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Hierarhies

of

Convexity

of

Functions

2015

I want to thank S. S. Dragomir, P. T. Mocanu, J. E. Pečarić, I. Raşa, I. Şerb and Silvia Toader, the coautors of some of the included papers.

Introduction

Let us consider the sets of continuous, convex, starshaped, and superadditive functions on $[a,b]$ given by:

$$C[a,b] = \{f : [a,b] \to \mathrm{R}, f \quad \text{continuous}\},$$

$$K[a,b] = \{f \in C[a,b]; f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \forall x,y \in [a,b], t \in [0,1]\},$$

$$S^*[a,b] = \left\{f \in C[a,b]; \frac{f(x)-f(a)}{x-a} \leq \frac{f(y)-f(a)}{y-a},\ a < x < y \leq b\right\},$$

and

$$S[a,b] = \{f \in C[a,b]; f(x) + f(y) \leq f(x+y-a) + f(a), \forall x,y,x+y-a \in [a,b]\},$$

respectively.

For $a = 0$ we denote by $C(b), K(b), S^*(b),$ and $S(b)$ respectively, the corresponding set of functions, restricted also under the condition $f(0) = 0$. A.M. Bruckner and E. Ostrow have proven in [1] the strict inclusions:

$$K(b) \subset S^*(b) \subset S(b).$$

These inclusions were extended with some results of preservation of the above properties by the arithmetic integral mean

$$A(f)(x) = \frac{1}{x}\int_0^x f(t)dt.$$

A function f is said to have the property "P" in the mean if $A(f)$ has the property "P". Denoting by $MK(b), MS^*(b)$ and $MS(b)$ the sets of functions which are convex, starshaped, respectively superadditive in the mean, in [1] was proved that

$$K(b) \subset MK(b) \subset S^*(b) \subset S(b) \subset MS^*(b) \subset MS(b),$$

which was named in [2] as the **hierarchy of convexity**.

Simple proofs and generalizations of the results of [1] may be found in [6]. In [5] was considered a more general integral mean A_g defined by

$$A_g(f)(x) = \frac{1}{g(x)}\int_0^x g'(t)f(t)dt.$$

In [6] was proved that if A_g preserves the convexity (the starshapedness or the superadditivity) then the function g is of the form

$$g(x) = kx^u, u > 0, k \neq 0.$$

Making the substitution $t = xs^{1/u}$, the mean A_g, denoted now A_u, becomes

$$A_u(f)(x) = \int_0^1 f(xs^{1/u})ds.$$

It was also proved that for all $b, u > 0$, the following inclusions

$$\begin{array}{ccccccc}
K(b) & \subset & M^uK(b) & \subset & S^*(b) & \subset & S(b) \\
 & & & & \cap & & \cap \\
 & & & & M^uS^*(b) & \subset & M^uS(b),
\end{array}$$

hold.

In [4], one of the many generalizations on the convexity of functions - called m - convexity - was introduced. The set of m - convex functions is defined by:

$$K_m[a,b] = \{f \in C[ma,b]; f(tx+m(1-t)y) \leq tf(x)+m(1-t)f(y),$$

$$\forall x,y \in [a,b], t \in [0,1]\},\ m \in [0,1].$$

If $a = 0$ and $f(0) \leq 0$, we also obtain a hierarchy of m - convexity:

$$K(b) \subset K_m(b) \subset K_n(b) \subset S^*(b), \text{ for } 1 > m > n > 0.$$

Taking it into consideration, in [3] was defined the order of star-convexity of a function $f \in S^*(b)$ by

$$m^*(f) = \sup\{m : f \text{ is } m-\text{convex}\}.$$

As was shown in [9], for every $p \in [0,1]$ there is a polynomial P of degree four such that $m^*(P) = p.$

The preservation of m - convexity by the integral mean A_u was proved in [7]. It was shown that for $u > 0$ and $0 < n < m < 1,$ the following inclusions

$$\begin{array}{ccccccc} K(b) & \subset & K_m(b) & \subset & K_n(b) & \subset & S^*(b) \\ \cap & & \cap & & \cap & & \cap \\ M^uK(b) & \subset & M^uK_m(b) & \subset & M^uK_n(b) & \subset & M^uS^*(b) \end{array}$$

hold.

Assuming $m \neq 0,$ in [8] were defined the following sets of functions:

$$S_m^*[a,b] = \left\{f \in C[a,b]; \frac{f(x)-mf(a)}{x-ma} \geq \frac{f(z)-mf(a)}{z-ma},\ a \leq z < x \leq b\right\},$$

called m – starshaped functions;

$$S_m[a,b] = \{f \in C[a,b]; f(x)+f(x) \leq f(x+y-ma)+mf(a), \forall x,y \in [a,b]\},$$

called m – superadditive functions;

$$J_m^*[a,b] = \left\{f \in C[a,b]; f(2x-ma)-mf(a) \geq 2[f(x)-mf(a)],\ a \leq x \leq b\right\},$$

called Jensen m – starshaped functions;

$$J_m[a,b] = \left\{f \in C[a,b]; f\left(\frac{m(x+y)}{1+m}\right) \leq \frac{m[f(x)+f(y)]}{1+m}, \forall x,y \in [a,b]\right\},$$

called m – Jensen convex functions;

$$H_m[a,b] = \{f \in C[a,b]; f(tx) \leq tf(x), a \leq x \leq b, m \leq t \leq 1\},$$

called m – subhomogenous functions, and

$$H_m^*[a,b] = \left\{f \in C[a,b]; f\left(\frac{2mx}{1+m}\right) \leq \frac{2m}{1+m}f(x),\ a \leq x \leq b\right\},$$

called Jensen m – subhomogenous functions.

In fact, to assure that all the definitions and results that follow are valid we will assume that the functions are defined on $[ma, 2b-ma].$ For these sets, we have the following main results.

Theorem *The following inclusions*

$$K_m[a,b] \subseteq S_m^*[a,b] \subseteq S_m[a,b] \subseteq J_m^*[a,b]$$

and

$$H_m^*[a,b] \supseteq H_m[a,b] \supseteq K_m[a,b] \subseteq J_m[a,b]$$

hold.

I am also the author or coauthor of other thirty papers with subject related to the hierarchy of convexity of functions. Most of those papers were published with more than twenty years ago, in Romanian of other less known journals. As I got many demands of copies of some of these papers, I decided to offer them with open access.

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PAPERS

Proc. Colloqu. Approx. Optim., Cluj-Napoca, 1984, 329-338

SOME GENERALIZATIONS OF THE CONVEXITY

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In this paper we present some generalizations of the convexity of real functions and propose a new one. First of all let us recall four equivalent definitions of the convexity. Although the discussion may be done in more general cases, we content ourselves to consider only real functions defined on a convex (real) set C.

Definition 0.1. *The function $f : C \longrightarrow \mathbb{R}$ is said to be convex if it satisfies one of the following conditions:*

$$E_f = \{(x,y); x \in C,\ y \geq f(x)\} \textit{ is a convex set} \tag{1}$$

$$[x,y,z;f] \geq 0 \textit{ for any } x,y,z \in C \tag{2}$$

where $[x,y,z;f]$ represents a divided difference;

$$(z-y)\cdot f(x) + (x-z)\cdot f(y) + (y-x)\cdot f(z) \geq 0,\ \textit{for } x<y<z; \tag{3}$$

$$a\cdot f(x) + b\cdot f(y) - f(ax+by) \geq 0,\ \textit{for any } x,y \in C,\ (a,b) \in J \tag{4}$$

where $J = \{(a,b) : a,b \geq 0, a+b=1\}$.

Each of these relations has led to some generalizations.

1. First of all let us recall more directions in which was generalized the notion of convex set.

A. *A* set C is said to be convex if for any points $x,y \in C$ the segment xy is in C. Some generalizations replace the segment xy by a joint set $J(x,y)$. Systems of axioms for the joint sets were given by A. Ghika [15], W. Prenowitz [39], V.W. Bryant and R.J. Webster [9].

B. The segment xy is the set of points $tx+(1-t)y$, for $t \in I = [0,1]$.
In other generalizations one considers, instead of the combination given before, a "mixture" $< t,x,y >$ which satisfies some axioms. Such systems of axioms gave T. Swirszck [44], S.P. Gudder [18] and L.A. Skorniakov [41].

C. Sometimes the combination $tx + (1-t)y$ is only replaced by another. So is, for example, the k-convexity (D.K. Kulshrestha [31]):$x, y \in C,\ t \in I \Rightarrow t^k x + (1-t)^k y \in C$.

D. Fixing one of the ends of the segment, say x at x_0, we find another generalization of convexity, the stellarity with respect to x_0. In the complex plane, this was generalized by G.M. Goluzin [16]. The idea was taken again by I. Marusciac in [27] defining the polygonal convexity: a set X is π-convex if for any $x, y \in X$ ther is a polygonal line between x and y, $\pi_{xy} \subset X$. If any such polygonal line may be taken to have at most m edges, the set X is called π_m-convex.

E. Another generalization is similarly to the definition of the topology a family of sets defines a convexity on a space X if it satisfies some axioms (see V.P. Soltan [43]).

F. A way for other generalizations was given by F.A. Valentine [47] defining a three point convexity: with any three points x, y, z, the set contains at least one of the segments xy, yz, zx. Later was considered a m point convexity, a (m, n)-convexity (see M. Breen [7]) and a m-segment convexity (M. Breen [8]).

G. One considers also some discrete convexities as: p-convexity (see I.Munteanu [30]), S-convexity (see I.Oprea [32]) and strong convexity (L. Lupsa [25]).

2. The various generalizations of the relation (2) may be found in the book of T. Popoviciu [38].

3. Let $g : I^2 \longrightarrow I$ be such that $g(x, y) > 0$ for $y > x$. The relation (3) led to the following generalization:

Definition 0.2. *The function f is called g-convex if:*

$$g(y,z) \cdot f(x) + g(z,x) \cdot f(y) + g(x,y) \cdot f(z) \geq 0,\ for\ x \leq y \leq z.$$

In a particular case this definition was given by I.E. Ovčarenko [34] and in this form by D.M. Vasić and J.D. Keckić in [48].

4. It seems (see [38] that the convex functions were introduced by O. Stolz in 1893 in the study of the derivatives, considering the relation:

$$f(x+h) - 2f(x) + f(x-h) \geq 0$$

that is

$$\frac{f(x) + f(y)}{2} - f\Big(\frac{x+y}{2}\Big) \geq 0. \tag{5}$$

J.L.W.V. Jensen [21] is the first who studied them systematically.The relation (5) is in fact a generalization of (4). Let us see other generalizations.

A. The geometrical interpretation of the relation is that on the segment xy, the function f takes values below the chord that connects the values of f at x and y. Starting from this, in some generalizations the chords are replaced by families of functions with some propertiee (see the book of E. Popoviciu [35] and the paper of V.P. Soltan [42]).

B. Many generalizations refer to the convexity with respect to a given function h (that is the function $h \circ f$ to be convex). So, for $h(x) = \log x$ we have the log-convexity (P. Montel [29]), for $h(x) = exp(rx)$ the r-convexity defined by B. Martos (see [17]) and by M. Avriel [1], for $h(x) = x^{\alpha}$ the αm-convexity defined by M. Avriel, I. Zang [2] and by I. Marusciac [26].

C. Other generalizations are of the form:

$$g(a) \cdot f(x) + h(b) \cdot f(y) - f(ax + by) \geq 0. \tag{6}$$

So for $g(a) = h(a) = a^s$, we have the s-convexity defined by W.W. Breckner [6] and D.K. Kulshrestha [24] and $g(a) = \lambda a$ and $h(1 - a) = 1 - \lambda a$ the λ-convexity of Chi Zong Tao and Qi Li Qun [10] if $\lambda \in I$ is constant and $f(x) \leq f(y)$, or the weak convexity of C.R. Bector [4] if $\lambda = \lambda(x, y, a)$.

D. A well known generalization is the quasi-convexity, defined by:

$$f(ax + by) \leq Cmax\{f(x), f(y)\} \tag{7}$$

with $(a, b) \in J$ and $C = 1$. Such functions were introduced by T. Popoviciu [37] as being unimodal. Then B. de Finetti [13] considered function with convex level sets:

$$L_y = \{x \in C : f(x) \leq y\}$$

finding again the class of quasi-convex functions. I. Kolumban [22] has enlarged the class by admitting $C \neq 1$ and (a, b) in a subset of J. Many authors have defined families of functions between that of convex and of quasi-convex functions. Let us to mention E.F. Beckenbach (see [5]), B. Martos (see [17]), I. Oprea [31] and Chi Zong Tao and Qi Li Qun [10].

In the last case, λ-convexity means convexity for $\lambda = 1$ and quasi-convexity for $\lambda = 0$. Many generalizations of quasi-convex function may be found in the survey paper of H.J. Greenberg and W.J. Pierskalla [17]. Other generalizations are in the book of J.M. Ortega and W.C. Rheiboldt [33] and in the papers of M. Avriel and I. Zang [3], V.P. Soltan [43] and I. Marusciac [27]. Quasi-convexities of high order are defined by E. Popoviciu [36].

E. A wide generalization is given by A. Guerraggio and L. Paganoni in [19] a function f is said to be (H, K)-convex if:

$$f(H(x,y)) \leq K(f(x), f(y)),\ for\ x, y \in C.$$

F. M.Kuczma [23] has relaxed the relation (4) requiring it only almost everywhere, that is defining almost convex functions.

G. Another relaxation is given by:

$$af(x) + bf(y) - f(ax + by) \in P. \tag{8}$$

In the case of convexity P is $[0, \infty)$. D.H. Hyers and S.M. Ulam [20] and Gh. Cimoca and I.Serb [11] have considered $P = [-\epsilon, \infty)$, obtaining approximately convex functions. Arbitrary sets P have used D. Duca [14] for complex functions and Gh. Toader [46] for sequences in a group.

H. In their book [40], A.W. Roberts and D.E. Varberg have proposed, as an independent study project, to replace J in (4) by an arbitrary set M.
Call this M-convexity. So, the Jensen-convexity, given by (5) coresponds to $M = \{(1/2, 1/2)\}$, the p-convexity of E. Deak [12] to $M = \{(p, 1-p)\}$, the subadditivity to $M = \{(1,1)\}$ and the stellarity to $M = I \times \{0\}$. In what follows, we shall introduce another notion of convexity of this type. In the case of complex functions, P.T. Mocanu [28], has introduced α-convexity, a notion intermediate to convexity ($\alpha = 1$) and stellarity ($\alpha = 0$). We have transpose in [45] this notion to sequences and now we want to do it in the case of real functions.

Definition 0.3. *The function $f : C \longrightarrow \mathbb{R}$ is said to be m-convex if for any $x, y \in C$ and any $t \in I$ it satisfies:*

$$t \cdot f(x) + m \cdot (1-t) \cdot f(y) - f(tx + m(1-t)y) \geq 0. \tag{9}$$

Remark 0.1. *The relation (4) is requested to be verified for any (a, b) on the segment joining $(1, 0)$ with:*

- $(0, 1)$ *in the case of convexity;*

- $(0, 0)$ *in the case of stellarity;*

- $(0, m)$ *in the case of* m*-convexity.*

Remark 0.2. *Let us denote for* $y < x$ *the points* $A(x, f(x)), B(y, f(y))$ *and* $P(my, mf(y))$*. Then* $f(z)$ *is under the chord:*

- - BA, *for* $z \in (y, x)$, *if* f *is convex;*
- - OA, *for* $z \in (0, x)$, *if* f *is starshaped;*
- - PA, *for* $z \in (my, x)$, *if* f *is* m*-convex.*

Remark 0.3. *Obviously,* m*-convexity for* $m = 1$ *and stellarity for* $m = 0$*. To obtain a hierarchy of* m*-convexities, we shall prove first some relations in the general case of* M*-convexity.*

Definition 0.4. *Two sets* M_1 *and* M_2 *are in the relation* $M_1 \leq M_2$ *if for any* $(a, b_1) \in M_1$ *there is a point* $(a, b_2) \in M_2$ *such that* $b_2 \geq b_1$*. By* $M \geq 0$ *we mean* $M \geq I \times \{0\}$*.*

Lemma 0.1. *If* $M \geq 0$*, the function* f *is* M*-convex and* $f(0) \leq 0$ *then it is starshaped.*

Proof. For any $a \in I$, there is an $(a, b) \in M$ with $b \geq 0$. Thus

$$f(ax) = f(ax + b0) \leq a \cdot f9x) + b \cdot f(0) \leq a \cdot f(x0.$$

□

Theorem 0.1. *If* $0 \leq M_1 \leq M_2$*, then any* M_2*-convex function* f *is* M_1*-convex.*

Proof. For any $(a, b_1) \in M_1$, there is an $(a, b_2) \leq M_2$ with $b_2 \geq b_1$.Thus:

$$f(ax + b_1 y) = f(ax + b_2 \cdot \frac{b_1}{b_2} y) \leq af(x) + b_2 f\left(\frac{b_1}{b_2} y\right) \leq af(x) + b_2 \frac{b_1}{b_2} f(y).$$

□

Theorem 0.2. *For any* $M \subset I^2$*, closed,* $M \geq 0$*, there is a function* $g : I \longrightarrow I$ *such that* M*-convexity be equivalent with* G*-convexity, where* G *is the graph of* g*.*

Proof. It is enough to define:

$$g(x) = max\{y : (x, y) \in M\}.$$

□

Remark 0.4. *All these properties may be proved in more general cases but this only complicates the enounces.*

Theorem 0.3. *If* $0 \leq m_1 \leq m_2 \leq 1$ *then:*

$$\textit{convexity} \implies m_2 - \ \textit{convexity} \implies m_1 \ \textit{convexity} \implies \ \textit{stellarity}.$$

Lemma 0.2. *The function* f *is* m*-convex on* $[a, b]$ *if and only if the function:*

$$f_m(x) = \frac{f(x) - m \cdot f(y)}{x - my}$$

is increasing on $my, b) \cap [a, b)$ *for any* $y \in [a, b)$*.*

Proof. The relation (9) is equivalent to:

$$\frac{f(x) - mf(y)}{x - my} \geq \frac{f(tx + m(1-t)y) - mf(y)}{t(x - my)}$$

that is, denoting $z = tx + m(1-t)y$ (or $t = (z - my) : (x - my)$ if it is given an $z \leq x$), we have:

$$f_m(z) \leq f_m(x), \ for \ z \leq x.$$

□

Lemma 0.3. *The function* $f \in C^1[a, b]$ *is* m*-convex on* $[a, b]$ *if and only if* $x > y$ *implies:*

$$f'(x) \geq \frac{f(x) - mf(y)}{x - my}. \tag{10}$$

Proof. By Lemma 2, we have $f_m(x)$ increasing, that is $f'_m(x) \geq 0$, which gives (10). □

Remark 0.5. *These results combine known properties of convex and of starshaped functions. Instead to give more such properties, we look for relation between the* m*-convexity and other convexities.*

M-convex function may be defined on a more general set.

Definition 0.5. *A set* D *(in a linear space) is called:*

i) M*-convex if for any* $x, y \in D$ *and* $(a, b) \in M,$ *we have* $ax + by \in D;$

ii) m*-convex if for any* $x, y \in D,$ *any* $t \in I,$ *we have* $tx + m(1-t)y \in D.$

Lemma 0.4. *A function defined on a* M*-convex set is* M*-convex if and only if its epigraph* E_f *is* M*-convex.*

Lemma 0.5. *A* m*-convex set* D *is* $\pi 2$*-convex.*

Proof. For any $x, y \in D$, the segment joining x with my and that joining y with mx are in D. They meet in the point $(x + y) \cdot m/(1 + m)$, that is D contains a 2-polygonal line joining x with y. □

Remark 0.6. *In spite of the lemmas 4 and 5, a m-convex function is not $\pi 2$-convex (as it is defined in [27]). Also it can be proved that the m-convexity is not a g-convexity in the sense of [48]. The M-convexity is also independent of the order generalizations of (4) so that it can be combined with any of them.*

We give here only:

Definition 0.6. *A function f defined on a M-convex set D is said to be M-quasi-convex if it satisfies (7), with $C = 1$ for any $x, y \in D$ and any $(a, b) \in M$.*

Lemma 0.6. *A function f is M-quasi-convex if and only if all his level sets are M-convex.*

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Buletinul Ştiinţific al Institutului Politehnic Cluj-Napoca
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AN INTEGRAL MEAN THAT PRESERVES SOME FUNCTION CLASSES

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Abstract. Se demonstrează că singurele medii integrale de forma (6) care păstrează clasa funcţiilor convexe, stelate, sau supraaditive se obţin pentru funcţia pondere de forma: $g(x) = x^u$, cu $u > 0$ arbitrar.

In [4] we have shown that the sequence $(X_n)_{n\geq 0}$ given by:

$$X_n = \frac{p_0x_0 + \cdots + p_nx_n}{p_0 + \cdots + p_n} \tag{1}$$

is convex for any convex sequence $(x_n)_{n\geq 0}$ iff there is an $u > 0$ such that the weights p_n be given by:

$$p_n = p_0\binom{u+n-1}{n} \tag{2}$$

where:

$$\binom{v}{n} = \frac{v(v-1)\dots(v-n+1)}{n!},\ \text{for } n \geq 1,\ \binom{v}{0} = 1. \tag{3}$$

A similar result we have proved in [5] for starshaped sequences and so we have obtained an improvement of the hierarchy of convexity from [3]. More generally, we have proved in [6] the following:

Theorem 1. *For some functions $a, b, c, d : \mathbb{R} \to \mathbb{R}$, let us consider the expression:*

$$t(x_n) = a(n)x_{n+2} + b(n)x_{n+1} + c(n)x_n + d(n)x_0 \tag{4}$$

and the set:

$$K = \{(x_n)_{n\geq 0} : \; t(x_n) \geq 0, \; \forall \; n \geq 0\}. \tag{5}$$

If for any $(x_n)_{n\geq 0} \in K$, the sequence $(X_n)_{n\geq 0}$ given by (1) is also in K and if the sequence $(kn)_{n\geq 0}$ belongs to K for any real k, then the weights p_n are given by (2) with some $u > 0$.

In what follows we establish similar properties for functions improving some results from [1].

In [2] is considered the integral mean:

$$F_g(f)(x) = \frac{1}{g(x)} \int_0^x f(s)g'(x)ds \tag{6}$$

and for fixed function g have a given sufficient conditions on f such that $F_g(f)$ have a prescribed property. We are interested in finding those functions g which furnish by (6) integral means that preserve a given function class: that of convex functions, of starshaped functions, or of superadditive functions. If g is such a function, then kg has the same property for any real $k \neq 0$. As a convention, we consider always $k = 1$.

For some fixed functions $a, b, c, d \; : \; [0, q] \; \to \; \mathbb{R}$, let us consider the operator:

$$Ty(x) = a(x)y''(x) + b(x)y'(x) + c(x)y(x) + d(x)y(0) \tag{7}$$

and the set of functions:

$$S = \{f : [0, q] \to \mathbb{R}, \; Tf(x) \geq 0, \; \forall \; x\}. \tag{8}$$

It may be easily proved the following:

Lemma. *The set S contains the functions $f(x) = kx$ for any real k iff:*

$$b(x) = -xc(x), \ \forall \ x \in [0, q]. \tag{9}$$

Theorem 2. *If it is satisfied (9) and for any $f \in S$, the transformation (6) gives a function $F_g(f)$ in S, then g must be of the form:*

$$g(x) = \exp\left(\int \frac{dx}{h(x)}\right) / h(x), \tag{10}$$

with

$$h(x) = c_1 x + c_2 x \int \frac{1}{x^2} \exp\left(\int \frac{xc(x)}{a(x)} dx\right)$$

if $a \neq 0$, or:

$$g(x) = x^u, \quad u > 0 \tag{11}$$

if $a = 0$.

Proof. From (9) we have that $f(x) = kx$ is in S for any real k. So, by hypothesis,

$$F_g(f)(x) = k \frac{1}{g(x)} \int_0^x t g'(t) dt$$

is also in S. But:

$$F(x) = F_g(f)(x) = \frac{k}{g(x)} \left[x g(x) - \int_0^x g(t) dt \right]$$

and denoting:

$$G(x) = \frac{1}{g(x)} \int_0^x g(t) dt \tag{12}$$

we have:

$$F(x) = k[x - G(x)],$$

and so

$$TF(x) = -k[a(x)G''(x) - xc(x)G'(x) + c(x)G(x)].$$

Hence $TF(x) \geq 0$ for any real k iff:

$$a(x)G''(x) - c(x)[xG'(x) - G(x)] = 0. \tag{13}$$

If $a = 0$, (13) becomes:

$$xG'(x) - G(x) = 0$$

or $G(x) = Cx$, which gives g of the form (11). If $a \neq 0$, we put:

$$z(x) = xG'(x) - G(x) \tag{14}$$

and the relation (13) becomes:

$$a(x)z'(x) - xc(x)z(x) = 0 \tag{15}$$

that is:

$$z(x) = c_2 \exp\left(\int \frac{xc(x)}{a(x)} dx\right)$$

and by the usual methods:

$$G(x) = c_1 x + c_2 x \int \frac{1}{x^2} \exp\left(\int \frac{xc(x)}{a(x)} dx\right) dx$$

which gives (10).

We may obtain some consequences.

Theorem 3. *The function $F_g(f)$ is convex for any convex f if and only if there is an $u > 0$ such that $g(x) = x^u$.*

Proof. If a convex function f is twice differentiable then $f''(x) \geq 0$. So if F_g preserves any convex function it also preserves the set (8) which correspond to the operator (7) with:

$$a = 1, \quad b = c = d = 0. \tag{16}$$

By the theorem 2:

$$g(x) = \frac{1}{c_1 x + c_2} \exp\left(\int \frac{dx}{c_1 x + c_2}\right) \tag{17}$$

which gives $g_1(x) = x^u$, for $c_1 \neq 0$, and $g_2(x) = \exp(ux)$, for $c_1 = 0$. But the function $f(x) = -\sin(ux)$ is convex on $[0, \pi/u]$, while $F''_{g_2}(f)(0) = -u^2$, that is g_2 don't preserve the convexity. That g_1 preserves the convexity for any $u > 0$ is proved in [2]. Namely, making in (6), for $g(x) = x^u$, the substitution: $s = xt^{1/u}$, it becomes:

$$F_u(f)(x) = \int_0^1 f(xt^{1/u})dt \tag{6'}$$

and the conclusion follows easily.

Theorem 4. *The function $F_g(f)$ is starshaped for any starshaped function f iff $g(x) = x^u$ for some $u > 0$.*

Proof. As it is proved in [1], if the starshaped function f is differentiable, then:

$$-xf'(x) + f(x) - f(0) \geq 0. \tag{18}$$

So, if F_g preserves the starshaped functions, it preserves also the set S which correspond to the operator (7) with:

$$a = 0, \quad c = 1 \tag{19}$$

and so g must be given by (11). That for such functions g, the transformation F_g really preserves starshaped functions is also proved in [2], using the relation (6').

Remark 1. In [1] it is considered also the class of superadditive functions, that is of functions $f : [0, q] \to \mathbb{R}$, which satisfy:

$$f(x + y) + f(0) \geq f(x) + f(y),\ \forall\ x, y, x + y \in [0, q]. \tag{20}$$

Although the relation (20) is not of the form asked by the theorem 2, the method of proof may be used to obtain:

Theorem 5. *The function $F_g(f)$ is superadditive for any superadditive f iff $g(x) = x^u$, for some $u > 0$.*

Proof. As in the proof of the theorem 2, because $f(x) = kx$ is superadditive for any real k, the function $F(x) = k[x - G(x)]$ also must be so. Hence

$$k[G(x) + G(y) - G(x + y)] \geq 0,\ \text{for } x, y, x + y \in [0, q]$$

that is, k being of arbitrary sign, the function G must satisfy the functional equation of Cauchy. In very large hypothesis, this implies: $G(x) = Cx$, which gives $g(x) = x^{(1-c)/C}$. That the condition is sufficient also follows from (6').

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Mathematica – Revue d'Analyse Numérique
et de Théorie de l'Approximation
Mathematica, Tome 27(50), no. 1, 1985, pp.53-57

GENERALIZED FINITE DIFFERENCES

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1. Finite differences may be easily expressed by means of divided differences (see [7]), but they are not simply divided differences with equidistant knots. In this paper, we consider a modified expression instead of the usual divided difference, which reduces exactly at finite difference in the case of equidistant knots. This expression is taken as definition for finite differences with respect to a Tchebysheff system.

2. We begin by presenting some definitions and results which we need in what follows.

Definition 1. The system U_{n+1} of real functions $(u_0, \ldots, u_n)$, will be called a *Tchebysheff system* (or a *T-system*) of the set E if the determinant:

$$V\binom{u_0, \ldots, u_n}{x_0, \ldots, x_n} = \begin{vmatrix} u_0(x_0) & \ldots & u_0(x_n) \\ \ldots & \ldots & \ldots \\ u_n(x_0) & \ldots & u_n(x_n) \end{vmatrix} \tag{1}$$

does not vanish for any system of different points $x_0, \ldots, x_n$ in E.

Definition 2. The system U_{n+1} of real functions $(u_0, \dots, u_n)$ will be called a *Markov system* (or an *M-system*) on E if every subsystem $(u_0, \dots, u_k)$, for $k = 0, \dots, n$, is a T-system on E.

Definition 3. Let U_{n+1} be a T-system on E and $x_0, \dots, x_n$ a set of different knots from E. The expression:

$$[U_{n+1}; x_0, \dots, x_n; f] = \frac{V\begin{pmatrix} u_0, \dots, u_{n-1}; f \\ x_0, \dots, x_{n-1}, x_n \end{pmatrix}}{V\begin{pmatrix} u_0, \dots, u_n \\ x_0, \dots, x_n \end{pmatrix}} \tag{2}$$

is said to be a *generalized divided difference* of the function f.

In [8] the following is proved:

Lemma 1. *If U_n is an M-system on E, then for any real function f on E and any set of different knots $x_0, \dots, x_n$ in E, we have:*

$$V\begin{pmatrix} u_0, \dots, u_{n-1}, f \\ x_0, \dots, x_{n-1}, x_n \end{pmatrix} \tag{3}$$

$$= C_n\{[U_n; x_1, \dots, x_n; f] - [U_n; x_0, \dots, x_{n-1}; f]\}$$

where

$$C_n = V\begin{pmatrix} u_0, \dots, u_{n-1} \\ x_1, \dots, x_n \end{pmatrix} V\begin{pmatrix} u_0, \dots, u_{n-1} \\ x_0, \dots, x_{n-1} \end{pmatrix} / V\begin{pmatrix} u_0, \dots, u_{n-2} \\ x_1, \dots, x_{n-1} \end{pmatrix}. \tag{4}$$

Remark 1. For the validity of relations (3) and (4) in the case $n = 1$, in what follows we make the convention:

$$V\begin{pmatrix} \dots u_{-1} \\ \dots \end{pmatrix} = 1 \tag{5}$$

Remark 2. From (3) and (4) we can obtain recurrence relations for divided differences in the form given by Silvia Toader in [8], as well as in the form given by G. Mülbach in [5]. Also they may be written as a recurrence relation for the determinants V:

$$(3') \qquad V\begin{pmatrix} u_0, \dots, u_{n-1}, f \\ x_0, \dots, x_{n-1}, x_n \end{pmatrix} = A_n V\begin{pmatrix} u_0, \dots, u_{n-1}, f \\ x_1, \dots, x_{n-1}, x_n \end{pmatrix}$$
$$-B_n V\begin{pmatrix} u_0, \dots, u_{n-2}, f \\ x_0, \dots, x_{n-2}, x_{n-1} \end{pmatrix}$$

where

$$(4') \qquad A_n = V\begin{pmatrix} u_0, \dots, u_{n-1} \\ x_0, \dots, x_{n-1} \end{pmatrix} / V\begin{pmatrix} u_0, \dots, u_{n-2} \\ x_1, \dots, x_{n-1} \end{pmatrix}$$

and

$$(4'') \qquad B_n = V\begin{pmatrix} u_0, \dots, u_{n-1} \\ x_1, \dots, x_n \end{pmatrix} / V\begin{pmatrix} u_0, \dots, u_{n-2} \\ x_1, \dots, x_{n-1} \end{pmatrix}.$$

Definition 4. *Finite differences in the knot x and with step h* of the function f are given successively by:

$$(6) \quad \Delta_h^1 f(x) = f(x+h) - f(x), \quad \Delta_h^{n+1} f(x) = \Delta_h^n f(x+h) - \Delta_h^n f(x).$$

3. For the system of functions $u_i(x) = x^i$ and the knots $x_i = x + ih$ $(i = 0, \dots, n)$, one obtains (see [3]):

$$(7) \qquad [U_{n+1}; x_0, \dots, x_n; f] = 1/(n!h^n)\Delta_h^n f(x).$$

But, if we consider instead of the divided difference (2), the expression:

$$(8) \qquad \Delta[U_n; x_0, \dots, x_n; f] = V\begin{pmatrix} u_0, \dots, u_{n-1}, f \\ x_0, \dots, x_{n-1}, x_n \end{pmatrix} / V\begin{pmatrix} u_0, \dots, u_{n-1} \\ x_0, \dots, x_{n-1} \end{pmatrix}$$

for the same system of functions and of knots, we have:

$$(9) \qquad \Delta[U_n; x, x+h, \dots, x+nh; f] = \Delta_h^n f(x).$$

This suggests the definition that follows. Let us suppose that the set E contains the knots:

$$(10) \qquad x_i = x + ih, \quad i = 0, \dots, n.$$

Definition 5. We call *finite difference of order n of the function f in the knot x, with step h, in respect to the T-system U_n on E*, the expression:

$$\Delta_h^{U_n} f(x) = \Delta[U_n; x, x+h, \dots, x+nh; f]. \tag{11}$$

From (2) and (8) we have:

Lemma 2. *For any M-system U_{n+1}, any function f and any system of different knots $x_0, \dots, x_n$, we have:*

$$\Delta[U_n; x_0, \dots, x_n; f] \tag{12}$$
$$= [U_{n+1}; x_0, \dots, x_n; f] V\begin{pmatrix} u_0, \dots, u_n \\ x_0, \dots, x_n \end{pmatrix} / V\begin{pmatrix} u_0, \dots, u_{n-1} \\ x_0, \dots, x_{n-1} \end{pmatrix}.$$

Lemma 3. *If U_n is an M-system on E, then for any function f and any system of different knots $x_0, \dots, x_n \in E$, the following recurrence relation is valid:*

$$\Delta[U_n; x_0, \dots, x_n; f] = \Delta[U_{n-1}; x_1, \dots, x_n; f] \tag{13}$$
$$-D_n \Delta[U_{n-1}; x_0, \dots, x_{n-1}; f]$$

where:

$$D_n = D(U_n; x_0, \dots, x_n) = \frac{V\begin{pmatrix} u_0, \dots, u_{n-2} \\ x_0, \dots, x_{n-2} \end{pmatrix} V\begin{pmatrix} u_0, \dots, u_{n-1} \\ x_1, \dots, x_n \end{pmatrix}}{V\begin{pmatrix} u_0, \dots, u_{n-2} \\ x_1, \dots, x_{n-1} \end{pmatrix} V\begin{pmatrix} u_0, \dots, u_{n-1} \\ x_0, \dots, x_{n-1} \end{pmatrix}}. \tag{14}$$

Proof. It follows from Lemma 1 and 2 (or from relations (3') and (8)) with a simple computation. The relations holds also for $n = 1$, if we make the conventions (5) and

$$\Delta[U_0; x; f] = f(x). \tag{15}$$

Particularly, if the knots are equidistant, we have:

Theorem 1. *If U_n is a M-system on a set E which contains the knots (10), then the following recurrence relation holds:*

$$\Delta_h^{U_n} f(x) = \Delta_h^{U_{n-1}} f(x+h) - D(U_n; x, h) \cdot \Delta_h^{U_{n-1}} f(x) \tag{16}$$

where $D(U_n; x, h)$ is given by (14) for the knots (10).

Remark 3. If $u_i(x) = x^i$ $(i = 0, \ldots, n-1)$ we have $D(U_n; x, h) = 1$ and (16) reduces to the usual relation (6) of definition of "ordinary" finite differences. In [6] we found some references at the paper [1] in which the author, I. Aldanondo, defines generalized finite differences for sequences. Let us give a sequence of real numbers $(d_n)_{n\geq 1}$. One defines finite differences of a sequence $(a_n)_{n\geq 1}$ by:

$$\Delta^p a_n = \Delta^{p-1} a_{n+1} - d_n \cdot \Delta^{p-1} a_n, \text{ with } \Delta^0 a_n = a_n \tag{16'}$$

which has the same form as (16). We shall return to this problem in [2].

Example 1. On $[0, \pi)$ let us have the system $C_n = (c_0, \ldots, c_{n-1})$ given by $c_k(x) = \cos kx$, for $k = 0, \ldots, n-1$, with $n \geq 2$. One proves (see [4]) that:

$$V\begin{pmatrix} c_0, \ldots, c_{n-1} \\ x_0, \ldots, x_{n-1} \end{pmatrix} = 2^{(n-1)(n-2)/2} \prod_{k=1}^{n-1} \prod_{l=0}^{k-1} (\cos x_k - \cos x_1)$$

so that

$$D(C_n; x_0, \ldots, x_n) = \prod_{k=0}^{n-2} \frac{\cos x_n - \cos x_{k+1}}{\cos x_{n-1} - \cos x_k}$$

and

$$D(C_n; x, h) = \frac{\sin(x + nh - h)\sin(x + nh - h/2)}{\sin(x + nh/2 - h/2)\sin(x + nh/2)}.$$

Thus (16) becomes:

$$\Delta_h^{C_n} f(x) = \Delta_h^{C_{n-1}} f(x+h) \tag{17}$$

$$-\frac{\cos(2x+2nh-3h/2)-\cos h/2}{\cos(2x+nh-h/2)-\cos h/2}\Delta_h^{C_{n-1}} f(x).$$

Example 2. On the same set $[0,\pi)$ let us have the system $S_n = (s_1,\ldots,s_n)$, where $s_k(x)=\sin kx$, for $k=1,\ldots,n$. In this case we have (see [4]):

$$V\begin{pmatrix} s_1,\ldots,s_n \\ x_1,\ldots,x_n \end{pmatrix} = 2^{n(n-1)/2}\prod_{k=1}^{n}\sin x_k \prod_{k=2}^{n}\prod_{l=1}^{k}(\cos x_k - \cos x_l)$$

so that:

$$D(S_n;x_0,\ldots,x_n) = D(C_n;x_0,\ldots,x_n)\sin x_n/\sin x_{n-1}$$

and (16) becomes:

$$\Delta_h^{S_n} f(x) = \Delta_n^{S_{n-1}} f(x+h) \tag{18}$$

$$-\frac{\sin(x+nh)\sin(x+nh-h/2)}{\sin(x+nh/2-h/2)\sin(x+nh/2)}\Delta_h^{S_{n-1}} f(x).$$

Example 3. Let q be such a function that the system $Q_n = (q_0,\ldots,q_{n-1})$, where $q_k(x)=q^k(x)$, for $k=0,\ldots,n-1$, is an M-system on a set E. We have:

$$D(Q_n;x_0,\ldots,x_n) = \prod_{i=1}^{n-1}\frac{q(x_n)-q(x_i)}{q(x_{n-1})-q(x_{i-1})}.$$

In particular, if $q(x)=p^x$ and $x_i = x+ih$, we have:

$$D(Q_n;x;h) = p^{(n-1)h}$$

that is, it is independent of x.

4. In contrast with the recurrence formula of G. Mühlbach from [5], relation (3) has permitted the construction (in [8]) of the generalized

Lagrange interpolation formula in Newton's form:

$$f(x)=\sum_{i=0}^{n}[U_{i+1};x_0,\dots,x_i;f]\frac{V\begin{pmatrix}u_0,\dots,u_{i-1},u_i\\ x_0,\dots,x_{i-1},x\end{pmatrix}}{V\begin{pmatrix}u_0,\dots,u_{i-1}\\ x_0,\dots,x_{i-1}\end{pmatrix}}\tag{19}$$

$$+[U_{n+2};x_0,\dots,x_n,x;f]\frac{V\begin{pmatrix}u_0,\dots,u_n,u_{n+1}\\ x_0,\dots,x_n,x\end{pmatrix}}{V\begin{pmatrix}u_0,\dots,u_n\\ x_0,\dots,x_n\end{pmatrix}}.$$

Taking into account relations (12), we may write (19) as:

$$f(x)=\sum_{i=0}^{n}\Delta[U_i;x_0,\dots,x_i;f]\frac{V\begin{pmatrix}u_0,\dots,u_{i-1},u_i\\ x_0,\dots,x_{i-1},x\end{pmatrix}}{V\begin{pmatrix}u_0,\dots,u_i\\ x_0,\dots,x_i\end{pmatrix}}\tag{20}$$

$$+\Delta[U_{n+1};x_0,\dots,x_n,x;f]\frac{V\begin{pmatrix}u_0,\dots,u_{n-1},u_n\\ x_0,\dots,x_{n-1},x\end{pmatrix}}{V\begin{pmatrix}u_0,\dots,u_n\\ x_0,\dots,x_n\end{pmatrix}}.$$

From here we obtain the generalized Newton's polynomial with equidistant knots:

$$N_{U_n}(f,x_0,h;x)=\sum_{i=0}^{n}\Delta_h^{U_i}f(x_0)\frac{V\begin{pmatrix}u_0,\dots,u_{i-1},u_i\\ x_0,\dots,x_0+ih-h,x\end{pmatrix}}{V\begin{pmatrix}u_0,\dots,u_i\\ x_0,\dots,x_0+ih\end{pmatrix}}\tag{21}$$

where, by convention, the first term is $f(x_0)$.

Example 4. For the system C_n from example 1, we have:

$$N_{C_n}(f,x_0,h;x)=f(x_0)+\sum_{i=0}^{n}\Delta_h^{C_i}f(x_0)\prod_{j=0}^{i-1}\frac{\cos x-\cos(x_0+jh)}{\cos(x_0+ih)-\cos(x_0+jh)}$$

where $\Delta_h^{C_i}f(x_0)$ are given by (17).

Remark 4. The system S_n from example 2, or the system Q_n from example 3 may be treated similarly. Other systems will be analyzed in [2].

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Proceedings of the Conference on Differential Equations
Cluj-Napoca, November 21-23, 1985
"Babeş-Bolyai" University Preprint 3(1986), 311-316

The resolution of some inequations with finite differences

Gh. Toader

Let us consider the linear equation with finite differences:

$$L_p(x_n) = \sum_{i=0}^{p} c_i \Delta^i x_n = \sum_{j=0}^{p} d_j x_{n+j} = 0, \quad n \geq 0, (1)$$

where d_p and d_0 does not vanish. As one knows (see [1]), the resolution of this equation is related to the solutions of the algebraic equation:

$$L_p(t^n)/t^n = \sum_{i=0}^{p} d_i t^i = d_p \prod_{i=1}^{p} (t - t_i). (2)$$

In what follows, we shall deal with the set of convex sequences in respect to the operator L_p, that is:

$$K_m(t_1, \ldots, t_p) = \{(x_n)_{n=0}^m : \ L_p(x_n) \geq 0, \ 0 \leq n \leq m - p\}$$

or:

$$K(t_1, \ldots, t_p) = \{(x_n)_{n\geq 0} : \ L_p(x_n) \geq 0, \ n \geq 0\}.$$

The case $t_1 = \ldots = t_p = 1$, corresponds to the usual convexity of order p (that is $L_p = \Delta^p$). In [8] we have proved that a sequence $x = (x_n)_{n\geq 0}$ is convex of order p if and only if it may be represented by:

$$x_n = \sum_{i=0}^{n} \binom{n+p-i-1}{p-1} y_i, y_i \geq 0 i \geq p.$$

Such representations were also given in [10], in the case $p = 2$, for any t_1 and t_2. We want to extend this result to the general case.

A leading part will be played by the sequence $(u_n)_{n\geq 0}$ defined by:

$$L_p(u_n) = 0, \ \forall \ n \geq 0; \ u_0 = \ldots = u_{p-2} = 0, \ u_{p-1} = 1/d_p.$$

For example, if $t_1 = \ldots = t_p$, then $u_n = (t_1^n/d_p)\binom{n}{p-1}$ and if $t_i \neq t_j$ for $i \neq j$, then

$$u_n = (1/d_p)\sum_{k=1}^{p}\left[t_k^n / \prod_{\substack{i=1\\ i\neq k}}^{p}(t_k - t_i) \right].$$

Lemma 1. *If:*

$$x_n = \sum_{i=0}^{n} u_{n+p-i-1}y_i \quad (2)$$

then:

$$L_p(x_n) = y_{n+p}.$$

Proof. From (1) and (2) we have:

$$L_p(x_n) = \sum_{j=0}^{p} d_j \sum_{i=0}^{n+j} u_{n+j+p-i-1}y_i$$

$$= \sum_{i=0}^{n}\left(\sum_{j=0}^{p} d_j u_{n+j+p-i-1}\right)y_i + \sum_{i=n+1}^{n+p}\left(\sum_{j=n+1}^{p} d_j u_{n+j+p-i-1}\right)y_i$$

$$= \sum_{i=0}^{n} L_p(u_{n+p-i-1})y_i - \sum_{k=1}^{p-1}\left(\sum_{j=0}^{k-1} d_j u_{p+j-k-1}\right)y_{n+k} + d_p u_{p-1}y_{n+p} = y_{n+p}.$$

Remark 1. As from (2) we obtain:

$$y_n = d_p\left[x_n - x_{n-1} - \sum_{i=0}^{n-1}(u_{n+p-i-1} - u_{n+p-i-2})y_i \right]$$

it results the following:

Lemma 2. *Let $P \subset \mathbb{R}$. We have $L_p(x_n) \in P$ for every $n \geq 0$, if and only if $(x_n)_{n\geq 0}$ is represented by (2) with $y_i \in P$ for $i \geq p$.*

Lemma 3. *The sequence $(x_n)_{n\geq 0}$ verifies the equation:*

$$L_p(x_n) = z_n, \quad n \geq 0$$

if and only if it is represented by (2) with $y_i = z_{i-p}$ for $i \geq p$.

Theorem 1. *The sequence $(x_n)_{n\geq 0}^{m}$ belongs to $K_m(t_1, \ldots, t_p)$ if and only if it may be represented by (2) with $y_i \geq 0$ for $p \leq i \leq m-p$.*

Remark 2. Some other sequences can also be represented using (2). For example, in [9] we have given the following definition: the sequence $x = (x_n)_{n\geq 0}$ is starshaped of order p if $\Delta^{p-1}((x_{n+1} - x_0)/(n+1)) \geq 0$, for $n \geq 0$. So, the sequence x is starshaped of order p if and only if it may be represented by:

$$x_n = y_0 + n\sum_{k=1}^{n}\binom{n+p-k-2}{p-2}y_k, y_k \geq 0 k \geq p.$$

Remark 3. In what follows, we are interested in the determination of the dual cone of $K_m(t_1,\ldots,t_p)$, i.e.

$$K_m^*(t_1,\ldots,t_p) = \left\{(a_n)_{n=0}^m : \sum_{n=0}^{m} a_n x_n \geq 0, \ \forall\ x \in K_m(t_1,\ldots,t_p)\right\}.$$

As it is stated even in [2], results of this nature were obtained for the first time by T. Popoviciu (see [7]).

Theorem 2. *The sequence* $(a_n)_{n=0}^m$ *belongs to* $K_m^*(t_1,\ldots,t_p)$ *if and only if it satisfies the relations:*

$$\sum_{n=k}^{m} a_n u_{n+p-k-1} = 0 k = 0,\ldots,p-1 (3)$$

and

$$\sum_{n=k}^{m} a_n u_{n+p-k-1} \geq 0 k = p,\ldots,m. (4)$$

Proof. From (2) we have:

$$\sum_{n=0}^{m} a_n x_n = \sum_{n=0}^{m} a_n \sum_{k=0}^{n} u_{n+p-k-1} y_k = \sum_{k=0}^{m} y_k \sum_{n=k}^{m} u_{n+p-k-1} a_n \geq 0. (5)$$

As y_k is of arbitrary sign for $k = 0,\ldots,p-1$, but it is nonnegative for $k = p$, the relation (5) is equivalent with (3) and (4).

Remark 4. For $L_p = \Delta^p$ the result may be find in [2] (in the special case $p = 2$) and in [6] (in the general case). In [10] we have put the result in a more convenient form. We want to do the same thing for the general case. For this we need the operator:

$$L_p^*(x_n) = \sum_{j=0}^{p} d_{p-j} x_{n+j}.$$

Theorem 3. *The sequence* $(a_n)_{n=0}^m$ *belongs to* $K_m^*(t_1,\ldots,t_p)$ *if and only if it may be represented by:*

$$a_n = L_p^*(b_n), n = 0,\ldots,m (6)$$

with

$$b_n \geq 0 p \leq n \leq m; \quad b_n = 0 n \leq p-1 n > m. (7)$$

Proof. If we put:

$$\sum_{n=k}^{m} u_{n+p-k-1} a_n = b_k (8)$$

from (3) and (4) we have (7). But (8) may be written as:

$$\begin{bmatrix} u_{p-1} & u_p & u_{p+1} & \dots & u_{p+m-1} \\ 0 & u_{p-1} & u_p & \dots & u_{p+m-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & u_{p-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \dots \\ a_m \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \dots \\ b_m \end{bmatrix}$$

which gives:

$$\begin{bmatrix} a_0 \\ a_1 \\ \dots \\ a_m \end{bmatrix} = \begin{bmatrix} d_p & d_{p-1} & d_{p-2} & \dots & 0 \\ 0 & d_p & d_{p-1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_p \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \dots \\ b_m \end{bmatrix}$$

that is (6).

Remark 5. If $L_p = \Delta^p$, then $L_p^* = \nabla^p = (-1)^p \Delta^p$, and we get the result from [10]. The transition from the conditions (3) and (4) to (6) and (7) remind the Minkowski-Farkas Lemma [11], but it does not represent a simple consequence of it, we needing the conditions $b_n = 0$ for $n = 0, \dots, p-1$.

Remark 6. Let the triangular matrix $Q = (q_{n,k})_{\substack{n=0,1,\dots \\ k=0,\dots,n}}$. It defines a transformation in the set of sequences: to any sequence $x = (x_n)_{n \geq 0}$ corresponds the sequence $X = Q(x) = (X_n)_{n \geq 0}$ given by:

$$X_n = \sum_{k=0}^{n} q_{n,k} x_k. (9)$$

We have the following problem: what are the matrices Q with the property that $x \in K_m(t_1, \dots, t_p)$ implies $Q(x) \in K_m(t_1, \dots, t_p)$. For this we need:

$$L_p(X_n) = \sum_{i=0}^{p} d_i \sum_{k=0}^{n+i} q_{n+i,k} x_k$$

$$= \sum_{k=0}^{n} \left(\sum_{i=0}^{p} d_i q_{n+i,k} \right) x_k + \sum_{k=n+1}^{n+p} \left(\sum_{i=k-n}^{p} d_i q_{n_i,k} \right) x_k \geq 0$$

for any $0 \leq n \leq m-p$ if $x \in K_m(t_1, \dots, t_p)$. This means that the sequences $a^n = (a_k^n)_{k=0}^{n+p}$ given by:

$$a_k^n = \sum_{i=j}^{p} d_i q_{n+i,k}, k = 0,\ldots,n+p,\ j = \max\{0, k-n\}$$

belong to $K_{n+p}^*(t_1,\ldots,t_p)$. From (3) and (4) we have the following:

Theorem 4. *The sequence X given by (9) is in* $K_m(t_1,\ldots,t_p)$ *for any* $x \in K_m(t_1,\ldots,t_p)$ *if and only if:*

$$\sum_{i=0}^{p} d_i \sum_{k=1}^{n+i} u_{k+p-l-1} q_{n+i,k} = 0, \quad l = 0,\ldots,p-1$$

$$\sum_{i=j}^{p} d_i \sum_{k=1}^{n+i} u_{k+p-l-1} q_{n+i,k} \geq 0, \quad l = p,\ldots,n+p,\ j = \max\{0, 1-n\}$$

for every $0 \leq n \leq m-p$.

Remark 7. For $L_p = \Delta^p$ such results may be found in [3] and [4] and for L_2 arbitrary in [5]. We want to put the result in another form, using the theorem 3.

Theorem 5. *The matrix Q has the property* $Q(x) \in K_m(t_1,\ldots,t_p)$ *for any* $x \in K_m(t_1,\ldots,t_p)$ *if and only if, for every* $0 \leq n \leq m-p$, *there is a nonnegative sequence* $v^n = (v_k^n)_{k \geq 0}$ *such that* $v_k^n = 0$ *for* $k < p$ *and for* $k > n+p$, *with the property that:*

$$\sum_{i=j}^{p} d_i q_{n+i,k} = L_p^*(v_k^n), k = 0,\ldots,n+p,\ j = \max\{0, k-n\}.$$

Remark 8. So $q_{i,j}$ may be chosen arbitrarily for $i = 0,\ldots,p-1$ and $j = 0,\ldots,i$ and then, taking v^n as it is requested by the theorem 5, we can build, step by step, $q_{n,k}$ for $n = p, p+1,\ldots$ and $k = 0,\ldots,n$.

