$ which have been introduced for the fist time in [78].

Further we introduce the following classes of sequences:

Definition 4.41. *A zero sequence (a_k) belongs to the class $\widetilde{C}_{r,m}$, ($r = 0, 1, 2, \ldots$; $m = 1, 2, \ldots$), if for every $\varepsilon > 0$, there exists $\delta > 0$ independent on n and such that for all n,*

$$\int_0^\delta \left| \sum_{k=n}^{\infty} \Delta b_{k,m} D_k^{(r+m)}(x) \right| dx \le \varepsilon,$$

where $b_k = \frac{a_k}{k^m}$, $D_k^{(r+m)}(x)$ *denotes the* $(r+m)$*-th derivative of the Dirichlet kernel*

$$D_k(x) = \frac{1}{2} + \sum_{j=1}^{k} \cos jx = \frac{\sin\left(k+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}.$$

Remark 4.42. It is clear that $\widetilde{C}_{r+1,m} \subset \widetilde{C}_{r,m}$, $(r = 0,1,2,\ldots;\ m = 1,2,\ldots)$, however, the converse inclusion need not to be true in general as shows next example.

Example 4.43. Define $b_{n,m} = \sum_{k=n}^{\infty} \frac{1}{k^{r+m+2}}$, $(r = 0,1,2,\ldots;\ m = 1,2,\ldots)$, then $\Delta b_{n,m} = \frac{1}{n^{r+m+2}}$ and

$$a_n = nb_{n,m} = n\sum_{k=n}^{\infty} \frac{1}{k^{r+m+2}} \le \sum_{k=n}^{\infty} \frac{k}{k^{r+m+2}} = \sum_{k=n}^{\infty} \frac{1}{k^{r+m+2}} \to 0, \quad when \quad n \to \infty.$$

So, using Bernstein's inequality, the integral

$$\begin{aligned}\int_0^{\delta} \left|\sum_{k=n}^{\infty} \Delta b_{k,m} D_k^{(r+1+m)}(x)\right| dx &\le \int_0^{\pi} \left|\sum_{k=n}^{\infty} \Delta b_{k,m} D_k^{(r+1+m)}(x)\right| dx \\ &\le \sum_{k=n}^{\infty} |\Delta b_{k,m}| \int_0^{\pi} \left|D_k^{(r+1+m)}(x)\right| dx \\ &\le \sum_{k=n}^{\infty} \frac{k^{r+1+m}}{k^{r+m+2}} \int_0^{\pi} |D_k(x)|\, dx = \mathcal{O}\left(\sum_{k=1}^{\infty} \frac{\log k}{k}\right),\end{aligned}$$

is divergent, which means $(a_n) \notin \widetilde{C}_{r+1,m}$.

On the other side, the integral

$$\begin{aligned}\int_0^{\delta} \left|\sum_{k=n}^{\infty} \Delta b_{k,m} D_k^{(r+m)}(x)\right| dx &\le \int_0^{\pi} \left|\sum_{k=n}^{\infty} \Delta b_{k,m} D_k^{(r+m)}(x)\right| dx \\ &\le \sum_{k=n}^{\infty} |\Delta b_{k,m}| \int_0^{\pi} \left|D_k^{(r+m)}(x)\right| dx \\ &\le \sum_{k=n}^{\infty} \frac{k^{r+m}}{k^{r+m+2}} \int_0^{\pi} |D_k(x)|\, dx = \mathcal{O}\left(\sum_{k=1}^{\infty} \frac{\log k}{k^2}\right),\end{aligned}$$

is convergent, which means $(a_n) \in \widetilde{C}_{r,m}$.

Definition 4.44. *A zero sequence* (a_k) *belongs to the class* $\widetilde{S}_{r,m}$, $(r = 0,1,2,\ldots;$ $m = 1,2,\ldots)$, *if there exists a non-increasing sequence* (B_k) *so that,* $|\Delta b_{k,m}| \le B_k$, $\forall k \in \{1,2,\ldots\}$, *and* $\sum_{k=1}^{\infty} k^{r+m} B_k < \infty$, *where* $b_k = \frac{a_k}{k^m}$.

Remark 4.45. It is clear that $\widetilde{S}_{r+1,m} \subset \widetilde{S}_{r,m}$, $(r = 0,1,2,\ldots;\ m = 1,2,\ldots)$, however, the converse inclusion need not to be true as shows next example.

Example 4.46. Define $b_{n,m} = \sum_{k=n}^{\infty} \frac{1}{k^{r+m+2}}$, $(r = 0,1,2,\ldots; \; m = 1,2,\ldots)$, then $\Delta b_{n,m} = \frac{1}{n^{r+m+2}}$ and

$$a_n = n b_{n,m} = n \sum_{k=n}^{\infty} \frac{1}{k^{r+m+2}} \le \sum_{k=n}^{\infty} \frac{k}{k^{r+m+2}} = \sum_{k=n}^{\infty} \frac{1}{k^{r+m+1}} \to 0,$$

when $n \to \infty$.

Choosing $B_n = \frac{1}{n^{r+m+2}}$, $(r = 0,1,2,\ldots; \; m = 1,2,\ldots)$, then $B_n \downarrow 0$ and $|\Delta b_{n,m}| \le B_n$. Now, the series

$$\sum_{k=1}^{\infty} k^{r+m} B_k = \sum_{k=1}^{\infty} k^{r+m} \frac{1}{k^{r+m+2}} = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

is convergent, which means $(a_n) \in \widetilde{S}_{r,m}$.

However, the series

$$\sum_{k=1}^{\infty} k^{r+1+m} B_k = \sum_{k=1}^{\infty} k^{r+1+m} \frac{1}{k^{r+m+2}} = \sum_{k=1}^{\infty} \frac{1}{k}$$

is divergent, which means $(a_n) \notin \widetilde{S}_{r+1,m}$.

Definition 4.47. *A zero sequence (a_k) belongs to the class $\widetilde{BV}_{r,m}$, $(r = 0,1,2,\ldots; \; m = 1,2,\ldots)$, if*

$$\sum_{k=1}^{\infty} k^{r+m} |\Delta b_{k,m}| < \infty,$$

where $b_{k,m} = \frac{a_k}{k^m}$.

Remark 4.48. It is clear that $\widetilde{BV}_{r+1,m} \subset \widetilde{BV}_{r,m}$, $(r = 0,1,2,\ldots; \; m = 1,2,\ldots)$, however, the converse inclusion need not to be true.

Remark 4.49. We have to note that $\widetilde{C}_{r,m} \equiv \widetilde{C}_r$, $\widetilde{S}_{r,m} \equiv \widetilde{S}_r$, and $\widetilde{BV}_{r,m} \equiv \widetilde{BV}_r$ for $m = 1$, and $\widetilde{C}_{r,m} \equiv \widetilde{C}$, $\widetilde{S}_{r,m} \equiv \widetilde{S}$, and $\widetilde{BV}_{r,m} \equiv \widetilde{BV}$ for $m = 1$ and $r = 0$. These particular classes are introduced in [78].

Pertaining to $\widetilde{BV}_{r,m}$ class, $r \in \{0,1,2,\ldots\}$ and $m \in \{1,2,\ldots\}$, the following natural question can be raised: What about inclusion of classes $\widetilde{BV}_{r,m}$ with respect to m? The answer is given in next simple proposition.

Theorem 4.50. *If*

$$\sum_{k=1}^{\infty} (k+1)^r |\Delta a_k| < \infty,$$

then

$$\widetilde{BV}_{r,m} \subset \widetilde{BV}_{r,m+1},$$

for all $r \in \{0,1,2,\ldots\}$ and $m \in \{1,2,\ldots\}$.

Proof. We have

$$\begin{aligned}
\sum_{k=1}^{\infty} k^{r+m+1}|\Delta b_{k,m+1}| &\leq \sum_{k=1}^{\infty} k^{r+m+1}\left|\frac{a_k}{k^{m+1}} - \frac{a_{k+1}}{k(k+1)^m}\right| \\
&\quad + \sum_{k=1}^{\infty} k^{r+m+1}\left|\frac{a_{k+1}}{k(k+1)^m} - \frac{a_{k+1}}{(k+1)^{m+1}}\right| \\
&\leq \sum_{k=1}^{\infty} k^{r+m}\left|\frac{a_k}{k^m} - \frac{a_{k+1}}{(k+1)^m}\right| + \sum_{k=1}^{\infty} k^{r-1}\left|a_{k+1}\right| \\
&\leq \sum_{k=1}^{\infty} k^{r+m}\left|\Delta b_{k,m}\right| + \sum_{k=1}^{\infty} k^{r-1}\sum_{j=k+1}^{\infty}\left|\Delta a_j\right| \\
&= \sum_{k=1}^{\infty} k^{r+m}\left|\Delta b_{k,m}\right| + \sum_{j=1}^{\infty}\left|\Delta a_j\right|\sum_{k=1}^{j+1} k^{r-1} \\
&\leq \sum_{k=1}^{\infty} k^{r+m}\left|\Delta b_{k,m}\right| + \sum_{j=1}^{\infty}(j+1)^r\left|\Delta a_j\right|,
\end{aligned}$$

which implies that $\widetilde{BV}_{r,m} \subset \widetilde{BV}_{r,m+1}$, for all $r \in \{0,1,2,\ldots\}$ and $m \in \{1,2,\ldots\}$.

The proof is completed.

Theorem 4.51. *The following relation holds true* $\widetilde{S}_{r,m} \subset \widetilde{C}_{r,m} \cap \widetilde{BV}_{r,m}$ *for each* $r \in \{0,1,2,\ldots\}$ *and* $m \in \{1,2,\ldots\}$.

Proof. Let $(a_k) \in \widetilde{S}_{r,m}$, $(r = 0,1,2,\ldots; m = 1,2,\ldots)$. Then there exists a non-increasing sequence (B_k) of numbers so that, $|\Delta b_{k,m}| \leq B_k$, $\forall k \in \{1,2,\ldots\}$, and $\sum_{k=1}^{\infty} k^{r+m} B_k < \infty$. Whence, we clearly have

$$\sum_{k=1}^{\infty} k^{r+m}|\Delta b_{k,m}| \leq \sum_{k=1}^{\infty} k^{r+m} B_k < \infty, \tag{4.19}$$

which means that $\widetilde{S}_{r,m} \subset \widetilde{BV}_{r,m}$ for each $r \in \{0,1,2,\ldots\}$ and $m \in \{1,2,\ldots\}$.

So, it remains to prove the inclusion $\widetilde{S}_{r,m} \subset \widetilde{C}_{r,m}$ for each $r \in \{0,1,2,\ldots\}$ and $m \in \{1,2,\ldots\}$. Let $(a_k) \in \widetilde{S}_{r,m}$, then applying Abel's transformation we get

$$\begin{aligned}
\int_0^{\pi}\left|\sum_{k=n}^{\infty}\Delta b_{k,m} D_k^{(r+m)}(x)\right| dx &\leq \lim_{s\to\infty}\left[\sum_{k=n}^{s-1}\Delta B_k \int_0^{\pi}\left|\sum_{j=0}^{k}\frac{\Delta b_{j,m}}{B_j} D_j^{(r+m)}(x)\right| dx\right. \\
&\quad \left. + B_s \int_0^{\pi}\left|\sum_{j=0}^{s}\frac{\Delta b_{j,m}}{B_j} D_j^{(r+m)}(x)\right| dx\right.
\end{aligned}$$

$$+B_n \int_0^{\pi} \left| \sum_{j=0}^{n-1} \frac{\Delta b_{j,m}}{B_j} D_j^{(r+m)}(x) \right| dx \Bigg].$$

Applying, in last inequality, the well-known Bernstein's inequality and Lemma 1.34, we obtain

$$\begin{aligned}
\int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta b_{k,m} D_k^{(r+m)}(x) \right| dx \le \lim_{s\to\infty} \Bigg[&\sum_{k=n}^{s-1} k^{r+m} \Delta B_k \int_0^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta b_{j,m}}{B_j} D_j(x) \right| dx \\
&+ s^{r+m} B_s \int_0^{\pi} \left| \sum_{j=0}^{s} \frac{\Delta b_{j,m}}{B_j} D_j(x) \right| dx \\
&+ (n-1)^{r+m} B_n \int_0^{\pi} \left| \sum_{j=0}^{n-1} \frac{\Delta b_{j,m}}{B_j} D_j(x) \right| dx \Bigg] \\
\le C \lim_{s\to\infty} \Bigg[&\sum_{k=n}^{s-1} (k+1)^{r+m+1} \Delta B_k \\
&+ s^{r+m+1} B_s + n^{r+m+1} B_n \Bigg].
\end{aligned}$$

Since (B_k) is a non-increasing sequence and $\sum_{k=1}^{\infty} k^{r+m} B_k < \infty$, then $k^{r+m+1} B_k \to 0$ as $k \to \infty$, and thus

$$\begin{aligned}
\int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta b_{k,m} D_k^{(r+m)}(x) \right| dx &\le C \left[\sum_{k=n}^{\infty} (k+1)^{r+m+1} \Delta B_k + n^{r+m+1} B_n \right] \\
&\le C \left\{ \sum_{k=n}^{\infty} B_k \left[(k+1)^{r+m+1} - k^{r+m+1} \right] + n^{r+m+1} B_n \right\} \\
&\le C(r,m) \left\{ \sum_{k=n}^{\infty} k^{r+m} B_k + n^{r+m+1} B_n \right\} \le \frac{\varepsilon}{2},
\end{aligned}$$

for n big enough, say $s \ge n > n_0$.

Finally, using the fact that

$$\left| D_k^{(r+m)}(x) \right| = \left| \sum_{j=1}^{k} j^{(r+m)} \sin\left(jx + \frac{(r+m)\pi}{2} \right) \right| \le k^{r+m+1},$$

then for any $1 \le n \le s$ we can write as follows

$$\int_0^{\delta} \left| \sum_{k=n}^{s} \Delta b_{k,m} D_k^{(r+m)}(x) \right| dx \le \int_0^{\delta} \left| \sum_{k=n}^{n_0} \Delta b_{k,m} D_k^{(r+m)}(x) \right| dx$$

$$
\begin{aligned}
&+\int_0^\pi \left| \sum_{k=n_0+1}^{s} \Delta b_{k,m} D_k^{(r+m)}(x) \right| dx \\
&\leq \delta \sum_{k=n}^{n_0} k^{r+m+1} \left|\Delta b_{k,m}\right| \\
&+\int_0^\pi \left| \sum_{k=n_0+1}^{\infty} \Delta b_{k,m} D_k^{(r+m)}(x) \right| dx \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}
$$

for δ small enough. This means that $\widetilde{S}_{r,m} \subset \widetilde{C}_{r,m}$ for each $r \in \{0,1,2,\ldots\}$ and $m \in \{1,2,\ldots\}$.

The proof is completed.

For $m = 1$ we get the following corollary [78].

Corollary 4.52. *The following relation holds true* $\widetilde{S}_r \subset \widetilde{C}_r \cap \widetilde{BV}_r$ *for each* $r \in \{0,1,2,\ldots\}$.

Theorem 4.53. *Let* $(a_n) \in \widetilde{C}_m \cap \widetilde{BV}_m$ *for each* $m \in \{1,2,\ldots\}$, *and* $\lim_{n\to\infty} a_n \log n = 0$. *Then*

$$\lim_{n\to\infty} \|z_{n,m}^s - g\| = 0.$$

Proof. We have

$$
\begin{aligned}
z_{n,m}^s(x) &= \sum_{k=1}^{n} \left[\frac{a_{k+1}}{(k+1)^m} + \sum_{j=k}^{n} \Delta^2 \left(\frac{a_j}{j^m} \right) \right] k^m \sin kx \\
&= \sum_{k=1}^{n} a_k \sin kx + \left[\frac{a_{n+2}}{(n+2)^m} - \frac{a_{n+1}}{(n+1)^m} \right] \sum_{k=1}^{n} k^m \sin kx \\
&= S_n^s(x) - \Delta\left(b_{n+1,m}\right) \sum_{k=1}^{n} k^m \sin kx. \qquad (4.20)
\end{aligned}
$$

After some transformation we have found that

$$
S_n^s(x) = \begin{cases} -\sum_{k=1}^{n} b_{k,m} (\cos kx)^{(m)}, & \text{if} \quad m = 4p-3; \\ -\sum_{k=1}^{n} b_{k,m} (\sin kx)^{(m)}, & \text{if} \quad m = 4p-2; \\ +\sum_{k=1}^{n} b_{k,m} (\cos kx)^{(m)}, & \text{if} \quad m = 4p-1; \\ +\sum_{k=1}^{n} b_{k,m} (\sin kx)^{(m)}, & \text{if} \quad m = 4p, \end{cases} \qquad (4.21)
$$

and

$$
\sum_{k=1}^{n} k^m \sin kx = \begin{cases} -D_n^{(m)}(x), & \text{if} \quad m = 4p-3; \\ -\widetilde{D}_n^{(m)}(x), & \text{if} \quad m = 4p-2; \\ +D_n^{(m)}(x), & \text{if} \quad m = 4p-1; \\ +\widetilde{D}_n^{(m)}(x), & \text{if} \quad m = 4p, \end{cases} \qquad (4.22)
$$

where in all cases $p \in N$.

Combining (4.20) along with (4.21) and (4.22) we obtain

$$z^s_{n,m}(x) = \begin{cases} -\sum_{k=1}^n b_{k,m}(\cos kx)^{(m)} + \Delta\left(b_{n+1,m}\right) D_n^{(m)}(x), & \text{if} \quad m = 4p-3; \\ -\sum_{k=1}^n b_{k,m}(\sin kx)^{(m)} + \Delta\left(b_{n+1,m}\right) \widetilde{D}_n^{(m)}(x), & \text{if} \quad m = 4p-2; \\ +\sum_{k=1}^n b_{k,m}(\cos kx)^{(m)} - \Delta\left(b_{n+1,m}\right) D_n^{(m)}(x), & \text{if} \quad m = 4p-1; \\ +\sum_{k=1}^n b_{k,m}(\sin kx)^{(m)} - \Delta\left(b_{n+1,m}\right) \widetilde{D}_n^{(m)}(x), & \text{if} \quad m = 4p, \end{cases} \tag{4.23}$$

for all $p \in N$.

The use of Abel's transformation in (4.23), implies

$$z^s_{n,m}(x) = \begin{cases} -\sum_{k=1}^n \Delta b_{k,m} D_k^{(m)}(x) \\ \qquad -b_{n,m} D_n^{(m)}(x) + \Delta\left(b_{n+1,m}\right) D_n^{(m)}(x), & \text{if} \quad m = 4p-3; \\ -\sum_{k=1}^n \Delta b_{k,m} \widetilde{D}_k^{(m)}(x) \\ \qquad -b_{n,m} \widetilde{D}_n^{(m)}(x) + \Delta\left(b_{n+1,m}\right) \widetilde{D}_n^{(m)}(x), & \text{if} \quad m = 4p-2; \\ \sum_{k=1}^n \Delta b_{k,m} D_k^{(m)}(x) \\ \qquad +b_{n,m} D_n^{(m)}(x) - \Delta\left(b_{n+1,m}\right) D_n^{(m)}(x), & \text{if} \quad m = 4p-1; \\ \sum_{k=1}^n \Delta b_{k,m} \widetilde{D}_k^{(m)}(x) \\ \qquad +b_{n,m} \widetilde{D}_n^{(m)}(x) - \Delta\left(b_{n+1,m}\right) \widetilde{D}_n^{(m)}(x), & \text{if} \quad m = 4p, \end{cases} \tag{4.24}$$

for all $p \in N$.

Applying Abel's transformation in (4.21), we also get

$$S^s_n(x) = \begin{cases} -\sum_{k=1}^n \Delta b_{k,m} D_k^{(m)}(x) - b_{n,m} D_n^{(m)}(x), & \text{if} \quad m = 4p-3; \\ -\sum_{k=1}^n \Delta b_{k,m} \widetilde{D}_k^{(m)}(x) - b_{n,m} \widetilde{D}_n^{(m)}(x), & \text{if} \quad m = 4p-2; \\ +\sum_{k=1}^n \Delta b_{k,m} D_k^{(m)}(x) + b_{n,m} D_n^{(m)}(x), & \text{if} \quad m = 4p-1; \\ +\sum_{k=1}^n \Delta b_{k,m} \widetilde{D}_k^{(m)}(x) + b_{n,m} \widetilde{D}_n^{(m)}(x), & \text{if} \quad m = 4p, \end{cases} \tag{4.25}$$

for all $p \in N$.

Using Lemma 1.91 in (4.24) and (4.25), we have that

$$|z^s_{n,m}(x)| \le \mathcal{O}\left(x^{-1}\right) \left(\sum_{k=1}^n k^m |\Delta b_{k,m}| + |a_n| + |a_{n+1}| + |a_{n+2}| \right), \tag{4.26}$$

and

$$|S^s_n(x)| \le \mathcal{O}\left(x^{-1}\right) \left(\sum_{k=1}^n k^m |\Delta b_{k,m}| + |a_n| \right), \tag{4.27}$$

for all $m \in N$.

Whence, letting $n \to \infty$ in (4.26) and (4.27), and taking into account that $(a_k) \in \widetilde{BV}_{r,m}$, $m = 1, 2, \ldots)$, we conclude that series $\sum_{k=1}^\infty \Delta b_{k,m} D_k^{(m)}(x)$ and $\sum_{k=1}^\infty \Delta b_{k,m} \widetilde{D}_k^{(m)}(x)$ converge absolutely, and

$$\lim_{n\to\infty} z^s_{n,m}(x) = \lim_{n\to\infty} S^s_n(x) = g(x)$$

exists for all $x \in [\varepsilon, \pi]$, where $\varepsilon > 0$ as small as.

Now, we have

$$g(x) - z^s_{n,m}(x) = \begin{cases} -\sum_{k=n+1}^{\infty} \Delta b_{k,m} D_k^{(m)}(x) \\ \qquad +b_{n,m} D_n^{(m)}(x) - \Delta\left(b_{n+1,m}\right) D_n^{(m)}(x), & \text{if} \quad m = 4p-3; \\ -\sum_{k=n+1}^{\infty} \Delta b_{k,m} \widetilde{D}_k^{(m)}(x) \\ \qquad +b_{n,m} \widetilde{D}_n^{(m)}(x) - \Delta\left(b_{n+1,m}\right) \widetilde{D}_n^{(m)}(x), & \text{if} \quad m = 4p-2; \\ +\sum_{k=n+1}^{\infty} \Delta b_{k,m} D_k^{(m)}(x) \\ \qquad -b_{n,m} D_n^{(m)}(x) + \Delta\left(b_{n+1,m}\right) D_n^{(m)}(x), & \text{if} \quad m = 4p-1; \\ +\sum_{k=n+1}^{\infty} \Delta b_{k,m} \widetilde{D}_k^{(m)}(x) \\ \qquad -b_{n,m} \widetilde{D}_n^{(m)}(x) + \Delta\left(b_{n+1,m}\right) \widetilde{D}_n^{(m)}(x), & \text{if} \quad m = 4p, \end{cases} \tag{4.28}$$

for all $p \in N$.

Thus, based on (4.28), we have

$$\|g - z^s_{n,m}\| \le \begin{cases} \int_0^\pi \left|\sum_{k=n+1}^{\infty} \Delta b_{k,m} D_k^{(m)}(x)\right| dx \\ +|b_{n,m}| \int_0^\pi \left|D_n^{(m)}(x)\right| dx \\ +|\Delta\left(b_{n+1,m}\right)| \int_0^\pi \left|D_n^{(m)}(x)\right| dx, & \text{if} \quad m = 4p-3 \vee m = 4p-1; \\ \int_0^\pi \left|\sum_{k=n+1}^{\infty} \Delta b_{k,m} \widetilde{D}_k^{(m)}(x)\right| dx \\ +|b_{n,m}| \int_0^\pi \left|\widetilde{D}_n^{(m)}(x)\right| dx \\ +|\Delta\left(b_{n+1,m}\right)| \int_0^\pi \left|\widetilde{D}_n^{(m)}(x)\right| dx, & \text{if} \quad m = 4p-2 \vee m = 4p, \end{cases} \tag{4.29}$$

for all $p \in N$.

Let us estimate the terms in right hand side of (4.29). Namely, since $(a_n) \in \widetilde{C}_m \cap \widetilde{BV}_m$ for each $m \in \{1, 2, \ldots\}$, then for $\varepsilon > 0$ there exists $\delta > 0$, such that

$$\int_0^\delta \left|\sum_{k=n+1}^{\infty} \Delta b_{k,m} D_k^{(m)}(x)\right| dx \le \frac{\varepsilon}{2},$$

for all $n \ge 0$.

Consequently, for $m = 4p-3 \vee m = 4p-1$ and Bernstein's inequality we get

$$\int_0^\pi \left|\sum_{k=n+1}^{\infty} \Delta b_{k,m} D_k^{(m)}(x)\right| dx = \int_0^\delta \left|\sum_{k=n+1}^{\infty} \Delta b_{k,m} D_k^{(m)}(x)\right| dx + \int_\delta^\pi \left|\sum_{k=n+1}^{\infty} \Delta b_{k,m} D_k^{(m)}(x)\right| dx \tag{4.30}$$

$$\begin{aligned}
&\leq \frac{\varepsilon}{2} + \sum_{k=n+1}^{\infty} |\Delta b_{k,m}| \int_{\delta}^{\pi} \left| D_k^{(m)}(x) \right| dx \\
&\leq \frac{\varepsilon}{2} + \sum_{k=n+1}^{\infty} k^{m-1} |\Delta b_{k,m}| \int_{\delta}^{\pi} |D_k'(x)| \, dx \\
&\leq \frac{\varepsilon}{2} + C \sum_{k=n+1}^{\infty} k^{m} |\Delta b_{k,m}| \int_{\delta}^{\pi} \frac{dx}{x^2} \\
&\leq \frac{\varepsilon}{2} + \frac{C}{\delta} \sum_{k=n+1}^{\infty} k^{m} |\Delta b_{k,m}| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \qquad (4.31)
\end{aligned}$$

and in a similar way, for $m = 4p - 2 \vee m = 4p$,

$$\int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta b_{k,m} \widetilde{D}_k^{(m)}(x) \right| dx < \varepsilon. \qquad (4.32)$$

The other terms also tend to zero, since they can be estimated as $a_n \log n$ (using Lemma 1.84). Using these facts, (4.29), (4.30) and (4.32) we have proved that

$$\lim_{n\to\infty} \|z^s_{n,m} - g\| = 0.$$

The proof is completed.

For $m = 1$ we have the following result given in [78].

Corollary 4.54. *Let* $(a_n) \in \widetilde{C} \cap \widetilde{BV}$, *and* $\lim_{n\to\infty} a_n \log n = 0$. *Then*

$$\lim_{n\to\infty} \|z^s_n - g\| = 0.$$

Theorem 4.55. *Let* $(a_n) \in \widetilde{C}_m \cap \widetilde{BV}_m$ *for each* $m \in \{1, 2, \ldots\}$, *and* $\lim_{n\to\infty} a_n \log n = 0$. *Then*

$$\lim_{n\to\infty} \|S^s_n - g\| = 0.$$

Proof. Firstly, we have

$$\begin{aligned}
\|S^s_n - g\| &= \|S^s_n - z^s_{n,m} + z^s_{n,m} - g\| \\
&\leq \|z^s_{n,m} - S^s_n\| + \|z^s_{n,m} - g\|.
\end{aligned}$$

Using Theorem 4.53, equalities (4.24), and equalities (4.25), we get

$$\|S^s_n - g\| \leq \begin{cases} |\Delta (b_{n+1,m})| \int_0^{\pi} \left|D_n^{(m)}(x)\right| dx + o(1), & \text{if} \quad m = 4p-3 \wedge m = 4p-1; \\ |\Delta (b_{n+1,m})| \int_0^{\pi} \left|\widetilde{D}_n^{(m)}(x)\right| dx + o(1), & \text{if} \quad m = 4p-2 \wedge m = 4p; \end{cases} \qquad (4.33)$$

for all $p \in N$.

