A dictionary of inequalities

Supplement

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PURPOSE

The intention of this website is to allow the easy correction of and addition to:

A Dictionary of Inequalities, author *P S Bullen*;

AMS Classification 26Dxx
ISSN 0269-3666
ISBN 0 582 32748 2.

It was published in 1998 as #97 of the *Pitman Monographs and Surveys in Pure and Applied Mathematics*.
The publisher is *CRC Press LLC, 2000 NW Corporate Boulevard, Boca Raton, Florida, USA 33431*.

Reviews of this book can be found in:
Mathematical Reviews 2000e:26001,
Zentralblatt 934.26003.

PREFACE

It is in the nature of a book such as *A Dictionary of Inequalities* to have errors and omissions.
It is hoped that readers who note either will write to the author at:

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or send an e-mail to

bullen@math.ubc.ca.

All errors and omissions received, as well as errors, omissions and new inequalities, noted by the author will be posted on this site in a regular manner.
The format and notations used will follow those of the original book, but errors will be noted in red, running comment not meant to be part of the text in green. Additions and new inequalities will of course be in black, as will be the entry headings unless they are themselves errors. There are some inequalities that I cannot trace; these I have noted in purple, hoping someone will send me the reference. Finally there are hypertext links written in blue.

To avoid certain omissions being noted the following are extracts from the original Introduction.
" The . . . area of elementary geometric and trigonometric inequalities is omitted . . . Those interested are referred to. . . Geometric Inequalities, Bottema et al. . . . very few results from number theory are given. . . those interested are referred to . . . Inequalities in Number Theory, Mitrinović & Popadić."
" Most . . . inequalities . . . have often been put into an abstract form . . . this direction has not been pursued. Those interested. . . are referred to. . . Classical and New Inequalities in Analysis, Mitrinović et al, and Convex Functions, Partial Orderings and Statistical Applications, Pečarić et al."
In addition inequalities are usually discrete inequalities, with integral analogues being noted, although there are many exceptions to this.
Finally I wish to express my thanks to Professor Sever Dragomir and the whole of the staff of *Research Group in Mathematical Inequalities and Applications* for allowing this supplement to be posted on their site http://rgmia.vu.edu.au/, and for all the help in getting it into the proper format.

P.S.Bullen

Contents

Notations

3. Means

Add to the list in the first paragraph on page 4: **Heronian Mean Inequalities**

8. Probability and Statistics

Rewrite the last paragraph on page 12 as follows. This will result in changes in the entry **Error Function Inequalities**.

There are various functions involved with the *Gaussian, or normal distribution.*

$$E(x) = \frac{2}{\sqrt{\pi}}e^{-x^2},\ \operatorname{erf}(x) = \int_0^x E,\ \operatorname{erfc}(x) = \int_x^\infty E,\ \operatorname{mr}(x) = e^{x^2/2}\int_x^\infty e^{-t^2/2}\,\mathrm{d}t,$$

where $x \in \mathbb{R}$.

The functions $\operatorname{erf}, \operatorname{erfc}$ are the *error function* and the *complementary error function* respectively; and the last function is usually called the *Mills' Ratio*.

There are some simple relations between these functions:

$$\operatorname{erf}(0) = 0;\ \operatorname{erf}(\infty) = 1;\ \operatorname{erf} + \operatorname{erfc} = 1;\ \operatorname{erf}(-x) = -\operatorname{erf}(x);\ \operatorname{mr} = \frac{\sqrt{2}\operatorname{erfc}}{E}.$$

In addition there is the *normal distribution function.*

$$\Phi(x) = \frac{1}{2}\left(1 + \operatorname{erf}(x/\sqrt{2})\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}\,\mathrm{d}t.$$

For more details see [*CE, pp.561–563*]; [*EM Vol.7, pp.304–305*]; [*Abramowitz & Stegun pp.297–304*].

A

Aczél's Inequality

In the heading for this entry "Aczel "should be: "Aczél".

Add a sentence between the present two sentences in COMMENTS (I).

As this space is called a Lorentz space this inequality is sometimes called a *Lorentz inequality*.

Add the comment

(iii) The Aczél inequality was used to prove the **Aleksandrov-Fenchel Inequality**.

Aleksandrov[1]-Fenchel Inequality

Add the footnote.

Rewrite the entry as:

See **Permanent Inequalities** , **Mixed-volume Inequalities** EXTENSIONS.

Alzer's Inequalities

Add:

(d) If $a_1 \le \frac{a_2}{2} \le \cdots \le \frac{a_n}{n}$ then

$$A_n \le \frac{(n+1)a_n}{2}.$$

Add the comment:

(ii) Inequality (d) was used by Alzer in an very interesting extension of (C).

Change present COMMENTS (III) to: COMMENTS (IV).

Add the reference:

[*Alzer 1992*].

[1] А Д Александров. Also transliterated as Aleksandroff.

B

Bernoulli's Inequality

Add to the list in comment (x): **Favard's Inequalities** COMMENTS (IV).

Bernšteĭn's Inequality

Add.

(c)[COMPLEX POLYNOMIAL CASE] *If $p_n(z)$ is a complex polynomial of degree at most n then*

$$||p_n'||_{\infty,|z|=1} \le n||p_n||_{\infty,|z|=1},$$

with equality if and only if all the zeros of p_n are at the origin.

Add the new comment:
(v) (c) should be compared with **Erdös's Inequality**.
Rename comment (v) as:(vi).
Add the reference:
[*Govil & Nyuydinkong*]

Beta Function Inequalities

Call the present entry, starting at "if. . . ":(a), and number it as :(1).
Then add:

(b) If m, n, p, q are positive real numbers with $p - m$ and $q - n$ the same sign then

$$B(p,q)B(m,n) \le B(p,n)B(m,q); \tag{2}$$

in particular

$$B(m,p) \ge \sqrt{B(p,p)B(m,m)}.$$

If $p - m$ and $q - n$ are of opposite sign then (~ 2) holds.
(c)The beta function is log-convex in both variables on $]0,\infty[\times]0,\infty[$; that is if $p, q, m, n > O$, and $0 \le t \le 1$, then

$$B(\overline{1-t}\,p + tm, \overline{1-t}\,q + tn) \le B(p,q)^{1-t}B(m,n)^t.$$

Rewrite COMMENTS (I) as follows.

COMMENTS (i) Inequality (1) is a deduction from **Stolarsky's Inequality**.

Add the comments:
(ii) Inequality (2) can be deduced from a weighted form of **Čebisev's Inequality** INTEGRAL ANALOGUE.
(iii) The log-convexity is a consequence of the integral analogue of (H).
Renumber the present COMMENTS (II) as:(iv)
Add the extension and comment:

EXTENSIONS *If* $\underline{m}$ *is an* $n-$*tuple,* $n \geq 2$*, with* $\Re m_k > 0, 1 \leq k \leq n$*, then the* generalized Beta function of order n *is*

$$B_n(\underline{m}) = \frac{(m_1-1)!\dots(m_n-1)!}{(m_1+\cdots+m_n-1)!}.$$

(a) If $\underline{u} < \underline{v}$ *then* $B_n(\underline{u}) > B_n(\underline{v})$.
(b) B_n *is log-convex; that is if* $0 \leq t \leq 1$ *then*

$$B_n\big((1-t)\underline{u} + t\underline{v}\big) \leq B_n^{(1-t)}(\underline{u})B_n^t(\underline{v}).$$

COMMENTS (v) Clearly $B_2(m,n) = B(m,n)$.
Add the following references:
[*Dedić, Matić & Pečarić 2000b*], [*Dragomir, Agarwal & Barnett*], [*Sasvári*].

Bieberbach's Conjecture

Add the reference to the following book:
[*Gong*].

Binomial Function Inequalities

Add the inequality:

(ℓ)[LYONS][1] *If* $p \geq 1$ *then*

$$\left(\sum_{i=1}^{n} x^{i/p}(1-x)^{(n-i)/p}\right)\left(\frac{(\frac{n}{p})!}{\left(\frac{i}{p}\right)!\left(\frac{n-i}{p}\right)!}\right) \leq p^2$$

Add the new comment:
(ii) The proof of (ℓ) is quite complicated and it is conjectured that the right-hand side can be replaced by p.
Renumber the comments (ii)—(vi) as: (iii)—(vii).

[1] This is T Lyons.

Add to comment (vii), old comment (vi): **Bernoulli's Inequality**.
Add the reference:
[*Liu*].

Brascamp-Lieb-Luttinger Inequality

See **Rearrangement Inequalities** INTEGRAL ANALOGUES .

C

Carlson's Inequality

Add the following reference to the entry on p.46:
[*Barza*]

Chi Inequality

If $1 \le a_1 < a_2 < \cdots$ *is an* A*-sequence then*

$$(m+1)a_m + a_n \ge n(m+1);$$

and

$$\sum_{n=1}^{\infty} \frac{1}{a_n} < 4.$$

COMMENTS (i) $1 \le a_1 < a_2 < \cdots$ is an A*-sequence* if no element is the sum of two or more distinct earlier elements.
(ii) The sup of all the sums in the second inequality is also bounded by 4, and is bounded below by 2.

REFERENCES [*CE, pp. 5, 240*].

Complementary Error Function Inequalities

See **Error Function Inequalities**

Complex Number Inequalities

The case of equality in (2) is wrong it should be the same as for (1).

Concentration Inequality

What is this? See **Kahane-Hinčin Inequality**.