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Anal. Numer. Theor. Approx. **15**(1986)2, 167-172

ON THE HIERARCHY OF CONVEXITY OF FUNCTIONS

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In the first part of this paper we simplify the proof of the main theorem of A.M. Bruckner and E. Ostrow from [4]. In the second part we extend this result, simplyfing also some proofs from our paper [8].

Let us denote the classes of continuous, convex, starshaped, respectively superadditive functions, by:

$$C(b) = \{f : [0, b] \longrightarrow \mathbb{R},\ f(0) = 0, f\ continuous\}$$

$$K(b) = \{f \in C(b); f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$
$$\forall t \in (0, 1),\ \forall x, y \in [0, b]\}$$

$$S^*(b) = \{f \in C(b);\ f(tx) \leq tf(x),\ \forall t \in (0, 1),\ x \in [0, b]\}$$

$$S(b) = \{f \in C(b); f(x + y) \geq f(x) + f(y),\ \forall x, y, x + y \in [0, b]\}.$$

In what follows we need some well known results (see [4]). They are more general, but we prove only the form that we use.

Lemma 0.1. *If the convex function f is differentiable, then f' is nondecreasing.*

Proof. Let us suppose $x > y$. From the definition we have:

$$\frac{f(y + t(x-y)) - f(y)}{t(x-y)} \leq \frac{f(x) - f(y)}{x - y}$$

which gives:

$$f'(y) \leq \frac{f(x) - f(y)}{x - y}.$$

Replacing t by $1 - t$, we obtain similarly:

$$\frac{f(x) - f(y)}{x - y} \leq f'(x).$$

□

Lemma 0.2. *The function f is starshaped if and only if $f(x)/x$ is nondecreasing.*

Proof. If $0 < x < y$, from $f(ty) \leq tf(y)$ and $t = x/y$ we have: $f(x) \leq (x/y)f(y)$. Conversely, if $t \in (0,1), tx < x$ and so $f(tx)/(tx) \leq f(x)/x$ gives the starshapedness of f. □

Lemma 0.3. *If the function f is differentiable, then it is starshaped if and only if:$f'(x) \geq f(x)/x$.*

Proof. The function $f(x)/x$ is nondecreasing if and only if:

$$[f(x)/x']' = [f'(x)x - f(x)]x^2 \geq 0.$$

□

Lemma 0.4. *For any $b > 0$ hold the inclusions:*

$$K(b) \subset S^*(b) \subset S(b).$$

Proof. a) If $f \in K(b), t \in (0,1)$ and $x \in [0,b]$ then:

$$f(tx) = f(tx + (1-t)0) \leq tf(x) + (1-t)f(0) = tf(x)$$

that is $f \in S^*(b)$.

b) If $f \in S^*(b)$ and $x, y, x+y \in [0,b]$, then, by lemma 2, we have:

$$f(x+y) = x\frac{f(x+y)}{x+y} + y\frac{f(x+y)}{x+y} \geq x\frac{f(x)}{x} + y\frac{f(y)}{y}$$

and so, $f \in S(b)$.

□

Remark 0.1. *These simple inclusions were not always known. So, in [5] it is proved that if f is convex and subadditive then $f(x)/x$ is non-increasing. In fact it is constant (if $f(0) = 0$).*

Definition 0.1. *The function f has the property "P" in the mean, if the function:*

$$F(x) = \frac{1}{x}\int_0^x f(t)dt,\ x > 0;\ F(0) = 0 \tag{1}$$

has the property "P".

Let us denote by:$MK(b), MS^*(b)$ and $MS(b)$ the sets of functions which are convex, starshaped, respectively superadditive in the mean.
The main result from [4] is:

Theorem 0.1. *For any $b > 0$ hold the strict inclusions:*

$$K(b) \subset MK(b) \subset S^*(b) \subset S(b) \subset MS^*(b) \subset MS(b). \tag{2}$$

Proof. a) Making in (1) the change of variable:$t = xu$, it becomes (see [3]):

$$F(x) = \int_0^1 f(xu)du. \tag{3}$$

If $f \in K(b)$, then for every $t \in (0,1)$ and $x, y \in [0,b]$ we have:

$$F(tx+(1-t)y) = \int_0^1 f(txu+(1-t)yu)du \leq \int_0^1 (t\cdot f(xu)+(1-t)\cdot f(yu))du = tF(x)+(1-t)F(y)$$

that is $f \in MK(b)$.

b) From (1) we have:

$$f(x)/x = F'(x) + F(x)/x \tag{4}$$

and if F is convex F' is nondecreasing and by lemmas 4 and 2, $f \in S^*(b)$.

c) The inclusion $S^*(b) \subset S(b)$ was proved in Lemma 4. It implies also the inclusion: $MS^*(b) \subset MS(b)$.

d) Let $f \in S(b)$. Then, for every $x \in [0,b]$ and every $u \in (0,1)$:

$$f(x) = f(xu + (1-u)x) \geq f(xu) + f(1-u)x)$$

and so:

$$f(x) - 2F(x) = \int_0^1 (f(x) - 2f(xu))du \geq \int_0^1 (f((1-u)x)-$$
$$-f(xu))du = \int_0^1 f((1-u)x)du - \int_0^1 f(ux)du = 0.$$

But this, by Lemma 3 and by relation (4) is equivalent with $f \in MS^*(b)$.

The strictness of the inclusions (2) was proved in [3] by more examples. A beautiful proof of this fact was also given by E.F. Beckenbach in [2], showing that the function $f(x) = (1 + 1/x)exp(-1/x)$ is in $K(1/3), MK(1/2), S^*((5-1/2), S(0,8955\ldots), MS^*(1)$ and $MS(1/\log 2)$ (the values of b being in every case the greatest possible). ☐

Remark 0.2. *In [6] it was considered the more general mean:*

$$F_g(x) = \frac{1}{g(x)} \int_0^x g'(t)f(t)dt, \ F_g(0) = 0. \tag{5}$$

Related to it, we have given in [8] the following result, whose proof we want to simplify.

Theorem 0.2. *If the transformation (5) preserves the convexity (the starshapedness or the superadditivity) then the function g is of the form:*

$$g(x) = kx^a,\ a > 0,\ k \neq 0. \tag{6}$$

Proof. The function $f_0(x) = cx$ is in $K(b)$ for any $c \in \mathbb{R}$, and so by lemma 4:

$$F_0(x) = \frac{c}{g(x)} \int_0^x g'(t) t dt$$

must be in $S(b)$. But c being of arbitrary sign, this happens if and only if, for $c = 1$, it verifies:

$$F_0(x + y) = F_0(x) + F_0(y)$$

for any $x, y, x + y \in [0, b]$. Thus (see [1]):$F_0(x) = kx$ which gives (6) with $a \neq 0$. But, if $a < 0$, (5) is not defined for $f(t) = C$, thus we must take $a > 0$. □

Remark 0.3. *As was pointed out to me by prof. J.E. Pečarić, such a result was also proved by I.B. Lacković in his doctoral dissertation using:*

$$F_g(x) = \int_0^x g(t) f(t) dt / \int_0^x g(t) dt$$

instead of (5).

Remark 0.4. *Denoting by F_a the function (5) with g given by (6), we have:*

$$F_a(x) = \frac{a}{x^a} \int_0^x t^{a-1} f(t) dt \tag{7}$$

and so:

$$f(x) = F_a(x) + (x/a) F_a'(x). \tag{8}$$

If we make in (7) the substitution (see [6]): $t = xu^{1/a}$, it becomes:

$$F_a(x) = \int_0^1 f(xu^{1/a}) du. \tag{9}$$

In what follows we shall prove that the condition from theorem 2 is also sufficient. For this, let us denote by $M^a K(b), M^a S^*(b)$ and $M^a S(b)$, the sets of functions $f \in C(b)$ with the property that the corresponding functions F_a belong to $K(b), S^*(b)$ respectively $S(b)$.

Theorem 0.3. *For any $b > 0$ and any $a > 0$ hold the following inclusions:*

$$\begin{array}{ccccccc} K(b) & \subset & M^a K(b) & \subset & S^*(b) & \subset & S(b) \\ & & & & \cap & & \cap \\ & & & & M^a S^*(b) & \subset & M^a S(b). \end{array} \tag{10}$$

Proof. a) If $f \in D(b),\ t \in (0,1),\ x, y \in [0,b]$, then, by (9):

$$F_a(tx + (1-t)y) = \int_0^1 f(txu^{1/a} + (1-t)yu^{1/a})du \leq$$
$$\leq \int_0^1 (tf(xu^{1/a}) + (1-t)f(yu^{1/a}))du = tF_a(x) + (1-t)F_a(y)$$

thus $f \in M^aK(b)$.

b) If $f \in M^aK(b)$, taking into account (8), we have:

$$f(x)/x = F_a(x)/x + F_a'(x)/a$$

thus, by lemmas 1,2 and 4, $f \in S^*(b)$. Lemma 4 gives also the inclusions:

$$S^*(b) \subset S(b) \textit{ and } M^aS^*(b) \subset M^aS(b).$$

c) If $f \in S^*(b),\ t \in (0,1)$ and $x \in [0,b]$, using (9), we have:

$$F_a(tx) = \int_0^1 f(txu^{1/a})du \leq \int_0^1 tf(xu^{1/a})du = tF_a(x)$$

that is $f \in M^aS^*(b)$.

d) For $f \in S(b),\ x, y, x+y \in [0,b]$, we have also:

$$F_a(x+y) = \int_0^1 f((x+y)u^{1/a})du \geq \int_0^1 (f(xu^{1/a}) + f(yu^{1/a}))du = F_a(x) + F_a(y)$$

thus $f \in M^aS(b)$.

□

Remark 0.5. *To prove the strictness of the inclusions, we may proceed for $a \neq 1$ as was done in [2] for $a = 1$: let $F(x) = exp(-1)/x)$ for $x \neq 0$ and $F(0) = 0$. From (8) we get:$f(x) = (1+1/ax) \cdot exp(1-/x)$ for $x \neq 0$ and $f(0) = 0$. If we denote by $k, k_a, s^*, s_a^*, s, s_a$ the largest value of b, for what f belongs to $K(b), M^aK(b), S^*(b), M^aS^*(b), S(b)$ respectively $M^aS(b)$, we have from [2]:$k_a = 1/2, s_a^* = 1$ and $s_a = 1/\ln 2$. As $f''(x) \geq 0$ only for $x \in [(a-4-\sqrt{a^2+8})/(4a-4); (a-4+\sqrt{a^2+8})/(4a-4)]$, we have $k = 0$ if $0 < a < 1$ and $k = (a-4+\sqrt{a^2+8})/(4a-4) < 1/2$ if $a > 1$. Using Lemma 3 we have also $s^* = (a-2+\sqrt{a^2+4})/2a < 1$.*

Applying Bruckner's test (see [2]), we obtain also that s is the unique positive solution of the equation:

$$ax(exp(1/x) - 2) = 4 - exp(1/x)$$

thus:$1/\ln 4 < s < 1/\ln 2$. *So:*

$$k < k_a < s^* < s_a^* < s_a.$$

We remark also that $1/\ln 4 < 1 = s_a^*$, that is, for $0 < a < 1$ we can have $s < s_a^*$ and so $S(b) \nsubseteq M^a S^*(b)$.

Remark 0.6. *In [7] was proved that if* $0 < a < c$ *then:*

$$M^a K(b) \supset M^c K(b) \text{ and } M^a S^*(b) \supset M^c S^*(b).$$

Thus (10) extends to:

$$\begin{array}{ccccccccc}
K(b) & \subset & M^c K(b) & \subset & M^a K(b) & \subset & S^*(b) & \subset & S(b) \\
 & & & & & & \cap & & \cap \\
 & & & & & & M^c S^*(b) & \subset & M^c S(b) \\
 & & & & & & \cap & & \\
 & & & & & & M^a S^*(b) & \subset & M^a S(b).
\end{array}$$

Moreover, if $0 < a < 1$:

$$S(b) \subset M^1 S^*(b) = M S^*(b) \subset M^a S^*(b).$$

We do not know if it is true that:

$$M^c S(b) \subset M^a S(b).$$

We have proved also similar results for sequences (see [9]).

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ON A GENERAL TYPE OF CONVEXITY

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In their book [5], A.W. Roberts and S.E. Varberg have proposed, for an independent study project, the following general notion of convexity. Let S be a subset of $I \times I$ (where $I = [0,1]$) and $D = [0,b]$. The function $f : D \longrightarrow \mathbb{R}$ is said to be S-convex it is verifies the relation:

$$f(sx+ty) \leq s \cdot f(x) + t \cdot f(y) \tag{1}$$

for any $(s,t) \in S$ and any $x, y \in D$.

The set of all S-convex functions defined on D is denoted by $K(S)$. Theoretically S can be a subset of $\mathbb{R}^2$ and a S-convex function can be defined on some subsets of a linear space. But even in the case given before can appear some complications. For example, from (1) we can see that $s + t \leq 1$ for any $(s,t) \in S$. Otherwise b must be infinite because $(s \cdot t) \cdot x \in D$ for $x \in D$.

Apart from the well known examples of S-convexity given in [5], let us to mention here another one, given by us in [7]. For a given $m \in I$, we say that the function $f : D \longrightarrow \mathbb{R}$ is m-convex if:

$$f(sx + m(1-s)y) \leq s \cdot f(x) + m(1-s) \cdot f(y)$$

for any $x, y \in D$ and any $s \in I$. A function is m-convex if and only if it is S_m-convex, where:

$$S_m = \{(s,t) : s \in I,\ t = m(1-s)\}.$$

As follows from Lemma 2, m-convexity is a notion intermediate to convexity ($m = 1$) and starshapendness ($m = 0$). So, it may be considered similar to a notion given for complex functions by P.T. Mocanu in [4].

For $s = t = 0$, from (1) we have $f(0) \leq 0$, that we suppose to be valid for any function which appears in what follows.

To answer to some questions from [5], we consider the following relation between sets: $S < S'$ it for any $(s,t) \in S$ there is an $(s,t') \in S'$ such that $t \leq t'$. We put $0 < S$ for $I \times \{0\} < S$.

Lemma 0.1. *If $0 < S$, any S-convex function f is starshaped.*

Proof. For any $s \in I$, there is a $t \geq 0$ such that $(s,t) \in S$. So, for any $x \in D$, we have:

$$f(sx) = f(sx + t \cdot 0) \leq s \cdot f(x) + t \cdot f(0) \leq s \cdot f(x).$$

$\square$

Lemma 0.2. *If $0 < S < S'$, then $K(S) \supset K(S')$.*

Proof. Let f be in $K(S')$ and x, y in D. For any $(s,t) \in S$ there is a $(s,t') \in S$ such that $t' \in t$. Hence:

$$f(sx + ty) = f(sx + t'(t/t')y) \leq sf(x) + t'f((t/t')y) \leq sf(x) + tf(y).$$

$\square$

Remark 0.1. *As $s + t \leq 1$ for $(s,t) \in S$, we deduce that the usual convexity is the most restrictive.*

Corollary 0.1. *If $0 < S$ and $G \subset S$, where:*

$$G = \{(s, t_s) : s \in I,\ t_s = \inf\{t : (s,t) \in S\}\},$$

then $K(S) = K(G)$.

Remark 0.2. *This property gives an answer, at least partial, to the question on the minimality of the set S which determines a class $K(S)$.*

But our central objective in this note is a related to another problem. In [2] A.M. Bruckner and E. Ostrow have proved that the integral mean:

$$F(f)(x) = \frac{1}{x} \int_0^x f(v)dv$$

preserves the convexity, the starshapendness and the superadditivity of the function f. In [3] it is considered a more general mean:

$$F_g(f)(x) = \frac{1}{g(x)} \int_0^x g'(v)f(v)dv. \tag{2}$$

In [6] we have obtained a characterization of the weight-functions g which give integral means F_g that preserve the above properties. We want to extend now this characterization to the case of S-convexity.

Theorem 0.1. *The function $F_g(f)$ is S-convex for any S-convex function f if and only if the function g is of the form:*

$$g(x) = k \cdot x^a,\ k \in \mathbb{R},\ a > 0. \tag{3}$$

Proof. The function $f_0(x) = cx$ is S-convex for any real c. Hence so must be also the function:

$$F_0(x) = F_g(f_0)(x) = \frac{c}{g(x)} \int_0^x g'(v) \cdot v dv.$$

But, c being of arbitrary sign, this happens if and only if, for $c = 1$:

$$F_0(sx + t \cdot y) = s \cdot F_0(x) + t \cdot F_0(y)$$

for $(s, t) \in S;\ x, y \in D$. Thus (see [1]) $F_0(x) = bx$ and so g must be of the form (3). If $a > 0$, (2) is not defined for $f(x) = c$.

Conversely, if g is given by (3), then (2) becomes:

$$F_a(f)(x) = \frac{a}{x^a} \int_0^x v^{a-1} \cdot f(v) dv. \tag{4}$$

making the substitution (given in [3]):$v = x \cdot w^{1/a}$, from (4) we get:

$$F_a(f)(x) = \int_0^1 f(x \cdot w^{1/a}) dw.$$

If f is in $K(S)$, for any $(s, t) \in S$ and any $x, y \in D$, we have:

$$\begin{aligned} F_a(f)(sx + ty) = \int_0^1 f((sx + ty)w^{1/a}) dw \leq s \int_0^1 f(xw^{1/a}) dw + \\ + t \int_0^1 f(yw^{1/a}) dw = s \cdot F_a(f)(x) + t \cdot F_a(f)(y) \end{aligned} \tag{5}$$

that is $F_a(f)$ is also in $K(S)$. □

If we denote:

$$M^a K(S) = \{f : F_a(f) \in K(S)\}$$

we have thus the following:

Corollary 0.2. *If $0 < S < S'$ and $a > 0$, then:*

$$\begin{array}{ccc} K(S') & \subset & K(S) \\ \cap & & \cap \\ M^a K(S') & \subset & M^a K(S). \end{array}$$

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SOME GENERALIZATIONS OF JESSEN'S INEQUALITY

Gh.TOADER

1. The inequality of Jessen is a generalization of that of Jensen (see [1]). In what follows we want to extend this inequality by replacing the isotony required in Jessen's inequality with a weaker condition. This allows the passage to inequalities for convex functions of higher orders.

2. Let us recall some notations and definitions. We consider the set $C = C[a, b]$ of all continuous real functions defined on $[a, b]$ and the set K of convex functions (from C).

Let also $e_k(k = 0, 1, \dots)$ and w_c (with $c \in (a, b)$) be the functions defined by:

$$e_k(x) = x^k, \ \forall x \in [a, b]$$

respectively

$$w_c(x) = |x - c|, \ \forall x \in [a, b]$$

A functional $A : C \longrightarrow R$ is linear if:

$$A(af + bg) = aA(f) + bA(g), \ \forall f, g \in C; \ a, b \in R$$

and it is isotonic if:

$$A(f) \geq 0, \ \forall f \geq 0.$$

We consider the following form of Jessen's inequality:

Theorem 0.1. *The function $f \in C$ is convex if and only if for any isotonic linear functional A, with $A(e_0) = 1, f$ verifies:*

$$f(A(e_1)) \leq A(f) \tag{1}$$

Remark 0.1. *As w_c is convex for any c, we have also:*

$$w_c(A(e_1)) \leq A(w_c) \tag{2}$$

We want to prove that (2) can replace the condition of isotony of A in (1). For this we need the following theorem of K. Toda [6] and T.Popoviciu [4]:

Theorem 0.2. *Every function $f \in K$ is the uniform limit of a sequence $(g_m)_{m\geq 1}$, given by:*

$$g_m = p_m \cdot e_0 + q_m \cdot e_1 + \sum_{k=0}^{m} p_{k,m} \cdot w_{c_{k,m}} \tag{3}$$

where $p_m, q_m \in \mathbb{R}, p_{k,m} \geq 0, c_{k,m} \in [a,b]$.

Using this theorem, in [7] it is proved the following result.

Theorem 0.3. *Let A be a linear and continuous operator defined on C. Then,*

$$A(f) \geq 0,\ \forall f \in K$$

if and only if:

$$A(e_0) = A(e_1) = 0,\ A(w_c) \geq 0, \forall c \in [a,b].$$

Similarly we can prove the following generalization of Theorem 1.

We define by L^+ the set of liniar and continuous functionals A, which satisfy $A(e_0) = 1$ and the relation (2).

Theorem 0.4. *The function $f \in C$ is convex if and only if for any $A \in L^+, f$ verifies (1).*

In fact we can prove a stronger result. Let S^+ denote the set of all superadditive, positively homogeneous, upper semicontinuous functionals A, which satisfy (2) and $A(ae_0 + be_1) \geq a + b \cdot A(e_1)$.

Theorem 0.5. *The function $f \in C$ is convex if and only if for any $A \in S^+, f$ verifies (1).*

Proof. The sufficiency is obviously: take $A(f) = sf(x) + (1-s)f(y)$ with $s \in (0,1), x, y \in [a,b]$.

The necessity: for a given convex function f, let the sequence $(g_m)_{m\geq 1}$ given by (3), which converges uniformly to f. If $A \in S^+$, we have:

$$A(g_m) \geq p_m + q_m \cdot A(e_1) + \sum_{k=0}^{m} p_{k,m} \cdot A(w_{c_k,m}) \geq g_m(A(e_1)).$$

As A is upper semicontinuous it follows:

$$A(f) \geq \lim_{m\to\infty} A(g_m) \geq \lim_{m\to\infty} g_m(A(e_1)) = f(A(e_1))$$

We remark that the converse inequality of (1) may be also used for the characterization of the convexity. So, let S^- denote the set of all subadditive, positively homogeneous, lower semicontinuous functionals A, which satisfy $A(a \cdot e_0 + b \cdot e_1) \leq a + b \cdot A(e_1)$ and:

$$w_c(A(e_1)) \geq A(w_c) \tag{2'}$$

Theorem 0.6. *The function $f \in C$ is convex if and only if for any $A \in S^-$, f verifies:*

$$f(A(e_1)) \geq A(f) \tag{1'}$$

□

3. As we have proved in [5], the convexity of order two may be characterized by the same relation (1) valid for some linear functionals which verify the conditions

$$A(e_0) = 1,\ A(e_2) = [A(e_1)]^2$$

and, of course, are not isotonic. In what follows we want to transpose theorem 5 to convexity of higher order. We need the following result from [2] which generalizes Theorem 2.

Let us denote by w_c^n the function defined by:

$$w_c^n(x) = \begin{cases} 0 & \text{if } x < c \\ (x-c)^{n-1} & \text{if } x \geq c \end{cases}$$

by P_n, the set of polynomials of degree at most n and by $K_n = K_n[a,b]$ the set of all n-convex functions (convex of order n).

Theorem 0.7. *Every function from $K_n (n \geq 1)$ can be approximated uniformly on $[a,b]$ by spline functions of the form:*

$$g_{m,1}(x) = p_{m,n}(x) + \sum_{k=1}^{1-1} q_{m,1,n,k} \cdot w_{c_k}^n(x) \tag{3'}$$

where $p_{m,n} \in P_{n-1}$ and $q_{m,1,n,k} > 0$.

Using this result, we obtain a direct generalization of Theorem 4 in:

Theorem 0.8. *The function $f \in C$ is in K_n if and only if for any continuous linear functional $A : C \longrightarrow \mathbb{R}$ with the properties:*

$$A(p) \geq p(A(e_1)),\ \forall p \in P_{n-1} \tag{4}$$

and

$$w_c^n(A(e_1)) \leq A(w_c^n),\ \forall c \in (a,b) \tag{5}$$

the function f verifies:

$$f(A(e_1)) \leq A(f)$$

In fact, we can prove the following general result which extends also Theorem 5: let S^+ denote the set of all superadditive, positively homogeneous, upper semicontinuous functionals, $A : C \longrightarrow \mathbb{R}$, which satisfy (4) and (5).

Theorem 0.9. *The function $f \in C$ is in K_n if and only if for any $A \in S_n^+$, it verifies (1).*

Inequality (1') may be also used : let S^- denote the set of suadditive, positively homogeneous, lower semicontinuous functionals $A : C \longrightarrow \mathbb{R}$ which satisfy:

$$A(p) \le p(A(e_1)) \tag{4'}$$

and

$$w_c^n(A(e_1)) \ge A(w_c^n),\ \forall c \in (a,b). \tag{5'}$$

Theorem 0.10. *The function $f \in C$ is in K_n if and only if for any $A \in S_n^-$ it verifies (1').*

In the same manner, we can give the following generalization of the main result from [2], which extends also Theorem 3.

Theorem 0.11. *Let $B : C \longrightarrow \mathbb{R}$, be a superadditive, positively homogeneous, upper semicontinuous functional. In order that $B(f) \ge 0$ for every $f \in K_n (n \ge 1)$ it is necessary and sufficient that:*

$$B(p) \ge 0,\ \forall p \in P_{n-1} \tag{6}$$

and

$$B(w_c^n) \ge 0,\ \forall c \in (a,b). \tag{7}$$

Remark 0.2. *There is a strong connection between the functionals A from Theorem 9 and the functionals B from Theorem 10.*

If A satisfies (4) and (5), then

$$B(f) = A(f) - f(A(e_1))$$

verifies (6) and (7). Conversely, if B has properties (6) and (7) and $B(e_1) = 0$, then

$$A(f) = B(f) + f(B(e_1))$$

verifies (4) and (5).

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ON THE CONVEXITY OF ORDER TWO OF FUNCTIONS

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The convexity (of order one) of a function f is defined by:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),\ \forall t \in (0,1) \tag{1}$$

In what follows, we want to give analogous conditions for the convexity of order two.

Basic definitions and notations may be found in [2], [3] and [4].So, let f be defined on $[a,b]$ and $x_1 < x_2 < \cdots < x_n$ be points in this interval. The divided differences of the function f on these points is given by:

$$[x_1, x_2, \ldots, x_n; f] = \sum_{k=1}^{n} \frac{f(x_k)}{u'(x_k)}$$

where:

$$u(x) = \prod_{j=1}^{n}(x - x_j).$$

For $n = 3$, if $[x_1, x_2, x_3; f] \geq 0$, we get:

$$f(x_2) \leq \frac{x_3 - x_2}{x_3 - x_1} f(x_1) + \frac{x_2 - x_1}{x_3 - x_1} f(x_3)$$

or putting:

$$p_1 = (x_3 - x_2)/(x_3 - x_1)\ and\ p_3 = (x_2 - x_1)/(x_3 - x_1) \tag{2}$$

we have:

$$p_1, p_3 \geq 0, p_1 + p_3 = 1, p_1x_1 + p_3x_3 = x_2 \tag{3}$$

and so:

$$f(p_1x_1 + p_3x_3) \leq p_1f(x_1) + p_3f(x_3)$$

that is in (**??**). Continuing on this way, we get the well known result:

Theorem 0.1. *The following conditions are equivalent:*

a) *the function* f *is convex on* $[a,b]$*;*

b) *for any points:* $a \leq x_1 \leq x_2 \leq x_3 \leq b$, $[x_1, x_2, x_3; f] \geq 0$;

c) *for any points:*$a \leq x_1 \leq x_2 \leq b$ *and any numbers* $p_1 \leq 0, p_2 \geq 0$ *such that:*$p_1 + p_2 = 1, p_1x_1 + p_2x_2 \leq b$, *we have:*

$$f(p_1x_1 + p_2x_2) \geq p_1f(x_1) + p_2f(x_2);$$

d) *if:* $a \leq x_1 \leq x_3 \leq b, p_1, p_3 > 0, p_1 + p_3 = 1$, *then:*

$$f(p_1x_1 + p_3x_3) \leq p_1f(x_1) + p_3f(x_3)$$

;

e) *if:*$a \leq x_2 \leq x_3 \leq b, p_2 \geq 0, p_3 \leq 0, p_2 + p_3 = 1$ *and* $p_2x_2 + p_3x_3 \geq a$ *then:*

$$f(p_2x_2 + p_3x_3) \geq p_2f(x_2) + p_3f(x_3).$$

Remark 0.1. *As it is known, if* $M_1(x_1, y_1)$ *and* $M_2(x_2, y_2)$ *are two points in the plane, then* $M(p_1x_1 + p_2x_2, p_1y_1 + p_2x_2)$, *with* $p_1 + p_2 = 1$ *is an arbitrary point on the straight line determined by* M_1^1 *and* M_2.

Moreover, if $x_1 \leq x_2$ then:

$$p_1x_1 + p_2x_2 \leq x_1 \; iff \; p_2 \leq 0;$$
$$x_1 \leq p_1x_1 + p_2x_2 \leq x_2 \; iff \; p_1, p_2 \geq 0;$$
$$p_1x_1 + p_2x_2 \geq x_2 \; iff \; p_1 \leq 0.$$

Hence we get from Theorem 1 the well known geometric interpretation of the convexity.

Analogously, for $n = 4$, from $[x_1, x_2, x_3, x_4; f] \geq 0$, we have:

$$\begin{aligned} f(x_4) \geq & \frac{(x_4 - x_2)(x_4 - x_3)}{(x_2 - x_1)(x_3 - x_1)} f(x_1) - \frac{(x_4 - x_1)(x_4 - x_3)}{(x_2 - x_1)(x_3 - x_2)} f(x_2) + \\ & + \frac{(x_4 - x_1)(x_4 - x_2)}{(x_3 - x_1)(x_3 - x_2)} f(x_3) \end{aligned} \tag{4}$$

and putting:

$$\begin{aligned} p_1 = & \frac{(x_4 - x_2)(x_4 - x_3)}{(x_2 - x_1)(x_3 - x_1)}, p_2 = \frac{(x_4 - x_1)(x_4 - x_3)}{(x_2 - x_1)(x_2 - x_3)} \\ & p_3 = \frac{(x_4 - x_1)(x_4 - x_2)}{(x_3 - x_1)(x_3 - x_2)} \end{aligned} \tag{5}$$

we have:

$$p_1 > 0, p_2 < 0, p_3 > 0, p_1 + p_2 + p_3 = 1, p_1x_1 + p_2x_2 + p_3x_3 = x_4 \tag{6}$$

and hence:

$$f(p_1x_1 + p_2x_2 + p_3x_3) \leq p_1f(x_1) + p_2f(x_2) + p_3f(x_3). \tag{7}$$

Expressing x_1, x_2 respectively x_3, we get three other similar relations. But now (5) and (6) are not longer equivalent with (7) and (6). For example, we have also the relation:

$$p_1x_1^2 + p_2x_2^2 + p_3x_3^2 = x_4^2.$$

In fact, given the non-collinear points $M_k(x_k, y_k)$ for $k = 1, 2, 3$, a point $M(x, y)$ is on the parabola determined by these points if and only if:

$$y = p_1y_1 + p_2y_2 + p_3x_3 \tag{8}$$

where:

$$p_1 + p_2 + p_3 = 1 \tag{9}$$

$$p_1x_1 + p_2x_2 + p_3x_3 = x \tag{10}$$

$$p_1x_1^2 + p_2x_x^2 + p_3x_3^2 = x^2. \tag{11}$$

To precise the position of the point M on the parabola, we remark that any point $M(x, y)$ from the plane may be given by (8), (9) and (10).
Moreover, if we denote by M_k one of the points (that is M_1, M_2 or M_3) and by M_i and M_j the other two, we have p_k zero if M is on the straight line M_iM_j, positive if M and M_k are in the same semi-plane determined by M_iM_j andnegative if M and M_k are in opposite semi-planes. So, if $x_1 \leq x_2 \leq x_3$, the point M given by (8), (9) and (10) and (11) is between M_1 and M_2 iff $p_1, p_2 \geq 0$ and $p_3 \leq 0$. It is between M_2 and M_3 iff $p_1 \leq 0, p_2, p_3 \geq 0$. But it is before M_1 of after M_3, then $p_1 \geq 0, p_2 \leq 0$, $p_3 \geq 0$, thus these two cases cannot be separated on this way. Finally (taking the first equivalence as definition) we have:

Theorem 0.2. *The following conditions are equivalent:*

a) the function f is convex of order two on $[a, b]$;

b) for any points: $a \leq x_1 < x_2 < x_3 < x_4 \leq b$, $[x_1, x_2, x_3, x_4; f] \geq 0$;

c) for any points $a \leq x_1 < x_3 < x_4 \leq b$ and any numbers p_1, p_3, p_4 such that: $p_1 + p_3 + p_4 = 1, p_3 > 0, p_4 < 0$ and $p_1x_1 + p_3x_3 + p_4x_4)^2 = p_1x_1^2 + p_3x_3^2 + p_4x_4^2$, we have:

$$f(p_1x_1 + p_3x_3 + p_4x_4) \geq p_1f(x_1) + p_3f(x_3) + p_4f(x_4)$$

;

d) if:$a \le x_1 < x_2 < x_4 \le b$ *and* $p_1 + p_2 + p_4 = 1, p_1 < 0, p_2 > 0, p_4 > 0, (p_1 x_1 + p_2 x_2 + p_4 x_4)^2 = p_1 x_1^2 + p_2 x_2^2 + p_4 x_4^2$ *then:*

$$f(p_1 x_1 + p_2 x_2 + p_4 x_4) \ge p_1 f(x_1) + p_2 f(x_2) + p_4 f(x_4)$$

;

Remark 0.2. *If we denote by* e_k *the function given by* $e_k(x) = x^k$ $(k = 0, 1, 2, \dots)$*, by the point d) of Theorem 2, the convexity of order two may be characterized by the same relation:*$f(A(e_1)) \le A(f)$*, valid for some linear functionals which verify the conditions* $A(e_0) = 1$ *and* $A(e_2) = [A(e_1)]^2$*, but are not positive as they are in the inequality of Jessen [1].*

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SOME ASPECTS OF CONVEXITY OF FUNCTIONS AND OF SEQUENCES

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1. General definitions

Let E be a subset of $\mathbb{R}$ and f a real function defined on E. The divided difference (of order n) on the distinct points $x_0, x_1, \ldots, x_n$ is defined recurrently by:

$$[x_0; f] = f(x_0), \quad [x_0, x_1; f] = ([x_0; f] - [x_1; f])/(x_0 - x_1)$$

$$[x_0, \ldots, x_n; f] = ([x_0, \ldots, x_{n-1}; f] - [x_1, \ldots, x_n; f])/(x_0 - x_n).$$

The function f is said to be n-convex (or convex of order n) if it verifies: $[x_0, \ldots, x_{n+1}; f] \geq 0$ for any distinct points from E.

Following the book [20] of T. Popoviciu, the convex functions of order one on $[a, b]$ were defined in 1893 by O. Stolz in [24], but their systematic study begins with the paper [8] of J.L.W.V. Jensen from 1906. In 1916, L. Galvani has considered in [6] convex functions of order one on an arbitrary set E. The generalization to an order n, has appeared in two thesis: the first in 1926 of E. Hopf [7] for functions defined on $[a, b]$ and the second [16], in 1934 of T. Popoviciu for functions defined on an arbitrary set E. The generalization to a set E is essential as it is proved in [17] by T. Popoviciu. As a matter of fact, T. Popoviciu has many relevant results on convexity, which constantly preoccupied his activity. It is thus natural that some of the aspects which we analyse in what follows were initiated by him.

If we consider the points $x_0 < x_1 < \cdots < x_m$ $(m \geq n+1)$ from E, we have the following mean theorem of T. Popoviciu [20]: for any indices $0 \leq i_0 < i_1 < \cdots < i_{n+1} \leq m$ there are the constants $a_i > 0$ with $a_0 + \cdots + a_{m-n-1} = 1$ such that:

$$[x_{i_0}, \ldots, x_{i_{n+1}}; f] = \sum_{i=0}^{m-n-1} a_i [x_i, x_{i+1}, \ldots, x_{i+n+1}; f].$$

It results that if the set E is at most countable,

$$E = \{x_0, x_1, \ldots, x_{n+1}, \ldots\}$$

a function $f : E \to \mathbb{R}$ is n-convex iff it verifies $[x_k, \ldots, x_{k+n+1}; f] \geq 0$ for $k \geq 0$. This leads also to the definition of n-convex sequences.

The finite differences of the sequence $(x_n)_{n \geq 0}$ are defined by:

$$\Delta^0 x_m = x_m \quad \Delta^n x_m = \Delta^{n-1} x_{m+1} - \Delta^{n-1} x_m \text{ for } n \geq 1.$$

The sequence $(x_m)_{m \geq 0}$ is called n-convex (or convex of order n) if $\Delta^n x_m \geq 0$ for $m \geq 0$. For convex (of order two) sequences, in the book [23] of A.W. Roberts and D.E. Varberg, are given as basic references the books [1] and [39] on Fourier series (see also [5]). Let us remark that with this definition of n-convexity adopted for sequences (see for example

[11]) appears a difference of an unity between the order of convexity of a sequence if it is considered also as discrete function.

Before ending this introduction we remind that in the literature have appeared many generalizations of the convexity. We don't refer to them in what follows, but we can send to [14] and [26], where may be found most of them.

2. Representation theorems

In what follows we denote by $K_n(E)$ the cone of n-convex functions on E and by K_n the cone of n-convex sequences.

There are more approximation and representation theorems of convex functions. We remind some of them which are used in what follows. So, in [38] it is given the theorem of K. Toda [37] and T. Popoviciu [22].

Theorem 1. *a) Every function of the sequence:*

$$g_m(x) = px + q + \sum_{k=0}^{m} p_k|x - x_k|, \quad m \geq 1 \tag{1}$$

where $x, x_k \in [a, b]$, $p, q \in \mathbb{R}$, $p_k \geq 0$ $(k = 0, \ldots, n)$, belongs to $K_1[a, b]$.

b) Every function f from $K_1[a, b]$ is the uniform limit of a sequence of the form (1).

R. Bojanic and J. Roulier have given in [3] a generalization to an arbitrary order, using also some results of T. Popoviciu from [18]. Let us denote by w_c^n the function defined by:

$$w_c^n(x) = \begin{cases} 0 & \text{if} \quad x < c \\ (x-c)^{n-1} & \text{if} \quad x \geq c \end{cases}$$

and by P_n the set of polynomials of degree at most n.

Theorem 2. *Every function f from $K_n[a,b]$ $(n \geq 1)$ can be approximated uniformly on $[a,b]$ by spline functions of the form:*

$$g_{m,l}(x) = p_{m,n}(x) + \sum_{k=1}^{l-1} q_{m,l,n,k} w_{c_k}^n(x)$$

where $p_{m,n}$ belong to P_{n-1} and $q_{m,l,n,k}$ are positive constants.

In the study of n-convex sequences, we have used also more representation theorems. The simplest may be found for example in [29]:

Theorem 3. *A sequence $(x_m)_{m\geq 0}$ is in K_n if and only if it may be represented by:*

$$x_m = \sum_{k=0}^{m} \binom{m+n-k-1}{n-1} y_k$$

where $y_k \geq 0$ for $k \geq n$.

Using the method from [10] we can put this result in another form. In the vector space S of all sequences, it is considered the following metric d: for $x = (x_m)_{m\geq 0}$ and $y = (y_m)_{m\geq 0}$ we put:

$$d(x,y) = \sum_{m=0}^{\infty} 2^{-m} \frac{|x_m - y_m|}{1 + |x_m - y_m|}.$$

Let us denote by e_m the sequence with the components:

$$e_{m,k} = \binom{n-1+j-m}{n-1} \tag{2}$$

where $\binom{m}{k} = 0$ if $m < k$.

Theorem 4. *A sequence x is in K_n if and only if:*

$$x = \lim_{m\to\infty} y_0 e_0 + \cdots + y_m e_m$$

for $y_k \geq 0$ for $k \geq n$ and the limit is taken in (S,d).

3. Positive operators on $K_n(E)$

T. Popoviciu has begun in [19] the study of characterization of positive operators defined on $K_n(E)$. Some of his results may be found in [20]. Let us note the following one:

Theorem 5. *The inequality:*

$$\sum_{i=1}^{m} p_i f(x_i) \geq 0, \quad p_i \neq 0 \text{ for } i = 1, \ldots, m$$

is valid for any n-convex function defined on $x_1 < x_2 < \cdots < x_m$ if and only if:

$$\sum_{i=1}^{m} p_i x_i^k = 0, \text{ for } k = 0, 1, \ldots, n$$

$$\sum_{i=1}^{r} p_i (x_i - x_{r+1}) \ldots (x_i - x_{r+n}) \leq 0, \quad r = 1, \ldots, m-n-1.$$

In [3] it is used the representation from Theorem 2 to obtain a result of this type. It may be formulated more generally as follows. Let X be a topological vector space and $P \subset X$ a closed, convex cone in X.

Theorem 6. *Let $A : C[a,b] \to X$ be a continuous linear operator. In order that $A(f) \in P$ for every $f \in K_n[a,b]$ $(n \geq 1)$ it is necessary and sufficient that:*

i) $A(p) = 0$ for every $p \in P_{n-1}$

ii) $A(w_c^n) \in P$ for every $c \in (a,b)$.

In 1981, J.E. Pečarić has transposed in [13] the results of T. Popoviciu to sequences. Let us denote:

$$K_n(m) = \{(x_k)_{k=1}^m : \ \Delta^n x_k \geq 0, \ k = 1, \ldots, m-n-1\}$$

and

$$K_n^*(m) = \left\{ (p_k)_{k=1}^m : \sum_{k=1}^m p_k x_k \geq 0, \ \forall\, x = (x_k)_{k=1}^m \in K_n(m) \right\}.$$

Theorem 7. *The m-tuple $p = (p_k)_{k=1}^m$ belongs to $K_n^*(m)$ iff:*

$$\sum_{i=1}^m (i-1)^{(k)} p_i = 0, \ \text{for } k = 0, 1, \ldots, n-1$$

and

$$\sum_{i=k}^n (i-k+n-1)^{(n-1)} p_i \geq 0, \ \text{for } k = n+1, \ldots, m$$

where:

$$x^{(0)} = 1, \quad x^{(k)} = x(x-1)\ldots(x-k+1), \quad k \geq 1.$$

This is transposed in [36] as follows:

Theorem 8. *The m-tuple $p = (p_k)_{k=1}^m$ belongs to $K_n^*(m)$ iff:*

$$p_k = \nabla^n q_k = (-1)^n \Delta^n q_k, \quad k = 1, \ldots, m$$

where:

$$q_k = 0 \text{ for } k = 1, \ldots, n \text{ and } k = m+1, \ldots, m+n$$

and

$$q_k \geq 0 \text{ for } k = n+1, \ldots, m.$$

Using Theorem 4, in [34] it is given the following result, analogous with that of Theorem 6.

Theorem 9. *Let $A : S \to X$ be a continuous linear operator. In order that $A(x) \in P$ for every $x \in K_n$ it is necessary and sufficient that:*

i) $A(e_k) = 0$, for $k = 0, 1, \ldots, n-1$

ii) $A(e_k) \in P$, *for* $k \geq n$

where e_k *is given by (2).*

The results of the theorems 6 and 9 may be applied also for non-convex elements. Let us give an example:

Theorem 10. *Let* $A : S \to \mathbb{R}$ *be a continuous linear functional which verifies:*

$$A(e_k) = 0 \text{ for } k = 0, 1, \ldots, n-1, \quad A(e_k) \geq 0 \text{ for } k \geq n.$$

If $x = (x_k)_{k \geq 0} \in S$ *is such that:*

$$m \leq \Delta^n x_k \leq M, \ \forall \ k \geq 0$$

then

$$mA(w) \leq A(x) \leq MA(w)$$

where $w = (w_k)_{k \geq 0}$ *is given by* $w_k = \binom{k}{n}$ *for* $k \geq n$ *and* $w_k = 0$ *for* $k < n$.

4. Jessen's inequality

In 1931, B. Jessen has generalized in [9] the well known Jensen's inequality to isotonic linear functionals. Some aspects are analysed in [20] and more recently by P.R. Beesack and J.E. Pečarić in [2].

Let $E \neq \emptyset$ be a set and L be a linear class of functions $g : E \to \mathbb{R}$ such that $1 \in L$. A linear functional $B : L \to \mathbb{R}$ is said to be isotonic if $B(g) \geq 0$ for $g \geq 0$ on E.

Theorem 11. *If* f *is in* $K_1[a, b]$ *and* B *is any isotonic linear functional with* $B(1) = 1$, *then for all* $g \in L$ *such that* $f(g) \in L$, *we have* $B(g) \in [a, b]$ *and* $f(B(g)) \leq B(f(g))$.

Starting from [33] we have proposed to renounce at the isotony, which is proper to convexity of order one. So, in [34] and [35] we pass to convexity of higher order.

Theorem 12. *The function $f \in C[a,b]$ is in $K_n[a,b]$ if and only if for any continuous linear functional $B : C[a,b] \to \mathbb{R}$ with the properties:*

$$B(p) = p(B(e)),\ \forall\, p \in P_{n-1} \tag{3}$$

and

$$B(w_c^n) \geq w_c^n(B(e)),\ \forall\, c \in (a,b) \tag{4}$$

where $e(x) = x$, the function f verifies the inequality:

$$f(B(e)) \leq B(f).$$

Before ending, we must remark that there is a bijection between the functionals A from Theorem 6 (for $X = \mathbb{R}$ and $P = \mathbb{R}_+$) and the functionals B from Theorem 12. Indeed, if A satisfies the conditions:

$$A(p) = 0 \text{ for } p \in P_{n-1} \tag{5}$$

and

$$A(w_c^n) \geq 0 \text{ for } c \in (a,b) \tag{6}$$

then the functional B defined by:

$$B(f) = A(f) + f(0)$$

verifies (3) and (4). Conversely, if B has the properties (3) and (4) then:

$$A(f) = B(f) - f(B(e))$$

verifies (5) and (6).

5. Hierarchies of convexity

In 1962, A.M. Bruckner and E. Ostrow started in [4] a study on that is now called "hierarchy of convexity". Let us denote the classes of continuous, convex, starshaped, respectively superadditive functions by:

$C(b) = \{f : [0, b] \to \mathbb{R},\ f(0) = 0,\ f \text{ continuous}\}$

$K(b) = \{f \in C(b),\ f \in K_1[0, b)\}$

$S^*(b) = \{f \in C(B) :\ f(tx) \le tf(x),\ \forall\ t \in (0, 1),\ x \in [0, b]\}$

$S(b) = \{f \in C(b) :\ f(x + y) \ge f(x) + f(y),\ \forall\ x, y, x + y \in [0, b]\}$.

We say that the function f has the property "P" in the mean, if the function:

$$F(x) = \frac{1}{x}\int_0^x f(t)dt, \quad x > 0, \quad F(0) = 0$$

has the property "P". Let us denote by $MK(b)$, $MS^*(b)$ and $MS(b)$ the sets of functions which are convex, starshaped, respectively superadditive in the mean. The result from [4] may be formulated as:

Theorem 13. *For any $b > 0$ hold the strict inclusions:*

$$K(b) \subset MK(b) \subset S^*(b) \subset S(b) \subset MS^*(b) \subset MS(b).$$

In 1983, in [25] we have transposed this result to sequences. We call a sequence $(x_n)_{n\ge0}$ starshaped if:

$$\frac{x_{n-1} - x_0}{n - 1} \le \frac{x_n - x_0}{n},\ \forall\ n \ge 2$$

and superadditive if:

$$x_{n+m} + x_0 \ge x_n + x_m,\ \forall\ n, m > 0.$$

Let us denote by K, S^*, S the sets of convex, starshaped, respectively superadditive sequences. We say that the sequence $(x_n)_{n\geq 0}$ has the property "P" in the mean if the sequence $(x'_n)_{n\geq 0}$ given by:

$$x'_n = \frac{x_0 + \cdots + x_n}{n+1}$$

has the property "P". We denote by MK, MS^* and MS the sets of sequences which are convex, starshaped, respectively superadditive in the mean.

Theorem 14. *Hold the following strict inclusions:*

$$K \subset MK \subset S^* \subset S \subset MS^* \subset MS.$$

In [32] and [27] we have generalized this result as follows:

Theorem 15. *The sequence $(x'_n)_{n\geq 0}$ given by:*

$$x'_n = \frac{p_0x_0 + \cdots + p_nx_n}{p_0 + \cdots + p_n}, \quad p_n > 0,\ n \geq 0 \tag{7}$$

is in K (S^* or S) for any sequence $(x_n)_{n\geq 0}$ with the same property, if and only if there is an $u > 0$ such that:

$$p_n = p_0\binom{u+n-1}{n},\ \forall\ n \geq 1 \tag{8}$$

where

$$\binom{v}{0} = 1, \quad \binom{v}{n} = \frac{1}{n!}\prod_{k=0}^{n-1}(v-k), \quad n \geq 1,\ v \in \mathbb{R}.$$

In this case:

$$x'_n = x^u_n = \sum_{k=0}^{n}\binom{u+k-1}{k}x_k \Big/ \binom{u+n}{n}. \tag{9}$$

We say that the sequence $(x_n)_{n\geq 0}$ has the property "P" in the u-mean if the sequence $x^u = (x_n^u)_{n\geq 0}$ given by (9) has the property "P". We denote by M^uK, M^uS^* and M^uS the sets of sequences which are convex, starshaped, respectively superadditive in u-mean.

Theorem 16. *If $0 < v < u$, then hold the strict inclusions:*

$$\begin{array}{ccccccc} K \subset M^uK \subset M^vK \subset & S^* & \subset & M^uS^* & \subset & M^vS^* \\ & \cap & & \cap & & \cap \\ & S & & M^uS & & M^vS. \end{array}$$

These results are later generalized in [30] where it is defined a measure of convexity, of starshapedness and of superadditivity of a sequence.

Then I tried to transpose these results back to functions. In 1982, C. Mocanu has considered in [12] the weighted mean:

$$F_g(x) = \frac{1}{g(x)} \int_0^x g'(t)f(t)dt, \quad F_g(0) = 0. \tag{10}$$

Related to it, we have proven in [31] the following results:

Theorem 17. *If the transformation (10) preserves the convexity (the starshapedness or the superadditivity) then the function g is of the form:*

$$g(x) = kx^n, \quad u > 0,\ k \neq 0. \tag{11}$$

Denoting by F_u the function (10) with g given by (11) and by $M^uK(b)$, $M^uS^*(b)$ and $M^uS(b)$ the sets of functions $f \in C(b)$ with the property that the corresponding functions F_u belong to $K(b)$, $S^*(b)$ or $S(b)$, we have:

Theorem 18. *If $0 < v < u$, then hold the following strict inclusions:*

$$\begin{array}{ccccccc} K(b) \subset M^u K(b) \subset M^v K(b) \subset & S^*(b) & \subset & M^u S^*(b) & \subset & M^v S^*(b) \\ & \cap & & \cap & & \cap \\ & S(b) & \subset & M^u S(b) & & M^v S(b). \end{array}$$

For sequences we tried also to pass to the convexity of high order. So, in [28] we have considered the hierarchy of order three, giving the following definitions: the sequence $(x_n)_{n\geq 0}$ is said to be:

a) starshaped of order three if the sequence $((x_{n+1} - x_0)/(n+1))_{n\geq 0}$ is convex of order two;

b) superadditive of order three if:

$$x_{n+m+p} - x_{n+m} - x_{m+p} - x_{p+n} + x_n + x_m + x_p - x_0 \geq 0,\ \forall\ m, n, p > 0;$$

c) 2-starshaped of order three if it satisfies the relation:

$$\frac{x_{n+3} - x_0}{n+3} \geq \frac{x_{n+2} - x_1}{n+1}, \quad n \geq 0.$$

We denote by K_3, S_3^*, S_3 and S_3^{2*} the sets of convex, starshaped, superadditive, respectively 2-starshaped of order three sequences. In [28] are given the following results:

Theorem 19. *If the sequence $(x'_n)_{n\geq 0}$ defined by (7) is in K_3 (S_3^*, S_3 or S_3^{2*}) for any sequence $(x_n)_{n\geq 0}$ with the same property, then the sequence $(p_n)_{n\geq 0}$ must be given by (8).*

If we denote by $M^u K_3, M^u S_3^*, M^u S_3$ and $M^u S_3^{2*}$ the sets of sequences $(x_n)_{n\geq 0}$ with the property that $(x_n^u)_{n\geq 0}$ given by (9) is in K_3, S_3^*, S_3 respectively S_3^{2*}, we have also:

Theorem 20. *If* $0 < u < v$, *then:*

$$
(12)\qquad \begin{array}{ccccccc}
K_3 & \subset & S_3^* & \subset & S_3 & \subset & S_3^{2*} \\
\cap & & \cap & & & & \cap \\
M^vK_3 & \subset & M^vS_3^* & \subset & M^vS_3 & \subset & M^vS_3^{2*} \\
\cap & & \cap & & & & \cap \\
M^uK_3 & \subset & M^uS_3^* & \subset & M^uS_3 & \subset & M^uS_3^{2*}.
\end{array}
$$

For an arbitrary order, we have also given in [29] the following definitions: the sequence $(x_n)_{n\geq 0}$ is called:

a) $(p+1)$-starshaped of order r (with $p+1 < r$) if the sequence:

$$\left(\frac{(-1)^p}{p!}\sum_{i=0}^{p}(-1)^i\binom{p}{i}\frac{x_{n+p+1}-x_i}{n+p-i+1}\right)_{n\geq 0}$$

belongs to K_{r-p-1};

b) superadditive of order r if for any indices $n_1, \ldots, n_r > 0$ holds:

$$\sum_{k=0}^{r}(-1)^{r-k}\sum_{(i_1,\ldots,i_k)} x_{n_{i_1}+n_{i_k}} \geq 0$$

where, the second sum is extended to all choices of indices $i_1, \ldots, i_k$ from $1, \ldots, r$ and it reduces at a_0 for $k = 0$.