Now applying Lemma 1.84 we have

$$\|S_n^s - g\| \leq \begin{cases} C(n+1)^m |\Delta(b_{n+1,m})| \log(n+1) + o(1), & if \quad m = 4p-3 \wedge m = 4p-1; \\ (n+1)|\Delta(b_{n+1,m})| \times \Big(\int_0^\pi \big|D_n^{(m-1)}(x)\big| dx \\ + \int_0^\pi \big|F_n^{(m-1)}(x)\big| dx \Big) + o(1), & if \quad m = 4p-2 \wedge m = 4p; \end{cases} \quad (4.34)$$

for all $p \in N$.

The use of Bernstein's inequality in (4.34) gives

$$\|S_n^s - g\| \leq \begin{cases} C(n+1)^m |\Delta(b_{n+1,m})| \log(n+1) + o(1), & if \quad m = 4p-3 \wedge m = 4p-1; \\ (n+1)^m |\Delta(b_{n+1,m})| \times \Big(\int_0^\pi \big|D_n(x)\big| dx \\ + \int_0^\pi \big|F_n(x)\big| dx \Big) + o(1), & if \quad m = 4p-2 \wedge m = 4p; \end{cases} \quad (4.35)$$

for all $p \in N$.

Finally, using conditions of our theorem in (4.35) we obtain

$$\|S_n^s - g\| = \mathcal{O}(a_{n+1}\log(n+1) + a_{n+2}\log(n+2) + o(1)) = o(1),$$

as $n \to \infty$.

The proof is completed.

For $m = 1$ we have the following

Corollary 4.56. *Let* $(a_n) \in \widetilde{C} \cap \widetilde{BV}$ *and* $\lim_{n\to\infty} a_n \log n = 0$. *Then*

$$\lim_{n\to\infty} \|S_n^s - g\| = 0.$$

The following statements also hold true.

Theorem 4.57. *Let* $(a_n) \in \widetilde{C}_{r,m} \cap \widetilde{BV}_{r,m}$, $r \in \{0,1,\ldots\}$, $m \in \{1,2,\ldots\}$, *and* $\lim_{\to\infty} n^r a_n \log n = 0$. *Then*

$$\lim_{n\to\infty} \|[z_{n,m}^s]^{(r)} - g^{(r)}\| = 0.$$

Proof. The proof is similar to the Theorem 4.53. Therefore, we omit it.

For $r = 0$ we have the following

Corollary 4.58. *Let* $(a_n) \in \widetilde{C}_m \cap \widetilde{BV}_m$, $m \in \{1,2,\ldots\}$, *and* $\lim_{\to\infty} a_n \log n = 0$. *Then*

$$\lim_{n\to\infty} \|z_{n,m}^s - g\| = 0.$$

Using Theorem 4.55 and Theorem 4.57 we obtain next consequence.

Corollary 4.59. *Let* $(a_n) \in \widetilde{S}_{r,m}$, $r \in \{0,1,\ldots\}$, $m \in \{1,2,\ldots\}$, *and* $\lim_{n\to\infty} n^r a_n \log n = 0$. *Then*

(1) $\lim_{n\to\infty} \|[z^s_{n,m}]^{(r)} - g^{(r)}\| = 0$.
(2) $\lim_{n\to\infty} \|[S^s_n]^{(r)} - g^{(r)}\| = 0$.

For $m = 1$ in Corollary 4.59 we obtain next corollary proved in [78]:

Corollary 4.60. *Let* $(a_n) \in \widetilde{S}_r$, $r \in \{0,1,\ldots\}$, *and* $\lim_{n\to\infty} n^r a_n \log n = 0$. *Then*

(1) $\lim_{n\to\infty} \|[z^s_n]^{(r)} - g^{(r)}\| = 0$.
(2) $\lim_{n\to\infty} \|[S^s_n]^{(r)} - g^{(r)}\| = 0$.

5

L^1-convergence of $r-$th derivative of modified trigonometric sums

In this section we give all results regrading to L^1-convergence of r-th derivative of several modified trigonometric sums imposing generalized conditions in their coefficients.

5.1 L^1-convergence of $r-$th derivative of modified sums $f_n(x)$ with coefficients from the class S_r

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

and

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx.$$

We also write $f^{(r)}(x) = \lim_{n\to\infty}[S_n^c(x)]^{(r)}$, where $[S_n^c(x)]^{(r)}$ denotes the $r-$th derivative of the sum $S_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$.

Theorem 5.1. *Let $\{a_k\} \in \mathbf{S}_r$, then $f_n^{(r)}(x)$ converges to $f^{(r)}(x)$ in L^1-norm.*

Proof. First we have

$$f_n(x) = S_n^c(x) - a_{n+1}D_n(x),$$

where

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx.$$

It is clear that

$$f_n^{(r)}(x) = [S_n^c(x)]^{(r)} - a_{n+1}D_n^{(r)}(x),$$

where $f_n^{(r)}(x)$ and $D_n^{(r)}(x)$ are r-th derivative of $f_n(x)$ and $D_n(x)$ respectively.

Since $\{a_k\}$ is a null sequence and $D_n^{(r)}(x)$ is bounded in $(0,\pi]$, then

$$\lim_{n\to\infty} f_n^{(r)}(x) = \lim_{n\to\infty} [S_n^c(x)]^{(r)} = f^{(r)}(x).$$

For $x \neq 0$, it follows that

$$f^{(r)}(x) - f_n^{(r)}(x) = \sum_{k=n+1}^{\infty} a_k k^r \cos\left(kx + \frac{r\pi}{2}\right) + a_{n+1} D_n^{(r)}(x).$$

Applying the Abel's transformation, we get

$$f^{(r)}(x) - f_n^{(r)}(x) = \sum_{k=n+1}^{\infty} \Delta a_k D_k^{(r)}(x).$$

So, we can write

$$\int_0^{\pi} |f^{(r)}(x) - f_n^{(r)}(x)| dx = \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx.$$

Now, using Lemma 1.86 we have

$$\begin{aligned}
\int_0^{\pi} |f^{(r)}(x) - f_n^{(r)}(x)| dx &= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx \\
&= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k^{(r)}(x) \right| dx \\
&= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta A_k \sum_{\mu=1}^{k} \frac{\Delta a_\mu}{A_\mu} D_\mu^{(r)}(x) \right| dx \\
&\leq \sum_{k=n+1}^{\infty} \Delta A_k \int_0^{\pi} \left| \sum_{\mu=1}^{k} \frac{\Delta a_\mu}{A_\mu} D_\mu^{(r)}(x) \right| dx \\
&\leq C \sum_{k=n+1}^{\infty} (k+1)^{r+1} \Delta A_k = o(1) \quad as \quad n \to \infty,
\end{aligned}$$

taking into account that $\{a_k\} \in \mathbf{S_r}$.

So, we have obtained

$$\lim_{n\to\infty} \int_0^{\pi} |f^{(r)}(x) - f_n^{(r)}(x)| dx = 0.$$

The proof is completed.

Corollary 5.2. *Let $\{a_k\} \in \mathbf{S}_r$. The series*

$$f^{(r)}(x) = \sum_{k=1}^{\infty} k a_k \cos\left(kx + \frac{r\pi}{2}\right)$$

converges in L^1-norm if and only if

$$\lim_{n\to\infty} n^r |a_{n+1}| \log n = 0.$$

Proof. We notice that

$$\begin{aligned}\int_0^{\pi} |f^{(r)}(x) - [S_n^c(x)]^{(r)}|dx &\le \int_0^{\pi} |f^{(r)}(x) - f_n^{(r)}(x)|dx + \int_0^{\pi} |f_n^{(r)}(x) - [S_n^c(x)]^{(r)}|dx \\ &= \int_0^{\pi} |f^{(r)}(x) - f_n^{(r)}(x)|dx + |a_{n+1}| \int_0^{\pi} |D_n^{(r)}(x)|dx,\end{aligned}$$

and

$$\begin{aligned}\int_0^{\pi} |a_{n+1} D_n^{(r)}(x)|dx &= \int_0^{\pi} |f_n^{(r)}(x) - [S_n^c(x)]^{(r)}|dx \\ &\le \int_0^{\pi} |f_n^{(r)}(x) - f^{(r)}(x)|dx + \int_0^{\pi} |f^{(r)}(x) - [S_n^c(x)]^{(r)}|dx.\end{aligned}$$

So,

$$\|f^{(r)} - [S_n^c]^{(r)}\|_{L^1} = o(1) \ as\ n \to \infty$$

if and only if

$$|a_{n+1}| n^r \log n = o(1) \ as\ n \to \infty,$$

since $\int_0^{\pi} |a_{n+1} D_n^{(r)}(x)|dx$ behaves as $|a_{n+1}| n^r \log n$ for large values n.

The proof is completed.

5.2 L^1-convergence of $r-$th derivative of modified sums $g_n^c(x)$ with coefficients from the class $\mathbf{S_r^{**}}$

Let us consider the cosine series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

and modified cosine sums

$$g_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx.$$

We also write $f^{(r)}(x) = \lim_{n\to\infty} [S_n^c(x)]^{(r)}$, where $[S_n^c(x)]^{(r)}$ denotes the $r-$th derivative of the partial sums $S_n^c(x)$ of cosine series.

Theorem 5.3. *Let* $\{a_k\} \in \mathbf{S_r^{**}}$, $r \in \{0,1,2,\ldots\}$. *Then* $[g_n^c(x)]^{(r)}$ *converges to* $f^{(r)}(x)$ *in* L^1*-norm if and only if* $n^r|a_{n+1}|\log n = o(1)$ *as* $n \to \infty$.

Proof. Firstly we have

$$g_n^c(x) = S_n(x) - \frac{a_{n+1}}{n+1}\widetilde{D}'_n(x).$$

Then we have,

$$[g_n^c(x)]^{(r)} = [S_n^c(x)]^{(r)} - \frac{a_{n+1}}{n+1}\widetilde{D}_n^{(r+1)}(x),$$

where $\widetilde{D}_n^{(r+1)}(x)$ represents $r-$th derivative of the conjugate Dirichlet's kernel.

Since, $\{a_k\} \in \mathbf{S_r^{**}}$, $r \in \{0,1,2,\ldots\}$, then

$$\lim_{n\to\infty}[g_n^c(x)]^{(r)} = \lim_{n\to\infty}[S_n^c(x)]^{(r)} = f^{(r)}(x) \quad for \quad x \in (0,\pi].$$

This implies

$$f^{(r)}(x) - [g_n^c(x)]^{(r)} = \sum_{k=n+1}^{\infty} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right) + \frac{a_{n+1}}{n+1}\widetilde{D}_n^{(r+1)}(x).$$

Applying Abel's transformation, we have

$$f^{(r)}(x) - [g_n^c(x)]^{(r)} = \sum_{k=n+1}^{\infty} \Delta a_k D_k^{(r)}(x) - a_{n+1}D_n^{(r)}(x) + \frac{a_{n+1}}{n+1}\widetilde{D}_n^{(r+1)}(x).$$

Thus,

$$\begin{aligned}\|f^{(r)} - [g_n^c]^{(r)}\| = \int_0^\pi \Bigg| \sum_{k=n+1}^{\infty} \Delta a_k D_k^{(r)}(x) - a_{n+1}D_n^{(r)}(x) \\ + \frac{a_{n+1}}{n+1}\widetilde{D}_n^{(r+1)}(x)\Bigg| dx \le \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k^{(r)}(x)\right| dx \\ + \int_0^\pi \left|a_{n+1}D_n^{(r)}(x)\right| dx + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}_n^{(r+1)}(x)\right| dx.\end{aligned}$$

Since $\int_0^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}_n^{(r+1)}(x)\right| dx \sim n^r|a_{n+1}|\log n$, by Zygmund's theorem, $\{a_n\} \in S_r^{**}$ and Lemma 1.85, we get

$$\begin{aligned}\|f^{(r)} - [g_n^c]^{(r)}\| &= \mathcal{O}\left(\sum_{k=n+1}^{\infty} k^{r+1}\Delta a_k\right) + o\left(n^r|a_{n+1}|\log n\right) + o\left(n^r|a_{n+1}|\log n\right) \\ &= o(1) + o\left(n^r|a_{n+1}|\log n\right).\end{aligned}$$

Consequently, $\|f^{(r)} - [g_n^c]^{(r)}\| = o(1)$ as $n \to \infty$ if and only if $n^r|a_{n+1}|\log n = o(1)$ as $n \to \infty$.

The proof is completed.

Remark 5.4. For $r=0$ this Theorem reduces to the Theorem 3.15.

Corollary 5.5. *Let* $\{a_k\} \in \mathbf{S}_{\mathbf{r}}^{**}$. *Then* $[S_n^c(x)]^{(r)}$ *converges to* $f^{(r)}(x)$ *in* L^1-*norm if and only if* $n^r|a_{n+1}|\log n = o(1)$ *as* $n \to \infty$.

Proof. We note that,

$$\begin{aligned}\|f^{(r)} - [S_n^c]^{(r)}\| &\le \|f^{(r)} - [g_n^c]^{(r)} + [g_n^c]^{(r)} - [S_n^c]^{(r)}\| \\ &\le \|f^{(r)} - [g_n^c]^{(r)}\| + \|[g_n^c]^{(r)} - [S_n^c]^{(r)}\| \\ &= \|[g_n^c]^{(r)} - [g_n^c]^{(r)}\| + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}_n^{(r+1)}(x)\right| dx.\end{aligned}$$

Once again, since

$$\int_0^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D}_n^{(r+1)}(x)\right| dx$$

behaves like $n^r|a_{n+1}|\log n$ for large values of n, and by Theorem 5.3 we get

$$\lim_{n\to\infty} \|f^{(r)} - [S_n^c]^{(r)}\| = 0$$

if and only if $n^r|a_{n+1}|\log n = o(1)$ as $n \to \infty$.

The corollary is proved.

Remark 5.6. For $r=0$ this Corollary reduces to the Corollary 3.16.

5.3 L^1-convergence of $r-$th derivative of modified sums $j_n^c(x)$ and $j_n^s(x)$ with coefficients from the class $\mathbf{SJ}_r$

We consider together trigonometric series

$$\phi(x) = \sum_{k=0}^{\infty} a_k \phi_k(x),$$

where $\phi_k(x)$ is $\cos kx$ or $\sin kx$ and $\phi(x)$ is $f(x)$ or $g(x)$ respectively ($\cos 0 \cdot x := \frac{1}{2}$).

We also consider modified cosine and sine sums

$$j_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(a_j \cos jx\right)$$

and

$$j_n^s(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(a_j \sin jx\right).$$

We prove the following.

Theorem 5.7. *Let* $\{a_k\} \in \mathbf{SJ}_r$, $r \in \{0, 1, 2, \ldots\}$. *Then*

(i) $\lim_{n\to\infty} t_n^{(r)}(x) = t^{(r)}(x)$ *exists for all* $x \in (0, \pi]$, *where* $t_n^{(r)}(x)$ *is* r*-th derivative of either* $j_n^c(x)$ *or* $j_n^s(x)$,
(ii) $t^{(r)}(x) \in L^1(0, \pi]$, *and*
(iii) $\|t^{(r)} - S_n^{(r)}(t)\| = o(1)$ *as* $n \to \infty$.

Proof. We proof this statement only for cosine sums since for the sine sums the proof can be done with the same arguments. Noting that

$$j_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(a_j \cos jx\right) = S_n(x) - na_{n+1}\cos(n+1)x,$$

we have

$$[j_n^c(x)]^{(r)} = S_n^{(r)}(x) - n(n+1)^r a_{n+1} \cos\left((n+1)x + \frac{r\pi}{2}\right).$$

Since $A_k \downarrow 0$ as $k \to \infty$ and $\sum_{k=1}^{\infty} k^r A_k < \infty$, then we have $k^{r+1}A_k \to 0$ as $k \to \infty$ and therefore

$$n^{r+1}a_n = n^{r+2}\sum_{k=n}^{\infty} \Delta\left(\frac{a_k}{k}\right) \le \sum_{k=n}^{\infty} k^{r+2}\left(\frac{A_k}{k}\right) = o(1) \quad as \quad n \to \infty.$$

Also $\cos\left((n+1)x + \frac{r\pi}{2}\right)$ is finite in $(0, \pi]$ and therefore

$$t^{(r)}(x) = \lim_{n\to\infty} [j_n^c(x)]^{(r)} = \lim_{n\to\infty}\left[\sum_{k=1}^{n} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right)\right].$$

Abel's transformation implies that

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right)\right] = \lim_{n\to\infty}\left[\sum_{k=1}^{n-1} \Delta\left(\frac{a_k}{k}\right)\widetilde{D}_k^{(r+1)}(x) + \frac{a_n}{n}\widetilde{D}_n^{(r+1)}(x)\right],$$

where $\widetilde{D}_k^{(r+1)}(x)$ denotes r-th derivative of the conjugate Dirichlet's kernel.

Based on our assumptions and Lemma 1.83 the series $\sum_{k=1}^{\infty} \Delta\left(\frac{a_k}{k}\right)\widetilde{D}_k^{(r+1)}(x)$ converges, and hence the limit-function $t^{(r)}(x)$ exists for $x \in (0, \pi]$ and subsequently the statement (i) holds true.

Moreover, for $x \neq 0$ we have

$$t^{(r)}(x) - [j_n^c(x)]^{(r)} = \sum_{k=n+1}^{\infty} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right)$$

$$+n(n+1)^r a_{n+1}\cos\left((n+1)x+\frac{r\pi}{2}\right)$$
$$=\lim_{m\to\infty}\left[\sum_{k=n+1}^{m} k^r a_k\cos\left(kx+\frac{r\pi}{2}\right)\right]$$
$$+n(n+1)^r a_{n+1}\cos\left((n+1)x+\frac{r\pi}{2}\right).$$

Applying Abel's transformation, we obtain

$$\begin{aligned}
t^{(r)}(x)-[j_n^c(x)]^{(r)} &= \sum_{k=n+1}^{\infty}\Delta\left(\frac{A_k}{k}\right)\widetilde{D}_k^{(r+1)}(x)-\frac{a_{n+1}}{n+1}\widetilde{D}_n^{(r+1)}(x)\\
&\quad +n(n+1)^r a_{n+1}\cos\left((n+1)x+\frac{r\pi}{2}\right)\\
&= \sum_{k=n+1}^{\infty}\left(\frac{A_k}{k}\right)\frac{\Delta\left(\frac{a_k}{k}\right)}{\frac{A_k}{k}}\widetilde{D}_k^{(r+1)}(x)-\frac{a_{n+1}}{n+1}\widetilde{D}_n^{(r+1)}(x)\\
&\quad +n(n+1)^r a_{n+1}\cos\left((n+1)x+\frac{r\pi}{2}\right)\\
&= \sum_{k=n+1}^{\infty}\Delta\left(\frac{A_k}{k}\right)\sum_{j=1}^{k}\frac{\Delta\left(\frac{a_j}{j}\right)}{\frac{A_j}{j}}\widetilde{D}_j^{(r+1)}(x)\\
&\quad +\left(\frac{A_{n+1}}{n+1}\right)\sum_{j=1}^{n}\frac{\Delta\left(\frac{a_j}{j}\right)}{\frac{A_j}{j}}\widetilde{D}_j^{(r+1)}(x)-\frac{a_{n+1}}{n+1}\widetilde{D}_n^{(r+1)}(x)\\
&\quad +n(n+1)^r a_{n+1}\cos\left((n+1)x+\frac{r\pi}{2}\right).
\end{aligned}$$

Now applying Abel's transformation, Lemma 1.85 and Lemma 1.86 we get

$$\begin{aligned}
&\int_0^{\pi}|t^{(r)}(x)-[j_n^c(x)]^{(r)}|dx\\
&\le \int_0^{\pi}\left|\sum_{k=n+1}^{\infty}\Delta\left(\frac{A_k}{k}\right)\sum_{j=1}^{k}\frac{\Delta\left(\frac{a_j}{j}\right)}{\frac{A_j}{j}}\widetilde{D}_j^{(r+1)}(x)\right|dx\\
&\quad +\int_0^{\pi}\left|\left(\frac{A_{n+1}}{n+1}\right)\sum_{j=1}^{n}\frac{\Delta\left(\frac{a_j}{j}\right)}{\frac{A_j}{j}}\widetilde{D}_j^{(r+1)}(x)\right|dx\\
&\quad +\left|\frac{a_{n+1}}{n+1}\right|\int_0^{\pi}\left|\widetilde{D}_n^{(r+1)}(x)\right|dx+|n(n+1)^r a_{n+1}|\int_0^{\pi}\left|\cos\left((n+1)x+\frac{r\pi}{2}\right)\right|dx\\
&\le \sum_{k=n+1}^{\infty}\left|\Delta\left(\frac{A_k}{k}\right)\right|\int_0^{\pi}\left|\sum_{j=1}^{k}\frac{\Delta\left(\frac{a_j}{j}\right)}{\frac{A_j}{j}}\widetilde{D}_j^{(r+1)}(x)\right|dx\\
&\quad +\frac{|a_{n+1}|}{n+1}\mathcal{O}((n+1)^{r+1}\log n)+(n+1)^{r+1}|a_{n+1}|\cdot\frac{2}{n+1}
\end{aligned}$$

$$= \mathcal{O}\left(\sum_{k=n+1}^{\infty}(k+1)^{r+2}\left|\Delta\left(\frac{A_k}{k}\right)\right| + (n+1)^{r+1}|a_{n+1}|\right),$$

since

$$\int_0^{\pi}\left|\cos\left((n+1)x+\frac{r\pi}{2}\right)\right|dx \leq \frac{2}{n+1}.$$

By assumption $\{a_n\} \in \mathbf{SJ}_r$, then we can prove that the series

$$\sum_{k=1}^{\infty}(k+1)^2\Delta\left(\frac{A_k}{k}\right)$$

converges and $n^{r+1}a_n = o(1)$ as $n \to \infty$. So, it follows that $\|t^{(r)} - [j_n^c]^{(r)}\| = o(1)$ as $n \to \infty$.

Also, the fact that $t_n^{(r)}(x)$ is a trigonometric polynomial implies $t^{(r)} \in L^1(0,\pi]$. This conclusion verifies completely statement (ii).

Now,

$$\begin{aligned}\|t^{(r)} - [S_n^c]^{(r)}\| &\leq \|t^{(r)} - t_n^{(r)} + t_n^{(r)} - [S_n^c]^{(r)}\| \\ &\leq \|t^{(r)} - t_n^{(r)}\| + \|t_n^{(r)} - [S_n^c]^{(r)}\| \\ &= \|t^{(r)} - t_n^{(r)}\| + \int_0^{\pi}\left|n(n+1)^r a_{n+1}\cos\left((n+1)x+\frac{r\pi}{2}\right)\right|dx \\ &= o(1) + n(n+1)^r|a_{n+1}|\cdot\frac{2}{n+1} = o(1) \quad as \quad n\to\infty,\end{aligned}$$

which completes the proof of (iii).

The proof of theorem is completed.

Remark 5.8. Putting $r = 0$ in the Theorem 5.7, we immediately obtain Theorem 4.1.

5.4 L^1-convergence of $r-$th derivative of modified sums $f_n(x)$ with generalized quasi-convex coefficients

We consider cosine series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}a_k\cos kx,$$

modified cosine sums,

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n}\Delta(a_k) + \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta(a_j)\cos jx$$

and

$$g_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k\cos kx,$$

and generalized quasi-convex sequences.

Definition 5.9. *A sequence $\{u_k\}$ is said to be generalized quasi-convex sequence if*

$$\sum_{k=1}^{\infty} k^{r+1}|\Delta^2 a_k| < \infty,$$

for all $r \in \{0, 1, 2, \ldots\}$.

Now we prove the following.

Theorem 5.10. *Let $\{a_k\}$ be a generalized quasi-convex sequence. Then*

(i) $\lim_{n\to\infty} \|f^{(r)} - f_n^{(r)}\| = 0$,
(ii) $\lim_{n\to\infty} \|f^{(r)} - f_n^{(r)}\| = 0 \Longrightarrow \lim_{n\to\infty} \|f^{(r)} - [g_n^c]^{(r)}\| = 0$ *as* $\lim_{n\to\infty} n^r a_n = 0$
(iii) $\lim_{n\to\infty} \|f^{(r)} - S_n^{(r)}\| = 0$ *if and only if* $\lim_{n\to\infty} n^r a_{n+1}\log n = 0$,

where $r \in \{0, 1, 2, \ldots\}$.

Proof. (i) We have

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta(a_k) + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta(a_j)\cos jx = S_n(x) - a_{n+1}D_n(x).$$

Then

$$f_n^{(r)}(x) = S_n^{(r)}(x) - a_{n+1}D_n^{(r)}(x).$$

Clearly,

$$\lim_{n\to\infty} f_n^{(r)}(x) = \lim_{n\to\infty} S_n^{(r)}(x) = f^{(r)}(x),\ x \in (0, \pi].$$

Therefore,

$$\begin{aligned} f^{(r)}(x) - f_n^{(r)}(x) &= \sum_{k=n+1}^{\infty} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right) + a_{n+1}D_n^{(r)}(x) \\ &= \lim_{m\to\infty} \sum_{k=n+1}^{m} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right) + a_{n+1}D_n^{(r)}(x). \end{aligned}$$

Applying Abel's transformation twice we have

$$\begin{aligned} &f^{(r)}(x) - f_n^{(r)}(x) = \sum_{k=n+1}^{\infty} \Delta a_k D_k^{(r)}(x) + a_{n+1}D_n^{(r)}(x) \\ &= \lim_{m\to\infty}\left[\sum_{k=n+1}^{m-1} \Delta a_k D_k^{(r)}(x) + a_m D_m^{(r)}(x) - a_{n+1}D_n^{(r)}(x)\right] + a_{n+1}D_n^{(r)}(x) \end{aligned}$$

$$= \lim_{m\to\infty}\left[\sum_{k=n+1}^{m-1} \Delta a_k D_k^{(r)}(x) + a_m D_m^{(r)}(x)\right]$$

$$= \lim_{m\to\infty}\left[\sum_{k=n+1}^{m-2} (k+1)\Delta^2 a_k K_k^{(r)}(x) + m\Delta a_{m-1} K_{m-1}^{(r)}(x)\right.$$

$$\left. -(n+1)\Delta a_{n+1} K_n^{(r)}(x) + a_m D_m^{(r)}(x)\right]$$

$$= \sum_{k=n+1}^{\infty} (k+1)\Delta^2 a_k K_k^{(r)}(x) - (n+1)\Delta a_{n+1} K_n^{(r)}(x).$$

Thus,

$$\|f^{(r)} - f_n^{(r)}\| \le \sum_{k=n+1}^{\infty} (k+1)|\Delta^2 a_k| \int_0^\pi |K_k^{(r)}(x)|dx$$
$$+(n+1)|\Delta a_{n+1}| \int_0^\pi |K_n^{(r)}(x)|dx.$$

Since,

$$|\Delta a_{n+1}| = \left|\sum_{k=n+1}^{\infty} \Delta^2 a_k\right| \le \sum_{k=n+1}^{\infty} \frac{k+1}{k+1}\left|\Delta^2 a_k\right|$$
$$\le \frac{1}{n+1}\sum_{k=n+1}^{\infty} (k+1)\left|\Delta^2 a_k\right|,$$

then

$$(n+1)|\Delta a_{n+1}| \int_0^\pi |K_n^{(r)}(x)|dx$$
$$\le (n+1)\cdot\frac{1}{n+1}\sum_{k=n+1}^{\infty}(k+1)\left|\Delta^2 a_k\right| n^r \int_0^\pi |K_n(x)|dx \le \pi \sum_{k=n+1}^{\infty} k^{r+1}\left|\Delta^2 a_k\right|,$$

by Zygmund's theorem.