Conte's Inequality

If $x > 0$ *then*

$$\left(x + \frac{x^2}{24} + \frac{x^3}{12}\right) e^{-3x^2/4} < e^{-x^2} \int_0^x e^{t^2}\, \mathrm{d}t \le \frac{\pi^2}{8x}\left(1 - e^{-x^2}\right).$$

COMMENTS (i) This is said to be related to the Mills' ratio.
(ii) The right-hand side has been refined in the reference.

References [*AI, p. 180*]; [*Qi, Cui & Xu*].

Convex Function Inequalities

Add to the list in comments(xii): **s- Convex Function Inequalities**.

Copson's Inequality

Add the reference:
[*CE, p. 330*].

Copula Inequalities

Add the references:
[*CE, p. 330*]; [*Nelsen*].

Counter Harmonic Mean Inequalities

Add to comments (iii): An extensive generalization has been given by Liu & Chen.
Add the reference:
[*Liu & Chen*].

Cutler-Olsen Inequality *If $E \subseteq \mathbb{R}^n$ is a Borel set then $\dim E \le \sup_\mu \underline{R}(\mu)$, where $\dim E$ is the Hausdorff dimension of E, $\underline{R}(\mu)$ is the lower Renyi dimension of the probability measure μ on E, and the sup is over all such μ.*

References [*Zindulka*].

D

de la Vallée Poussin's Inequality *If on the interval $[a,b]$ we have that*

$$y'' + gy' + fy = 0, \quad y(a) = y(b) = 0$$

then

$$1 < 2||g||_{\infty,[a,b]}(b-a) + ||f||_{\infty,[a,b]}\frac{(b-a)^2}{2}.$$

COMMENTS (i) This result has been extended by many writers.

REFERENCES [*Agarwal & Pang*], [*Mitrinović, Pečarić & Fink*]; [*Brown, Fink & Hinton*]

Determinant Inequalities

Add.

(e) If $C = (c_{ij})_{1\le i,j\le n} = (a_i^{r_j})_{1\le i,j\le n}$, where $\underline{a}, \underline{r}$ are strictly decreasing n-tuples, $\underline{a}$ positive, then $\det C > 0$.
Further if for some k, $1 < k \le n$, $c'_{ij} = c_{ij}$, $i \ne k-1, c'_{k-1j} = r_k c_{kj}$ and $C' = (c'_{ij})_{1\le i,j\le n}$ then $\det C' > 0$.

Add these comments and examples before the extensions.

COMMENTS (i) In (e) $\det C'$ is just $a_k \dfrac{\partial \det C}{\partial a_k}\Big|_{a_{k-1}=a_k}$.
In other words the $(k-1)$th column of C is replaced by the column $(r_j a_i^{r_j})_{1\le j\le n}$. This covers the case when $\underline{a}$ is strictly decreasing except that $a_{k-1} = a_k$. Further extensions to general decreasing positive n-tuples can easily be made.
(ii) The inequality (e) is due to Good but Ursell's proof allows for extension to general decreasing n-tuples.

EXAMPLES (i) If $a > b > 0$ then

$$\begin{vmatrix} \pi^2 a^\pi & \pi a^\pi & a^\pi & \pi b^\pi & b^\pi \\ e^2 a^e & e a^e & a^e i & e b^e & b^e \\ 4a^2 & 2a^2 & a^2 & 2b^2 & b^2 \\ 0 & 0 & 1 & 0 & 1 \\ 2a^{-\sqrt{2}} & -\sqrt{2}a^{-\sqrt{2}} & a^{-\sqrt{2}} & -\sqrt{2}b^{-\sqrt{2}} & b^{-\sqrt{2}} \end{vmatrix} > 0.$$

Renumber comments (i) and (ii) as: (iii), (iv).

Add the references:
[*Good*], [*Ursell*].

Digamma Function Inequalities

Change (a) and (b) to:(b), (c).
Add the following.

(a) The digamma function is increasing and concave on $]0,\infty[$, *that is if* $x, y, u, v > 0$ *with* $x < y$, *and if* $0 \le t \le 1$, *then*

$$\Psi(x) \le \Psi(y), \quad \Psi(\overline{1-t}\,u + tv) \ge (1-t)\Psi(u) + t\Psi(v),$$

respectively.

Add the reference:
[*Dragomir, Agarwal & Barnett*].

Distortion Theorems

Add the reference:
[*Gong*].

Duff's Inequality

Line 3 on page 74 has an extra |; it should read: $\int_0^1 H \circ |f'|$.

E

Elliptic Integral Inequalities

The integral

$$K(m) = \int_0^{\pi/2} \frac{1}{\sqrt{1-mt^2}}\,\mathrm{d}t$$

is called the *complete elliptic integral of the first kind*; the quantity $m = \sin^2\alpha$ is called the *parameter*, α the *modular angle*; in particular then $0 \le m \le 1$. The usage complete refers to the upper limit , or *amplitude*, being $\pi/2$; other amplitudes, or upper limits, give what is called the *incomplete elliptic integral of the first kind*. The elliptic integrals of the second and third kind are also defined. These integrals occur in consideration of the arc-length of ellipses, and surface areas of ellipsoids. In addition they are the basis for the definition of the Jacobian elliptic functions.

$$1 + \left(\frac{\pi}{\log 16} - 1\right)(1-m) < \frac{K(m)}{\log(4/\sqrt{1-m})} < 1 + \frac{1}{4}(1-m);$$

the constants $\dfrac{\pi}{\log 16}$ *and* $\dfrac{1}{4}$ *are sharp.*

References [*Abramowitz & Stegun, pp. 589–590*]; [*Alzer, 1998*].

Erdös's Inequality

Change "roots" to:" zeros" .

Call the present entry (a) and add the following:

(b) If $p_n(z)$ *is a complex polynomial that does not vanish in* D *then*

$$||p_n'||_{\infty,|z|=1} \le \frac{1}{2}n||p_n||_{\infty,|z|=1},$$

with equality if and only if $p_n(z) = \alpha + \beta z^n$, *where* $|\alpha| = |\beta|$.

Rewrite the comments as follows.

Comments (i) (a) should be compared with **Markov's Inequality** , and (b) with **Bernšteĭn's Inequality** (c)

(ii) Another inequality where the zeros are restricted, the polynomial being required to be zero at ± 1, has been given by Schur.

Add the following references.

[*Milovanović, Mitrinović & Rassias, p. 628*]; [*Govil & Nyuydinkong*], [*Nikolov*].

Erdös-Mordell Inequality *If P is interior to a triangle then the sum of its distances from the sides of the triangle is at most one half of the sum of its distances from the vertices of the triangle.*

EXTENSIONS *If P is interior to a convex n-gon then the sum of its distances from the sides of the polygon is at most $\cos \pi/n$ times the sum of its distances from the vertices of the polygon*

COMMENTS (i) The Erdös -Mordell inequality is the case $n = 3$ of the extension.
(ii) The extension is obtained using the **Fejes Tóth Inequality**.

REFERENCES [*Abi-Khuzam*].

Error Function Inequalities

The inequality (a) is wrong; replace by the following:

$$\sqrt{1 - e^{-ax^2}} \le \sqrt{\frac{2}{\pi}} \int_0^x e^{-t^2/2}\, dt \le \sqrt{1 - e^{-bx^2}} \iff 0 \le a \le 1/2, \text{ and } \quad b \ge 2/\pi.$$

See changes made in **Notations**.
Add the following inequalities— and then change (a), (b) to :(c), (d).

(a) $If x, y \ge 0,$

$$\operatorname{erf}(x) \operatorname{erf}(y) \ge \operatorname{erf}(x) + \operatorname{erf}(y) - \operatorname{erf}(x+y).$$

(b) *If $x > 0$,*

$$\operatorname{erfc} x < \min \left\{ e^{-x^2}, \frac{e^{-x^2}}{x\sqrt{\pi}} \right\}.$$

Call the present comment COMMENTS (I), and add the comment.
(ii) See also **Conte's Inequality**.
Add the reference:
[*Qi, Cui & Xu*].

Exponential Function Inequalities

Add at the end of the Real Inequalities.

(ℓ) *If $0 \le x \le y$ then*

$$1 \le \frac{e^{-x} - xe^{-1/x}}{1-x} \le \frac{3}{e},$$

where both bounds are best possible. Further the function in the centre is strictly increasing.
(m) *If $a > b > d$, $a > c > d$ and $s < t, s, t \ne 0$, then*

$$\frac{e^{as} - e^{bs}}{e^{cs} - e^{ds}} < \frac{e^{at} - e^{bt}}{e^{ct} - e^{dt}}.$$

Then add the comments before Complex Inequalities
(v) The inequality (ℓ) is due to Alzer and the final comment is in the second reference.
(vi) (m) is a consequence of the mean value theorem of differentiation.
Renumber COMMENTS (v) as : (vii)
Add a new extension.

(e) If $\alpha \geq 1$, $0 \leq \beta \leq e - 2$, and $-1 < x < 1$, then

$$\frac{e}{\alpha+2}\left(\alpha + \frac{2}{\sqrt{1-x^2}}\right) \leq \left(\frac{1+x}{1-x}\right)^{1/x} \leq \frac{e}{\beta+2}\left(\beta + \frac{2}{\sqrt{1-x^2}}\right)$$

Renumber comment (vi) as: (viii).
Add to the list in comments (ix): **Conte's Inequality**.
Add the references:
[*Good*], [*Janous*], [*Mond & Pečarić 2000*], [*Qi*].

Extended Mean Inequalities

Add at the beginning: Extended means are also known as *Stolarsky means*.

F

Factorial Function Inequalities

Remove (m) and place in a separate entry, [see below]: **Minc-Sarthre Inequality**.