For functions, the first definition may be found in the paper [15] of Elena Popoviciu while the second was used by T. Popoviciu in [21].

Denoting by $S_r^{(p+1)*}$ and S_r the sets of all $(p+1)$-starshaped respectively superadditive of order r sequences, in [29] we have proved:

Theorem 21. *For any order* $r \geq 2$ *hold the inclusions:*

$$K_r \subset S_r^{1*} \subset S_r^{2*} \subset \cdots \subset S_r^{(r-1)*}.$$

We also remark that for functions, T. Popoviciu has proved in [21] the inclusion: $K_r \subset S_r$. For sequences we haven't yet find the place of S_r in this chain, as it appears in (12) for $r = 3$.

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Babeş-Bolyai University, Preprint 6(1987)

GENERALIZED FINITE DIFFERENCES FOR FUNCTIONS OF MORE VARIABLES

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1. Let the functions $u_0, \ldots, u_n$ be defined on some set $E \subset \mathbb{R}$ and $x_0, \ldots, x_n$ be a system of points from E. On denote

$$V\binom{u_0, \ldots, u_n}{x_0, \ldots, x_n} = \begin{vmatrix} u_0(x_0) & \ldots & u_0(x_n) \\ \ldots & \ldots & \ldots \\ u_n(x_0) & \ldots & u_n(x_n) \end{vmatrix}.$$

The system of functions $U_{n+1} = (u_0, \ldots, u_n)$ is called a T-system (or Tchebycheff system) on E if:

$$V\binom{u_0, \ldots, u_n}{x_0, \ldots, x_n} \neq 0$$

for any set of different points $x_0, \ldots, x_n$ from E. It is called a M-system (or Markov system) on E if every subsystem $U_{k+1} = (u_0, \ldots, u_k)$, for $k = 0, \ldots, n$ is a T-system on E.

If U_{n+1} is a set of functions which form a M-system on E and $X_{n+1} = (x_0, \dots, x_n)$ a set of different points from E, we shall use the notations:

$$V\begin{pmatrix} u_0, \dots, u_n \\ x_0, \dots, x_n \end{pmatrix} = V(U_{n+1}, X_{n+1})$$

and

$$V\begin{pmatrix} u_0, \dots, u_{n-1}, f \\ x_0, \dots, x_{n-1}, x_n \end{pmatrix} = V(U_n, X_{n+1}, f)$$

so that the generalized divided differences of the function f on the knots X_{n+1} with respect to the system U_{n+1} may be defined by:

$$[U_{n+1}, X_{n+1}; f] = V(U_n, X_{n+1}, f)/V(U_{n+1}, X_{n+1}). \tag{1}$$

For more reasons, in [7], it is considered also the definition:

$$\Delta[U_n, X_{n+1}; f] = V(U_n, X_{n+1}, f)/V(U_n, X_n)$$

which gives the generalized finite difference, taking $X_{n+1} = X_{n+1}^h = (x_0, x_0 + h, \dots, x_0 + nh)$:

$$\Delta_h^{U_n} f(x) = \Delta[U_n, X_{n+1}^h; f].$$

In what follows, we need some recurrence formulas from [7] and [8]. The main formula is:

$$V(U_n, X_{n+1}, f) = A_n V(U_{n-1}, X_n', f) - B_n V(U_{n-1}, X_n, f) \tag{2}$$

where:

$$X_n' = (x_1, \dots, x_n)$$

$$A_n = A_n(U_n, X_{n+1}) = V(U_n, X_n)/V(U_{n-1}, X_{n-1}')$$

and

$$B_n = B_n(U_n, X_{n+1}) = V(U_n, X_n')/V(U_{n-1}, X_{n-1}').$$

From (2) we have:

$$[U_{n+1}, X_{n+1}; f] = C_n([U_n, X'_n; f] - [U_n, X_n; f])$$

with:

$$C_n = C_n(U_{n+1}, X_{n+1}) = B_n V(U_n, X_n)/V(U_{n+1}, X_{n+1}) \tag{3}$$

and

$$\Delta[U_n, X_{n+1}; f] = \Delta[U_{n-1}, X'_n; f] - D_n\Delta[U_{n-1}, X_n; f]$$

with:

$$D_n = D_n(U_n, X_{n+1}) = B_n(U_n, X_{n+1})V(U_{n-1}, X_{n-1})/V(U_n, X_n). \tag{4}$$

as well as:

$$\Delta_h^{U_n} f(x) = \Delta_h^{U_{n-1}} f(x+h) - D_n^h \Delta_n^{U_{n-1}} f(x)$$

where $D_n^h = D_n(U_n, X_{n+1}^h)$.

We remark that (2) gives also the recurrence given in [4]:

$$[U_{n+1}, X_{n+1}; f] = \frac{[U_n, X'_n; f] - [U_n, X_n; f]}{[U_n, X'_n; u_n] - [U_n, X_n; u_n]}.$$

2. Passing to functions of more variables, as it is known from [3], there is no T-system or any domain, so that we cannot generalize (1) in a simple way.

One of the ways used for this generalization is the composition of more divided differences, each acting on a single variable. For example, if f is a function of two variables, $f : [a, b] \times [c, d] \to \mathbb{R}$, U_{n+1} is a M-system on $[a, b]$ and $W_{n+1} = (w_0, \ldots, w_m)$ a M-system on $[c, d]$, we can consider for any sets of distinct points $X_{n+1} = (x_0, \dots, x_n)$ from $[a,b]$ and $Y_{m+1} = (y_0, \dots, y_m)$ from $[c,d]$ the functions:

$$g(y) = V(U_n, X_{n+1}, f(\cdot, y)) \text{ and } h(x) = V(W_m Y_{m+1}, f(x, \cdot)).$$

We have:

$$V(U_n, X_{n+1}, h) = V(W_m, Y_{m+1}, g)$$

and we denote the common value by: $V(U_n, X_{n+1}; W_m, Y_{m+1}; f)$. From (2) we have the recurrence formula:

$$V(U_n, X_{n+1}; W_m, Y_{m+1}; f) \tag{5}$$

$$= A_n(U_n, X_{n+1}) A_m(W_m, Y_{m+1}) V(U_{n-1}, X'_n; W_{m-1}, Y'_m; f)$$
$$- A_n(U_n, X_{n+1}) B_m(W_m, Y_{m+1}) V(U_{n-1}, X'_n; W_{m-1}, Y_m; f)$$
$$- B_n(U_n, X_{n+1}) A_m(W_m, Y_{m+1}) V(U_{n-1}, X_n; W_{m-1}, Y'_m; f)$$
$$+ B_n(U_n, X_{n+1}) B_m(W_m, Y_{m+1}) V(U_{n-1}, X_n; W_{m-1}, Y_m; f)$$

where, as before, $X'_n = (x_1, \dots, x_n)$ and $Y'_m = (y_1, \dots, y_m)$.

We may consider not only the generalized divided difference:

$$[U_{n+1}, X_{n+1}; W_{m+1}, Y_{m+1}; f] \tag{6}$$

$$= V(U_n, X_{n+1}; W_m, Y_{m+1}; f) / (V(U_{n+1}, X_{n+1}) V(W_{m+1}, Y_{m+1}))$$

but also:

$$\Delta[U_n, X_{n+1}; W_m, Y_{m+1}; f] \tag{7}$$

$$= V(U_n, X_{n+1}; W_m, Y_{m+1}; f) / (V(U_n, X_n) V(W_m, Y_m))$$

to get the generalized finite difference:

$$\Delta_{h,k}^{U_n, W_m} f(x,y) = \Delta[U_n, X_{n+1}^h; W_m, Y_{m+1}^k; f] \tag{8}$$

where $X^h_{n+1} = (x, x+h, \dots, x+nh)$ and $Y^k_{m+1} = (y, y+k, \dots, y+mk)$.

From (5) we have the recurrence relations:

$$[U_{n+1}, X_{n+1}; W_{m+1}, Y_{m+1}; f]$$

$$= C_n(U_{n+1}, X_{n+1}) C_m(W_{m+1}, Y_{m+1})([U_n, X'_n; W_m, Y'_m; f]$$

$$-[U_n, X'_n; W_m, Y_m; f] - [U_n, X_n; W_m, Y'_m; f] + [U_n, X_n; W_m, Y_m; f])$$

and

$$\Delta[U_n, X_{n+1}; W_m, Y_{m+1}; f] = \Delta[U_{n-1}, X'_n; W_{m-1}, Y'_m; f]$$

$$-D_n(U_n, X_{n+1})\Delta[U_{n-1}, X_n; W_{m-1}, Y'_m; f]$$

$$-D_m(W_m, Y_{m+1})\Delta[U_{n-1}, X'_n; W_{m-1}, Y_m; f]$$

$$+D_n(U_n, X_{n+1})D_m(W_m, Y_{m+1})\Delta[U_{n-1}, X_n; W_{m-1}, Y_m; f]$$

which gives also:

$$\Delta^{U_n, W_m}_{h,k} f(x,y) = \Delta^{u_{n-1}, W_{m-1}}_{h,k} f(x+h, y+k)$$

$$-D_n(U_n, X^h_{n+1})\Delta^{U_{n-1}, W_{m-1}}_{h,k} f(x, y+k)$$

$$-D_m(W_m, Y^k_{m+1})\Delta^{U_{n-1}, W_{m-1}}_{h,k} f(x+h, y)$$

$$+D_n(U_n, X^h_{n+1})D_m(W_m, Y^k_{m+1})\Delta^{U_{n-1}, W_{m-1}}_{h,k} f(x, y).$$

Examples may be obtained from those given in [7] and [8].

3. Starting from the interpretation given in [6] to Bernstein polynomial, in [5] it is proposed a modification to the schema of Gontcharoff and then a generalization of divided differences. This may be yet pushed further. Let U_{n+1} a system of functions (of one or more variables), $P_{n+1} = (p_0, \ldots, p_n)$ a system of (linear) functionals defined on U_{n+1}, such that:

$$V(U_{n+1}, P_{n+1}) = \begin{vmatrix} p_0(u_0) & \ldots & p_0(u_n) \\ \ldots & \ldots & \ldots \\ p_n(u_0) & \ldots & p_n(u_n) \end{vmatrix} \neq 0.$$

Let F be a set of functions (of one or more variables, without any relation U_{n+1}) and $Q_{n+1} = (q_0, \ldots, q_n)$ a system of functionals defined on F. Then we can define a generalized divided difference of a function f from F with respect to the systems U_{n+1}, P_{n+1} and Q_{n+1} by:

$$[U_{n+1}, P_{n+1}, Q_{n+1}; f] = V(U_n, P_{n+1}, Q_{n+1}; f)/V(U_{n+1}, P_{n+1})$$

where:

$$V(U_n, P_{n+1}, Q_{n+1}; f) = \begin{vmatrix} p_0(u_0) & \ldots & p_0(u_{n-1}) & q_0(f) \\ \ldots & \ldots & \ldots & \ldots \\ p_n(u_0) & \ldots & p_n(u_{n-1}) & q_n(f) \end{vmatrix}.$$

Also, making some natural changements in the hypothesis, we can define:

$$\Delta[U_n, P_{n+1}, Q_{n+1}; f] = V(U_n, P_{n+1}, Q_{n+1}; f)/V(U_n, P_n).$$

For example U_{n+1} may be a M-system and the functionals p_k may be defined by $p_k(u_j) = u_j(x_k)$, where $X_{n+1} = (x_0, \ldots, x_n)$ is a set of distinct knots from $[a, b]$. Then the recurrence relations are:

$$(9) \qquad [U_{n+1}, X_{n+1}, Q_{n+1}; f] = C_n([U_n, X'_n, Q'_n; f] - [U_n, X_n, Q_n, f])$$

where C_n is given by (3) and $Q'_n = (q_1, \ldots, q_n)$; and also:

$$(10)\ \ \Delta[U_n, X_{n+1}, Q_{n+1}; f] = \Delta[U_{n-1}, X'_n, Q'_n; f] - D_n \Delta[U_{n-1}, X_n, Q_n; f]$$

with D_n given by (4).

Here, no relation between the functionals Q_{n+1} and the knots X_{n+1} is requested. But, in [2] it is given a special case in which such a relation exists. We want now to generalize it for M-systems.

Let $\mathbb{R}^p_+$ denotes the set of those $x \in \mathbb{R}^p$ whose first non-zero coordinate is positive. We write $x < y$ iff $y - x \in \mathbb{R}^p_+$ and so we can define the function sign. Let $D \subset \mathbb{R}^p$ be a convex set, $f : D \to \mathbb{R}$ be a function and $X_{n+1} = (x_0, \ldots, x_n)$ be a system of distinct collinear points in D. Put $h = (x_n - x_0)/|x_n - x_0|\mathrm{sign}(x_n - x_0)$, so that $h > 0$ and $x_i = x_0 + t_i h$ $(i = 0, 1, \ldots, n)$. For any M-system of functions U_{n+1} we consider the determinant:

$$V(U_n, X_{n+1}, f) = \begin{vmatrix} u_0(t_0) & \ldots & u_{n-1}(t_0) & f(x_0) \\ \ldots & \ldots & \ldots & \ldots \\ u_0(t_n) & \ldots & u_{n-1}(t_n) & f(x_n) \end{vmatrix}$$

the generalized divided difference:

$$[U_{n+1}, X_{n+1}; f] = V(U_n, X_{n+1}, f)/V(U_{n+1}, T_{n+1})$$

with $T_{n+1} = (t_0, \ldots, t_n)$ and also the expression:

$$\Delta[U_n, X_{n+1}; f] = V(U_n, X_{n+1}, f)/V(U_n, T_n)$$

which gives the generalized finite differences taking $X_{n+1} = X^h_{n+1} = (x_0, x_0 + h, \ldots, x_0 + nh)$. Of course, the recurrence formulas (9) and (10) may be rewrited in this case.

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ON SOME INEQUALITIES FOR CONVEX FUNCTIONS

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1 Introduction

Almost all of the known inequalities for convex functions may be expressed by the positivity of some linear functionals on the cone K of convex functions. As it was stated in [1] this point of view was initiated by T.Popoviciu and many related results may be found in his book [5]. Characterizations of the dual cone K^* of K, that is of the set of linear functionals which are positive on K, may be found in [1] and [10].

But we have also nonlinear functionals which are positive on K.
For example:$A(f) = min\{L_1(f), L_2(f)\}$ with $L_1, L_2 \in K*$which is superadditive. That is why we consider the generalized dual cone K^+ of all functionals which are positive on K. Such a generalization appears sometimes usefully. For example, in[6] I.Singer has used the dual space in the problem of the characterization of the elements of best approximation in normed spaces. But in linear metric spaces the method don't work and it was necessary to replace the dual space with a cone of subadditive functionals (see [4]).

We have obtained in [8] the characterization of a great part of K^+.
To give the result, we present some notions and notations.

Let C be4 the set of all continuous real functions defined on $I = [a, b]$. We denote by e_k and w_c the functions defined for $k \in N$ and $c \in (a, b)$ by:

$$e_k(x) = x^k,\ x \in I$$

respectively

$$w_c(x) = \begin{cases} x - c & \text{for } x \in (c, b] \\ 0 & \text{for } x \in [a, c] \end{cases}$$

A functional $A : C \longrightarrow \mathbb{R}$ is said to be:

a) superadditive if: $A(f + g) \geq A(f) + A(g),\ \forall f, g \in C$;

b) positively upper homogeneous if: $A(af) \geq aA(f),\ \forall f \in C,\ a \geq 0$;

c) upper semicontinuous if: $A(\lim_{n\to\infty} f_n) \geq \lim_{n\to\infty} A(f_n),\ \forall (f_n)$ convergent.

Theorem 1.1. *A supperadditive , positively upper homogeneous, upper semicontinuous functional A belongs to K^+ if and only if:*

$$A(\pm e_k) \geq o \; for \; k = 0, 1 \tag{1}$$

and

$$A(w_c) \geq 0,\ \forall c \in (a, b). \tag{2}$$

We remark that if A is homogeneous, the condition (1) becomes:

$$A(e_k) = 0 \; for \; k = 0, 1. \tag{3}$$

In fact, the results from [8], as well as those from [1] and [10], are given for convexity of order n. We have given here the result only in this form because in what follows we deal only with convex functions.

2 The Jensen-Steffensen's inequality

One of the simplest linear functional is of the form:

$$A'(f) = \sum_{k=1}^{n+1} p_k f(x_k),\ p_k \in \mathbb{R},\ x_k \in I. \tag{4}$$

From theorem 1.1 we have the following result which may be found even in [5]:

Consequence 2.1. *The functional (4) is positive for any convex function f if and only if:*

$$\sum_{k=1}^{n+1} p_k = 0,\ \sum_{k=1}^{n+1} p_k \cdot x_k = 0,\ \sum_{k=1}^{n+1} p_k \cdot w_c(x_k) \geq 0,\ \forall c \in I. \tag{5}$$

The first two relations from (5) give:

$$p_{n+1} = -\sum_{k=1}^{n} p_k \neq 0,\ \sum_{k=1}^{n} p_k x_k / \sum_{k=1}^{n} p_k$$

thus, supposing:

$$\sum_{k=1}^{n} p_k > 0 \tag{6}$$

and dividing in (4) by it, we have equivalently the functional:

$$A(f) = \sum_{k=1}^{n} p_k f(x_k) / \sum_{k=1}^{n} p_k - f(\sum_{k=1}^{n} p_k x_k / \sum_{k=1}^{n} p_k). \tag{7}$$

This is positive on K if and only if it verifies (2).

In what follows we want to give a characterization of the weights $p_1, \ldots, p_n$ for what the functional given by (7) is positive on K for any knots $x_1 \leq x_2 \leq \cdots \leq x_n$.We need the following results of Popoviciu's type:

Lemma 2.1. *The inequality:*

$$\sum_{k=1}^{m} q_k y_k \geq 0 \tag{8}$$

is valid for any positive increasing sequence $(y_k)_{k=1}^m$ if and only if holds:

$$\sum k = j^m q_k \geq 0,\ j = 1, \ldots, m. \tag{9}$$

Proof. If the sequence $(y_k)_{k=1}^m$ is positive and increasing, it may be represented by:

$$y_k = \sum_{j=1}^{k} v_j \textit{ with } v_j \geq 0,\ \forall j \geq 1.$$

The inequality (8) becomes:

$$\sum_{k=1}^{m} q_k \sum_{j=1}^{k} v_j = \sum_{j=1}^{m} (\sum_{k=j}^{m} q_k) v_j \geq 0$$

and it is valid for any $v_j \geq 0$ if and only if holds (9). □

Lemma 2.2. *The inequality (8) is valid for any positive decreasing sequence if and only if holds:*

$$\sum_{k=1}^{1} q_k \geq 0,\ j = 1, \ldots, m. \tag{10}$$

The proof can be deduced from Lemma 1 because if $(y_k)_{k=1}^m$ is decreasing, $(y_{m-k+1})_{k=1}^m$ is increasing.

Theorem 2.1. *The functional given by (7) is positive for any convex function f and any knots:*

$$a \leq x_1 \leq \cdots \leq x_n \leq b$$

if and only if:

$$p_n = \sum_{k=1}^{n} p_k > 0 \tag{11}$$

and

$$0 \leq \sum_{k=1}^{1} p_k \leq p_n,\ j = 1, \ldots, n. \tag{12}$$

Proof. We must have $p_n \neq 0$ in (7) and we have supposed $p_n > 0$ in (6). Let us denote:

$$X = \sum_{k=1}^{n} p_k x_k / p_n.$$

If $b \geq X \geq x_n$ then, for $x_n \leq c < X$ it results $A(w_c) = -(X - c) < 0$. Thus we must have $X \leq x_n$ which is equivalent with:

$$\sum_{k=1}^{n} p_k \cdot (x_n - x_k) \geq 0$$

and $(x_n - x_k)_{k=1}^{n}$ beeing positive and decreasing, from Lemma 2 we must have:

$$\sum_{k=1}^{1} p_k \geq 0 \; for \; j = 1, \ldots, n-1. \tag{13}$$

If $a \leq I \leq x_1$, for $X \leq c < x_1$, the condition $A(w_c) \geq 0$ implies:

$$\sum_{k=1}^{n} p_k \cdot (x_k - c) \geq 0$$

and so Lemma 1 gives:

$$\sum_{k=j}^{n} p_k \geq 0, \; j = 1, \ldots, n. \tag{14}$$

But then:

$$\sum_{k=1}^{n} p_k (x_k - x_1) \geq 0$$

that is $I \geq x_1$. Conversely, $x_1 \leq I$ gives (14). Thus we always have (13) and (14) which are equivalent with (12).

To prove the sufficiency of the conditions (11) and (12), we have to prove (2) for any $c \in (a, b)$. If $c > x_n$ or $c < x_1$, $A(w_c) = 0$.
If : $x_j \leq c < x_{j+1}$ $(j = 1, \ldots, n-1)$ we have two possibilities:

a) If $X \leq c$ then:

$$A(w_c) = \sum_{k=j+1}^{n} p_k (x_k - c)/p_n \geq 0$$

because $(x_k - c)_{k=j+1}^{n}$ is increasing and we have (14).

b) If $X > c$, then:

$$A(w_c) = \sum_{k=j+1}^{n} p_k (x_k - c)/p_n - (\sum_{k=1}^{n} p_k x_k / p_n - c) =$$
$$= \sum_{k=1}^{1} p_k (c - x_k)/p_n \geq 0$$

because $(c - x_k)_{k=1}^{j}$ is descreasing and we have (13).

$\square$

Remark 2.1. *If $p_k > 0,\ k = 1, \dots, n$, the conditions (12) are satisfied and we get the Jensen's inequality. The sufficiecncy of the conditions (11) and (12) was proved by J.F.Steffenen in [7] but we don't found anywhere stated their necessity (see also [2]). Examples if weights $(p_k)_{k=1}^n$ which satisfy (11) and (12) are given by the inequalities of Szegö, Bellman, Brunk, etc. (see [2]).*

3 Convex sequences

If the weights $(p_k)_{k=1}^n$ from (4) are imposed, the conditions (5) characterize the knots $(x_k)_{k=1}^n$ which give positive functionals on K. We start to analize this point of view by the following simple problem.

A sequence $(x_k)_{k\geq 1}$ is said to be convex if it verifies the conditions:

$$x_{k+2} - 2x_{k+1} + x_k \geq 0 \; for \; k \geq 1.$$

We want to characterize the increasing sequences $(x_k)_{k\geq 1}$ with the property that $(f(x_k))_{k\geq 1}$ is a convex sequence for any convex function f. From (4) we obtain the following result:

Theorem 3.1. *The sequence $(f(x_k))_{k\geq 1}$ is convex sequence for any convex function f if and only if the sequence $(x_k)_{k\geq 1}$ is an arithmetic progression.*

4 The inequalities of Toda and of Nanson

Another example of inequality with imposed weights is that proved by K.Toda in [9] for any convex function f and any points:

$$x_1 \leq x_2 \leq \cdots \leq x_n$$

we have:

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k) \geq \frac{1}{n-1}\sum_{j=1}^{n-1} f(y_j)$$

where y_j are the roots of the derivative of the polynomial:

$$\prod_{k=1}^{n}(x - x_k).$$

In what follows we want to characterize the knots for which such an inequality holds.

Theorem 4.1. *The functional:*

$$A(f) = \frac{1}{n}\sum_{k=1}^{n} f(x_{2k-1} - \frac{1}{n-1}\sum_{j=1}^{n-1} f(x_{2j})$$

with:

$$x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_{2n-1}$$

is positive for any convex function f if and only if:

$$\frac{1}{n}\sum_{k=1}^{n} x_{2k-1} = \frac{1}{n-1}\sum_{j=1}^{n-1} x_{2j} \tag{15}$$

and

$$\frac{1}{n}\sum_{k=i+1}^{n} x_{2k-1} + \frac{(n-i)x_{2i-1}}{n(n-1)} \geq \frac{1}{n-1}\sum_{j=i}^{n-1} x_{2j},\ j = 2, \ldots, n-1. \tag{16}$$

Proof. The condition (15) is the second relation from (5). We have to check the last condition from (5). If:$x_{2i-1} \leq c < x_{2i} (1 \leq i < n)$ then:

$$A(w_c) = \frac{1}{n}\sum_{k=i+1}^{n} (x_{2k-1} - c) - \frac{1}{n-1}\sum_{j=i}^{n-1} (x_{2j} - c) \geq 0$$

if and only if holds (16). Analogously, for $x_{2i-2} \leq c < x_i,\ A(w_c) = 0$ because of (15). If $x_{2n-2} \leq c < x_{2n-1}, A(w_c) = (x_{2n-1} - c) > 0$ and if $x_{2n-1} \leq c \leq b, A(w_c) = 0$. □

Consequence 4.1. *If $(x_k)_{k=1}^{2n-1}$ is an increasing arithmetic progression then for any convex function f holds:*

$$\frac{1}{n}\sum_{k=1}^{n} f(x_{2k-1}) \geq \frac{1}{n-1}\sum_{j=1}^{n-1} f(x_{2j}). \tag{17}$$

This is a Nanson's type inequality for functions. It also can be deduced from theorem 3 and the Nanson's inequality for sequences [3].

We remark that (17) can be iterated:

Consequence 4.2. *If $(x_k)_{k=1}^{2n-1}$ is an increasing arithmetic progression, then for any convex function f holds:*

$$\frac{1}{n}\sum_{k=1}^{n} f(x_{2k-1}) \geq \frac{1}{n-1}\sum_{k=1}^{n-1} f(x_{2k}) \geq \frac{1}{n-2}\sum_{k=2}^{n-1} f(x_{2k-1}) \geq \cdots \geq f(x_n).$$

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ON A GENERALIZATION OF THE CONVEXITY

Gh.TOADER

1 For complex functions, P.T. Mocanu has defined in [5] a general notion of convexity which is intermediary to usual convexity and to starlikeness. It has inspired my definitions for sequences from [6] and for real functions from [7]. Although based on different ideas, they also introduce an infinity of classes of sequences (respectively of functions) between those of convex and of starshaped ones.

In the following paragraph we give the definition of m-convexity for sets in a linear space. In the paragraph 3 we remaind the definition of m-convex functions (from [7]) and give some properties of them. In the last paragraph, we study the conservation of m-convexity of functions by some integral means considered in [4].

2 Let X be a linear space, $I = [0, 1]$ and $m \geq 0$ a fixed real number.

Definition 0.1. *A set $D \subset X$ is called m-convex if for any $x, y \in D$ and any $t \in I$ we have:*

$$tx + m(1 - t)y \in D. \tag{1}$$

Lemma 0.1. *If $m > 1, 0 \in D$ and D is m-convex then for any $x \in D,\ t \geq 0$ we have $tx \in D$.*

Proof. If $x \in D$ and $t \in I$, then $tx = tx + m(1-t) \cdot 0 \in D$. Also, $mx = 0 \cdot 0 + m(1-0)x \in D$. Thus for any $t \in I$ and $n \in N,\ (t \cdot m^n)x \in D$. □

Remark 0.1. *Taking into account this property, in what follows we shall consider only $m \in I$. The value $m = 1$. The value $m = 1$ corresponds to convexity and $m = 0$ to starshapendness. If $0 < m < 1$ and $x \in D$, then for any $t \in I,\ [t + m(1 - t)]x \in D$ that is $sx \in D$ for $s \in [m, 1]$ and so, step by step, $sx \in D$ for $s \in (0, 1]$. Also, if $x, y \in D$ for any $t, s \in I$:*

$$tx + m(1 - t)y \in D \text{ and } sy + m(1 - s)x \in D.$$

These points coincide for $t = s = m/(m + 1)$ which gives:

$$[m/(1 + m)] \cdot (x + y) \in D.$$

So, D is m-convex if and only if for any $x, y \in D$ the convex hull of the set $\{0, x, y, [m/(1+m)] \cdot (x+y)\}$ is contained in $D \cup \{0\}$. Thus the m-convexity so defined is relative to the origin (as the starshapendness). To be relative to another point x_0, we must replace (1) by:

$$tx + (1-t)[my + (1-m)x_0] \in D.$$

We can see that only for $m = 1$ this is independent of x_0. In what follows we consider only the case $x_0 = 0$ and suppose that $0 \in D$.So, as $m/(1+m)$ is increasing, we have:

Lemma 0.2. *If D is m-convex and $0 \leq n < m \leq 1$, then D is also n-convex.*

Remark 0.2. *Any m-convex set D is $\pi 2$-convex in the sense of [3]. Indeed, if we denote by $[x, y]$ the line segment joining the points x and y, then D contains $[x, (m/(1+m)(x+y)]$ and $[(m/(1+m))(x+y), y]$.*

3 Let D be a m-convex set, with $m \in I$.

Definition 0.2. *A function $f : D \longrightarrow \mathbb{R}$ is said to be m-convex if for every $x, y \in D$ and $t \in I$ it verifies:*

$$f(tx + m(1-t)y) \leq t \cdot f(x) + m(1-t) \cdot f(y). \tag{2}$$

Remark 0.3. *If we write (2) as:*

$$f(ax + by) \leq a \cdot f(x) + b \cdot f(y) \tag{3}$$

then this relation must be verified for any (a, b) on the segment joining $(1, 0)$ with $(0, m)$. This last point becomes $(0, 1)$ in the case of convexity and $(0, 0)$ in the case of stellarity.

Another geometric interpretation of (2) is the following: let us denote the points $A(x, f(x))$, $B(y, f(y))$, $P(mx, mf(x))$ and $Q(my, mf(y))$; then f is m-convex if and only if the point $M(z, f(z))$ is under the chord BP for $z \in [y, mx]$ and also under the chord AQ for $z \in [x, my]$.

Taking into account the remark 1 it is natural to suppose:

$$0 \in D \text{ and } f(0) \leq 0 \tag{4}$$

otherwise the relation (2) should be modified. With this convention we obtain:

Lemma 0.3. *The function $f : D \longrightarrow \mathbb{R}$ is m-convex if and only if the set:*

$$epif = \{(x, y) \in D \times R; y \geq f(x)\}$$

is m-convex.

Lemma 0.4. *If f is m-convex then it is starshaped.*

Proof. For any $x \in D$ and $t \in I$:

$$f(tx) = f(tx + m(1-t) \cdot 0) \leq t \cdot f(x) + m(1-t) \cdot f(0) \leq t \cdot f(x).$$

$\square$

Theorem 0.1. *If f is m-convex and $0 \leq n < m \leq 1$, then f is n-convex.*

Proof. It results from the Lemmas 2 and 3 but also from Lemma 4: if $x, y \in D$ and $t \in I$, then:

$$f(tx + n(1-t)y) = f(tx + m(1-t)(n/m)y) \leq$$
$$\leq t \cdot f(x) + m(1-t) \cdot f((n/m)y) \leq t \cdot f(x) + m(1-t)(n/m) \cdot f(y).$$

In what follows we consider only functions defined on the real interval $[0, b]$ and denote by $K_m(b)$ the set of m-convex functions on $[0, b]$ such that $f(0) \leq 0$. $\square$

Lemma 0.5. *The function f is in $K_m(b)$ if and only if:*

$$f_m(x) = \frac{f(x) - m \cdot f(y)}{x - my} \text{ is increasing on } (my, b];\ y \in [0, b]. \tag{5}$$

Proof. The relation (2) may be written as:

$$\frac{f(x) - m \cdot f(y)}{x - my} \geq \frac{f(tx + m(1-t)y) - m \cdot f(y)}{t(x - my)} \tag{6}$$

and denoting $z = tx + m(1-t)y$ we have $z \leq x$ and $f_m(z) \leq f_m(x)$. Conversely, for $z \leq x$ we take $t = (z - my)/(x - my)$. $\square$

Lemma 0.6. *If f is differentiable in $[0, b]$ then $f \in K_m(b)$ if and only if:*

$$f'(x) \geq \frac{f(x) - m \cdot f(y)}{x - my},\ \text{for } x > my. \tag{7}$$

Proof. From (5) we have $f'_m(x) \geq 0$, which gives (7).

These results generalize and unify some results known for convex and for starshaped functions (see [2]). $\square$

4 In [2] was proved the conservation of the convexity and starshapendness by the integral mean, that is, if f is convex or starshaped, then so is also:

$$F(x) = \frac{1}{x} \int_0^x f(t)dt.$$

In [4] it was considered a more general mean:

$$F_g(x) = \frac{1}{g(x)} \int_0^x g'(t) \cdot f(t)dt. \tag{8}$$

For this we can prove the following:

Lemma 0.7. *If F_g given by (8) is m-convex for every m-convex function f, then there is a real k and an $u > 0$ such that:*

$$g(x) = k \cdot x^u. \tag{9}$$

Proof. The function $f_0(x) = c \cdot x$ is in $K_m(b)$ for every $c \in R$. So:

$$F_g(x) = \frac{c}{g(x)} \int_0^x g'(t) \cdot t \cdot dt = c \cdot G(x)$$

is also in $K_m(b)$. So, for every $x, y \in [a, b]$ and $t \in I$:

$$c[G(tx + m(1-t)y) - t \cdot G(x) - m(1-t) \cdot G(y)] \leq 0$$

and taking $c = \pm 1$, we have:

$$G(tx + m(1-t)y) = t \cdot G(x) + m(1-t) \cdot G(y)$$

which gives (see [1]): $G(x) = a \cdot x$, that is:

$$x \cdot g'(x) = a[g(x) + xg'(x)]$$

and so (9). If $u \leq 0$, then (8) is not defined for $f(x) = c$. $\square$

Lemma 0.8. *If g is given by (9), with $u > 0$, then F_g is in $K_m(b)$ for every $f \in K_m(b)$.*

Proof. From (8) and (9) we have:

$$F_g(x) = F_u(x) = \frac{u}{x''} \int_0^x t^{u-1} f(t) dt \tag{10}$$

and making the substitution $t = x \cdot s^{1/u}$ (given in [4]), we get:

$$F_u(x) = \int_0^1 f(x \cdot s^{1/u}) ds. \tag{11}$$

So, if $f \in K_m(b); x, y \in [0, b]; t \in I$:

$$F_u(tx + m(1-t)y) = \int_0^1 f(txs^{1/u} + m(1-t)ys^{1/u}) ds \leq$$

$$\leq \int_0^1 [t \cdot f(xs^{1/u}) + m(1-t) \cdot f(ys^{1/u})] ds = t \cdot F_u(x) + m(1-t) \cdot F_u(y).$$

$\square$

If we denote by $M^u K_m(b)$ the set of functions $f : [0, b] \longrightarrow \mathbb{R}$ with the property that F_u given by (10) is in $K_m(b)$, we have thus:

Theorem 0.2. *If* $0 < n < m < 1$ *and* $u > 0$ *then hold the following inclusions:*

$$\begin{array}{ccccccc} K_1(b) & \subset & K_m(b) & \subset & K_n(b) & \subset & K_0(b) \\ \cap & & \cap & & \cap & & \cap \\ M^u K_1(b) & \subset & M^u K_m(b) & \subset & M^u K_n(b) & \subset & M^u K_0(b). \end{array}$$

Lemma 0.9. *The function* f *belongs to* $M^u K_m(b)$ *if and only if:*

$$f(x) \geq \frac{[(1+u)\cdot x - m\cdot u\cdot y]F_u(x) - m\cdot x\cdot F_u(y)}{u\cdot(x-my)} \; for\; x > my \tag{12}$$

Proof. From (10)we have:

$$F'_u(x) = (u/x)[f(x) - F_u(x)]$$

and from (7) we get (12). □

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A GENERALIZED HIERARCHY OF CONVEXITY OF FUNCTIONS

Gh.TOADER

1 Introduction

Let us consider the classes of continuous, convex, starshaped respectively superadditive functions defined on the interval $I = [0, b]$:

$$C(b) = \{f : I \longrightarrow \mathbb{R},\ f(0) = 0,\ f \ continuous\}$$
$$K(b) = \{f \in C(b); f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$
$$\forall t \in (0,1),\ \forall x, y \in I\}$$
$$S*(b) = \{f \in C(b);\ f(tx) \leq tf(x)\ \forall t \in (1,0), \forall x \in I\}$$
$$S(b) = \{f \in C(b); f(x+y) \geq f(x) + f(y),\ \forall x, y, x+y \in I\}.$$

It is known that:

$$K(b) \subset S*(b) \subset S(b). \tag{1}$$

In [1], A.M. Bruckner and E.Ostrow have extended these inclusions as follows. The Cesáro operator $A : C(b) \longrightarrow C(b)$ is defined by:

$$A(f)(x) = \frac{1}{x}\int_0^x f(t)dt,\ A(f)(0) = 0.$$

We denote by $MK(b), MS*(b)$ and $MS(b)$ the sets of functions f with the property that $A(f)$ is in $K(b)$, in $S*(b)$ respectively in $S(b)$. The result from [1] is that for any $b \geq 0$, hold the strict inclusions:

$$K(b) \subset MK(b) \subset S*(b) \subset S(b) \subset MS*(b) \subset MS(b). \tag{2}$$

Using the Cesáro type operator $A_u : C(b) \longrightarrow C(b)$ defined by:

$$A_u(f)(x) = \frac{u}{x^u}\int_0^x t^{u-1} f(t)dt$$

we have given in [5] a generalization of (2). Let us denote by $M^uK(b), M^uS*(b)$ and $M^uS(b)$ the sets of functions $f \in C(b)$ with the property that $A_u(f)$ belongs to $K(b), S{*}(b)$ respectively $S(b)$. Then, for any $b \geq 0$ and say $0 \leq u \leq v$, hold the inclusions:

$$\begin{array}{ccccccccc} K(b) & \subset & M^vK(b) & \subset & M^uK(b) & \subset & S*(b) & & S(b) \\ & & & & & & \cap & & \cap \\ & & & & & & M^vS*(b) & & M^vS(b) \\ & & & & & & \cap & & \\ & & & & & & M^uS*(b) & & M^uS(b). \end{array}$$

To obtain a further generalization we can use the following nonlinear operator, studied, for example by C.Mocanu in [3] and [4]:

$$A_{g,p}(f)(x) = \Big[\frac{1}{g(x)} \int_0^x f^p(t)g'(t)dt\Big]^{1/p}.$$

If p is an arbitrary positive real number, the function f must be positive. We will denote by $C_+(b), K_+(b), S_+*(b)$ and $S_+(b)$ the sets of positive functions from the corresponding classes. We remark that for $p \geq 0, A_{g,p} : C_+(b) \longrightarrow C_+(b)$ but for
$n \in \mathbb{N}, A_{g,p} : C(b) \longrightarrow C(b)$.

Lemma 1.1. *If $A_{g,n}(f) \in K(b)$ for any $f \in K(b)$, then there is an $u \geq 0$ and a real c such that:*

$$g(x) = cx^u. \tag{3}$$

Proof. As $f_0(x) = ax$ is convex for any a, so must be also:

$$F_0(x) = a\Big[\frac{1}{g(x)} \int_0^x t^n g'(t)dt\Big]^{1/n}$$

By (1) F_0 is also superadditive and a being of arbitrary sign, this means that F_0 is additive (for $a = 1$). Thus:

$$\Big[\frac{1}{g(x)} \int_0^x t^n g'(t)dt\Big]^{1/n} = kx$$

which gives (3).

For g given by (3), we denote the operator $A_{g,p}$ by $A_{u,p}$ that is:

$$A_{u,p}(f)(x) = \Big[\frac{u}{x^u} \int_0^x t^{u-1} f^p(t)dt\Big]^{1/p} \tag{4}$$

As it is done in [3], making the substitution:

$$t = ks^{1/u}$$

the relation (4) becomes:

$$A_{u,p}(f)(x) = \Big[\int_0^1 f^p(xs^{1/u})ds\Big]^{1/p}. \tag{5}$$

We need some well known results (see [5]): ☐

Lemma 1.2. *If the convex function f is differentiable, then f' is nondecreasing.*

Lemma 1.3. *The function f is starshaped if and only if $f/1_I$ is nondecreasing.*

So we can prove the following:

Theorem 1.1. *For any $b, u \geq 0$ and any $p \geq 1$, hold the inclusions:*

$$\begin{array}{ccccccc} K_(b) & \subset & M^{u,p}K_+(b) & \subset & S*_+(b) & \subset & S_+(b) \\ & & & & \cap & & \\ & & & & M^{u,p}S*_+(b) & \subset & M^{u,p}S_+(b). \end{array}$$

If p is replaced a natural number $n \geq 1$, we can renounce at the lower index $+$.

Proof. 1. If $f \in K_+(b)$, then for any $x, y \in I$ and any $t \in (0,1)$, we have by (5):

$$A_{u,p}(f)(tx+(1-t)y) \leq [\int_0^1 f^p((tx+(1-t)y)s^{1/u})ds]^{1/p} \leq$$
$$\leq [\int_0^1 [tf(xs^{1/u})+(1-t)f(ys^{1/u})]^p ds]^{1/p} \leq$$
$$\leq t \cdot A_{u,p}(f)(x) + (1-t)A_{u,p}(f)(y)$$

which means that $f \in M^{u o p}K_+(b)$. We have used Mnkowski's inequality (see [2]), for what we need $p \geq 1$.

2. From (4) we have:

$$x^{-1}f(x) = [(x^{-1}A_{u,p}(f)(x))^p + pu^{-1}(x^{-1}A_{u,p}(f)(x))^{p-1} \cdot$$
$$A'_{u,p}(f)(x)]^{1/p}.$$

So, if $f \in M^{u,p}K_+(b)$ by Lemmas 2 and 3, $f/1_I$ is nondecreasing that is $f \in S*_+(b)$.

3. The inclusions $S*_+(b) \subset S_+(b)$ and $M^{u,p}S*_+(b) \subset M^{u,p}S_+(b)$ follow from (1).

4. If $f \in S*_+(b), t \in (0,1)$ and $x \in I$ we have:

$$A_{u,p}(f)(x) = [\int_0^1 f^p(txs^{1/u})ds]^{1/p} \leq tA_{u,p}(f)(x)$$

that is $f \in M^{u,p}S*_+(b)$.

□

Remark 1.1. *The inclusion: $S_+(b) \subset M^{u,p}S_+(b)$ is valid for $p \leq 1$.*

Indeed, if $f \in S_+(b)$:

$$A_{u,p}(f)(x+y) = [\int_0^1 f^p((x+y)s^{1/u})ds]^{1/p} \geq$$

$$= [\int_0^1 [f(xs^{1/u}) + f(ys^{1/u})]^{1/p} \geq A_{u,p}(f)(x) + A_{u,p}(f)(y).$$

To apply the Minkovski's inequality in this sense we need $0 \leq p \leq 1$.

Remark 1.2. *From some results proved in [4] we deduce inclusion relations between the classes $M^{u,p}S*_+(b)$ if u or p decreases. We don't know if similar results hold for other classes.*

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On a Theorem of Tiberiu Popoviciu

Gh.TOADER

In [4] T.Popoviciu has proved that if the function $f : [0, b] \longrightarrow \mathbb{R}$ satisfies the conditions:

(i) $f(0) = 0$

(ii) f has the $(n-1)$ th derivative

(iii) $(-1)^{n-1} f^{(n-1)}$ is increasing

then for any $x_1, \ldots, x_n \in [0, b]$ distinct and such that $x_1 + \cdots + x_n \leq b$ it verifies:

$$\sum_{k=1}^{n} 9-1)^{k-1} \sum_{(k)} f(x_{i_1} + \cdots + x_{i_k}) \geq 0 \tag{1}$$

where $\sum\limits_{(k)} f(x_{i_1} + \cdots + x_{i_k})$ denotes the sum over all the combination of class k of $x_1, \ldots, x_n$.

For some values of n the result was generalized for n-convex functions. This was proved by M.Petrović in [3] for $n = 2$, by P.M. Vasić in [6] for $n = 3$ and by J.D.Kečkié in [2] for $n = 4$.

On the other hand, for $n = 2$, the result was again generalized by A.M. Bruckner and E.Ostrow in [1], proving (1) for starshaped functions (a simple proof may be also found in [5]). In this paper we want to prove similar results for $n = 3$ and 4. Also we pass to an arbitrary interval $[a, b]$.

Let us remind that the divided difference (of order n) on the distinct points $(x_0, x_1, \ldots, x_n$ is defined recurrently by:

$$[x_0; f] = f(x_0), [x_0, \ldots, x_n; f] - [x_1, \ldots, x_n; f])/(x_0 - x_n).$$

We consider the set of n-convex functions (or convex of order n):

$$K_n[a, b] = \{f : [a,b] \longrightarrow \mathbb{R},\ [x_0, x_1, \ldots, x_n; f] \geq 0,$$
$$\forall x_0, x_n \in [a, b] \ distinct\ \}$$

and that of n-starshaped functions (or starshaped of order n) on $[a, b]$:

$$S_n^*[a,b] = \{f : [a,b] \longrightarrow \mathbb{R},\ [a, x_1, \ldots, x_n; f] \geq 0, \\ \forall x_1, \ldots, x_n \in [a,b] \ \mathit{distinct}\ \}$$

By analogy with the definition for $n = 2$ and generalizing the inequality (1), we define also the class of n-superadditive functions (or superadditive of order n) on $[a, b]$:

$$S_n[a,b] = \{f :[a,b] \longrightarrow \mathbb{R},\ \forall x_1, \ldots, x_n \in (a,b] \ \mathit{distinct} \\ \mathit{such\ that}\ x_1 + \cdots + x_n - na \leq b - a \\ \sum_{k=1}^{n} (-1)^{n-k} \sum_{(k)} f(x_{i_1} + \cdots + x_{i_k} - (k-1)a) + (-1)^n f(a) \geq 0\}.$$

We see that if $a = 0$ and $f(0) = 0$ the relation of definition of n-superadditive functions reduces at (1) multiplied by $(-1)^{n+1}$.

The result of M. Petrović, P.M. Vasić and J.D.Kečkić means that:

$$K_n[0,b] \subset S_n[0,b], \text{ for } n = 2, 3, 4.$$

We prove the stronger result:

Theorem 0.1. *For any interval* $[a, b]$ *hold the inclusions:*

$$K_n[a,b] \subset S_n^*[a,b] \subset S_n[a,b], \ \mathit{for}\, n = 2, 3, 4.$$

Proof. The first inclusion is obvious for every n. To prove the second inclusion, for $n = 2$ we have:

$$f(x+y-a) - f(x) - f(y) + f(a) = \\ (x+y-2a)\frac{f(x+y-a) - f(a)}{x+y-2a} - (x-a)\frac{f(x) - f(a)}{x-a} - \\ -(y-a)\frac{f(y) - f(a)}{y-a} = (x-a)([a, x+y-a; f] = [a, x; f]) + \\ +(y-a)([a, x+y-a; f] - [a, y; f]) = (x-a)(y-a) \cdot \\ \cdot([a, x, x+y-a; f] + [a, y, x+y-a; f])$$

and so $f \in S_2^*[a, b]$ implies $f \in S_2[a, b]$.

For $n = 3$ we deduce:

$$f(x+y+z-2a) - f(x+y-a) - f(x+z-a) - f(y+z-a) +$$
$$+ fx) + f(y) = f(z) - f(a) = (f(x+y+z-2a) - f(x+y-a) - f(z) + f(a)) - (f(x+z-a) -$$
$$- f(x) - f(z) + f(a)) - (f(y+z-a) - f(y) - f(z) + f(a)) =$$
$$= (x+y-2a)(z-a)([a, x+y-a, x+y+z-2a; f] +$$
$$+ [a, z, x+y+z-2a; f]) - (x-a)(z-a)([a, x, x+z-a; f] +$$
$$+ [a, z, x+z-a; f]) - (y-a)(z-a)([a, y, y+z-a; f] +$$
$$+ [a, z, y+z-a; f]) = (x-a)(z-a)([a, x+y-a, x+y+z-2a; f] -$$
$$- [a, x+y-a, x+z-a; f] + [a, x+y-a, x+z-a; f] -$$
$$- [a, x, x+z-a; f] + [a, z, x+y+z-2a; f] - [a, z, x+z-a; f]) +$$
$$+ (y-a)(z-a)([a, x+y-a, x+y+z-2a; f] - [a, x+y-a, y+z-a; f] +$$
$$+ [a, x+y-a, y+z-a; f] - [a, y, y+z-a; f] + [a, z, x+y+z-2a; f] -$$
$$- [a, z, y+z-a; f]) = (x-a)(y-a)(z-a) \cdot$$
$$\cdot ([a, x+y-a, x+z-a, x+y+z-2a; f] + [a, x, x+y-a, x+z-a; f] +$$
$$+ [a, z, x+z-a, x+y+z-2a; f] + [a, x+y-a, y+z-a, x+y+z-2a; f] +$$
$$+ [a, y, x+y-a, y+z-a; f] + [a, z, y+z-a, x+y+z-2a; f])$$
$$= (x-a)(y-a)(z-a)([a, x, x+y-a, x+z-a; f] +$$
$$+ [a, y, x+y-a, y+z-a; f] + [a, z, x+z-a, y+z-a; f] +$$
$$+ [a, x+y-a, y+z-a, x+y+z-2a; f] + [a, x+y-a, y+$$
$$z-a, x+y+z-2a; f] + [a, x+z-a, y+z-a, x+y+z-2a; f])$$

because

$$[a, z, x+z-a, x+y+z-2a; f] + [a, z, y+z-a, x+y+z-2a; f] =$$
$$= [a, z, x+z-a, y+z-a; f] + [a, x+z-a, y+z-a, x+y+z-2a; f]$$

as :

$$[a, x+z-a, y+z-a, x+y+z-2a; f] - [a, z, y+z-a, x+y+z-2a; f] =$$
$$= [a, z, x+z-a, x+y+z-2a; f] - [a, z, x+z-a, y+z-a; f] =$$
$$= (x-a)[a, z, x+z-a, y+z-a, x+y+z-2a; f].$$

Thus $f \in S_3^*[a, b]$ implies $f \in S_3[a, b]$.