So, we have obtained

$$\lim_{n\to\infty} \|f^{(r)} - f_n^{(r)}\| = 0.$$

(ii) Subtracting

$$[g_n^c(x)]^{(r)} = S_n^{(r)}(x) - \frac{a_{n+1}}{n+1}\widetilde{D}_n^{(r+1)}(x)$$

from

$$f_n^{(r)}(x) = S_n^{(r)}(x) - a_{n+1} D_n^{(r)}(x),$$

and using the equality

$$\widetilde{D}_n^{(r+1)}(x) = (n+1)D_n^{(r)}(x) - (n+1)K_n^{(r)}(x),$$

we have

$$f_n^{(r)}(x) - [g_n^c(x)]^{(r)} = -a_{n+1}K_n^{(r)}(x),$$

which implies

$$\|f_n^{(r)} - (g_n^c)^{(r)}\| = |a_{n+1}|\|K_n^{(r)}(x)\|.$$

Whence, by Zygmund's theorem

$$\begin{aligned}\|f^{(r)} - (g_n^c)^{(r)}\| &\le \|f^{(r)} - f_n^{(r)}\| + \|f_n^{(r)} - (g_n^c)^{(r)}\| \\ &= o(1) + |a_{n+1}|\|K_n^{(r)}(x)\| \\ &= o(1) + n^r|a_{n+1}|\|K_n(x)\| = o(1) + \mathcal{O}(n^r|a_{n+1}|) \\ &= o(1),\ as\ n \to \infty.\end{aligned}$$

(iii) We have

$$\begin{aligned}\|S_n^{(r)} - f^{(r)}\| &= \|S_n^{(r)} - f_n^{(r)} + f_n^{(r)} - f^{(r)}\| \\ &\le \|S_n^{(r)} - f_n^{(r)}\| + \|f_n^{(r)} - f^{(r)}\| \\ &= \|a_{n+1}D_n^{(r)}(x)\| + \|f_n^{(r)} - f^{(r)}\| \\ &= \mathcal{O}(n^r|a_{n+1}|\log n) + o(1) = o(1),\ as\ n \to \infty.\end{aligned}$$

Conversely, we have

$$\begin{aligned}\mathcal{O}(n^r|a_{n+1}|\log n) \sim \|a_{n+1}D_n^{(r)}(x)\| &= \|S_n^{(r)} - f_n^{(r)}\| \\ &\le \|S_n^{(r)} - f_n^{(r)}\| + \|f_n^{(r)} - f^{(r)}\| \\ &= o(1) + o(1) = o(1),\ as\ n \to \infty,\end{aligned}$$

by part (i) of the theorem.

The proof is completed.

Remark 5.11. It should be noted here that, if the part (i) can be proved for any other classes of sequences, then the parts (i) and (ii) will always be true.

5.5 L^1-convergence of $r-$th derivative of modified sums $f_n(x)$ and $g_n^c(x)$ with coefficients from the class $\mathrm{S}_{p\alpha}$

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta\,(a_k) + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\,(a_j)\cos jx,$$

$$g_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k\cos kx,$$

and $\{a_k\}$ any sequence from the class $S_{p\alpha}$.

Definition 5.12. *The zero-sequence* $\{u_n\}$ *is said to be in the class* $\mathbf{S}_{p\alpha}$*, if there exists a sequence* $\{A_n\}$ *such that*

(i) $A_n \downarrow 0$ *as* $n \to \infty$*,*
(ii) $\sum_{n=0}^{\infty} n^{\alpha} A_n < \infty$*, for some* $\alpha \geq 0$*, and*
(iii) $\frac{1}{n}\sum_{k=1}^{n} \frac{|\Delta u_k|^p}{A_k^p} = \mathcal{O}(1)$*,* $1 < p \leq 2$*, as* $n \to \infty$*.*

Next statement holds true.

Theorem 5.13. *Let* $\{a_k\} \in \mathbf{S}_{p\alpha}$*,* $\alpha \geq 0$*,* $r \in \{0, 1, 2, \ldots, [\alpha]\}$ *and* $n^r|a_{n+1}|\log n = o(1)$ *as* $n \to \infty$*. Then*

(i) $\lim_{n\to\infty} n^{\alpha-r}\|f^{(r)} - f_n^{(r)}\| = 0$*,*
(ii) $\lim_{n\to\infty} \|f^{(r)} - [g_n^c]^{(r)}\| = 0$ *as* $\lim_{n\to\infty} n^r a_n = 0$*.*

Proof. (i) We have

$$f_n^{(r)}(x) = S_n^{(r)}(x) - a_{n+1}D_n^{(r)}(x),$$

and

$$f^{(r)}(x) - f_n^{(r)}(x) = \sum_{k=n+1}^{\infty} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right) + a_{n+1}D_n^{(r)}(x).$$

By making use of Abel's transformation, get

$$\begin{aligned}
&\int_0^{\pi} |f^{(r)}(x) - f_n^{(r)}(x)|dx \\
&\quad\leq \int_0^{\pi}\left|\sum_{k=n+1}^{\infty} \Delta a_k D_k^{(r)}(x)\right| dx + |a_{n+1}|\int_0^{\pi} |D_n^{(r)}(x)|dx \\
&\quad\leq \int_0^{\pi}\left|\sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k^{(r)}(x)\right| dx + |a_{n+1}|\int_0^{\pi} |D_n^{(r)}(x)|dx \\
&\quad\leq \int_0^{\pi}\left|\sum_{k=n+1}^{\infty} \Delta A_k \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x)\right| dx \\
&\qquad + A_n \int_0^{\pi}\left|\sum_{j=1}^{n} \frac{\Delta a_j}{A_j} D_j^{(r)}(x)\right| dx + |a_{n+1}|\int_0^{\pi} |D_n^{(r)}(x)|dx \\
&\quad\leq \sum_{k=n+1}^{\infty} |\Delta A_k| \int_0^{\pi}\left|\sum_{j=1}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x)\right| dx
\end{aligned}$$

$$+A_n \int_0^\pi \left| \sum_{j=1}^{n} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx$$

$$+|a_{n+1}| \int_0^\pi |D_n^{(r)}(x)| dx := I_1 + I_2 + I_3.$$

For I_1 we can write

$$I_1 := \sum_{k=n+1}^{\infty} |\Delta A_k| \int_0^{\frac{\pi}{k}} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx$$

$$+ \sum_{k=n+1}^{\infty} |\Delta A_k| \int_{\frac{\pi}{k}}^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx := I_{11} + I_{12}.$$

Since $\{a_k\} \in \mathbf{S_{p}}_\alpha$, $\alpha \geq 0$, $r \in \{0, 1, 2, \ldots, [\alpha]\}$, then it can be proved that (see [41])

$$I_{11} = o(n^{r-\alpha}) \quad and \quad I_{12} = o(n^{r-\alpha}).$$

Also in the similar way we can get $I_2 = o(n^{r-\alpha})$ and

$$I_3 := \int_0^\pi \left| a_{n+1} D_n^{(r)}(x) \right| dx = |a_{n+1}| \int_0^\pi \left| D_n^{(r)}(x) \right| dx$$

$$\leq n^r |a_{n+1}| \int_0^\pi |D_n(x)| \, dx \leq n^r |a_{n+1}| \log n = o(1) \ and \ n \to \infty,$$

by given hypothesis.

Whence,

$$\|f^{(r)} - f_n^{(r)}\| = o(n^{r-\alpha}) \ and \ n \to \infty.$$

(ii) Now we are going to prove that

$$\|f^{(r)} - f_n^{(r)}\| = o(n^{r-\alpha}) \Longrightarrow \|f^{(r)} - [g_n^c]^{(r)}\| = o(1)$$

as $n^r a_n = 0, \ n \to \infty$.

As we have discussed in Theorem 5.10, part (ii), we have

$$f_n^{(r)}(x) - [g_n^c(x)]^{(r)} = -a_{n+1} K_n^{(r)}(x),$$

which implies

$$\|f_n^{(r)} - (g_n^c)^{(r)}\| = |a_{n+1}| \|K_n^{(r)}(x)\|.$$

Thus, by Zygmund's theorem

$$\begin{aligned}
\|f^{(r)} - (g_n^c)^{(r)}\| &= \|f^{(r)} - f_n^{(r)} + f_n^{(r)} - (g_n^c)^{(r)}\| \\
&\leq \|f^{(r)} - f_n^{(r)}\| + \|f_n^{(r)} - (g_n^c)^{(r)}\| \\
&= o(1) + |a_{n+1}| \|K_n^{(r)}(x)\| \\
&= o(1) + n^r |a_{n+1}| \|K_n(x)\| \\
&= o(1) + \mathcal{O}(n^r |a_{n+1}|) \\
&= o(1), \ as \ n \to \infty,
\end{aligned}$$

by given hypothesis.

The proof is completed.

5.6 L^1-convergence of r-th derivative of modified sums $K_n^s(x)$ with coefficients from the class $(\mathrm{SC})^r$

Let

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

$$K_n^s(x) = \frac{1}{2\sin x} \sum_{k=1}^{n} \sum_{j=k}^{n} (\Delta a_{j-1} - \Delta a_{j+1}) \sin kx,$$

and $\{a_k\}$ any sequence from the class $(\mathbf{SC})^{\mathbf{r}}$, $r \in \{0, 1, 2, \ldots\}$.

Theorem 5.14. *Let $\{a_n\}$ be a semi-convex zero sequence of order r, $r \in \{0, 1, 2, \ldots\}$. Then $[K_n^s(x)]^{(r)}$ converges to $g^{(r)}(x)$ in L^1-norm.*

Proof. We have

$$\begin{aligned} K_n^s(x) &= \frac{1}{2\sin x} \sum_{k=1}^{n} \sum_{j=k}^{n} (\Delta a_{j-1} - \Delta a_{j+1}) \sin kx \\ &= \frac{1}{2\sin x} \left[\sum_{k=1}^{n} (a_{k-1} - a_{k+1}) \sin kx - (a_n - a_{n+2}) \widetilde{D}_n(x) \right]. \end{aligned}$$

Applying Abel's transformation, we get

$$\begin{aligned} K_n^s(x) &= \frac{1}{2\sin x} \sum_{k=1}^{n} (\Delta a_{k-1} - \Delta a_{k+1}) \widetilde{D}_k(x) \\ &= \frac{1}{2\sin x} \sum_{k=1}^{n} \left(\Delta^2 a_{k-1} + \Delta^2 a_k\right) \widetilde{D}_k(x). \end{aligned}$$

Therefore,

$$[K_n^s(x)]^{(r)} = \sum_{k=1}^{n} \left(\Delta^2 a_{k-1} + \Delta^2 a_k\right) \left(\frac{\widetilde{D}_k(x)}{2\sin x} \right)^{(r)}. \tag{5.1}$$

On the other side we have

$$S_n(x) = \frac{1}{\sin x} \sum_{k=1}^{n} a_k \cos kx \sin x$$

$$= \frac{1}{2\sin x} \sum_{k=1}^{n} a_k \left[\sin(k+1)x - \sin(k-1)x\right]$$

$$= \frac{1}{2\sin x}\sum_{k=1}^{n}(a_{k-1}-a_{k+1})\sin kx + a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x}$$

$$= \frac{1}{2\sin x}\sum_{k=1}^{n}(\Delta a_{k-1}+\Delta a_k)\sin kx + a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x}. \quad (5.2)$$

Applying Abel's transformation to the equality (5.2) we get

$$S_n(x) = \sum_{k=1}^{n}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)\frac{\widetilde{D}_k(x)}{2\sin x}$$
$$+ (a_n - a_{n+2})\frac{\widetilde{D}_n(x)}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x}.$$

Thus

$$S_n^{(r)}(x) = \sum_{k=1}^{n}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)\left(\frac{\widetilde{D}_k(x)}{2\sin x}\right)^{(r)} + (a_n - a_{n+2})\left(\frac{\widetilde{D}_n(x)}{2\sin x}\right)^{(r)}$$
$$+a_{n+1}\left(\frac{\sin nx}{2\sin x}\right)^{(r)} + a_n\left(\frac{\sin(n+1)x}{2\sin x}\right)^{(r)}. \quad (5.3)$$

By Lemma 1.88 and since (a_n) is semi-convex null sequence of order r, we have

$$\left|(a_n - a_{n+2})\left(\frac{\widetilde{D}_n(x)}{2\sin x}\right)^{(r)}\right|$$
$$= \mathcal{O}_{r,\epsilon}\left(\left|(n+1)^{r+1}(a_n - a_{n+2})\right|\right)$$
$$= \mathcal{O}_{r,\epsilon}\left(\left|(n+1)^{r+1}\sum_{k=n}^{\infty}(\Delta a_k - \Delta a_{k+2})\right|\right)$$
$$= \mathcal{O}_{r,\epsilon}\left(\left|(n+1)^{r+1}\sum_{k=n+1}^{\infty}(\Delta a_{k-1} - \Delta a_{k+1})\right|\right)$$
$$= \mathcal{O}_{r,\epsilon}\left(\sum_{k=n+1}^{\infty}k^{r+1}|\Delta^2 a_{k-1}+\Delta^2 a_k|\right) = o(1), \quad n\to\infty. \quad (5.4)$$

Also after some elementary calculations and by virtue of Lemma 1.88 we obtain

$$a_{n+1}\left(\frac{\sin nx}{2\sin x}\right)^{(r)} + a_n\left(\frac{\sin(n+1)x}{2\sin x}\right)^{(r)}$$
$$= a_{n+1}\left[\left(\frac{\widetilde{D}_n(x)}{2\sin x}\right)^{(r)} - \left(\frac{\widetilde{D}_{n-1}(x)}{2\sin x}\right)^{(r)}\right]$$

$$
\begin{aligned}
&+a_n\left[\left(\frac{\widetilde{D}_{n+1}(x)}{2\sin x}\right)^{(r)}-\left(\frac{\widetilde{D}_n(x)}{2\sin x}\right)^{(r)}\right]\\
&=a_{n+1}\mathcal{O}_{r,\epsilon}\left(n^{r+1}+(n-1)^{r+1}\right)+a_n\mathcal{O}_{r,\epsilon}\left((n+1)^{r+1}+n^{r+1}\right)\\
&=\mathcal{O}_{r,\epsilon}\left((n+1)^{r+1}\left(a_n+a_{n+1}\right)\right)\\
&=\mathcal{O}_{r,\epsilon}\left((n+1)^{r+1}\left[(a_n-a_{n+2})+(a_{n+1}-a_{n+3})+(a_{n+2}-a_{n+4})+\cdots\right]\right)\\
&=\mathcal{O}_{r,\epsilon}\left(\sum_{k=n+1}^{\infty}k^{r+1}|\Delta^2a_{k-1}+\Delta^2a_k|+\sum_{k=n+2}^{\infty}k^{r+1}|\Delta^2a_{k-1}+\Delta^2a_k|+\cdots\right)\\
&=o(1), n\to\infty. \qquad (5.5)
\end{aligned}
$$

Because of (5.4) and (5.5), when we pass on limit as $n\to\infty$ to (5.1) and (5.3), we get

$$
\begin{aligned}
g^{(r)}(x)&=\lim_{n\to\infty}S_n^{(r)}(x)\\
&=\lim_{n\to\infty}[K_n^s(x)]^{(r)}=\sum_{k=1}^{\infty}\left(\Delta^2a_{k-1}+\Delta^2a_k\right)\left(\frac{\widetilde{D}_k(x)}{2\sin x}\right)^{(r)}. \qquad (5.6)
\end{aligned}
$$

Using Lemma 1.88, from (5.3), (5.4) and (5.6), we obtain

$$
\begin{aligned}
&\int_{-\pi}^{\pi}\left|g^{(r)}(x)-[K_n^s(x)]^{(r)}\right|dx\\
&\quad=2\int_0^{\pi}\sum_{k=n+1}^{\infty}\left|\Delta^2a_{k-1}+\Delta^2a_k\right|\left|\left(\frac{\widetilde{D}_k(x)}{2\sin x}\right)^{(r)}\right|dx\\
&\quad=\mathcal{O}_{r,\epsilon}\left(\sum_{k=n+1}^{\infty}k^{r+1}\left|\Delta^2a_{k-1}+\Delta^2a_k\right|\right)=o(1),\quad n\to\infty.
\end{aligned}
$$

The proof is completed.

Remark 5.15. If we take $r=0$ in Theorem 5.14, then we obtain Theorem 4.7 and Corollary 4.8.

Corollary 5.16. *Let $(a_n)\in(\mathbf{SC})^r$, then the sufficient condition for L^1-convergence of the r-th derivative of the cosine series is $n^ra_n\log n=o(1)$ as $n\to\infty$.*

Proof. We have

$$
\left\|g^{(r)}(x)-S_n^{(r)}(x)\right\|\le\left\|g^{(r)}(x)-[K_n^s(x)]^{(r)}\right\|+\left\|[K_n^s(x)]^{(r)}-S_n^{(r)}(x)\right\|
$$

$$
\begin{aligned}
&= o(1) + \left\| (a_n - a_{n+2}) \left(\frac{\widetilde{D}_n(x)}{2\sin x} \right)^{(r)} \right\| \\
&\quad + \left\| a_{n+1} \left(\frac{\sin nx}{2\sin x} \right)^{(r)} + a_n \left(\frac{\sin(n+1)x}{2\sin x} \right)^{(r)} \right\| \\
&\le o(1) + a_n \left\| \widetilde{D}_n^{(r)}(x) \right\| \\
&= o(1) + O\left(n^r a_n \log n\right) = o(1), \ \ n \to \infty.
\end{aligned}
$$

taking into account the Theorem 5.14 and Lemma 1.88.

The proof is completed.

5.7 L^1-convergence of $r-$th derivative of modified sums $g_n^c(x)$ with coefficients from the class S_r

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

and

$$g_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left(\frac{a_j}{j} \right) k \cos kx.$$

We also write $f^{(r)}(x) = \lim_{n\to\infty} [S_n^c(x)]^{(r)}$, where $[S_n^c(x)]^{(r)}$ denotes the $r-$th derivative of the sum $S_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$.

Theorem 5.17. *Let $\{a_k\} \in \mathbf{S}_r$, $r \in \{0,1,2,\ldots\}$, and $n^r |a_{n+1}| \log n = o(1)$ as $n \to \infty$. Then $[g_n^c(x)]^{(r)}$ converges to $f^{(r)}(x)$ in L^1-norm.*

Proof. First we have

$$g_n^c(x) = S_n^c(x) - \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x),$$

where $\widetilde{D}'_n(x)$ denotes the first derivative of the conjugate Dirichlet's kernel.

So,

$$[g_n^c(x)]^{(r)} = [S_n^c(x)]^{(r)} - \frac{a_{n+1}}{n+1} \widetilde{D}_n^{(r+1)}(x),$$

where $[g_n^c(x)]^{(r)}$ and $\widetilde{D}_n^{(r+1)}(x)$ are r-th derivative of $g_n^c(x)$ and $\widetilde{D}'_n(x)$ respectively.

Since $\{a_k\} \in \mathbf{S}_r$, $r \in \{0,1,2,\ldots\}$ is a zero sequence and $\widetilde{D}_n^{(r+1)}(x)$ is bounded in $(0,\pi]$, then

$$\lim_{n\to\infty} [g_n^c(x)]^{(r)} = \lim_{n\to\infty} [S_n^c(x)]^{(r)} = f^{(r)}(x).$$

For $x \neq 0$, it follows that

$$f^{(r)}(x) - [g_n^c(x)]^{(r)} = \sum_{k=n+1}^{\infty} a_k k^r \cos\left(kx + \frac{r\pi}{2}\right) + \frac{a_{n+1}}{n+1}\widetilde{D}_n^{(r+1)}(x).$$

Applying the Abel's transformation, we get

$$\begin{aligned} f^{(r)}(x) - [g_n^c(x)]^{(r)} = &\sum_{k=n+1}^{\infty} \Delta a_k \widetilde{D}_k^{(r)}(x) \\ &-a_{n+1}\widetilde{D}_n^{(r)}(x) + \frac{a_{n+1}}{n+1}\widetilde{D}_n^{(r+1)}(x). \end{aligned}$$

So, we can write

$$\begin{aligned} \int_0^\pi |f^{(r)}(x) - [g_n^c(x)]^{(r)}|dx \le &\int_0^\pi \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} \widetilde{D}_k^{(r)}(x) \right| dx \\ &+ \int_0^\pi \left| a_{n+1}\widetilde{D}_n^{(r)}(x) \right| dx + \int_0^\pi \left| \frac{a_{n+1}}{n+1}\widetilde{D}_n^{(r+1)}(x) \right| dx. \end{aligned}$$

Now, applying Abel's transformation again and using Lemma 1.85 and Lemma 1.86, we obtain

$$\begin{aligned} \int_0^\pi |f^{(r)}(x) - [g_n^c(x)]^{(r)}|dx &\le \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta A_k \sum_{\mu=1}^{k} \frac{\Delta a_\mu}{A_\mu} \widetilde{D}_\mu^{(r)}(x) \right| dx \\ &\le \sum_{k=n+1}^{\infty} \Delta A_k \int_0^\pi \left| \sum_{\mu=1}^{k} \frac{\Delta a_\mu}{A_\mu} \widetilde{D}_\mu^{(r)}(x) \right| dx \\ &\quad + \mathcal{O}(n^r |a_{n+1}| \log n) + \mathcal{O}(n^r |a_{n+1}| \log n) \\ &\le C \sum_{k=n+1}^{\infty} (k+1)^{r+1} \Delta A_k + \mathcal{O}(n^r |a_{n+1}| \log n) = o(1), \end{aligned}$$

as $n \to \infty$, taking into account that $\{a_k\} \in \mathbf{S}_r$ and given hypothesis.

So, we have obtained

$$\lim_{n\to\infty} \int_0^\pi |f^{(r)}(x) - [g_n^c(x)]^{(r)}|dx = 0.$$

The proof is completed.

Corollary 5.18. *Let* $\{a_k\} \in \mathbf{S}_r$, $r \in \{0, 1, 2, \ldots\}$, *and* $n^r |a_{n+1}| \log n = o(1)$ *as* $n \to \infty$. *Then* $[S_n^c(x)]^{(r)}$ *converges to* $f^{(r)}(x)$ *in* L^1*-norm.*

Proof. We notice that

$$\begin{aligned}\int_0^\pi |f^{(r)}(x) - [S_n^c(x)]^{(r)}|dx &\le \int_0^\pi |f^{(r)}(x) - [g_n^c(x)]^{(r)}|dx \\ &\quad + \int_0^\pi |[g_n^c(x)]^{(r)} - [S_n^c(x)]^{(r)}|dx \\ &= \int_0^\pi |f^{(r)}(x) - [g_n^c(x)]^{(r)}|dx + \left|\frac{a_{n+1}}{n+1}\right| \int_0^\pi |\widetilde{D}_n^{(r+1)}(x)|dx.\end{aligned}$$

So,

$$\|f^{(r)}(x) - [S_n^c(x)]^{(r)}\|_{L^1} = o(1) \ as \ n \to \infty,$$

if

$$|a_{n+1}|n^r \log n = o(1) \ as \ n \to \infty,$$

since $\int_0^\pi \left|a_{n+1}\widetilde{D}_n^{(r+1)}(x)\right| dx$ behaves as $|a_{n+1}|n^r \log n$ for large values n.

The proof is completed.

5.8 L^1-convergence of $r-$th derivative of modified sums $N_n(x)$ with r-quasi convex coefficients

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

and

$$N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \sum_{k=1}^{n}\sum_{j=k}^{n} (\Delta^2 a_{j-1} - \Delta^2 a_j)\cos kx + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}.$$

We also write $f^{(r)}(x) = \lim_{n\to\infty}[S_n^c(x)]^{(r)}$, where $[S_n^c(x)]^{(r)}$ denotes the $r-$th derivative of the sum $S_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$.

Theorem 5.19. *Let $\{a_n\}$ be a r-quasi convex null sequence, then $N_n^{(r)}(x)$ converges to $f^{(r)}(x)$ in L^1 norm.*

Proof. From definition of $S_n(x)$ we have:

$$\begin{aligned}S_n(x) &= \frac{1}{\left(2\sin\frac{x}{2}\right)^2} \sum_{k=1}^{n} a_k \cos kx \left(2\sin\frac{x}{2}\right)^2 \\ &= -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \sum_{k=1}^{n} a_k[\cos(k+1)x - 2\cos kx + \cos(k-1)x] \\ &= -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \sum_{k=1}^{n} (a_{k-1} - 2a_k + a_{k+1})\cos kx\end{aligned}$$

$$-\frac{a_0\cos x}{\left(2\sin\frac{x}{2}\right)^2}+\frac{a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2}+\frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}-\frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2}$$

$$=-\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n}\Delta^2 a_{k-1}\cos kx-\frac{a_0\cos x}{\left(2\sin\frac{x}{2}\right)^2}$$
$$+\frac{a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2}+\frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}-\frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2}.$$

Applying Abel's transformation, we get

$$S_n(x)=-\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n-1}(\Delta^2 a_{k-1}-\Delta^2 a_k)\widetilde{D}_k(x)+\frac{\Delta^2 a_{n-1}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}$$
$$-\frac{a_0\cos x}{\left(2\sin\frac{x}{2}\right)^2}+\frac{a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2}+\frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}-\frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2}.$$

Thus,

$$S_n^{(r)}(x)=-\sum_{k=1}^{n-1}(\Delta^2 a_{k-1}-\Delta^2 a_k)\left(\frac{\widetilde{D}_k(x)}{4\sin^2\frac{x}{2}}\right)^{(r)}$$
$$+\Delta^2 a_{n-1}\left(\frac{\widetilde{D}_n(x)}{4\sin^2\frac{x}{2}}\right)^{(r)}+a_n\left(\frac{\cos(n+1)(x)}{4\sin^2\frac{x}{2}}\right)^{(r)}$$
$$+a_1\left(\frac{1}{4\sin^2\frac{x}{2}}\right)^{(r)}-a_{n+1}\left(\frac{\cos nx}{4\sin^2\frac{x}{2}}\right)^{(r)}.$$

On the other hand, applying the Abel's transformation to the equality

$$N_n(x)=-\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta^2 a_{j-1}-\Delta^2 a_j)\cos kx+\frac{a_1}{\left(2\sin\frac{x}{2}\right)^2},$$

we have

$$N_n(x)=-\sum_{k=1}^{n-1}(\Delta^2 a_{k-1}-\Delta^2 a_k)\frac{\widetilde{D}_k(x)}{4\sin^2\frac{x}{2}}$$
$$+\frac{\Delta^2 a_{n-1}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}+\frac{\Delta^2 a_n\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}+\frac{a_1}{\left(2\sin\frac{x}{2}\right)^2},$$

whose r-th derivative is the following

$$N_n^{(r)}(x)=-\sum_{k=1}^{n-1}(\Delta^2 a_{k-1}-\Delta^2 a_k)\left(\frac{\widetilde{D}_k(x)}{4\sin^2\frac{x}{2}}\right)^{(r)}$$
$$+\Delta^2 a_{n-1}\left(\frac{\widetilde{D}_n(x)}{4\sin^2\frac{x}{2}}\right)^{(r)}+\Delta^2 a_n\left(\frac{\widetilde{D}_n(x)}{4\sin^2\frac{x}{2}}\right)^{(r)}+a_1\left(\frac{1}{4\sin^2\frac{x}{2}}\right)^{(r)}.$$

Since $\{a_n\}$ is r-quasi convex null sequence, then we have

$$\begin{aligned}
\left|\Delta^2 a_n \left(\frac{\tilde{D}_n(x)}{2\sin^2\frac{x}{2}}\right)^{(r)}\right| &= O_{r,\epsilon}\left(\left|n^r\left(\Delta^2 a_n\right)\right|\right) \\
&= O_{r,\epsilon}\left(\left|n^r\sum_{k=n}^{\infty}\left(\Delta^2 a_k - \Delta^2 a_{k+1}\right)\right|\right) \\
&= O_{r,\epsilon}\left(\sum_{k=n}^{\infty} k^{r+1}|\Delta^2 a_k - \Delta^2 a_{k+1}|\right) \\
&= O_{r,\epsilon}\left(\sum_{k=n}^{\infty} k^{r+1}|\Delta a_k - \Delta a_{k+1}|\right) \\
&\quad + O_{r,\epsilon}\left(\sum_{k=n}^{\infty} k^{r+1}|\Delta a_{k+1} - \Delta a_{k+2}|\right) = o(1), \quad n\to\infty.
\end{aligned}$$

Also, after some elementary calculations, we obtain

$$\begin{aligned}
& a_n\left(\frac{\cos(n+1)(x)}{4\sin^2\frac{x}{2}}\right)^{(r)} - a_{n+1}\left(\frac{\cos nx}{4\sin^2\frac{x}{2}}\right)^{(r)} \\
&= a_n\left[\left(\frac{D_{n+1}(x)}{4\sin^2\frac{x}{2}}\right)^{(r)} - \left(\frac{D_n(x)}{4\sin^2\frac{x}{2}}\right)^{(r)}\right] \\
&\quad - a_{n+1}\left[\left(\frac{D_n(x)}{4\sin^2\frac{x}{2}}\right)^{(r)} - \left(\frac{D_{n-1}(x)}{4\sin^2\frac{x}{2}}\right)^{(r)}\right] \\
&= a_n O_{r,\epsilon}\left((n+1)^{r+1} - n^{r+1}\right) - a_{n+1}O_{r,\epsilon}\left(n^{r+1} - (n-1)^{r+1}\right) \\
&= O_{r,\epsilon}\left(n^r\left(a_n - a_{n+1}\right)\right) \\
&= O_{r,\epsilon}\left(n^r\sum_{k=n}^{\infty}\left(\Delta a_k - \Delta a_{k+1}\right)\right) \\
&= O_{r,\epsilon}\left(\sum_{k=n}^{\infty} k^r|\Delta a_k - \Delta a_{k+1}|\right) = o(1), \quad n\to\infty,
\end{aligned}$$

and respectively

$$\begin{aligned}
f^{(r)}(x) &= \lim_{n\to\infty} N_n^{(r)}(x) = \lim_{n\to\infty} S_n^{(r)}(x) \\
&= -\sum_{k=1}^{\infty}\left(\Delta^2 a_{k-1} - \Delta^2 a_k\right)\left(\frac{\widetilde{D}_k(x)}{4\sin^2\frac{x}{2}}\right)^{(r)} + a_1\left(\frac{1}{4\sin^2\frac{x}{2}}\right)^{(r)}.
\end{aligned}$$

Now using the above equalities we obtain

$$\begin{aligned}
&\int_{-\pi}^{\pi}\left|f^{(r)}(x)-N_n^{(r)}(x)\right|dx\\
&=2\int_0^{\pi}\sum_{k=n}^{\infty}\left|\Delta^2 a_k-\Delta^2 a_{k+1}\right|\left|\left(\frac{\widetilde{D}_k(x)}{4\sin^2\frac{x}{2}}\right)^{(r)}\right|dx\\
&=O_{r,\epsilon}\left(\sum_{k=n}^{\infty}k^{r+1}\left|\Delta^2 a_k-\Delta^2 a_{k+1}\right|\right)=o(1),\quad n\to\infty.
\end{aligned}$$

The proof is completed.