As a result add to the list in comments (iii): **Minc-Sarthre Inequality** , and take out the reference: [*Alzer 1993*$^{(3)}$].

Add the following:

(n) $\big|z!\big| \le \big|\Re z!\big|$

(o) If m, n, p, q *are positive real numbers with* $p-m$ *and* $q-n$ *the same sign then*

$$(p+n)!(q+m)! \le (p+q)!(m+n)!.$$

The opposite inequality holds if $p-m$ *and* $q-n$ *are of opposite sign.*
In particular

$$(p+m)! \ge \sqrt{(2p)!(2m)!}.$$

(p) If $p > -1, |q| < p+1$ *then* $(p!)^2 \le (p-q)!(p+q)!$.

(q) $\log x!$ *is superadditive if* $x > 0$, *and log-convex if* $x > -1$; *that is if* $0 \le t \le 1$

$$\log(x+y)! \ge \log x! + \log y!, \qquad (\overline{1-t}\,x + ty)! \le (x!)^{1-t}(y!)^t,$$

respectively.

(r)

$$e\left(\frac{n}{e}\right) \le n! \le en\left(\frac{n}{e}\right)$$

(s) If $\alpha = 0.21609\ldots$, *with* $\underline{a}, \underline{w}$ *positive* n*-tuples,* $n \ge 2$, *with* $\underline{a} \le \alpha$, *and* $W_n = 1$ *then*

$$(\alpha!)^{\min \underline{w}} \le \frac{\mathfrak{G}_n\big((\underline{a}-1)!;\underline{w}\big)}{\big(\mathfrak{G}_n(\underline{a};\underline{w})-1\big)!} \le 1$$

Add the following comments:

(iii) Inequalities (o) and (p) can be deduced from a weighted form of **Čebisev's Inequality** INTEGRAL ANALOGUE.

(iv) Inequality (q) follows form the integral analogue of (H).

(v) (r) is a very simple form of the inequality in **Stirling's Formula**.

(vi) The right-hand side of (s) is due to Lucht and the left-hand side to Alzer.

Renumber comments (iii) as:(vii).
Add the following references.
[*Cloud & Drachman, p. 77*], [*Klambauer, p. 410*]; [*Alzer 2000*], [*Dragomir, Agarwal & Barnett*], [*Elezović, Giordano & Pečarić*].

Favard's Inequalities

Call the present extension: (a); then add the new extension:

(b) If f is a non-negative concave function on $[0,1]$ and $-1 < r \le s$, then

$$(r+1)^{1/r}||f||_r \ge (s+1)^{1/s}||f||_s.$$

Add the comment:
(iv) Extension (b) has been used to extend (B):

$$1 + \alpha x \le \left(\frac{||f||_x}{||f||_{\alpha x}}\right)^{\alpha x}(1+x)^{\alpha},$$

where f is as in (b), and x, α are as in (B). If the conditions of (∼B) hold then this last inequality is reversed.
Add the reference:
[*Alzer 1991*a].

Fejér-Jackson Inequality

Add to the list in comment (iii): **Sine Integral Inequalities**.

Fejes Tóth Inequality *If $x_i, \delta_i,\ 1 \le i \le n$, are two sets of positive numbers, and if $\sum_{i=1}^{n} \delta_i = \pi$,, then*

$$\sum_{i=1}^{n} x_i x_{i+1} \cos\delta_i \le \cos\frac{\pi}{n}\sum_{i=1}^{n} x_i^2$$

COMMENTS (i) This was conjectured by Fejes Tóth and proved for general n by Lenhard. where $x_{n+1} = x_1$.
(ii) The interest of this inequality is that it gave an extension of the **Erdös-Mordell Inequality**.

REFERENCES [*Abi-Khuzam*]

Fibonacci Number Inequalities

The elements of sequence $F_1, F_2, F_3, \ldots$ of positive integers defined by

$$F_1 = F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2},\ n \ge 3,$$

are called *Fibonacci numbers*.

If n is even

$$F_n < \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^n < F_n + \frac{1}{3};$$

if n is odd

$$F_n - \frac{1}{3} < \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^n < F_n.$$

COMMENTS (i) More precisely we have *Binet's formula*; for all $n \geq 1$,

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{\sqrt{5}+1}{2}\right)^n - (-1)^n\left(\frac{\sqrt{5}-1}{2}\right)^n\right). \tag{1}$$

(ii) It follows from the above results that F_n is the closest integer to $\left(\dfrac{1+\sqrt{5}}{2}\right)^n$.

(iii) The second fraction inside the brackets in (1) is called the *Golden mean, or section*; the first is its reciprocal.

REFERENCES [*EM, vol.4, pp. 1–3*], [*Körner, pp. 250–257*].

Function Inequalities

Add to to the list in COMMENTS (II): **Incomplete Beta**, **Incomplete Factorial**, **s-Convex**, **Sine Integral**.

Renumber comments (ii) as: (iii).

Furuta's Inequality *If A, B are positive bounded linear operators on the Hilbert space X and if $r \geq 0, p \geq 0, q \geq 1$ with $(1+r)q \geq p+r$ then*

$$\left(B^{r/2}A^pB^{r/2}\right)^{1/q} \geq \left(B^{r/2}B^pB^{r/2}\right)^{1/q},$$

and

$$\left(A^{r/2}A^pA^{r/2}\right)^{1/q} \geq \left(AB^{r/2}B^pA^{r/2}\right)^{1/q}.$$

COMMENTS (i) This reduces to the **Löwner-Heinz Inequality** on putting $r = 0, \alpha = p/q$.

REFERENCES [*EM Supp., pp. 260–262*].

G

Gauss's Inequality

Add to the list in COMMENTS (II): **Pólya's Inequality**.
Add the references:
[*Pečarić 1989*], [*Varošanec & Pečarić*], [*Varošanec, Pečarić & Šunde*].

Gautschi's Inequality

See **Factorial Function Inequalities** (a), **Incomplete Factorial Function Inequalities** COMMENTS (II).

Geometric-Arithmetic Mean Inequality

Take out the final period in COMMENTS (VI) and then add: , or by using **Geometric Mean Inequalities** (1). The right-hand side is an immediate consequence of (GA).
Add to the Converse Inequalities

(c) [CARTWRIGHT & FIELD] *If $\underline{a}$ and $\underline{w}$ are as in (a), $\underline{a}$ is not constant, then,*

$$\frac{1}{2M}\sum_{k=1}^{n} w_k\left(a_k - \mathfrak{A}_n(\underline{a};\underline{w})\right)^2 \leq \mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w}) \leq \frac{1}{2m}\sum_{k=1}^{n} w_k\left(a_k - \mathfrak{A}_n(\underline{a};\underline{w})\right)^2$$

(d) [ZHUANG] *If $0 < \alpha \leq a \leq A$, $0 < \beta \leq b \leq B$, $p > 1$, and q the conjugate index then*

$$\frac{a^p}{p} + \frac{b^q}{q} \leq Kab,$$

where

$$K = \max\left\{\frac{\alpha^p/p + B^q/q}{\alpha B}, \frac{A^p/p + \beta^q/q}{A\beta}\right\}.$$

(e) [KOBER-DIANANDA] *If $\underline{a}$ is a non-constant non-negative n-tuple and $\underline{w}$ is an n-tuple with $W_n = 1$ then*

$$\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w}) \leq \frac{1}{2w}\sum_{i,j=1}^{n} w_i w_j\left(\sqrt{a_i} - \sqrt{a_j}\right)^2,$$

with equality if and only if either $n = 2$ or $n > 2$ and the a_i with minimum weight is zero and all the other a_i are equal and positive.

Add the comment:

COMMENTS (x) The inequality of Cartwright & Field has been generalized by Alzer, replacing the arithmetic mean on the left-hand side by the geometric mean.
(xi) This same inequality has been used to give a simple proof of Alzer's additive analogue of **Ky Fan's Inequality**.
Renumber COMMENTS (x) and COMMENTS (xi)as: COMMENTS (xii) and COMMENTS (xiii).
Change present COMMENTS (xi) from . . . EXTENSIONS to : . . . EXTENSIONS(a).
Insert into the list in the present comments (xii) : **Korovkin's Inequality** COMMENTS (i),(iii).
Add the references:
[*MI, p. 124*]; [*Alzer, 1997*], [*Dianada*], [*Zhuang*].

Geometric Inequalities

Add to comments (ii): **Erdös-Mordell Inequality**, **Jarnick's Inequality**.

Geometric Mean Inequalities

In comment (iv) **Binomila** should be: **Binomial**.

Gini-Dresher Inequalities

Take out the year in the reference:[*Losonczi & Páles*].

Godunova & Levin's[1] Inequality

See **Opial's Inequality** COMMENTS (i).

Gronwall's Inequality

Add the reference: [*EM. Supp.II, pp. 51–52*].

Grunsky's Inequality

Add to COMMENTS (iv): " and the recent book by Gong."
Add the reference:
[*Gong*].

[1] Е К Годунова, В И Левин.

H

Haber's Inequality

Change [MERCER] to: [MERCER A MCD]

Halmos's Inequality *If A and B are bounded linear operators on a Hilbert space such that A, or B, commutes with $AB - BA$ then*

$$||I - (AB - BA)|| \geq 1.$$

REFERENCES [*Maher*]

Hardy's Inequality

Add the references:
[*Marcus, Mizel & Pinchover*], [*Sobolevskiĭ*].