For $n = 4$ we can continue on this way or we may remark that the function f is n-superadditive on $[a, b]$ if and only if the function f_a, defined by $f_a(x) = f(a + x)$, is

n-superadditive on $[0, b-a]$. Thus it is enough to prove the property for $a = 0$. Thus:

$$
\begin{aligned}
& f(x+y+z+w) - f(x+y+z) - f(x+y+w) - f(x+z+w) - \\
& - f(y+z+w) + f(x+y) + f(x+z) + f(x+w) + f(y+z) + \\
& + f(y+w) + f(z+w) - f(x) - f(y) - -f(z) - f(w) + f(0) = \\
& = (f(x+y+z+w) - f(x+y+z) - f(x+y+w) - f(z+w) + \\
& + f(x+y) + f(z) + f(w) - f(0)) - (f(x+z+w) - f(x+z) - \\
& - f(x+w) - f(z+w) + f(x) + f(z) + f(w) - f(0)) - (f(y+z+w) - \\
& - f(y+z) - f(y+w) - f(z+w) + f(y) + f(z) + f(w) - f(0)) = \\
& = (x+y)zw([0, x+y, x+y+z, x+y+w; f] + [0, z, x+y+z, z+w; f] + \\
& + [0, w, x+y+w, w+z; f] + [0, x+y+z, x+y+w, x+y+z+w; f] + \\
& + [0, x+y+z, z+w, x+y+z+w; f] + [0, x+y+w, z+w, x+y+z+w; f]) - \\
& - xzw([0, x, x+z, x+w; f] + [0, z, x+z, z+w; f] + [0, w, x+w, z+w; f] + \\
& + [0, x+z, x+w, x+z+w; f] + [0, x+z, z+w, x+z+w; f] + [0, x+w, z+w, x+z+w; f]) \\
& - yzw([0, y, y+z, y+w; f] + [0, z, y+z, z+w; f] + [0, w, y+w, z+w; f] + \\
& + [0, y+z, y+w, y+z+w; f] + [0, z+y, z+w, y+z+w; f] + [0, y+w, z+w, y+z+w; f]) \\
& = xzw([0, x+y, x+y+z, x+y+w; f] + [0, x, x+y+z, z+w; f] + [0, w, x+y+w, w+z; f] + \\
& + [0, w, x+y+w, z+w; f] + [0, x+y+z, x+y+w, x+y+z+w; f] + [0, z+w, x+y+z, x+y+z \\
& + w; f] + [0, z+w, x+y+w, x+y+z+w; f] - [0, x, x+z, z+w; f] - [0, z, z+x, z+w; f] - \\
& - [0, w, x+w, z+w; f] - [0, x+z, x+w, x+z+w; f] - [0, x+z, z+w, x+z+w; f] - \\
& - [0, w, x+w, z+w; f] - [0, x+z, x+w, x+z+w; f] - [0, x+z, z+w, x+z+w; f] - \\
& - [0, x+y, z+w, x+z+w; f]) + yzw([0, x+z, x+y+z, x+y+w; f] + [0, z, z+w, x+y+z \\
& + w; f] + [0, w, z+w, x+y+w; f] + [0, x+y+z, x+y+w, x+y+z+w; f] + [0, z+w, x+y+z, \\
& x+y+z+w; f][0, z+w, x+y+w, x+y+z+w; f] - [0, y, y+z, y+w; f] \\
& - [0, w, y+w, z+w; f] - [0, y+z, y+w, y+z+w; f] - [0, y+z, z+w, y+z+w] - [0, y+w, z+ \\
& + w, y+z+w]) = xzw([0, x+y+z, x+y+w, x+y+z+w; f] - [0, x+y+z, x+y+wx+z+w; f] \\
& [0, x+y+z, x+y+w, x+y+z+w; f] - [0, x+z, x+y+w, x+z+w; f] + [0, x+z, x+y+w, x+z+ \\
& w; f][0, x+z, x+w, x+z+w; f] + [0, z+w, x+y+z, x+y+z+w; f] - [0, z+w, x+y+z, x+z+w; f] \\
& + [0, z+w, x+y+z, x+z+w; f] - [0, z+w, x+z, x+z+w; f] + [0, z+w, x+y+w, x+y+z+w; f] \\
& - [0, z+w, x+y+w, x+z+w; f] + [0, z+w, x+y+w, x+z+w; f] - [0, z+w, x+w, x+z+w; f] \\
& + [0, w, z+w, x+y+w; f] - [0, w, z+w, x+w; f] + [0, z, z+w, x+y+z; f] - [0, z, z+w, x+z; f] \\
& + [0, x+y, x+y+z, x+y+w; f] - [0, x+y, x+z, x+y+w; f] + [0, x+y, x+z, x+y+w; f] - \\
& - [0, x, x+z, x+y+w] + [0, x, x+z, x+y+w] - [0, x, x+z, x+w; f] + yzw \cdot \\
& \cdot ([0, x+y+z, x+y+w, x+y+z+w; f] - [0, x+y+z, x+y+w, x+y+z+w; f] + [0, x+y+z, x+ \\
& y+w, y+z+w; f] - [0, y+z, x+y+w, y+z+w; f] + [0, y+z, x+y+w, y+z+w; f] - [0, y+z, y+w, \\
& y+z+w; f] + [0, z+w, x+y+z, x+y+z+w; f] - [0, z+w, x+y+z, y+z+w; f] + [0, z+w,
\end{aligned}
$$

$[0,x+y,x+y+z,x+y+w;f]-[0,x+y,y+z,x+y+w;f]+[0,x+y,y+z,x+y+w;f]-$
$[0,x+y,y+z,y+w;f]+[0,x+y,y+z,y+w]-[0,y,y+z,y+w;f]+[0,z,z+w,$
$x+y+z;f]-[0,z,z+w,y+z;f]+[0,w,z+w,x+y+w;f]-[0,w,z+w,y+w;f])=$
$xyzw([0,x+y+z,x+y+wx+z+w,x+y+z+w;f]+[0,x+z,x+y+z,x+y+w;f]$
$[0,x+z,x+w,x+y+w,x+z+w;f]+[0,z+w,x+y+z,x+z+w,x+y+z+w;f]+[0,z+$
$w,x+z,x+y+z,x+z+w;f]+[0,z+w,x+y+w,x+z+w,x+y+z+w;f]+[0,x+w,z+w,$
$x+y+w,x+z+w;f]+[0,w,z+w,x+w,x+y+w;f]+[0,z,x+z,z+w,x+y+z;f]+$
$[0,x+y,x+z,x+y+z,x+y+w;f]+[0,x,x+y,x+z,x+y+w;f]+[0,x,x+z,x+w,x+y+w;f]$
$+[0,x+y+z,x+y+w,y+z+w,x+y+z+w;f]+[0,x+y,x+z,x+y+z,x+y+w;f]+[0,y+z,$
$x+y+z,x+y+w,y+z+w;f]+[0,y+z,y+w,x+y+w,y+z+w;f]+[0,z+w,x+y+z,y+z$
$+w,x+y+z+w;f]+[0,z+w,y+z,x+y+z,x+y+z+w;f]+[0,z+w,x+y+w,z+y+w,$
$x+y+z+w;f]+[0,z++w,y+w,x+y+w,y+z+w;f]+[0,x+y,y+z,x+y+z,x+y+z+w;f]+$
$[0,x+y,y+z,y+w,x+y+w;f]+[0,y,x+y,y+z,y+w;f]+[0,z,y+z,z+w,x+y+z;f]+[0,w,$
$y+w,z+w,x+y+w;f])$.

Hence $f\in S_4^z[0,b]$ implies $f\in S_4[0,b]$. $\square$

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CONVEXITY AND FUNCTIONALS

Gh.TOADER

1 Convexity

There are many generalizations of the convexity of real functions. Some of them are surveyed in [7]. Let us recall here those which we use in what follows. We denote by:

$$C[a,b] = \{f : [a,b] \longrightarrow \mathbb{R},\ continuous\}$$
$$K[a,b] = \{f \in C[a,b];\ f(tx + (1-t)y) \le tf(x) + (1-t)f(y),$$
$$\forall t \in I,\ \forall x, y \in [a,b]\},\ where\ I = [0,1]$$
$$Q[a,b] = \{f \in C[a,b];\ f(tx + (1-t)y \le max\ (f(x), f(y)),$$
$$\forall y, x \in [a,b],\ \forall t \in I\}$$
$$C(b) = \{f \in C[0,b],\ f(0) = 0\}$$
$$K_p(b) = \{f \in C(b);\ f(tx + p(1-t)y) \le tf(x) + p(1-t)f(y),\ \forall t \in I,$$
$$\forall x, y \in [0,b]\}\ for\ p \in I$$
$$K_j(b) =\}f \in C(b);\ f(tx + sy) \le tf(x) + sf(y),\ \forall (t,s) \in J,$$
$$\forall x, y \in [0,b]\},\ with\ J \subset I \times I$$
$$S^*(b) = K_0(b)$$
$$S(b) = \{f \in C(b);\ f(x+y) \ge f(x) + f(y),\ \forall x, y,\ x + y \in (0,b]\}$$

the sets of continuous, convex, quasi-convex functions on $[a,b]$, respectively continuous, p-convex, j-convex, starshaped, superadditive functions on $[0,b]$ with $f(0) = 0$.

Taking into account all these classes of functions, we are led naturally to the following general definition.

Let L be a set of functionals defined on a set M of functions.

Definition 1.1. *A function $f \in M$ is said to be convex with respect to the set L (or L-convex) if:*

$$A(f) \ge 0,\ \forall A \in L.$$

We denote by L^+M the set of L-convex functions from M.

Remark 1.1. *A similar definition is given in [2] and [3] for the elements of a vector space but having in view other problems.*

It is easy to indicate the sets of functionals which define each of the above classes. We use mainly the functional of evaluation, given by:

$$E_x(f) = f(x).$$

Then, the sets of convex, p-convex, J-convex, starshaped, superadditive and quasi-convex functions are defined respectively by the sets of functionals:

$$\begin{aligned}
&K = \{tE_x + (1-t)E_y - E_{tx+(1-t)y};\ t \in I,\ x, y \in [a, b]\}\\
&K_p = \{tE_x + p(1-t)E_y - E_{tx+p(1-t)y};\ t \in I,\ x, y \in [0, b]\}\\
&K_j = \{tE_x + sE_y - E_{tx+sy};\ (t, s) \in J,\ x, y \in [0, b]\}\\
&S^* = \{tE_x - E_{tx};\ t \in I,\ x \in [0, b]\}\\
&S = \{E_{x+y} - E_x - E_y;\ x, y,\ x + y \in [0, b]\}\\
&Q = \{max(E_x, E_y) - E_{tx+(1-t)y};\ t \in I,\ x, y \in [a, b]\}.
\end{aligned}$$

We have thus:

$$K^+C[a, b] = K[a, b],\ K_p^+C(b) = K_p(b),$$

and so on .

2 Inequalities

As it is known for $K[a, b]$ (see [14] and [10] the set of functionals:

$$L^+ = \{A : M \longrightarrow \mathbb{R};\ A(f) \geq 0,\ \forall f \in L^+M\}$$

may be generally much more rich than L.

What we can say in general about it? We shall indicate three way for construction of elements from L^+.

If we consider the convex conical span of the set L:

$$\begin{gathered}
\textit{cone}\ (L) = \{A : M \longrightarrow \mathbb{R};\ \exists t_1, \ldots, t_n \geq 0,\ \exists A_1, \ldots, A_n \in L,\\
A = t_1A_1 + \cdots + t_nA_n\}
\end{gathered}$$

we have easily the following:

Lemma 2.1. *For every set of functionals L, holds the inclusion:*

$$cone(L) \subset L^+.$$

Also we can consider a generalized adherence of cone (L) by:

$$clcone(L) = \{A : M \to \mathbb{R};\ \forall n \in N,\ \exists A_n \in cone(L),\ \forall f \in M$$
$$A(f) = \lim_{n\to\infty} inf A_n(f)\}.$$

Then we have also:

Lemma 2.2. *For every set L, holds:*

$$clcone(L) \subseteq L^+.$$

Definition 2.1. *Two sets of functionals L and L' are in relation $L' \geq L$ it for every $B \in L'$ there is an $A \in L$ such that $B(f) \geq A(f)$ for every $f \in M$.*

Lemma 2.3. *If $L' \geq L$ then $L' \subset L^+$.*

There are many papers which prove inequalities for convex functions. In fact, they establish the belonging of some functional to the corresponding set L^+. We want to analyse some of them from this point of view.

We begin with the most familiar of them, the inequality of Jensen:if f is a convex function on $[a,b], x_1,\ldots,x_n \in [a,b]$ and $c_1,\ldots,c_n$ are positive constants, then:

$$f\Big(\sum_{k=1}^{n} c_k x_k / \sum_{k=1}^{n} c_k\Big) \leq \sum_{k=1}^{n} c_k f(x_k) / \sum_{k=1}^{n} c_k.$$

This is equivalent with the belonging to K^+ of the functional:

$$J_n = \sum_{k=1}^{n} c_k E_{x_k} / \sum_{k=1}^{n} c_k - E_{x_n}$$

where we have denoted:

$$X_m = \sum_{k=1}^{m} c_k x_k / \sum_{k=1}^{m} c_k.$$

Writing J_n as:

$$J_n = \sum_{k=2}^{n} [(c_1 + \cdots + c_{k-1}) E x_k + c_k E_{x_k} - (c_1 + \cdots + c_k) E x_k] / \sum_{i=1}^{n} c_i$$

it result that:

$$J_n \in cone(K).$$

Another well-known inequality is that of Hadamard: for $f \in K[a,b]$ hold the relations:

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f9x)dx \le \frac{f(a)+f(b)}{2}.$$

To prove it we show that:

$$(E_a + E_b)/2 - M_{a,b},\ M_{a,b} - E_{(a+b)/2} \in clcone(K)$$

where we have denoted by $M_{a,b}$ the functional defined by:

$$M_{a,b}(f) = \frac{1}{b-a}\int_a^b f(x)dx.$$

But $(E_a + E_b)/2 - M_{a,b}$ is the (punctual) limit of the following sequence of functionals:

$$\frac{1}{2}E_a + \frac{1}{2}E_b - \frac{1}{n}\sum_{i=1}^{n} E_{a+(1-1/2)(b-a)/n} =$$

$$= \frac{1}{n}\sum_{i=1}^{n}\Big[\frac{n-i+1/2}{n}E_a + \frac{i-1/2}{n}E_b - \frac{E_{n-i+1/2}}{n}a + \frac{i-1/2}{n}b\Big]$$

while $M_{a,b} - E_{(a+b)/2}$ may be obtained as the limit of the sequence:

$$\frac{1}{2n}\Big(\sum_{i=o}^{n-1} E_{a+ih} + E_{a-ih}\Big) - E_{(a+b)/2} =$$

$$= \frac{1}{2n}\sum_{i=0}^{n-1}(E_{a+th} + E_{b-ih} - 2E(a+b)/2)$$

where $h = (b-a)/2n$. As these are elements of $cone(K)4$, the results follow.

3 Hierarchies of convexity

On this way we can also easily compare two convexity classes.

Lemma 3.1. *We have $L_1^+M \subset L_2^+M$ if and only if $L_2 \subset L_1^+$.*

Remark 3.1. *Lemmas $1-3$ offer methods for obtaining such subsets L_2.*

Definition 3.1. *For the two subsets J and H of $I \times I$, the relation $J \ge H$ means that for any $(t,s) \in H$ there is an $(i,r) \in J$ with $s \le r$. Also $H \ge 0$ means that $H \ge I \times \{0\}$.*

Theorem 3.1. *Hold the following relations:*

(a) $S^* \subset K$ on $C(b)$

(b) $S^* \subset K_p$ on $C(b)$

(c) $S^* \subset K_j$ *on* $C(b)$ if $J \geq 0$

(d) $Q \geq K$ *in* $C[a, b]$

(e) if $p \leq q$ *then* $K_p \subset K_q$ *on* $C(b)$

(f) if $J \geq H \geq 0$ *then* $K_H \subset K_J$ *on* $C(b)$

(g) $S \subset cone(S^*)$ *on* $C(b)$.

Proof. (a), (b) and (c) follow by taking $y = 0$.
(d) We have for any $f \in C[a, b]$:
$max(E_x(f), E_y(f)) - E_{tx+(1-t)y}(f) = [t + (1 - t)]max(E - x(f),$
$E_y(f)) - E_{tx+(1-t)y}(f) \geq tE_x(f) + (1 - t)E_y(f) - E_{tx+(1-t)y}(f)$
(e) For $p \leq q$, it follows:

$$tE - x + p(1 - t)E_y - E_{tx+(1-t)y} =$$
$$tE_x + (1 - t)E_{(p/q)y} - E_{tx+q(1-t)(p/q)y} + q(1 - t)((p/q)E_v - E_{(p/q)y})$$

that is, it belongs to cone (K_q) because $S^* \subset K_q$ by (b).
(f) If $J \geq H$, then for $(t, s) \in H$ there is an $(t, r) \in J$ with $r \geq s$. So:

$$tE_x + sE_y - E_{tx+sy} = tE_x + rE_{(s/r)y} - E_{tx+(s/r)y} + r\left(\frac{s}{r}E_y - E_{(s/r)y}\right)$$

and the conclusion follows because $S^* \subset K_J$.
(g) If $x, y,\ x + y \in [0, b]$:

$$E_{x+y} - E_x - E_y = \frac{x}{x+y}E_{x+y} - E_x + \frac{y}{x+y}E_{x+y} - E_y =$$
$$= \left[\frac{x}{x+y}E_{x+y} - E_{\frac{x}{x+y}(x+y)}\right] + \left[\frac{y}{x+y}E_{x+y} - E_{\frac{y}{x+y}(x+y)}\right].$$

These relations give the following known results: □

Theorem 3.2. *Hold the following inclusions:*

$$K_1(b) = K(b) \subset S^*(b) \subset S(b),\ (see[1]and[8])$$
$$K_q(b) \subset K_p(b)\ if\ q \geq p,\ (see[7]or[11])$$
$$K_j(b) \subset K_H(b)\ if\ J \geq H \geq 0\ (see[7]or[9])$$
$$K[a, b] \subset Q[a, b].$$

In [12] we have generalized the first chain of inclusions for convexity of higher order. Let us remind that the divided differences (on the distinct points $x_0, x_1, \ldots$) are defined recurrently by:

$$[x_0; f] = f(x_0), [x_0, \ldots, x_n; f] = ([x_0, \ldots, x_{n-1}; f]-$$
$$- [x_1, \ldots, x_n; f])/(x_0 - x_n).$$

One considers the set of functions convex of order n:

$$K_n[a,b] = \{f : [a,b] \to \mathbb{R},\ [x_0,\dots,x_n;f] \geq 0,$$
$$\forall x_0,\dots,x_n \in [a,b] distinct\}$$

the set of starshaped of order n functions:

$$S_n^*[a,b] = \{f : [a,b] \to \mathbb{R},\ [a,x_1,\dots,x_n;f] \geq 0,$$
$$\forall x_1,\dots,x_n \in [a,b] distinct\}$$

and that of functions superadditive of order n:

$$S_n[a,b] = \{f : [a,b] \to \mathbb{R},\ \forall x_1,\dots,x_n \in (a,b] \text{ distinct and}$$
$$x_1 + \cdots + x_n - na \leq b-a \text{ implies } \sum_{k=0}^{n} (-1)^{n-h} \cdot$$
$$\cdot \sum_{(k)} f(x_{i_1} + \cdots + x_{i_k} - (k-1)a) \geq 0\}$$

where $\sum_{(k)}$ means the sum over all the combinations of indices for $k > 0$ and $f(a)$ for $k = 0$.

These sets of functions are defined by the sets of functionals:

$$K_n = \{[x_0,x_1,\dots,x_n;\cdot]; x_0,x_1,\dots,x_n \in [a,b] \textit{ distinct }\}$$

$$S_n^* = \{[a,x_1,\dots,x_n;\cdot]; x_1,\dots,x_n \in (a,b] \textit{ distinct }\}$$

respectively:

$$S_n = \{\sum_{k=0}^{n} (-1)^{n-k} \sum_{(k)} E_{x_{i_1}} + \cdots + x_{i_k} - k - 1)a;$$
$$x_1,\dots,x_n \in [a,b] \textit{ distinct }, x_1 + \cdots + x_n - na \leq b-a\}.$$

Obviously:

$$S_n^* \subset K_n,\ \forall n.$$

In [12] we have prove that:

$$E_{x+y-a} - E_x - E_y + E_a = (x-a)(y-a)([a,x,x+y-a;\cdot]+$$
$$+ [a,y,x+y-a;\cdot])$$

$$E_{x+y+z-2a} - E_{x+y-a} - E_{x+z-a} - E_{y+z-a} + E_x + E_y + E_z - E_a =$$
$$= (x-a)(y-a)(z-a)([a,x,x+y-a,x+z-a;\cdot]+$$
$$+ [a,y,x+y-a,y+z-a;\cdot] + [a,z,x+z-a,y+z-a;\cdot]+$$
$$+ [a,x+y-a,x+z-a,x+y+z-2a;\cdot]+$$
$$+ [a,x+y-a,y+z-a,x+y+z-2a;\cdot]+$$
$$+ [a,x+z-a,y+z-a,x+y+z-2a;\cdot])$$

and analogously every functional from S_4 can be expressed as a sum of 4! functionals from S_n^*. Thus:

$$S_n \subset cone(S_n^*) \; for \; n = 2, 3, 4.$$

So we have proved:

Theorem 3.3. *For every interval $[a, b]$ hold the inclusions:*

$$K_n[a, b] \subset S_n^*[a, b] \subset S_n[a, b], \; for \; n = 2, 3, 4.$$

These inclusions generalize the relations:

$$K_n[a, b] \subset S_n[a, b]$$

proved for $n = 2$ by M.Petrovič in [5], for $n = 3$ by P.M. Vasič in [13] and for $n = 4$ by J.D. Kcĕkié in [4]. So they also generalize the corresponding results of T.Popoviciu from [6].

Taking into account these results, we can remark that in some cases an inequality is valid in more general conditions than those in which it is given. So are those form [5], [13] and [4]. We remind here that from [4] which generalizes all of them: if $f \in K_n[0, b]$, then for every $m \geq n, \; x_i \in [0, b], \; i = 1, \dots, m, x_1 + \dots + x_m \leq b$, holds:

$$f(x_1 + \dots + x_m) + \sum_{k=0}^{n-1} (-1)^{n-k} \binom{m-k-1}{n-k-1} \sum_{(k)} f(x_{i_1} + \dots + x_{i_k}) \geq 0.$$

In fact this is valid for every function f from $S_n[0, b]$ and it follows even from the original proof.

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Proceedings of the Third Sympo. Math. Appl., Timisoara, 1989, 9-12

A Hierarchy of convexity of order three of functions

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The divided differences on the distinct points $x_0, x_1, \dots$ are defined recurrently by:

$$[x_0; f] = f(x_0), [x_0, x_1, \dots, x_n; f] = ([x_0, \dots, x_{n-1}; f] - [x_1, \dots, x_n; f])/(x_0 - x_n).$$

Using them we can define the following sets:

$$\begin{aligned} C(b) &= \{f : [0,b] - \mathbb{R}, f(0) = 0, \text{ continuous}\} \\ K_n(b) &= \{f \in C(b),\ [x_0, \dots, x_n; f] \geq 0,\ \forall x_0, \dots, x_n \in [0,b]\} \\ S_n^*(b) &= \{f \in C(b),\ [0, x_1, \dots, x_n; f] \geq 0,\ \forall x_1, \dots, x_n \in [0,b]\} \\ S_n(b) &= \{f \in C(b),\ \forall x_1, \dots, x_n \in [0,b] \text{ distinct},\ x_1 + \dots + x_n \leq b, \\ &\sum_{k=1}^{n} (-1)^{n-k} \sum_{(k)} f(x_{i_1} + x_{i_k}) \geq 0\} \end{aligned}$$

that is of continuous, n-convex, n-starshaped respectively n-superadditive functions. By $\sum_{(k)} f(x_{i_1} + x_{i_k})$ we have denoted the sum over all the combinations of indices $1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n$.

We study the problem of transformation of functions by the weighted mean $A_g : C(b) \longrightarrow C(b)$, defined by:

$$A_g(f)(x) = \frac{1}{g(x)} \int_0^\pi g'(t) f(t) dt,\ A_g(f)(0) = 0 \tag{1}$$

where g has a continuous first derivative and $g(0) = 0$. An important particular case is given by the Cesáro type operator defined by:

$$A_u(f)(x) = \frac{u}{x^u} \int_0^x t^{u-1} f(t) dt,\ A_u(f)(0) = 0. \tag{1'}$$

We can consider the sets $M^u K_n(b)$, $M^u S_n^*(b)$ and $M^u S_n(b)$ of functions f with the property that $A_u(f)$ belongs to $K_n(b), S_n^*(b)$ respectively $S_n(b)$.

For $n = 2$ and $u = 1$ it was proved in [2] that:

$$K_2(b) \subset M^1 K_2(b) \subset S_2^*(b) \subset S_2(b) \subset M^1 S_2^*(b) \subset M^1 S_n(b).$$

This was called "hierarchy of convexity". Using some results from [3] we have extended in [4] these results proving that for $n = 2$ and $0 \leq u \leq v$ hold the inclusions:

$$\begin{array}{ccccc}
K_2(b) & \subset M^v K_2(b) & \subset M^u K_2(b) & \subset S_2^*(b) & \subset S_2(b) \\
 & & & \cap & \cap \\
 & & & M^v S_2^*(b) & \subset M^v S_2(b) \\
 & & & \cap & \\
 & & & M^u S_2^*(b) & \subset M^u S_2(b).
\end{array}$$

In this paper we study analogous property for greather n starting from the result proved in [5] that for $n = 3$ and 4 hold the inclusions:

$$K_n(b) \subset S_n^*(b) \subset S_n(b). \tag{2}$$

Theorem 0.1. *If $A_g(f) \in K_n(b)$ ($S_n^*(b)$ or $S_n(b)$) for any f of $K_n(b)$ ($S_n^*(b)$ respectively $S_n(b)$) with $n = 3$ or 4, then there is an $u \geq 0$ and a real c such that:*

$$g(x) = c \cdot x^u \tag{3}$$

that is $A_g = A_u$.

Proof. Let us denote by P_n the set of all polynomials of degree at most n. As $\pm p \in K_n(b)$ ($S_n^*(b)$ or $S_n(b)$) for any $p \in p_{n-1}$, by (2) it means that for $n = 3$ (or $n = 4$) $A_g(\pm p) = \pm A_g(p) \in S_n(b)$, that is (see [1]) $A_g(p) \in P_{n-1}$. We denote $e_k(x) = x^k$ and $A_g(e_k) = p_k$. From (1) we deduce:

$$g'(x)/g(x) = p_k'(x)/(x^k - p_k(x)). \tag{4}$$

If $n = 3$, we have $k = 1$ and 2 and (4) gives:

$$p_1'(x)/(x - p_1(x)) = p_2'(x)/(x^2 - p_2(x)). \tag{5}$$

We denote: $p_k(x) = a_k + b_k x + c_k x^2$ and multiplying in (5) and equalizing the coefficients of x^3 we get $c_1 - 0$. For $n = 4$, we have $k = 1, 2$ and 3 hence (4) gives:

$$p_1'(x)/(x - p_1(x)) = p_2'(x)/(x62 - p_2(x)) = p_3'(x)/(x^3 - p_3(x)).$$

Denoting: $p_k(x) = a_k + b_k x + c_k x^2 + d_k x^3$, from the first and the last report we get $d_1 = 0$, from the second and the third report we get $d_2 = 0$ and then, from the first and the second report we get $c_1 = 0$.

So, for $k = 1$ and $n = 3$ or 4, the relation (4) becomes:

$$g'(x)/g(x) = b_1/(x - b_1x - a_1)$$

that is:

$$g(x) = [kx(1 - b_1) - a_1]^{b_1/(1-b_1)}.$$

As $g(0) = 0$, we get (3). $\square$

Theorem 0.2. *For any $b, u > 0$ and any n, hold the inclusions:*

$$K_n(b) \subset M^u K_n(b),\ S_n^*(b) \subset M^u S_n^*(b), S_n(b) \subset M^u S_n(b).$$

Proof. Making (as in [3]) the substitution:

$$t = x \cdot s^{1/u}$$

(1') becomes:

$$A_u(f)(x) = \int_0^1 f(xs^{1/u})ds.$$

As we know:

$$[x_0, x_1, \dots, x_n; f] = \sum_{k=0}^{n} \frac{f(x_k)}{p_n'(x_k)}$$

where $p_n(x) = (x - x_0)(x - x_1)\dots(x - x_n)$. So:

$$[x_0, x_1, \dots, x_n; A_u(f)] = \sum_{k=0}^{n} \frac{A_u(f)(x_k)}{p_n'(x_k)} =$$
$$\sum_{k=0}^{n} \frac{1}{p_n'(x_k)} \int_0^1 f(x_k s^{1/u})ds = \int_0^1 \sum_{k=0}^{n} \frac{f(x_k s^{1/u})}{p_n'(x_k)} ds =$$
$$= \int_0^1 [x_0 s^{1/u}, \dots, x_n s^{1/u}; f]ds.$$

Thus $A_u(f) \in K_n(b)$ if $f \in K_n(b)$ and $A_u(f) \in S_n^*(b)$ if $f \in S_n^*(b)$. Also:

$$\sum_{k=0}^{n} (-1)^{n-k} \sum_{(k)} A_u(f)(x_{i_1} + \dots + x_{i_k}) = \sum_{k=0}^{n} (-1)^{n-k} \sum_{(k)} \int_0^1 f(s^{1/u})\cdot$$
$$(x_{i_1} + \dots + x_{i_k}))ds = \int_0^1 \sum_{k=0}^{n} (-1)^{n-k} \sum_{(k)} f(s^{1/u}(x_{i_1} + \dots + x_{i_k}))ds$$

hence $A_n(f) \in B_n(f)$ if $f \in S_n(b)$. $\square$

Theorem 0.3. *For any $b, u > 0$ and $n = 3$ or 4, hold the inclusions:*

$$\begin{array}{ccc} K_n(b) & \subset S_n^*(b) & \subset S_n(b) \\ \cap & \cap & \cap \\ M^u K_n(b) & \subset M^u S_n^*(b) & \subset M^u S_n(b). \end{array}$$

Proof. The inclusions from the first line is given in (2) and these of the second line follows from these. The other inclusions are proved in Theorem 3. ☐

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Bull. Appl. Math. (Budapest) **743**(1991)(LIX), 99-102

ON SOME INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS

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The integral inequalities can be proved starting from the interpretation of definite integral as a summation process. In some cases it may be used an arbitrary integral summ and apply a corresponding discrete result.

Such are, for example, the integral inequality of Jensen or the inequality between two integral quasi-arithmetic means (see [1]). In more complicated cases we prove that the discrete result is valid for a sequence of divisions having the norm tending to zero. So we have proved in [7] the Hermite-Hadamard inequality (see [5]).

This paper we want to prove analogously a generalization of this inequality given by L.Fejér in [3] and also to improve an integral inequality from [4]. It is interesting to analise how the hypotheses are used in these demonstrations. The result will also follow for integrable Jensen convex functions, not only for convex functions.

Theorem 0.1. *If $f : [a,b] \longrightarrow \mathbb{R}$ is (Jensen) convex and the function $h : [a,b] \longrightarrow \mathbb{R}$ is positive and symmetric with respect to $(a+b)/2$, then:*

$$f\Big(\frac{a+b}{2}\Big) \leq \int_a^b f(x)h(x)dx / \int_a^b h(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

Proof. The inequalities (1) are equivalent with:

$$\int_a^b (f(x) - f\Big(\frac{a+b}{2}\Big))h(x)dx \geq 0 \tag{2}$$

and

$$\int_a^b (f(a)+f(b)-2f(x))h(x)dx \geq 0. \tag{3}$$

To prove them, we use for every n the equidistant division with $2n$ knots. Denoting $k = (b-a)/2n$ we consider the sum:

$$S_1 = k\sum_{\substack{i=0\\ i\neq n}}^{2n}(f(a+ki) - f\Big(\frac{a+b}{2}\Big))h(a+ki).$$

As:$h(a+ki)=h(b-ki)$, we get:

$$S_1 = k\sum_{i=0}^{n-1}(f(a+ki)+f(b-ki)-2f\Big(\frac{a+b}{2}\Big))h(a+ki)\geq 0,$$

because f is Jensen convex and h positive. Analogously, using the sum:

$$\begin{aligned}
S_2 &= k\sum_{\substack{i=0\\ i\neq n}}^{2n}(f(a)+f(b)-2f(a+ki))h(a+ki) = \\
&= 2k\sum_{i=0}^{n-1}(f(a)+f(b)-f(a+ki)-f(b-ki))h(a+ki) = \\
&= 2k\sum_{i=0}^{n-1}(((\frac{2n-i}{2n})f(a)+\Big(\frac{i}{2n}\Big)f(b)-f\Big(\frac{2n-i}{2n}a+\frac{i}{2n}b\Big))+ \\
&+\Big(\frac{i}{2n}f(a)+\frac{2n-i}{2n}f(b)-f\Big(\frac{i}{2n}a+\frac{2n-i}{2n}b\Big)\Big))h(a+ki)\geq 0,
\end{aligned}$$

we get (3). $\square$

Remark 0.1. *This is the inequality of L.Fejér for convex functions. Taking $h(x)=1$ we get the inequality of Hermite-Hadamard.*

Theorem 0.2. *If the function $f:[a,b]\longrightarrow[c,d]$ is increasing and (Jensen) convex and $g,h:[c,d)\longrightarrow[0,\infty)$ are such that g/h is increasing, then:*

$$\int_a^b g(f(x))dx/\int_a^b h(f(x))dx \leq \int_{f(a)}^{f(b)} g(x)dx/\int_{f(a)}^{f(b)} h(x)dx. \tag{4}$$

Proof. For every n we consider the equidistant knots $(x_k)_{k=0}^n$, that is $x_{k+1}-x_k=(b-a)/n$. We use them for the integrals of the left part of the inequality while for those of the right port we use the knots $(f(x_k))_{k=0}^n$ f being increasing. To prove (4) we have so to show that:

$$\frac{\frac{b-a}{n}\sum\limits_{k=0}^{n-1}g(f(x_k))}{\frac{b-a}{n}\sum\limits_{k=0}^{n-1}h(f(x_k))}\frac{\sum\limits_{k=0}^{n-1}g(f(x_k))(f(x_{k+1})-f(x_k))}{\sum\limits_{k=0}^{n-1}h(f(x_k))(f(x_{k+1})-f(x_k))}. \tag{5}$$

We use Abel's identity:

$$\sum_{k=0}^{n-1}a_k\cdot b_k = a_0(b_0+\cdots+b_{n-1}+\sum_{k=1}^{n-1}(a_k-a_{k-1})(b_k+\cdots+b_{n-1})$$

and the simple equivalences valid for positive numbers:

$$\frac{px+qy}{pz+qw}\geq\frac{x}{z} \quad iff \quad \frac{y}{w}\geq\frac{x}{z}$$

and
$$\frac{px+qy}{pz+qw} \geq \frac{x}{z} \quad iff \quad \frac{y}{w} \leq \frac{x}{z}.$$
Denoting $f(x_{k+1}) - f(x_k) = \triangle f(x_k)$ and $\triangle f(x_{k+1} - \triangle f(x_k) = \triangle^2 f(x_k)$, we get:
$$\frac{\sum\limits_{k=0}^{n-1} g(f(x_k))\triangle f(x_k)}{\sum\limits_{k=0}^{n-1} g(f(x_k))\triangle f(x_k)} = \frac{\triangle f(x_0) \sum\limits_{k=0}^{n-1} g(f(x_k)) + \sum\limits_{i=1}^{n-1} \triangle^2 f(x_{i-1}) \sum\limits_{k=i}^{n-1} g(f(x_k))}{\triangle f(x_0) \sum\limits_{k=0}^{n-1} h(f(x_k)) \sum\limits_{i=1}^{n-1} \triangle^2 f(x_{i-1}) \sum\limits_{k=i}^{n-1} g(f(x_k))} \geq$$
$$\geq \frac{\sum\limits_{k=0}^{n-1} g(f(x_k))}{\sum\limits_{k=0}^{n-1} h(f(x_k))},$$
because:
$$\frac{g(f(x_{n-1}))}{hg(f(x_{n-1}))} \geq \frac{g(f(x_{n-2})) + g(f(x_{n-1}))}{h(f(x_{n-2})) + h(f(x_{n-1}))} \geq \cdots \geq \frac{g(f(x_0)) + \cdots + g(f(x_{n-1}))}{h(f(x_0)) + \cdots + h(f(x_{n-1}))}$$
and $\triangle^2 f(x_i) \geq 0$, the function f being Jensen convex. □

Remark 0.2. *The inequality (4) was proposed as a problem in [5] for $g(x) = x^2$ and $h(x) = x$. It was proved in [2] for $g(x) = x^r$ and $h(x) = x^s$ with $r \geq s$. In this form it was proved in [4] but under the assumption of differentiability of f.*

Finally we note that the inequality (4) can give some results analogous with those of H. Thunsdorff generalized by L.Berwald or those of A.M.Fink (see [6]).

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MEANS AND CONVEXITY

Gh.TOADER

1 Introduction

In this paper we consider a notion of convexity with respect to a power mean called-r-convexity. We generalize Hermite-Hadamard's inequality for functions with r-convexe inverse. Then we apply it for the study of the monotony of the "relative growth" of generalized logarithmic means. We try to analyse so the position of the mean values of two numbers between those numbers.

As moust of the definitions and results which we need may be found in the book of P.S. Bullen, D.S.Mitrinovič and P.M. Vasič [1] we content ourself to refer mainly at it.

2 Means

We shall use in what follows some means of two positive numbers $0 < a < b$. They all belong to the familly of extended mean values defined by K.B. Stolarsky (see [1], p.345) for $r \neq s,\ rs \neq 0$ by:

$$E_{rs}(a,b) = ((r/s)(b^2 - a^2)/(b^r - a^r))^{1/(s-r)}$$

the definition for other values being obtained by taking limits.
As special cases we have the power means:

$$P_r = E_{r,2r} \text{ for } r \neq 0$$

and

$$P_0(a,b) = G(a,b) = (a \cdot b)^{1/2}$$

then the generalized logarithmic means defined by:

$$L_r = E_{1,r+1}, \text{ for } r \neq 1,\ r \neq 0$$

but

$$L_{-1}(a,b) = L(a,b) = (b-a)/(\log b - \log a)$$

and

$$L_0(a,b) = r(a,b) = (1/e)(b^b/a^a)^{1/(b-a)}.$$

Also we use weighted power means defined for $0 \le t \le 1$ by:

$$P_{rt}(a,b) = (ta^r + (1-t)b^r)^{1/r} \text{ if } r \neq 0$$

and

$$P_{0t}(a,b) = G_t(a,b) = a^t b^{1-t}.$$

For $t = 1/2$ we get the usual power means and for $r = 1$ the weighted arithmetic mean $P_{rt} = A_t$.

Among the properties of these means we are interested in their monotony with respect to the parameter. So we have (see [1], p.159) for $r < s$:

$$P_{rt}(a,b) < P_{st}(a,b),\ 0 < t < 1 \tag{1}$$

and also (see [1] p.347):

$$L_r(a,b) < L_s(a,b). \tag{2}$$

3 r-Convexity

Let us consider the following notion: we said that the positive function $f : [a,b] \longrightarrow \mathbb{R}$ is r-convex if:

$$f(A_t(X,Y)) \le P_{rt}(f(X), f(Y)),\ \forall X, Y \in [a,b],\ t \in [0,1].$$

As we can remark, this notion differs from a similar one given in [1] called r-mean convexity.

From [1] we deduce that if f is r-convex then it is also s-convex for every $s > r$. Also from the definition we deduce that f is r-convex if and only if:

a) f^r is convex, for $r > 0$;

b) $\log f$ is convex, for $r = 0$ and

c) f^r is concave for $r < 0$

Thus 0-convexity is in fact logarithmic convexity.

The paper [3] deals with functions which have logarithmic convex inverse. We consider also functions with r-convex inverse. Let us denote by $K_r^-[a,b]$ the set of positive, strictly increasing functions with r-convex inverse defined on $[a,b]$. We have:

$$K_r^-[a,b] \subset K_s^-[a,b],\ \text{for } r < s. \tag{3}$$

It is also easy to check the following:

Lemma 3.1. *If the positive function f is twice differentiable then it belongs to $K_r^-[a,b]$ if and only if:*

$$f'(x) > 0 \text{ and } 1 + xf''(x)/f'(x) \leq r,\ \forall x \in [a,b] \tag{4}$$

Integrating the differential equation obtained from (4) we get functions which can be considered to be r-linear. As a special case we have:

Lemma 3.2. *The function f_r defined by:*

$$f_r(x) = \begin{cases} x^r - a^r, & r > 0 \\ \log x - \log a, & r = 0 \\ a^r - x^r, & r < 0 \end{cases} \tag{5}$$

has the properties:

$$f_r(x) \geq 0,\ f_r'(x) > 0,\ 1 + xf_r''(x)/f_r'(x) = r,\ \forall x \geq a.$$

4 Hermite-Hadamard's inequality

For a function $f : [a,b] \longrightarrow \mathbb{R}$ consider the integral arithmetic mean defined by:

$$A(f;a,b) = \int_a^b f(x)dx/(b-a).$$

Hermite-Hadamard's inequality (see [1], p.30) gives for a concave function f the evaluation:

$$(f(a) + f(b))/2 \leq A(f;a,b) \leq f((a+b)/2). \tag{6}$$

Also H.-J. Seiffert proved in [3] that for a function f from $K_0^-[a,b]$ holds:

$$A(f;a,b) \leq f(I(a,b)). \tag{7}$$

We remark that from (2) it follows:

$$I(a,b) = L_0(a,b) < L_1(a,b) = (a+b)/2$$

thus (7) improves the right side of (6) for this special case.
We can do the same thing for functions from $K_0^-[a,b]$ with $r \neq 0$.

In the proof of the relation (7) it is used the following result, proposed as a problem by R.Euler in [2]:

$$\lim_{n\to\infty} \Big(\prod_{i=1}^{n} (c + (i-1)/n)\Big)^{1/n} = I(c, c+1),\ \forall c > 0. \tag{8}$$

The expression from the first member of (8) is a geometric mean (of n numbers). We can prove a similar relation to (8) for an arbitrary power mean.

Lemma 4.1. *If $r \neq 0$ and $c > 0$ then:*

$$\lim_{n\to\infty}\Big(\sum_{i=1}^{n}(c+\frac{(\frac{i-1}{n})^r}{n}\Big)^{1/r} = L_r(c,c+1). \tag{9}$$

Proof. If $r > 0$, the mean value theorem of the differential calculus applied to the function $f(x) = (x+1)^{r+1},\ x > 0$, gives:

$$((x+1)^{r+1} - x^{r+1})/(r+1) < (x+1)^r < ((x+2)^{r+1} - (x+1)^{r+1})/(r+1). \tag{10}$$

For $n > 1/c$, we get by addition:

$$L_r\Big(c-\frac{1}{n}, c+1-\frac{1}{n}\Big) < \Big(\sum_{i=1}^{n}\Big(c+\frac{(\frac{i-1}{n})^r}{n}\Big)\Big)^{1/r} < L_r(c,c+1)$$

hence (9). For $r < 0, r \neq -1$, we have to do minor changes in the proof, while for $r = -1$ we must replace (10) by:

$$\log(x+2) - \log(x+1) < (x+1)^{-1} < \log(x+1) - \log x.$$

Finally we remark that the case $r = 0$, excepted from (9), is contained in (8).

Replacing (8) by (9) in the proof of (7) given in [3] we get: $\square$

Theorem 4.1. *If the function f belongs to $K_r^-[a,b]$ then:*

$$A(f;a,b) \leq f(L_r(a,b)). \tag{11}$$

Let us remark that the function f_r defined by (5) verifies:

$$a(f_r;a,b) = f_r(L_r(a,b)). \tag{12}$$

We can improve also the left inequality from (6) for the same class of functions.

Theorem 4.2. *If the function f belongs to $K_r^-[a,b]$ then:*

$$\begin{aligned} &A(f;a,b) \geq \\ &\geq (f(a)(b^r - L_r^r(a,b)) + f(b)(L_r^r(a,b) - a^r))/(b^r - a^r). \end{aligned} \tag{13}$$

if $r \neq 0$ and

$$A(f;a,b) \geq (f(a)(L(a,b) - a) + f(b)(b - L(a,b)))/(b-a) \tag{14}$$

if $r = 0$.

Proof. For $t \in [a,b]$ we have:

$$f(t) = \frac{f(b)-f(t)}{f(b)-f(a)}f(a) + \frac{f(t)-f(a)}{f(b)-f(a)}f(b). \tag{15}$$

So, if $r > 0$, $(f^{-1})^r$ being convex:

$$t^r \leq \frac{f(b)-f(t)}{f(b)-f(a)}a^r + \frac{f(t)-f(a)}{f(b)-f(a)}b^r$$

or

$$f(t) \geq \frac{f(b)-f(a)}{b^r-a^r}t^r + \frac{b^r f(a) - a^r f(b)}{b^r - a^r}.$$

It is also valid for $r < 0$. by integration we get (13). For $r = 0$; $\log(f^{-1})$ is convex and (15) gives:

$$\log t \leq \frac{f(b)-f(t)}{f9b)-f(a)}\log a + \frac{f(t)-f(a)}{f(b)-f(a)}\log b.$$

Isolating $f(t)$ and integrating we get (14). □

5 The relative growth

We consider the following expression:

$$D_r 9a,b) = \begin{cases} \frac{L_r^r(a,b)-a^r}{b^r-a^r}, & r \neq 0 \\ \frac{b-L(a,b)}{b-a}, & r = 0. \end{cases}$$

which we call relative growth of L_r. It is easy to see that:

$$0 \leq D_r(a,b) \leq 1, \ \forall r; \ D_1(a,b) = 1/2.$$

Theorem 5.1. *If $r < s$ and $0 < a, b$ then:*

$$D_r(a,b) \geq D_s(a,b). \tag{16}$$

Proof. As the function f_r given by (5) belongs to $K_r^-[a,b]$ and $r < s$, from (3) it follows that it is also in $K_s^-[a,b]$ and so (12), (13) and (14) implies:

$$A(f_r;a,b) = f_r(L_r(a,b)) \geq f_r(b)D_s(a,b)$$

which gives (16). In fact we must consider separately the cases: $0 < r < s$, $0 = r < s$, $r < s = 0$ and $r < s < 0$. □

Remark 5.1. *From (2) it follows that the evaluation given by (11) is improved by decreasing the value of the parameter r. The same conclusion is valid for (13) and (14) if we take into account (16). On the other hand, from (4) we deduce that for a strictly increasing and continuously twice differentiable function, there is a sufficiently large r for which (11) and (13) be valid.*

Remark 5.2. *An inequality similar to (16) for power means was proved by A.J.Goldman (see [1], p.203). On the other hand we remark that (16) contains many inequalities between means. For example, for* $r > 1$ *it is equivalent with* $L_r(a,b) \geq P_r(a,b)$ *and for* $0 < r < 1$ *it gives* $L_r(a,b) \leq P_r(a,b)$. *For* $r < 0 < s$ *we get:*

$$E_{r,r+1}(a,b) \leq L(a,b) \leq E_{s,s+1}(a,b).$$

All these relations may be found in [1]. We also have:

$$L_{r,r+1}(a,b)L(a,b) \leq G^2(a,b), \text{ for } r < -1$$

but the converse inequality for $-1 < r < 0$.

Remark 5.3. *From* $0 \leq D_r(a,b) \leq 1$ *we deduce that it may be preferable to use instead* D_r *the differences* $D_r - 1/2$, *that is:*

$$\frac{L_r^r(a,b) - P_r^r(a,b)}{b^r - a^r} \text{ for } r \neq 0; \ \frac{A(a,b) - L(a,b)}{b-a}\ ,\ r = 0$$

where $A = A_{1/2}$. *These are between* $-1/2$ *and* $1/2$ *and are decreasing upon* r, *as* D_r *is.*

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A NEW IMPROVEMENT OF JENSEN'S INEQUALITY

S.S.Dragomir and Gh.TOADER

Refinements of Jensen's discrete inequality and applications for arithmetic mean - geometric mean inequality are given.

1 INTRODUCTION

Let X be a real linear space, C be a convex subset of X and $f : C \longrightarrow \mathbb{R}$ a convex mapping on C. The following inequality is known in literature as Jensen's inequality:

$$A_n(f(y); p) \geq f(A_n(y; p)) \tag{1}$$

where $y = (y_1, \ldots, y_n) \in C^n,\ p = (p_1, \ldots, p_n) \in R^n_+$ with $P_n = p_1, \ldots, p_n > 0, f(y) = (f(y_1, \ldots, f(y_n)) \in R^n$ and

$$A_n(y; p) = \sum_{i=1}^{n} p_i y_i / p_n.$$

In the paper [6], the first author has established the following refinement of (1):

Theorem 1.1. *Let f, p, y be as above and $q = (q_1, \ldots, q_k) \in R^k_+$ with $Q_k = q_1, \ldots, q_k > 0$ for $1 \leq k \leq n$. Then:*

$$A_n(f(y); p) \geq \sum_{i1,\ldots,i_k=1}^{n} p_{i_1} \ldots p_{i_k} f\Big(\sum_{j=1}^{k} q_j x_{i_j} / Q_k\Big) / P_n^k \geq f(A_n(y; p)).$$

For $q_k = 1 (1 \leq k \leq n)$ the result was proved in [9]. In this paper we will point out other refinement of (1) and also we will apply these results to improve some well-known inequalities (see also [1], [4-6] and [9].

2 THE MAIN RESULTS

Suppose that f and p are as above and let $t = (t_1, \ldots, t_n) \in [0, 1]^n$. We define the following weighted means:

$$A_n^{(3)}(f(x); p; t) := \sum_{i,j,k=1}^{n} p_i p_j p_k f(t_k x_i + (1 - t_k) x_j) / p_n^3$$

$$A_n^{(2)}(f(x);p;t) := \sum_{i,j,k=1}^{n} p_i p_j f(A_n(t;p)x_i + (1 - A_n(t;p)x_j)/p_n^2$$

and

$$A_n^{(1)}(f(x);p;t) := \sum_{i=1}^{n} p_i f(A_n(t;p)x_i + (1 - A_n(t;p)A_n(x;p))/p_n.$$

We prove the following theorem which improves Jensen's result (1).

Theorem 2.1. *Let f be a real convex mapping on the convex set C, $x \in C^n$, $p \in R_+^n$ with $P_n > 0$ and $t \in [0,1]^n$. Then we have the inequalities:*

$$\begin{aligned} A_n(f(x);p) \geq A_n^{(3)}(f(x);p;t) \geq A_n^{(2)}(f(x);p;t) \geq \\ \geq A_n^{(13)}(f(x);p;t) \geq f(A_n(x;p)). \end{aligned} \tag{2}$$

Proof. Since f is convex on C, hence for all $t_k \in [0,1]$ and $x_i, x_j \in C$ we have the inequality:

$$t_k f(x_i) + (1 - t_k) f \geq f(t_k x_i + (1 - t_k) x_j).$$

Multiplying with $p_i p_j p_k \geq 0$ and summing after i, j, k we derive:

$$\sum_{i,j,k=1}^{n} p_i p_j p_k (t_k f(x_i) + (1 - t_k) f(x_j)) \geq \sum_{i,j,k=1}^{n} p_i p_j p_k f(t_k x_i + (1 - t_k) x_j).$$

As:

$$\sum_{i,j,k=1}^{n} (t_k f(x_i) + (1 - t_k) f(x_j)) p_i p_j p_k = p_n^2 \sum_{i=1}^{n} p_i f(x_i),$$

the first inequality in (2) is proven.

For the second inequality we remark that:

$$A_n^{(2)}(f(x);p;t) := \sum_{i,j=1}^{n} p_i p_j f\Big(\sum_{k=1}^{n} p_k (t_k + (1 - t_k) x_j)/P_n\Big)/P_n^2.$$

By Jensen's inequality (1) for $y_k = t_k x_i + (1 - t_k) x_j$, we obtain:

$$\sum_{k=1}^{n} p_k f(t_k x_i + (1 - t_k) x_j/P_n \geq f\Big(\sum_{k=1}^{n} p_k (t_k x_i + (1 - t_k) x_j/P_n\Big).$$

Multiplying with $p_i p_j \geq 0$ and summing after i and j, we have:

$$A_n^{(3)}(f(x);p;t) \geq A_n^{(2)}(f(x);p;t).$$

Now, observe that, by Jensen's inequality for $y_j = A_n(t;p)x_i + (1 - A_n(t;p))x_j$, we have:

$$\begin{aligned} &\sum_{j=1}^{n} p_j f(A_n(t;p)x_i + (1 - A_n(t;p)x_j)/P_n \geq f\Big(\sum_{j=1}^{n} p_j f(A_n(t;p)x_i + \\ &+ (1 - A_n(t;p)x_j)/P_n\Big) = f(A_n(t;p)x_i + (1 - A_n(t;p)A_n(x;p)) \end{aligned}$$

which implies the third inequality from (5).

The last inequality is also obvious from Jensen's inequality (1) applied for $y_1 = A_n(t;p)x_i + (1 - A_n(t;p))A_n(t;p)$.

In the theory of inequalities, the famous inequality between the arithmetic mean and the geometric mean:

$$a_n(x) := \sum_{i=1}^{n} x_i/n \geq \prod_{i=1}^{n} x_i^{1/n} = g_n(x) \tag{3}$$

valid for every sequence x of positive real numbers, occupies a central place. Many authors have tried to establish (3) in a variety of ways and also to find different extensions, refinements and counterparts (see[7-8]). $\square$

Further on we shall consider the weighted arithmetic and geometric means of x with weights p:

$$A_n(x;p) = \sum_{i=1}^{n} p_i x_i/P_n, \ G_n(x;p) = (\prod_{i=1}^{n} x_i^{p_i})^{1/p_n} \quad where \quad p_i > 0 \ (i = 1, \ldots, n)$$

and we will point out a refinement of the following arithmetic-geometric inequality:

$$A_n(f(x);p) \geq G_n(f(x);p) \tag{4}$$

for a class of real functions f.

Corollary 2.1. *Let f be a strictly positive convex mapping on C which is also logarithmically concave on C (that is in f is concave on C).*

Then one has the inequalities:

$$\begin{aligned} &A_n(f(x);p) \geq A_n^{(3)}(f(x);p;t) \geq A_n^{(2)}(f(x);p;t) \geq \\ &A_n^{(1)}(f(x);p;t) \geq f(A_n(x;p)) \geq G_n^{(1)}(f(x);p;t) \geq \\ &G_n^{(2)}(f(x);p;t) \geq G_n^{(3)}(f(x);p;t) \geq G_n(f(x);p) \end{aligned} \tag{5}$$

where

$$G_n^{(1)}(f(x);p;t) := (\prod_{i=1}^{n}(f(A_n(t;p)x_i + (1 - A_n(t;p))A_n(x;p))^{p_i})^{1/P_n}$$

$$G_n^{(2)}(f(x);p;t) := (\prod_{i,j=1}^{n}(f(A_n(t;p)x_i + (1 - A_n(t;p))x_j(x;p))^{p_i p_j})^{1/P_n^2}$$

$$G_n^{(3)}(f(x);p;t) := (\prod_{i,j,k=1}^{n}(f(t_k x_i + (1 - t_k)x_j)^{p_i p_j p_k})^{1/P_n^3}$$

respectively, where $t \in [0,1]^n$.