Corollary 5.20. *Let $\{a_n\}$ be a r-quasi convex null sequence, then the necessary and sufficient condition for L^1-convergence of the $r-th$ derivative of the cosine series is $n^{r+1}|a_n|=o(1)$ as $n\to\infty$.*

Proof. We have

$$\begin{aligned}
&\left\|f^{(r)}(x)-S_n^{(r)}(x)\right\|\\
&\le\left\|f^{(r)}(x)-N_n^{(r)}(x)\right\|+\left\|N_n^{(r)}(x)-S_n^{(r)}(x)\right\|\\
&=o(1)+\left\|a_n\left(\frac{\cos{(n+1)x}}{4\sin^2\frac{x}{2}}\right)^{(r)}\right\|+\left\|a_{n+1}\left(\frac{\cos nx}{4\sin^2\frac{x}{2}}\right)^{(r)}\right\|\\
&\quad+\left\|\Delta^2 a_n\left(\frac{\widetilde{D}_n(x)}{4\sin^2\frac{x}{2}}\right)^{(r)}\right\|\\
&\le o(1)+\left\|a_n\left(\frac{\cos{(n+1)x}}{4\sin^2\frac{x}{2}}\right)^{(r)}\right\|+\left\|a_{n+1}\left(\frac{\cos nx}{4\sin^2\frac{x}{2}}\right)^{(r)}\right\|\\
&=o(1)+O\left((n+1)^{r+1}|a_n|\right)+O\left((n+1)^{r+1}|a_{n+1}|\right)=o(1),\quad n\to\infty.
\end{aligned}$$

The proof is completed.

6

L^1-convergence of single complex modified sums

In this section we have collected all results regrading to L^1-convergence of complex modified trigonometric sums whose coefficients belong to some specific classes of real sequences.

6.1 L^1-convergence of complex modified sums $g_n(C;x)$ with coefficients from the class $\mathbf{R}^*$

We consider the complex series

$$\sum_{|k|\leq\infty}^{\infty} c_k e^{ikx},$$

its partial sums

$$S_n(C;x) = \sum_{|k|\leq n} c_k e^{ikx},$$

and the complex modified sums

$$g_n(C;x) = S_n(C;x) + \frac{i}{n+1}\left[c_{n+1}E_n'(x) - c_{n+1}E_{-n}'(x)\right],$$

with coefficients from the class $\mathbf{R}^*$.

Regarding to this class we prove the following.

Theorem 6.1. *Let $\{c_k\} \in \mathbf{R}^*$. Then there exists $f(x)$ such that:*

(i) $\lim_{n\to\infty} g_n(C;x) = f(x)$ *for all* $0 < |x| \leq \pi$,
(ii) $f \in L^1$ *and* $\|g_n(C;x) - f(x)\| = o(1)$ *as* $n \to \infty$, *and*
(iii) $\lim_{n\to\infty} \|S_n(C;x) - f(x)\| = 0$ *if and only if* $\lim_{|n|\to\infty} c_n \log|n| = 0$.

Proof. (i) Using Abel's transformation, we have

$$\begin{aligned} g_n(C;x) &= S_n(C;x) + \frac{i}{n+1}\left[c_{n+1}E'_n(x) - c_{n+1}E'_{-n}(x)\right] \\ &= 2\sum_{k=1}^{n} \Delta\left(\frac{c_k}{k}\right)\widetilde{D}'_k(x) + \sum_{k=1}^{n}\Delta\left(\frac{c_{k-1}-c_k}{k}\right) iE'_k(x). \end{aligned}$$

By Lemma 1.91 and changing the order of summation, we get

$$\begin{aligned} \sum_{k=1}^{\infty}\left|\Delta\left(\frac{c_k}{k}\right)\widetilde{D}'_k(x)\right| &\le \frac{C}{|x|}\sum_{k=1}^{\infty} k\left|\Delta\left(\frac{c_k}{k}\right)\right| \\ &\le \frac{C}{|x|}\sum_{k=1}^{\infty}\sum_{j=k}^{\infty} k\left|\Delta^2\left(\frac{c_j}{j}\right)\right| \\ &= \frac{C}{|x|}\sum_{j=1}^{\infty}\left(\sum_{k=1}^{j} k\right)\left|\Delta^2\left(\frac{c_j}{j}\right)\right| \\ &= \mathcal{O}\left[\frac{1}{|x|}\sum_{j=1}^{\infty} j^2\left|\Delta^2\left(\frac{c_j}{j}\right)\right|\right] < \infty, \end{aligned}$$

and

$$\begin{aligned} \sum_{k=3}^{\infty}\left|\Delta\left(\frac{c_{k-1}-c_k}{k}\right) iE'_k(x)\right| &\le \frac{C_1}{|x|}\sum_{k=3}^{\infty} k\left|\Delta\left(\frac{c_{k-1}-c_k}{k}\right)\right| \\ &= \mathcal{O}\left[\frac{1}{|x|}\sum_{k=3}^{\infty} k\log k\left|\Delta\left(\frac{c_{k-1}-c_k}{k}\right)\right|\right] < \infty, \end{aligned}$$

where C_1 is a suitable constant.

These imply that

$$f(x) = 2\sum_{k=1}^{\infty}\Delta\left(\frac{c_k}{k}\right)\widetilde{D}'_k(x) + i\sum_{k=1}^{\infty}\Delta\left(\frac{c_{k-1}-c_k}{k}\right) iE'_k(x)$$

exists and consequently (i) follows.

(ii) For $x \neq 0$ and applying the Abel's transformation, we have

$$\begin{aligned} f(x) - g_n(C;x) &= 2\sum_{k=n+1}^{\infty}\Delta\left(\frac{c_k}{k}\right)\widetilde{D}'_k(x) + i\sum_{k=n+1}^{\infty}\Delta\left(\frac{c_{k-1}-c_k}{k}\right) iE'_k(x) \\ &= 2\sum_{k=n+1}^{\infty}(k+1)\Delta^2\left(\frac{c_k}{k}\right)\widetilde{K}'_k(x) - 2(n+1)\Delta\left(\frac{c_{n+1}}{n+1}\right)\widetilde{K}'_n(x) \\ &\quad + i\sum_{k=n+1}^{\infty}\Delta\left(\frac{c_{k-1}-c_k}{k}\right) iE'_k(x). \end{aligned}$$

Whence,

$$\int_{-\pi}^{\pi} |f(x) - g_n(C;x)|\,dx \le 2 \sum_{k=n+1}^{\infty} (k+1) \left|\Delta^2 \left(\frac{c_k}{k}\right)\right| \int_{-\pi}^{\pi} \left|\widetilde{K}_k'(x)\right| dx$$
$$+2(n+1) \left|\Delta \left(\frac{c_{n+1}}{n+1}\right)\right| \int_{-\pi}^{\pi} \left|\widetilde{K}_n'(x)\right| dx$$
$$+ \sum_{k=n+1}^{\infty} \left|\Delta \left(\frac{c_{k-1} - c_k}{k}\right)\right| \int_{-\pi}^{\pi} |E_k'(x)|\,dx.$$

By Lemma 1.94, $\int_{-\pi}^{\pi} |\widetilde{K}_k'(x)| dx = o(k)$. Also,

$$\left|\Delta \left(\frac{c_{n+1}}{n+1}\right)\right| = \left|\sum_{k=n+1}^{\infty} \Delta^2 \left(\frac{c_{n+1}}{n+1}\right)\right|$$
$$\le \sum_{k=n+1}^{\infty} \frac{k^2}{k^2} \left|\Delta^2 \left(\frac{c_{n+1}}{n+1}\right)\right|$$
$$\le \frac{1}{(n+1)^2} \sum_{k=n+1}^{\infty} k^2 \left|\Delta^2 \left(\frac{c_{n+1}}{n+1}\right)\right|$$
$$= o((n+1)^{-2}),$$

by the hypothesis of the theorem.

Since Lemma 1.84 and Lemma 1.85 imply

$$\int_{-\pi}^{\pi} |\widetilde{E}_{-k}'(x)| dx = o(k \log k),$$

then

$$\|f(x) - g_n(C;x)\| = o\left(\sum_{k=n+1}^{\infty} (k+1) \left|\Delta^2 \left(\frac{c_k}{k}\right)\right|\right) + o(1)$$
$$+o\left(\sum_{k=n+1}^{\infty} \left|\Delta \left(\frac{c_{k-1} - c_k}{k}\right)\right| k \log k\right) = o(1),$$

by hypothesis of the theorem.

Since $g_n(C;x)$ is a polynomial, it follows that $f \in L^1$, which proves (ii).

(iii) We can write

$$\|f(x) - S_n(C;x)\| \le \|f(x) - g_n(C;x)\| + \|g_n(C;x) - S_n(C;x)\|$$
$$= \|f(x) - g_n(C;x)\|$$
$$+ \left\|\frac{i}{n+1} \left[c_{n+1} E_n'(x) - c_{n+1} E_{-n}'(x)\right]\right\|$$

and

$$\left\| \frac{i}{n+1} \left[c_{n+1} E'_n(x) - c_{-(n+1)} E'_{-n}(x) \right] \right\| = \|g_n(C;x) - S_n(C;x)\|$$
$$\leq \|f(x) - S_n(C;x)\| + \|f(x) - g_n(C;x)\|.$$

Since $\|f(x) - g_n(C;x)\| = o(1)$ as $n \to \infty$, by (ii) and by Lemma 1.95, then

$$\lim_{n\to\infty} \left\| \frac{1}{n+1} \left[c_{n+1} E'_n(x) - c_{-(n+1)} E'_{-n}(x) \right] \right\| = 0$$

if and only if

$$\lim_{|n|\to\infty} c_n \log |n| = 0.$$

The proof is completed.

6.2 L^1-convergence of complex modified sums $g_n(C;x)$ with coefficients from the class $\mathbf{K}^*$

In this unit we consider the complex series

$$\sum_{|k|\leq\infty}^{\infty} c_k e^{ikx},$$

its partial sums

$$S_n(C;x) = \sum_{|k|\leq n} c_k e^{ikx},$$

and the complex modified sums

$$g_n(C;x) = S_n(C;x) + \frac{i}{n+1} \left[c_{n+1} E'_n(x) - c_{n+1} E'_{-n}(x) \right],$$

with coefficients from the class $\mathbf{K}^*$ (see Definition 1.68).

Next, we prove the following theorem.

Theorem 6.2. *Let $\{c_k\} \in \mathbf{K}^*$. Then for $f \in L^1(\mathbf{T})$, $\mathbf{T} = \frac{R}{2\pi Z}$, we have:*

(i) $\|g_n(C;x) - f(x)\| = o(1)$ as $n \to \infty$, and
(ii)$\lim_{n\to\infty} \|S_n(C;x) - f(x)\| = 0$ if and only if $\lim_{|n|\to\infty} c_{n+1} \log |n| = 0$.

Proof. Let $\lambda > 1$ and $n > 1$, then we have

$$V_n^\lambda(C;x) - f(x) = \frac{[\lambda n] + 1}{[\lambda n] - n} \left[\sigma_{[\lambda n]}(x) - f(x) \right] - \frac{n+1}{[\lambda n] - n} \left[\sigma_n(x) - f(x) \right],$$

where

$$V_n^\lambda(C,x) = \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]} S_k(C;x)$$

is known as truncated Cesàro means, and

$$\sigma_n(C;x) = \frac{1}{n+1}\sum_{k=0}^{n} S_k(C;x).$$

Moreover, the difference $V_n^\lambda(C;x) - f(x)$ can be written as

$$V_n^\lambda(C;x) - f(x) = \sum_{|k|=n+1}^{[\lambda n]} \frac{[\lambda n]-k+1}{[\lambda n]-n} c_k e^{ikx}.$$

Next, we have

$$\begin{aligned} g_n(C;x) - V_n^\lambda(C;x) = &- \sum_{|k|=n+1}^{[\lambda n]} \frac{[\lambda n]-k+1}{[\lambda n]-n} c_k e^{ikx} \\ &+ \frac{i}{n+1}\left[c_{n+1}E'_n(x) - c_{n+1}E'_{-n}(x)\right]. \end{aligned}$$

Using summation by parts, we get

$$\begin{aligned} -\sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n]-k+1}{[\lambda n]-n} \ c_k e^{ikx} = -i\Bigg\{ &\sum_{k=n+1}^{[\lambda n]-1} \frac{[\lambda n]-k+1}{[\lambda n]-n} \Delta\left(\frac{c_k}{k}\right) E'_k(x) \\ &+ \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]} \left(\frac{c_k}{k}\right) E'_k(x) - \frac{c_{n+1}}{n+1}E'_n(x)\Bigg\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} -\sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n]-k+1}{[\lambda n]-n} \ c_{-k} e^{-ikx} = -i\Bigg\{ &\sum_{k=n+1}^{[\lambda n]-1} \frac{[\lambda n]-k+1}{[\lambda n]-n} \Delta\left(\frac{c_{-k}}{k}\right) E'_{-k}(x) \\ &+ \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]} \left(\frac{c_{-k}}{k}\right) E'_{-k}(x) - \frac{c_{n+1}}{n+1}E'_{-n}(x)\Bigg\}. \end{aligned}$$

Therefore,

$$\begin{aligned} g_n(C;x) - V_n^\lambda(C;x) = i\Bigg[&\sum_{k=n+1}^{[\lambda n]-1} \frac{[\lambda n]-k+1}{[\lambda n]-n} \Delta\left(\frac{c_k}{k}\right) E'_k(x) \\ &- \sum_{k=n+1}^{[\lambda n]-1} \frac{[\lambda n]-k+1}{[\lambda n]-n} \Delta\left(\frac{c_{-k}}{k}\right) E'_{-k}(x)\Bigg] \end{aligned}$$

$$+i\left[\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left(\frac{c_k}{k}\right)E'_k(x) - \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left(\frac{c_{-k}}{k}\right)E'_{-k}(x)\right].$$

Based on equality

$$E'_n(x)-E'_{-n}(x)=2i\widetilde{D}'_n(x),$$

we can write

$$\begin{aligned} g_n(C;x)-V_n^\lambda(C;x) &= i\left[\sum_{k=n+1}^{[\lambda n]-1}\frac{[\lambda n]-k}{[\lambda n]-n}\Delta\left(\frac{c_k}{k}\right)2i\widetilde{D}'_k(x)\right. \\ &\quad \left.+\sum_{k=n+1}^{[\lambda n]-1}\frac{[\lambda n]-k}{[\lambda n]-n}\Delta\left(\frac{c_k-c_{-k}}{k}\right)E'_{-k}(x)\right] \\ &\quad +i\left[\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left(\frac{c_k}{k}\right)2i\widetilde{D}'_k(x)\right. \\ &\quad \left.+\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left(\frac{c_k-c_{-k}}{k}\right)E'_{-k}(x)\right]. \end{aligned}$$

Whence,

$$\begin{aligned} \|g_n(C;x)-f(x)\| &\le \|g_n(C;x)-V_n^\lambda(C;x)\|+\|V_n^\lambda(C;x)-f(x)\| \\ &\le \frac{[\lambda n]+1}{[\lambda n]-n}\|\sigma_{[\lambda n]}(C;x)-f(x)\| \\ &\quad +\frac{n+1}{[\lambda n]-n}\|\sigma_n(C;x)-f(x)\| \\ &\quad +\left\|2\sum_{k=n+1}^{[\lambda n]-1}\frac{[\lambda n]-k}{[\lambda n]-n}\Delta\left(\frac{c_k}{k}\right)\widetilde{D}'_k(x)\right. \\ &\quad \left.+\sum_{k=n+1}^{[\lambda n]-1}\frac{[\lambda n]-k}{[\lambda n]-n}\Delta\left(\frac{c_k-c_{-k}}{k}\right)E'_{-k}(x)\right\| \\ &\quad +\left\|\frac{2}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left(\frac{c_k}{k}\right)\widetilde{D}'_k(x)\right. \\ &\quad \left.+\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left(\frac{c_k-c_{-k}}{k}\right)E'_{-k}(x)\right\| \end{aligned}$$

$$\leq \frac{[\lambda n]+1}{[\lambda n]-n}\|\sigma_{[\lambda n]}(C;x)-f(x)\| + \frac{n+1}{[\lambda n]-n}\|\sigma_n(C;x)-f(x)\| + \mathbf{I}_1 + \mathbf{I}_2.$$

Let us estimate first the quantity $\mathbf{I}_2$. Namely, we have

$$\mathbf{I}_2 \leq \frac{2}{[\lambda n]-n}\left\|\sum_{k=n}^{[\lambda n]}\left(\frac{c_k}{k}\right)\widetilde{D}'_k(x)\right\| + \frac{1}{[\lambda n]-n}\left\|\sum_{k=n}^{[\lambda n]}\left(\frac{c_k-c_{-k}}{k}\right)E'_{-k}(x)\right\|$$
$$= \mathbf{J}_{11} + \frac{1}{[\lambda n]-n}\left\|\sum_{k=n}^{[\lambda n]}\left(\frac{c_k-c_{-k}}{k}\right)E'_{-k}(x)\right\|,$$

where

$$\mathbf{J}_{11} := \frac{2}{[\lambda n]-n}\left\|\sum_{k=n}^{[\lambda n]}\left(\frac{c_k}{k}\right)\widetilde{D}'_k(x)\right\|.$$

Since,

$$(k+1)K_k(x) = (k+1)D_k(x) - \widetilde{D}'_k(x),$$

where $K_k(x) = \frac{1}{k}\sum_{j=0}^{k} D_j(x)$ is the Fejér kernel, and $K_k(x) \geq 0$, then

$$\widetilde{D}'_k(x) \leq (k+1)D_k(x).$$

This fact implies

$$\mathbf{J}_{11} \leq \frac{2}{[\lambda n]-n}\left\|\sum_{k=n}^{[\lambda n]}(k+1)\left(\frac{c_k}{k}\right)D_k(x)\right\|$$
$$\leq \frac{2([\lambda n]+1)}{[\lambda n]-n}\left\|\sum_{k=n}^{[\lambda n]}\left(\frac{c_k}{k}\right)D_k(x)\right\|.$$

Now, we have

$$\int_0^{\pi}\left|\sum_{k=n}^{[\lambda n]}\left(\frac{c_k}{k}\right)D_k(x)\right|dx = \int_0^{\frac{\pi}{[\lambda n]}}\left|\sum_{k=n}^{[\lambda n]}\left(\frac{c_k}{k}\right)D_k(x)\right|dx + \int_{\frac{\pi}{[\lambda n]}}^{\pi}\left|\sum_{k=n}^{[\lambda n]}\left(\frac{c_k}{k}\right)D_k(x)\right|dx := \mathbf{I}_n + \mathbf{J}_n.$$

Since, for $\mathbf{I}_n$ we have

$$\mathbf{I}_n \leq C(p)\sum_{k=n}^{[\lambda n]}\left|\frac{c_k}{k}\right| \leq C(p)[\lambda n]\left(\frac{1}{[\lambda n]}\sum_{k=n}^{[\lambda n]}\left|\frac{c_k}{k}\right|^p\right)^{\frac{1}{p}},$$

then

$$\mathbf{I}_{11} \le C(p)[\lambda n]\left(\frac{1}{[\lambda n]}\sum_{k=n}^{[\lambda n]}\left|\frac{c_k}{k}\right|^p\right)^{\frac{1}{p}},$$

where $C(p)$ is a positive constant.

On the other hand, we have

$$\mathbf{J}_n = \int_{\frac{\pi}{[\lambda n]}}^{\pi}\left|\sum_{k=n}^{[\lambda n]}\left(\frac{c_k}{k}\right)D_k(x)\right|dx = \int_{\frac{\pi}{[\lambda n]}}^{\pi}\frac{1}{2\sin\frac{x}{2}}\left|\sum_{k=n}^{[\lambda n]}\left(\frac{c_k}{k}\right)\sin\left(k+\frac{1}{2}\right)x\right|.dx$$

After applying the Hölder's inequality $(1 < p \le 2)$ we obtain

$$\mathbf{J}_n \le \left[\int_{\frac{\pi}{[\lambda n]}}^{\pi}\left(\frac{1}{2\sin\frac{x}{2}}\right)^q dx\right]^{\frac{1}{q}}\left[\int_0^{\pi}\left|\sum_{k=n}^{[\lambda n]}\left(\frac{c_k}{k}\right)\sin\left(k+\frac{1}{2}\right)x\right|^p dx\right]^{\frac{1}{p}},$$

and then the Hausdorff-Young inequality, we have

$$\mathbf{J}_n \le C(p)[\lambda n]^{\frac{1}{q}}\left(\sum_{k=n}^{[\lambda n]}\left|\frac{c_k}{k}\right|^p\right)^{\frac{1}{p}} = C(p)[\lambda n]\left(\frac{1}{[\lambda n]}\sum_{k=n}^{[\lambda n]}\left|\frac{c_k}{k}\right|^p\right)^{\frac{1}{p}},$$

where $C(p)$ is a positive constant depending only in p and $p+q=pq$.

Lemma 1.84 and Lemma 1.85 imply

$$\|E'_{-k}(x)\| = \mathcal{O}(k\log k),$$

and hence

$$\mathbf{I}_2 \le \frac{C_1[\lambda n]}{[\lambda n]-n}\left(\frac{1}{[\lambda n]}\sum_{k=n}^{[\lambda n]}\left|\frac{c_k}{k}\right|^p\right)^{\frac{1}{p}} + \frac{C_2[\lambda n]}{[\lambda n]-n}\left(\frac{1}{[\lambda n]}\sum_{k=n}^{[\lambda n]}\left(\left|\frac{c_k-c_{-k}}{k}\right|k\log k\right)\right),$$

where C_1, C_2 are positive constants.

Similarly, for $\mathbf{I}_1$, we have

$$\mathbf{I}_1 \le C_3\sum_{k=n}^{[\lambda n]}\left|\Delta\left(\frac{c_k-c_{-k}}{k}\right)\right|k\log k + C_4\left(\sum_{k=n}^{[\lambda n]}k^{p-1}\left|\frac{c_k}{k}\right|^p\right)^{\frac{1}{p}},$$

where C_1, C_2 are positive constants.

Combining the above estimations, we get

$$\begin{aligned}
\|g_n(C;x)-f(x)\| &\le \frac{[\lambda n]+1}{[\lambda n]-n}\|\sigma_{[\lambda n]}(C;x)-f(x)\| \\
&\quad + \frac{n+1}{[\lambda n]-n}\|\sigma_n(C;x)-f(x)\| + \mathbf{I}_1 + \mathbf{I}_2 \\
&\le \frac{[\lambda n]+1}{[\lambda n]-n}\|\sigma_{[\lambda n]}(C;x)-f(x)\| \\
&\quad + \frac{n+1}{[\lambda n]-n}\|\sigma_n(C;x)-f(x)\| \\
&\quad + \frac{C_1[\lambda n]}{[\lambda n]-n}\left(\frac{1}{[\lambda n]}\sum_{k=n}^{[\lambda n]}\left|\frac{c_k}{k}\right|^p\right)^{\frac{1}{p}} \\
&\quad + \frac{C_2[\lambda n]}{[\lambda n]-n}\left(\frac{1}{[\lambda n]}\sum_{k=n}^{[\lambda n]}\left(\left|\frac{c_k-c_{-k}}{k}\right| k\log k\right)\right) \\
&\quad + C_3\sum_{k=n}^{[\lambda n]}\left|\Delta\left(\frac{c_k-c_{-k}}{k}\right)\right| k\log k \\
&\quad + C_4\left(\sum_{k=n}^{[\lambda n]} k^{p-1}\left|\frac{c_k}{k}\right|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Since, $\{c_k\}$ is a null sequence and $\lambda > 1$, we have

$$\frac{[\lambda n]}{[\lambda n]-n} \sim \frac{\lambda}{\lambda-1}, \quad n\to\infty,$$

and it follows that

$$\overline{\lim_{n\to\infty}} \frac{[\lambda n]}{[\lambda n]-n} c_n = 0.$$

Based on this fact the quantity

$$\frac{[\lambda n]}{[\lambda n]-n}\left(\frac{1}{[\lambda n]}\sum_{k=n}^{[\lambda n]}\left|\frac{c_k}{k}\right|^p\right)^{\frac{1}{p}} \to 0, \quad n\to\infty.$$

Also, since $f\in L^1(\mathbf{T})$, it implies

$$\|\sigma_n(C;x)-f(x)\| \to 0, \quad n\to\infty.$$

Now, using condition $\{c_k\}\in\mathbf{K}^*$, we obtain

$$\begin{aligned}
\overline{\lim_{n\to\infty}}\|g_n(C;x)-f(x)\| &\le C_3\overline{\lim_{n\to\infty}}\sum_{k=n}^{[\lambda n]}\left|\Delta\left(\frac{c_k-c_{-k}}{k}\right)\right| k\log k \\
&\quad + C_4\overline{\lim_{n\to\infty}}\left(\sum_{k=n}^{[\lambda n]} k^{p-1}\left|\frac{c_k}{k}\right|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

In the last inequality we take the limit as $\lambda \to 1$ and since $\{c_k\} \in \mathbf{K}^*$, then

$$\overline{\lim_{n\to\infty}} \|g_n(C;x) - f(x)\| = 0.$$

In the sequel, we notice that

$$\begin{aligned}\|S_n(C;x) - f(x)\| &\le \|S_n(C;x) - g_n(C;x)\| + \|g_n(C;x) - f(x)\| \\ &= \left\| \frac{i}{n+1} \left[c_{n+1}E'_n(x) - c_{n+1}E'_{-n}(x)\right] \right\| + \|g_n(C;x) - f(x)\|,\end{aligned}$$

and

$$\begin{aligned}\left\| \frac{i}{n+1} \left[c_{n+1}E'_n(x) - c_{n+1}E'_{-n}(x)\right] \right\| &= \|S_n(C;x) - g_n(C;x)\| \\ &\le \|g_n(C;x) - f(x)\| + \|S_n(C;x) - f(x)\|.\end{aligned}$$

Finally, since $\|g_n(C;x) - f(x)\| = o(1)$, as $n \to \infty$, and with some slight modifications in Lemma 1.95 we can show that

$$\lim_{n\to\infty} \left\| \frac{i}{n+1} \left[c_{n+1}E'_n(x) - c_{n+1}E'_{-n}(x)\right] \right\| = 0 \Longleftrightarrow \lim_{|n|\to\infty} c_{n+1} \log|n| = 0.$$

Subsequently, we have proved that

$$\lim_{n\to\infty} \|S_n(C;x) - f(x)\| = 0 \Longleftrightarrow \lim_{|n|\to\infty} c_{n+1} \log|n| = 0,$$

which implies the assertion (ii).