Heisenberg-Weyl Inequality

In comment (ii) "form" should be:"from".
Add the reference:
[*Gao*]

HELP Inequalities

Add the reference:
[*Evans, Everitt, Hayman & Jones*].

Heronian Mean Inequalities

A particular case of the Extended Mean, see **Logarithmic Mean Inequalities**, is known as the *Heronian Mean*:

$$\mathfrak{H}_e(a,b) = \mathfrak{E}_{1/2,3/2}(a,b) = \frac{a + \sqrt{ab} + b}{3}, \quad 0 < a < b.$$

[JANOUS]

$$\mathfrak{M}_2^{[\log 2/\log 3]}(a,b) < \mathfrak{H}_e(a,b) < \mathfrak{M}_2^{[2/3]}(a,b),$$

and the values $\log 2/\log 3, 2/3$ *are best possible in that the first cannot be increased and the second cannot be decreased.*

EXTENSIONS (i) Noting that $\mathfrak{H}_e(a,b) = \frac{2}{3}\mathfrak{A}_2(a,b) + \frac{1}{3}\mathfrak{G}_2(a,b)$ it is natural to define the *generalized Heronian mean* as

$$\mathfrak{H}_e^{[t]}(a,b) = (1-t)\mathfrak{A}_2(a,b) + t\mathfrak{G}_2(a,b), \quad 0 \le 1 \le t.$$

So that $\mathfrak{H}_e^{[1/3]}(a,b) = \mathfrak{H}_e(a,b)$, and $\mathfrak{H}_e^{[1/2]}(a,b) = \mathfrak{M}_2^{[1/2]}(a,b)$.

[JANOUS] *(a) If* $0 < t \le 1/2$ *then*

$$\mathfrak{M}_2^{[(\log 2(/(\log 2 - \log(1-t))]}(a,b) < \mathfrak{H}_e^{[t]}(a,b) < \mathfrak{M}_2^{[1-t]}(a,b);$$

if $1/2 \le t < 1$ *the exponents in the power means are reversed. In both cases the exponents are, as in the previous result, best possible*
(b)

$$\mathfrak{H}_e^{[1]}(a,b) < \mathfrak{L}(a,b) < \mathfrak{H}_e^{[2/3]}(a,b), \tag{1}$$

the value 1 *cannot be decreased, and the value* $2/3$ *cannot be increased.*
(c)

$$\mathfrak{H}_e^{[1/3]}(a,b) < \mathfrak{I}(a,b) < \mathfrak{H}_e^{[(e-2)/2]}(a,b);$$

again the number$1/3$ *cannot be decreased, nor the value* $(e-2)/e$ *increased.*

COMMENTS (i) Inequality (1) should be compared with **Logarithmic Mean Inequalities** (2).

REFERENCES [MI, p. 350]; [Janous].

Hilbert's Inequalities

Add the reference:
[*Gao & Yang*].

Hinčin-Kahane Inequality *If* $f(x) = \sum_{n=1}^{\infty} c_n r_n(x)$, $0 \le x \le 1$, *where* $r_n(x) = \operatorname{sign} \circ \sin(2^n \pi x)$, $0 \le x \le 1$, $n = 1, 2 \ldots$, *and* $||\underline{c}||_2 < \infty$, *then for any* $p > 0$

$$A_p ||\underline{c}||_2 \le ||f||_{p,[0.1]} \le B_p ||\underline{c}||_2,$$

where $B_p = O(\sqrt{p})$ *as* $p \to \infty$.

COMMENTS (i) The orthonormal system of functions r_n, $n = 1, 2, \ldots$, is called the *Rademacher system*, and the series for f in the above inequality is called its *Rademacher series*.

REFERENCES [*EM vol.5, pp. 267–268*]; [*Zygmund vol.I, p. 213*].

Hlawka's Inequality

Call the present Extensions entry (a) and add the following.

(b) [Djoković] *With the notation of (a)*

$$\sum_{k=1}^{p}|\underline{a}_k| + (p-2)\left|\sum_{k=1}^{p}\underline{a}_k\right| \geq \sum_{i=1}^{p}|\underline{a}_1 + \cdots \underline{a}_{i-1} + \underline{a}_{i+1} + \cdots + \underline{a}_p|.$$

Add the comment:
(v) All these results have been placed in a very general setting; see [*Takahashi et al*].
Change comment (V) to: (vi).
Add the references:
[*Jiang & Cheng*], [*Takahashi, Takahashi & Wada*].

Hölder's Inequality

Add to [Other forms] (a), after (a) : [Renyi's Inequality].
Add the following after comment (xi):

Converse Inequalities [Nehari] *If* $n \geq 1$, $r_i > 1$, $1 \leq i \leq n$, *with* $\sum_{i=1}^{n} 1/r_i = 1$, *and if the functions* f_i, $1 \leq i \leq n$, *are continuous and concave on* $[0,1]$ *then*

$$\int_0^1 \prod_{i=1}^{n} f_i \geq \frac{\left[\frac{n}{2}\right]!\left(n - \left[\frac{n}{2}\right]\right)!}{(n+1)!}\left(\prod_{i=1}^{n}(1+p_i)^{1/p_i}\right)\left(\prod_{i=1}^{n}||f_i||_{p_i}\right).$$

There is equality if and only if $\left[\frac{n}{2}\right]$ *of the functions are equal to* x *and the rest are equal to* $1-x$.

Add the following comment:

Comments (xii) An error in Nehari's original proof is corrected by Choi.
Change present comments (xii) and (xiii) to: (xiii) and (xiv), and add the comment now numbered (xiii) "In addition Maligranda has pointed out that there seems a good reason why (H) should be called *Rogers' Inequality*.
Add following page to the present reference [*MI*]: 65;and add the references:
[*Choi*], [*Maligranda*], [*Sun Xie-Hua*].

Hua's Inequality

Add to the second reference: [*,1997*]

Hyperbolic Functions

Add a new entry as (a): (a) $\sinh x \leq \cosh x$.
Rename the present entries as : (b)—(e).
Add the reference:
[*Cloud & Drachman, p. 14*].

I

Incomplete Beta Function Inequalities

See **Vietoris's Inequality**.

Incomplete Factorial Function Inequalities

The *incomplete factorial, or Gamma, function* is defined as:

$$\Gamma(a+1,x) = \int_x^\infty u^a e^{-u}\,\mathrm{d}u, \quad a, x > 0$$

COMMENTS (i) Slightly different definitions can be found in various references.

(a) If $a > a + c^{-1}$ *then*

$$\Gamma(a,x) < cx^a e^{-x}.$$

(b) [ALZER] *If* $p > 1, x > 0$ *, and if* $\alpha = \big((p^{-1})!\big)^{-p}$ *then*

$$(p^{-1})!(1-e^{-x^p})^{1/p} < \int_0^x e^{-t^p}\,\mathrm{d}t < (p^{-1})!(1-e^{-\alpha x^p})^{1/p},$$

(c) [EBERT & LAFORGIA] *If* $x \geq 0$ *and* $p > 1.87705\ldots$ *then*

$$\int_0^x e^{-t^p}\,\mathrm{d}t \int_x^\infty e^{-t^p}\,\mathrm{d}t < \frac{1}{4}.$$

COMMENTS (ii) The integral in (b) can be expressed as $p^{-1}\big((\Gamma(p^{-1}) - \Gamma(p^{-1}, x^p)\big)$. The inequality generalizes one due to Gautschi.

REFERENCES [*CE, pp. 696–700*], [*EM vol.5, pp. 32–33*]; [*Elbert & Laforgia*], [*Natalini & Palumbo*]

Incomplete Gamma Function Inequalities

See **Incomplete Factorial Function Inequalities**.

Ingham-Jessen Inequality

See **Jessen's Inequality** COMMENTS (I).

Integral Inequalities

Add the new entry as (a) *(a) If* $f \in \mathcal{C}([a,b])$, $f \geq 0$ *and positive somewhere then* $\int_a^b f > 0$.

Rename the present entries (a)—(f) as: (b)—(g).

In the present comment (i) change (a) to: (b).

Add to the list in the present comment (iv): **Persistence of Inequalities** INTEGRAL ANALOGUES.

Iyengar's Inequality

What is this?[*Čulak & Elezović*].

J

Jarnick's Inequality *If a closed convex region has area A, and perimeter L then*

$$|n - A| < L,$$

where n is the number of lattice points in the region.

COMMENTS (i) A point is a *lattice point* if it has integer coordinates.

[NOSARZEWSKA] *Under the same conditions,*

$$-\frac{L}{2} < n - A \leq 1 + \frac{L}{2}.$$

REFERENCES [*CE, pp. 952, 1248*]

Jensen-Steffensen Inequality

Omission in (a). Add: $n \geq 3$.
Add a new comment: (iv) If $n = 2$ and a negative weight is allowed then ($\sim$J) holds.

Jessen's Inequality

Add to COMMENTS (I):This inequality is sometimes called the *Ingham-Jessen Inequality*.
Add the reference:
[*AI, p. 285*].

Jørgensen Inequality

What is this? See M.R., 00h 30034

K

Kahane-Hinčin Inequality

What is this? See paper by Guédon.[*Guédon On Kahane-Khinchine inequalities for negative exponent. Mathematatika, 46 (1999), 165–173.*]

Kantorović's Inequality

Take out the final period and add to COMMENTS (III): . . . , and **Variance Inequalities** COMMENTS (II).

Kolmogorov's Inequalities

The spelling of Kolmogorov in the comments (ii) is wrong.
Add to COMMENTS (I), after ". . . Kurepa." and before "See also...": " Kolmogorov's result is the case $p = q = r = \infty$, the general case is due to Stein. So this inequality is sometimes called the *Kolmogorov-Stein Inequality*."
Add the reference:
[*Bang & Le*].