The proof of the second part of (5) follows by the above theorem for the concave mapping in f.We will omit the details.

3 APPLICATIONS

1. Let x and p be as above. Then the following refinement of the arithmetic mean-geometric mean inequality is valid:

$$A_n(x;p) \geq G_n^{(1)}(x;p;t) \geq G_n^{(2)}(x;p;t) \geq G_n^{(3)}(x;p;t) \geq G_n(x;p)$$

for all $t \in [0,1]^n$.

2. If $x \in R^n$ and p is above, then for all $s \geq 1$ we have inequalities:

$$\begin{aligned} &A_n(|x|^s;p) \geq A_n^{(3)}(|x|^s;p) \geq A_n^{(2)}(|x|^s;p) \geq \\ &A_n^{(1)}(|x|^s;p) \geq |A_n(x;p)|^s \end{aligned}$$

for all $t \in [0,1]^n$.

3. In [11] C.L.wang has proved the following inequality:

$$A_n(x;p)/A_n(1-x;p) \geq G_n(x;p)/G_n(1-x;p)$$

where $x_i \in (0,1/2)(i=1,\ldots,n)$, which shows that Ky Fan's inequality (3,p.5):

$$a_n(x)/a_n(1-x) \geq g_n(x)/g_n(1-x),\ 0 < x_1 \leq 1/2$$

also holds for weighted means.

On the other hand, by Chebyshev's inequality (see [8]), it is clear that:

$$A_n(x/(1-x);p) \geq A_n(x;p)/A_n(1-x;p)$$

which shows that Wang's inequality may be regarded as a refiment of arithmetic mean-geometric mean inequality (4) for the mapping $f(x) := x/91-x)$ on the interval (0,1/2).

Now let us consider the mapping $f : (0,1/2] \longrightarrow (0,\infty)$ given by $f(x) = (x/(1-x))^r,\ r \geq 1$. It is clear that this mapping is convex and logarithmically concave, therefore we have the following is convex of inequalities:

$$\begin{aligned} &A_n((x/(1-x))^r;p) \geq A_n^{(3)}((x/(1-x))^r;p;t) \geq \\ &\geq A_n^{(2)}((x/(1-x))^r;p;t) \geq A_n^{(1)}((x/(1-x))^r;p;t) \geq \\ &\geq (A_n(x;p)/A_n(1-x;p))^r \geq (G_n^{(1)}(x;p;t)/G_n^{(1)}(1-x;p;t))^r \geq \\ &\geq (G_n^{(2)}(x;p;t)/G_n^{(2)}(1-x;p;t))^r \geq \\ &\geq G_n^{(3)}(x;p;t)/G_n^{(3)}(1-x;p;t))^r \geq (G_n(x;p)/G_n(1-x;p))^r \end{aligned}$$

for all t in $[0,1]^n$.

Remark 3.1. *The above inequalities contain refinements of Wang's inequality and thus of Ky Fan's result.*
For other improvements of this well-known result see [6].

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Revue d'Analyse Numérique et de Théorie de l'Approximation
Tome 21, no. 1, 1992, pp. 83-88

INTEGRAL AND DISCRETE INEQUALITIES

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1. Introduction

In [8] we have proved some integral inequalities showing that the inequalities are valid for a sequence of integral sums with norm tending to zero. In this paper, starting from some integral inequalities, we prove discrete versions.

To avoid complications related to the integrability, we suppose all the functions which appear in what follows to be continuous. The following results were considered in [8]:

Theorem A. *If the function $f : [a,b] \to \mathbb{R}$ is Jensen convex, $h : [a,b] \to \mathbb{R}$ is positive and symmetric with respect to $(a+b)/2$, then:*

$$f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)h(x)dx / \int_a^b h(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Theorem B. *If the function $f : [a, b] \to [c, d]$ is increasing and Jensen convex and $g, h : [c, d] \to [0, \infty)$ are such that g/h is increasing, then:*

$$(1) \qquad \int_a^b g(f(x))dx \Big/ \int_a^b h(f(x))dx \leq \int_{f(a)}^{f(b)} g(x)dx \Big/ \int_{f(a)}^{f(b)} h(x)dx.$$

Remark 1. The first theorem was proved by L. Fejér in [2] and for $h(x) = 1$ it gives the inequality of Hermite-Hadamard. The inequality (1) was proposed as a problem by A. Lupaş in [5] for $g(x) = x^2$ and $h(x) = x$. It was proved by L. Daia in [1] for $g(x) = x^r$ and $h(x) = x^s$ with $r > s$. In the form (1) it was given by I. Gavrea in [3] but under the assumption of differentiability of f. We have shown in [8] that the inequality is valid without this last condition.

The following result was given by J. Kolumban and C. Mocanu in [4].

Theorem C. *If the functions $f, g, h : [a, b] \to \mathbb{R}$ are positive, g is increasing and differentiable, $g(a) > 0$ and:*

$$\int_a^x f^q(t)h(t)dt \leq \int_a^x g^q(t)h(t)dt, \ \forall \ x \in [a, b]$$

then for $0 < p < q$:

$$\int_a^x f^p(t)h(t)dt \leq \int_a^x g^p(t)h(t)dt, \ \forall \ x \in [a, b].$$

2. Finite differences

For a sequence $(x_k)_{k=1}^n$, we consider the finite differences of order one:

$$\Delta_p^1 x_k = x_{k+p} - x_k, \quad 1 \leq k < k + p \leq n$$

and of order two:

$$\Delta^2_{pq} x_k = x_{k+p+q} - (1+q/p)x_{k+p} + (q/p)x_k, \quad 1 \le k < k+p < k+p+q \le n$$

We denote simply $\Delta^1_1 = \Delta^1$ and $\Delta^2_{11} = \Delta^2$.

A sequence $(x_k)_{k=1}^n$ is increasing if $\Delta^1 x_k \ge 0$ for $1 \le k \le n-1$, but this is equivalent with the condition:

$$\Delta^1_p x_k \ge 0, \text{ for } 1 \le k < k+p \le n$$

as:

$$\Delta^1_p x_k = \sum_{i=1}^{p} \Delta^1 x_{k+i-1}. \tag{2}$$

Analogously, the sequence $(x_k)_{k=1}^n$ is said to be convex if $\Delta^2 x_k \ge 0$ for $1 \le k \le n-2$ and this is equivalent with:

$$\Delta^2_{pq} x_k \ge 0, \text{ for } 1 \le k < k+p < k+p+q \le n$$

because we have:

Lemma 1. *For every k, p and q:*

$$\Delta^2_{pq} x_k = \sum_{i=1}^{q} i\Delta^2 x_{k+p+q-i-1} + (q/p)\sum_{j=1}^{p-1} j\Delta^2 x_{k+j-1} \tag{3}$$

hold.

Proof. We have:

$$\Delta^2_{pq} x_k = \Delta^1_q x_{k+p} - (q/p)\Delta^1_p x_k$$

and using (2):

$$\Delta^2_{pq} x_k = \sum_{i=1}^{q} \Delta^i x_{k+p+i-1} - (q/p)\sum_{j=1}^{p} \Delta^1 x_{k+j-1}. \tag{4}$$

Applying Abel's identities:

$$\sum_{i=1}^{q} y_i = \sum_{i=1}^{q-1} i\Delta^1 y_{q-i} + qy_1$$

respectively

$$\sum_{j=1}^{p} z_j = pz_p - \sum_{j=1}^{p-1} j\Delta^1 z_j$$

for the two sums of (8), we get:

$$\Delta^2_{pq} x_k = \sum_{i=1}^{q-1} i\Delta^2 x_{k+p+q-i-1} + q\Delta^1 x_{k+p}$$

$$-(q/p)\left(p\Delta^1 x_{k+p-1} - \sum_{j=1}^{p-1} j\Delta^2 x_{k+j-1}\right)$$

thus (3).

Remark 2. Relation (3) is similar with that given by T. Popoviciu in [7] for divided differences.

3. Discrete inequalities

We begin with a discrete version of Fejer's inequality. We say that the sequence $(p_i)_{i=1}^n$ is symmetric if:

$$p_i = p_{n-i+1} \text{ for } 1 \leq i \leq n.$$

Theorem 1. *If the sequence $(x_i)_{i=1}^n$ is convex and $(p_i)_{i=1}^n$ is symmetric and positive, then:*

$$(5) \qquad (x_{[(n+1)/2]} + x_{[(n+2)/2]})/2 \leq \sum_{i=1}^{b} x_i p_i / \sum_{i=1}^{n} p_i \leq (x_1 + x_n)/2$$

where $[a]$ denotes the integer part of a.

Proof. As $\Delta^2_{i-1,n-i}x_1 \geq 0$, we have:

$$(n-1)x_i \leq (i-1)x_n + (n-i)x_1.$$

Putting $n-i+1$ instead i we get:

$$(n-1)x_{n-i+1} \leq (n-i)x_n + (i-1)x_1$$

and by addition:

$$x_i + x_{n-i+1} \leq x_1 + x_n.$$

Multiplying by $p_i = p_{n-i+1}$ and adding for $i = 1, \ldots, n$ we get the second part of (5). For the first part we consider separately the case of n odd or n even. So, if $n = 2m+1$, as $\Delta^2_{m-i+1,m-i+1}x_i \geq 0$, we have:

$$2x_{m+1} \leq x_i + x_{2m+2-t}.$$

Multiplying by $p_i = p_{2m+2-i}$ and adding for $i = 1, \ldots, 2m+1$ we get:

$$\sum_{i=1}^{2m+1} x_i p_i / \sum_{i=1}^{2m+1} p_i \geq x_{m+1} = (x_{[(2m+2)/2]} + x_{[(2m+3)/2]})/2.$$

For $n = 2m$, we have:

$$(m-i)\Delta^2_{m-i,m-i+1}x_i + (m-i+1)\Delta^2_{m-i+1,m-i}x_i \geq 0$$

hence

$$x_m + x_{m+1} \leq x_i + x_{2m-i+1}.$$

Multiplying by $p_i = p_{2m-i+1}$ and adding for $i = 1, \ldots, 2m$, we obtain:

$$\sum_{i=1}^{2m} x_i p_i / \sum_{i=1}^{2m} p_i \geq (x_m + x_{m+1})/2 = (x_{[(2m+1)/2]} + x_{[(2m+2)/2]})/2.$$

Remark 3. For $p_i = 1$ $(i = 1, \ldots, n)$ we get a discrete variant of Hermite-Hamard inequality. On the other hand, inequality (5) can be used for the proof of Fejer's integral inequality.

Passing to theorem B, we can see that inequality (1) holds if and only if for every natural n, denoting:

$$x_i = a + (i-1)(b-a)/n, \quad i = 1, \ldots, n+1$$

we have the inequality:

(6)
$$\sum_{i=1}^{n} g(f(x_i)) / \sum_{i=1}^{n} h(f(x_i)) \leq \sum_{i=1}^{n} g(f(x_i))\Delta^1 f(x_i) / \sum_{i=1}^{n} h(f(x_i))\Delta^1 f(x_i)$$

But we can prove a much stronger result which generalizes also Cauchy's inequality and Chebyshev's inequality (see [6]). We say that the sequences $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ are synchrone if:

$$(a_i - a_j)(b_i - b_j) \geq 0, \quad i \leq i, j \leq n.$$

Theorem 2. *If the sequences* $(y_i)_{i=1}^n$ *and* $(q_i)_{i=1}^n$ *are strictly positive and* $(x_i/y_i)_{i=1}^n$ *and* $(p_i/q_i)_{i=1}^n$ *the synchrone, then:*

$$\sum_{i=1}^{n} x_i p_i \sum_{i=1}^{n} y_i q_i \geq \sum_{i=1}^{n} x_j q_i \sum_{i=1}^{n} y_i p_i. \tag{7}$$

Proof. As:

$$(x_i/y_i - x_j/y_j)(p_i/q_i - p_j/q_j) \geq 0$$

we have:

$$x_i p_i y_j q_j - x_j q_j y_i p_i - x_i q_i y_j p_j + x_j p_j y_i q_i \geq 0$$

and adding consecutively for $i = 1, \ldots, n$ and then for $j = 1, \ldots, n$ we get (7).

Remark 4. This is a discrete variant of an integral inequality of M. Fujiwara (see [6]). For $p_i = x_i$ and $q_i = y_i$, $i = 1, \ldots, n$, we have Cauchy's inequality and for $y_i = q_i = 1$, $i = 1, \ldots, n$, we have Chebyshev's inequality.

If the sequence $(p_i)_{i=1}^{n+1}$ is convex, then the sequence $(\Delta^1 p_i)_{i=1}^{n}$ is increasing and taking $q_i = 1$ for $i = 1, \ldots, n$, we have the following result which also implies (6):

Consequence. If the sequence $(y_i)_{i=1}^{n}$ is strictly positive, $(x_i/y_i)_{i=1}^{n}$ is increasing and $(p_i)_{i=1}^{n+1}$ convex, then:

$$\sum_{i=1}^{n} x_i \Delta^1 p_i \sum_{i=1}^{n} y_i \geq \sum_{i=1}^{n} y_i \Delta^1 p_i \sum_{i=1}^{n} x_i.$$

To prove a discrete version of theorem C we need the following:

Lemma 2. *If the sequence $(b_i)_{i=1}^{n}$ is positive and decreasing, then:*

$$\sum_{i=1}^{k} a_i \geq m, \ \forall \ k \leq n$$

implies:

$$\sum_{i=1}^{n} a_i b_i \geq m b_1.$$

Proof. Using Abel's identity, we have:

$$\sum_{i=1}^{n} a_i b_i = \sum_{k=1}^{n-1} \left(\sum_{i=1}^{k} a_i \right) (b_k - b_{k+1}) + \sum_{i=1}^{n} a_i b_n$$

$$\geq m \left(\sum_{k=1}^{n-1} (b_k - b_{k+1}) + b_n \right) = m b_1.$$

Theorem 3. *If the sequences $(x_i)_{i=1}^n$ and $(z_i)_{i=1}^n$ are positive and $(y_i)_{i=1}^n$ is strictly positive and increasing, then:*

$$\sum_{i=1}^k x_i^q z_i \le \sum_{i=1}^k y_i^q z_i,\ \forall\ k = 1, \dots, n$$

implies:

$$\sum_{i=1}^k x_i^p z_i \le \sum_{i=1}^k y_i^p z_i,\ \forall\ k = 1, \dots, n$$

for $0 < p < q$.

Proof. We use Hölder's inequality:

$$\sum_{k=1}^n a_k b_k \le \left(\sum_{k=1}^n a_k^r\right)^{1/r} \left(\sum_{k=1}^n b_k k^s\right)^{1/s}, \quad r > 1,\ 1/r + 1/s = 1$$

for $r = q/p$ and $s = q/(q-p)$. So:

$$\left(\sum_{i=1}^k x_i^p z_i\right)^q = \left(\sum_{i=1}^k (x_i^p z_i^1/r/y_i^{p/s}) y_i^{p/s} z_i^{1/s}\right)^q$$

$$\le \left(\sum_{i=1}^k x_i^{rp} z_i / y_i^{pr/s}\right)^{q/r} \left(\sum_{i=1}^k y_i^p z_i\right)^{q/s} = \left(\sum_{i=1}^k x_i^q z_i / y_i^{q-p}\right)^p \left(\sum_{i=1}^k y_i^p z_i\right)^{q-p}$$

$$= \left(\sum_{i=1}^k y_i^p z_i - \sum_{i=1}^k z_i (y_i^q - x_i^q)/y_i^{q-p}\right)^p \left(\sum_{i=1}^k y_i^p z_i\right)^{q-p} \le \left(\sum_{i=1}^k y_i^p z_i\right)^q$$

because, by hypothesis

$$\sum_{i=1}^k (y_i^q - x_i^q) z_i \ge 0,\ \forall\ k$$

and $(1/y_i^{q-p})_{i=1}^n$ is decreasing, hence, by Lemma 2:

$$\sum_{i=1}^k (y_i^q - x_i^q) z_i / y_i^{q-p} \ge 0.$$

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Facta Univ. Nis **7**(1992), 43-48

FUJIWARA'S INEQUALITY FOR FUNCTIONALS

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Abstract

Fujiwara's inequality generalizes both Cauchy's and Chebyshev's inequalities. Here we give its variant for linear isotonic functionals. Also we prove the monotonicity and the superadditivity of an operator related to the inequality, obtaining so some refinements of it.

1 Introduction

>From [4] we can deduce an inequality proved by M.Fujiwara in [1]: If f_1/f_2 and g_1/g_2 are monotone in the same sense on (a,b) then

$$A(f_1g_1)A(f_2g_2) \geq A(f_1g_2)A(f_2g_1)$$

where

$$A(f) = \int_a^b f(x)dx.$$

Taking $f_2 = g_2 = e$, where $e(x) = 1$ for $x \in [a,b]$, we get Chebyshev's inequality, while for $g_1 = f_1$ and $g_2 = f_2$ we have Cauchy's inequality.

In what follows we want to generalize Fujiwara's inequality for arbitrary linear positive functionals A. We also prove in this general case some properties of monotonicity and of superadditivity of the operator D' defined by

$$D'(A)(f_1,f_2,g_1,g_2) = A(f_1g_1)A(f_2g_2) - A(f_1g_2)A(f_2g_1)$$

which are known in some special cases (see for example [3] and [7]).

2 Functionals

Let E be an arbitrary set, $F(E)$ be the set of real-valued functions defined on E and denote also $F_+(E)$ the subset of positive functions from $F(E)$.

Usually one considers linear positive functionals on $F(E)$ but we take the apparently weaker conditions of sublinearity and isotony. So we consider the set of functionals

$$M_+(E) = \{A : F(E) \longrightarrow \mathbb{R}; A(tf + sg) \leq tA(f) + sA(g),$$
$$\forall t, s \in \mathbb{R}, \forall f, g \in F(E) \quad and \quad A(f) \geq 0, \forall f \in F_+(E)\}.$$

As ordinary examples of such functionals we can consider

$$A(f) = \sum_{i=1}^{n} p_i f(x_i), \quad with \quad p > 0 \quad and \quad x_i \in E \tag{1}$$

and

$$A(f) = \int_a^b p(x)f(x)dx, \quad with \quad p \geq 0 \quad on \quad [a, b] \tag{2}$$

but also others deduced by Minkowski's inequality or by Mulholand's inequality (see [4]).

We shall use in what follows an order relation on $M_+(E)$ defined by

$$A \geq B \quad if \quad A(f) \geq B(f),\ \forall f \in F_+(E).$$

Lemma 2.1. *If $A \geq B$ and $f, g \in F_+(E)$, $f \geq g$, then $A(f) \geq B(g)$.*

Proof. As $f - g \in F_+(E)$ we get

$$0 \leq A(f - g) \leq A(f) - A(g)$$

and from $A \geq B$ and $g \geq 0$ we have also $A(g) \geq B(g)$. □

3 Fujiwara's Inequality

The condition from Chebyshev's inequality that the functions f and g are monotone in the same sense can be weakned by the following definition: The functions $f, g \in F(E)$ are synchrone if

$$(f(x) - f(y))(g(x) - g(y)) \geq 0, \quad \forall x, y \in E.$$

By analogy, we shall say that the pairs of functions (f_1, f_2) and (g_1, g_2) are synchrone if

$$\begin{vmatrix} f_1(x) & f_1(y) \\ f_2(x) & f_2(y) \end{vmatrix} \cdot \begin{vmatrix} g_1(x) & g_1(y) \\ g_2(x) & g_2(y) \end{vmatrix} \geq 0, \quad \forall x, y \in E. \tag{3}$$

Theorem 3.1. *If the pairs of functions (f_1, f_2) and (g_1, g_2) are synchrone and $A, B \in M_+(E)$ then*

$$A(f_1g_1)B(f_2g_2) + A(f_2g_2)B(f_1g_1) \geq A(f_1g_2)B(f_2g_1) + A(f_2g_1)B(f_1g_2). \tag{4}$$

Proof. The inequality (3) gives

$$\begin{aligned} & f_1(x)g_1(x)f_2(y)g_2(y) + f_2(x)g_2(x)f_1(y)g_1(y) \\ & - f_1(x)g_2(x)f_2(y)g_1(y) - f_2(x)g_1(x)f_1(y)g_2(y) \geq 0. \end{aligned} \tag{5}$$

Considering that A maps the functions of variable x and then B those of variable y, we get (4). □

Consequence 3.1. *If the pairs of functions* (f_1, f_2) *and* (g_1, g_2) *are synchrone and* $A \in M_+(E)$ *then*

$$A(f_1g_1)A(f_2g_2) \geq A(f_1g_2)A(f_2g_1). \tag{6}$$

Consequence 3.2. *(Fujiwara's inequality). If* f_1/f_2 *and* g_1/g_2 *are monotone in the same sense,* $f_2, g_2 > 0$ *and* $A \in M_+(E)$ *then (6) holds.*

Consequence 3.3. *(Cauchy's inequality). If* $A \in M_+(E)$ *then*

$$A(f^2)A(g^2) \geq (A(fg))^2, \ \forall f, g \in F(E).$$

Consequence 3.4. *(Chebyshev's inequality). If the functions* $f, g \in F(E)$ *are synchrone and* $A \in M_+(E)$*, then*

$$A(fg)A(e) \geq A(f)A(g).$$

Proof. To get these consequences we need to consider in (4) the special case $A = B$. Moreover $f_1 = g_1 = f$ and $f_2 = g_2 = g$ for Consequence 3.3 respectively $f_1 = f, g_1 = g$ and $f_2 = g_2 = e$ for Consequence 3.4. The hyphothesis of Consequence 3.2 assure the synchronism of the pairs (f_1, f_2) and (g_1, g_2).

Usually the consequences are stated for A given by (1) or (2). □

Consequence 3.5. *If the differentiable function* f *is convex and* g/h *is increasing then*

$$A(f'g)A(h) \geq A(f'h)A(g), \ \forall A \in M_+(E). \tag{7}$$

Consequence 3.6. *If the differentiable function* $f : [a, b] \longrightarrow \mathbb{R}$ *is convex and increasing and the functions* g *and* h *are defined on* $[f(a), f(b)]$ *and* g/h *is increasing, then*

$$\int_a^b g(f(x))dx \int_{f(a)}^{f(b)} h(x)dx \geq \int_a^b h(f(x))dx \int_{f(a)}^{f(b)} g(x)dx. \tag{8}$$

Proof. The inequality (7) follows from (6) as f'/e is increasing. Then (8) may be obtained by taking in (7)

$$A(g) = \int_{f(a)}^{f(b)} g(x)dx.$$

□

As we have proved in [8], the inequality (8) is valid without the differentiability of f. There we give also its history.

4 The Operator D

To formulate and to prove the following results we introduce the operators D and $D^{'}$ with expressions deduced from (4) respectively (6).

$$\begin{aligned} D(A,B)(f_1,f_2,g_1,g_2) &= A(f_1g_1)B(f_2g_2) + A(f_2g_2)B(f_1g_1) \\ &- A(f_1g_2)B(f_2g_1) - A(f_2g_1)B(f_1g_2) \end{aligned}$$

and

$$D^{'}(A)(f_1,f_2,g_1,g_2) = (1/2)D(A,A)(f_1,f_2,g_1,g_2).$$

Theorem 4.1. *If the functionals $A, A^{'}, B, B^{'} \in M_+(E)$ are such that $A \geq A^{'}$ and $B \geq B^{'}$ then*

$$D(A,B)(f_1,f_2,g_1,g_2) \geq D(A^{'},B^{'})(f_1,f_2,g_1,g_2) \tag{9}$$

for every synchrone pairs of functions (f_1,f_2) and (g_1,g_2).

Proof. >From (5) and $A \geq A^{'}$ we have

$$\begin{aligned} & A(f_1g_1)f_2(y)g_2(y) + A(f_2g_2)f_1(y)g_1(y) - A(f_1g_2)f_2(y)g_1(y) \\ & - A(f_2g_1)f_1(y)g_2(y) = A^{'}(f_1g_1)f_2(y)g_2(y) + A^{'}(f_2g_2)f_1(y)g_1(y) \\ & - A^{'}(f_1g_2)f_2(y)g_1(y) - A^{'}(f_2g_1)f_1(y)g_2(y) \geq 0 \end{aligned}$$

and then Lemma 1 applied for $B = B^{'}$ gives (9). $\square$

Consequence 4.1. *If the functionals $A, A^{'} \in M_+(E)$ are comparable, $A \geq A^{'}$, then*

$$D^{'}(A)(f_1,f_2,g_1,g_2) \geq D^{'}(A^{'})(f_1,f_2,g_1,g_2) \tag{10}$$

for every synchrone pairs of functions (f_1,f_2) and (g_1,g_2).

Example 1. If for fixed $p_i \geq 0$ and $x_i \in E, i = 1,\dots,n$, we denote

$$A_n(f) = \sum_{i=1}^{n} p_i f(x_i)$$

we have

$$A_k \geq A_{k-1} \quad for \quad k = 2,\dots,n$$

thus, by (10)

$$\begin{aligned} D^{'}(A_n)(f_1,f_2,g_1,g_2) &\geq D^{'}(A_{n-1})(f_1,f_2,g_1,g_2) \\ &\geq \cdots \geq D^{'}(A_2)(f_1,f_2,g_1,g_2) \geq 0 \end{aligned}$$

for the synchrone pairs (f_1,f_2) and (g_1,g_2).

This result is known in some special cases as one can see in [7].

Theorem 4.2. *If* $A, A^{'}, B, B^{'} \in M_{+}(E)$ *then*

$$\begin{aligned} D(A+A^{'},B+B^{'})(f_1,f_2,g_1,g_2) \geq D(A,B)(f_1,f_2,g_1,g_2) \\ + D(A^{'},B^{'})(f_1,f_2,g_1,g_2) \end{aligned} \tag{11}$$

for every synchrone pairs of functions (f_1, f_2) *and* (g_1, g_2)*.*

Proof. As

$$\begin{aligned} D(A+A^{'},&B+B^{'})(f_1,f_2,g_1,g_2) - D(A,B)(f_1,f_2,g_1,g_2) - D(A^{'},B^{'})(f_1,f_2,g_1,g_2) \\ &= D(A,B^{'})(f_1,f_2,g_1,g_2) + D(A^{'},B)(f_1,f_2,g_1,g_2) \end{aligned}$$

the relation (11) follows from (4). □

Consequence 4.2. *If* $A, A^{'} \in M_{+}(E)$ *then*

$$D^{'}(A+A^{'})(f_1,f_2,g_1,g_2) \geq D^{'}(A)(f_1,f_2,g_1,g_2) + D^{'}(A^{'})(f_1,f_2,g_1,g_2) \tag{12}$$

for every synchrone pairs of functions (f_1, f_2) *and* (g_1, g_2)*.*

Example 2. For a finite index set I and fixed $p_i > 0$ and $x_i \in E$ for $i \in I$, we denote

$$A_I(f) = \sum i \in I p_i f(x_i).$$

From (12) we deduce that if I and J are disjoint then

$$D^{'}(A_{I\cup J})(f_1,f_2,g_1,g_2) \geq D^{'}(A_I)(f_1,f_2,g_1,g_2) + D^{'}(A_J)(f_1,f_2,g_1,g_2)$$

for (f_1, f_2) and (g_1, g_2) synchrone pairs. This result is also known in some special cases (see [3], [7]).

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ON CHEBYSHEV'S INEQUALITY FOR FUNCTIONALS

Gh.TOADER

1 Introduction

Let E be an arbitrary set and $F(E)$ be the set of real-valued functions defined on E. Let also $L_+(E)$ be the set of all isotonic linear functionals defined on $F(E)$, thus of all the functionals $A : F(E) \longrightarrow \mathbb{R}$ with the properties:

$$A(tf + sg) = tA(f) + sA(g),\ \forall t, s \in \mathbb{R},\ \forall f, g \in F(E)$$

and

$$A(f) \geq 0,\ \forall f \in F(E),\ f \geq 0.$$

Common examples of elements of $L_+(E)$ are:

$$A(f) = \sum_{i \in I} p_i f(x_i),\ p_i \geq 0, x_i \in E, i \in I \tag{1}$$

and

$$A(f) = \int_a^b p(x)f(x)dx,\ p(x) \geq 0,\ x \in [a, b] = E. \tag{2}$$

Two functions $f, g \in F(E)$ are said to be synchrone if:

$$(f(x) - f(y))(g(x) - g(y)) \geq 0,\ \forall x, y \in E. \tag{3}$$

The following result is known as Chebyshev's inequaltiy.

Theorem 1.1. *If A is an isotonic linear functional, the functions f, g are syncrone and p is positive, then:*

$$A(pfg)A(p) \geq A(pf)A(pg). \tag{4}$$

Proof. Multiplying (3) by $p(x)p(y) \geq 0$, we have:

$$p(x)f(x)g(x)p(y) + p(x)p(y)f(y)g(y) - p(x)f(x)p(y)g(y)-$$
$$- p(x)g(x)p(y)f(y) \geq 0.$$

Applying A to the functions of variable x and then again to those of variable y, we get (4).

The classical inequality was given by taking A of the form (1) or (2) and the functions f and g both increasing.In what follows we obtain similar results for starshaped or convex functions. We improve so some results from [1], [3], [4] and [5]. $\square$

2 Chebyshev's inequality for starshaped functions

Let E be a set of positive numbers and the functions $e_k \in F(E)$ be defined by:

$$e_k(x) = x^k,\ x \in E,\ k = 0, 1, 2, \ldots$$

A function $f \in F(E)$ is said to be starshaped if f/e_1 is increasing (see [2]). If $0 \in E$ we must have, of course, $f(0) = 0$.

In [5] it is proved that if A from $L_+(E)$ and f and g are starshaped, then:

$$A(fg)A(e_2) \geq A(e_1 f)A(e_1 g).$$

We want to put this inequality in another form, more similar to (4).

Theorem 2.1. *If $A \in L_+(E), p$ is positive and f and g are starshaped then:*

$$A(pfg)A^2(pe_1) \geq A(pe_2)A(pf)A(pg). \tag{5}$$

Proof. As f/e_2 and g/e_1 are increasing, (4) with the weight function pe_2 gives:

$$A(pfg)A(pe_2) \geq A(pe_1 f)A(pe_1 g).$$

Again (4) with the weight function pe_1 and the increasing functions f/e_1 and e_1 gives:

$$A(pfe_1)A(pe_1) \geq A(pf)A(pe_2)$$

and so we get (5). $\square$

Remark 2.1. *>From (4) we have also:*

$$A(pe_2)A(p) \geq A^2(pe_1)$$

so that (5) is stronger then (4) for starshaped functions.

3 Dunkel's and Anderson's inequalities

In [3], O.Dunkel passed in Chebyshev's inequality at more functions.

We can obtain it easily from (4) by mathematical induction.

Theorem 3.1. *If the functions* $f_1, \dots, f_m \in F(E)$ *are increasing,* p *is positive and* $A \in L_+(E)$*, then:*

$$A(pf_1 \dots f_m)A^{m-1}(p) \geq A(pf_1) \dots A(pf_m). \tag{6}$$

>From it we obtain also the following generalization for starshaped functions:

Theorem 3.2. *If the function* p *is positive,* $f_1, \dots, f_m \in F(E)$ *are starshaped and* A *is an isotonic linear functional, then:*

$$A(pf_1 \dots f_m)A^m(pe_1) \geq A(pe_m)A(pf_1) \dots A(pf_m) \tag{7}$$

holds.

Proof. Taking (6) the weight function pe_m and the increasing functions $f_1/e_1, \dots, f_m/e_1$ we have:

$$A(pf_1 \dots f_m)A^{m-1}(pe_m) \geq A(pe_{m-1}f_1) \dots A(pe_{m-1}f_m). \tag{8}$$

Applying (4) to the weight function pe_1 and the increasing functions f_i/e_1 and e_{m-1} we get:

$$A(pe_{m-1}f_i)A(pe-1) \geq A(pe_m)A(pf_i).$$

so that we obtain (7). □

Remark 3.1. *As was proved in [2], every convex function is starshaped. So, the inequality (7) is valid also for convex functions. Thus the theorem 3 generalize a result from [4] which contains Anderson's inequality from [1].*

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A HIERARCHY OF CONVEXITY OF HIGHER ORDER OF FUNCTIONS

Gh.TOADER

For the beginning, let us consider the following classes of continuous functions:

$$\begin{aligned}
K_2[a,b] =& \{f \in C[a,b] : [x,y,z;f] \geq 0,\ \forall x,y,z \in [a,b]\}\\
S_2^*[a,b;c] =& \{f \in C[a,b] : [c,x,y;f] \geq 0, \forall x,y \in [a,b]\}\\
S_2[a,b;c] =& \{f \in C[a,b] : (f(x+y-c) - f(x) - f(y) +\\
& f(c))(x-c)(y-c) \geq 0,\ \forall x,y \in [a,b]\}\\
J_2^*[a,b;c] =& \{f \in C[a,b] : [c,(x+c)/2,x;f] \geq 0,\ \forall x \in [a,b]\}
\end{aligned}$$

that is of convex, starshaped, superadditive respectively J-starshaped (Jensen starshaped) of order two relative to the point $c \in [a,b]$. In the definition are used divided differences which are given recurrently by:

$$[x_0;f] = f(x_0),\ [x_0,x_1,\ldots,x_n,x_{n+1};f] =$$
$$= ([x_1,\ldots,x_{n+1};f] - [x_0,\ldots,x_n;f])/(x_{n+1}-x_0).$$

Lemma 0.1. *For any $c \in [a,b]$ there hold the inclusions:*

$$K_2[a,b] \subset S_2^*[a,b;c] \subset S_2[a,b;c] \subset J_2^*[a,b;c].$$

Proof. The first inclusion is obvious. For the second inclusion we use the relation (see [10]):

$$f(x+y-c) - f(x) - f(y) + f(c))/(x-c)(y-c) =$$
$$[c,x,x+y-c;f] + [c,y,x+y-c;f].$$

The last inclusion follows from the remark that:

$$[c,(x+c)/2,x;f] = 2(f(x) - 2f((x+c)/2) + f(c))/(x-c)^2 =$$
$$\frac{1}{2}\Big(f\Big(2\frac{x+c}{2} - c\Big) - 2f\Big(\frac{x+c}{2} + f(c)\Big)\Big/\Big(\frac{x+c}{2} - c\Big)^2.$$

Thus it can be obtained if we take for x and y the value $(x+c)/2$ in the definition of superadditive functions. $\square$

Remark 0.1. *For $a = c = 0$ the first two inclusions were proved in [2]. To our knowlege, the consideration of the J-starshaped functions and thus the last inclusion is new. In [2] there is also studied (for $a = c = 0,\ f(0) = 0$) the problem of preservation of the properties of functions by the arithmetic integral mean:*

$$A(f)(x) = \frac{1}{x}\int_0^x f(t)dt,\ A(f)(0) = 0. \tag{1}$$

If we denote by MF the set of functions f with the property that $A(f)$ belongs to the class F, the result of [2] is:

$$K_2[0,b] \subset MK_2[0,b] \subset S_2^*[0,b;0] \subset MS_2^*[0,b;0] \subset$$
$$\subset MS_2[0,b;0]$$

which is called hierarchy of convexity. Starting from [6], in [9] we have extended this result considering transformations more general than that given by (1). In what follows we want to do same thing for the convexity of higher order.

To avoid some complications, we consider the case $a = c = 0$. Thus, let us denote by:

$$C(b) = \{f : [0,b] \longrightarrow \mathbb{R},\ f(0) = 0,\ f \quad continue\}$$
$$K_n(b) = \{f \in C(b) : [x_0,\dots,x_n;f] \geq 0,\ \forall x_0,\dots,x_n \quad distinct \quad in \quad [0,b]\}$$
$$S_n^*(b) = \{f \in C(b) : [0,x_0,\dots,x_n;f] \geq 0,\ \forall x_1,\dots,x_n \quad distinct \quad in \quad [0,b]\}$$
$$S_n(b) = \{f \in C(b) : \forall x_1,\dots,x_n \in (0,b],$$
$$\sum_{k=1}^{n}(-1)^{n-k}\sum_{(k)} f(x_{i_1} + \dots + x_{i_k}) \geq 0\}$$
$$J_n^*(b) = \{f \in C(b) : \left[0, \frac{x}{n}, \frac{2x}{n}, \dots, x; f\right] \geq 0,\ \forall x \in (0,b]\},$$

the sets of continuous, convex, starshaped, superadditive respectively J-starshaped of order n-functions. By $\sum\limits_{(k)} f(x_{i_1} + \dots + x_{i_k})$ we denote the sum over all the combinations of class k of $x_1,\dots,x_n$.

Lemma 0.2. *For every $n \geq 2$ and every $b \geq 0$ the inclusions hold:*

$$K_n(b) \subset S_n^*(b) \subset S_n(b) \subset J_n^*(b) > \tag{2}$$

Proof. The first inclusion follows by the definitions. The second was proved in a weaker form by T.Popoviciu in [8]. For $n = 3$ it was proved in [4] and for $n \leq 4$ in [10]. For an arbitrary n it was proved in [4] and for $n \leq 4$ in [10]. For an arbitrary n it was proved in the unpublished dissertation of I.B.Lacković and restarted without proof in [7]. A proof will appear also in [3]. The last inclusion may be obtained from the condition:

$$\sum_{k=1}^{n}(-1)^{n-k}\sum_{(k)} f(x_{i_1} + \dots + x_{i_n}) \geq 0$$

which for $x_1 = \cdots = x_n = x/n$ becomes:

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}f\Big(\frac{kx}{n}\Big) = \Big(\frac{n}{x}\Big)^n\frac{1}{n!}[0, x/n, \ldots, x; f] \geq 0.$$

$\square$

Remark 0.2. *Like in [9], we want to find the differentiable functions* $g : [0, b] \longrightarrow \mathbb{R},\ g(0) = 0$ *such that the weighted arithmetic integral mean:*

$$W_g(f)(x) = \frac{1}{g(x)}\int_0^x g^{'}(t)f(t)dt \tag{3}$$

preserves the classes of functions defined above.

Theorem 0.1. *If $W_g(f) \in K_n(b)$ $(S_n^*(b) \quad or \quad S_n(b))$ for any $f \in K_n(b)$ $(S_n^*(b) \quad respectively \quad S_n(b))$ then there is a $u > 0$ and a real c such that:*

$$g(x) = c \cdot x^u. \tag{4}$$

Proof. Let us denote by P_n the set of all polynomials of degree at most n. As $\pm p$ belongs to $K_n(b)$ $(S_n^*(b) \quad or \quad S_n(b))$ for any $p \in P_{n-1}$, it follows by (2) that $W_g(\pm p) = \pm W_g(p) \in S_n(b)$. Thus, by the functional characterization of polynomials given by M.Fréchet (see [1]), $W_g(p) \in P_{n-1}$.

Let us write $e_k(x) = x^k$ and $W_g(e_k) = p_k$. From (3) we deduce:

$$\frac{g^{'}(x)}{g(x)} = \frac{p_k^{'}(x)}{e_k(x) - p_k(x)},\ k = 1, \ldots, n-1 \tag{5}$$

or, if we consider:

$$p_k(x) = \sum_{j=0}^{n-1} a_{kj}x^j,$$

we have for $1 \leq k < m \leq n-1$:

$$\frac{\sum\limits_{j=1}^{n-1} ja_{ki}x^{j-1}}{x^k - \sum\limits_{j=0}^{n-1} a_{kj}x^j}\frac{\sum\limits_{j=1}^{n-1} ja_{mj}x^{j-1}}{x^m - \sum\limits_{j=0}^{n-1} a_{mj}x^j}.$$

Thus:

$$\Big(x^m - \sum_{j=0}^{n-1} a_{mj}x^j\Big)\sum_{j=1}^{n-1} ja_{ki}x^{j-1}\Big(x^k - \sum_{j=0}^{n-1} a_{kj}x^j\Big)\sum_{j=1}^{n-1} ja_{mj}x^{j-1}.$$

For $m = n-1$ equalizing the coefficients of x^{2n-3} we get:

$$a_{k,n-1} = 0 \quad for \quad k < n-1.$$

Then, for $m = n-2$ and the power $2n-5$, we deduce also:

$$a_{k,n-2} = 0 \quad for \quad k < n-2.$$

and generally, step by step:

$$a_{kj} = 0 \quad for \quad k < j.$$

Thus:

$$p_1(x) = a_{10} + a_{11}x$$

and from (5), with $k = 1$, we have:

$$\frac{g^{'}(x)}{g(x)} = \frac{a_{11}}{x - (a_{10} + a_{11}x)}$$

hence, as $g(0) = 0$, we get (4). $\square$

Remark 0.3. *As concerns the class $J_n^*(b)$, the condition $\pm W_g(p) \in J_n^*(b)$ leads to a functional equation in a single variable which may have non-polynomial solutions without auxiliary conditions (see [2]). However, for $n = 2$ we get Schröder's equation:*
$h(2x) = 2h(x)$ which has the unique continuously differentiable solution $h(x) = cx$, that is we get again (4).

Remark 0.4. *For $g(x) = c \cdot x^u$ the transformation (3) becomes a Cesáro type operator:*

$$A_u(f)(x) = \frac{u}{x^u} \int_0^x t^{u-1} f(t) dt. \tag{6}$$

We consider the sets $M^uK_n(b)$, $M^uS_n^*(b), M^uS_n(b)$ and $M^uJ_n^*(b)$ of functions f with the property that $A_u(f)$ belongs to $K_n(b), S_n^*(b), S_n(b)$, respectively $J_n^*(b)$. The following results prove that the condition (4) is also sufficient for the preservation of the above classes of functions.

Theorem 0.2. *For every $b, u > 0$ and every $n \geq 2$, the inclusions hold:*

(a) $K_n(b) \subset M^uK_n(b)$

(b) $S_n^*(b) \subset M^uS_n^*(b)$

(c) $S_n(b) \subset M^uS_n(b)$

(d) $J_n^*(b) \subset M^uJ_n^*(b)$.

Proof. Making (as in [6] the substitution:$t = x \cdot s^{1/u}$, (6) becomes:

$$A_u(f)(x) = \int_0^1 f(xs^{1/s}) ds.$$

As:

$$[x_0, x_1, \ldots, x_n; f] = \sum_{k=0}^{n} \frac{f(x_k)}{p'(x_k)}$$

where

$$p(x) = (x - x_0)(x - x_1) \ldots (x - x_n)$$

we have:

$$[x_0, x_1, \ldots, x_n; A_u(f)] = \sum_{k=0}^{n} \frac{1}{p'(x_k)} \int_0^1 f(x_k s^{1/u}) ds =$$
$$= \int_0^1 s^{n/u} [x_0 s^{1/u}, \ldots, x_n s^{1/u}; f] ds$$

which proves the inclusions (a),(b) and (d). The inclusion (c) follows from the relation:

$$\sum_{k=1}^{n} (-1)^{n-k} \sum_{(k)} A_u(f)(x_{i_1} + \cdots + x_{i_k}) = \sum_{k=0}^{n} (-1)^{n-k} \sum_{(k)} \int_0^1 f((x_{i_1} +$$
$$\cdots + x_{i_k}) s^{1/u}) ds = \int_0^1 \sum_{k=0}^{n} (-1)^{n-k} \sum_{(k)} f((x_{i_1} + \cdots + x_{i_k}) s^{1/u}) ds.$$

□

*Consequence.*For every $b, u > 0$ and $n \geq 2$, the inclusions hold:

$$\begin{array}{ccccccc} K_n(b) & \subset & S_n^*(b) & \subset & S_n(b) & \subset & J_n^*(b). \\ \cap & & \cap & & \cap & & \cap \\ M^u K_n(b) & \subset & M^u S_n^*(b) & \subset & M^u S_n(b) & \subset & M^u J_n^*(b). \end{array}$$

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Studia Univ. Babes-Bolyai. Mathematica **38**(1993), 1, 21-28

SOME INEQUALITIES FOR m-CONVEX FUNCTIONS

S.S. DRAGOMIR and Gh.TOADER

Rezumat -Câteva inegalităţi pentru funcţii m-convexe. Funcţiile m-convexe au fost definite în [4]. Ele au alură intermediară celei de convexitate şi celei de stelaritate. Pentru aceste funcţii în lucrare se demonstrează inegalităţi de tip Jensen şi de tip Hermite-Hadamard.

1 Introduction

We will follow the paper [5].

Let X be a real linear space, $I = [Q, 1]$ and $m \geq 0$ a fixed real number.

Definition 1.1. *A set $D \subseteq X$ will be called m-convex if for any $x, y \in S$ and any $t \in I$ we have $tx + m(1-t)y \in D$.*

The following two lemmas which describe some properties of m-convex sets hold.

Lemma 1.1. *If $m > 1, 0 \in D$ and D is m-convex, then for any $x \in D$, $t \geq 0$ we have $tx \in D$.*

Taking into account this property, in what follows we shall consider only $m \in I$. The value $m = 1$ corresponds to convexity and $m = 0$ to starshapendness.

Lemma 1.2. *If D is m-convex and $0 \leq m \leq 1$, then D is also n-convex.*

Now, let D be a m-convex set in the linear space X with $m \in I$. Transporting the idea from [3] to the real case, in [4] it was introduced the following class of functions.

Definition 1.2. *A function $f : D \longrightarrow \mathbb{R}$ is said to be m-convex if for every $x, y \in D$ and $t \in I$ it verifies the condition:*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

Here again, $m = 1$ gives convex functions and $m = 0$ starshaped functions.

As it is shown in [5], it is natural to suppose $0 \in D$ and $f(0) \leq 0$.

Now we recall some fundamental properties of m-convex functions (see[5]).

Lemma 1.3. *The function $f : D \longrightarrow \mathbb{N}$ is m-convex if and only if the set:*

$$epi(f) = \{(x, y) \in D \times \mathbb{R},\ y \geq f(x)\}$$

is m-convex.

Lemma 1.4. *If f is m-convex then it is starshaped.*

Theorem 1.1. *If f is m-convex and $0 \leq n \leq 1$ then f is n-convex.*

2 Jensen's inequality for m-convex functions.

We will prove the following inequality of Jensen's type.

Theorem 2.1. *Let X be a linear space, $m \in [0, 1]$ and $D \subseteq X$ is a m-convex set in X. If $f : D \longrightarrow \mathbb{R}$ is a m-convex function, then for all $p_i > 0$ and $x_i \in D (i = 1, \ldots, n)$ we have:*

$$\sum_{i=1}^{n} p_i m^{i-1} x_i \ /\ P_n \in D, \quad where \quad P_n = \sum_{i=1}^{n} p_i$$

and the following inequality:

$$f\Big(\sum_{i=1}^{n} p_i m^{i-1} x_i \ /\ P_n\Big) \leq \sum_{i=1}^{n} p_i m^{i-1} f(x_i) \ /\ P_n \tag{1}$$

holds.

Proof. We proceed by mathematical induction. If $n = 2$, the statement follows by the definition. Suppose that (1)holds for "$n - 1$", i.e.

$$f\Big(\sum_{i=1}^{n-1} q_i m^{i-1} y_i \ /\ Q_{n-1}\Big) \leq \sum_{i=1}^{n-1} q_i m^{i-1} f(y_i) \ /\ Q_{n-1}$$

where $\sum\limits_{i=1}^{n-1} q_i m^{i-1} y_i \ /\ Q_{n-1}$ is assumed to be in D, provided that $q_i > 0,\ y_i \in D$ and $Q_{n-1} = \sum\limits_{i=1}^{n-1} q_i$. Now:

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i m^{i-1} x_i = \frac{p_1}{P_n} x_1 + \Big(1 - \frac{p_1}{P_n}\Big) \sum_{i=2}^{n} p_i m^{i-2} x_i \ /\ \sum_{i=2}^{n} p_i$$

and since:

$$\sum_{i=2}^{n} p_i m^{i-2} x_i \ /\ \sum_{i=2}^{n} p_i \in D \quad itfollows \quad \sum_{i=1}^{n} p_i m^{i-1} x_i \ /\ P_n \in D.$$

By the above considerations we have that:

$$f\Big(\sum_{i=1}^{n} p_i m^{i-1} x_i \;/\; P_n\Big) = f\Big(\frac{p_1}{P_n} x_1 + m\Big(1-\frac{p_1}{P_n}\Big)\sum_{i=2}^{n} p_i m^{i-2} x_i \;/\; \sum_{i=2}^{n} p_i\Big) \le$$
$$\frac{p_1}{P_n} f(x_1) + \Big(1-\frac{p_1}{P_n}\Big) f\Big(\sum_{i=2}^{n} p_i m^{i-2} x_i \;/\; \sum_{i=2}^{n} p_i\Big) \le \frac{p_1}{P_n} f(x_1) +$$
$$m\frac{1}{P_n}\sum_{i=2}^{n} p_i \sum_{i=2}^{n} p_i m^{i-2} f(x_i) \;/\; \sum_{i=2}^{n} p_i = \sum_{i=1}^{n} p_i m^{i-1} f(x_i) \;/\; P_n.$$

and the theorem is proved. □

Corollary 2.1. *In the above assumptions for D, f, m and $x_i (i = 1, \dots, n)$ we have that $\sum_{i=1}^{n} m^{i-1} x_i \;/\; n \in D$ and:*

$$f\Big(\sum_{i=1}^{n} m^{i-1} x_i \;/\; n\Big) \le \sum_{i=1}^{n} m^{i-1} f(x_i) \;/\; n.$$

Application 1. Let $m \in I$ and $x_i, p_i > 0$ for $i = 1, \dots, n$).
The one has the inequalities:

$$\Big(\sum_{i=1}^{n} p_i m^{i-1} x_i\Big)^q \le p_n^{q-1} \sum_{i=1}^{n} p_i m^{i-1} x_i^q, \ \forall q \ge 1$$

and

$$1 + \frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1} x_i \ge \Big(\prod_{i=1}^{n} (x_i+1) m^{i-1} p_i\Big)^{1/P_n}.$$

The proof of the above inequalities follows by (1) choosing the functions $f : [0,\infty) \longrightarrow [0,\infty),\ f(x) = x^q$ respectively $f : [0,\infty) \longrightarrow (-\infty, 0],\ f(x) = -\ln(x+1)$ which are m-convex. A second result is contained in the next theorem.

Theorem 2.2. *Let X be a linear space, $m \in I$ and D a m-convex set in X. If $f : D \longrightarrow \mathbb{R}$ is a m-convex function, then for all $p_i > 0, x_i \in D$, one has the inequalities:*

$$f\Big((t + m(1-t)\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1} \frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1} x_i\Big) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1} \cdot$$
$$\cdot \Big(t\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1} x_i + m(1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_i m^{j-1} x_j\Big) \le \frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j \cdot$$
$$m^{i+j-2} f(tx_i + m(1-t)x_j) \le (t + m(1-t))\frac{1}{P_n^2}\sum_{i=1}^{n} p_i m^{i-1} \sum_{i=1}^{n} p_i m^{i-1} f(x_i).$$

Proof. By the definition of m-convex functions, one has:

$$f(tx_1 + m(1-t)x_i) \le t \cdot f(x_i) + m(1-t) \cdot f(x_j) \quad for \quad all \quad i, j \in \{1, \dots, n\}.$$

By multiplying with $m^{j-1}p_j \geq 0$ and summing over j to 1 at n, one has:

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1} f\Big(t\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1} x_i + m(1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1} x_j\Big) \leq$$

$$\leq \frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j m^{i+j-2} f(tx_i + m(1-t)x_j) \leq t\frac{1}{P_n}\sum_{j=1}^{n} m^{j-1} p_j \cdot$$

$$\cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1} f(x_i) + m(1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1} f(x_j)\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1} =$$

$$= (t + m(1-t))\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1}\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1} f(x_i).$$

On the other hand, by Jensen's inequality for m-convex functions, we deduce:

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1} f\Big(t\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1} x_i + m(1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1} x_j\Big) \geq$$

$$f\Big(\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1}\Big(t\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1} x_j + m(1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j m^{j-1} x_j\Big)\Big) =$$

$$= f\Big((t + m(1-t))\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1}\frac{1}{P_n}\sum_{i=1}^{n} p_i m^{i-1} x_i\Big)$$

and the theorem is proved. □

Remark 2.1. *If we assume that $m = 1$, we obtain a refinement of Jensen's inequality established in [2].*

Corollary 2.2. *In the above assumptions for D, f and x_i we have for all $m \in [0, 1)$ the inequalities:*

$$f\Big(\frac{(t + m(1-t))(1-m^n)}{1-m}\frac{1}{n}\sum_{i=1}^{n} m^{i-1} x_i\Big) \leq$$

$$\leq \frac{1}{n}\sum_{i=1}^{n} m^{i-1} f\Big(t\frac{m^n - 1}{m-1} x_i + m(1-t)\frac{1}{n}\sum_{j=1}^{n} m^{j-1} x_j\Big) \leq$$

$$\leq \frac{1}{n^2}\sum_{i,j=1}^{n} m^{i+j-2} f(tx_i + m(1-t)x_j) \leq \frac{(t + m(1-t))(1-m^n)}{n(1-m)}\sum_{i=1}^{n} m^{i-1} f(x_i).$$

The following applications also hold.