The proof is completed.

6.3 L^1-convergence of complex modified sums $k_n(C;x)$ with coefficients from the class $\mathbf{J}^*$

Here we consider the complex series

$$\sum_{|k|\le\infty}^{\infty} c_k e^{ikx},$$

its partial sums

$$S_n(C;x) = \sum_{|k|\le n} c_k e^{ikx},$$

and the complex modified sums (which indeed are the complex form of the modified sums $k_n^c(x)$ and $k_n^s(x)$)

$$k_n(C;x) = S_n(C;x) + \frac{i}{2\sin x}\Big[\; c_n e^{i(n+1)x} - c_{-n}e^{-i(n+1)x} \\ + c_{n+1}e^{inx} - c_{-(n+1)}e^{-inx} \\ + (n+1)(c_n - c_{n+2})e^{i(n+1)x} \\ + (n+1)(c_{-(n+2)} - c_{-n})e^{-i(n+1)x}\Big],$$

with coefficients from the class $\mathbf{J}^*$ (see Definition 1.69).

Next, we prove the following theorem.

Theorem 6.3. *Let $\{c_k\} \in \mathbf{J}^*$. Then there exists $f(x)$, $x \in \mathbf{T} = \frac{R}{2\pi Z}$, so that:*

(i) $\lim_{n\to\infty} k_n(C;x) = f(x)$ *for* $|x| \in (0,\pi]$,
(ii) $f \in L^1(\mathbf{T})$ *and* $\|k_n(C;x) - f(x)\| = o(1)$ *as* $n \to \infty$,
(iii) $\|S_n(C;x) - f(x)\| = o(1)$ *as* $|n| \to \infty$.

Proof. Since $\left|\frac{e^{inx}}{2\sin x}\right| \le M_\varepsilon$ for $0 < \varepsilon \le x \le \pi$, and $\{c_k\}$ is a zero sequence, then

$$\lim_{n\to\infty} k_n(C;x) = \lim_{n\to\infty} S_n(C;x) \\ + \lim_{n\to\infty} \frac{iM_\varepsilon(n+1)}{2}\Big[(c_n - c_{-n}) - (c_{n+2} - c_{-(n+2)})\Big],$$

where $M_\varepsilon > 0$ is a constant depending only in $\varepsilon > 0$.

However, for $n \ge 1$ we have

$$n(n+1)\frac{c_n - c_{-n}}{n} = n(n+1)\sum_{k=n}^{\infty} \Delta\left(\frac{c_k - c_{-k}}{k}\right) \\ \le \sum_{k=n}^{\infty} k(k+1)\left(\frac{A_k}{k}\right) = o(1),\ \textit{as}\ n \to \infty,$$

by given hypothesis.

Therefore,

$$\lim_{n\to\infty} k_n(C;x) = \lim_{n\to\infty} S_n(C;x) = f(x).$$

Now, we are going to show that $f(x)$ exists in $(0,\pi]$. First, we can write

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} = c_0 + \sum_{k=1}^{n}\left(\frac{c_k}{k}ke^{ikx} + \frac{c_{-k}}{k}ke^{-ikx}\right).$$

Applying the summation by parts, we obtain

$$S_n(x) = c_0 + \sum_{k=1}^{n-1} \Delta\left(\frac{c_k}{k}\right)(-iE'_k(x)) + \frac{c_n}{n}(-iE'_n(x))$$

$$
\begin{aligned}
&+\sum_{k=1}^{n-1} \Delta\left(\frac{c_{-k}}{k}\right)(-iE'_{-k}(x)) + \frac{c_{-n}}{n}(iE'_{-n}(x)) \\
&= c_0 - i\sum_{k=1}^{n-1}\left[\Delta\left(\frac{c_k}{k}\right)E'_k(x) - \Delta\left(\frac{c_{-k}}{k}\right)E'_{-k}(x)\right] \\
&-i\frac{c_nE'_n(x) - c_{-n}E'_{-n}(x)}{n}.
\end{aligned}
$$

Using Lemma 1.91 and $\{c_k\} \in \mathbf{J}^*$, we obtain

$$
\begin{aligned}
\sum_{k=1}^{\infty}\left|\Delta\left(\frac{c_k}{k}\right)E'_k(x) - \Delta\left(\frac{c_{-k}}{k}\right)E'_{-k}(x)\right| &\le \frac{M_\varepsilon}{|x|}\sum_{k=1}^{\infty} k\left|\Delta\left(\frac{c_k - c_{-k}}{k}\right)\right| \\
&\le \frac{M_\varepsilon}{|x|}\sum_{k=1}^{\infty}\frac{kA_k}{k} = \frac{M_\varepsilon}{|x|}\sum_{k=1}^{\infty}A_k < \infty,
\end{aligned}
$$

since by assumption $\sum_{k=1}^{\infty} kA_k < \infty$ and from obvious inequality $A_k \le kA_k$ it holds that $\sum_{k=1}^{\infty} A_k < \infty$.

Also, we have

$$
\left|-i\frac{c_nE'_n(x) - c_{-n}E'_{-n}(x)}{n}\right| \le \frac{nM_\varepsilon}{|x|}\left|\frac{c_n - c_{-n}}{n}\right| = o(1), \quad n \to \infty.
$$

Whence $f(x) = \lim_{n\to\infty} S_n(C;x)$ exists and consequently (i) follows.

Further, for $x \ne 0$, consider

$$
\begin{aligned}
f(x) - k_n(C;x) = &\sum_{|k|>n}^{\infty} c_k e^{ikx} \\
&-\frac{i}{2\sin x}\Big[c_n e^{i(n+1)x} - c_{-n}e^{-i(n+1)x} \\
&+c_{n+1}e^{inx} - c_{-(n+1)}e^{-inx} \\
&+(n+1)(c_n - c_{n+2})e^{i(n+1)x} \\
&+(n+1)(c_{-(n+2)} - c_{-n})e^{-i(n+1)x}\Big].
\end{aligned}
$$

Using Abel's transformation, again, we get

$$
\begin{aligned}
f(x) - k_n(C;x) = i &\sum_{k=n+1}^{\infty}\left[\Delta\left(\frac{c_k}{k}\right)E'_k(x) - \Delta\left(\frac{c_{-k}}{k}\right)E'_{-k}(x)\right] \\
&-i\frac{c_{n+1}E'_n(x) + c_{-n}E'_{-n}(x)}{n+1} \\
&-\frac{i}{2\sin x}\Big[c_n e^{i(n+1)x} - c_{-n}e^{-i(n+1)x}
\end{aligned}
$$

$$
\begin{aligned}
&+c_{n+1}e^{inx} - c_{-(n+1)}e^{-inx} \\
&+(n+1)(c_n - c_{n+2})e^{i(n+1)x} \\
&+(n+1)(c_{-(n+2)} - c_{-n})e^{-i(n+1)x}\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
&\|f(x) - k_n(C;x)\| \\
&\le \int_0^\pi \sum_{k=n+1}^\infty \left|\Delta\left(\frac{c_k - c_{-k}}{k}\right)\right| \frac{M_\varepsilon k}{|x|}dx + \int_0^\pi \frac{M_\varepsilon n}{|x|}\left|\frac{c_{n+1} - c_{-(n+1)}}{n+1}\right| dx \\
&+ \int_0^\pi \left|\frac{(c_{n+1} - c_{-(n+1)}) + (n+2)(c_n - c_{-n}) - (n+1)(c_{n+2} - c_{-(n+2)})}{\sin x}\right| dx.
\end{aligned}
$$

Therefore, based on the assumption $\{c_k\} \in \mathbf{J}^*$, we find that

$$
\begin{aligned}
&\|f(x) - k_n(C;x)\| \\
&\qquad \le \int_0^\pi \sum_{k=n+1}^\infty \frac{A_k}{k}\frac{M_\varepsilon k}{|x|}dx + \int_0^\pi \frac{M_\varepsilon}{|x|}\left|c_{n+1} - c_{-(n+1)}\right| dx \\
&\qquad + \int_0^\pi \frac{\left|(c_{n+1} - c_{-(n+1)}) + (c_{n+1} - c_{-(n+1)})\right|}{|\sin x|}dx \\
&\qquad + \int_0^\pi \frac{\left|(n+1)(c_n - c_{-n}) - (n+1)(c_{n+2} - c_{-(n+2)})\right|}{|\sin x|}dx \\
&\qquad = \mathcal{O}\left(\sum_{k=n+1}^\infty A_k \log k\right) \\
&\qquad + o(\log n(c_n - c_{-n})) + o(\log n(c_n - c_{-n})) + o(\log n(c_n - c_{-n})).
\end{aligned}
$$

For all $n \ge 1$ we have that

$$
\begin{aligned}
\log n(c_n - c_{-n}) &\le n^2 \frac{c_n - c_{-n}}{n} \\
&\le \sum_{k=n}^\infty k^2 \Delta\left(\frac{c_k}{k}\right) \\
&\le \sum_{k=n}^\infty k^2 \left(\frac{A_k}{k}\right) = o(1), \quad n \to \infty.
\end{aligned}
$$

Subsequently, $\|f(x) - k_n(C;x)\| = o(1)$ as $n \to \infty$ and since $k_n(C;x)$ is a polynomial, it follows that $f \in L^1(0,\pi]$, which proves the assertion (ii).

Further, we notice that

$$
\begin{aligned}
\|f(x) - S_n(C;x)\| &= \|f(x) - k_n(C;x) + k_n(C;x) - S_n(C;x)\| \\
&\le \|f(x) - k_n(C;x)\| + \|k_n(C;x) - S_n(C;x)\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|f(x) - k_n(C;x)\| + \\
&\Bigg\| \frac{i}{2\sin x}\Big[c_n e^{i(n+1)x} - c_{-n}e^{-i(n+1)x} \\
&\quad + c_{n+1}e^{inx} - c_{-(n+1)}e^{-inx} \\
&\quad + (n+1)(c_n - c_{n+2})e^{i(n+1)x} \\
&\quad + (n+1)(c_{-(n+2)} - c_{-n})e^{-i(n+1)x}\Big]\Bigg\|.
\end{aligned}
$$

Finally, using (ii) and some of above estimations, we obtain the conclusion of the assertion (iii).

The proof is completed.

6.4 L^1-convergence of complex modified sums $K_n(C;x)$ with coefficients from the class $\mathcal{K}$

Here we consider the complex series

$$\sum_{|k|\leq\infty}^{\infty} c_k e^{ikx},$$

its partial sums

$$S_n(C;x) = \sum_{|k|\leq n} c_k e^{ikx},$$

and the complex modified sums (which indeed are the complex form of the modified sums $k_n^c(x)$ and $k_n^s(x)$)

$$
\begin{aligned}
K_n(C;x) = S_n(C;x) + \frac{i}{2\sin x}\Big[&c_n e^{i(n+1)x} - c_{-n}e^{-i(n+1)x} \\
&+ c_{n+1}e^{inx} - c_{-(n+1)}e^{-inx} \\
&+ (c_n - c_{n+2})E_n(x) \\
&+ (c_{-(n+2)} - c_{-n})E_{-n}(x)\Big],
\end{aligned}
$$

with coefficients from the class $\mathcal{K}$ (see Definition 1.70), where

$$E_n(x) = \sum_{k=1}^{n} e^{ikx}, \quad E_{-n}(x) = \sum_{k=1}^{n} e^{-ikx}.$$

We prove the following theorem.

Theorem 6.4. *Let* (c_k) *belongs to the class* $\mathcal{K}$. *Then:*

(i) $\lim_{n\to\infty} K_n(C;x) = f(x)$ *exists for* $|x| \in (0,\pi]$,

(ii) $f \in L^1(0,\pi]$ *and* $\|K_n(C;x) - f(x)\| \to 0$, *as* $n \to \infty$,
(iii) $\|S_n(f;x) - f(x)\| \to 0$ *as* $n \to \infty$.

Proof. Firstly, we will show that $f(x)$ exists on $(0,\pi]$. Indeed, it is clear that we can write

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} = c_0 + \sum_{k=1}^{n} \left(\frac{c_k}{k} k e^{ikx} + \frac{c_{-k}}{k} k e^{-ikx} \right)$$

Applying the Abel's transformation we obtain

$$S_n(x) = c_0 - i \sum_{k=1}^{n-1} \left[\Delta\left(\frac{c_k}{k}\right) E'_k(x) - \Delta\left(\frac{c_{-k}}{k}\right) E'_{-k}(x) \right]$$
$$-i \left[\frac{c_n}{n} E'_n(x) - \frac{c_{-n}}{n} E'_{-n}(x) \right].$$

Based on Lemma 1.91 we clearly have

$$|S_n(x)| \le |c_0| + \sum_{k=1}^{n-1} \left[\left| \Delta\left(\frac{c_k}{k}\right) \right| |E'_k(x)| + \left| \Delta\left(\frac{c_{-k}}{k}\right) \right| |E'_{-k}(x)| \right]$$
$$+ \frac{|c_n|}{n} |E'_n(x)| + \frac{|c_{-n}|}{n} |E'_{-n}(x)|$$
$$\le |c_0| + \frac{M_{r\varepsilon}}{|x|} \left\{ \sum_{k=1}^{n-1} k \left[\left| \Delta\left(\frac{c_k}{k}\right) \right| + \left| \Delta\left(\frac{c_{-k}}{k}\right) \right| \right] + |c_n| + |c_{-n}| \right\}$$
$$\le |c_0| + \frac{M_{r\varepsilon}}{|x|} \left\{ 2 \sum_{k=1}^{n-1} A_k + 2 \sum_{k=n}^{\infty} A_k \right\}$$
$$\le |c_0| + \frac{2M_{r\varepsilon}}{|x|} \sum_{k=1}^{\infty} k A_k < +\infty,$$

since $(c_k) \in \mathcal{K}$, and

$$|c_{\pm n}| = n \left| \frac{c_{\pm n}}{n} \right| \le n \sum_{k=n}^{\infty} \left| \Delta\left(\frac{c_{\pm k}}{k}\right) \right| \le \sum_{k=n}^{\infty} k \left| \Delta\left(\frac{c_{\pm j}}{j}\right) \right| \le \sum_{j=n}^{\infty} A_k.$$

Subsequently, $\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} K_n(C;x) = f(x)$ exists, because of the boundedness of the functions $\frac{e^{inx}}{\sin x}$, $\frac{E_n(x)}{\sin x}$, $\frac{E_{-n}(x)}{\sin x}$ on $(0,\pi]$, and thus (i) holds true.

Now we are going to prove (ii). Indeed, for $x \ne 0$ we have

$$f(x) - K_n(C;x) = \sum_{k=n+1}^{\infty} \left(\frac{c_k}{k} k e^{ikx} + \frac{c_{-k}}{k} k e^{-ikx} \right)$$
$$- \frac{i}{2\sin x} [c_n e^{i(n+1)x} - c_{-n} e^{-i(n+1)x} + c_{n+1} e^{inx}$$
$$- c_{-(n+1)} e^{-inx} + (c_n - c_{n+2}) E_n(x) + (c_{-(n+2)} - c_{-n}) E_{-n}(x)].$$

Again, applying the Abel's transformation to the above equality we obtain

$$\begin{aligned}
&f(x) - K_n(C;x) \\
&\quad = -i \lim_{p\to\infty} \left\{ \sum_{k=n+1}^{p-1} \left[\Delta\left(\frac{c_k}{k}\right) E'_k(x) - \Delta\left(\frac{c_{-k}}{k}\right) E'_{-k}(x) \right] \right. \\
&\qquad \left. + \frac{c_p}{p} E'_p(x) - \frac{c_{-p}}{p} E'_{-p}(x) - \frac{c_{n+1}}{n+1} E'_n(x) + \frac{c_{-(n+1)}}{n+1} E'_{-n}(x) \right\} \\
&\qquad - \frac{i}{2\sin x} [c_n e^{i(n+1)x} - c_{-n} e^{-i(n+1)x} + c_{n+1} e^{inx} \\
&\qquad - c_{-(n+1)} e^{-inx} + (c_n - c_{n+2}) E_n(x) + (c_{-(n+2)} - c_{-n}) E_{-n}(x)] \\
&\quad = -i \left\{ \sum_{k=n+1}^{\infty} \left[\Delta\left(\frac{c_k}{k}\right) E'_k(x) - \Delta^2\left(\frac{c_{-k}}{k}\right) E'_{-k}(x) \right] \right. \\
&\qquad \left. - \frac{c_{n+1}}{n+1} E'_n(x) + \frac{c_{-(n+1)}}{n+1} E'_{-n}(x) \right\} - \frac{i}{2\sin x} [c_n e^{i(n+1)x} - c_{-n} e^{-i(n+1)x} \\
&\qquad + c_{n+1} e^{inx} - c_{-(n+1)} e^{-inx} + (c_n - c_{n+2}) E_n(x) + (c_{-(n+2)} - c_{-n}) E_{-n}(x)].
\end{aligned}$$

Thus, using Lemmas 1.91 we get

$$\begin{aligned}
|f(x) - K_n(C;x)| &\le \sum_{k=n+1}^{\infty} \left[\left| \Delta\left(\frac{c_k}{k}\right) \right| |E'_k(x)| + \left| \Delta\left(\frac{c_{-k}}{k}\right) \right| |E'_{-k}(x)| \right] \\
&\quad \left. + \left| \frac{c_{n+1}}{n+1} \right| |E'_n(x)| + \left| \frac{c_{-(n+1)}}{n+1} \right| |E'_{-n}(x)| \right\} \\
&\quad + \frac{1}{2|\sin x|} [|c_n| + |c_{-n}| + |c_{n+1}| + |c_{-(n+1)}| \\
&\quad + (|c_n| + |c_{n+2}|)|E_n(x)| + (|c_{-(n+2)}| + |c_{-n}|)|E_{-n}(x)|] \\
&\le \frac{M_{r\varepsilon}}{|x|} \left\{ \sum_{k=n+1}^{\infty} k \left[\left| \Delta\left(\frac{c_k}{k}\right) \right| + \left| \Delta\left(\frac{c_{-k}}{k}\right) \right| \right] \right. \\
&\quad \left. + \left(|c_{n+1}| + |c_{-(n+1)}| \right) \right\} + \frac{|c_n| + |c_{-n}| + |c_{n+1}| + |c_{-(n+1)}|}{2|\sin x|} \\
&\quad + (|c_n| + |c_{n+2}|) \left| \frac{E_n(x)}{2\sin x} \right| + (|c_{-(n+2)}| + |c_{-n}|) \left| \frac{E_{-n}(x)}{2\sin x} \right| \\
&\le \frac{M_{r\varepsilon}}{|x|} \left[2 \sum_{k=n}^{\infty} A_k + \left(|c_{n+1}| + |c_{-(n+1)}| \right) \right] \\
&\quad + \frac{|c_n| + |c_{-n}| + |c_{n+1}| + |c_{-(n+1)}|}{2|\sin x|} \\
&\quad + (|c_n| + |c_{n+2}|) \left| \frac{E_n(x)}{2\sin x} \right| + (|c_{-(n+2)}| + |c_{-n}|) \left| \frac{E_{-n}(x)}{2\sin x} \right|.
\end{aligned}$$

Therefore, using Lemma 1.96 we obtain

$$\begin{aligned}
&\|f(x) - K_n(C;x)\|_{L^1} \\
&\leq M_{r\varepsilon}\left[2\sum_{k=n}^{\infty} A_k \int_0^{\pi} \frac{dx}{|x|} + \left(|c_{n+1}| + \left|c_{-(n+1)}\right|\right) \int_0^{\pi} \frac{dx}{|x|}\right] \\
&\quad + \frac{|c_n| + |c_{-n}| + |c_{n+1}| + |c_{-(n+1)}|}{2} \int_0^{\pi} \frac{dx}{|\sin x|} \\
&\quad + (|c_n| + |c_{n+2}|) \int_0^{\pi} \left|\frac{E_n(x)}{2\sin x}\right| dx + (|c_{-(n+2)}| + |c_{-n}|) \int_0^{\pi} \left|\frac{E_{-n}(x)}{2\sin x}\right| dx \\
&\leq M_{r\varepsilon}\left[2\sum_{k=n}^{\infty} A_k o(\log k) + \left(|c_{n+1}| + \left|c_{-(n+1)}\right|\right) o\left(\log n\right)\right] \\
&\quad + \frac{|c_n| + |c_{-n}| + |c_{n+1}| + |c_{-(n+1)}|}{2} o\left(\log n\right) \\
&\quad + (|c_n| + |c_{n+2}|) o(n) + (|c_{-(n+2)}| + |c_{-n}|) o(n).
\end{aligned}$$

Now we note that

$$\sum_{k=n}^{\infty} A_k \log k \leq \sum_{k=n}^{\infty} k^2 A_k = o(1),$$

and for $m = n, n+1, n+2$ we get

$$\begin{aligned}
|c_{\pm m}| \log m &\leq m^2 \left|\frac{c_{\pm m}}{m}\right| \leq m^2 \sum_{j=m}^{\infty} \left|\Delta\left(\frac{c_{\pm j}}{j}\right)\right| \\
&\leq \sum_{j=m}^{\infty} j^2 \left|\Delta\left(\frac{c_{\pm j}}{j}\right)\right| \leq \sum_{j=m}^{\infty} j^2 \frac{A_j}{j} = \sum_{j=m}^{\infty} j A_j = o(1)
\end{aligned}$$

as $n \to \infty$.

Subsequently, we get

$$\|f(x) - K_n(C;x)\|_{L^1} = o(1) \quad as \quad n \to \infty.$$

Using the latest equality and the fact that $K_n(C;x)$ is a polynomial it follows that $f \in L^1(0,\pi]$. We proved (ii) entirely.

Finally, we will prove (iii). Namely, using some facts used above we have

$$\begin{aligned}
&\|f(x) - S_n(x)\|_{L^1} \leq \int_0^{\pi} |f(x) - K_n(C;x)| dx + \int_0^{\pi} |K_n(C;x) - S_n(x)| dx \\
&\leq \int_0^{\pi} |f(x) - K_n(C;x)| dx + \int_0^{\pi} \left|\frac{i}{2\sin x}[c_n e^{i(n+1)x} - c_{-n} e^{-i(n+1)x} + c_{n+1} e^{inx}\right. \\
&\quad \left. - c_{-(n+1)} e^{-inx} + (c_n - c_{n+2}) E_n(x) + (c_{-(n+2)} - c_{-n}) E_{-n}(x)]\right| dx
\end{aligned}$$

$$\leq \int_0^\pi |f(x) - K_n(C;x)|dx + [|c_n| + |c_{-n}| + |c_{n+1}| + |c_{-(n+1)}|] \int_0^\pi \frac{dx}{2|\sin x|}$$
$$+(|c_n| + |c_{n+2}|) \int_0^\pi \left|\frac{E_n(x)}{2\sin x}\right| dx + (|c_{-(n+2)}| + |c_{-n}|) \int_0^\pi \left|\frac{E_{-n}(x)}{2\sin x}\right| dx$$
$$\leq \int_0^\pi |f(x) - K_n(C;x)|dx + [|c_n| + |c_{-n}| + |c_{n+1}| + |c_{-(n+1)}|] o(\log n)$$
$$+[|c_n| + |c_{n+2}| + |c_{-(n+2)}| + |c_{-n}|] o(n)$$
$$= o(1) + o(1) + o(1) = o(1),\ n \to \infty.$$

The proof is completed.

6.5 L^1-convergence of complex modified sums $K_n(C;x)$ with coefficients from the class $\mathcal{K}^2$

We consider the complex series

$$\sum_{|k|\leq\infty}^{\infty} c_k e^{ikx},$$

its partial sums

$$S_n(C;x) = \sum_{|k|\leq n} c_k e^{ikx},$$

and the complex modified sums

$$K_n(C;x) = S_n(C;x) + \frac{i}{2\sin x}\Big[c_n e^{i(n+1)x} - c_{-n} e^{-i(n+1)x} + c_{n+1} e^{inx} - c_{-(n+1)} e^{-inx} + (c_n - c_{n+2}) E_n(x) + (c_{-(n+2)} - c_{-n}) E_{-n}(x)\Big],$$

with coefficients from the class $\mathcal{K}^2$ (see Definition 1.71), where

$$E_n(x) = \sum_{k=1}^{n} e^{ikx}, \quad E_{-n}(x) = \sum_{k=1}^{n} e^{-ikx}.$$

We prove the following theorem.

Theorem 6.5. *Let (c_k) belongs to the class $\mathcal{K}^{\mathbf{2}}$. Then:*

(i) $\lim_{n\to\infty} K_n(C;x) = f(x)$ *exists for* $|x| \in (0,\pi]$,
(ii) $f \in L^1(0,\pi]$ *and* $\|K_n(C;x) - f(x)\| \to 0$, *as* $n \to \infty$,
(iii) $\|S_n(f;x) - f(x)\| \to 0$ *as* $n \to \infty$.