Kolmogorov-Stein Inequality

See **Kolmogorov's Inequality** COMMENTS (I).

Korovkin's Inequality

There is an ommission in the statement of the result and of extension (a). Add the condition: if $\underline{a}$ is a positive n-tuple,
Add to COMMENTS (I): Putting $x = \sqrt{a}/\sqrt{b}$ we see that this inequality is just **Geometric-Arithmetic Inequality** (3). Inequality (1) can be used to prove the equal weight case of (GA).
Add the following extension:

(c) [TENG] *If $\underline{a}$ is a positive n-tuple, $n \geq 2$ then*

$$a_1 + \cdots + a_n + \frac{1}{a_1 \ldots a_n} \geq n + 1,$$

with equality if and only if $\underline{a} = \underline{e}$.

Add the comment: (iii) In fact the extension (c) is equivalent to the equal weight case of (GA).

Add the reference:
[*Hering*].

Ky Fan's Inequality

Change the first inequality to:

$$\frac{\mathfrak{H}_n(\underline{a},\underline{w})}{\mathfrak{H}_n(1-\underline{a},\underline{w})} \le \frac{\mathfrak{G}_n(\underline{a},\underline{w})}{\mathfrak{G}_n(1-\underline{a},\underline{w})} \le \frac{\mathfrak{A}_n(\underline{a},\underline{w})}{\mathfrak{A}_n(1-\underline{a},\underline{w})}$$

Change comment (i) as follows:

COMMENTS (i) The equal weight case of the right hand inequality is due to Ky Fan. The left inequality was added by Wang W L & Wang P F. These two inequalities have been the subject of much research and there are many generalizations.
Add the following extensions:

(c) [ALZER] *If $\underline{w}$ is a positive n-tuple and if $0 \le a_i \le 1/2$, $1 \le i \le n$, then*

$$\frac{\min \underline{a}}{1-\min \underline{a}} \le \frac{\mathfrak{A}_n(1-\underline{a};\underline{w}) - \mathfrak{G}_n(1-\underline{a};\underline{w})}{\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w})} \le \frac{\max \underline{a}}{1-\max \underline{a}},$$

with equality if and only if $\underline{a}$ is constant.
(d) With the notation of the main result

$$\frac{\mathfrak{G}_2(a,b)}{\mathfrak{G}_2(1-a,1-b)} \le \frac{\mathfrak{I}(a,b)}{\mathfrak{I}(1-a,1-b)} \le \frac{\mathfrak{A}_2(a,b)}{\mathfrak{A}_2(1-a,1-b)},$$

Then add the comments:

COMMENTS (iii) The original proof of (c) by Alzer has been considerably shortened by Mercer by making an ingenious use of a result of Cartwright & Field; see **Geometric-Arithmetic Mean Inequality** CONVERSE INEQUALITIES (c).
(iv) Inequalities (c) imply the original Ky Fan inequality.
(v) (d) had been extended to n-variables.
(vi) In the literature it is common to write $\mathfrak{A}'_n(\underline{a};\underline{w}) = \mathfrak{A}_n(1-\underline{a};\underline{w})$, and similarly for the other means.
Change comments (iii), (iv) to: (vii), (viii).
Change the last sentence in comment (vii) to: For other inequalities due to Ky Fan see **Determinant Inequalities** EXTENSIONS (a), **Trace Inequalities** (d).(d).
Add the references:
[*Mercer P R*], [*Sándor&Trif*].

Ky Fan-Taussky-Todd Inequalities

Add:

CONVERSE INEQUALITIES [ALZER] *With the above notations,*

$$\sum_{i=0}^{n}(\Delta a_i)^2 \le 2\Big(1+\cos\frac{\pi}{n+1}\Big)\sum_{i=1}^{n}a_i^2; \qquad \sum_{i=0}^{n-1}(\Delta a_i)^2 \le 2\Big(1+\cos\frac{2\pi}{2n+1}\Big)\sum_{i=1}^{n}a_i^2.$$

The constants on the right-hand sides are best possible.

Add the references:
[*BB p.183*]; [*Alzer 1991b*].

L

Lieb-Thiring Inequality *If $H = -\nabla^2 + V$, the Schrödinger operator on $\mathcal{L}^2(\mathbb{R}^n)$, has negative eigenvalues $\lambda_1 \le \lambda_2 \le \cdots < 0$, then for suitable $\gamma \ge 0$, and constants $L(\gamma, n)$,*

$$\sum_{k\ge 1} |\lambda_k|^\gamma \le L(\gamma, n) \int_{\mathbb{R}^n} (\max\{-V(x), 0\})^{\gamma+n/2}\, \mathrm{d}x.$$

COMMENTS (i) If $n = 1$ we must have $\gamma \ge 1/2$, if $n = 2$ we must have $\gamma > 0$, and if $n \ge 3$ we must have $\gamma \ge 0$

REFERENCES [*EM Supp.II. pp. 11-313*].

Littlewood's Conjecture *Given a positive integer m, and if $n_k, 1 \le k \le m-1$, is any strictly increasing set of positive integers, $c_k \in \mathbb{C}, |c_k| = 1, 1 \le k \le m-1$, then writing $z = e^{i\theta}$,*

$$\frac{1}{2\pi}\int_0^{2\pi} |1 + c_1 z^{n_1} + \cdots + c_{m-1} z^{n_{m-1}}|\, \mathrm{d}\theta \ge \frac{1}{2\pi}\int_0^{2\pi} |1 + z + \cdots + z^{m-1}|\, \mathrm{d}\theta$$

COMMENTS (i) More precisely this is called the *sharp Littlewood conjecture.*
(ii) The conjecture has been verified in the case that $c_k = \pm 1$, and $n_k = k, 1 \le k \le m-1$.
EXTENSIONS The same conjecture can be made using the M_p norm, instead of M_1 as above, $0 \le p < 2$, and with the inequality reversed if $p > 2$. In the case $p = 2$ there is equality.

COMMENTS (ii) This extension is known under the restrictions of COMMENTS (II), provided that $0 \le p \le 4$.

REFERENCES [*Klemes*]

Logarithmic Function Inequalities

Add :

(k) [WANG] *If $x \ne 1$ then*

$$\log x < n(x^{1/n} - 1) < x^{1/n} \log x,\ n = 1, 2, \cdots .$$

Add to the list in comment (vi): **Napier's Inequality**.

Logarithmic Mean Inequalities

In the last line of page 161 change $[-r], [r]$ to: $[-r-1], [r-1]$, respectively. This error was caused by various forms of the definition.
Add to the first set of results:

(e) [ALZER] *If* $0 < m \le a \le b \le M$ *and if* $r \le s$ *then*

$$\frac{\mathfrak{L}_r(m,M)}{\mathfrak{L}_s(m,M)} \le \min_{a,b} \frac{\mathfrak{L}_r(a,b)}{\mathfrak{L}_s(a,b)} \le \max_{a,b} \frac{\mathfrak{L}_r(a,b)}{\mathfrak{L}_s(a,b)} \le 1.$$

Take out the period at the end of comment (ii) and add: ", and should be compared with **Heronian Mean Inequalities** (1)."

Add the new comment: (iii) The right-hand of Alzer's result is an immediate corollary of (1).

Renumber comments (iii)—(vi): (iv)—(vii).

Add to the extensions:

(c) *If* $s_1 > s_2 \ge \alpha > 0$ *then* $\mathfrak{E}_{2\alpha-s_1,s_1}(a,b) \le \mathfrak{E}_{2\alpha-s_2,s_2}(a,b)$.

(d) *If* $0 < r_1 \le \beta \le s_1, 0 < r_2 \le \beta \le s_2$, *with* $\mathfrak{L}(r_1,s_1) = \mathfrak{L}(r_2,s_2) = \beta$, *and if* $s_1 \le s_2$ *then* $\mathfrak{E}_{2\alpha-s_1,s_1}(a,b) \le \mathfrak{E}_{2\alpha-s_2,s_2}(a,b)$.

Rewrite present COMMENTS (VI)replacing : "Neuman has. . . " by "Pittenger and Neuman have. . . ".

Then add the following comment:

(viii) It can be shown that

$$\mathfrak{E}_{r,s}(a,b) = \mathfrak{M}_{[0,1]}^{[s-r]}\big(\mathfrak{M}_2^{[r]}(a,b;t,1-t)\big),$$

which has led to further generalizations with the power mean being replaced by quasi-arithmetic means, see **Quasi-arithmetic mean Inequalities**.

Renumber comment (vii) as: (ix)

Add the references:

[*Alzer, 1995*], [*Pearce, Pečarić & Šimić*], [*Pittenger*].

Lorenz's Inequality

See **Aczél's Inequality** COMMENTS (II).

Löwner-Heinz Inequality

Change comment (iii) to :(iv). Then add: Many equivalent forms of this inequality have been given by Furuta in the reference.

Insert the comment :

(iii) A beautiful extension to this inequality can be found in **Furuta's Inequality**.

The spelling of "Furata" in present comment is wrong (iii); it should be:"Furuta".

Add the reference:

[*Furuta*]

Lyons's[1] Inequality

If $0 < \alpha \le 1$, $x, y > 0$ *then*

$$\alpha \sum_{k=0}^{n} \binom{\alpha n}{\alpha k} x^{\alpha k} y^{\alpha(n-k)} \le \frac{1}{\alpha}(x+y)^{\alpha n}.$$

[1] This is T Lyons.