Application 2. Let $x_i, p_i > 0, q \geq 1$ and $m \in I$. Then one has the inequalities:

$$
\begin{aligned}
&(t+m(1-t))^q \Big(\sum_{i=1}^n p_i m^{i-1}\Big)^q \Big(\sum_{i=1}^n p_i m^{i-1} x_i\Big)^q \leq \\
&\leq P_n^{q-1} \sum_{i=1}^n p_i m^{i-1} \Big(t \sum_{j=1}^n p_j m^{j-1} x_i + m(1-t) \sum_{j=1}^n p_j m^{j-1} x_j\Big)^q \leq \\
&\leq P_n^{2q-2} \sum_{i,j=1}^n p_i p_j m^{i+j-2} x_j (t x_i + m(1-t) x_j)^q \leq \\
&\leq (t+m(1-t)) P_n^{2q-2} \sum_{i=1}^n m^{i-1} p_i \sum_{j=1}^n m^{j-1} p_j x_j^q.
\end{aligned}
$$

Application 3. Let $x_i, p_i > 0$ and $m \in (0,1]$. Then one has the inequalities:

$$
\begin{aligned}
&(t+m(1-t)) \frac{1}{P_n^2} \sum_{i=1}^n p_i m^{i-1} \sum_{i=1}^n p_i m^{i-1} x_i + 1 \geq \\
&\geq \Big(\prod_{i=1}^n \Big(\frac{t x_i}{P_n} \sum_{j=1}^n p_j m^{j-1} + \frac{m(1-t)}{P_n} \sum_{j=1}^n p_j m^{j-1} x_j + 1 \Big)^{p_i m^{i-1}} \Big)^{1/P_n} \geq \\
&\geq \Big(\prod_{i,j=1}^n (t x_i + m(1-t) x_j + 1)^{p_i p_j m^{i+j-2}} \Big)^{1/P_n^2} \geq \\
&\geq \Big(\prod_{i=1}^n (x_i+1)^{p_i m^{i-1}} \Big)^{(1+m(1-t) \frac{1}{P_n^2} \sum_{i=1}^n m^{i-1} p_i}.
\end{aligned}
$$

The proofs follow by Theorem 3 applied to the m-convex functions $f : [0,\infty) \longrightarrow [0,\infty),\ f(x) = x^q\ (q \geq 1)$ respectively $f : [0,\infty) \longrightarrow (-\infty, 0],\ f(x) = -\ln(x+1)$.

Note that Theorem 2.1 and Theorem 2.2 give also some interesting inequalities in a normed linear space.

Application 4. Let $(X, ||\cdot||)$ be a normed space, $p_i \geq 0$ with $p_n > 0,\ x_i \in X,\ m \in I$ and $q \geq 1$. Then one has the inequalities:

$$
\Bigg|\Bigg| \sum_{i=1}^n p_i m^{i-1} x_i \Bigg|\Bigg|^q \leq P_n^{q-1} \sum_{i=1}^n p_i m^{i-1} ||x_i||^q
$$

and

$$
\begin{aligned}
&(t+m(1-t))^q \Big(\sum_{i=1}^n p_i m^{i-1}\Big)^q \Bigg|\Bigg| \sum_{i=1}^n p_i m^{i-1} x_i \Bigg|\Bigg|^q \leq \\
&\leq P_n^{q-1} \sum_{i=1}^n p_i m^{i-1} \Bigg|\Bigg| t \sum_{j=1}^n p_j m^{j-1} x_i + m(1-t) \sum_{j=1}^n p_j m^{j-1} x_j \Bigg|\Bigg|^q \leq \\
&\leq P_n^{2q-2} \sum_{i,j=1}^n p_i p_j m^{i+j-2} ||t x_i + m(1-t) x_j||^q \leq \\
&\leq (t+m(1-t)) P_n^{2q-2} \sum_{i=1}^n m^{i-1} p_i \sum_{i=1}^n m^{i-1} p_i ||x_i||^q.
\end{aligned}
$$

The proofs follow by the above theorems for the m-convex function $f : X \longrightarrow \mathbb{R},\ f(x) = ||x||^q$.

3 Some integral inequalities for m-convex functions.

In what follows we consider only functions defined on the real interval $[0, b]$ and denote by $K_m(b)$ the set of m-convex functions on $[0, b]$ such that $f(0) \le 0$ (see also [5]).

The following lemmas hold:

Lemma 3.1. *The function f is in $K_m(b)$ if and only if*

$$f_m(x) = \frac{f(x) - mf(y)}{x - my}$$

is increasing on $(my, b]$ for $y \in [0, b]$.

Lemma 3.2. *If f is differentiable in $[0, b]$ then $f \in K_m(b)$ if and only if:*

$$f^{'}(x) \ge \frac{f(x) - mf(y)}{x - my} \quad for \quad x > my.$$

The following integral inequality for m-convex functions holds.

Theorem 3.1. *Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be a m-convex integrable function with $m \in (0, 1]$ and $0 \le a < b < \infty$. Then one has the inequality:*

$$\frac{1}{b - a} \int_a^b f(x)dx \le \frac{1}{4}(f(a) + f(b) + m(f(a/m) + f(b/m))) \tag{2}$$

Proof. Since f is m-convex, we have:

$$f(tx + m(1 - t)y) \le tf(x) + m(1 - t)f(y),\ \forall x, y > 0$$

which gives:

$$f(ta + (1 - t)b) \le tf(a) + m(1 - t)f(b/m)$$

and

$$f(tb + (1 - t)a) \le tf(b) + m(1 - t)f(a/m)$$

for all $t \in I$. Integrating on I we get:

$$\int_0^1 f(ta + (1 - t)b)dt \le (f(a) + mf(b/m))/2$$

and

$$\int_0^1 f(tb + (1 - t)a)dt \le (f(b) + mf(a/m))/2.$$

but

$$\int_0^1 f(ta + (1 - t)b)dt = \int_0^1 f(tb + (1 - t)a)dt = \frac{1}{b - a} \int_a^b f(x)dx.$$

□

Thus, adding the above inequalities,we obtain (2).

Remark 3.1. >*From the proof we deduce that holds also a better evaluation:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq min\{(f(a)+mf(b/m))/2; (f(b)+mf(a/m))/2\}.$$

Theorem 3.2. *Let $f : [0,\infty) \longrightarrow \mathbb{R}$ be a m-convex differentiable function with $m \in 90, 1]$. Then for all $0 \leq a < b$ one has the inequalities:*

$$\frac{f(mb)}{m} - \frac{b-a}{2}f^{'}(mb) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{(b-ma)f(b)-(a-mb)f(a)}{2(b-a)}. \tag{3}$$

Proof. Using Lemma 6, we have for all $x, y \geq 0$ with $x \geq my$ that:

$$(x-my)f^{'}(x) \geq f(x) - mf(y). \tag{4}$$

Choosing in the above inequality $x = mb$ and $a \leq y \leq b$, then $x \geq my$ and:

$$(mb-my)f^{'}(mb) \geq f(mb) - mf(y).$$

Integrating over y on $[a, b]$, we get:

$$m\frac{(b-a)^2}{2}f^{'}(mb) \geq (b-a)f(mb) - m\int_a^b f(y)dy$$

thus the first inequality of (3). Putting in (4) $y = a$ and then integrating on $[a, b]$ one gets the second inequality of (3). □

Remark 3.2. *The second inequality from (3) is also valid for $m = 0$, while (2) is not. For $m = 1$ it is identical with (2) and represents a part of Hermite-Hadamard's inequality.*

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Real Analysis Exchange 20(2), 1994/5, pp. 1-5

THE PRESERVATION OF THE CONVEXITY OF FUNCTIONS

Josip E. Pećarič Gh. Toader

1 Introduction

Let us consider the classes of continuous, convex, starshaped and superadditive functions defined respectively by:

$$C(b) = \{f : [0, b] \to \mathbb{R},\ f(0) = 0,\ f \text{ continuous}\}$$

$$K(b) = \{f \in C(b); f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \forall t \in (0,1),$$
$$\forall x, y \in [0, b]\}$$

$$S^*(b) = \{f \in C(b) \mid f(tx) \leq tf(x),\ \forall t \in [0,1],\ x \in [0, b]\}$$

$$S(b) = \{f \in C(b) \,|f(x+y) \geq f(x) + f(y),\ \forall x, y,\ x + y \in [0, b]\}.$$

In [2] it is proved that all these classes are preserved by the arithmetic integral mean A defined by

$$A(f)(x) = \frac{1}{x}\int_0^x f(t)dt, \text{ for } x > 0,\ A(f)(0) = 0.$$

Moreover, if for a given set F of functions we denote by:

$$MF = \{f \in C(b) \,|A(f) \in F\},$$

in [2] it is proved that for any positive b the following strict inclusions hold:

$$K(b) \subset MK(b) \subset S^*(b) \subset S(b) \subset MS^*(b) \subset MS(b).$$

Simple proofs of these relations are also given in [5].

References [3] and [4] consider the integral operator W_g, defined by

$$W_g(f)(x) = \frac{1}{g(x) - g(0)} \int_0^x g^{'}(t) f(t) dt, \; W_g(f)(0) = f(0) \tag{1}$$

where g is a given differentiable function. In [5] it is proved that if W_g preserves one of the classes $K(b), S^*(b)$, or $S(b)$, then the function g is necessarily of the form $g(x) = kx^u$ for some $u > 0$ and some $k \neq 0$. If we denote the resulting operator by A_u:

$$A_u(f)(x) = \frac{u}{x^u} \int_0^x t^{u-1} f(t) dt \tag{2}$$

and if for a given set F of functions we set $M^u F = \{f \in C(b) \; : \; A_u(f) \in F\}$, then it is proved that for any positive numbers b and u hold the inclusions:

$$\begin{array}{ccccccc} K(b) & \subset & M^u K(b) & \subset & S^*(b) & \subset & S(b) \\ & & & & \cap & & \cap \\ & & & & M^u S^*(b) & \subset & M^u S(b) \end{array}$$

A similar result was proved for some classes of generalized convexity of order two in [6] and [7] and for convexity, starshapedness and superadditivity of higher order in [8].

Analyzing all these results, we can produce a general scheme that we want to consider in what follows.

2 A Class of Generalized Convex Functions

Let $D = (d_{jk})_{n,m}$ be a $n \times m$ matrix and $C = (c_j)_n$ be a given n vector with the property that $c_1 + \cdots + c_n = 0$. Let

$$D(b) = X = \left\{ (x_k)_m \mid \sum_{k=1}^{m} d_{jk} x_k \in [0, b], \; j = 1, \ldots, n \right\}$$

and then, for any X from $D(b)$, the functional $L_{CD}(\cdot)(X) \; : \; C(b) \longrightarrow \mathbb{R}$ defined by

$$L_{CD}(f)(X) = \sum_{j=1}^{n} c_j f \left(\sum_{k=1}^{m} d_{jk} x_k \right)$$

Using them, we can define a general class of convex functions

$$K_{CD}(b) = \{f \in C[0, b] \mid L_{CD}(f)(X) \geq 0, \; \forall X \in D(b)\}.$$

By adequate choice of C and D we get the sets of Jensen convex functions and of superadditive functions, usual or generalized, and of any order. For example the condition of superadditivity of $f \in C[0,b]$ is

$$f(x_1+x_2)-f(x_1)-f(x_2)+f(0)\geq 0,\ \forall x_1,x_2,\ x_1+x_2\in[0,b]$$

and it becomes that given in the definition of $S(b)$ for f from $C(b)$. In [8] we have considered also superadditivity of order $n>2$. For example $f\in C[0,b]$ is said to be superadditive of order 3 if

$$f(x_1+x_2+x_3)-f(x_1+x_2)-f(x_1+x_3)-f(x_2+x_3)+f(x_1)+$$

$$+f(x_2)+f(x_3)-f(0)\geq 0,\ \forall x_1,x_2,x_3,\ x_1+x_2+x_3\in[0,b].$$

For convexity and starshapedness we must refer at Remark 3.

The condition on C assures that the class $K_CD(b)$ is nonempty because it contains the constant functions. But we need a more precise condition. For this, let us denote by P_q the set of polynomials of degree at most q.

Definition 2.1. *The class $K_{CD}(f)=0$ if and only if $f\in P_q$.*

Remark 2.1. The determination of the value of q for C and D given is a problem of functional equations. Of course, necessary conditions are $L_{CD}(e_k)=0$ for $k=0,\dots,q$, and $L_{CD}(e_{q+1})\neq 0$ where $e_k(x)=x^k$ for $k\geq 0$. But it is a difficult problem to prove that they are also sufficient or to find simpler conditions. For some results and references see [1, pages. 129-131]. For example if $L_{CD}(f)(X)=\sum_{j=1}^{n}c_jf(x_1+(j-1)x_2)$, the value of q is less than 1 plus the order of multiplicity of the root $t=1$ in the equation $c_1+c_2t+\cdots+c_nt^{n-1}=0$.

3 Main Results

We want to determine those functions g that give an integral operator W_g, defined by (1), which preserves the class $K_{CD}(b)$. We have the following result:

Theorem 3.1. *If the class of functions $K_{CD}(b)$ is well defined and W_g preserves it, then there is a positive number u such that $g(x)=vx^u\ \forall x\in[0,b]$.*

PROOF. For any p from P_q, because p and $-p$ belong to $K_{CD}(b)$, we have $W_g(p)$ and $W_g(-p)$ also in $K_{CD}(b)$ and this is equivalent to $L_{CD}(W_g(p))(X)=0$ $\forall X \in D(b)$. Thus $W_g(p)$ is in P_q as $K_{CD}(b)$ is well defined. Let $W_g(e_k)=p_k$ for $k=1,\dots,q$. Differentiating these relations we get

$$\frac{g^{'}(x)}{g(x)-g(0)}=\frac{p_k^{'}}{e_k(x)-p_k(x)} \quad \text{for} \quad x\in(0,b],\ k=1,\dots,q \tag{3}$$

or, if we set $p_k(x)=a_{k0}+a_{k1}x+\cdots+a_{kq}x^q$, we have for $1\le k<h\le q$

$$\left(x^h-\sum_{j=0}^{q}a_{hj}x^j\right)\sum_{j=1}^{q}ja_{kj}x^{j-1}=\left(x^k-\sum_{j=1}^{q}a_{kj}x^j\right)\sum_{j=1}^{q}ja_{hj}x^{j-1}.$$

For $h=q$ equating the coefficients of x^{2q-1} we get $a_{kq}=0$ for $k>q$. Then for $h=q-1$ and the power $2q-3$, we deduce also $a_{k,q-1}=0$ for $k<q-1$ and by induction $a_{kj}=0$ for $k<j$. Thus $p_1(x)=a_{10}+a_{11}x$ and from (3) with $k=1$, we have

$$\frac{g^{'}(x)}{g(x)-g(0)}=\frac{a_{11}}{x-(a_{10}+a_{11}x)}$$

which gives the result.

Using such a weight function we denote the resulting operator by A_u. It is given by (2). Also we introduce the following class of functions

$$M^uK_{CD}(b)=\{f\in C(b)\mid A_u(f)\in K_{CD}(b)\}.$$

Theorem 3.2. *If tX belongs to $D(b)$ for any $t\in[0,1]$ and any $X\in D(b)$, there for any positive u we have $K_{CD}(b)\subset M^uK_{CD}b$.*

PROOF.Substituting $t=xs^{1/u}$ in $A_u(f)$ we get $A_u(f)(x)=\int\limits_0^1 f(xs^{1/u}ds$. So, for any X from $D(b)$

$$\begin{aligned}
L_{CD}(A_u(f))(X) &= \sum_{j=1}^{n}c_jA_u(f)\left(\sum_{k=1}^{m}d_{jk}x_k\right)\\
&= \sum_{j=1}^{n}c_j\int_0^1 f\left(s^{1/u}\sum_{k=1}^{m}d_{jk}x_k\right)ds\\
&= \int\limits_0^1\sum_{j=1}^{n}c_jf\left(\sum_{k=1}^{m}d_{jk}x_ks^{1/u}\right)ds\\
&= \int\limits_0^1 L_{CD}(f)(Xs^{1/u})ds\ge 0
\end{aligned}$$

because f is from $K_{CD}(b)$ and $s^{1/u}X$ from $D(b)$.

Remark 3.1. *The condition* $[0,1] \times D(b) \subset D(b)$ *holds, for example, if the matrix* D *is positive.*

Remark 3.2. *If instead* C *and* D *we use families of vectors* C *and of matrice* D*, all the above results remain valid. So we obtained similar theorems for various sets of convex or of starshaped functions. For example, the function* $f \in C[0,b]$ *is starshaped if* $tf(x) - ft(x) + (1-t)f(0) \geq 0 \ \forall x \in [0,b]$ *for every* $t \in [0,1]$ *that is we have a set of conditions.*

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SUPERADDITIVITY AND HERMITE-HADAMARD'S INEQUALITIES

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Rezumat -Superaditivitate şi inegalităţile lui Hermite-Hadamard. Îmbunătăţim în anumite sensuri inegalităţile lui Hermite-Hadamard, valabile pentru funţii convexe pe $[a, b]$

$$f(A(a, b)) \leq A(f; a, b) \leq A(f(a), f(b))$$

unde $A(f; a, b)$ reprezintă media aritmetică integrală a funcţiei f pe $[a, b]$, iar $A(a, b)$ media aritmetică a numerelor a şi b. De exemplu $A(f; a, b)$ se înlocuieşte cu o funcţională liniară izotonă, simetrică într-un anume sens. De asemenea, inegalităţile se demonstrează pentru clase mai largi de funcţii, care le includ pe cele convexe.

1 An inequality for superadditive functions.

Let us consider the sets of continuous, convex, starahaped respectively superadditive functions on $[a, b]$ given by:

$$C[a, b] = \{f : [a, b] \longrightarrow \mathbb{R}, f \quad continuous\}$$

$$K[a, b] = \{f \in C[a, b]; f(tx + (1 - t)y) \leq tf(x) + \quad (1 - t)f(y), \\ \forall x, y \in [a, b], \ \forall t \in [0, 1]\}$$

$$St[a, b] = \{f \in C[a, b]; (f(x) - f(a))/(x - a) \leq \\ (f(y) - f(a))/(y - a), \ a < x < y \leq b\}$$

respectively

$$S[a, b] = \{f \in C[a, b]; f(x) + f(y) \leq f(x + y - a) + f(a), \\ \forall x, y, x + y - a \in [a, b]\}.$$

For $a = 0$ we denote by $C(b), K(b), St(b)$ respectively $S(b)$ the corresponding sets of functions, submitted also to the condition $f(0) = 0$. A.M. Bruckner and E.Ostrow have proved in [1] the strict inclusions:

$$K(b) \subset St(b) \subset S(b)$$

Simple proofs and generalizations of the results of [1] may be found in [5].

Starting from some properties of superadditive sequences (see [6]) at the 31th international Symposium on Functional Equations (August 22-28, 1993, Debrecen, Hungary) we have proposed the following problem: find some positive functions p of $C[a, b]$, different from the identity function, with the property that the inequality:

$$\int_0^x p(t)\left[\frac{f(x)}{x} - \frac{f(t)}{t}\right] dt \geq 0, \ \forall x \in [0, b] \tag{1}$$

hold for every $f \in S(b)$.

Of course, for $f \in St(b)$ the inequality (1) is valid for all positive p. On the other side, for the identity function, $p(x) = x$, (1) is valid for all $f \in S(b)$. Indeed we have:

Lemma 1.1. *For every $f \in S(b)$ holds the inequality:*

$$\int_0^x f(t)dt \leq \frac{xf(x)}{2}, \ \forall x \in (0, b] \tag{2}$$

Proof. We have:

$$f(t) + f(x - t) \leq f(x), \ \forall t \in [0, x].$$

Integrating on $[0, x]$ we get (2). □

Remark 1.1. *We can write (2) as:*

$$\frac{1}{x}\int_0^x f(t)dt \leq \frac{f(x) + f(0)}{2} \tag{3}$$

which is one of Hermite-Hadamard's inequalities, as we see at once.

2 Hermite-Hadamard's inequalities

Let us denote by $A(f; a, b)$ and $A(a, b)$ the integral arithmetic mean of f on $[a, b]$ respectively the arithmetic mean of a and b given by:

$$A(f; a, b) = \frac{1}{b - a}\int_a^b f(x)dx$$

and

$$A(a, b) = \frac{a + b}{2}.$$

The inequalities of Hermite-Hadamard, valid for every function f from $K[a,b]$ are:

$$f(A(a,b)) \leq A(f;a,b) \leq A(f(a),f(b)). \tag{4}$$

In (3) we see that the second inequality of (4) holds for all f in $S(b)$. In fact it is valid for all superadditive functions, even of a weak kind.

Definition 2.1. *The function f is called weakly superadditive on $[a,b]$ if it verifies:*

$$f(a+t)+f(b-t) \leq f(a)+f(b),\ \forall t \in [0,(b-a)/2]. \tag{5}$$

Let us denote by $wS[a,b]$ the set of all these functions.

Theorem 2.1. *The inequality*

$$A(f;a,b) \leq A(f(a),f(b)) \tag{6}$$

is valid for every f of $wS[a,b]$.

Proof. Integrating (5) on $0, b-a]$, where it is valid in fact, we get (6). □

Similarly we can extend the set of functions for which the first inequality of (4) is valid.

Definition 2.2. *The function f is weakly Jensen convex on $[a,b]$ if:*

$$\frac{f(a+t)+f(b-t)}{2} \geq f\left(\frac{a+b}{2}\right),\ \forall t \in \left[0,\frac{b-a}{2}\right] \tag{7}$$

We denote by $wJ[a,b]$ the set of all such functions.

Theorem 2.2. *If $f \in wJ[a,b]$ then:*

$$A(f;a,b) \geq f(A(a,b)). \tag{8}$$

Proof. In fact (7) is valid for $t \in [0,b-a]$ and integrating on this interval, we get (8). □

We can characterize the functions from $wS[a,b]$ and those from $wJ[a,b]$. For this we begin with the following:

Lemma 2.1. *For every function $f \in C[a,b]$ we can determine two functions $f_1, f_2 : [0,(b-a)/2] \longrightarrow \mathbb{R}$ such that:*

$$f(x) = \begin{cases} f_1(x-a) & , for \quad x \in \left[a, \dfrac{a+b}{2}\right] \\ f_1\left(\dfrac{b-a}{2}\right) + f_2\left(\dfrac{b-a}{2}\right) - f_1(b-x) & , for \quad x \in \left(\dfrac{a+b}{2}, b\right] \end{cases} \tag{9}$$

Proof. Of course:

$$f_1(t) = f(a+t) \quad for \quad t \in [0,(b-a)/2]$$

and

$$f_2(t) = f((b-a)/2) + c - f(b-t) \quad for \quad t \in [0,(b-a)/2]$$

where c is an arbitrary real number. □

Using it we can obtain the desired characterizations, which permit also the construction of such functions.

Theorem 2.3. *The function f belongs to:*

a) $wS[a,b]$ *if and only if*

$$f_1(t) - f_1(0) \leq f_2(t) - f_2(0);$$

b) $wJ[a,b]$ *if and only if*

$$f_1(t) - f_1((b-a)/2) \geq f_2(t) - f_2((b-a)/2).$$

Remark 2.1. *If we take in (9) $f_1 = f_2$ arbitrary, we get a function f with the property:*

$$f(a+t) + f(b-t) = f(a) + f(b) = 2f((a+b)/2),\ \forall t \in [0,(b-a)/2]$$

thus it is contained in $wS[a,b] \cap wJ[a,b]$, as are also all the convex functions.

3 Symmetric linear functionals

The inequalities (4) were generalized in [3] replacing the integral arithmetic mean $A(f;a,b)$ by an arbitrary isotonic linear functional but also with the modification of the first and of the last terms. In what follows we want to do the same change of $A(f;a,b)$ but with the preservation of the inequalities (4). And this will be done, as in the previous paragraph, not only for convex functions.

Let $L(\cdot,a,b) : C[a,b] \longrightarrow \mathbb{R}$ be an isotonic linear functional, that is, for $t,s \in \mathbb{R}$, $f,g \in C[a,b]$:

$$L(f;a,b) \geq 0 \quad if \quad f \geq 0$$

$$L(tf+sg;a,b) = tL(f;a,b) + sL(g;a,b).$$

Analysing the proofs of Theorem 1 (or Lemma 2) and Theorem 2 we see that for our intention we can use a special type of functionals. If $f \in C[a,b]$ we denote by f_- the function defined by:

$$f_-(x) = f(a+b-x) \quad for \quad x \in [a,b].$$

Definition 3.1. *The functional* $L(\cdot,a,b)$ *is symmetric if:*

$$L(f_-;a,b) = L(f;a,b),\ \forall f \in C[a,b].$$

Theorem 3.1. *If* $L(\cdot,a,b)$ *is a symmetric isotonic linear functional, with* $L(1;a,b)=1$, *then:*

$$L(f;a,b) \leq A(f(a),f(b)),\ \forall f \in wS[a,b]$$

and

$$L(f;a,b) \geq f(A(a,b)),\ \forall wJ[a,b].$$

Proof. Indeed (5) is equivalent with:

$$f(x)+f_-(x) \leq f(a)+f(b) \quad for \quad x \in [a,b]$$

and (9) with:

$$f(x)+f_-(x) \geq 2f(A(a,b)) \quad for \quad x \in [a,b]$$

and we have only to apply the functional $L(\cdot,a,b)$. ☐

Remark 3.1. *If* $g \in C[a,b]$ *is symmetric with respect to* $A(a,b)$, *the functional defined by:*

$$L(f;a,b) = \int_a^b f(x)g(x)dx \ / \ \int_a^b g(x)dx$$

satisfies all the hypothesis of Theorem 4. So we get a generalization of Hermite-Hadamard's inequalities which include the result of L.Fejér from [2] (established also only for convex functions).

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Periodica Mathematica Hungarica **30**(1995),2, 165-170

ON AN INEQUALITY OF SEITZ

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1 Introduction

In [2] we find the following inequality proved by G.Seitz in 1937 which contains both Cauchy's and Chebyshev's inequalities: let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n), z = (z_1, \ldots, z_n)$ and $u = (u_1, \ldots, u_n)$ be given sequences of real numbers and let $a_{ij}(i, j = 1, \ldots, n)$ be given real numbers. If for every pair of numbers $i, j (i < j)$ and for every pair $r, s (r < s)$

$$\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \cdot \begin{vmatrix} z_r & z_s \\ u_r & u_s \end{vmatrix} \geq 0 \quad and \quad \begin{vmatrix} a_{ri} & a_{rj} \\ a_{si} & a_{sj} \end{vmatrix} \geq 0, \tag{1}$$

then:

$$\frac{\sum\limits_{i,j=1}^{n} a_{ij} x_i z_j}{\sum\limits_{i,j=1}^{n} a_{ij} x_i u_j} \geq \frac{\sum\limits_{i,j=1}^{n} a_{ij} y_i z_j}{\sum\limits_{i,j=1}^{n} a_{ij} y_i u_j} \tag{2}$$

In what follows we want to generalize this inequality for positive linear functionals as it was done in [1] in the case of Grüss'inequality and in [4] in that of Chebyshev's.

2 Functionals

Let E be an arbitrary set and $F(E)$ be the set of real-valued functions defined on E.

Instead of linearity and positivity of functional we consider the apparently weaker conditions of sublinearity and isotony, that is the set of functionals:

$$M_+(E) = \{A : F(E) \longrightarrow \mathbb{R} | A(tf + sg) \leq tA(f) + sA(g), \ \forall t, s \in \mathbb{R}, \ \forall f, g \in F(E)$$
$$A(f) \geq 0, \ \forall f \in F(E), f \geq 0\}.$$

Usually one takes:

$$A(f) = \sum_{i=1}^{n} f(x_i), \ x_i \in E, \ i = 1, \ldots, n \tag{3}$$

or

$$A(f) = \int_a^b f(x)dx, \; E = [a, b]. \tag{4}$$

We shall use an order relation on $M_+(E)$:

Definition 2.1. *The functional $A \in M_+(E)$ is called to be greater than $B \in M_+(E)$ and write $A \geq B$ if $A(f) \geq B(f)$ for every positive function $f \in F(E)$.*

It is easy to check the validity of the following:

Lemma 2.1. *If $A \geq B$ and $f \geq g \geq 0$ then $A(f) \geq B(g)$.*

Finally we consider a kind of product of two functionals. If $A \in M_+(E_1)$ and $B \in M_+(E_2)$ we denote by AB the functional defined as follows: for $p \in F(E_1 \times E_2)$ we have $B(p) = q \in F(E_1)$ where $q(x) = B(p(x, .))$ and then $AB(p) = A(q)$. We get so a functional from $M_+(E_1 \times E_2)$.

3 Inequalities

Taking into account the condition (1) we consider the following:

Definition 3.1. *We say that the functions p, f_1, f_2, g_1, g_2 are synchrone on $E_1 \times E_2$ if $p \in F(E_1 \times E_2), f_1, f_2 \in F(E_1), g_1, g_2 \in F(E_2)$ and:*

$$\begin{vmatrix} p(x,s) & p(y,s) \\ p(x,t) & p(y,t) \end{vmatrix} \cdot \begin{vmatrix} f_1(x) & f_1(y) \\ f_2(x) & f_2(y) \end{vmatrix} \cdot \begin{vmatrix} g_1(s) & g_1(t) \\ g_2(s) & g_2(t) \end{vmatrix} \geq 0 \tag{5}$$

for every $x, y \in E_1$ and $s, t \in E_2$.

Theorem 3.1. *If the functions p, f_1, f_2, g_1, g_2 are synchrone on $E_1 \times E_2$ and $A, B \in M_+(E_1), C, D \in M_+(E_2)$ then:*

$$T(A, B, C, D)(p, f_1, f_2, g_1, g_2) \geq 0 \tag{6}$$

where:

$$\begin{aligned} T(A, B, C, D)(p, f_1, f_2, g_1, g_2) &= AC(pf_1g_1)BD(pf_2g_2) + AC(pf_2g_2)BD(p, f_1, g_1) \\ &- AC(pf_1g_2)BD(pf_2g_1) - AC(pf_2g_1)BD(pf_1g_2) \\ &+ AD(p, f_1g_1)BC(pf_2g_2) + AD(pf_2g_2)BC(pf_1g_1) \\ &- AD(pf_1g_2)BC(pf_2g_1) - AD(pf_2g_1)BC(pf_1g_2). \end{aligned}$$

Proof. >From (5) we have:

$$\begin{aligned}&p(x,s)f_1(x)g_1(s)p(y,t)f_2(y)g_2(t)+p(x,t)f_2(x)g_2(t)p(y,s)f_1(y)g_1(s)\\&-p(x,t)f_1(x)g_2(t)p(y,s)f_2(y)g_1(s)-p(x,s)f_2(x)g_1(s)p(y,t)f_1(y)g_2(t)\\&+p(x,t)f_1(x)g_1(t)p(y,s)f_2(y)g_2(s)-p(x,s)f_2(x)g_2(s)p(y,t)f_1(y)g_1(t)\\&-p(x,s)f_1(x)g_2(s)p(y,t)f_2(y)g_1(t)-p(x,t)f_2(x)g_1(t)p(y,s)f_1(y)g_2(s)\geq 0.\end{aligned}\tag{7}$$

Applying successively the functionals C,D,A,B to functions of variable s,t,x respectively y we get (6). ☐

Consequence 3.1. *If the functions p,f_1,f_2,g_1,g_2 are synchrone on $E_1\times E_2$ and $A\in M_+(E_1), C\in M_+(E_2)$ then:*

$$S(A,C)(p,f_1,f_2,g_1,g_2)\geq 0 \tag{8}$$

where:

$$S(A,C)(p,f_1,f_2,g_1,g_2)=AC(pf_1g_1)AC(pf_2g_2)-AC(pf_1g_2)AC(pf_2g_1)$$

.

Remark 3.1. *It follows from (6) because:*

$$S(A,C)(p,f_1,f_2,g_1,g_2)=T(A,A,C,C)(p,f_1,f_2,g_1,g_2)/4.$$

This result is still a large generalization of the inequality (2). Even if we take A and C of the form (3) they can be different.

Consequence 3.2. *If the functions p,f_1,f_2,g_1,g_2 are synchrone on $E_1\times E_2$ and $A\in M_+(E)$ then:*

$$AA(pf_1g_1)AA(pf_2g_2)\geq AA(pf_1g_2)AA(pf_2g_1).$$

Remark 3.2. *This is a direct generalization of (2). If we take now:*

$$p(x,y)=\begin{cases}q(x), & if\ y=x\\ 0, & if\ y\neq x,\end{cases}$$

the above results can be transposed to functions of a single variable.

Definition 3.2. *The functions $q,f_1,f_2,g_1,g_2\in F(E)$ are synchrone on E if:*

$$q(x)q(y)\begin{vmatrix}f_1(x) & f_1(y)\\ f_2(x) & f_2(y)\end{vmatrix}\cdot\begin{vmatrix}g_1(x) & g_1(y)\\ g_2(x) & g_2(y)\end{vmatrix}\geq 0,\ \forall x,y\in E.$$

Consequence 3.3. *If the functions q, f_1, f_2, g_1, g_2 are synchrone on E and $A \in M_+(E)$ then:*

$$A(q, f_1 g_1) A(q f_2 g_2) \geq A(q f_1 g_2) A(q f_2 g_1).$$

Consequence 3.4. *If the functions f, g are increasing and q is positive on E then:*

$$A(qfg)A(q) \geq A(qf)A(gq)$$

for $A \in M_+(E)$.

Consequence 3.5. *If the function q is positive and $A \in M_+(E)$ then:*

$$A(qf^2)A(qg^2) \geq (A(qfg))^2.$$

Remark 3.3. *For A given by (4) the Consequence 3.3 gives an inequality of Fujiwara (see [2]). Their particular cases given by the consequences 4 and 5 represent Chebyshev's respectively Cauchy's inequalities (see also [2]).*

4 The operators T and S

We prove now some properties of monotony of the operators T and S generalizing known results.

Theorem 4.1. *If the functions p, f_1, f_2, g_1, g_2 are synchrone on $E_1 \times E_2$ and the functionals $A, B, A', B' \in M_+(E_1), C, D, C', D' \in M_+(E_2)$ are such that:*

$$A \geq A', \; B \geq B' \; C \geq C', \; D \geq D',$$

then:

$$T(A, B, C, D)(p, f_1, f_2, g_1, g_2) \geq T(A', B', C', D')(p, f_1, f_2, g_1, g_2).$$

Proof. We start by using the relation $C \geq C'$ for the function of variable s given by (7). Then we use the lemma for the pairs $D \geq D', A \geq A'$ respectively $B \geq B'$ for successively resulting functions of variable t, x and y. $\square$

Consequence 4.1. *If the functions p, f_1, f_2, g_1, g_2 are synchrone on $E_1 \times E_2$ and $A, A' \in M_+(E_1), C, C' \in M_+(E_2)$ are such that:*

$$A \geq A', \; C \geq C',$$

then

$$S(A, C)(p, f_1, f_2, g_1, g_2) \geq S(A', C')(p, f_1, f_2, g_1, g_2).$$

EXAMPLE 1 If we denote:

$$A_n(f) = \sum_{i=1}^{n} f(x_i),\ x_i \in E_1,\ i = 1, \dots, n$$

and

$$C_m(g) = \sum_{j=1}^{n} g(y_i),\ y_i \in E_2,\ j = 1, \dots, m,$$

we have

$$A_k \geq A_{k-1} \quad for \quad k = 2, \dots, n$$

and

$$C_h \geq C_{h-1} \quad for \quad h = 2, \dots, m$$

thus:

$$S(A_k, C_h)(p, f_1, f_2, g_1, g_2) \geq S(A_i, C_j)(p, f_1, f_2, g_1, g_2)$$

for every set of synchrone functions p, f_1, f_2, g_1, g_2 on $E_1 \times E_2$ and indices: $1 \leq i \leq k \leq n$ and $1 \leq j \leq h \leq m$. For example, taking equal indices we get:

$$\begin{aligned} S(A_n, C_n)(p, f_1, f_2, g_1, g_2) &\geq S(A_{n-1}, C_{n-1})(p, f_1, f_2, g_1, g_2) \geq \dots \\ &\geq S(A_2, C_2)(p, f_1, f_2, g_1, g_2) \geq 0 \end{aligned}$$

which gives a refinement of (2). Such results are known relative to the inequality of Chebyshev (see [3]).

In what follows we consider the special case $E_1 = E_2 = E$ and we denote $S(A, A)$ by $S(A)$. Thus:

$$S(A)(p, f_1, f_2, g_1, g_2) = AA(pf_1g_1)AA(pf_2g_2) - AA(pf_1g_2)AA(pf_2g_1).$$

Theorem 4.2. *If the functions p, f_1, f_2, g_1, g_2 are synchrone on $E \times E$ and $A, B \in M_+(E)$ then:*

$$S(A+B)(p, f_1, f_2, g_1, g_2) \geq S(A)(p, f_1, f_2, g_1, g_2) + S(B)(p, f_1, f_2, g_1, g_2). \tag{9}$$

Proof. This follows by (6) because, if we omit the argument (p, f_1, f_2, g_1, g_2), i.e. if we write $S(A)$ for $S(A)(p, f_1, f_2, g_1, g_2)$ and $T(A, B, C, D)$ for $T(A, B, C, D)(p, f_1, f_2, g_1, g_2)$, we have:

$$
\begin{aligned}
S(A+B)-S(A)-s(B) &= \frac{1}{4}(T(A,A,A,B)+T(A,A,B,A)+T(A,B,A,A)\\
&+T(B,A,A,A)+T(A,A,B,B)+T(A,B,A,B)\\
&+T(A,B,B,A)+T(B,A,A,B)+T(B,A,B,A)\\
&+T(B,B,A,A)+T(A,B,B,B)+T(B,A,B,B)\\
&+T(B,B,A,B)+T(B,B,B,A))=\\
&=\frac{1}{4}(2T(A,A,A,B)+2T(B,A,A,A)+T(A,A,B,B)\\
&+4T(A,B,A,B)+T(B,B,A,A)+2T(A,B,B,B)\\
&+2T(B,B,B,A))\geq 0.
\end{aligned}
$$

□

EXAMPLE 2 For a finite index set I and fixed points $x_i \in E,\ i \in I$, we denote:

$$A_I(f) = \sum_{i\in I} f(x_i).$$

From (9) we deduce that if the index sets I and J are disjoint then:

$$S(A_{A\cup J})(p,f_1,f_2,g_1,g_2) \geq S(A_I)(p,f_1,f_2,g_1,g_2)+S(A_J)(p,f_1,f_2,g_1,g_2)$$

for the functions p, f_1, f_2, g_1, g_2 which are synchrone on $E \times E$. Such properties are also known for other inequalities (see [3]).

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Demonstratio Mathematica **28** (1995), 2, 329-334

REFINEMENTS OF JENSSEN'S INEQUALITY

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1 Jessen's inequality

Let f be a real convex function defined on $[a,b]$. The classical Hermite-Hadamard's inequality (see [9]) asserts that:

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

This inequality was generalized (see [1], [7], and [10]) for an arbitrary isotonic linear functional, i.e., a functional $A : C[a,b] \longrightarrow \mathbb{R}$ with the properties:

(i) $A(tf+sg) = tA(f)+sA(g)$ for $t,s \in \mathbb{R},\ f,g \in C[a,b]$;

(ii) $A(f) \geq 0$ if $f(x) \geq 0$ for all $x \in [a,b]$.

The result from [7] is: if f is convex and A is an isotonic linear functional with $A(1) = 1$, then

$$f(A(e)) \leq A(f(e)) \leq [(b-A(e))f(a) + (A(e)-a)f(b)]/(b-a) \tag{2}$$

where $e(x) = x$ for $x \in [a,b]$.
Note that taking in (2)

$$A(f) := \frac{1}{b-a}\int_a^b f(x)dx, \tag{3}$$

we get (1), so the inequality (2) generalizes, for isotonic linear functionals, the well known Jessen's inequality.

In turn, the inequality (2) was generalized in [1] where the function e was replaced by an arbitrary one.

2 Some refinements

The following lemma is proved in [10]:

Lemma 2.1. *Let X be a real linear space and $C \subset X$ be a convex subset. If $f : C \longrightarrow \mathbb{R}$ is convex then for all $x, y \in C$ the mapping $g_{x,y}(t) := f(tx + (1-t)y)$ is convex on $[0,1]$.*

Using this result, the authors proved a generalization of (2) for functions defined on an arbitrary linear space.

Another result of this type was established in [6].

Analogously we can prove the following lemma:

Lemma 2.2. *If $f : [a,b] \longrightarrow \mathbb{R}$ is convex, then for every $t \in [0,1]$ and every $y \in [a,b]$, the function $g_{t,y} : [a,b] \longrightarrow \mathbb{R}$ given by $g_{t,y}(x) := f(tx + (1-t)y)$ is convex.*

Further on we will use the following convention:

if the functional A acts on the function

$$g(x_i) = f(x_1, \ldots, x_i, \ldots, x_n),$$

where all the variables except for x_i are fixed, then we denote

$$A(g) = Ax_i(f(x_1, \ldots, x_i, \ldots, x_n)).$$

Applying the inequality (2) to the convex function $g_{t,y}$ from Lemma 2, we get:

Theorem 2.1. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous convex function and A be an isotonic linear functional with $A(1) = 1$. Then for every $t \in [0,1]$ and for every $y \in [a,b]$ the inequalities*

$$\begin{aligned} &f(tA(e) + (1-t)y) \leq A_x(f(tx + (1-t)y)) \leq \\ &\leq [(b - A(e))f(ta + (1-t)y) + (A(e) - a)f(tb + (1-t)y)]/(b-a) \end{aligned} \tag{4}$$

hold.

We can obtain another variant of (4) generalizing the method used in [3] (see also [5]).

Lemma 2.3. *Assume that the function $f : [a,b] \longrightarrow \mathbb{R}$ is continuous convex and the functional A is linear and isotonic. Then the function $H_y : [0,1] \longrightarrow \mathbb{R}$ defined by*

$$H_y(t) := A_x[f(tx + (1-t)y)], \quad y \in [a,b]$$

is convex on $[0,1]$.

Proof. Let $x, y \in [a,b]$ and $t, su, v \in [0,1]$ and $u + v = 1$, then we have

$$f((ut+vs)x + (1-ut-vs)y) = f(u(tx+(1-t)y) +$$
$$+v(sx+(1-s)y)) \leq uf(tx+(1-t)y) + vf(sx+(1-s)y),$$

because f is convex. Because the functional A is linear and isotonic it is increasing and so

$$H_y(ut+vs) \leq uH_y(t) + vH_y(s).$$

□

Now we prove

Theorem 2.2. *If the function $f : [a,b] \longrightarrow \mathbb{R}$ is continuous convex and the functional A is linear, isotonic with $A(1) = 1$, then the function $H_0 : [0,1] \longrightarrow \mathbb{R}$ defined by*

$$H_0(t) := A_x(f(tx + (1-t)A(e))$$

has the following properties:

(i) H_0 is convex on $[0,1]$;

(ii) it has the bounds

$$\sup_{t\in[0,1]} H_0(1) = A(f(e))$$

and

$$\inf_{t\in[0,1]} H_0(t) = H_0(0) = f(A(e));$$

(iii) H_0 is nondecreasing on $[0,1]$.

Proof. (i) It follows from Lemma 3 by taking $y = A(e)$.
In order to get (ii) let us notice that

$$f(tx + (1-t)A(e)) \leq tf(x) + (1-t)f(A(e)), \quad t \in [0,1]$$

and so

$$H_0(t) \le tA(f) + (1-t)f(A(e)) \le A(f) = H_0(1)$$

Because from (2) we have $f(A(e)) \le A(f)$. On the other hand the function $h : [a,b] \longrightarrow \mathbb{R}$, given by $h(x) := f(tx + (1-t)A(e))$, is convex for every fixed $t \in [0,1]$ and so, again by (2)

$$H_0(t) = A(h) \ge h(A(e)) = f(A(e)) = H_0(0)$$

what gives (ii).

(iii) Let $0 < t_1 < t_2 < 1$. Then by the convexity argument for H_0 and by (ii) one has:

$$[H_0(t_2) - H_0(t_1)]/(t_2 - t_1) \ge [H_0(t_1) - H_0(0)]/t_1 \ge 0$$

what shows that H_0 is increasing on $(0,1)$ and by (ii) also in $[0,1]$. □

Remark 2.1. *Obviously the above theorem gives a generalization of the result from [3] (see also [5]). On the other hand the statement (ii) can be written as:*

$$f(A(e)) \le A_x[f(tx + (1-t)A(e))] \le A(f) \tag{5}$$

which represent a refinement of Jessen's inequality.

APPLICATION.If the function $f : [a,b] \longrightarrow \mathbb{R}$ is convex, $x_1, \ldots, x_n \in [a,b]$ and $p_1, \ldots, p_n$ are strictly positive weights, then denoting

$$m := \sum_{k=1}^{n} p_k x_k / \sum_{k=1}^{n} p_k,$$

we have the inequality

$$\sum_{k=1}^{n} p_k f(x_k) \ge \sum_{k=1}^{n} p_k f((x_k + m)/2).$$

Indeed, it follows from (5) for $A(f) := \sum_{k=1}^{n} p_k f(x_k) / \sum_{k=1}^{n} p_k$ and $t = 1/2$.

Remark 2.2. *Notice that, this inequality follows also from an inequality of Fuchs (see also [8]), so we get another proof of it.*

3 Iteration of Jessen's inequality

We will start with the following lemma:

Lemma 3.1. *If the function $f : [a, b] \longrightarrow \mathbb{R}$ is continuous convex and the functional A is linear and isotonic, then the function $G_t : [a, b] \longrightarrow \mathbb{R}$ given by*

$$G_t(x) := A_y[f(tx + (1 - t)y)]$$

is convex for all $t \in [0, 1]$.

The proof is similar to that one of Lemma 2.3 and we will omit the details.

Theorem 3.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous convex functions and A, B are two isotonic linear functionals with $A(1) = 1$ and $B(1) = 1$. Then one has the inequalities*

$$\begin{aligned} f(tA(e) + (1-t)B(e)) &\le B_y(f(tA(e) + (1-t)y)) \le \\ \le B_y(A_x(f(tx + (1-t)y)) &\le tA(f) + (1-t)B(f) \le \\ \le [(b - B(e))f(a) + (B(e) - a)f(b)]/(b - a) &+ \\ +t(B(e) - A(e))(f(a) - f(b))/(b - a). \end{aligned} \tag{6}$$

Proof. Applying the inequality (2) to the convex functions given by the previous lemmas we have:

$$A_x(f(tx + (1-t)y)) \ge f(tA(e) + (1-t)y)$$

and then

$$B_y(A_x(f(tx + (1-t)y))) \ge B_y(f(tA(e) + (1-t)y)) \ge f(tA(e) + (1-t)B(e)).$$

Thus, we get the first and the second inequality in (6).

Further on, from

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

we deduce succesively that

$$A_x(f(tx + (1-t)y)) \le tA(f) + (1-t)f(y)$$

and

$$B_y(A_x(f(tx + (1-t)y))) \le tA(f) + (1-t)B(f)$$

getting so the second inequality from (2). ∎

Corollary 3.1. *If $f : [a,b] \longrightarrow \mathbb{R}$ is a continuous convex function and A an isotonic linear functional with $A(1) = 1$, then*

$$f(A(e)) \leq A_y(f(tA(e) + (1-t)y)) \leq A_y(A_x(f(tx + (1-t)y)) \leq$$
$$\leq A(f) \leq [(b - A(e))f(a) + (A(e) - a)f(b)]/(b-a)$$

for all $t \in [0,1]$.

Remark 3.1. *These inequalities also give a refinement of Jenssen's inequality. they generalize some results from [1-5], given for the mapping from (3).*

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GENERAL MATHEMATICS
Vol.3, nr.3-4, 1995, 113-119

ON SOME INEQUALITIES OF A. M. FINK

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Abstract. A result from [1] is extended and some applications are given.

1991 Mathematical Subject Classification. Primary 26D15.

1. Let $g \in C[0,\infty)$ be such that $g(0)=0$, $g'(0+)$ exists and $g>0$ on $(0,\infty)$. Denote

$$A_1 = \{f \in C^1[0,\infty): \ f(0)=0, \ f'>0\}$$

and

$$A_2 = \{f \in C^2[0,\infty): \ f(0)=0, \ f' \geq 0, \ f''>0\}.$$

A.M. Fink has considered in [1] the inequalities

$$\frac{1}{g(f(t))}\int_0^t g(f(s))ds \leq K_j(g)\frac{1}{f(t)}\int_0^t f(s)ds \tag{1}$$

where $K_j(g)$ denotes the best possible constant for which (1) holds for all $f \in A_j$, $j = 1$ or $j = 2$. He proved that

$$K_1(g) = \sup_{u>0} \left\{ \frac{u}{g(u)} \sup_{0<v\le u} \frac{g(v)}{v} \right\} \tag{2}$$

and

$$K_2(g) = \sup_{u>0} \left\{ \frac{2u}{g(u)} \sup_{0<v\le u} \frac{G(v)}{v^2} \right\} \tag{3}$$

where $G(v) = \displaystyle\int_0^v g(s)ds$. If $K_j(g) = \infty$, there is no inequality (1).

This paper contains some extensions of these results and some applications to other inequalities.

2. Consider the inequalities

$$\frac{1}{g_1(f(t))} \int_0^t h_1(f(s))ds \le K_j \frac{1}{g_2 f(t)} \int_0^t h_2(f(s))ds \tag{4}$$

where K_j denotes the best possible constant for which (4) holds for all $f \in A_j$.

We proceed as in [1]. Let $u = f(t)$ and change the variables in the integrals by $v = f(s)$. One gets

$$\int_0^u \left[\frac{h_1(v)}{g_1(u)} - K_1 \frac{h_2(v)}{g_2(u)} \right] \frac{1}{f'(f^{-1}(v))} dv \le 0.$$

If f is in A_j, then $(f^{-1})'$ belongs to B_j, where

$$B_1 = \{h \in C[0,\infty),\ h > 0\}$$

and

$$B_2 = \{h \in C^1[0,\infty),\ h > 0,\ h' \le 0\}.$$

The following lemmas were also proved in [1].

Lemma 1. *Let p be a continuous function on $[a,b]$. Then*

$$\int_a^b p(v)h(v)dv \leq 0, \ \forall \ h \in B_1$$

if and only if

$$p(v) \leq 0, \ \forall \ v \in [a,b].$$

Lemma 2. *Let p be continuous. Then*

$$\int_a^b p(v)h(v)dv \leq 0, \ \forall \ h \in B_2$$

if and only if

$$p(v) = \int_a^b p(s)ds \leq 0, \ \forall \ v \in [a,b].$$

Using them we can deduce the values of K_j in (4).

Theorem 1. *Let $g_1, g_2, h_1, h_2 \in C[0,\infty)$ be positive on $(0,\infty)$. Then*

$$K_1 = \sup_{u>0} \left\{ \frac{g_2(u)}{g_1(u)} \sup_{0<v\leq u} \frac{h_1(v)}{h_2(v)} \right\}$$

and

$$K_2 = \sup_{u>0} \left\{ \frac{g_2(u)}{g_1(u)} \sup_{0<v\leq u} \frac{H_1(v)}{H_2(v)} \right\}$$

where

$$H_j(v) = \int_0^v h_j(s)ds.$$

If $K_j = \infty$, the inequality (4) does not exist.

Remark 1. Multivariate versions of the above results can also be obtained, but we omit the details.

Remark 2. Consider the reverse inequality in (1):

$$\frac{1}{f(t)}\int_0^t f(s)ds \le k_j(g)\frac{1}{g(f(t))}\int_0^t g(f(s))ds, \quad j=1,2.$$

Apply Theorem 1 with $g_1(t)=h_1(t)=t$, $g_2(t)=h_2(t)=g(t)$. We deduce that

$$k_1(g)=\sup_{u>0}\left\{\frac{g(u)}{u}\sup_{0<v\le u}\frac{v}{g(v)}\right\}$$

and

$$k_2(g)=\sup_{u>0}\left\{\frac{g(u)}{2u}\sup_{0<v\le u}\frac{v^2}{G(v)}\right\}.$$

In particular, let $g(t)=t^a$. If $0<a\le 1$, then $k_1(g)=1$ and $k_2(g)=\frac{1+a}{2}$. It is known from [1] that for $a\ge 1$,

$$K_1(g)=1 \text{ and } K_2(g)=\frac{2}{1+a}.$$

More generally, we have the following

Corollary 1. *If $a>b$, then for every $f\in A_j$ $(j=1,2)$ holds*

$$\frac{1}{f^a(t)}\int_0^t f^a(s)ds \le K_j\frac{1}{f^b(t)}\int_0^t f^b(s)ds,$$

where $K_1=1$ and $K_2=\frac{b+1}{a+1}$.