Proof. Firstly, we will show that $f(x)$ exists on $(0,\pi]$. Indeed, it is clear that we can write

$$S_n(x)=\sum_{k=-n}^{n}c_k e^{ikx}=c_0+\sum_{k=1}^{n}\left(\frac{c_k}{k}ke^{ikx}+\frac{c_{-k}}{k}ke^{-ikx}\right)$$

Applying twice the Abel's transformation we obtain

$$\begin{aligned}S_n(x)=c_0-i&\left[\sum_{k=1}^{n-2}\Delta^2\left(\frac{c_k}{k}\right)\overline{E}'_k(x)+\Delta\left(\frac{c_{n-1}}{n-1}\right)\overline{E}'_n(x)\right]\\&+i\left[\sum_{k=1}^{n-2}\Delta^2\left(\frac{c_{-k}}{k}\right)\overline{E}'_{-k}(x)+\Delta\left(\frac{c_{-(n-1)}}{n-1}\right)\overline{E}'_{-n}(x)\right]\\&+i\frac{c_{-n}}{n}E'_{-n}(x)-i\frac{c_n}{n}E'_n(x).\end{aligned}$$

Based on Lemmas 1.91 and 1.92 we clearly have

$$\begin{aligned}|S_n(x)|\le|c_0|+\sum_{k=1}^{n-2}&\left[\left|\Delta^2\left(\frac{c_k}{k}\right)\right||\overline{E}'_k(x)|+\left|\Delta^2\left(\frac{c_{-k}}{k}\right)\right||\overline{E}'_{-k}(x)|\right]\\&+\left[\left|\Delta\left(\frac{c_{n-1}}{n-1}\right)\right||\overline{E}'_n(x)|+\left|\Delta\left(\frac{c_{-(n-1)}}{n-1}\right)\right||\overline{E}'_{-n}(x)|\right]\\&+\frac{|c_n|}{n}|E'_n(x)|+\frac{|c_{-n}|}{n}|E'_{-n}(x)|\\&\le|c_0|+\frac{M_{r\varepsilon}}{|x|}\left\{\sum_{k=1}^{n-2}k^2\left[\left|\Delta^2\left(\frac{c_k}{k}\right)\right|+\left|\Delta^2\left(\frac{c_{-k}}{k}\right)\right|\right]\right.\\&\left.+n^2\left[\left|\Delta\left(\frac{c_{n-1}}{n-1}\right)\right|+\left|\Delta\left(\frac{c_{-(n-1)}}{n-1}\right)\right|\right]+|c_n|+|c_{-n}|\right\}\\&\le|c_0|+\frac{2M_{r\varepsilon}}{|x|}\left\{\sum_{k=1}^{n-2}A_k\right.\\&\left.+2\sum_{k=n-1}^{\infty}k^2\left[\left|\Delta^2\left(\frac{c_k}{k}\right)\right|+\left|\Delta^2\left(\frac{c_{-k}}{k}\right)\right|\right]+2\overline{M}\right\}\\&\le|c_0|+\frac{2M_{r\varepsilon}}{|x|}\left\{5\sum_{k=1}^{\infty}k^2A_k+2\overline{M}\right\}<+\infty,\end{aligned}$$

since $(c_k)\in\mathcal{K}^2$, where $\overline{M}$ is a positive constant.

Subsequently, $\lim_{n\to\infty}S_n(x)=\lim_{n\to\infty}K_n(C;x)=f(x)$ exists, because of the boundedness of the functions $\frac{e^{inx}}{\sin x}$, $\frac{E_n(x)}{\sin x}$, $\frac{E_{-n}(x)}{\sin x}$ on $(0,\pi]$, and thus (i) holds true.

Now we are going to prove (ii). Indeed, for $x\neq 0$ we have

$$
\begin{aligned}
f(x) - K_n(C;x) = \sum_{k=n+1}^{\infty} &\left(\frac{c_k}{k} k e^{ikx} + \frac{c_{-k}}{k} k e^{-ikx}\right) \\
&- \frac{i}{2\sin x}[c_n e^{i(n+1)x} - c_{-n} e^{-i(n+1)x} \\
&+ c_{n+1} e^{inx} - c_{-(n+1)} e^{-inx} \\
&+ (c_n - c_{n+2}) E_n(x) + (c_{-(n+2)} - c_{-n}) E_{-n}(x)].
\end{aligned}
$$

Again, applying twice the Abel's transformation to the above equality we obtain

$$
\begin{aligned}
f(x) - K_n(C;x) = -i \lim_{p\to\infty} \Bigg\{ &\sum_{k=n+1}^{p-2} \left[\Delta^2\left(\frac{c_k}{k}\right)\overline{E}'_k(x) - \Delta^2\left(\frac{c_{-k}}{k}\right)\overline{E}'_{-k}(x)\right] \\
&+ \Delta\left(\frac{c_{p-1}}{p-1}\right)\overline{E}'_{p-1}(x) - \Delta\left(\frac{c_{-(p-1)}}{p-1}\right)\overline{E}'_{-(p-1)}(x) \\
&+ \frac{c_p}{p} E'_p(x) - \frac{c_{-p}}{p} E'_{-p}(x) \\
&- \Delta\left(\frac{c_n}{n}\right)\overline{E}'_n(x) + \Delta\left(\frac{c_{-n}}{n}\right)\overline{E}'_{-n}(x) - \frac{c_{n+1}}{n+1} E'_n(x) \\
&+ \frac{c_{-(n+1)}}{n+1} E'_{-n}(x)\Bigg\} - \frac{i}{2\sin x}[c_n e^{i(n+1)x} - c_{-n} e^{-i(n+1)x} \\
&+ c_{n+1} e^{inx} - c_{-(n+1)} e^{-inx} \\
&+ (c_n - c_{n+2}) E_n(x) + (c_{-(n+2)} - c_{-n}) E_{-n}(x)] \\
= -i\Bigg\{ &\sum_{k=n+1}^{\infty} \left[\Delta^2\left(\frac{c_k}{k}\right)\overline{E}'_k(x) - \Delta^2\left(\frac{c_{-k}}{k}\right)\overline{E}'_{-k}(x)\right] \\
&- \Delta\left(\frac{c_n}{n}\right)\overline{E}'_n(x) + \Delta\left(\frac{c_{-n}}{n}\right)\overline{E}'_{-n}(x) \\
&- \frac{c_{n+1}}{n+1} E'_n(x) + \frac{c_{-(n+1)}}{n+1} E'_{-n}(x)\Bigg\} \\
&- \frac{i}{2\sin x}[c_n e^{i(n+1)x} - c_{-n} e^{-i(n+1)x} + c_{n+1} e^{inx} - c_{-(n+1)} e^{-inx} \\
&+ (c_n - c_{n+2}) E_n(x) + (c_{-(n+2)} - c_{-n}) E_{-n}(x)].
\end{aligned}
$$

Hence, using Lemmas 1.91 and 1.92 we get

$$
\begin{aligned}
|f(x) - K_n(C;x)| \le \sum_{k=n+1}^{\infty} &\left[\left|\Delta^2\left(\frac{c_k}{k}\right)\right| |\overline{E}'_k(x)| + \left|\Delta^2\left(\frac{c_{-k}}{k}\right)\right| |\overline{E}'_{-k}(x)|\right] \\
&+ \left|\Delta\left(\frac{c_n}{n}\right)\right| |\overline{E}'_n(x)| + \left|\Delta\left(\frac{c_{-n}}{n}\right)\right| |\overline{E}'_{-n}(x)| \\
&+ \left|\frac{c_{n+1}}{n+1}\right| |E'_n(x)| + \left|\frac{c_{-(n+1)}}{n+1}\right| |E'_{-n}(x)|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2|\sin x|}[|c_n| + |c_{-n}| + |c_{n+1}| \\
&+|c_{-(n+1)}| + (|c_n| + |c_{n+2}|)|E_n(x)| \\
&+(|c_{-(n+2)}| + |c_{-n}|)|E_{-n}(x)|] \\
\le\ & \frac{M_{r\varepsilon}}{|x|}\left\{ \sum_{k=n+1}^{\infty} k^2 \left[\left|\Delta^2\left(\frac{c_k}{k}\right)\right| + \left|\Delta^2\left(\frac{c_{-k}}{k}\right)\right|\right]\right. \\
&+n^2\left[\left|\Delta\left(\frac{c_n}{n}\right)\right| + \left|\Delta\left(\frac{c_{-n}}{n}\right)\right|\right] \\
&\left.+(n+1)\left(|c_{n+1}| + \left|c_{-(n+1)}\right|\right)\right\} \\
&+\frac{|c_n| + |c_{-n}| + |c_{n+1}| + |c_{-(n+1)}|}{2|\sin x|} \\
&+(|c_n| + |c_{n+2}|)\left|\frac{E_n(x)}{2\sin x}\right| \\
&+(|c_{-(n+2)}| + |c_{-n}|)\left|\frac{E_{-n}(x)}{2\sin x}\right| \\
\le\ & \frac{M_{r\varepsilon}}{|x|}\left[2\sum_{k=n}^{\infty} A_k + (n+1)\left(|c_{n+1}| + \left|c_{-(n+1)}\right|\right)\right] \\
&+\frac{|c_n| + |c_{-n}| + |c_{n+1}| + |c_{-(n+1)}|}{2|\sin x|} \\
&+(|c_n| + |c_{n+2}|)\left|\frac{E_n(x)}{2\sin x}\right| \\
&+(|c_{-(n+2)}| + |c_{-n}|)\left|\frac{E_{-n}(x)}{2\sin x}\right|.
\end{aligned}
$$

Therefore, using Lemma 1.96 we obtain

$$
\begin{aligned}
\|f(x) - K_n(C;x)\| \le\ & M_{r\varepsilon}\left[2\sum_{k=n}^{\infty} A_k \int_0^{\pi} \frac{dx}{|x|}\right. \\
&\left.+(n+1)\left(|c_{n+1}| + \left|c_{-(n+1)}\right|\right)\int_0^{\pi}\frac{dx}{|x|}\right] \\
&+\frac{|c_n| + |c_{-n}| + |c_{n+1}| + |c_{-(n+1)}|}{2}\int_0^{\pi}\frac{dx}{|\sin x|} \\
&+(|c_n| + |c_{n+2}|)\int_0^{\pi}\left|\frac{E_n(x)}{2\sin x}\right| dx \\
&+(|c_{-(n+2)}| + |c_{-n}|)\int_0^{\pi}\left|\frac{E_{-n}(x)}{2\sin x}\right| dx
\end{aligned}
$$

$$\leq M_{r\varepsilon}\left[2\sum_{k=n}^{\infty} A_k \log k \right.$$
$$\left. +(n+1)\left(|c_{n+1}| + \left|c_{-(n+1)}\right|\right) o\left(\log n\right)\right]$$
$$+\frac{|c_n| + |c_{-n}| + |c_{n+1}| + |c_{-(n+1)}|}{2} o\left(\log n\right)$$
$$+(|c_n| + |c_{n+2}|)o(n) + (|c_{-(n+2)}| + |c_{-n}|)o(n).$$

Now we note that

$$\sum_{k=n}^{\infty} A_k \log k \leq \sum_{k=n}^{\infty} k^2 A_k = o(1),$$

and

$$(n+1)\left|c_{\pm(n+1)}\right| \log n \leq (n+1)^3 \left|\frac{c_{\pm(n+1)}}{n+1}\right| \leq (n+1)^3 \sum_{k=n+1}^{\infty} \left|\Delta\left(\frac{c_{\pm k}}{k}\right)\right|$$
$$\leq (n+1)^3 \sum_{k=n+1}^{\infty} \sum_{j=k}^{\infty} \left|\Delta^2\left(\frac{c_{\pm j}}{j}\right)\right|$$
$$= (n+1)^3 \sum_{j=n+1}^{\infty} (j-n) \left|\Delta^2\left(\frac{c_{\pm j}}{j}\right)\right|$$
$$\leq \sum_{j=n+1}^{\infty} j^4 \left|\Delta^2\left(\frac{c_{\pm j}}{j}\right)\right|$$
$$\leq \sum_{j=n+1}^{\infty} j^4 \frac{A_j}{j^2} = \sum_{j=n+1}^{\infty} j^2 A_j = o(1)$$

as $n \to \infty$. Subsequently, we get

$$\|f(x) - K_n(C;x)\| = o(1) \quad as \quad n \to \infty.$$

Using the latest equality and the fact that $K_n(C;x)$ is a polynomial it follows that $f \in L^1(0,\pi]$.

Finally, we will prove (iii). Namely, using some facts used above we have

$$\int_0^{\pi} |f(x) - S_n(x)|dx \leq \int_0^{\pi} |f(x) - K_n(C;x)|dx + \int_0^{\pi} |g_n^c(x) - S_n(x)|dx$$
$$\leq \int_0^{\pi} |f(x) - K_n(C;x)|dx + \int_0^{\pi} \left|\frac{i}{2\sin x}[c_n e^{i(n+1)x}\right.$$
$$-c_{-n}e^{-i(n+1)x} + c_{n+1}e^{inx} - c_{-(n+1)}e^{-inx}$$
$$\left.+(c_n - c_{n+2})E_n(x) + (c_{-(n+2)} - c_{-n})E_{-n}(x)]\right|dx$$

$$\begin{aligned}
&\leq \int_0^\pi |f(x) - K_n(C;x)|dx \\
&+[|c_n| + |c_{-n}| + |c_{n+1}| + |c_{-(n+1)}|] \int_0^\pi \frac{dx}{2|\sin x|} \\
&+(|c_n| + |c_{n+2}|) \int_0^\pi \left|\frac{E_n(x)}{2\sin x}\right|dx \\
&+(|c_{-(n+2)}| + |c_{-n}|) \int_0^\pi \left|\frac{E_{-n}(x)}{2\sin x}\right|dx \\
&\leq \int_0^\pi |f(x) - K_n(C;x)|dx + [|c_n| + |c_{-n}| + |c_{n+1}| \\
&+|c_{-(n+1)}|]o\left(\log n\right) \\
&+[|c_n| + |c_{n+2}| + |c_{-(n+2)}| + |c_{-n}|]o\left(n\right) \\
&= o(1) + o(1) + o(1) = o(1),\ \ n \to \infty.
\end{aligned}$$

The proof is completed.

6.6 L^1-convergence of r-th derivative of the modified sums $g_n(C;x)$ with coefficients from the class $\mathrm{R}^*(r)$

We consider the complex series

$$\sum_{|k|\leq\infty}^{\infty} c_k e^{ikx},$$

its partial sums

$$S_n(C;x) = \sum_{|k|\leq n} c_k e^{ikx},$$

the complex modified sums

$$g_n(C;x) = S_n(C;x) + \frac{i}{n+1}\left[c_{n+1}E'_n(x) - c_{n+1}E'_{-n}(x)\right],$$

and their the r-th derivative

$$g_n^{(r)}(C;x) = S_n^{(r)}(C;x) + \frac{i}{n+1}\left[c_{n+1}E_n^{(r+1)}(x) - c_{n+1}E_{-n}^{(r+1)}(x)\right],$$

with coefficients from the class $\mathbf{R}^*(r)$, $r = 0, 1, 2, \ldots$.

We prove the following.

Theorem 6.6. *Let $\{c_k\} \in \mathbf{R}^*(r)$, $r = 0, 1, 2, \ldots$. Then there exists $f(x)$ such that:*

(i) $\lim_{n\to\infty} g_n^{(r)}(C;x) = f^{(r)}(x)$ *for all* $0 < |x| \leq \pi$,

(ii) $f^{(r)} \in L^1$ *and* $\|g_n^{(r)}(C;x) - f^{(r)}(x)\| = o(1)$ *as* $n \to \infty$, *and*
(iii) $\lim_{n\to\infty} \|S_n^{(r)}(C;x) - f^{(r)}(x)\| = 0$ *if and only if* $\lim_{|n|\to\infty} |n|^r c_n \log|n| = 0$.

Proof. (i) Applying Abel's transformation, we have

$$\begin{aligned} g_n^{(r)}(C;x) &= S_n^{(r)}(C;x) + \frac{i}{n+1}\left[c_{n+1}E_n^{(r+1)}(x) - c_{n+1}E_{-n}^{(r+1)}(x)\right] \\ &= 2\sum_{k=1}^{n} \Delta\left(\frac{c_k}{k}\right)\widetilde{D}_k^{(r+1)}(x) + \sum_{k=1}^{n}\Delta\left(\frac{c_{k-1}-c_k}{k}\right) iE_k^{(r+1)}(x). \end{aligned}$$

By Lemma 1.91 and changing the order of summation, we get

$$\begin{aligned} \sum_{k=1}^{\infty}\left|\Delta\left(\frac{c_k}{k}\right)\widetilde{D}_k^{(r+1)}(x)\right| &\le \frac{C}{|x|}\sum_{k=1}^{\infty} k^{r+1}\left|\Delta\left(\frac{c_k}{k}\right)\right| \\ &\le \frac{C}{|x|}\sum_{k=1}^{\infty}\sum_{j=k}^{\infty} k^{r+1}\left|\Delta^2\left(\frac{c_j}{j}\right)\right| \\ &= \frac{C}{|x|}\sum_{j=1}^{\infty}\left(\sum_{k=1}^{j} k^{r+1}\right)\left|\Delta^2\left(\frac{c_j}{j}\right)\right| \\ &= \mathcal{O}\left[\frac{1}{|x|}\sum_{j=1}^{\infty} j^{r+2}\left|\Delta^2\left(\frac{c_j}{j}\right)\right|\right] < \infty, \end{aligned}$$

and

$$\begin{aligned} \sum_{k=3}^{\infty}\left|\Delta\left(\frac{c_{k-1}-c_k}{k}\right)E_k^{(r+1)}(x)\right| &\le \frac{C_1}{|x|}\sum_{k=3}^{\infty} k^{r+1}\left|\Delta\left(\frac{c_{k-1}-c_k}{k}\right)\right| \\ &= \mathcal{O}\left[\frac{1}{|x|}\sum_{k=3}^{\infty} k^{r+1}\log k\left|\Delta\left(\frac{c_{k-1}-c_k}{k}\right)\right|\right] < \infty, \end{aligned}$$

where C_1 is a suitable constant.

Consequently,

$$f^{(r)}(x) = 2\sum_{k=1}^{\infty}\Delta\left(\frac{c_k}{k}\right)\widetilde{D}_k^{(r+1)}(x) + i\sum_{k=1}^{\infty}\Delta\left(\frac{c_{k-1}-c_k}{k}\right) iE_k^{(r+1)}(x)$$

exists and consequently (i) follows.

(ii) Now for $x \neq 0$, we have

$$\begin{aligned} &f^{(r)}(x) - g_n^{(r)}(C;x) \\ &= 2\sum_{k=n+1}^{\infty}\Delta\left(\frac{c_k}{k}\right)\widetilde{D}_k^{(r+1)}(x) + i\sum_{k=n+1}^{\infty}\Delta\left(\frac{c_{k-1}-c_k}{k}\right) iE_k^{(r+1)}(x) \end{aligned}$$

$$= 2\sum_{k=n+1}^{\infty}(k+1)\Delta^2\left(\frac{c_k}{k}\right)\widetilde{K}_k^{(r+1)}(x) - 2(n+1)\Delta\left(\frac{c_{n+1}}{n+1}\right)\widetilde{K}_n^{(r+1)}(x)$$
$$+i\sum_{k=n+1}^{\infty}\Delta\left(\frac{c_{k-1}-c_k}{k}\right)iE_k^{(r+1)}(x).$$

Then,

$$\left\|f^{(r)}(x) - g_n^{(r)}(C;x)\right\| \le 2\sum_{k=n+1}^{\infty}(k+1)\left|\Delta^2\left(\frac{c_k}{k}\right)\right|\int_{-\pi}^{\pi}\left|\widetilde{K}_k^{(r+1)}(x)\right|dx$$
$$+2(n+1)\left|\Delta\left(\frac{c_{n+1}}{n+1}\right)\right|\int_{-\pi}^{\pi}\left|\widetilde{K}_n^{(r+1)}(x)\right|dx$$
$$+\sum_{k=n+1}^{\infty}\left|\Delta\left(\frac{c_{k-1}-c_k}{k}\right)\right|\int_{-\pi}^{\pi}\left|E_k^{(r+1)}(x)\right|dx.$$

By Lemma 1.94, it holds $\int_{-\pi}^{\pi}|\widetilde{K}_k'(x)|dx = \mathcal{O}(k)$, and by Bernstein's inequality we obtain

$$\int_{-\pi}^{\pi}|\widetilde{K}_k^{(r)}(x)|dx = \mathcal{O}(k^r).$$

Also,

$$\left|\Delta\left(\frac{c_{n+1}}{n+1}\right)\right| = \left|\sum_{k=n+1}^{\infty}\Delta^2\left(\frac{c_{n+1}}{n+1}\right)\right|$$
$$\le \sum_{k=n+1}^{\infty}\frac{k^{r+2}}{k^{r+2}}\left|\Delta^2\left(\frac{c_{n+1}}{n+1}\right)\right|$$
$$\le \frac{1}{(n+1)^{r+2}}\sum_{k=n+1}^{\infty}k^{r+2}\left|\Delta^2\left(\frac{c_{n+1}}{n+1}\right)\right|$$
$$= o((n+1)^{-r-2})$$

as $n \to \infty$.

Since Lemma 1.84 and Lemma 1.85 imply

$$\int_{-\pi}^{\pi}|\widetilde{E}_{-k}^{(r+1)}(x)|dx = o(k^{r+1}\log k),$$

then

$$\|f^{(r)}(x) - g_n^{(r)}(C;x)\| = o\left(\sum_{k=n+1}^{\infty}(k+1)\left|\Delta^2\left(\frac{c_k}{k}\right)\right|\right) + o(1)$$
$$+o\left(\sum_{k=n+1}^{\infty}\left|\Delta\left(\frac{c_{k-1}-c_k}{k}\right)\right|k^{r+1}\log k\right) = o(1),$$

by hypothesis of the theorem.

Since $g_n^{(r)}(C;x)$ is a polynomial, it follows that $f^{(r)} \in L^1$, which proves (ii).

(iii) We can write

$$\begin{aligned}\|f^{(r)}(x) - S_n^{(r)}(C;x)\| &\le \|f^{(r)}(x) - g_n^{(r)}(C;x)\| + \|g_n^{(r)}(C;x) - S_n^{(r)}(C;x)\| \\ &= \|f^{(r)}(x) - g_n^{(r)}(C;x)\| \\ &\quad + \left\| \frac{i}{n+1} \left[c_{n+1} E_n^{(r+1)}(x) - c_{n+1} E_{-n}^{(r+1)}(x) \right] \right\|\end{aligned}$$

and

$$\begin{aligned}\left\| \frac{i}{n+1} \left[c_{n+1} E_n^{(r+1)}(x) - c_{-(n+1)} E_{-n}^{(r+1)}(x) \right] \right\| &= \|g_n^{(r)}(C;x) - S_n^{(r)}(C;x)\| \\ &\le \|f^{(r)}(x) - S_n^{(r)}(C;x)\| + \|f^{(r)}(x) - g_n^{(r)}(C;x)\|.\end{aligned}$$

Since $\|f^{(r)}(x) - g_n^{(r)}(C;x)\| = o(1)$ as $n \to \infty$, by (ii) and by Lemma 1.95, then

$$\lim_{n\to\infty} \left\| \frac{1}{n+1} \left[c_{n+1} E_n^{(r+1)}(x) - c_{-(n+1)} E_{-n}^{(r+1)}(x) \right] \right\| = 0$$

if and only if

$$\lim_{|n|\to\infty} |n|^r c_n \log |n| = 0.$$

The proof is completed.

7

L^1-convergence of double modified trigonometric sums

In this section we have written only few results regrading to L^1-convergence of some double modified trigonometric sums whose coefficients belong to some classes of real sequences.

7.1 L^1-convergence of double modified trigonometric sums $X_{m,n}(x,y)$

For a function $f_1(x,y)$ with two independent variables x and y we write $f_1 \in L^1(T^2)$ if

$$\|f_1\| = \iint_{T^2} |f_1(x,y)|dxdy < +\infty,$$

where $Q := [0,\pi] \times [0,\pi]$.

Let us consider double cosine series of the form

$$f_1(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k} \cos jx \cos ky, \tag{7.1}$$

with its partial sums

$$S_{m,n}^{\cos}(x,y) := \sum_{j=1}^{m}\sum_{k=1}^{n} a_{j,k} \cos jx \cos ky, \quad m,n \geq 1,$$

and

$$f_1(x,y) = \lim_{m+n\to\infty} S_{m,n}^{\cos}(x,y).$$

Then, we use double modified cosine sums

$$X_{m,n}(x,y) = \sum_{j=1}^{m}\sum_{k=1}^{n}\left(\sum_{i=j}^{m}\sum_{\ell=k}^{n} \Delta_{1,1}\left(a_{i,\ell}\cos ix \cos \ell y\right)\right), \tag{7.2}$$

the partial sums of the series (7.1), and

$$f_1(x,y) = \lim_{m+n\to\infty} S^{\cos}_{m,n}(x,y).$$

Moreover, we use the following class of numerical sequences:

Definition 7.1. *A double null sequence $\{a_{j,k}\}$ of positive numbers is said to belong to the class $\mathbf{J}_d$ if there exists a double sequence $\{A_{j,k}\}$ such that*

$$A_{j,k} \downarrow 0, \quad j+k \to \infty, \tag{7.3}$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} jkA_{j,k} < \infty, \tag{7.4}$$

and

$$\left|\Delta_{p,q}\left(\frac{a_{j,k}}{jk}\right)\right| \leq \frac{A_{j,k}}{jk}, \quad 1 \leq p+q \leq 2 \tag{7.5}$$

for any non-negative integers p,q and $j,k \in \{1,2,3,\ldots\}$.

Now, we prove the following.

Theorem 7.2. *If a double sequence $\{a_{j,k}\}$ belongs to the class $\mathbf{J}_d$, then $\|f_1 - X_{m,n}\| \to 0$ as $j+k\to\infty$.*

Proof. Firstly, after some simple calculations we have

$$\begin{aligned}
X_{m,n}(x,y) &= \sum_{j=1}^{m}\sum_{k=1}^{n}\left(\sum_{i=j}^{m}\sum_{\ell=k}^{n}\Delta_{1,1}\left(a_{i,\ell}\cos ix\cos \ell y\right)\right)\\
&= \sum_{j=1}^{m}\sum_{k=1}^{n}\left\{\sum_{i=j}^{m}\left[\sum_{\ell=k}^{n}\Delta_{0,1}\left(\Delta_{1,0}\left(a_{i,\ell}\cos ix\cos \ell y\right)\right)\right]\right\}\\
&= \sum_{j=1}^{m}\sum_{k=1}^{n}\left\{\sum_{i=j}^{m}\big(\Delta_{1,0}\big(a_{i,k}\cos ix\cos ky\big)\right.\\
&\qquad \left. -\Delta_{1,0}\big(a_{i,n+1}\cos ix\cos(n+1)y\big)\big)\right\}\\
&= \sum_{j=1}^{m}\sum_{k=1}^{n}\big\{a_{j,k}\cos jx\cos ky - a_{m+1,k}\cos(m+1)x\cos ky\\
&\qquad -a_{j,n+1}\cos jx\cos(n+1)y + a_{m+1,n+1}\cos(m+1)x\cos(n+1)y\big\}\\
&= S^{\cos}_{m,n}(x,y) - \sum_{j=1}^{m}\sum_{k=1}^{n}\big\{a_{m+1,k}\cos(m+1)x\cos ky\\
&\qquad +a_{j,n+1}\cos jx\cos(n+1)y\big\}
\end{aligned}$$

$$+\sum_{j=1}^{m}\sum_{k=1}^{n} a_{m+1,n+1}\cos(m+1)x\cos(n+1)y$$
$$=\sum_{j=1}^{m}\sum_{k=1}^{n} a_{j,k}\cos jx\cos ky - m\cos(m+1)x\sum_{k=1}^{n} a_{m+1,k}\cos ky$$
$$-n\cos(n+1)y\sum_{j=1}^{m} a_{j,n+1}\cos jx$$
$$+mna_{m+1,n+1}\cos(m+1)x\cos(n+1)y. \qquad (7.6)$$

Moreover, the equality (7.2) can be rewritten as follows

$$X_{m,n}(x,y) = \sum_{j=1}^{m}\sum_{k=1}^{n}\left(\frac{a_{j,k}}{jk}\right)(\sin jx)'(\sin ky)'$$
$$-m\cos(m+1)x\sum_{k=1}^{n} a_{m+1,k}\cos ky$$
$$-n\cos(n+1)y\sum_{j=1}^{m} a_{j,n+1}\cos jx$$
$$+mna_{m+1,n+1}\cos(m+1)x\cos(n+1)y. \qquad (7.7)$$

Applying double summation by parts to (7.7) we obtain

$$X_{m,n}(x,y) = \sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\Delta_{11}\left(\frac{a_{j,k}}{jk}\right)\widetilde{D}'_j(x)\widetilde{D}'_k(y) + \sum_{j=1}^{m-1}\Delta_{10}\left(\frac{a_{j,n}}{jn}\right)\widetilde{D}'_j(x)\widetilde{D}'_n(y)$$
$$+\sum_{k=1}^{n-1}\Delta_{01}\left(\frac{a_{m,k}}{mk}\right)\widetilde{D}'_m(x)\widetilde{D}'_k(y) + \frac{a_{m,n}}{mn}\widetilde{D}'_m(x)\widetilde{D}'_n(y)$$
$$-m\cos(m+1)x\sum_{k=1}^{n} a_{m+1,k}\cos ky - n\cos(n+1)y\sum_{j=1}^{m} a_{j,n+1}\cos jx$$
$$+mna_{m+1,n+1}\cos(m+1)x\cos(n+1)y = \sum_{s=1}^{7} R_s(x,y).$$

Based on Lemma 1.91 we have $|\widetilde{D}'_m(u)| \le \frac{C_\varepsilon m}{u}$, $0 < u \le \pi$, therefore using (7.4) and (7.5) we clearly have

$$|R_1(x,y)| \le \sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\left|\Delta_{11}\left(\frac{a_{j,k}}{jk}\right)\right||\widetilde{D}'_j(x)||\widetilde{D}'_k(y)|$$
$$\le \frac{C_\varepsilon}{xy}\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\left(\frac{A_{j,k}}{jk}\right)jk \le \frac{C_\varepsilon}{xy}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} A_{j,k} < +\infty,$$

for all x and y such that $0 < x \le \pi$, $0 < y \le \pi$.