COMMENTS (i) Lyons conjectures that the factor $1/\alpha$ on the right-hand side can be replaced by 1; this has been proved by Love in the cases $\alpha = 2m^{-1}$, $m = 1, 2, \ldots$.
(ii) For another inequality of Lyons see **Binomial Function Inequalities (ℓ).**

REFERENCES [*Love, 1998*]

M

Markov's Inequality

Add the reference:
[*Milovanović, Mitrinović & Rassias, pp. 527–623*].

Maroni's Inequality

See **Opial's Inequality** COMMENTS (i).

Mason-Stothers's Inequality *If $f, g, h = f + g$ are relatively prime polynomials of degrees λ, μ, ν respectively, and if k is the number of distinct roots of fgh then*

$$\max\{\lambda, \mu, \nu\} \le k - 1.$$

COMMENTS (i) This result can be used to give a simple proof of Fermat's Last Theorem for polynomials.

REFERENCES [*Lang*]

Matrix Inequalities

Insert into the list in COMMENTS (iii): **Halmos's Inequality**.

Mid-point Inequality

See **Quadrature Inequalities**(b).

Mills' Ratio Inequalities

Add: **Conte's Inequality** COMMENTS (i).

Minc[1]-Sarthre Inequality *If $r > 0$ and $n \ge 1$ then*

$$\frac{n}{n+1} < \left(\frac{\dfrac{1}{n}\displaystyle\sum_{i=1}^{n} i^r}{\dfrac{1}{n+1}\displaystyle\sum_{i=1}^{n+1} i^r} \right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{1}$$

[1] This is pronounced as "mints".

COMMENTS (i) The outer inequality is the original Minc-Sarthre inequality; the right-hand inequality is due to Martins. The final form above is due to Alzer.
(ii) Considered as bounds on the central ratio the extreme terms are best possible as was pointed out by Alzer.
(iii) Martins has pointed out that in the case of $a_i = i^r$, $i \geq 1$, his inequality improves the equal weight case of **Popoviciu's Geometric-Arithmetic Mean Inequality** (1), being just

$$\frac{\mathfrak{A}_{n+1}(\underline{a})}{\mathfrak{G}_{n+1}(\underline{a})} > \frac{\mathfrak{A}_n(\underline{a})}{\mathfrak{G}_n(\underline{a})}.$$

This lead Alzer to further observe that the following improvement of **Rado's Geometric-Arithmetic Mean Inequality** (1) holds in this special case:

$$\mathfrak{A}_{n+1}(\underline{a}) - \mathfrak{G}_{n+1}(\underline{a}) > \mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a}).$$

EXTENSIONS *(a)* [FENG QI] *If $r > 0$ and $n, m \geq 1, k \geq 0$ then*

$$\frac{n+k}{n+m+k} < \left(\frac{\dfrac{1}{n} \displaystyle\sum_{i=k+1}^{n+k} i^r}{\dfrac{1}{n+m} \displaystyle\sum_{i=k+1}^{n+m+k} i^r} \right)^{1/r}.$$

(b) [KUANG] *If f is strictly increasing and convex, or concave, on $]0,1]$ then*

$$\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{k}{n}\right) > \frac{1}{n+1}\sum_{i=1}^{n+1} f\left(\frac{k}{n+1}\right) > \int_0^1 f.$$

COMMENTS (iv) Taking $f(x) = \log(1+x)$ in the left inequality in (b) gives the outer inequality in (1), the Minc-Sathre inequality.

REFERENCES [*Qi 2000*].

Minkowski's Inequality

Replace "$p > 1$ or $p < 0$" by:" $p > 1$", and "$0 < p < 1$" by:"$p < 1, p \neq 0$".
On page 176 line 1 " adn" should be:"and"

Mixed Mean Inequalities

Add to Special Cases:

(c) [Kuczma]

$$\mathfrak{M}_n(2,0;2;\underline{a}) \le \mathfrak{M}_n(0,1;n-1;\underline{a}),$$
$$\mathfrak{M}_n(n-1,0;2;\underline{a}c) \le \mathfrak{M}_n(0,1;n-1;\underline{a}).$$

Add new comment (iv) and change present comment (iv) to: (v)
(iv) Both inequalities in (c) include (3), which is $\mathfrak{M}_3(2,0;2;a,b,c) \le \mathfrak{M}_3(0,1;2;a,b,c)$. All are special cases of a conjecture of Carlson, Meany & Nelson: *if* $r+m>n$ *then* $\mathfrak{M}_n(r,0;r;\underline{a}) \le \mathfrak{M}_n(0,1;m;\underline{a})$.
Add reference:
[*Kuczma*].

Moment Inequalities

Call the present entry (a) and add the following

(b) [Good] *If* $m<n, m,n \in \mathbb{N}$, $X>0$, *and the integrals exist then*

$$\frac{\int_X^\infty f(x)x^m\,\mathrm{d}x}{\int_0^\infty f(x)x^m\,\mathrm{d}x} < \frac{\int_X^\infty f(x)x^n\,\mathrm{d}x}{\int_0^\infty f(x)x^n\,\mathrm{d}x}.$$

In comment (ii) the space between “Function” and “Integral’ is missing.
Add the reference:
[*Good*].

N

Nanson's Inequality

Remove the period in comment (ii) and add: ", and Alzer."
Add the reference:
[*Alzer1995*].

Napier's Inequality

If $0 < x < y$ *then*

$$\frac{1}{y} < \frac{\log y - \log x}{y - x} < \frac{1}{x}.$$

COMMENTS (i) This is an immediate deduction from the Mean Value Theorem of Differential Calculus.

REFERENCES [*CE, p. 1214*].

Nehari's Inequality

See **Hölder's Inequality** CONVERSE INEQUALITIES.

Newman's[1] Conjecture

If p_n *is a complex polynomial of degree* n *with coefficients* ± 1 *then if* n *is large enough*

$$M_1(p;1) \leq n$$

COMMENTS (I) In any case by (C) $M_1(p;1) \leq M_2(p;1) = n+1$ and Newman proves that the 1 can be replaced by 0.97. In the reference this is improved to 0.8250041 for n large enough .

REFERENCES [*Habsieger*].

Nosarzewska's Inequality

See **Jarnick's Inequality**.

Number Theory Inequalities

Call the present inequality (a) and add:

[1] This is D Newman.

(b) If $x > 10^6$ *then*

$$\pi(x) < \frac{x}{\log x - 1.08366};$$

while if $x \geq 4$,

$$\pi(x) < \frac{x}{\log x - 1.11}.$$

Add the comment:

(iii) See also **Chi Inequality**.

Add after comment (iii)

A different kind of result is the following.

If N *is a positive integer,* $N = \sum_{1=0}^{n} a_i 10^i$, $n > 2$, a_i *an integer,* $0 \leq a_i \leq 9$, $a_n \neq 0$, $0 \leq i \leq 9$, *then*

$$N \geq \frac{1\overbrace{0\ldots0}^{k+1}\overbrace{9\ldots9}^{n-k}}{1 + 9(n-k-1)} A_n,$$

where $A_n = \sum_{i=0}^{n} a_i$, *and* k *is unique integer such that*

$$2 + \sum_{i=0}^{k} 10^i \leq n \leq 2 + \sum_{i=0}^{k+1} 10^i.$$

Add the references:

[*Gu & Liu*], [*Panaitopol*].

O

Operator Inequalities

See **Furuta's Inequality**, **Halmos's Inequality**, Heinz-Kato-Furuta Inequality, Löwner-Heinz Inequality.

Opial's Inequality

Add to COMMENTS (I): " Some of the generalizations are known by other names; for instance, *Maroni's Inequality, Godunova-Levin Inequality* ".
Add the references:
[*Agarwal & Pang*], [*Mitrinović, Pečarić & Fink*].

Ostrowski's Inequalities

Add.

(d) If f is absolutely continuous and if $\lambda \geq 1$ then

$$\int_0^1 \int_0^1 \left| \frac{f(x) - f(y)}{x - y} \right|^\lambda \mathrm{d}x \, \mathrm{d}y \leq \log 4 \int_0^1 |f'|^\lambda.$$

If $\lambda = 1$ the constant is best possible

Add the comment before the extensions:
(iii) Inequality (d) has been extended to lower bounds, and upper bounds of the averages of the n-th divided difference, see **n-Convex Function Inequalities** COMMENTS (II).
Renumber comments (iii) and (iv) as: (iv), (v).
Add the references:
:[*Fink, 2000*], [*Dedić L, Matić M & Pečarić J E 2000a*], [*Matić M, Pečarić J & Ujević N*].

Ozeki's Inequalities

Rewrite (b) as follows:

(b) Let $p > 0$ and $\underline{a}$ an n-tuple, $n \geq 2$ with $\min_{i \neq j} |a_i - a_j| = 1$, then

$$\sum_{i=1}^n |a_i|^p \geq C_{n,p},$$

where

$$C_{n,p} = \begin{cases} 2\sum_{i=1}^{(n-1)/2} i^p & \text{if } n \text{ is odd,} \\ 2\sum_{i=1}^{(n/2)-1} i^p + (n/2)^p & \text{if } n \text{ is even and } 0 < p < 1, \\ 2\sum_{i=1}^{n/2} (i - \frac{1}{2})^p & \text{if } n \text{ is even and } p > 1. \end{cases}$$

Add

(c) If $0 \le a \le \underline{a} \le A$, $0 \le b \le \underline{a} \le B$ *then*

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \big(\sum_{i=1}^n a_i b_i\big)^2 \le \begin{cases} \dfrac{n^2}{3}(AB - ab)^2 & \text{if } n = 0 \pmod 3, \\ \dfrac{(n-1)^2}{3}(AB - ab)^2 & \text{if } n = \pm 1 \pmod 3. \end{cases}$$

The constants on the right-hand side are best possible.