3. We also obtain an interesting consequence of the Theorem 1 if we choose $g_j=H_j$ for $j=1,2$.

Corollary 2. *Let $h_1,h_2\in C[0,\infty)$ be positive on $(0,\infty)$. Then for all $f\in A_j$ holds*

$$\frac{\int_0^t h_1(f(s))ds}{\int_0^{f(t)} h_1(s)ds} \le K_j\frac{\int_0^t h_2(f(s))ds}{\int_0^{f(t)} h_2(s)ds}$$

where the best possible constants K_j are

$$K_1 = \sup_{u>0} \left\{ \frac{H_2(u)}{H_1(u)} \sup_{0<v\leq u} \frac{h_1(v)}{h_2(v)} \right\}$$

respectively

$$K_2 = \sup_{u>0} \left\{ \frac{H_2(u)}{H_1(u)} \sup_{0<v\leq u} \frac{H_1(v)}{H_2(v)} \right\}$$

where

$$H_j(u) = \int_0^u h_j(u)dv, \quad j = 1, 2.$$

We remark that if $\frac{H_1}{H_2}$ is increasing, then $K_2 = 1$. This happens, for instance, if $\frac{h_1}{h_2}$ is increasing. Indeed, we have

$$\left[\frac{H_1(v)}{H_2(v)}\right]' = \frac{h_2(v)}{H_2(v)} \left[\frac{h_1(v)}{h_2(v)} - \frac{H_1(v)}{H_2(v)}\right] \geq 0.$$

4. We can apply the same method to produce analogous inequalities on an arbitrary interval $[a, b]$. We use only the sufficiency part of the lemmas, so that the constants are not necessarily optimal.

Let $h_i : [c, d] \to \mathbb{R}$ for $i = 1, 2$, be continuous functions. Denote

$$A_1(a, b) = \{f : [a, b] \to [c, d];\ f' > 0\}$$

and

$$A_2(a, b) = \{f : [a, b] \to [c, d];\ f' > 0,\ f'' \geq 0\}.$$

We have

Theorem 2. *If $h_2 > 0$, then for all $f \in A_j(a, b)$, $j = 1, 2$, holds*

$$\int_a^b h_1(f(s))ds \leq M_j \int_a^b h_2(f(s))ds$$

where

$$M_1 = \sup_{c \le v \le d} \frac{h_1(v)}{h_2(v)}$$

and

$$M_2 = \sup_{c \le v \le d} \frac{H_1(v)}{H_2(v)}.$$

Proof. As in the proof of Theorem 1, making the change of variable $v = f(s)$, we have

$$\begin{aligned} D_j &= \int_a^b h_1(f(s))ds - M_j \int_a^b h_2(f(s))ds \\ &= \int_{f(a)}^{f(b)} [h_1(v) - M_j h_2(v)][f^{-1}(v)]' dv. \end{aligned}$$

So $D_1 \le 0$ if $h_1(v) \le M_1 h_2(v)$ for all v, which gives the value of M_1. As in the proof of Lemma 2, for $f \in A_2(a,b)$, the mean value theorem gives an $u \in (a,b)$ such that

$$\begin{aligned} D_2 &= \frac{1}{f'(a)} \int_{f(a)}^{f(u)} [h_1(v) - M_2 h_2(v)]dv \\ &\quad + \frac{1}{f'(b)} \int_{f(u)}^{f(b)} [h_1(v) - M_2 h_2(v)]dv \\ &= \left[\frac{1}{f'(a)} - \frac{1}{f'(b)}\right] \int_{f(a)}^{f(u)} [h_1(v) - M_2 h_2(v)]dv \\ &\quad + \frac{1}{f'(b)} \int_{f(a)}^{f(b)} [h_1(v) - M_2 h_2(v)]dv. \end{aligned}$$

Thus, $D_2 \le 0$ if

$$\int_{f(a)}^{f(u)} h_1(v)dv \le M_2 \int_{f(a)}^{f(u)} h_2(v)dv$$

which justifies the given value of M_2. Following the lines of the above proof, we deduce also the next

Corollary 3. *If* $\dfrac{H_1}{H_2}$ *is increasing, then for every* $f \in A_2(a,b)$ *holds*

$$\frac{\displaystyle\int_a^b h_1(f(s))ds}{\displaystyle\int_{f(a)}^{f(b)} h_1(s)ds} \leq \frac{\displaystyle\int_a^b h_2(f(s))ds}{\displaystyle\int_{f(a)}^{f(b)} h_2(s)ds}.$$

Remark 3. As we have seen in the paragraph 3, if $\dfrac{h_1}{h_2}$ is increasing, so is also $\dfrac{H_1}{H_2}$. Thus Corollary 3 improves an inequality from [3], where it is given also a history of this result which started as a problem proposed in [2] for a special case.

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Some inequalities with integral means

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Abstract

We generalize some inequalities of Seiffert, Pearce, Pecarić and Dragomir.

AMS Subject Classification 26D15

1 Introduction

For $0 < a < b$, let us denote by

$$G(a,b) = (ab)^{\frac{1}{2}} \quad , \quad A(a,b) = \frac{a+b}{2}$$

and

$$H(a,b) = \frac{2ab}{a+b} \ ,$$

the geometric mean, the arithmetic mean respectively the harmonic mean of a and b.

In [5] H.J. Seiffert proved the following result.

Theorem A. *Let $f : [a,b] \to \mathcal{R}$ be a Riemann–integrable positive function and $g : [G(a,b), A(a,b)] \to \mathcal{R}$ a strictly positive* *increasing function. Then the inequality*

$$g(G(a,b)) < \frac{\int\limits_a^b f(t)g(h(t))dt}{\int\limits_a^b f(t)dt} < g(A(a,b))$$

holds, where $h(t) = G(t, a+b-t)$.

This result was generalized in [3] where, in the definition of h, the geometric mean was replaced by other concrete means (power means, extended logarithmic means or integral power means). We go further with this generalization considering some abstract means and also replacing the integral by a functional. So the results remember those from [6].

We deal also with some results of other type. In [1] some inequalities related to Hermite–Hadamard's inequality are proved.

Theorem B. *Let* f *be a differentiable convex function on* $[a, b]$.

i) If $f'(G(a,b)) \geq 0$, *then*

$$\frac{1}{b-a}\int\limits_a^b f(x)dx \geq f(G(a,b)) \ .$$

ii) If $f'(H(a,b)) \geq 0$ *then*

$$\frac{1}{b-a}\int\limits_a^b f(x)dx \geq f(H(a,b)) \ .$$

In what follows we generalize also these results in more directions.

2 A property of some means

Let $p : [a, b] \to \mathcal{R}$ be a positive Riemann–integrable function. As it is known, the expression

$$m_p(a, b) = \frac{1}{b-a} \int_a^b p(t)dt$$

represents the integral arithmetic mean of p on $[a, b]$. If p is strictly increasing, then

$$M_p(a, b) = p^{-1}(m_p(a, b))$$

defines a mean of a and b. For example, taking $p(t) = t^r$ we get the extended logarithmic mean.

We want to study the functions:

$$h_p(x) = m_p(x, a + b - x)$$

and

$$H_p(x) = M_p(x, a + b - x) \ .$$

They are symmetric, so that we study them only on $[a, A(a, b)]$.

Theorem 1 . *If p is a convex function on $[a, b]$ then*

$$p(A(a, b)) \leq h_p(x) \leq m_p(a, b) \quad , \quad a \leq x \leq b \ .$$

Proof. We have

$$h_p'(x) =$$

$$= \frac{2}{a+b-2x}\left[\frac{1}{a+b-2x}\int_x^{a+b-x} p(t)dt - \frac{p(a+b-x)+p(x)}{2}\right] .$$

It follows from Hermite–Hadamard's inequality (see [6]) that h' is negative on $[a, A(a, b)]$ for p convex. Thus h_p is decreasing on $[a, A(a, b)]$ which gives the desired inequalities.

Corollary 1 . *If p is a strictly increasing convex function then*

$$A(a,b) \le H_p(x) \le M_p(a,b) \quad , \textit{ for all } x \ .$$

Let us also considere some means defined otherwise. Let q be a positive function on $[a,b]$. The expression

$$n_q(a,b) = \frac{q(a)+q(b)}{2}$$

represents the discrete arithmetic mean of q on [a,b]. If a is strictly increasing, then

$$N_q(a,b) = q^{-1}(n_q(a,b))$$

defines a quasi–arithmetic mean of the numbers a and b. We study also the functions

$$k_q(x) = n_q(x, a+b-x)$$

and

$$K_q(x) = N_q(x, a+b-x) \ .$$

Theorem 2 . *If q is a convex function on $[a,b]$, then*

$$q(A(a,b)) \le k_q(x) \le n_q(a,b) \quad , \quad a \le x \le b \ .$$

Proof. For the first inequality we have

$$q(A(a,b)) = q\left(\frac{x+a+b-x}{2}\right) \le$$

$$\frac{q(x)+q(a+b-x)}{2} = n_q(x, a+b-x) \ .$$

The second inequality follows from

$$q(x) \le \frac{b-x}{b-a} q(a) + \frac{x-a}{b-a} q(b)$$

and

$$q(a+b-x) \le \frac{x-a}{b-a} q(a) + \frac{b-x}{b-a} q(b) \ .$$

Corollary 2 *If q is a strictly increasing convex function, then*

$$A(a,b) \le K_q(x) \le N_q(a,b) \quad , \quad a \le x \le b \ .$$

3 The result of Seiffert

We generalize the result of Theorem A for an increasing functional, i.e. a functional $L : C[a,b] \to R$ with the property

$$L(f) \le L(g) \quad \text{if} \quad f(x) \le g(x) \quad , \quad a \le x \le b.$$

Common examples of such functionals are the isotonic linear functionals (see the next paragraph) but here we have other examples.

The monotony of L implies that if

$$m \le f(x) \le M \quad , \quad \text{for} \ \ a \le x \le b$$

then

$$L(m) \le L(f) \le L(M) \ .$$

So we have the following properties.

Corollary 3 *If L is an increasing functional on $C[a,b]$ and p a convex function on $[a,b]$, then*

$$L(p(A(a,b))) \le L(h_p) \le L(m_p(a,b)) \ .$$

Corollary 4 *If L is an increasing functional on $C[a,b]$ and q a convex function on $[a,b]$, then*

$$L(q(A(a,b))) \le L(k_q) \le L(n_q(a,b)) \ .$$

Corollary 5 *If L is an increasing functional on $C[a,b]$ and p a strictly increasing convex function on $[a,b]$, then*

$$L(A(a,b)) \leq L(H_p) \leq L(M_p(a,b)) \ .$$

Corollary 6 *If L is an increasing functional on $C[a,b]$ and q is a strictly increasing convex function on $[a,b]$, then*

$$L(A(a,b)) \leq L(K_q) \leq L(N_q(a,b)) \ .$$

Example. Let f be a Riemann integrable positive function on $[a,b]$ and g be a strictly increasing continuous function on $[c,d]$. If for continuous functions $h:[a,b]\to[c,d]$ we define

$$L_g(h) = \frac{\int_a^b f(t)g(h(t))dt}{\int_a^b f(t)dt}$$

we get an increasing functional L_g. So Corollary 3 gives the following result. If p is a convex function on $[a,b]$, then

$$g(p(A(a,b))) \leq L_g(h_p) \leq g(m_p(a,b)) \ . \tag{1}$$

If moreover p is supposed to be strictly increasing, replacing g by $g(p^{-1})$, we get the inequality

$$g(A(a,b)) \leq L_g(H_p) \leq g(M_p(a,b)) \ .$$

For $p(x) = x^r$ with $r > 1$ we get the Theorem 2 from [3]. If we take in (1) $p(x) = w^r(x)$ and replace $g(x)$ by $g(x^{\frac{1}{r}})$, we have Theorem 3 from [3]. Analogously, from Corollary 4, we deduce that if q is a convex function on $[a,b]$, then

$$g(q(A(a,b))) \leq L_g(k_q) \leq g(n_q(a,b)) \ .$$

If q is also strictly increasing, replacing g by $g(q^{-1})$, we have

$$g(A(a,b)) \leq L_g(K_q) \leq g(N_q(a,b)) \ ,$$

which for $q(x) = x^r$, $r > 1$, gives an improvement of Theorem 1 from [3] because we have renonced at the assumption of differentialbility of g.

Remark 1 *If the convexity and/or the increasing monotony is repleaced by concavity respectively by decreasing monotony, we get the same or the reversed inequalities.*

4 The result of Dragomir

First of all let us generalize the result from [1] for an isotonic linear functional L. That is, let L be a functional defined on $C[a, b]$ with the properties

$$L(\alpha f+\beta g) = \alpha L(f)+\beta L(g)\ ,\ \forall\alpha, \beta \in \mathcal{R}, \beta \in \mathcal{R}, \forall f, g \in C[a, b]$$

and

$$L(f) \geq 0 \quad , \quad \forall f \in C[a, b]\ ,\ f \geq 0\ .$$

We make also the unessential assumption $L(1) = 1$, where the first 1 is the constant function with the values 1. We have the following improvement of of Jessenś inequality (see [2]).

Theorem 3 *Il L is an isotonic linear functional on $C[a, b]$ and f is a diferentiable convex function on $[a, b]$, then*

$$L(f) \geq \sup\{f(t) : [L(e) - t]f'(t)\} \geq f(L(e))$$

where $e(x) = x$, for x in $[a, b]$.

Proof. We use the same basic inequality as in [1]. The function f being differentiable convex, then

$$f(x) \geq f(t) + (x - t)f'(t) \quad , \quad \forall x, t \in [a, b].$$

Of course L is increasing and if we apply it for the functions of variable x we get

$$L(f) \geq f(t) + [L(e) - t]f'(t)$$

which gives the desired result.

Corollary 7 *If L and f satisfy the above conditions and $t \leq L(e)$, $f'(t) \geq 0$ then $L(f) \geq f(t)$.*

We remark that for

$$L(f) = A(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

we have $L(e) = A(a,b)$. So if we denote by P_r the power mean defined by

$$P_r = P_r(a,b) = \left(\frac{a^r + b^r}{2}\right)^{\frac{1}{r}}$$

for $r \neq 0$ and $P_0 = G$, it is known that for $r < 1$, $P_r < A$. Thus

$$f'(P_r(a,b)) \geq 0 \quad \Rightarrow \quad A(f) \geq f(P_r(a,b)).$$

For $r = 0$ and $r = -1$ we get Theorem B.

For this last case of $A(f)$, we give also another generalization of this theorem. It is based on the following result from [4].

Theorem C. *Let f be a differentiable convex function on $[a,b]$ and*

$$c = \inf\{x \in [a,b] : f'(x) \geq 0\} \ .$$

Then

$$A(f) = \frac{1}{b-a}\int_a^b f(x)dx \geq \max\{f(x) : x \in I_f\}$$

where

$$I_f = \left[\frac{a+b}{2} - \frac{(b-c)^2}{2(b-a)}\ , \ \frac{a+b}{2} + \frac{(c-a)^2}{2(b-a)}\right] \ .$$

We deduce the following consequence.

Theorem 4 *Let f be a differentiable convex function on $[a,b]$. If for $r<1$,*

$$f'(b-\sqrt{2(b-a)(A-P_r)})\geq 0$$

or for $r>1$,

$$f'(a+\sqrt{2(b-a)(P_r-A)})\geq 0$$

then

$$A(f)\geq f(P_r(a,b))\ .$$

Proof. The conditions assure that $P_r\in I_f$ and so the result follows from Theorem C.

Corollary 8 *Let f be a differentiable convex function on $[a,b]$. Then*

$$f'(b-\sqrt{b-a}(\sqrt{b}-\sqrt{a}))\geq 0 \quad \Rightarrow \quad A(f)\geq f(G(a,b)).$$

and

$$f'\left(b-(b-a)\sqrt{\frac{b-a}{b+a}}\right)\geq 0 \quad \Rightarrow \quad A(f)\geq f(H(a,b))\ .$$

Remark 2 *As*

$$b-\sqrt{b-a}(\sqrt{b}-\sqrt{a})\geq\sqrt{ab}$$

and

$$b-(b-a)\sqrt{\frac{b-a}{b+a}}\geq\frac{2ab}{(a+b)},$$

the corollary 8 improves Theorem B.

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STUDIA UNIV. "BABES-BOLYAI", MATH., 42(1997), 2,103-108.

ON THE INEQUALITY OF HERMITE-HADAMARD

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Abstract. One considers a notion of convexity with respect of a function h, called h- convexity. One improves the Hermite-Hadamard inequality for functions with h-convex inverse, generalizing a result of H.-J.Seiffert.

1 Introduction

We consider a notion of convexity with respect to a function h, called h-convexity.We improve Hermite-Hadamard's inequality for functions with h-convex inverse. For $h(x) = x^r$ we obtain the result from [9] which includes the results of H.-J.Seiffert from [8] and that of H.Alzer from [1].The proof is like that of [1] and not like those of [8] and [9].

To formulate them we need more definitions on functionals and means.

2 Functionals

Let E be a nonempty set and $F(E)$ be a linear space of real-valued functions defined on E. A functional $T : F(E) \longrightarrow \mathbb{R}$ is linear if:

$$T(tf + sg) = tT(f) + sT(g),\ \forall t, s \in \mathbb{R},\ f, g \in F(E).$$

It is isotonic if:

$$T(f) \geq 0,\ \forall f \in F(E),\ f \geq 0.$$

We shall suppose also that $T(1) = 1$, where the first "1" denotes the constant function $f(x) = 1,\ \forall x \in E$. Here as in what follows, if we use $T(f)$, we assume that $f \in F(E)$.

Common examples of such functionals are given by:

$$T(f) = \int_E f dm / \int_E dm$$

and

$$T(f) = \sum_{k=1}^{n} p_k f(x_k) / \sum_{k=1}^{n} p_k$$

where m is a positive measure on E and $p_k > 0,\ x_k \in E$ for $k = 1, \ldots, n$.

A.Lupas has generalized in [4] Hermite-Hadamard's inequality for isotonic linear functionals but we need it in a more general form given in [2].

Theorem 2.1. *If the function f is convex on $[c, d]$ and the functional T is isotonic and linear on $F(E)$, with $T(1) = 1$, then for every function $g : E \longrightarrow [c, d]$ we have $T(g) \in [c, d]$ and:*

$$f(T(g)) \leq T(f(g)) \leq [(d - T(g))f(c) + (T(g) - c)f(d)]/(d - c). \qquad ((1))$$

For $E = [a, b] = [c, d]$ and $g(x) = x$ we get the result from [4]. In the special case when T is the integral arithmetic mean W, defined for a continuous on $[a, b]$ function f by:

$$W(f; a, b) = \frac{1}{b - a} \int_a^b f(x) dx$$

the inequality (1) becomes Hermite-Hadamard's inequality:

$$f((a + b)/2) \leq W(f; a, b) \leq [f(a) + f(b)]/2 \qquad ((2))$$

3 Means

In what follows we use some quasi-arithmetic means. If h is a positive strictly monotone function defined on the set of positive numbers and $t \in [0, 1]$, we denote:

$$A_{h,t}(x, y) = h^{-1}(th(x) + (1 - t)h(y)).$$

If h is the identity function we get the usual weighted arithmetic mean A_t, which for $t = 1/2$ becomes the arithmetic mean A.

For $h(x) = e_r(x) = x^r,\ r \neq 0$, we have the power means:

$$P_{r,t}(x,y) = (tx^r + (1-t)y^r)^{1/r}.$$

For $r = 0$ one takes $h(x) = e_0(x) = \log x$, getting the (weighted) geometric mean:

$$P_{0,t}(x,y) = G_t(x,y) = x^t y^{1-t}.$$

It is easy to verify (see [3]) that:

$$A_{h,t}(x,y) \leq A_{g,t}(x,y),\ \forall x,y > 0,\ 0,t \in [0,1] \qquad ((3))$$

if and only if:

i) g is increasing and $g(h^{-1})$ is convex, or:

ii) g is decreasing and $g(h^{-1})$ is concave.

As shown by J.G.Mikusinski (see [3],p.31) if g and h are twice differentiable and $g^{'}, h^{'}$ are never zero, then the above conditions hold if and only if:

$$g^{''}/g^{'} \geq h^{''}/h^{'}.$$

In the special case of the power means we see that they are increasing, that is:

$$P_{r,t}(x,t) < P_{s,t}(x,y) \quad if \quad r < s,\ t \in (0,1),\ x \neq y.$$

We use also the family of generalized logarithmic means defined for r different from -1 and 0 by:

$$L_r(x,y) = [(y^{r+1} - x^{r+1})/((r+1)(y-x))]^{1/r}$$

but

$$L_0(x,y) = I(x,y) = (1/e)(y^y/x^x)^{1/(y-x)}$$

is the identical mean, and

$$L_{-1}(x,y) = L(x,y) = (y-x)/(\log y - \log x)$$

the logarithmic mean. For $y = x$ all the means have the value x. This family is also increasing:

$$L_r(x-y) < L_s(x,y) \quad if \quad r < s, x \neq y. \qquad ((4))$$

4 Generalized convexity

Using the quasi-arithmetic means we can define a notion of convexity generalizing the logarithmic convexity.

Definition 4.1. *The positive function $f \in C[a,b]$ is called h-convex if:*

$$f(A_t(x,y)) \leq A_{h,t}(f(x), f(y)), \ \forall x, y \in [a,b].$$

In addition to the usual convexity (with $h = e_1$) and the logarithmic convexity (where $h = \log = e_0$), C.Das has considered in his Ph.D.Thesis (see [5]) the case of harmonic convexity by taking $h = e_{-1}$. The notion of e_r-convexity was considered in [9] under the name of r-convexity.

Of course, the function f is h-convex if and only if $h(f)$ is convex for h increasing and concave for h decreasing. So (3) holds if and only if h^{-1} is g-convex. Thus, the above definition is in concordance with that of logarithmic convexity but differs from a definition accepted in [3, pp.30-31].

>From the above remarks we deduce that every h-convex function is also g-convex if and only if h^{-1} is g-convex. In the special case of the power means it follows that if $r < s$ every e_r-convex function is also e_s- convex. This is generalizes the relation between logarithmic convexity and convexity.

5 A result of Seiffert

In what follows we suppose that $0 < a < b$. In [8] H.-J. Seiffert proved that if $f' \in C[a,b]$ is strictly increasing and f^{-1} is log-convex then:

$$W(f,a,b) \leq f(I(a,b)). \tag{(5)}$$

We remark that if f^{-1} is log-convex then f is also concave but (5) improves the corresponding inequality from (2) because by (4):

$$I = L_0 < L_1 = A.$$

Also, H. Alzer proved in [1] a related result: if $f \in C[a,b]$ is strictly increasing and $1/f^{-1}$ is convex, then:

$$W(f;a,b) \geq f(L(a,b)).$$

The result of H.-J. Seiffert is related to e_0-convexity and that of H. Alzer to e_{-1}-concavity. In what follows we shall generalize these results.

Theorem 5.1. *If the function $f : [a,b] \longrightarrow [c,d]$ and $h : [a,b] \longrightarrow \mathbb{R}$ are strictly increasing, f^{-1} is h-convex and the functional T is isotonic and linear on $F([a,b])$, with $T(1)=1$, then:*

$$h(a) \leq T(h) \leq h(b)$$

and

$$\frac{f(a)[h(b)-T(h)]+f(b)[T(h)-h(a)]}{h(b)-h(a)} \leq T(f) \leq f(h^{-1}(T(h))). \qquad ((6))$$

Proof. In (1) we put $c=f(a), d=f(b), h(f^{-1})$ for f and f for g obtaining:

$$h(f^{-1}(T(f))) \leq T(h) \leq \frac{[f(b)-T(f)]h(a)+[T(f)-f(a)]h(b)}{f(b)-f(a)}.$$

Extracting from each inequality $T(f)$ we get (6).□

Remark 5.1. *If h^{-1} is g-convex, (6) gives:*

$$\frac{T(h)-h(a)}{h(b)-h(a)} \geq \frac{T(g)-g(a)}{g(b)-g(a)}$$

and

$$h^{-1}(T(h)) \leq g^{-1}(T(g)).$$

So if we pass from g-convexity to h-convexity the class of functions for which (6) is valid is diminished but the evaluations are improved.

Consequence 5.1. *If the function $f \in C[a,b]$ is strictly increasing and f^{-1} is log-convex then:*

$$\frac{f(a)[L(a,b)-a]+f(b)[b-L(a,b)]}{b-a} \leq W(f;a,b) \leq f(I(a,b)). \qquad ((7))$$

Proof. We have $h=\log$ and

$$W(\log;a,b) = \frac{b\log b - a\log a}{b-a} = 1$$

so that (6) gives (7).

We remark that (7) offers a companion inequality to Seiffert's inequality (5). □

Consequence 5.2. *If the function $f \in C[a,b]$ is strictly increasing and f^{-1} is e_r-convex, with $r \neq 0$, then:*

$$\frac{f(a)[b^r - L_r^r(a,b)] + f(b)[L_r^r(a,b) - a^r]}{b^r - a^r} \leq W(f;a,b) \leq f(L_r(a,b)).$$

Remark 5.2. *This result was proved otherwise in [9]. As we have shown there, the conditions of the consequences are catisfied by twice differentiable functions f if and only if:*

$$f'(x) > 0 \quad \textit{and} \quad 1 + \frac{xf''(x)}{f'(x)} \leq r, \ \forall x \in [a,b]. \tag{(8)}$$

We obtain so a class of functions which can be very interesting because the second relation of (8) is analogous with that satisfied by complex convex functions (see [7], pp.255-256).

Also the relation (8) shows that an inequality of J.D.Kečkic and I.B.Lackovic (see [6], pp.367-368) can be deduced from (6).

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STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA
Volume XLII, Number 3, September 1997, 65-80

REAL STAR-CONVEX FUNCTIONS

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ABSTRACT. This paper contains a survey of the properties of a class of real functions, which is intermediate between the class of convex functions and the class of starshaped functions. We present some known as well as new results or new proofs and examples.

1991 Mathematics Subject Classification: 26A51, 26A16.
Keywords and phrases: starlikeness, convexity.

1. INTRODUCTION

Let $\mathbb{R}$ be the real axis and let $I \subseteq \mathbb{R}$ be an interval (closed or not, bounded or not). A function $f : I \to \mathbb{R}$ is said to be *convex* on I if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \tag{1}$$

for all $x, y \in I$ and all $\lambda \in [0,1]$.

The function f is called *starshaped* on I if

$$f(\lambda x) \leq \lambda f(x), \tag{2}$$

for all $x \in I$ and all $\lambda \in [0,1]$. For $\lambda = 0$ we get $f(0) \leq 0$, which also implies $0 \in I$.

The aim of this survey paper is to analyze an intermediate concept, which connects the property of convexity with that of starshapedness by means of a parameter $\alpha \in [0,1]$. This concept was introduced in [9] and it was inspired by the notion of α-convexity defined for complex functions in [3]. We shall present here some results obtained in [1], [4], [7], [9] and [10] as well as some new results or new proofs.

2. α-Star-convex functions

We begin with the definition and some general properties of α-star-convex functions.

Definition 1. [9] Given $\alpha \in [0,1]$, the function $f : I \to \mathbb{R}$ is said to be *α-star-convex* on I if

$$f(\lambda x + (1-\lambda)\alpha y) \leq \lambda f(x) + (1-\lambda)\alpha f(y), \tag{3}$$

for all $x, y \in I$ and all $\lambda \in [0,1]$.

Remark 1. If $\alpha = 1$, then (3) reduces to (1), i.e. an 1-star-convex function is convex. If $\alpha = 0$, then (3) reduces to (2), i.e. a 0-star-convex function is starshaped. As in this last case, in [10] it was shown that it is natural to put the conditions

$$0 \in I \text{ and } f(0) \leq 0. \tag{4}$$

In fact, taking $x = y = 0$ from (3) we get the second part of (4) but only for $\alpha \neq 1$. Remark that $y \in I$ implies $\alpha y \in I$ and so for $\alpha \in (0,1)$ we have $(0, y] \subseteq I$. This gives

Lemma 1. *If f is α-star-convex on I, $0 \in I$, then f is starshaped on I.*

Proof. For any $x \in I$ and $\lambda \in [0,1]$ we have

$$f(\lambda x) = f(\lambda x + (1-\lambda)\alpha \cdot 0) \leq \lambda f(x) + (1-\lambda)\alpha f(0) \leq \lambda f(x). \quad \square$$

Theorem 1. *If f is α-star-convex on I, $0 \in I$ and $0 \leq \beta \leq \alpha$, then f is also β-star-convex.*

Proof. If $x, y \in I$ and $\lambda \in [0,1]$, then by using Lemma 1 we deduce

$$f(\lambda x + (1-\lambda)\beta y) = f\left(\lambda x + (1-\lambda)\alpha \frac{\beta y}{\alpha}\right)$$

$$\leq \lambda f(x) + (1-\lambda)\alpha f\left(\frac{\beta}{\alpha} y\right) \leq \lambda f(x) + (1-\lambda)\beta f(y). \quad \square$$

Remark 2. A.W. Roberts and D.E. Varberg [6] defined the class of functions $f : I \to \mathbb{R}$ that satisfi the condition

$$f(sx + ty) \leq sf(x) + tf(y)$$

for all $x, y \in I$ and all (s,t) in a given set M. Note that for example Jensen convexity corresponds to $M = \{(1/2, 1/2)\}$, superadditivity corresponds to $M = \{(1,1)\}$ and α-star-convexity is also of this type, with M given by the segment joining the points $A(1,0)$ and $B(0,\alpha)$.

Remark 3. The concept of α-star-convexity has the following geometric interpretation. If $y \in I$ is fixed and if we consider the point $M = M(\alpha y, \alpha f(y))$, then for all $x \in I$ the graph Γ_f of the function f in the interval $[x, \alpha y]$ or $[\alpha y, x]$ lies under the segment MP, where $P = P(x, f(x))$. This means that Γ_f is starshaped with respect to the point M (see Figure 1).

In view of Theorem 1, in [4] it was given the following definition.

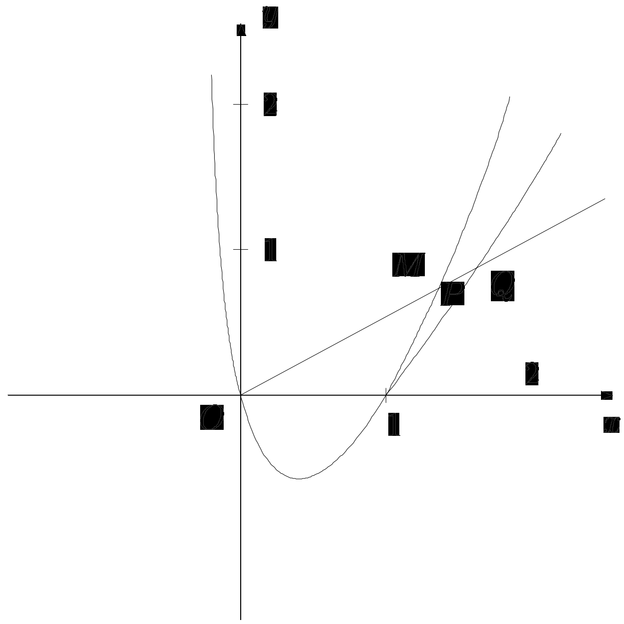

FIGURE 1

Definition 2. Given a starshaped function $f : I \to \mathbb{R}$ we define the *order of star-convexity* of f by

$$\alpha = \alpha^*[f] = \sup\{\beta : f \text{ is } \beta - \text{star-convex on } I\}. \tag{5}$$

In this case we say that f is *star-convex of order* α.

Remark 4. The geometric interpretation mentioned in Remark 3 allows us to obtain the order of star-convexity of the function f given by (5) in the following way. Take a point $P \in \Gamma_f$ and starting from $O = O(0, 0)$ let consider the point M on the segment OP at a longest distance from O with the property that the graph Γ_f is starshaped with respect to M.

Then

$$\alpha = \alpha^*[f] = \inf\left\{\frac{OM}{OP} : \ P \in \Gamma_f\right\}. \tag{6}$$

Given $\alpha \in [0,1]$ a natural problem is to find a function f such that $\alpha^*[f] = \alpha$. The answer to this problem is given by the following simple example [4].

Example 1. Let $\alpha \in (0,1]$ and let $f : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -x, & \text{if } \ 0 \le x \le 1; \\ x-2, & \text{if } \ 1 \le x \le 2; \\ \dfrac{\alpha}{2-\alpha}(x-2), & \text{if } \ 2 \le x \le \dfrac{2+\alpha}{\alpha}, \\ 1 + a\left[x - \dfrac{2+\alpha}{\alpha}\right], & \text{if } \ \dfrac{2+\alpha}{\alpha} \le x, \ a > 1. \end{cases}$$

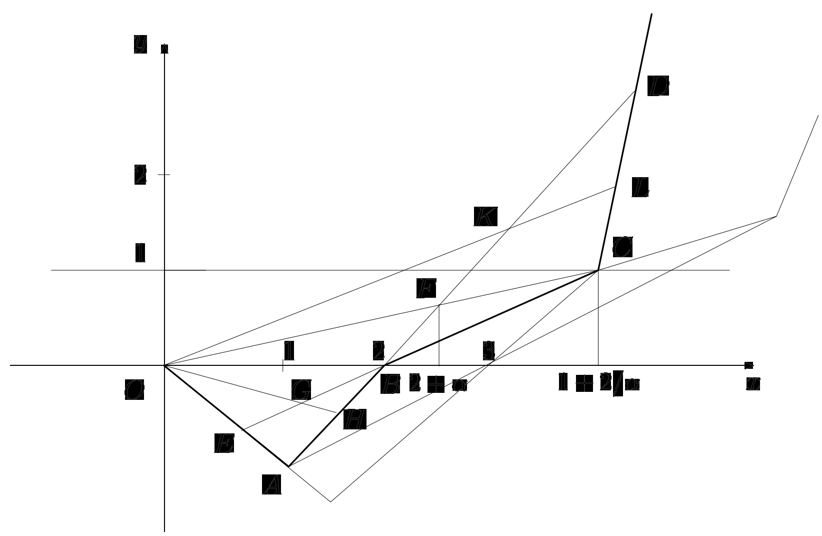

FIGURE 2

If $\alpha = 0$, then we take $f(x) = 0$, for $x \geq 2$. The graph Γ_f is given in Figure 2.

By using (5) and some elementary geometric considerations we easily find that $\alpha^*[f] = \alpha$. In Figure 2 we have $OE/OA = OF/OC = \alpha$ and $OK/OL > OF/OC$, $OG/OH > OE/OA$.

Remark 5. If $\alpha > 1$ the only functions with $f(0) = 0$ which are α-star-convex are of the form $f_0(x) = ax$ and in this case $\alpha^*[f_0] = \infty$. Hence the significant range for α in Definition 1 is the interval $[0, 1]$.

Remark 6. As in the case of convex functions, in [1] the following inequality of Jensen type is given: If $f : I \to \mathbb{R}$ is an α-star-convex function with condition (4) then for all $p_i \geq 0$, with $\sum_{i=0}^{n} p_i = 1$ and all $x_i \in I$, $i = 0, 1, \ldots, n$, we have

$$f(p_0x_0 + \alpha p_1x_1 + \cdots + \alpha^n p_nx_n) \leq p_0f(x_0) + \alpha p_1f(x_1) + \cdots + \alpha^n p_nf(x_n).$$

3. The boundedness of star-convex functions

It is known that a convex function is bounded on every compact interval but a starshaped function is not. Let us study the boundedness of α-star-convex functions.

Lemma 2. *If the function f is starshaped on $[0, b]$, then it is bounded from above by $M = \max\{0, f(b)\}$.*

Proof. For every $x \in [0, b]$, there is a $t \in [0, 1]$ such that $x = tb$. So we have

$$f(x) \leq tf(b) \leq M. \quad \square$$

Analogously we can prove the boundedness from above on $[a, 0]$ and thus to deduce

Theorem 2. *If the function f is α-star-convex on I, with $\alpha \in [0,1]$, then it is bounded from above on every closed interval of I.*

It is easy to find examples of starshaped functions which are not bounded from below, but for α strictly positive we have the following result.

Theorem 3. *If the function $f : I \to \mathbb{R}$ is α-star-convex, with $\alpha \in (0,1]$, then it is also bounded from below on every closed interval $[a,b] \subseteq I$.*

Proof. We have

$$\begin{aligned} f\left(\frac{a+\alpha b}{2}\right) &= f\left(\frac{1}{2}(a+\alpha t) + \frac{1}{2}\alpha(b-t)\right) \\ &\leq \frac{1}{2} f(a+\alpha t) + \frac{1}{2}\alpha f(b-t). \end{aligned}$$

If $t \in [0, b-a]$, we have $a + \alpha t \in [a, a+\alpha(b-a)] \subseteq [a,b]$, so that if we denote by M the upper bound of f on $[a,b]$, we get

$$\begin{aligned} f(b-t) &\geq \frac{2}{\alpha}\left[f\left(\frac{a+\alpha b}{2}\right) - \frac{1}{2} f(a+\alpha t)\right] \\ &\geq \frac{2}{\alpha}\left[f\left(\frac{a+\alpha b}{2}\right) - \frac{M}{2}\right] = m, \end{aligned}$$

hence m is a lower bound of f on $[a,b]$. □

4. The Lipschitz continuity of α-star-convex functions

It is easy to observe that a function $f : [a,b] \to \mathbb{R}$, with $0 \in [a,b]$ is starshaped on $[a,b]$ if and only if f can be written in the form

$$f(x) = \begin{cases} x g_+(x), & \text{if} \quad x \in (0,b], \\ f(0), & \text{if} \quad x = 0, \\ x g_-(x), & \text{if} \quad x \in [a,0), \end{cases}$$

where $f(0) \leq 0$, $g_+ : (0,b] \to \mathbb{R}$ and $g_- : [a,0) \to \mathbb{R}$ are increasing functions on $(0,b]$ and $[a,0)$ respectively. From this representation it immediately follows that f has at most a countable number of discontinuity of the first kind. Moreover, the point $x = 0$ can be a discontinuity point of the second kind. We shall show that if α is strictly positive an α-star-convex function is Lipschitz on certain interval.

Theorem 4. *Let $\alpha \in (0,1]$ and let $a < b$ with $0 \in [a,b]$. If the function $f : [a,b] \to \mathbb{R}$, is α-star-convex on $[a,b]$, then f is Lipschitz continuous on each compact interval $K = [a_1, a_2] \subseteq (\alpha a, \alpha b)$, where $a_1 < a_2$.*

Proof. Since $K \subseteq (\alpha a, \alpha b)$, there exists $h > 0$ such that $K_h = [a_1 - \alpha h, a_2 + \alpha h] \subseteq (\alpha a, \alpha b)$, and hence $K_h^1 = [a_1/\alpha - h, a_2/\alpha + h] \subseteq (a,b)$. Let m_h be the greatest lower bound of f on K_h and let M_h be the least upper bound of f on K_h^1. From the definition of the least upper bound there is a sequence $(\varepsilon_n)_{n\geq 1}$, with $\varepsilon_n \searrow 0$ and a corresponding sequence $(x_n)_{n\geq 1}$, $x_n \in K_h^1$, such that $M_h - \varepsilon_n = f(x_n)$. Since $\alpha x_n \in K_h$ we have

$$M_h - \varepsilon_n = f(x_n) = f\left(\frac{1}{\alpha}\alpha x_n\right) \geq \frac{1}{\alpha} f(\alpha x_n) \geq \frac{1}{\alpha} m_h,$$

hence $\alpha M_h \geq m_h$.

Let denote by $\overline{f}'(x_0+0)$, $\overline{f}'(x_0-0)$, $\underline{f}'(x_0+0)$, and $\underline{f}'(x_0-0)$ the upper-right, upper-left, lower-right and lower-left Dini derivatives at $x_0 \in K$ respectively. If in (3) we let $x = x_0$, $y = x_0/\alpha + h$ and divide by $(1-\lambda)\alpha h$, we deduce

$$\lambda \frac{f(x_0 + (1-\lambda)\alpha h) - f(x_0)}{(1-\lambda)\alpha h}$$

$$\leq \frac{\alpha f(x_0/\alpha + h) - f(x_0 + (1-\lambda)\alpha h)}{\alpha h} \leq \frac{\alpha M_h - m_h}{\alpha h},$$

and by letting $\lambda \nearrow 1$ we obtain

$$\overline{f}'(x_0+0) \leq \frac{\alpha M_h - m_h}{\alpha h}, \ \forall\, x_0 \in K.$$

Analogously, if in (3) we let $x = x_0$ and $y = x_0/\alpha - h$ we deduce

$$\underline{f}'(x_0-0) \geq \frac{m_h - \alpha M_h}{\alpha h}, \ \forall\, x_0 \in K.$$

If in (3) we let $x = x_0 - (1-\lambda)\alpha h$, $y = x_0/\alpha + \lambda h$ and divide by $(1-\lambda)\alpha h$, then we get

$$\lambda \frac{f(x_0) - f(x_0 - (1-\lambda)\alpha h)}{(1-\lambda)\alpha h}$$

$$\leq \frac{\alpha f(x_0/\alpha + \lambda h) - f(x_0)}{\alpha h} \leq \frac{\alpha M_h - m_h}{\alpha h},$$

and by letting $\lambda \nearrow 1$, we deduce

$$\overline{f}'(x_0-0) \leq \frac{\alpha M_h - m_h}{\alpha h}, \ \forall\, x_0 \in K.$$

Analogously, if in (3) we let $x = x_0 + (1-\lambda)\alpha h$ and $y = x_0/\alpha - \lambda h$, we obtain

$$\underline{f}'(x_0+0) \geq \frac{m_h - \alpha M_h}{\alpha h}, \ \forall\, x_0 \in K.$$

Therefore we deduce that f satisfies the Lipschitz condition with the constant $(\alpha M_h - m_h)/(\alpha h)$ on $K \subseteq (\alpha a, \alpha b)$. $\square$

Corollary 1. [7] *If $f : [a,b] \to \mathbb{R}$, with $a < b$, $0 \in [a,b]$, is α-star-convex, where $\alpha \in (0,1]$, then f is continuous on $(\alpha a, \alpha b)$. In particular, if $f : \mathbb{R} \to \mathbb{R}$ is α-star-convex then f is continuous on $\mathbb{R}$, and Lipschitz continuous on each compact interval of $\mathbb{R}$.*

5. Other characterizations of α-star-convex functions

Suppose now that the function f has a right-hand derivative $f'(x+0)$ and a left-hand derivative $f'(x-0)$ at each point $x \in I$. If we let $u = \lambda x + (1-\lambda)\alpha y$, then from (1) we obtain

$$f(u) - f(x) \leq (1-\lambda)[\alpha f(y) - f(x)].$$

If $u > x$, i.e. $\alpha y > x$, then we have

$$\frac{f(u)-f(x)}{u-x} \leq \frac{\alpha f(y) - f(x)}{\alpha y - x},$$

and if we let $\lambda \nearrow 1$ we deduce

$$f'(x+0) \leq \frac{\alpha f(y) - f(x)}{\alpha y - x},$$

hence

$$f(y) \geq \frac{f(x)}{\alpha} + f'(x+0)\left(y - \frac{x}{\alpha}\right), \ \forall\, y > \frac{x}{\alpha}.$$

In a similar way we obtain

$$f(y) \geq \frac{f(x)}{\alpha} + f'(x-0)\left(y - \frac{x}{\alpha}\right), \ \forall\, y < \frac{x}{\alpha}.$$

The above results have the following geometric interpretation [4]: Take a point $P = P(x, f(x)) \in \Gamma_f$ and consider the point Q on the ray OP such that $OP/OQ = 1/\alpha$ (see Figure 1). Then the graph Γ_f lies above the reunion of the half lines

$$Y = \frac{f(x)}{\alpha} + f'(x+0)\left(X - \frac{x}{\alpha}\right), \quad X > \frac{x}{\alpha},$$

and

$$Y = \frac{f(x)}{\alpha} + f'(x-0)\left(X - \frac{x}{\alpha}\right), \quad X < \frac{x}{\alpha}$$

(see Figure 2).

From the geometric interpretation mentioned in Remark 3 we deduce the following characterization of an α-star-convex function [9].

Theorem 5. *The function $f : I \to \mathbb{R}$, with condition (4) is α-star-convex on I if and only if for all $y \in I$ the function $\varphi_y : I \setminus \{\alpha y\} \to \mathbb{R}$ defined by*

$$\varphi_y(x) = \frac{f(x) - \alpha f(y)}{x - \alpha y} \tag{7}$$

is increasing on each interval $\{x \in I : \ x < \alpha y\}$ and $\{x \in I : \ x > \alpha y\}$.

If we suppose that the function f is differentiable on I, then from Theorem 5 and (7) we deduce that f is α-star-convex on I if and only if for each $x, y \in I$ the following inequality holds

$$f'(x)(x - \alpha y) - [f(x) - \alpha f(y)] \geq 0$$

or

$$xf'(x) - f(x) - \alpha[yf'(x) - f(y)] \geq 0. \tag{8}$$

Since an α-star-convex function is necessarily starshaped we have $xf'(x) - f(x) \geq 0$. If $yf'(x) - f(y) \leq 0$ then (8) holds for all positive α. If we suppose that $yf'(x) - f(y) > 0$, then from (8) we deduce

$$\alpha \leq \frac{xf'(x) - f(x)}{yf'(x) - f(y)} \equiv \Phi(x, y).$$

From this inequality we obtain in (5) of Definition 2 the following formula [4]:

$$\alpha^*[f] = \inf \left\{ \frac{xf'(x) - f(x)}{yf'(x) - f(y)} : \ yf'(x) - f(y) > 0, \ x, y \in I \right\}.$$

If there exist $x_0, y_0 \in I$ such that $x_0 f'(x_0) = f(x_0)$ and $y_0 f'(x_0) - f(y_0) > 0$, then $\alpha^*[f] = 0$.

Suppose now that $xf'(x) - f(x) > 0$ for all $x \in I \setminus \{0\}$ (i.e. f is strictly starshaped on I) and that f is twice differentiable on I. Then the system

$$\frac{\partial \Phi}{\partial x} = 0, \quad \frac{\partial \Phi}{\partial y} = 0$$

is equivalent to

$$f''(x) = 0, \quad f'(x) = f'(y). \tag{10}$$

Hence in certain cases $\alpha^*[f]$ given by (9) can be obtained by solving the system (10).

Example 2. [4] Let $f : \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$f(x) = x^4 - 5x^3 + 9x^2 - 5x.$$

If we let $g(x) = f(x)/x$, then $g'(x) = 3x^2 - 10x + 9 > 0$, hence f is strictly starshaped on $\mathbb{R}$. We also have

$$f'(x) = 4x^3 - 15x^2 + 18x - 5$$

and

$$f''(x) = 6(2x^2 - 5x + 3).$$

The equation $f''(x) = 0$ has the roots $x_1 = 1$ and $x_2 = 3/2$. For $x_1 = 1$ equation $f'(y) = f'(x_1)$ has the root $y_1 = 7/4$ and we have $\Phi(x_1, y_1) = 512/539 \approx 0.949\ldots$ For $x_2 = 3/2$ equation $f'(y) = f'(x_2)$ has the root $y_2 = 3/4$ and we have $\Phi(x_2, y_2) = 16/17 = 0.941\ldots$ Hence from (9) we deduce $\alpha^*[f] = 16/17$.

The graph of the function f is given in Figure 1.

6. Hermite-Hadamard inequalities

It is known that if f is convex on $[a,b]$ then the following Hermite-Hadamard inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{11}$$

hold. A variant of (11) for α-star-convex functions was given in [1]. We give here another one.

Theorem 6. *If the function f is α-star-convex on $[a,b]$ with $\alpha \in (0,1]$, then*

$$\frac{1}{\alpha b-a}\int_a^{\alpha b} f(x)dx \leq \frac{f(a)+\alpha f(b)}{2}. \tag{12}$$

Proof. Integrating

$$f(ta+\alpha(1-t)b) \leq tf(a)+\alpha(1-t)f(b)$$

for $t \in [0,1]$ we get (12). □

Theorem 7. *If the function f is α-star-convex on $[a,b]$, $a<b$ with $\alpha \in (0,1]$ then*

$$f\left(\frac{a+\alpha b}{2}\right) \leq \frac{1+\alpha}{2\alpha(b-a)}\int_a^{\frac{a+\alpha b}{1+\alpha}} f(x)dx + \alpha\frac{1+\alpha}{2(b-a)}\int_{\frac{a+\alpha b}{1+\alpha}}^b f(x)dx. \tag{13}$$

Proof. We have

$$\begin{aligned} f\left(\frac{a+\alpha b}{2}\right) &= f\left[\frac{1}{2}(a+\alpha t)+\frac{1}{2}\alpha(b-t)\right] \\ &\leq \frac{1}{2}f(a+\alpha t)+\frac{1}{2}\alpha f(b-t) \end{aligned}$$

and integrating for $t \in [0,(b-a)/(1+\alpha)]$ we get (13). □

Note that if we take $\alpha = 1$ in (12) and (13) then we obtain (11).

7. Weighted arithmetic means

In [10] it was studied the problem of the conservation of α-star-convexity by a weighted arithmetic mean of the form

$$A_g[f](x) = \frac{1}{g(x)} \int_0^x g'(t) f(t) dt. \tag{14}$$

Let us denote by $K_\alpha(b)$ the set of α-star-convex functions on $[0, b]$, such that $f(0) = 0$. In [10] the following results were obtained.

Theorem 8. *If $A_g[f] \in K_\alpha(b)$ for all $f \in K_\alpha(b)$ then*

$$g(x) = kx^\gamma,$$

for some $k \neq 0$ and $\gamma > 0$. In this case

$$A_g[f](x) = A_\gamma[f](x) = \frac{\gamma}{x^\gamma} \int_0^x t^{\gamma-1} f(t) dt = \int_0^1 f(xs^{1/\gamma}) ds.$$

If we denote by $M^\gamma K_\alpha(b)$ the set of functions f with the property that $A_\gamma[f] \in K_\alpha(b)$, then we have

Theorem 9. *If $0 < \alpha < \beta < 1$ and $\gamma > 0$ then the following inclusions*

$$\begin{array}{ccccccc}
K_1(b) & \subseteq & K_\beta(b) & \subseteq & K_\alpha(b) & \subseteq & K_0(b) \\
\cap & & \cap & & \cap & & \cap \\
M^\gamma K_1(b) & \subseteq & M^\gamma K_\beta(b) & \subseteq & M^\gamma K_\alpha(b) & \subseteq & M^\gamma K_0(b)
\end{array}$$

hold.

In fact an α-star-convex function can be mapped onto a β-star-convex function with $\beta > \alpha$, as was shown in [4] by the following example, for $\gamma = 1$.

Example 3. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$f(x) + 5x^4 - 20x^3 + 27x^2 - 10x$$

and let

$$F(x) = \frac{1}{x}\int_0^x f(t)dt = x^4 - 5x^3 + 9x^2 - 5x,$$

which is the function given in Example 2. By using the system (10) we obtain $\alpha^*[f] = 0.302\ldots$, while $\alpha^*[F] = 16/17 = 0.941\ldots$

8. Star-convexity and Bernstein polynomials

For a function $f : [0,1] \to \mathbb{R}$ let us denote by $B_n(f)$ the Bernstein polynomial of order n of f defined by

$$B_n(f)(x) = \sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1].$$

A well known result of classical analysis (see D.D. Stancu [8] p.264) asserts that if f is convex in $[0,1]$ then:

$$B_n(f)(x) \geq f(x), \ \forall\ x \in [0,1]. \tag{15}$$

First we consider an example of a starshaped function not verifying the inequality (15).

Example 4. Let $f : [0,1] \to \mathbb{R}$, be given by

$$f(x) = \begin{cases} -x, & \text{if } \ x \in [0,1/3] \\ 2x, & \text{if } \ x \in (1/3,2/3] \\ 4x, & \text{if } \ x \in (2/3,1]. \end{cases}$$

We have

$$B_2(f)(x) = 2x + 2x^2 < 4x = f(x), \ \forall\ x \in (2/3,1).$$

But the function $x \to f(x)/x$ being increasing on $(0,1]$, f will be starshaped on $[0,1]$.

In the following lemma of independent interest we give a generalization of inequality (15) for α-star-convex functions, $\alpha \in [0,1]$. Particularly, for $\alpha = 1$ one obtains again (15).

Lemma 3. *Given* $\alpha \in [0,1]$, *let us denote by* S_n^α *the real function defined on* $[0,1]$ *by*

$$S_n^\alpha(x) = \alpha^{nx}, \ \forall \ x \in [0,1].$$

If f *is* α*-star-convex on* $[0,1]$ *then:*

$$B_n(S_n^\alpha \cdot f)(x) \geq f(B_n(S_n^\alpha \cdot J)(x)), \ \forall \ x \in [0,1],$$

where J *is the identity mapping on* $[0,1]$.