Also, we have

$$\begin{aligned}
|R_2(x,y)| &\le \sum_{j=1}^{m-1} \left| \Delta_{10}\left(\frac{a_{j,n}}{jn}\right)\right| |\widetilde{D}'_j(x)||\widetilde{D}'_n(y)| \\
&\le \frac{C_\varepsilon}{xy} \sum_{j=1}^{m-1} \left| \sum_{k=n}^{\infty} \Delta_{11}\left(\frac{a_{j,k}}{jk}\right)\right| jn \\
&\le \frac{C_\varepsilon}{xy} \sum_{j=1}^{m-1} \sum_{k=n}^{\infty} \left| \Delta_{11}\left(\frac{a_{j,k}}{jk}\right)\right| jn \\
&\le \frac{C_\varepsilon}{xy} \sum_{j=1}^{m-1} \sum_{k=n}^{\infty} \frac{A_{j,k}}{jk} jn \\
&\le \frac{C_\varepsilon}{xy} \sum_{j=1}^{m-1} \sum_{k=n}^{\infty} \frac{A_{j,k}}{n} n \\
&= \frac{C_\varepsilon}{xy} \sum_{j=1}^{m-1} \sum_{k=n}^{\infty} A_{j,k} \to 0, \quad as \quad n \to \infty
\end{aligned}$$

uniformly in m and for all x and y such that $0 < x \le \pi$, $0 < y \le \pi$.

Similarly,

$$|R_3(x,y)| \le \frac{C_\varepsilon}{xy} \sum_{j=m}^{\infty} \sum_{k=1}^{n-1} A_{j,k} \to 0, \quad as \quad m \to \infty$$

uniformly in n and for all x and y such that $0 < x \le \pi$, $0 < y \le \pi$.

Then based on Lemma 1.91 and on the fact that $\{a_{jk}\}$ is a double null sequence we have

$$|R_4(x,y)| = \frac{a_{m,n}}{mn} \left|\widetilde{D}'_m(x)\right| \left|\widetilde{D}'_n(y)\right| \le \frac{C_\varepsilon}{xy} \frac{a_{m,n}}{mn} mn = \frac{C_\varepsilon}{xy} a_{m,n} \to 0,$$

as $m + n \to \infty$, for all x and y such that $0 < x \le \pi$, $0 < y \le \pi$.

Next, since $\{a_{jk}\} \in J_d$ we have

$$\begin{aligned}
|R_5(x,y)| \le m \sum_{k=1}^{n} |a_{m+1,k}| &= m(m+1) \sum_{k=1}^{n} \left| \frac{a_{m+1,k}}{(m+1)k} \right| k \\
&= m(m+1) \sum_{k=1}^{n} \left| \sum_{j=m+1}^{\infty} \Delta_{10}\left(\frac{a_{j,k}}{jk}\right) \right| k \\
&\le m(m+1) \sum_{k=1}^{n} \sum_{j=m+1}^{\infty} \left| \Delta_{10}\left(\frac{a_{j,k}}{jk}\right) \right| k
\end{aligned}$$

$$\le m(m+1)\sum_{j=m+1}^{\infty}\sum_{k=1}^{n}\frac{A_{j,k}}{jk}k$$
$$\le m(m+1)\sum_{j=m+1}^{\infty}\sum_{k=1}^{n}\frac{A_{j,k}}{m+1}$$
$$\le \frac{1}{2}\sum_{j=m+1}^{\infty}\sum_{k=1}^{n} jA_{j,k}\to 0 \quad as \quad m\to\infty \tag{7.8}$$

uniformly in n and for all x and y such that $0<x\le\pi$, $0<y\le\pi$.

Similarly, we have verified that

$$|R_6(x,y)|\le \frac{1}{2}\sum_{j=1}^{m}\sum_{k=n+1}^{\infty} kA_{j,k}\to 0 \quad as \quad n\to\infty \tag{7.9}$$

uniformly in m and for all x and y such that $0<x\le\pi$, $0<y\le\pi$.

Doing almost the same reasoning we have proved that

$$|R_7(x,y)|\le \frac{1}{4}\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty} jkA_{j,k}\to 0 \quad as \quad m+n\to\infty \tag{7.10}$$

for all x and y such that $0<x\le\pi$, $0<y\le\pi$.

Subsequently,

$$\lim_{m+n\to\infty} X_{m,n}(x,y)=\lim_{m+n\to\infty} S^{\cos}_{m,n}(x,y)=f_1(x,y)$$

exists and $f_1(x,y)\in L^1(T^2)$.

Now, we consider

$$\|f_1-X_{m,n}\|\le \int_0^\pi\!\!\int_0^\pi\left|\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}\Delta_{11}\left(\frac{a_{j,k}}{jk}\right)\widetilde{D}'_j(x)\widetilde{D}'_k(y)\right|dxdy$$
$$+\int_0^\pi\!\!\int_0^\pi\left|\sum_{j=1}^{m}\Delta_{10}\left(\frac{a_{j,n}}{jn}\right)\widetilde{D}'_j(x)\widetilde{D}'_n(y)\right|dxdy$$
$$+\int_0^\pi\!\!\int_0^\pi\left|\sum_{k=1}^{n}\Delta_{01}\left(\frac{a_{m,k}}{mk}\right)\widetilde{D}'_m(x)\widetilde{D}'_k(y)\right|dxdy$$
$$+\int_0^\pi\!\!\int_0^\pi\left|\frac{a_{m,n}}{mk}\widetilde{D}'_m(x)\widetilde{D}'_n(y)\right|dxdy$$
$$+\int_0^\pi\!\!\int_0^\pi\left|\sum_{j=1}^{m} na_{j,n+1}\cos jx\cos(n+1)y\right|dxdy$$
$$+\int_0^\pi\!\!\int_0^\pi\left|\sum_{k=1}^{n} ma_{m+1,k}\cos(m+1)x\cos ky\right|dxdy$$

$$
\begin{aligned}
&+mn|a_{m+1,n+1}|\int_0^\pi\int_0^\pi |\cos(m+1)x\cos(n+1)y|\,dxdy\\
&\le \int_0^\pi\int_0^\pi\left|\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}\left(\frac{A_{j,k}}{jk}\right)\widetilde{D}'_j(x)\widetilde{D}'_k(y)\right|dxdy\\
&+\int_0^\pi\int_0^\pi\left|\sum_{j=1}^{m}\left(\frac{A_{j,n}}{jn}\right)\widetilde{D}'_j(x)\widetilde{D}'_n(y)\right|dxdy\\
&+\int_0^\pi\int_0^\pi\left|\sum_{k=1}^{n}\left(\frac{A_{m,k}}{mk}\right)\widetilde{D}'_m(x)\widetilde{D}'_k(y)\right|dxdy\\
&+\int_0^\pi\int_0^\pi\left|\frac{a_{m,n}}{mk}\widetilde{D}'_m(x)\widetilde{D}'_n(y)\right|dxdy \qquad (7.11)\\
&+\int_0^\pi\int_0^\pi\left|\sum_{j=1}^{m}\sum_{k=n+1}^{\infty}jk^2\frac{A_{j,k}}{jk}\cos jx\cos(n+1)y\right|dxdy\\
&+\int_0^\pi\int_0^\pi\left|\sum_{j=m+1}^{\infty}\sum_{k=1}^{n}jk^2\frac{A_{j,k}}{jk}\cos(m+1)x\cos ky\right|dxdy\\
&+mn|a_{m+1,n+1}|\int_0^\pi\int_0^\pi |\cos(m+1)x\cos(n+1)y|\,dxdy.
\end{aligned}
$$

Applying Lemma 1.85, (7.4), and (7.5), we get

$$\int_0^\pi\int_0^\pi\left|\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}jk\left(\frac{A_{j,k}}{j^2k^2}\right)\widetilde{D}'_j(x)\widetilde{D}'_k(y)\right|dxdy\to 0 \quad as \quad m+n\to\infty.$$

Thus, from these arguments and the given hypothesis all terms on the right hand side of (7.11) tend to zero.

So,

$$\|f_1 - X_{m,n}\|\to 0 \quad as \quad m+n\to\infty.$$

The proof is completed.

Corollary 7.3. *If a double sequence* $\{a_{j,k}\}$ *belongs to the class* J_d*, then* $\|S^{\cos}_{m,n} - f_1\|\to 0$ *as* $j+k\to\infty$.

Proof. According to (2.1) it is clear that

$$
\begin{aligned}
\|S^{\cos}_{m,n} - f_1\| &= \|S^{\cos}_{m,n} - X_{m,n} + X_{m,n} - f_1\|\\
&\le \|f_1 - X_{m,n}\| + \|X_{m,n} - S^{\cos}_{m,n}\|\\
&\le \|f_1 - X_{m,n}\|\\
&\quad + \int_0^\pi\int_0^\pi |m\cos(m+1)x\sum_{k=1}^{n}a_{m+1,k}\cos ky|dxdy
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^\pi\int_0^\pi |n\cos(n+1)y\sum_{j=1}^m a_{j,n+1}\cos jx|dxdy \\
&+\int_0^\pi\int_0^\pi |mna_{m+1,n+1}\cos(m+1)x\cos(n+1)y|dxdy \\
\leq\ & \|f_1 - X_{m,n}\| \\
&+\int_0^\pi\int_0^\pi |m\sum_{k=1}^n a_{m+1,k}|dxdy \\
&+\int_0^\pi\int_0^\pi |n\sum_{j=1}^m a_{j,n+1}|dxdy + \int_0^\pi\int_0^\pi |mna_{m+1,n+1}|dxdy.
\end{aligned}
$$

Note that the first term tends to zero based on Theorem 7.2 as well as the second, third and fourth terms according to some parts of the proof of Theorem 7.2.

The proof is completed.

7.2 L^1-convergence of double modified trigonometric sums $f^d_{m,n}(x,y)$

In this unit we consider double cosine series of the form

$$\sum_{j=0}^\infty\sum_{k=0}^\infty \lambda_j\lambda_k q^d_{j,k}\cos jx\cos ky \tag{7.12}$$

on the positive quadrant $Q := [0,\pi]\times[0,\pi]$ of the two-dimensional torus, where $\lambda_0 = \frac{1}{2}$, $\lambda_j = \lambda_k = 1$, if $j \geq 1, k \geq 1$, and $q^d_{j,k}$ are real coefficients.

The rectangular partial sums of the series (7.12) are

$$S^d_{m,n}(x,y) := \sum_{j=0}^m\sum_{k=0}^n \lambda_j\lambda_k q^d_{j,k}\cos jx\cos ky, \quad m,n \geq 0,$$

and let

$$f^d(x,y) = \lim_{m,n\to\infty} S^d_{m,n}(x,y).$$

The differences $\Delta_{22}q^d_{j,k}$ are defined by

$$\Delta_{22}q^d_{j,k} := \sum_{i=0}^2\sum_{\ell=0}^2 (-1)^{i+\ell}\binom{2}{i}\binom{2}{\ell}q^d_{j+i,k+\ell}.$$

we will use the following class of double numerical sequences:

Definition 7.4. *A double sequence* $\{q_{j,k}^d\}$ *of real numbers is said to belong to the class* $\mathbf{S}_{jk}$ *if*

$$q_{j,k}^d \to 0, \quad as \quad \max(j,k) \to \infty,$$

and

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(j+1)(k+1)|\Delta_{22}q_{j,k}^d|\ln(j+2)\ln(k+2) < \infty. \tag{7.13}$$

Example 7.5. Let us define $\Delta_{22}q_{j,k}^d := \frac{1}{(jk)^4}$ for all $j,k \geq 1$. It is obvious that $\{q_{j,k}^d\}$ belongs to the class $\mathbf{S}_{jk}$.

In the sequel we will use the following double modified sine sums

$$f_{m,n}^d(x,y) = \frac{1}{4\sin x \sin y}\sum_{j=1}^{m}\sum_{k=1}^{n} \times \sum_{r=j}^{m}\sum_{\ell=k}^{n}\Delta_{1,1}\left[\left(\Delta_{1,1}\left(q_{r-1,\ell-1}^d\right)\sin rx \sin \ell y\right)\right], \tag{7.14}$$

where $q_{j,k}^d := 0$ for either $j=0$ or $k=0$.

Some properties of the a sequence $\{q_{j,k}^d\}$ that belongs to the class $\mathbf{S}_{jk}$ are given in next statement.

Theorem 7.6. *If* $\{q_{j,k}^d\} \in \mathbf{S}_{jk}$, *then the following hold true:*

(i)

$$\sum_{j=0}^{m}(j+1)|\Delta_{20}q_{j,n}^d|\ln(j+2)\ln(n) \to 0 \ \ as \ \ n \to \infty$$

(ii)

$$\sum_{k=0}^{n}(k+1)|\Delta_{02}q_{m,k}^d|\ln(m)\ln(k+2) \to 0 \ \ as \ \ m \to \infty$$

(iii)

$$m^s n^r \Delta_{sr}q_{m,n}^d \ln(m)\ln(n) \to 0 \ \ as \ \ \min(m,n) \to \infty,$$

where $s,r \in \{0,1\}$.

Proof. (i) Applying the discrete summation by parts on the right side of the equality

$$\sum_{j=0}^{m}(j+1)|\Delta_{20}q_{j,n}^d|\ln(j+2)\ln(n) = \ln(n)\sum_{j=0}^{m}(j+1)\sum_{k=n}^{\infty}|\Delta_{21}q_{j,k}^d|\ln(j+2),$$

we have

$$
\begin{aligned}
&\ln(n)\sum_{j=0}^{m}\sum_{k=n}^{\infty}(j+1)(k+1)|\Delta_{22}q^d_{j,k}|\ln(j+2) \\
&\qquad\qquad\qquad + \ln(n)\sum_{j=0}^{m} n(j+1)|\Delta_{21}q^d_{j,n}|\ln(j+2) \\
&\le \sum_{j=0}^{m}\sum_{k=n}^{\infty}(j+1)(k+1)|\Delta_{22}q^d_{j,k}|\ln(j+2)\ln(k+2) \\
&\qquad\qquad\qquad + n\ln(n)\sum_{j=0}^{m}\sum_{k=n}^{\infty}(j+1)|\Delta_{22}q^d_{j,k}|\ln(j+2) \\
&\le 2\sum_{j=0}^{m}\sum_{k=n}^{\infty}(j+1)(k+1)|\Delta_{22}q^d_{j,k}|\ln(j+2)\ln(k+2).
\end{aligned}
$$

Now, the right side of last equality tends to zero, since $\{q^d_{j,k}\} \in \mathbf{S}_{jk}$. Consequently,

$$
\sum_{j=0}^{m}(j+1)|\Delta_{20}q^d_{j,n}|\ln(j+2)\ln(n) \to 0
$$

independent on m as $n \to \infty$.

(ii) The proof can be done in the same lines as the proof of (i).

(iii) *Case 1.* When $s=1$ and $r=1$ we have

$$
\begin{aligned}
\Delta_{11}q^d_{m,n}\ln(m)\ln(n) &= \ln(m)\ln(n)\sum_{j=m}^{\infty}\Delta_{21}q^d_{j,n} \\
&= \ln(m)\ln(n)\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}\Delta_{22}q^d_{j,k} \\
&\le \frac{\ln(m)\ln(n)}{mn}\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}(j+1)(k+1)|\Delta_{22}q^d_{j,k}|.
\end{aligned}
$$

So,

$$
mn\Delta_{11}q^d_{m,n}\ln(m)\ln(n) \le \ln(m)\ln(n)\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}(j+1)(k+1)|\Delta_{22}q^d_{j,k}| \to 0
$$

as $\min(m,n) \to \infty$.

Case 2. When $s=1, r=0$ or $s=0, r=1$, then it is clear that

$$
q^d_{m,n}\ln(m)\ln(n) = \ln(m)\ln(n)\sum_{j=m}^{\infty}\Delta_{10}q^d_{j,n} = \ln(m)\ln(n)\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}\Delta_{11}q^d_{j,k}.
$$

Performing summation by parts we have

$$q^d_{m,n}\ln(m)\ln(n) \le \ln(m)\ln(n)\left[\sum_{j=m}^{\infty}\sum_{k=n}^{\infty} k|\Delta_{12}q^d_{j,k}| + \sum_{j=m}^{\infty} n|\Delta_{11}q^d_{j,n}|\right]$$
$$= \ln(m)\ln(n)\left[\sum_{j=m}^{\infty}\sum_{k=n}^{\infty} k|\Delta_{12}q^d_{j,k}| + n\sum_{j=m}^{\infty}\sum_{k=n}^{\infty} |\Delta_{12}q^d_{j,n}|\right].$$

In last two series we apply summation by parts again in order to obtain

$$q^d_{m,n}\ln(m)\ln(n) \le \ln(m)\ln(n)\Bigg[\sum_{j=m}^{\infty}\left|\sum_{k=n}^{\infty} jk\Delta_{22}q^d_{j,k} - mk\Delta_{12}q^d_{j,k}\right|$$
$$+n\sum_{j=m}^{\infty}\left|\sum_{k=n}^{\infty} j\Delta_{22}q^d_{j,k} - m\Delta_{12}q^d_{m,k}\right|\Bigg]$$
$$\le 4\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}(j+1)(k+1)|\Delta_{22}q^d_{j,k}|\ln(j+2)\ln(k+2) \to 0$$

as $\min(m,n)\to\infty$.

The proof is completed.

Now we prove the main statement of this unit.

Theorem 7.7. *If a double sequence $\{q^d_{j,k}\} \in \mathbf{S}_{jk}$, then for $0 < x, y < \pi$ the following assertions hold true:*

(i) The limit $\lim_{m,n\to\infty} f^d_{m,n}(x,y) = f^d(x,y)$ exists,
(ii) $f^d(x,y) \in L^1(Q)$, and
(iii) $\|f^d_{m,n} - f^d\| \to 0$ as $\min(m,n)\to\infty$.

Proof. (i) The modified sums

$$f^d_{m,n}(x,y) = \frac{1}{4\sin x\sin y}\sum_{j=1}^{m}\sum_{k=1}^{n}$$
$$\times\sum_{r=j}^{m}\sum_{\ell=k}^{n}\Delta_{1,1}\left[\left(\Delta_{1,1}\left(q^d_{r-1,\ell-1}\right)\sin rx\sin \ell y\right)\right],$$

which can rewritten as follows

$$f^d_{m,n}(x,y) = \frac{1}{4\sin x\sin y}$$
$$\times\Bigg[\sum_{j=1}^{m}\sum_{k=1}^{n}\left(q^d_{j-1,k-1} - q^d_{j+1,k-1} - q^d_{j-1,k+1} + q^d_{j+1,k+1}\right)\sin jx\sin ky$$
$$-\sum_{j=1}^{m}\sum_{k=1}^{n}\left(q^d_{j-1,n} - q^d_{j+1,n} - q^d_{j-1,n+2} + q^d_{j+1,n+2}\right)\sin jx\sin(n+1)y$$

$$- \sum_{j=1}^{m} \sum_{k=1}^{n} \left(q^d_{m,k-1} - q^d_{m+2,k-1} - q^d_{m,k+1} + q^d_{m+2,k+1}\right) \sin(m+1)x \sin ky$$

$$+ \sum_{j=1}^{m} \sum_{k=1}^{n} \left(q^d_{m,n} - q^d_{m+2,n} - q^d_{m,n+2} + q^d_{m+2,n+2}\right) \sin(m+1)x \sin(n+1)y \Bigg]$$

$$:= I_1 - I_2 - I_3 + I_4.$$

Moreover, the quantities I_1, I_2, I_3 and I_4 can be written as follows

$$\begin{aligned}
I_1 =& \frac{1}{4\sin x \sin y}\Bigg[\sum_{j=1}^{m}\sum_{k=1}^{n}(\sin(k+1)y - \sin(k-1)y)q^d_{j-1,k}\sin jx \\
&- \sum_{j=1}^{m}\sum_{k=1}^{n}(\sin(k+1)y - \sin(k-1)y)q^d_{j+1,k}\sin jx \\
&- \sum_{j=1}^{m}(q^d_{j-1,n} - q^d_{j+1,n})\sin jx \sin(n+1)y \\
&- \sum_{j=1}^{m}(q^d_{j-1,n+1} - q^d_{j+1,n+1})\sin jx \sin ny\Bigg] \\
=& \frac{1}{2\sin x}\sum_{j=1}^{m}\sum_{k=1}^{n}(q^d_{j-1,k} - q^d_{j+1,k})\sin jx \cos ky \\
&- \frac{\sin(n+1)y}{4\sin x \sin y}\Bigg[\sum_{j=1}^{m}(\sin(j+1)x - \sin(j-1)x)q^d_{j,n} \\
&\qquad - q^d_{m,n}\sin(m+1)x - q^d_{m+1,n}\sin mx\Bigg] \\
&- \frac{\sin ny}{4\sin x \sin y}\Bigg[\sum_{j=1}^{m}(\sin(j+1)x - \sin(j-1)x)q^d_{j,n+1} \\
&\qquad - q^d_{m,n+1}\sin(m+1)x - q^d_{m+1,n+1}\sin mx\Bigg] \\
=& \frac{1}{2\sin x}\Bigg[\sum_{j=1}^{m}\sum_{k=1}^{n}(\sin(j+1)x - \sin(j-1)x)q^d_{j,k}\sin jx \cos ky \\
&- \sum_{k=1}^{n} q^d_{m,k}\sin(m+1)x\cos ky - \sum_{k=1}^{n} q^d_{m+1,k}\sin mx \cos ky\Bigg] \\
&- \frac{\sin(n+1)y}{4\sin x \sin y}\Bigg[\sum_{j=1}^{m}(\sin(j+1)x - \sin(j-1)x)q^d_{j,n}
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\left. -q^d_{m,n}\sin(m+1)x - q^d_{m+1,n}\sin mx\right] \\
&-\frac{\sin ny}{4\sin x\sin y}\left[\sum_{j=1}^{m}(\sin(j+1)x-\sin(j-1)x)q^d_{j,n+1}\right. \\
&\qquad\qquad\left. -q^d_{m,n+1}\sin(m+1)x - q^d_{m+1,n+1}\sin mx\right] \\
=&\sum_{j=1}^{m}\sum_{k=1}^{n} q^d_{j,k}\cos jx\cos ky \\
&-\frac{\sin(m+1)x}{2\sin x}\sum_{k=1}^{n} q^d_{m,k}\cos ky - \frac{\sin mx}{2\sin x}\sum_{k=1}^{n} q^d_{m+1,k}\cos ky \\
&-\frac{\sin(n+1)y}{2\sin y}\sum_{j=1}^{m} q^d_{j,n}\cos jx - \frac{\sin ny}{2\sin y}\sum_{j=1}^{m} q^d_{j,n+1}\cos jx \\
&+\frac{\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}q^d_{m,n} + \frac{\sin mx\sin(n+1)y}{4\sin x\sin y}q^d_{m+1,n} \\
&+\frac{\sin(m+1)x\sin ny}{4\sin x\sin y}q^d_{m,n+1} + \frac{\sin mx\sin ny}{4\sin x\sin y}q^d_{m+1,n+1},
\end{aligned}
$$

$$
\begin{aligned}
I_2 =& \frac{n\sin(n+1)y}{4\sin x\sin y}\left[\sum_{j=1}^{m}(\sin(j+1)x-\sin(j-1)x)q^d_{j,n}\right. \\
&-\sin(m+1)xq^d_{m,n} - \sin mxq^d_{m+1,n} \\
&-\sum_{j=1}^{m}(\sin(j+1)x-\sin(j-1)x)q^d_{j,n+2} \\
&\left.+\sin(m+1)xq^d_{m,n+2} + \sin mxq^d_{m+2,n+2}\right] \\
=& \frac{n\sin(n+1)y}{4\sin x\sin y}\sum_{j=1}^{m}(q^d_{j,n}-q^d_{j,n+2})\cos jx \\
&-\frac{n\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m,n+2}) \\
&-\frac{n\sin mx\sin(n+1)y}{4\sin x\sin y}(q^d_{m+1,n}-q^d_{m+1,n+2}),
\end{aligned}
$$

$$
I_3 = \frac{m\sin(m+1)y}{4\sin x\sin y}\sum_{k=1}^{n}(q^d_{m,k}-q^d_{m+2,k})\cos ky
$$

$$-\frac{m\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m+2,n})$$
$$-\frac{m\sin(m+1)x\sin ny}{4\sin x\sin y}(q^d_{m,n+1}-q^d_{m+2,n+1}),$$

and finally

$$I_4=\frac{mn\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m+2,n}-q^d_{m,n+2}+q^d_{m+2,n+2}).$$

Combining all the terms of I_1, I_2, I_3, and I_4 we obtain

$$\begin{aligned} f^d_{m,n}(x,y) = {} & S^d_{m,n}(x,y) \\ & -\frac{\sin(m+1)x}{2\sin x}\sum_{k=1}^{n} q^d_{m,k}\cos ky-\frac{\sin mx}{2\sin x}\sum_{k=1}^{n} q^d_{m+1,k}\cos ky \\ & -\frac{\sin(n+1)y}{2\sin y}\sum_{j=1}^{m} q^d_{j,n}\cos jx-\frac{\sin ny}{2\sin y}\sum_{j=1}^{m} q^d_{j,n+1}\cos jx \\ & +\frac{\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}q^d_{m,n}+\frac{\sin mx\sin(n+1)y}{4\sin x\sin y}q^d_{m+1,n} \\ & +\frac{\sin(m+1)x\sin ny}{4\sin x\sin y}q^d_{m,n+1}+\frac{\sin mx\sin ny}{4\sin x\sin y}q^d_{m+1,n+1} \\ & -\frac{n\sin(n+1)y}{4\sin x\sin y}\sum_{j=1}^{m}(q^d_{j,n}-q^d_{j,n+2})\cos jx \\ & +\frac{n\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m,n+2}) \\ & +\frac{n\sin mx\sin(n+1)y}{4\sin x\sin y}(q^d_{m+1,n}-q^d_{m+1,n+2}) \\ & -\frac{m\sin(m+1)y}{4\sin x\sin y}\sum_{k=1}^{n}(q^d_{m,k}-q^d_{m+2,k})\cos ky \\ & +\frac{m\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m+2,n}) \\ & +\frac{m\sin(m+1)x\sin ny}{4\sin x\sin y}(q^d_{m,n+1}-q^d_{m+2,n+1}) \\ & +\frac{mn\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m+2,n}-q^d_{m,n+2}+q^d_{m+2,n+2}). \end{aligned} \tag{7.15}$$

Taking into account that $\frac{\sin mx\sin ny}{\sin x\sin y}$ is bounded in $(0,\pi)\times(0,\pi)$, then we conclude that

$$\frac{\sin(m+1)x\sin ny}{4\sin x\sin y}q^d_{m,n+1}\to 0,\quad \frac{\sin mx\sin ny}{4\sin x\sin y}q^d_{m+1,n+1}\to 0,$$

$$\frac{\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}q^d_{m,n}\to 0,\quad \frac{\sin mx\sin(n+1)y}{4\sin x\sin y}q^d_{m+1,n}\to 0$$

as $\min(m,n)\to\infty$.