Add the comment: (ii) This is an additive analogue of the **Polyá & Szegö Inequality**
Renumber the present comment (ii) as:(iii).
Add the references:
[*Mitrinović, Pečarić & Fink p. 121*]; [*Alzer 2000a*], [*Izumino, Mori & Seo*].

P

Permanent Inequalities

Add

(d) [Aleksandrov-Fenchel Inequality] *If A is an $n \times n$ positive matrix and A_1 is obtained from A by replacing the second column by the first, so that the first two columns are the same, and A_2 is obtained from A by replacing the first column by the second, so that again the first two columns are the same, then*

$$\mathrm{per}^2(A) \le \mathrm{per}(A_1)\mathrm{per}(A_2),$$

with equality if and only if the first two columns are dependent.

Rewrite comment (ii) as:

Comments (ii) The Aleksandrov-Fenchel inequality was used in solving **van der Waerden's Conjecture**.

Add the reference:

[*Wang C L, 1984*].

Persistence of Inequalities

(a) If $\underline{a}, \underline{b}$ are real sequences and $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty b_n} = B$ and if $\underline{a} \le \underline{b}$ then $A \le B$.
(b) If $a_1 \le a_2 \le \cdots$ then $\mathfrak{A}_2(a_1, a_2) \le \mathfrak{A}_3(a_1.a_3, a_3) \le \cdots$.

Comments (i) The result in (a) cannot be improved to strict inequality as $a_n = 0,\ b_n = 1/n, n \in \mathbb{N}^*$, shows.

Extensions (a) Clearly (b) can be extended to more general means. More interestingly the hypothesis of increasing can be generalized to $\underline{a}$ being n-convex, with the analogous generalization in the conclusion; see **n-Convex Sequence inequalities** (b).

Integral Analogues *If f is an increasing function on $[a, b]$ then $\displaystyle\frac{1}{v-u}\int_u^v f$ is an increasing function of both u and v, $a \le u, v \le b$.*

References [*AI p.9, MI p.127–128*], [*Cloud & Drachman, p. 6*].

Pólya's Inequality

Add the comments below, numbering the present comment as Comments (i):

(ii) This inequality can be written using geometric means which then have been replaced by more general means to obtain other inequalities of the same type.
(iii) See also **Gauss's Inequality**.
Add the reference
[*Pearce, Pečarić & Varošanec*].

Pólya & Szegö's Inequality

The spelling of Pólya is wrong in the heading of this entry.
Add the comment:
(v) An additive analogue of this inequality can be found in **Ozeki's Inequalities** (c).

Pólya-Vinogradov Inequality *If* χ *is a primitive character* $(\mathrm{mod}\ p)$, $k > 2$, *and if* $s(\chi) = \max_{r\geq 1} \sum_{n=1}^{r} |\chi(n)|$, *then*

$$\frac{s(\chi)}{\sqrt{k}\log k} \leq \begin{cases} \dfrac{1}{2\pi} + o(1), & \text{if} \quad \xi(-1) = 1, \\ \dfrac{1}{\pi} + o(1), & \text{if} \quad \xi(-1) = -1. \end{cases}$$

Extensions (i) The following extension is due to Simalrides:

$$s(\chi) \leq \begin{cases} \dfrac{1}{\pi}\sqrt{k}\log k + \sqrt{k}\big(1 - \dfrac{\log 2}{\pi}\big), & \text{if} \quad \xi(-1) = 1, \\ \dfrac{1}{\pi}\sqrt{k}\log k + \sqrt{k} + \dfrac{1}{2}, & \text{if} \quad \xi(-1) = -1. \end{cases}$$

References [*Simalrides*]

Polynomial Inequalities

Add to the list in comment (v):**Mason-Stothers's Inequality**, **Newman's Conjecture**, **Schoenberg's Conjecture**, **Zeros of a Polynomial**.

Popoviciu's Geometric-Arithmetic Mean Inequality

Add and extra comment after the extensions entry:

Comments (iv) See also **Minc-Sathre Inequality** Comments (iii).

Popoviciu's Inequality

See **Variance Inequalities** Comments (ii).

Power Sums Inequalities

Add to the list in comment (v): **Ozeki's Inequalities** (b).

Q

Quadrature Inequalities

These inequalities arise from estimates of the remainders of various quadrature rules. The trapezoidal rule, the mid-point rule, Simpson's rule etc.

(a) [TRAPEZOIDAL INEQUALITY] *If $f:[a,b]\mapsto\mathbb{R}$ has a bounded continuous second derivative derivative on $]a,b[$, with bound M_2, then*

$$\left|\int_a^b f-\frac{b-a}{2}\left(\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right)\right|\leq\frac{M_2}{48}(b-a)^3.$$

(b) [MID-POINT INEQUALITY] *If $f:[a,b]\mapsto\mathbb{R}$ has a bounded continuous second derivative derivative on $]a,b[$, with bound M_2, then*

$$\left|\int_a^b f-\frac{b-a}{2}\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\right|\leq\frac{M_2}{96}(b-a)^3.$$

(c) [SIMPSON'S INEQUALITY] *If $f:[a,b]\mapsto\mathbb{R}$ has a bounded continuous fourth derivative on $]a,b[$, with bound M_4, then*

$$\left|\int_a^b f-\frac{b-a}{3}\left(\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right)\right|\leq\frac{M_4}{2880}(b-a)^5.$$

COMMENTS (i) Similar inequalities have been obtained with much weaker conditions on the function f

REFERENCES [*Davis & Rabinowitz*]; [*Dragomir, Agarwal & Cerone*].

Quasi-arithmetic Mean Inequalities

Add the reference:
[*Páles, 1999*].

R

Rado's Geometric-Arithmetic Mean Inequality

Add to the list in COMMENTS (v): **Minc-Sathre Inequality** COMMENTS (iii).

Rearrangement Inequalities

Add the following comment before extensions and renumber COMMENTS (i) as: (ii).

COMMENTS (i) A simple application of (1) gives:

$$A_n B_n \leq n \sum_{i=1}^{n} a_i b_i,$$

where $\underline{a}, \underline{b}$ are increasing real n-tuples.
Add to INTEGRAL ANALOGUES after calling the present entry (a):

(b)[BRASCAMP-LIEB-LUTTINGER] *Let f_k, $1 \leq k \leq m$, be non-negative functions on $\mathbb{R}^n$, vanishing at infinity. Let $k \leq m$ and let $B = (b_{ij})_{\substack{1\leq i\leq k\\ 1\leq j\leq m}}$ be a $k \times m$ matrix, and define*

$$I(f_1, \ldots, f_n) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j \left(\sum_{i=1}^{k} b_{ij} x_i \right) \mathrm{d}x_1 \cdots \mathrm{d}x_k.$$

Then $I(f_1, \ldots, f_n) \leq I(f_1^{(*)}, \ldots, f_n^{(*)})$.

COMMENTS (iii) A function $f : \mathbb{R}^n \mapsto \mathbb{C}$ is said to *vanish at infinity* if $|\{\underline{x}; |f(\underline{x})| > t\}| < \infty$ for all t.
(iv) This inequality is used to obtain the exact constants in **Titchmarsh's Theorem** and **Young's Convolution Inequality**.
Renumber COMMENTS (ii) as:(v).
Add the reference:
[*Lieb & Loss, p. 85*]; [*Atanassov*].

Renyi's Inequality

This is a variant of (H); see **Hölder's Inequality** [OTHER FORMS] (a), inequality (7).

Reverse Inequalities

The terms $\dfrac{1}{p_1}$, $w_1 a_1$ in the first expression in (1) should be: $\dfrac{1}{w_1}$, $a_i w_i$ respectively.
The definition of $\mathfrak{g}_n(\underline{a};\underline{w})$is wrong it should be:

$$\mathfrak{g}_n(\underline{a};\underline{w}) = \frac{a_1^{W_n/w_1}}{\prod_{i=2}^n a_i^{w_i/w_1}}$$

In both comments Aczel should be: Aczél

Riesz Mean Value Theorem

An inequality sign is missing; there should be a "≥ 0 " after the first integral in both the main result and the extension, that is $\ldots, 0 \leq x \leq y$ should be:$\ldots \geq 0, 0 \leq x \leq y$
This was noted by reviewer in the Mathematical Reviews.

Rogers' Inequality

See **Hölder's Inequality** COMMENTS (xii).

Root and Ratio Test Inequalities

If $a_n, n \in \mathbb{N}$, is a positive sequence then

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

COMMENTS (i) All the inequalities above can be strict; consider the sequence $a_{2n} = 3^{-n}$, $a_{2n-1} = 2^{-n}, n \geq 1$.
(ii) The name comes from the fact that the sequence of ratios and of roots are used as tests for the convergence of series. Further if $a_n, n \in \mathbb{N}$, is the complex sequence of coefficients of a power series then the radius of convergence of that series is $1/\limsup_{n\to\infty} \sqrt[n]{|a_n|}$; with suitable interpretations if this upper limit is either zero or infinity.

REFERENCES [*Rudin, 1964, pp. 57–61*].

S

Schoenberg's Conjecture

If $p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = a_n \prod_{k=1}^n (z - z_k)$, $a_n \neq 0$, define the *quadratic mean radius of* p_n by: $R(p_n) = \sqrt{\frac{1}{n} \sum_{k=1}^n |z_k|^2} = \mathfrak{M}_n^{[2]}(|z_1|, \ldots, |z_n|)$.

If p_n is a monic polynomial with the sum of its zeros being zero, equivalently $a_{n-1} = 0$, then

$$R(p_n') \le \sqrt{\frac{n-2}{n-1}} R(p_n).$$

COMMENTS (i) In the case of real zeros this is easy to verify.

REFERENCES [*De Bruin, Ivanov & Sharma*]

Schur's Inequality

Add to the list in comment (iv): "**Erdös's Inequality** COMMENTS (II)"

s-Convex Function Inequalities *(a) If $0 < s \le 1$ and f is an s-convex function of type one, or of the first kind, defined on $[0, \infty[$ then for all $x, y \ge 0$, and $0 \le t \le 1$,*

$$f\left((1-t)^{1/s} x + t^{1/s} y\right) \le (1-t) f(x) + t f(y).$$

(b) If $0 < s \le 1$ and f is an s-convex function of type two , or of the second kind, defined on $[0, \infty[$ then for all $x, y \ge 0$, and $0 \le t \le 1$,

$$f\left((1-t)x + ty\right) \le (1-t)^s f(x) + t^s f(y).$$

COMMENTS (i) These are just the definition of s-convexity of types one and two respectively. Both reduce to the ordinary convexity of $s = 1$, see **Convex Function Inequalities** (1).
(ii) If $0 < s < 1$ an s-convex function of type one is increasing on $]0, \infty[$, while an s-convex function of type two if non-negative.
(iii) If $f(0) = 0$ and f is an s-convex function of type two then it is also an s-convex function of type one, but not conversely.

REFERENCES [Hudzik & Maligranda]

Seitz's Inequality

What is it ? There is a paper by Toader that generalizes it; [*Toader 1995*].

Sierpinski's Inequality

Add:

COMMENTS (i) Extension (a) has been further extended by Pečaríc.
Add the reference:
[*Pečarić 1988*].

Simpson's Inequality

See **Quadrature Inequalities**(c).

Sine Integral Inequalities

The *sine integral* is the function

$$\mathrm{Si}(x) = \int_0^x \frac{\sin u}{u}\, \mathrm{d}u.$$

$$(a) \qquad \text{If } 0 < x < \pi \text{ then} \qquad 0 < \sum_{k=1}^{n} \frac{\sin kx}{k} \le \mathrm{Si}(\pi) = 1.8519\ldots < \frac{\pi}{2} + 1,$$

$$(b) \qquad 1 < \mathrm{Si}(\pi/2) < \pi/2.$$

COMMENTS (i) The left-hand side of (a) is the **Fejér-Jackson inequality**.
(ii) It is well- known that $\mathrm{Si}(\infty) = \pi/2$.
(ii) Various refinements of (a) and (b) can be found in the references.

REFERENCES [*MPF, p. 613*]; [*Qi, Cui & Xu*].

Statistical Inequalities

Add: See also **Variance Inequalities**.

Steffensen's Inequalities

Before INTEGRAL ANALOGUES put the following.

[DISCRETE ANALOGUES] *(a) Let $\underline{a}$ be a decreasing n-tuple and $\underline{b}$ an n-tuple with $0 \le \underline{b} \le 1$. If $1 \le n_2 \le B_n \le n_1 \le n$ then*

$$\sum_{k=n-n_2+1}^{n} a_k \le \sum_{k=1}^{n} a_k b_k \le \sum_{k=1}^{n_1} a_k.$$

(b) [PEČARIĆ] *Let* $p \geq 1$, *and* $\underline{a}, \underline{b}$ *a real* n*-tuples,* $\underline{a}$ *non-negative and decreasing. If* $k, 1 \leq k \leq n$, *is such that such that* $k^{1/p} \geq B_n$, *and* $b_i B_n \leq 1, 1 \leq i \leq k$, *then*

$$\sum_{i=1}^{n} a_i b_i \leq \Big(\sum_{i=1}^{k} b_i^p\Big)^{1/p}.$$

The first reference should be: [*AI, pp.107–119*]
Add the reference:
[*Gauchmann 2000*a,*b*].

Stolarsky Mean Inequalities

See **Extended Mean Inequalities**.

Sums of Integer Powers Inequalities

Add

(e)

$$\frac{1}{4n} < \sum_{i=n+1}^{2n} \frac{1}{i^2} < \frac{1}{n}.$$

(f)

Add the reference:
[*Cloud & Drachman, p.11*].

Symmetric Mean Inequalities

In Extensions POPVICIU should be: POPOVICIU.

T

Titchmarsh's Theorem

See **Fourier Transform Inequalities** TITCHMARSH'S THEOREM.

Trace Inequalities

Add:

(d) [KY FAN]*If A, B are $n \times n$ real symmetric matrices with eigenvalues, in decreasing order, $\lambda(A) = \{\lambda_1(A), \ldots \lambda_n(A)\}, \lambda(B) = \{\lambda_1(B), \ldots \lambda_n(B)\}$ then,*

$$\operatorname{tr}(AB) \le \lambda(A)^T \lambda(B). \tag{1}$$

There is equality in (1) if and only if there is an orthogonal matrix U such that

$$A = U^T \operatorname{Diag}\big(\lambda(A)\big)U, \text{ and } B = U^T \operatorname{Diag}\big(\lambda(B)\big)U. \tag{2}$$

Call the present comment: COMMENTS (i)
Add to COMMENTS (I): "If X is a row or column matrix then $\operatorname{Diag}(X)$ is the diagonal matrix with entries along the diagonal those of X".
Add the comments:
(ii) The condition (2) is called *simultaneous ordered spectral decomposition*; it implies that the matrices commute.
(iii) Inequality (1) can be regarded as a refinement of (C) in the space of real symmetric matrices.
Add the reference:
[*Borwein & Lewis*].

Trapezoidal Inequality

See **Quadrature Inequalities**(a).

Trigonometric Function Inequalities

Add:

(s) If $x \neq y$ then

$$\mathfrak{G}_2(x,y) < \frac{x-y}{2\arcsin\left(\dfrac{x-y}{x+y}\right)} < \mathfrak{A}_2(x,y) < \frac{x-y}{2\arctan\left(\dfrac{x-y}{x+y}\right)} < \mathfrak{M}_2^{[2]}(x,y).$$

(t) If $f \in \mathcal{C}^\infty(0,\pi/2),\ f(0)=0, |f(\pi/2)| \leq 1$ and $\max_{n\geq 1}|f^{(n)}| \leq 1$, then

$$|f(x)| \leq \sin x, \quad 0 \leq x \leq \frac{\pi}{2}.$$

In particular if $f \in \mathcal{C}^\infty(0,\pi/2),\ f(0)=0$, then

$$\int_0^{\pi/2} |f| \leq \max_{n\geq 0} \max |f^{(n)}|,$$

Add the comment:

(xi) Inequalities (s) are due to Seiffert and have been generalized by Toader; see the reference.

Renumber comment (xi) as : (xii).

Add to the list in the present comment (xi): **Sine Integral Inequalities**.

Add the references:

[*Toader 1999*], [*Wright*],

U-V

Variance Inequalities *If $\underline{a}$ is an n-tuple with $m \le \underline{a} \le M$, then*

$$\frac{1}{n}\sum_{i=1}^{n}\big(a_i - \mathfrak{A}_n(\underline{a})\big)^2 \le \big(M - \mathfrak{A}_n(\underline{a})\big)\big(\mathfrak{A}_n(\underline{a}) - m\big),$$

with equality if and only if all the elements of $\underline{a}$ are equal to either M or m.

COMMENTS (i) This implies the following inequality known as *Popoviciu's inequality*

$$\frac{1}{n}\sum_{i=1}^{n}\big(a_i - \mathfrak{A}_n(\underline{a})\big)^2 \le \frac{(M-m)^2}{4}.$$

Here there is equality if and only if n is even, and half of the elements of $\underline{a}$ are equal to m and half are equal to M.
(ii) This result has been given a simple proof in an abstract setting in the reference where a connection with **Kantorović's Inequality** is made.

REFERENCES [*Bhatia & Davis*]

W

Weierstrass's inequalities

Rewrite the entry to insert the footnote : [1].

Y-Z

Young's Convolution Inequality

Add after the DISCRETE ANALOGUE.

[AN ELEMENTARY CASE]

$$\int_{\mathbb{R}^2} f(x)g(y)h(x-y)k(x-y)\mathrm{d}x\mathrm{d}y \leq \frac{1}{\sqrt{2}}||f||_2||g||_2||h||_2||k||_2.$$

Add the comments:

COMMENTS (iv) The constant is best possible as can be seen with the functions: $f(x) = g(x) = e^{-2x^2}, h(x) = k(x) = e^{-x^2}$.
(v) This is an easy consequence of **Cauchy's Inequality** INTEGRAL ANALOGUES and Fubini's theorem on the interchange of order on integration.
Renumber COMMENTS (IV):(vi)

Zeros of a Polynomial *If $p(z) = a_0 + \cdots + a_n z^n$, $a_n \neq 0$, where the coefficients are complex, then all of the zeros ζ of this polynomial satisfy*

$$|\zeta| \leq 1 + \frac{n}{a_n} \max_{0\leq k\leq n-1} |a_k|$$

REFERENCES [*Cloud & Drachman, pp. 9–10*].

[1] Also written Weierstraß.

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[EM] "Encyclopedia" should be "Encyclopaedia".

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