Applying twice the summation by parts we get

$$\begin{aligned}\frac{\sin(m+1)x}{2\sin x}\sum_{k=1}^{n}q^d_{m,k}\cos ky &= \frac{\sin(m+1)x}{2\sin x}\Bigg[\sum_{k=1}^{n}(k+1)\Delta_{02}q^d_{m,k}F_k(y)\\ &\quad +(n+1)\Delta_{01}q^d_{m,n+1}F_n(y)-q^d_{m,n+1}D_n(y)\Bigg]\\ &\le \frac{|\sin(m+1)x|}{2|\sin x|}\Bigg[\sum_{k=1}^{n}(k+1)|\Delta_{02}q^d_{m,k}||F_k(y)|\\ &\quad +(n+1)|\Delta_{01}q^d_{m,n+1}||F_n(y)|+|q^d_{m,n+1}||D_n(y)|\Bigg],\end{aligned}$$

where $F_n(y)$ and $D_n(y)$ are bounded on $(0,\pi)$.

The use of Theorem 7.6 implies that all the above terms on the right side of last inequality tend to zero. Thus

$$\frac{\sin(m+1)x}{2\sin x}\sum_{k=1}^{n}q^d_{m,k}\cos ky\to 0\quad as\quad \min(m,n)\to\infty.$$

Similarly, has been shown that

$$\frac{\sin mx}{2\sin x}\sum_{k=1}^{n}q^d_{m+1,k}\cos ky\to 0,\quad \frac{\sin(n+1)y}{2\sin y}\sum_{j=1}^{m}q^d_{j,n}\cos jx\to 0,$$

and

$$\frac{\sin ny}{2\sin y}\sum_{j=1}^{m}q^d_{j,n+1}\cos jx\to 0\quad as\quad \min(m,n)\to\infty.$$

Further, putting $\gamma_n(x):=\frac{n\sin(n+1)y}{4\sin x\sin y}$, and applying twice the summation by parts to the equality

$$\gamma_n(x)\sum_{j=1}^{m}(q^d_{j,n}-q^d_{j,n+2})\cos jx=\gamma_n(x)\sum_{j=1}^{m}(\Delta_{01}q^d_{j,n}+\Delta_{01}q^d_{j,n+1})\cos jx$$

we have

$$\begin{aligned}\gamma_n(x)\sum_{j=1}^{m}(q^d_{j,n}-q^d_{j,n+2})\cos jx &= \gamma_n(x)\Bigg[\sum_{j=1}^{m}(j+1)(\Delta_{21}q^d_{j,n}+\Delta_{21}q^d_{j,n+1})F_j(x)\\ &\quad +(m+1)(\Delta_{11}q^d_{m+1,n}+\Delta_{11}q^d_{m+1,n+1})F_m(x)\end{aligned}$$

$$
\begin{aligned}
&+(\Delta_{01}q^d_{m+1,n}+\Delta_{01}q^d_{m+1,n+1})D_m(x)\Bigg] \\
&\le C\Bigg[\sum_{j=1}^{m}\sum_{k=n}^{\infty}(j+1)(k+1)|\Delta_{22}q^d_{j,k}+\Delta_{22}q^d_{j,k+1}| \\
&+(m+1)n|\Delta_{11}q^d_{m+1,n}+\Delta_{11}q^d_{m+1,n+1}| \\
&+n|\Delta_{01}q^d_{m+1,n}+\Delta_{01}q^d_{m+1,n+1}|\Bigg]\to 0,
\end{aligned}
$$

as $\min(m,n)\to\infty$ (by given hypothesis).

In the same lines we have concluded that

$$\frac{m\sin(m+1)y}{4\sin x\sin y}\sum_{k=1}^{n}(q^d_{m,k}-q^d_{m+2,k})\cos ky\to 0$$

as $\min(m,n)\to\infty$.

Also, by Theorem 7.6, we have

$$
\begin{aligned}
&\frac{n\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m,n+2}) \\
&\qquad=\frac{n\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(\Delta_{01}q^d_{m,n}+\Delta_{01}q^d_{m,n+1})\to 0
\end{aligned}
$$

as $\min(m,n)\to\infty$.

Reasoning in the same way we get

$$\frac{n\sin mx\sin(n+1)y}{4\sin x\sin y}(q^d_{m+1,n}-q^d_{m+1,n+2})\to 0,$$

$$\frac{m\sin(m+1)y}{4\sin x\sin y}\sum_{k=1}^{n}(q^d_{m,k}-q^d_{m+2,k})\cos ky\to 0,$$

$$\frac{m\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m+2,n})\to 0,$$

and

$$
\begin{aligned}
&\frac{mn\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m+2,n}-q^d_{m,n+2}+q^d_{m+2,n+2}) \\
&\quad=\frac{mn\sin(m+1)x\sin(n+1)y}{4\sin x\sin y} \\
&\qquad\times(\Delta_{11}q^d_{m,n}+\Delta_{11}q^d_{m,n+1}+\Delta_{11}q^d_{m+1,n}+\Delta_{11}q^d_{m+1,n+1})\to 0
\end{aligned}
$$

as $\min(m,n)\to\infty$.

Now (considering $q^d_{0,k}=q^d_{j,0}=0$, $\forall j,k$), we apply double summation by parts to

$$S^d_{m,n}(x,y) = \sum_{j=0}^{m}\sum_{k=0}^{n} \lambda_j \lambda_k q^d_{j,k} \cos jx \cos ky,$$

which implies

$$\begin{aligned} S^d_{m,n}(x,y) &= \sum_{j=0}^{m}\sum_{k=0}^{n} \Delta_{11} q^d_{j,k} D_j(x) D_k(y) + \sum_{k=0}^{n} \Delta_{01} q^d_{m+1,k} D_m(x) D_k(y) \\ &\quad + \sum_{j=0}^{m} \Delta_{10} q^d_{j,n+1} D_j(x) D_n(y) + q^d_{m+1,n+1} D_m(x) D_n(y). \end{aligned}$$

Once again, we apply double summation by parts to the last equality to obtain

$$\begin{aligned} S^d_{m,n}(x,y) &= \sum_{j=0}^{m}\sum_{k=0}^{n} (j+1)(k+1) \Delta_{22} q^d_{j,k} F_j(x) F_k(y) \\ &\quad + \sum_{k=0}^{n} (k+1)(j+1) \Delta_{12} q^d_{m+1,k} F_m(x) F_k(y) \\ &\quad + \sum_{j=0}^{m} (j+1)(n+1) \Delta_{21} q^d_{j,n+1} F_j(x) F_n(y) \\ &\quad + (m+1)(n+1) \Delta_{11} q^d_{m+1,n+1} F_m(x) F_n(y) \\ &\quad + \sum_{k=0}^{n} (k+1) \Delta_{02} q^d_{m+1,k} D_m(x) F_k(y) + \sum_{j=0}^{m} (j+1) \Delta_{20} q^d_{j,n+1} F_j(x) D_n(y) \\ &\quad + (n+1) \Delta_{01} q^d_{m+1,n+1} D_m(x) F_n(y) + (m+1) \Delta_{01} q^d_{m+1,n+1} F_m(x) D_n(y) \\ &\quad + q^d_{m+1,n+1} D_m(x) D_n(y) \\ &\le \sum_{j=0}^{m}\sum_{k=0}^{n} (j+1)(k+1) |\Delta_{22} q^d_{j,k}| |F_j(x)| |F_k(y)| \\ &\quad + \sum_{k=0}^{n} (k+1)(j+1) |\Delta_{12} q^d_{m+1,k}| |F_m(x)| |F_k(y)| \\ &\quad + \sum_{j=0}^{m} (j+1)(n+1) |\Delta_{21} q^d_{j,n+1}| |F_j(x)| |F_n(y)| \\ &\quad + (m+1)(n+1) |\Delta_{11} q^d_{m+1,n+1}| |F_m(x)| |F_n(y)| \\ &\quad + \sum_{k=0}^{n} (k+1) |\Delta_{02} q^d_{m+1,k}| |D_m(x)| |F_k(y)| + \sum_{j=0}^{m} (j+1) |\Delta_{20} q^d_{j,n+1}| |F_j(x)| |D_n(y)| \\ &\quad + (n+1) |\Delta_{01} q^d_{m+1,n+1}| |D_m(x)| |F_n(y)| + (m+1) |\Delta_{01} q^d_{m+1,n+1}| |F_m(x)| |D_n(y)| \\ &\quad + |q^d_{m+1,n+1}| |D_m(x)| |D_n(y)|. \end{aligned}$$

From the last inequality and the given assumptions we conclude that $S^d_{m,n}(x,y)$ converges to $f^d(x,y)$ as $\min(m,n)\to\infty$. So,

$$f^d(x,y)=\lim_{\min(m,n)\to\infty} S^d_{m,n}(x,y)=\lim_{\min(m,n)\to\infty} f^d_{m,n}(x,y)$$

exists.

(ii) In the equality

$$\int_0^\pi\int_0^\pi \left|f^d_{m,n}(x,y)\right| dxdy=\int_0^\pi\int_0^\pi \left|\sum_{j=0}^m\sum_{k=0}^n q^d_{j,k}\cos jx\cos ky\right| dxdy$$

we apply the double summation by parts to obtain

$$\int_0^\pi\int_0^\pi \left|f^d_{m,n}(x,y)\right| dxdy=\int_0^\pi\int_0^\pi \left|\sum_{j=0}^m\sum_{k=0}^n (j+1)(k+1)\Delta_{22}q^d_{j,k}F_j(x)F_k(y)\right| dxdy.$$

Since $\frac{1}{\pi}\int_{-\pi}^{\pi}F_j(x)dx=1$, then by given assumption

$$\int_0^\pi\int_0^\pi \left|f^d_{m,n}(x,y)\right| dxdy=\mathcal{O}\left(\sum_{j=0}^m\sum_{k=0}^n (j+1)(k+1)\left|\Delta_{22}q^d_{j,k}\right|\right)<\infty.$$

(iii) We have to show that under given assumptions $\|f^d_{m,n}-f^d\|\to 0$ as $\min(m,n)\to\infty$. Indeed, we can write

$$\begin{aligned}\|f^d_{m,n}-f^d\| = \Bigg\| &\sum_{j=m+1}^\infty\sum_{k=n+1}^\infty q^d_{j,k}\cos jx\cos ky\\
&+\frac{\sin(m+1)x}{2\sin x}\sum_{k=1}^n q^d_{m,k}\cos ky+\frac{\sin mx}{2\sin x}\sum_{k=1}^n q^d_{m+1,k}\cos ky\\
&+\frac{\sin(n+1)y}{2\sin y}\sum_{j=1}^m q^d_{j,n}\cos jx+\frac{\sin ny}{2\sin y}\sum_{j=1}^m q^d_{j,n+1}\cos jx\\
&-\frac{\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}q^d_{m,n}-\frac{\sin mx\sin(n+1)y}{4\sin x\sin y}q^d_{m+1,n}\\
&-\frac{\sin(m+1)x\sin ny}{4\sin x\sin y}q^d_{m,n+1}-\frac{\sin mx\sin ny}{4\sin x\sin y}q^d_{m+1,n+1}\\
&+\frac{n\sin(n+1)y}{4\sin x\sin y}\sum_{j=1}^m (q^d_{j,n}-q^d_{j,n+2})\cos jx\\
&-\frac{n\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m,n+2})\end{aligned}$$

$$
\begin{aligned}
&-\frac{n\sin mx\sin(n+1)y}{4\sin x\sin y}(q^d_{m+1,n}-q^d_{m+1,n+2})\\
&+\frac{m\sin(m+1)y}{4\sin x\sin y}\sum_{k=1}^{n}(q^d_{m,k}-q^d_{m+2,k})\cos ky\\
&-\frac{m\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m+2,n})\\
&-\frac{m\sin(m+1)x\sin ny}{4\sin x\sin y}(q^d_{m,n+1}-q^d_{m+2,n+1})\\
&-\frac{mn\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m+2,n}-q^d_{m,n+2}+q^d_{m+2,n+2})\Bigg\|\\
\le\ &\left\|\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}q^d_{j,k}\cos jx\cos ky\right\|\\
&+\left\|\frac{\sin(m+1)x}{2\sin x}\sum_{k=1}^{n}q^d_{m,k}\cos ky\right\|+\left\|\frac{\sin mx}{2\sin x}\sum_{k=1}^{n}q^d_{m+1,k}\cos ky\right\|\\
&+\left\|\frac{\sin(n+1)y}{2\sin y}\sum_{j=1}^{m}q^d_{j,n}\cos jx\right\|+\left\|\frac{\sin ny}{2\sin y}\sum_{j=1}^{m}q^d_{j,n+1}\cos jx\right\|\\
&+\left\|\frac{\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}q^d_{m,n}\right\|+\left\|\frac{\sin mx\sin(n+1)y}{4\sin x\sin y}q^d_{m+1,n}\right\|\\
&+\left\|\frac{\sin(m+1)x\sin ny}{4\sin x\sin y}q^d_{m,n+1}\right\|+\left\|\frac{\sin mx\sin ny}{4\sin x\sin y}q^d_{m+1,n+1}\right\|\\
&+\left\|\frac{n\sin(n+1)y}{4\sin x\sin y}\sum_{j=1}^{m}(q^d_{j,n}-q^d_{j,n+2})\cos jx\right\|\\
&+\left\|\frac{n\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m,n+2})\right\|\\
&+\left\|\frac{n\sin mx\sin(n+1)y}{4\sin x\sin y}(q^d_{m+1,n}-q^d_{m+1,n+2})\right\|\\
&+\left\|\frac{m\sin(m+1)y}{4\sin x\sin y}\sum_{k=1}^{n}(q^d_{m,k}-q^d_{m+2,k})\cos ky\right\|\\
&+\left\|\frac{m\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m+2,n})\right\|\\
&+\left\|\frac{m\sin(m+1)x\sin ny}{4\sin x\sin y}(q^d_{m,n+1}-q^d_{m+2,n+1})\right\|
\end{aligned}
$$

$$+\left\|\frac{mn\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m+2,n}-q^d_{m,n+2}+q^d_{m+2,n+2})\right\|$$
$$:=\sum_{s=1}^{16}\mathbf{I}_s.$$

Now, we have

$$\mathbf{I}_6=\left\|\frac{\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}q^d_{m,n}\right\|=\mathcal{O}(|q^d_{m,n}|\ln m\ln n)\to 0$$

as $\min(m,n)\to\infty$.

Similarly, we get

$$\mathbf{I}_7=\left\|\frac{\sin mx\sin(n+1)y}{4\sin x\sin y}q^d_{m+1,n}\right\|\to 0,$$

$$\mathbf{I}_8=\left\|\frac{\sin(m+1)x\sin ny}{4\sin x\sin y}q^d_{m,n+1}\right\|\to 0,$$

and

$$\mathbf{I}_9=\left\|\frac{\sin mx\sin ny}{4\sin x\sin y}q^d_{m+1,n+1}\right\|\to 0$$

as $\min(m,n)\to\infty$.

Moreover,

$$\begin{aligned}\mathbf{I}_{11}&=\mathcal{O}\left(|n(\Delta_{01}q^d_{m,n}+\Delta_{01}q^d_{m,n+1})|\ln m\ln n\right)\\&=\mathcal{O}\left(|n(\Delta_{01}q^d_{m,n}|+|\Delta_{01}q^d_{m,n+1})|\ln m\ln n\right)\to 0\end{aligned}$$

as $\min(m,n)\to\infty$.

In the same way we have verified that

$$\mathbf{I}_{12}=\left\|\frac{n\sin mx\sin(n+1)y}{4\sin x\sin y}(q^d_{m+1,n}-q^d_{m+1,n+2})\right\|\to 0,$$

$$\mathbf{I}_{14}=\left\|\frac{m\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m+2,n})\right\|\to 0,$$

and

$$\mathbf{I}_{15}=\left\|\frac{m\sin(m+1)x\sin ny}{4\sin x\sin y}(q^d_{m,n+1}-q^d_{m+2,n+1})\right\|\to 0$$

as $\min(m,n)\to\infty$.

For $\mathbf{I}_{16}$ we have

$$\mathbf{I}_{16} = \mathcal{O}\left(|mn(\Delta_{11}q^d_{m,n} + \Delta_{11}q^d_{m,n+1} + \Delta_{11}q^d_{m+1,n} + \Delta_{11}q^d_{m+1,n+1})| \ln m \ln n\right) \to 0$$

as $\min(m,n) \to \infty$.

Applying the summation by parts to $\mathbf{I}_2$ we obtain

$$\begin{aligned}
\mathbf{I}_2 &= \left\| \frac{\sin(m+1)x}{2\sin x} \left[\sum_{k=1}^{n} \Delta_{01}q^d_{m,k} D_k(y) + q^d_{m,n+1} D_n(y) \right] \right\| \\
&= \left\| \frac{\sin(m+1)x}{2\sin x} \left[\sum_{k=1}^{n} (k+1)\Delta_{02}q^d_{m,k} F_k(y) + (n+1)\Delta_{01}q^d_{m,n+1} F_n(y) + q^d_{m,n+1} D_n(y) \right] \right\| \\
&= \mathcal{O}\left(\ln m \sum_{k=1}^{n} (k+1)|\Delta_{02}q^d_{m,k}| + (n+1)\ln m |\Delta_{01}q^d_{m,n+1}| + |q^d_{m,n+1}| \ln m \ln n \right).
\end{aligned}$$

Consequently, the use of Theorem 7.6 implies

$$\mathbf{I}_2 = \left\| \frac{\sin(m+1)x}{2\sin x} \sum_{k=1}^{n} q^d_{m,k} \cos ky \right\| \to 0$$

as $\min(m,n) \to \infty$.

Similarly, we have

$$\mathbf{I}_3 = \left\| \frac{\sin mx}{2\sin x} \sum_{k=1}^{n} q^d_{m+1,k} \cos ky \right\| \to 0,$$

$$\mathbf{I}_4 = \left\| \frac{\sin(n+1)y}{2\sin y} \sum_{j=1}^{m} q^d_{j,n} \cos jx \right\| \to 0,$$

and

$$\mathbf{I}_5 = \left\| \frac{\sin ny}{2\sin y} \sum_{j=1}^{m} q^d_{j,n+1} \cos jx \right\| \to 0$$

as $\min(m,n) \to \infty$.

Now, for $\mathbf{I}_{10}$ can be written

$$\begin{aligned}
\mathbf{I}_{10} &= \mathcal{O}(n \ln n) \int_0^\pi \left| \sum_{j=1}^{m} (q^d_{j,n} - q^d_{j,n+2}) \cos jx \right| dx \\
&= \mathcal{O}(\ln n) \int_0^\pi \left| \sum_{j=1}^{\infty} \sum_{k=m}^{\infty} k(\Delta_{01}q^d_{j,k} - \Delta_{01}q^d_{j,k+2}) \cos jx \right| dx,
\end{aligned}$$

which, after using the summation by parts, takes the following form

$$\begin{aligned}
\mathbf{I}_{10} &= \mathcal{O}(\ln n) \int_0^\pi \left| \sum_{j=1}^{\infty} \sum_{k=m}^{\infty} k(\Delta_{11}q^d_{j,k} - \Delta_{11}q^d_{j,k+2}) D_j(x) \right| dx \\
&= \mathcal{O}(\ln n) \int_0^\pi \left| \sum_{j=1}^{\infty} \sum_{k=m}^{\infty} k(\Delta_{12}q^d_{j,k} + \Delta_{12}q^d_{j,k+2}) D_j(x) \right| dx.
\end{aligned}$$

Applying summation by parts, once again, we get

$$\begin{aligned}
\mathbf{I}_{10} &= \mathcal{O}(\ln n)\int_0^\pi \left|\sum_{j=1}^\infty\sum_{k=m}^\infty (j+1)k(\Delta_{22}q^d_{j,k}+\Delta_{22}q^d_{j,k+1})F_j(x)\right|dx\\
&= \mathcal{O}\left(\sum_{j=1}^\infty\sum_{k=m}^\infty (j+1)k\left|\Delta_{22}q^d_{j,k}+\Delta_{22}q^d_{j,k+1}\right|\ln(k+2)\right)\\
&= \mathcal{O}\left(\sum_{j=1}^\infty\sum_{k=m}^\infty (j+1)(k+1)\left|\Delta_{22}q^d_{j,k}\right|\ln(k+2)\right)\\
&\quad +\mathcal{O}\left(\sum_{j=1}^\infty\sum_{k=m}^\infty (j+1)(k+1)\left|\Delta_{22}q^d_{j,k+1}\right|\ln(k+2)\right)\to 0
\end{aligned}$$

as $\min(m,n)\to\infty$.

Similarly, we have verified that

$$\mathbf{I}_{13} = \left\|\frac{m\sin(m+1)y}{4\sin x\sin y}\sum_{k=1}^n (q^d_{m,k}-q^d_{m+2,k})\cos ky\right\|\to 0$$

as $\min(m,n)\to\infty$.

Finally, we have to show that

$$\mathbf{I}_1 = \left\|\sum_{j=m+1}^\infty\sum_{k=n+1}^\infty q^d_{j,k}\cos jx\cos ky\right\|\to 0$$

as $\min(m,n)\to\infty$.

Namely, after performing double summation by parts twice, we have

$$\begin{aligned}
\mathbf{I}_1 &= \left\|\sum_{j=m+1}^\infty\sum_{k=n+1}^\infty \Delta_{11}q^d_{j,k}D_j(x)D_k(y) - \sum_{k=n+1}^\infty \Delta_{01}q^d_{m+1,k}D_m(x)D_k(y)\right.\\
&\quad \left. - \sum_{j=m+1}^\infty \Delta_{10}q^d_{j,n+1}D_j(x)D_n(y) + q^d_{m+1,n+1}D_m(x)D_n(y)\right\|\\
&\le \left\|\sum_{j=m+1}^\infty\sum_{k=n+1}^\infty (j+1)(k+1)\Delta_{22}q^d_{j,k}F_j(x)F_k(y)\right\|\\
&\quad + \left\|\sum_{k=n+1}^\infty (m+1)(k+1)\Delta_{12}q^d_{m+1,k}F_m(x)F_k(y)\right\|\\
&\quad + \left\|\sum_{j=m+1}^\infty (j+1)(n+1)\Delta_{21}q^d_{j,n+1}F_j(x)F_n(y)\right\|
\end{aligned}$$

$$
\begin{aligned}
&+\left\|(m+1)(n+1)\Delta_{11}q^d_{m+1,n+1}F_m(x)F_n(y)\right\| \\
&+\left\|\sum_{k=n+1}^{\infty}(k+1)\Delta_{02}q^d_{m+1,k}D_m(x)F_k(y)\right\| \\
&+\left\|\sum_{j=m+1}^{\infty}(j+1)\Delta_{20}q^d_{j,n+1}F_j(x)D_n(y)\right\| \\
&+\left\|(n+1)\Delta_{01}q^d_{m+1,n+1}D_m(x)F_n(y)\right\| \\
&+\left\|(m+1)\Delta_{10}q^d_{m+1,n+1}F_m(x)D_n(y)\right\| \\
&+\left\|q^d_{m+1,n+1}D_m(x)D_n(y)\right\| \\
=\ &\mathcal{O}\Bigg(\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}(j+1)(k+1)|\Delta_{22}q^d_{j,k}| + (m+1)(n+1)|\Delta_{11}q^d_{m+1,n+1}| \\
&+\sum_{k=n+1}^{\infty}(k+1)|\Delta_{02}q^d_{m+1,k}|\ln m + \sum_{j=m+1}^{\infty}(j+1)|\Delta_{20}q^d_{j,n+1}|\ln n \\
&+(n+1)|\Delta_{01}q^d_{m+1,n+1}|\ln m + (m+1)|\Delta_{10}q^d_{m+1,n+1}|\ln n\Bigg) \\
&+|q^d_{m+1,n+1}|\ln m\ln n \to 0
\end{aligned}
$$

by given assumptions and when $\min(m,n)\to\infty$.

As a conclusion, combining all the above terms, we obtain $\|f^d_{m,n}-f^d\|\to 0$ as $\min(m,n)\to\infty$.

The proof is completed.

Remark 7.8. As we know that if a double trigonometric series converges in L^1-norm to a function $f^d\in L^1(Q)$, then it is a Fourier series of the function f^d. Consequently, according to Theorem 7.7 the series (7.12) is the Fourier series of f^d.

Corollary 7.9. *If a double sequence $\{q^d_{j,k}\}$ belongs to the class $\mathbf{S}_{jk}$, then $\|S^d_{m,n}-f^d\|\to 0$ as $\min(m,n)\to\infty$.*

Proof. It is clear that

$$
\begin{aligned}
\|f^d-S^d_{m,n}\| &= \|f^d-f^d_{m,n}+f^d_{m,n}-S^d_{m,n}\| \\
&\le \|f^d-f^d_{m,n}\|+\|f^d_{m,n}-S^d_{m,n}\| = \|f^d-f^d_{m,n}\| \\
&\quad+\left\|\frac{\sin(n+1)y}{2\sin y}\sum_{j=1}^{m}q^d_{j,n}\cos jx\right\| + \left\|\frac{\sin ny}{2\sin y}\sum_{j=1}^{m}q^d_{j,n+1}\cos jx\right\|
\end{aligned}
$$

$$
\begin{aligned}
&+\left\|\frac{\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}q^d_{m,n}\right\|+\left\|\frac{\sin mx\sin(n+1)y}{4\sin x\sin y}q^d_{m+1,n}\right\| \\
&+\left\|\frac{\sin(m+1)x\sin ny}{4\sin x\sin y}q^d_{m,n+1}\right\|+\left\|\frac{\sin mx\sin ny}{4\sin x\sin y}q^d_{m+1,n+1}\right\| \\
&+\left\|\frac{n\sin(n+1)y}{4\sin x\sin y}\sum_{j=1}^{m}(q^d_{j,n}-q^d_{j,n+2})\cos jx\right\| \\
&+\left\|\frac{n\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m,n+2})\right\| \\
&+\left\|\frac{n\sin mx\sin(n+1)y}{4\sin x\sin y}(q^d_{m+1,n}-q^d_{m+1,n+2})\right\| \\
&+\left\|\frac{m\sin(m+1)y}{4\sin x\sin y}\sum_{k=1}^{n}(q^d_{m,k}-q^d_{m+2,k})\cos ky\right\| \\
&+\left\|\frac{m\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m+2,n})\right\| \\
&+\left\|\frac{m\sin(m+1)x\sin ny}{4\sin x\sin y}(q^d_{m,n+1}-q^d_{m+2,n+1})\right\| \\
&+\left\|\frac{mn\sin(m+1)x\sin(n+1)y}{4\sin x\sin y}(q^d_{m,n}-q^d_{m+2,n}-q^d_{m,n+2}+q^d_{m+2,n+2})\right\|
\end{aligned}
$$

According to Theorem 7.7, all terms to the right side of last inequality tend to zero as $\min(m,n)\to\infty$. Subsequently, the conclusion of this corollary follows.

The proof is completed.

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