Proof. If we let in the Jensen type inequality (mentioned in Remark 6)

$$p_k = C_n^k x^k (1-x)^{n-k}, \quad x_k = k/n, \ k = 0,1,\ldots,n,$$

x being fixed in $[0,1]$ one obtains

$$f\left(\sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} \alpha^{nk/n} \frac{k}{n}\right) \leq \sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} \alpha^{nk/n} f\left(\frac{k}{n}\right),$$

and this yields the conclusion. □

In [2] it was proved that if the starshaped function $f : [0,1] \to \mathbb{R}$, verifies the properties $f(0) = 0$, $f(x) \geq 0$, $\forall \ x \in [0,1]$ and $f \in C[0,1]$, then $B_n(f)$ is starshaped, $B_n(f)(0) = 0$ and $B_n(f)(x) \geq 0$, $\forall \ x \in [0,1]$, $n = 1,2,\ldots$. The proof in [2] can be extended with some minor changes to a little more general setting. So, if f is an arbitrary starshaped function on $[0,1]$ then $B_n(f)$ is also starshaped on $[0,1]$, for all $n \geq 1$.

Now, a natural problem is: Given an $\alpha \in [0,1]$ and a starshaped function f on $[0,1]$ with $\alpha^*(f) = \alpha$, does it follows that $\alpha^*[B_n(f)] = \alpha$, $n = 1, 2, \ldots$? The answer is negative.

Example 5. Let f be defined on $[0,1]$ with $f(0) = 0$ a function such that $\alpha^*[f] = \alpha \in [0,1)$. Then: $B_1(f)(x) = f(1) \cdot x$, and so $B_1(f)$ is convex. On the other hand

$$B_2(f)(x) = 2f\left(\frac{1}{2}\right)x + \left[\frac{f(1)}{1} - \frac{f(1/2)}{1/2}\right]x^2$$

and $B_2(f)$ is also a convex function on $[0,1]$. However, f is not a convex function.

Example 6. In this example the function $f : [0,1] \to \mathbb{R}$ is starshaped and $B_3(f)$ is not convex. Letting

$$f(x) = 3x^4 - 10x^3 + 11x^2, \ \forall\, x \in [0,1],$$

we have that f is starshaped and because $f''(0) = 22 > 0$, $f''(1) = -2 < 0$, f is not convex on $[0,1]$. The third Bernstein polynomial

$$B_3(f)(x) = \frac{8}{3}x + \frac{20}{9}x^2 - \frac{8}{9}x^3$$

is starshaped but from $B_3(f)''(0) = 40/9 > 0$ and $B_3(f)''(1) = -8/9 < 0$, it follows the non-convexity of $B_3(f)$.

Let f be a continuous starshaped function on $[0,1]$. We will be interested to obtain informations on the order of star-convexity of $B_n(f)$, $n = 1, 2, \ldots$, when we know the order of star-convexity of f. For a particular case one obtains effectively $\alpha^*[B_n(f)]$. A comparison of this order to $\alpha^*[f] \in [0,1]$ will be made. The study of the asymptotic behaviour of the sequence $(\alpha^*[B_n(f)])_{n \geq 1}$ is our main purpose in the sequel.

Lemma 4. *Suppose that for the continuous function f on $[0,1]$, $\alpha^*[f] \leq 1$. Then*

$$\overline{\lim_{n\to\infty}} \alpha^*[B_n(f)] \leq \alpha^*[f], \quad n = 1, 2, \ldots$$

Proof. If $\alpha^*[f] = 1$, then f is convex on $[0,1]$ and from a well known result [5], $B_n(f)$ is convex on $[0,1]$, for all $n \leq 1$. Since $\alpha^*[f] = 1$, it follows that there exists $n_0 \in \mathbb{N}$ such that $degree(B_n(f)) \geq 2$, $\forall\, n \geq n_0$. Then $\alpha^*[B_n(f)] = 1$, $\forall\, n \geq n_0$ and in this case Lemma is proved. Let now suppose that $\alpha^*[f] < 1$ and $\varepsilon > 0$ be given. Because f isn't $(\alpha^*[f] + \varepsilon)$-star-convex this means that there exist $\lambda_0, x_0, y_0 \in [0,1]$ such that

$$f(\lambda_0 x_0 + (1-\lambda_0)(\alpha^*[f]+\varepsilon)y_0) - \lambda_0 f(x_0) - (1-\lambda_0)(\alpha^*[f]+\varepsilon)f(y_0) = d > 0.$$

From the uniform convergence of $(B_n(f))_{n\geq 1}$ to f it follows that

$$B_n(f)(\lambda_0 x_0 + (1-\lambda_0)(\alpha^*[f]+\varepsilon)y_0)$$

$$-\lambda_0 B_n(f)(x_0) - (1-\lambda_0)(\alpha^*[f]+\varepsilon)B_n(f)(y_0) \geq \frac{d}{2} > 0,$$

for all $n \geq n_0 \in \mathbb{N}$. This implies that $B_n(f)$ is not $(\alpha^*[f]+\varepsilon)$-star-convex, for $n \geq n_0$ and

$$\overline{\lim_{n\to\infty}} \alpha^*[B_n(f)] \leq \alpha^*[f] + \varepsilon, \ \forall\, \varepsilon > 0. \quad \Box$$

Remark 7. In particular it follows that if $\alpha^*[f] = 0$ then

$$\lim_{n\to\infty} \alpha^*[B_n(f)] = 0.$$

Example 7. Let $f : [0,1] \to \mathbb{R}$, be given by

$$f(x) = -2x^3 + 5x^2 + 6x.$$

After some simple computations one obtains that $\alpha^*[f] = 27/28$. Moreover the infimum in formula (9) giving $\alpha^*[f]$ is attained for $x = 1$ and $y = 2/3$.

The Bernstein polynomials $B_n(f)$ are

$$B_n(f)(x) = \frac{6n^2 + 5n - 2}{n^2}x + \frac{(n-1)(5n-6)}{n^2}x^2 - \frac{2(n-1)(n-2)}{n^2}x^3,$$

$n = 1, 2, \ldots$. It follows that $B_n(f)$, $n = 1, 2, \ldots$, is starshaped and

$$\alpha^*[B_n(f)] = \frac{27}{4} \cdot \frac{(n+2)(n-2)^2}{n^2(7n-18)}, \ \forall\ n \geq 6.$$

Moreover the sequence $(\alpha^*[B_n(f)])_{n\geq 6}$ is decreasing and

$$\lim_{n\to\infty} \alpha^*[B_n(f)] = 27/28 = \alpha^*[f].$$

Also $\alpha^*[B_n(f)] > \alpha^*[f]$, $\forall\ n \geq 1$ and the infimum in formula (9) giving $\alpha^*[B_n(f)]$ is attained for $x = 1$ and $y = 2n/(3n-6)$, $n \geq 6$. In this example we have that $\alpha^*[B_n(f)] \geq \alpha^*[f]$, $\forall\ n \geq 1$. We expect that generally

$$\varliminf_{n\to\infty} \alpha^*[B_n(f)] \geq \alpha^*[f].$$

Proposition 1. *Let $\alpha \in [0,1]$ be fixed and let $(f_n)_{n\geq 1}$ be a sequence of real functions on $[0,1]$. Suppose that $\alpha^*[f_n] \geq \alpha$, $\forall\ n \geq 1$ and that $f_n(x) \to f(x)$, for any $x \in [0,1]$. Then $\alpha^*[f] \geq \alpha$.*

Proof. Indeed, for a given pair $(x, y) \in [0,1]^2$ making $n \to \infty$ in the inequality

$$f_n(\lambda x + (1-\lambda)\alpha y) \leq \lambda f_n(x) + (1-\lambda)\alpha f_n(y),$$

one obtains that $\alpha^*[f] \geq \alpha$. $\square$

Example 8. Let $f_n : [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = (4/n)x(x-1),\ \forall\ x \in [0,1],\ n = 1, 2, \ldots$$

Then $\alpha^*[f_n] = 1$, $n = 1, 2, \ldots$ and the sequence $(f_n)_{n\geq 1}$ converges uniformly to the null function g_0 on $[0,1]$. But $\alpha^*[g_0] = \infty$.

Lemma 5. *a) Let $f \in C^1[0,1]$ be a strictly starshaped function. If $f(0) < 0$ then:*

$$\varliminf_{n\to\infty} \alpha^*[B_n(f)] \geq \alpha^*[f].$$

b) Let $f \in C^2[0,1]$ be a strictly starshaped function. If $f(0) = 0$ and $f''(0) \neq 0$, then:

$$\varliminf_{n\to\infty} \alpha^*[B_n(f)] \geq \alpha^*[f].$$

Proof. a) Suppose that

$$\varliminf_{n\to\infty} \alpha^*[B_n(f)] = a < \alpha^*[f].$$

Let $\varepsilon > 0$ be small enough such that $a + 2\varepsilon < \alpha^*[f]$. Then there exists a sequence of indices $(n_k)_{k\geq 1}$ such that

$$\lim_{k\to\infty} \alpha^*[B_{n_k}(f)] = a,$$

and for $k \geq k_0 \in \mathbb{N}$ we have

$$\alpha^*[B_{n_k}(f)] \in (a - \varepsilon, a + \varepsilon). \tag{16}$$

This means that for $k \geq k_0$, $B_{n_k}(f)$ is not $(a + 2\varepsilon)$-star-convex. There exist the sequences $(x_{n_k})_{k\geq 1}$, $(y_{n_k})_{k\geq 1}$ of reals in $[0,1]$ with the property

$$\begin{aligned} &x_{n_k} B_{n_k}(f)'(x_{n_k}) - B_{n_k}(f)(x_{n_k}) \\ &-(a + 2\varepsilon)(y_{n_k} B_{n_k}(f)'(x_{n_k}) - B_{n_k}(f)(y_{n_k})) < 0, \end{aligned} \tag{17}$$

for all $k \geq k_0$. We can suppose that the sequences $(x_{n_k})_{k\geq 1}$, $(y_{n_k})_{k\geq 1}$ are convergent.

Let $x = \lim\limits_{k\to\infty} x_{n_k}$, $y = \lim\limits_{k\to\infty} y_{n_k}$. Because $B_{n_k}(f) \rightrightarrows f$, $B_{n_k}(f)' \rightrightarrows f'$, from (16) we obtain

$$xf'(x) - f(x) - (a + 2\varepsilon)(yf'(x) - f(y)) \leq 0. \tag{18}$$

On the other hand from (16) and (17) it follows

$$y_{n_k} B_{n_k}(f)'(x_{n_k}) - B_{n_k}(f)(y_{n_k}) \geq 0, \ \forall\ k \geq k_0.$$

Then $yf'(x) - f(y) \geq 0$. But, f being $\alpha^*[f]$-star-convex, from (18) we have

$$xf'(x) - f(x) - (a + 2\varepsilon)(yf'(x) - f(y)) = 0 \tag{19}$$

and

$$xf'(x) - f(x) - \alpha^*[f](yf'(x) - f(y)) \geq 0.$$

From this and (19) we have:

$$(-\alpha^*[f] + a + 2\varepsilon)(yf'(x) - f(y)) \geq 0,$$

so

$$yf'(x) - f(y) = 0 \text{ and } xf'(x) - f(x) = 0, \tag{20}$$

which contradicts $f(0) < 0$ or the strict starshapedness of f.

b) Suppose now that $f \in C^2[0,1]$, $f(0) = 0$, $f''(0) \neq 0$. One observe that $f''(0) > 0$. Indeed

$$f''(0) = \lim_{x\searrow 0} \frac{f(2x) - 2f(x) + f(0)}{x^2} = \lim_{x\searrow 0} \frac{2}{x}\left[\frac{f(2x)}{2x} - \frac{f(x)}{x}\right] \geq 0.$$

Suppose that $f''(0) = d > 0$. Using the same arguments as in the case a) and supposing that $\underline{\lim}_{n\to\infty} \alpha^*[B_n(f)] = a < \alpha^*[f]$, we have again (20) with $x = \lim_{k\to\infty} x_{n_k}$ and $y = \lim_{k\to\infty} y_{n_k}$.

Now, from strictly starshapedness of f it follows that (20) yields $x = 0$. But $f'(0) = \lim_{x\searrow 0} f(x)/x < f(z)/z$, $\forall\ z \in (0,1]$. This means that $y = 0$ and $x = y = 0$. From $f''(0) = d > 0$ and from the continuity of f'' it follows that f is strictly convex on a neighbourhood of 0. More precise $f''(x) > d/2$, $\forall\ x \in [0,\delta]$ with $\delta > 0$ sufficiently small. From $B_{n_k}(f)'' \rightrightarrows f''$ it follows that for $k \geq k_1 \in \mathbb{N}$, $B_{n_k}(f)''(x) \geq d/4$, $\forall\ x \in [0,\delta]$. Then $B_{n_k}(f)$, is convex on $[0,\delta]$ for $k \geq k_1$. But for $k \geq k_2 \in \mathbb{N}$, $x_{n_k}, y_{n_k} \in [0,\delta]$ and (17) will be contradicted for all $k > k_3 = \max\{k_1, k_2\}$. $\square$

Theorem 10. *If f verifies the conditions a) or b) in Lemma 5 then*

$$\lim_{n\to\infty} \alpha^*[B_n(f)] = \alpha^*[f].$$

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Bullet. Applied & Comp. Math. (Budapest)
85-B(1998), BAM-1473, 347-350

THE ORDER OF A STAR-CONVEX FUNCTION

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1. Introduction

In the first part of this paper we characterize the polynomials of the fourth degree which are starshaped but not convex on $[0, \infty)$. In the second part, we determine the order of star-convexity of such a polynomial. As a conclusion follows the main result that for every $p \in (0, 1)$ there are polynomials with the order of star-convexity equal to p. Star-convex functions were defined in [3] and studied in [4], [1] and [2].

2. Starshaped polynomials

In what follows we consider real functions defined on $[0,\infty)$. It is known that such a function is called starshaped if it satisfies the condition

$$f(tx) \le tf(x)\ ,\ \forall t \in [0, 1]\ ,\ \forall x \ge 0.$$

Taking $t = 0$, it follows that $f(0) \le 0$, but we shall assume, as usual, that $f(0) = 0$. It is easy to see that f is starshaped if and only if the function g, defined by:

$$g(x) = f(x)\ /\ x\ , \forall x > 0$$

is increasing. It follows that if the function f is differentiable , it is starshaped if and only if

$$f^{'}(x) \ge f(x)\,/\,x\,, \forall x > 0.$$

Let us use this condition for a polynomial

$$f(x) = a_1x + a_2x^2 + ... + a_nx^n \tag{1}$$

We have

$$f^{'}(x) - f(x)/x = a_2x + 2a_3x^2 + ... + (n-1)a_nx^{n-1}$$

so that it is easy to deduce that a polynomial of degree two or three is starshaped if and only if $a_2 > 0$ respectively $a_3 > 0$, $a_2 \geq 0$. Also we have the following

Lemma 2.1. *A polynomial of degree four is starshaped if and only if its coefficients satisfy one of the conditions*

i) $a_4 > 0$ and $a_3^2 - 3a_2a_4 \leq 0$,

or

ii) $a_4 > 0$, $a_3 \geq 0$ and $a_2 \geq 0$.

Proof. In this case

$$f'(x) - f(x)/x = x(a_2 + 2a_3x + 3a_4x^2)$$

which is nonnegative for every positive x if $a_4 > 0$ and if the second degree factor has: i) at most one (double) real root, or ii) negative roots.

3. Convex polynomials

The function f is called convex on D if the following condition

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),\ \forall x, y \in D,\ \forall t \in [0,1]$$

holds. It is well known that if a function is two times differentiable, it is convex if and only if its second order derivative is nonnegative. So, the polynomial (1) is convex on $[0,\infty)$ if and only if

$$f^{''}(x) = 2a_2 + 6a_3x + ... + n(n-1)a_nx^{n-2}$$

is nonnegative for $x \geq 0$.

Again, a polynomial of degree two or three is convex on $[0,\infty)$ if and only if $a_2 > 0$, respectively $a_3 > 0$ and $a_2 \geq 0$. Further we have

Lemma 3.1. *A polynomial of degree four is convex on* $[0,\infty)$ *if and only if its coefficients satisfy one of the following conditions*

j) $a_4 > 0$ and $3a_3^2 - 8a_2a_4 \leq 0$;

or

jj) $a_4 > 0$, $a_3 \geq 0$ and $a_2 \geq 0$.

The proof is similar with that of Lemma 2.1.

Remark 3.2. We look for polynomials which are starshaped but not convex. This cannot happen for degrees less than four. But we have

Lemma 3.3. *A polynomial (1) of degree four is starshaped but not convex if and only if*

$$a_4 > 0, a_3 < 0 and 8a_2a_4 < 3a_3^2 < 9a_2a_4 \tag{2}$$

Proof. By lemmas 2.1 and 3.1 f is starshaped but not convex if its coefficients satisfy i) but not j) or jj). From i) we have $a_4 > 0$ and $a_3^2 \leq 3a_2a_4$. But we must to avoid j), for which we need $3a_3^2 > 8a_2a_4$ and to avoid jj) for which we need $a_3 < 0$, getting (2).

4. Star-convex polynomials

Let $p \in [0,1]$. A real function f defined on [0,∞) was called in [3] p-star-convex if

$$f(tx + (1-t)py) \leq tf(x) + (1-t)pf(y), \forall x, y \geq 0, \ \forall t \in [0,1].$$

Of course, for $p = 1$ we get convex functions and for $p = 0$ starshaped functions.

In [3] also was proved that for 0$\leq p \leq q \leq 1$, if f is q-star-convex then it is also p-star-convex. So the definition of the order of star-convexity of a function, given in [1], is well justified:

$$p(f) = \sup\{p \in [0,1] : f \ is \ p - star - convex\}.$$

In [3] and [4] it is proved that the function f is p-star-convex if and only if for all y, the function φ_y defined by

$$\varphi_y(x) = [f(x) - pf(y)] / (x - py)$$

is increasing on each interval [0,py) and (py,∞). This gives the following formula (see [1] or [2]):

$$p(f) = \inf\left\{\frac{xf'(x) - f(x)}{yf'(x) - f(y)} : yf'(x) > f(y), \ x, y > 0\right\},$$

or, if f is twice differentiable

$$p(f) = \inf\left\{\frac{xf'(x) - f(x)}{yf'(x) - f(y)}; f''(x) = 0, f'(x) = f'(y), x, y > 0\right\}. \tag{3}$$

We use this result to determine the order of star-convexity of a polynomial of fourth degree. First of all we remark that for each positive constant c, $p(cf) = p(f)$. So, we can choose an arbitrary coefficient for x^4.

Theorem 4.1. *If the polynomial*

$$f(x) = x^4/12 + a_3x^3 + a_2x^2 + a_1x \tag{1'}$$

is starshaped but not convex, that is its coefficients satisfy the conditions

$$a_3 < 0\ ,\ 8a_2 < 36a_3^2 < 9a_2, \tag{2'}$$

then its order of star-convexity is

$$p(f) = A_2/(A_2 + B)$$

where

$$A_2 = 27a_3^4 - (\Delta + a_2)^2 + 2a_3\sqrt{\Delta^3},$$
$$B = (3\Delta/2)^2$$

with

$$\Delta = 9a_3^2 - 2a_2 \tag{4}$$

Proof. The condition (2') are given by (2) for $a_4 = 1/12$. To use (3) we have to solve the system of equations

$$f''(x) = 0, f'(x) = f'(y).$$

The first equation

$$f''(x) = x^2 + 6a_3x + 2a_2 = 0 \tag{5}$$

has the roots

$$x_i = -3a_3 + (-1)^i\sqrt{\Delta}, i = 1, 2$$

where Δ is given by (4) and it is positive as it was assumed in (2'). The second equation becomes (for i=1,2)

$$f'(y) = f'(x_i)$$

or

$$y^2 + (x_i + 9a_3)y + 3a_3x_i + 4a_2 = 0$$

which has a solution x_i and a second solution

$$y_i = -2x_i - 9a_3.$$

Taking into account (5) we obtain

$$x_if'(x_i) - f(x_i) = 27a_3^4 - (\Delta + a_2)^2 + (-1)^i2a_3\sqrt{\Delta^3} = A_i,$$

then

$$f(y_i) = 4x_i^4/3 + 16a_3x_i^3 + 54a_3^2x_i^2 - 729a_3^4/4 + 4a_2x_i^2$$
$$+36a_2a_3x_i + 81a_2a_3^2 - 2a_1x_i - 9a_1a_3$$

giving

$$y_if'(x_i) - f(y_i) = -2x_i^4 - 25a_3x_i^3 - 8a_2x_i^2 - 81a_3^2x_i^2$$
$$-54a_2a_3x_i - 81a_2a_3^2 + (729/4)a_3^4$$
$$= 2a_3(9a_3^2 - 2a_2)x_i + 8a_2^2 + (729/4)a_3^4 - 75a_2a_3^2 = A_i + B$$

where

$$B = (3\Delta\ /2)^2.$$

Obviously $B > 0$and

$$A_1 - A_2 = -4a_3\sqrt{\Delta\ ^3} > 0.$$

We want to prove that $A_2 > 0$. This is equivalent with

$$27a_3^4 - (9a_3^2 - a_2)^2 \ > \ -2a_3\sqrt{(9a_3^2 - 2a_2)^3} \tag{6}$$

The second member is obviously positive. To prove that the first member is also positive we write it as

$$(3\sqrt{3}a_3^2 - 9a_3^2 + a_2)(3\sqrt{3}a_3^2 + 9a_3^2 - a_2)$$
$$= [(3\sqrt{3} - 5)a_3^2 + a_2 - 4a_3^2][3\sqrt{3}a_3^2 + (9a_3^2 - 2a_2) + a_2]$$

and make use of (2'). So (6) is equivalent with

$$(18a_2a_3^2 - a_2^2 - 54a_3^4)^2 \ > \ 4a_3^2(9a_3^2 - 2a_2)^3$$

which reduces at

$$a_2 > 4a_3^2$$

and this is true after (2'). Thus $0 < A_2 < A_1$ and so (3) gives

$$p(f) \ = \ \inf\left\{\frac{A_1}{A_1 + B}, \frac{A_2}{A_2 + B}\right\} \ = \ \frac{A_2}{A_2 + B}.$$

Remark 4.2. For

$$f(x) = (x^4 - 5x^3 + 9x^2 - 5x)/12$$

we get $p(f) = 16/17$ which was proved in [1] and [2]. There was also given for each $p \in (0, 1)$ a function f, such that $p(f) = p$. All these functions are polygonal lines. Using Theorem 4.1 we can replace these functions by polynomials (of fourth degree).

Theorem 4.3. *For each $p \in (0, 1)$there is a polynomial f, such that $p(f) = p$.*

Proof. We take f as in (1') and look after the coefficients a_2 and a_3 such that

$$A_2 = p(A_2 + B)$$

thus

$$27a_3^4 - 18\Delta\ a_3^2 + 8a_3\sqrt{\Delta^3} \ = \ \Delta^2(8p+1)/(1-p). \tag{7}$$

We add the condition

$$\Delta = 9a_3^2 - 2a_2 = 1.$$

This is possible if

$$a_2 = (9a_3^2 - 1)/2 > 0$$

or $a_3 < -1/3$. But the equation (7) reduces then at

$$27a_3^4 - 18a_3^2 + 8a_3 = (1+8p)/(1-p)$$

which has a solution a_3 even less than -1.

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Chebyshev-Anderson's inequality

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1 Introduction.

Let us consider the inequality:

$$A(f_1 \ldots f_m) \geq K_m A(f_1) \ldots A(f_m) \tag{1}$$

where

$$A(f) = \int_0^1 f(x)dx.$$

Classical Chebyshev's inequality asserts that (1) is valid for increasing functions $f_1, ..., f_m$, with $K_m = 1$. There are many papers that treat this inequality as we can see in the syntheses [6] and [7]. In [1], B. J. Anderson has proved that for convex functions (1) is valid with $K_m = 2^m/(m+1)$. We have shown in [10] that Anderson's inequality is in fact valid for starshaped functions.

Here we define general notions of convexity and of starshapedness which include many of the known generalizations and prove inequalities of type (1) for them. Moreover, we take for A an isotonic linear (or superlinear) functional.

2 Superlinear functionals

Let E be a non-empty set and L be a linear class of real-valued functions defined on E . More exactly we assume that L contains the constant functions and $f, g \in L$ implies that $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$.

It is well known the following definition: an isotonic linear functional is a functional $A : L \to \mathbb{R}$ satisfying the conditions

$$A(\alpha f + \beta g) = \alpha A(f) + \beta A(g), \text{ for all } f, g \in L \text{ and } \alpha, \beta \in \mathbb{R}$$

and

$$A(f) \geq 0 \text{ for all positive } f \in L.$$

Common examples are given by :

$$A(f) = \sum_{i=1}^{n} p_i f(x_i), p_i > 0, x_i \in E \text{ for } i = 1, ..., n,$$

and

$$A(f) = \int_a^b p(x) f(x) dx, p(x) \geq 0 \text{ for } x \in E = [a, b].$$

Many of the classical inequalities are given for isotonic linear functionals. We shall consider here the case of Chebyshev's inequality. Its classical variant refers to pairs of increasing functions. Now it is used more often the following generalization. The functions $f, g \in L$ are called synchronous if :

$$[f(t) - f(s)][g(t) - g(s)] \geq 0 \text{ for all } t, s \in E. \tag{2}$$

We remind now the following general inequality of Chebyshev type. We consider weight functions p , which is not essential but simplifies its later use. A proof can be find, for example, in [9] or [10] , but we give it here for comparison with the generalization what follows.

Theorem 1 *If $A : L \to \mathbb{R}$ is an isotonic linear functional, $p \in L$ is positive and $f, g \in L$ are synchronous, then :*

$$A(pfg)A(p) \geq A(pf)A(pg). \tag{3}$$

Proof. Multiplying (2) by $p(t)p(s) \geq 0$, we have :

$$p(t)f(t)g(t)p(s) + p(t)p(s)f(s)g(s) \geq p(t)f(t)p(s)g(s) + p(t)g(t)p(s)f(s).$$

Taking the functional A for functions of variable t , we get

$$p(s)A(pfg) + p(s)f(s)g(s)A(p) \geq p(s)g(s)A(pf) + p(s)f(s)A(pg).$$

Using again A for functions of variable s, we get (3). ■

Some of the classical inequalities can be also proved for a larger set of functionals, that of isotonic sublinear functionals (see, for example [3]). This cannot be done in our case, at least by following the same way as in the previous proof. That's why we consider another generalization of the linearity for which the proof can be adapted.

An isotonic superlinear functional is a functional $A : L \to \mathbb{R}$ satisfying the conditions

$$A(f + g) \geq A(f) + A(g), \text{ for all } f, g \in L ;$$

$$A(\alpha f) = \alpha A(f), \text{ for all } f \in L \text{ and } \alpha \geq 0$$

and

$$A(f) \geq 0 \text{ for all positive } f \in L.$$

Of course, if $f \geq g$ we have

$$A(f) = A(g + f - g) \geq A(g) + A(f - g) \geq A(g).$$

A typical example of isotonic superlinear functional is given by

$$A(f) = \min_{1 \leq i \leq n} \{A_i(f)\}$$

where $A_1, ..., A_n$ are arbitrary isotonic linear functionals. To get more examples, we have only to replace the maximum by minimum in the functionals considered in [3]. Generally we can follow the same ideas as those used in the case of sublinear functionals.

Let us denote by $\boldsymbol{\alpha}$ the constant function with value α , that is $\boldsymbol{\alpha}(x) = \alpha, \forall x \in E$. We shall use in what follows functionals with the property

$$A(-\mathbf{1}) = -A(\mathbf{1}).$$

Such a functional is homogenous on constant functions, that is

$$A(\alpha\mathbf{1}) = \alpha A(\mathbf{1}) \text{ for all } \alpha \in \mathbb{R}.$$

Moreover, it is also additive if one of the terms is a constant function. Indeed, by superadditivity we have

$$A(f + \mathbf{1}) \geq A(f) + A(\mathbf{1}),$$

but also

$$A(f) = A(f + \mathbf{1} - \mathbf{1}) \geq A(f + \mathbf{1}) + A(-\mathbf{1})$$

that is

$$A(f + \mathbf{1}) \leq A(f) + A(\mathbf{1}).$$

We can now prove the following Chebyshev type inequality.

Theorem 2 *If $A : L \to \mathbb{R}$ is an isotonic superlinear functional which is homogenous on constant functions and $f, g \in L$ are positive synchronous functions, then :*

$$A(fg)A(\mathbf{1}) \geq A(f)A(g). \tag{4}$$

Proof. The relation (2) can be written as:

$$f(t)g(t) + f(s)g(s) \geq f(t)g(s) + g(t)f(s).$$

Taking the functional A for functions of variable t , we get

$$A(fg) + f(s)g(s)A(\mathbf{1}) = A(.fg + f(s)g(s)\mathbf{1}) \geq$$

$$A(g(s)f + f(s)g) \geq g(s)A(f) + f(s)A(g).$$

Using again A for functions of variable s, we get

$$2A(fg)A(\mathbf{1}) = A(A(fg)\mathbf{1} + A(\mathbf{1})fg) \geq$$

$$A(A(f)g + A(g)f) \geq A(A(f)g) + A(A(g)f) = 2A(f)A(g)$$

which gives (4). ■

Remark 3 *We need the positivity of* f *and* g *for the last equality.*

3 Generalized convex functions.

Let us fix four functions g, h, α, β, defined as follows:

$$g, \beta : [c, d] \to \mathbb{R},\ h : [0, 1] \to \mathbb{R},\ \alpha : [a, b] \to [a, b].$$

Definition 4 *The function* $f : [a, b] \to [c, d]$ *is called* (g, h, α, β)*-convex if:*

$$g(f[tx + (1-t)\alpha(y)]) \leq h(t)g[f(x)] + [1 - h(t)]\beta[f(y)]$$

for all $t \in [0, 1]$ *and* $x, y \in [a, b]$ *.*

Choosing adequately the functions g, h, α, β,we get some known examples of generalized convexity. Let us denote by e_q the function defined (on different sets) by $e_q(x) = x^q$ .For $q = 1$ we write $e_1 = e$. We find that:

i) (e, e, e, e)-convexity means convexity, where $e(x) = x$;

ii) $(e, e, 0, 0)$-convexity means starshapedness;

iii) $(e, e_q, 0, 0)$-convexity means starshapedness with respect to e_q ;

iv) (e, e, me, me)-convexity means m-convexity as we defined in [8] and used in [4] ;

v) (e_p, e_q, me, me_p)-convexity means (p, q, m)-convexity defined in [5] ;

vi) (g, e, e, g)-convexity means convexity with respect to g (for example log-convexity) ;

vii) (e, pe, e, e)-convexity means p-convexity defined in [2].

To give a second definition, we make the same hypotheses on the functions g, α and β but assume that $h : [a - b, b - a] \to \mathbb{R}$ and it preserves the sign ϵ_- on the interval $[a - b, 0)$ and the sign ϵ_+ on the interval $(0, b - a]$.

Definition 5 *The function* $f : [a, b] \to [c, d]$ *is called strongly* (g, α, β)*-starshaped with respect to* h *if for any* $y \in [a, b]$*, the function* F_y *given by*

$$F_y(x) = \frac{g[f(x)] - \beta[f(y)]}{h[x - \alpha(y)]}$$

is increasing on the interval $[a, \alpha(y))$ *if* $\epsilon_- = -1$ *and on* $(\alpha(y), b]$ *if* $\epsilon_+ = +1$ *but it is decreasing on that interval in which this condition fails.*

We have the following relation between these two notions, which is well known in some of the special cases given above. We assume that the function h is defined on a set which includes the intervals $[0,1]$ and $[a-b, b-a]$.

Theorem 6 *i) If the function h is supermultiplicative, then every function f which is (g,h,α,β) - convex is also strongly (g,α,β)-starshaped with respect to h .*

ii) If the function h is submultiplicative, then every function f which is strongly (g,α,β)-starshaped with respect to h is also (g,h,α,β)-convex.

.iii) If the function h is multiplicative, then every function f is (g,h,α,β) - convex if and only if it is strongly (g,α,β)-starshaped with respect to h .

Proof. i) If f is (g,h,α,β)-convex, $\epsilon_+ = +1$ and $\alpha(y) < z < x \leq b$, then there is a $t \in (0,1)$ such that $z = tx + (1-t)\alpha(y)$. Thus:

$$F_y(z) = \frac{g[f(tx+(1-t)\alpha(y))] - \beta[f(y)}{h[tx+(1-t)\alpha(y) - \alpha(y)]} \leq$$

$$\frac{h(t)[g(f(x)) - \beta(f(y))]}{h[t(x-\alpha(y)]} \leq F_y(x).$$

For the last inequality we have used the positivity and the supermultiplicity of h . Other cases can be treated in a similar way.

ii) Let f be strongly (g,α,β)-starshaped with respect to h , $\epsilon_+ = +1$, $\alpha(y) < x \leq b$ and $t \in (0,1)$. Putting $z = tx + (1-t)\alpha(y)$ it follows that $\alpha(y) < z < x$ and so $F_y(z) \leq F_y(x)$ or

$$\frac{g[f(tx+(1-t)\alpha(y))] - \beta[f(y)]}{h[t(x-\alpha(y))]} \leq \frac{g[f(x)] - \beta[f(y)]}{h[x-\alpha(y)]}.$$

As h is submultiplicative, we have $h[t(x-\alpha(y))] \leq h(x-\alpha(y))h(t)$, which gives the (g,h,α,β)-convexity of f. The other cases can be treated in a similar way.

iii) It is a simple consequence of i) and ii). ■

Remark 7 *As it is known, under large hypotheses, the only multiplicative functions h are e_q with $q > 0$.*

4 Generalized starshaped functions.

If the conditions of the last definition are assumed only for $y = a$, the function f is called (g,α,β)-starshaped with respect to h. We assume now that $h : [a-\alpha(a), b-\alpha(a)] \to \mathbb{R}$ and it preserves the sign ϵ_- on the interval $[a-\alpha(a), 0)$ and the sign ϵ_+ on the interval $(0, b-\alpha(a)]$.

Definition 8 *The function $f : [a,b] \to [c,d]$ is called (g,α,β)-starshaped with respect to h if the function F given by*

$$F(x) = \frac{g[f(x)] - \beta[f(a)]}{h[x-\alpha(a)]}$$

is increasing on the interval $[a, \alpha(a))$ *if* $\epsilon_- = -1$ *and on* $(\alpha(a), b]$ *if* $\epsilon_+ = +1$ *but it is decreasing on every interval in which this condition fails.*

If f is $(e, 0, 0)$-starshaped with respect to h, we say simply that f is starshaped with respect to h. Of course, in this case $0 \in [a, b]$.

We have the following Chebyshev type inequalities for generalized starshaped functions. Consider $E \subseteq (\alpha(a), b]$ and assume that L contains the monotone functions on E. For h given as above, we denote by h^- the function defined on E by:

$$h^-(x) = h[x - \alpha(a)].$$

For the simplification of the exposition, we use only isotonic linear functionals on L and assume that h is positive on E, that is $\epsilon_+ = +1$. We can obtain similar results for superlinear functionals. We can also take $E \subseteq [a, \alpha(a))$.

Theorem 9 *Let* A *be an isotonic linear functional on* L *and* $h_1, h_2 \in L$ *be positive increasing functions. If* f_i *is* (g_i, α, β_i)*-starshaped with respect to* h_i *for* $i = 1, 2$ *, then it is valid the inequality:*

$$A(f_1^- f_2^-)A(h_1^-)A(h_2^-) \geq A(h_1^- h_2^-)A(f_1^-)A(f_2^-), \tag{5}$$

where $f_i^-(x) = g_i[f_i(x)] - \beta_i[f_i(a)]$ *for* $i = 1, 2$.

Proof. The functions $F_i = f_i^- / h_i^-$ are increasing. Chebyshev's inequality for F_1 and F_2 with weight $h_1^- h_2^-$ gives:

$$A(f_1^- f_2^-)A(h_1^- h_2^-) \geq A(h_2^- f_1^-)A(h_1^- f_2^-).$$

The same inequality for increasing functions F_1, h_2^- and weight h_1^- implies

$$A(h_2^- f_1^-)A(h_1^-) \geq A(h_1^- h_2^-)A(f_1^-).$$

Similarly we get:

$$A(h_1^- f_2^-)A(h_2^-) \geq A(h_1^- h_2^-)A(f_2^-).$$

Combining these inequalities we get (5). ∎

Corollary 10 *Let* A *be an isotonic linear functional on* L *and* $h_1, h_2 \in L$ *be positive increasing functions. If* f_i *is* $(g_i, \alpha, 0)$*-starshaped with respect to* h_i *for* $i = 1, 2$ *, then it is valid the inequality:*

$$A[(g_1 \circ f_1)(g_2 \circ f_2)]A(h_1^-)A(h_2^-) \geq A(h_1^- h_2^-)A(g_1 \circ f_1)A(g_2 \circ f_2).$$

Corollary 11 *Let* A *be an isotonic linear functional on* L *and* $h_1, h_2 \in L$ *be positive increasing functions. If* f_i *is* $(g_i, 0, 0)$*-starshaped with respect to* h_i *for* $i = 1, 2$ *, then it is valid the inequality:*

$$A[(g_1 \circ f_1)(g_2 \circ f_2)]A(h_1)A(h_2) \geq A(h_1 h_2)A(g_1 \circ f_1)A(g_2 \circ f_2).$$

Corollary 12 *If the continuous function* $f_i : [0,1] \to \mathbb{R}$ *is starshaped with respect to* e_{p_i} *for* $i = 1, 2$ *, then the following inequality:*

$$\int_0^1 f_1(x) f_2(x) dx \geq \frac{(p_1+1)(p_2+1)}{p_1+p_2+1} \int_0^1 f_1(x) dx \int_0^1 f_2(x) dx$$

holds.

The following generalization of Anderson's inequality can be also proved.

Corollary 13 *Let* A *be an isotonic linear functional on* L *and* $h \in L$ *be a positive increasing function. If* f_i *is starshaped with respect to* h *for each* $i = 1, 2, ..., m$*, then the following inequality is valid:*

$$A(f_1 ... f_m) A^m(h) \geq A(h^m) A(f_1) ... A(f_m).$$

Corollary 14 *If the continuous functions* $f_i : [0,1] \to \mathbb{R}$ *are starshaped with respect to* e_p *for* $i = 1, 2, ..., m$ *, then the following inequality:*

$$\int_0^1 f_1(x) \cdots f_m(x) dx \geq \frac{(p+1)^m}{mp+1} \int_0^1 f_1(x) dx \cdots \int_0^1 f_m(x) dx$$

holds.

Remark 15 *As in Anderson's inequality, all the above results are valid for corresponding generalized convex functions, but this is only a special case.*

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J. Math. Inequ. 3(2009), 3, 305-313

The hierarchy of convexity and some classic inequalities

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Abstract

In what follows, a hierarchy of m - convexity is considered: we define m - starshaped functions, m - superadditive functions, Jensen m - convex functions, weak Jensen m - convex functions, Jensen m - superadditive functions, and weak m - superadditive functions. Some inclusions between such classes of functions are established. We also analyze the validity of the Hermite-Hadamard inequality, and of the Chebyshev-Andersson inequality for m-convex functions.

Mathematics subject classification (2000): 26A51, 26D15

Key words and phrases: Hermite - Hadamard inequality, Chebyshev - Andersson inequality, hierarchy of m - convexity

1 Introduction

Let us consider the sets of continuous, convex, starshaped, and superadditive functions on $[a, b]$ given by:

$$C[a,b] = \{f : [a,b] \longrightarrow \mathbb{R}, f \quad \text{continuous}\},$$

$$K[a,b] = \{f \in C[a,b]; f(tx+(1-t)y) \leq tf(x)+(1-t)f(y), \forall x, y \in [a,b], t \in [0,1]\},$$

$$S^*[a,b] = \left\{ f \in C[a,b]; \frac{f(x)-f(a)}{x-a} \leq \frac{f(y)-f(a)}{y-a},\ a < x < y \leq b \right\},$$

and

$$S[a,b] = \{f \in C[a,b]; f(x)+f(y) \leq f(x+y-a)+f(a), \forall x, y, x+y-a \in [a,b]\},$$

respectively. For $a = 0$ we denote by $C(b), K(b), S^*(b)$, and $S(b)$ respectively, the corresponding set of functions, restricted also under the condition $f(0) = 0$. A.M. Bruckner and E. Ostrow have proven in [1] the strict inclusions:

$$K(b) \subset S^*(b) \subset S(b).$$

These inclusions, extended with some results of preservation of the above properties by the arithmetic integral mean, are collectively referred to in [6] as the **hierarchy of convexity**. Simple proofs and generalizations of the results of [1] may be found in [8].

Let us remark that we can also define a superadditive function by

$$f(x) + f(y) \leq f(x + y - a) + f(a), \forall x, y \in [a, b],$$

thus assuming $f \in C[a, 2b - a]$. This is the preferred layout for superadditive functions in what follows.

In [9], one of the many generalizations on the convexity of functions - called m - convexity - was introduced. The set of m - convex functions is defined by:

$$K_m[a, b] = \{f \in C[a, b]; f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y),$$

$$\forall x, y \in [a, b], t \in [0, 1]\},\ m \in [0, 1].$$

If $a = 0$ and $f(0) \leq 0$, we also obtain a hierarchy of convexity:

$$K[a, b] \subset K_m[a, b] \subset K_n[a, b] \subset S^*[a, b],\ \text{for } 1 > m > n > 0.$$

A much larger generalization of convexity was given in [12] : the function $f : [a, b] \to \mathbb{R}$ is called $(g, h, \lambda, \mu)-$convex if

$$g(f(tx + (1 - t)\lambda(y))) \leq h(t)g(f(x)) + [1 - h(t)]\mu(f(y)), \forall x, y \in [a, b], \forall t \in [0, 1].$$

It is shown that more interesting results can be obtained for $h(t) = t^\alpha$, with $\alpha \in [0, 1]$. This case was combined with the m - convexity in [5] giving the $(\alpha, m)-$ convexity. In the next paragraph we define a hierarchy of $(\alpha, m)-$ convexity. Taking $\alpha = 1$, we obtain a more fruitful hierarchy of $m -$ convexity. Finally we study the Fejér inequality (generalization of the Hermite-Hadamard inequality) and the Chebyshev-Andersson inequality for $m -$ convex functions.

2 A hierarchy of $(\alpha, m)-$convexity

The set of $(\alpha, m)-$ **convex functions** is defined by

$$K_{m,\alpha}[a, b] = \{f \in C[a, b]; f(tx + m(1 - t)y) \leq t^\alpha f(x) + m(1 - t^\alpha)f(y),$$

$$\forall x, y \in [a, b], t \in [0, 1]\},\ m, \alpha \in [0, 1].$$

Note that for $t = 0$ and $y = a$ we have the condition $f(ma) \leq mf(a)$ meaning that the function must be defined on $ma \leq a$. In fact, to assure that all the definitions and results that follow are valid we will assume that the functions are defined on $[ma, 2b - ma]$. Assuming $\alpha \neq 0, m \neq 0$, we define the following sets of functions:

$$S^*_{m,\alpha}[a, b] = \left\{f \in C[a, b]; \frac{f(x) - mf(a)}{(x - ma)^\alpha} \geq \frac{f(z) - mf(a)}{(z - ma)^\alpha},\ a < z < x \leq b\right\},$$

called $(\alpha, m)-$ **starshaped functions**;

$$S_{m,\alpha}[a,b] = \{f \in C[a,b]; [f(x) - mf(a)](x - ma)^{1-\alpha} + [f(y) - mf(a)]$$

$$\cdot (y - ma)^{1-\alpha} \leq [f(x + y - ma) - mf(a)](x + y - 2ma)^{1-\alpha}, \forall x, y \in [a,b]\},$$

called $(\alpha, m)-$ **superadditive functions**;

$$J^{*}_{m,\alpha}[a,b] = \{f \in C[a,b]; f(2x - ma) - mf(a) \geq 2^{\alpha}[f(x) - mf(a)], \forall x \in [a,b]\},$$

called **Jensen** $(\alpha, m)-$ **starshaped functions**;

$$J_{m,\alpha}[a,b] = \left\{ f \in C[a,b]; f\left(\frac{m^{\frac{1}{\alpha}}x + my}{1 + m^{\frac{1}{\alpha}}}\right)\right.$$

$$\left.\leq \frac{mf(x) + m\left[\left(1 + m^{\frac{1}{\alpha}}\right)^{\alpha} - m\right]f(y)}{\left(1 + m^{\frac{1}{\alpha}}\right)^{\alpha}}, \forall x, y \in [a,b]\right\},$$

called $(\alpha, m)-$ **Jensen convex functions**;

$$H_{m,\alpha}[a,b]\left\{= f \in C[a,b]; f(tx) \leq \left[m + (t - m)^{\alpha}(1 - m)^{1-\alpha}\right]f(x),\right.$$

$$\left. a \leq x \leq b, m \leq t \leq 1\right\},$$

called (α, m) $-$ **subhomogenous functions**;

$$H^{*}_{m,\alpha}[a,b] = \left\{ f \in C[a,b]; f\left(\frac{m + m^{\frac{1}{\alpha}}}{1 + m^{\frac{1}{\alpha}}}x\right)\right.$$

$$\left.\leq m\left[1 + \frac{1 - m}{\left(1 + m^{\frac{1}{\alpha}}\right)^{\alpha}}\right]f(x),\ a \leq x \leq b\right\},$$

called **Jensen** $(\alpha, m)-$ **subhomogenous functions;**

$$wS_{m,\alpha}[a,b] = \{f \in C[a,b]; [f(a + t) - mf(a)](a + t - ma)^{1-\alpha} + [f(b - t)$$

$$-mf(a)] \cdot (b - t - ma)^{1-\alpha} \leq [f(b + (1 - m)a) - mf(a)](a + b - 2ma)^{1-\alpha},$$

$$\forall t \in [0, (b - a)/2]\},$$

called **weak** $(\alpha, m)-$**superadditive**; and

$$wJ_{m,\alpha}[a,b] = \left\{ f \in C[a,b]; \frac{m}{\left(1 + m^{\frac{1}{\alpha}}\right)^{\alpha}}\left\{f(a + t) + \left[\left(1 + m^{\frac{1}{\alpha}}\right)^{\alpha} - m\right]\right.\right.$$

$$\left.\left. f(b - t)\right\} \geq f\left(\frac{m^{\frac{1}{\alpha}}(a + t) + m(b - t)}{1 + m^{\frac{1}{\alpha}}}\right), \forall t \in [0, (b - a)/2]\right\},$$

called **weak** $(\alpha, m)-$ **Jensen convex**.

For these sets, we have the following main results.

Theorem 1 *The following inclusions*

$$K_{m,\alpha}[a,b] \subseteq S^*_{m,\alpha}[a,b] \subseteq S_{m,\alpha}[a,b] \subseteq J^*_{m,\alpha}[a,b], S_{m,\alpha}[a,b] \subseteq wS_{m,\alpha}[a,b],$$

$$H^*_{m,\alpha}[a,b] \supseteq H_{m,\alpha}[a,b] \supseteq K_{m,\alpha}[a,b] \subseteq J_{m,\alpha}[a,b] \subseteq H^*_{m,\alpha}[a,b]$$

and

$$J_{m,\alpha}[a,b] \subseteq wJ_{m,\alpha}[a,b]$$

hold.

Proof. a) Taking $f \in K_{m,\alpha}[a,b]$ and $y = a$ we obtain

$$f(xt + m(1-t)y) - mf(a) \leq t^\alpha \left[f(x) - mf(a)\right].$$

Denoting $xt + m(1-t)y = z$ we prove that $f \in S^*_{m,\alpha}[a,b]$. b) Assuming that $f \in S^*_{m,\alpha}[a,b]$ we have

$$\left[f(x+y-ma) - mf(a)\right](x+y-2ma)^{1-\alpha} = \frac{f(x+y-ma) - mf(a)}{(x+y-2ma)^\alpha}$$

$$\cdot (x+y-2ma) = \frac{f(x+y-ma) - mf(a)}{(x+y-2ma)^\alpha}(x-ma) + \frac{f(x+y-ma) - mf(a)}{(x+y-2ma)^\alpha}\cdot$$

$$(y-ma) \geq \frac{f(x) - mf(a)}{(x-ma)^\alpha}(x-ma) + \frac{f(y) - mf(a)}{(y-ma)^\alpha}(y-ma),$$

thus $f \in S_{m,\alpha}[a,b]$. c) For $f \in S_{m,\alpha}[a,b]$ if we take $x = y$ we obtain

$$2\left[f(x) - mf(a)\right](x-ma)^{1-\alpha} \leq \left[f(2x-ma) - mf(a)\right](2x-2ma)^{1-\alpha},$$

implying that $f \in J^*_{m,\alpha}[a,b]$. d) For $f \in S_{m,\alpha}[a,b]$ if we take $x = a-t, y = b-t$ we obtain $f \in wS_{m,\alpha}[a,b]$. e) If $f \in K_{m,\alpha}[a,b]$ for $t = m^{1/\alpha}/\left(1+m^{1/\alpha}\right)$ we deduce that $f \in J_{m,\alpha}[a,b]$. f) For $f \in J_{m,\alpha}[a,b]$ if we take $x = y$ we obtain that $f \in H^*_{m,\alpha}[a,b]$. g) If $f \in K_{m,\alpha}[a,b]$ for $x = y$ we obtain

$$f(x(m+t(1-m)) \leq \left[t^\alpha + m(1-t^\alpha)\right]f(x)$$

and denoting $m + t(1-m) = s$ we deduce that $f \in H_{m,\alpha}[a,b]$. h) If $f \in H_{m,\alpha}[a,b]$, for $t = (m+m^{1/\alpha})/(1+m^{1/\alpha})$ it follows that $f \in H^*_{m,\alpha}[a,b]$. k) For $f \in J_{m,\alpha}[a,b]$ if we take $x = a-t, y = b-t$ we obtain that $f \in wJ_{m,\alpha}[a,b]$. ■

3 A hierarchy of $m-$convexity

For $\alpha = 1$ we obtain the following sets of functions:

$$S^*_m[a,b] = \left\{f \in C[a,b]; \frac{f(x) - mf(a)}{x-ma} \geq \frac{f(z) - mf(a)}{z-ma},\ a \leq z < x \leq b\right\},$$

called $m -$ starshaped functions;

$$S_m[a,b] = \{f \in C[a,b]; f(x) + f(x) \leq f(x+y-ma) + mf(a), \forall x, y \in [a,b]\},$$

called $m -$ superadditive functions;

$$J_m^*[a,b] = \{f \in C[a,b]; f(2x-ma) - mf(a) \geq 2\left[f(x) - mf(a)\right],\ a \leq x \leq b\},$$

called Jensen $m -$ starshaped functions;

$$J_m[a,b] = \left\{f \in C[a,b]; f\left(\frac{m\,(x+y)}{1+m}\right) \leq \frac{m\left[f(x)+f(y)\right]}{1+m}, \forall x, y \in [a,b]\right\},$$

called $m -$ Jensen convex functions;

$$H_m[a,b] = \{f \in C[a,b]; f(tx) \leq tf(x),\ a \leq x \leq b, m \leq t \leq 1\},$$

called $m -$ subhomogenous functions;

$$H_m^*[a,b] = \left\{f \in C[a,b]; f\left(\frac{2mx}{1+m}\right) \leq \frac{2m}{1+m} f(x),\ a \leq x \leq b\right\},$$

called Jensen $m -$ subhomogenous functions;

$$wS_m[a,b] = \{f \in C[a,b]; f(a+t) + f(b-t) \leq f(b+(1-m)a) + mf(a),$$
$$\forall t \in [0,(b-a)/2]\},$$

called **weak** m **- superadditive**; and

$$wJ_m[a,b] = \left\{f \in C[a,b]; \frac{m\left[f(a+t)+f(b-t)\right]}{1+m}\right.$$
$$\left.\geq f\left(\frac{m\,(a+b)}{1+m}\right), \forall t \in [0,(b-a)/2]\right\},$$

called **weak** m **- Jensen convex**.

From the hierarchy of $m -$ convexity we underline only some results.

Theorem 2 *The following inclusions*

$$K_m[a,b] \subseteq S_m^*[a,b] \subseteq S_m[a,b] \subseteq wS_m[a,b]$$

and

$$H_m^*[a,b] \supseteq H_m[a,b] \supseteq K_m[a,b] \subseteq J_m[a,b] \subseteq wJ_m[a,b]$$

hold.

Moreover, in this simple case $\alpha = 1$ we can characterize the functions from $wS_m[a,b]$ and those from $wJ_m[a,b]$. For this we begin with the following:

Lemma 3 *For every function $f \in C[a,b]$ we can determine two functions $f_1 : [a(1-m),(b+(1-2m)a)/2] \longrightarrow \mathbb{R}$ and $f_2 : [0,(b+(1-2m)a)/2] \longrightarrow \mathbb{R}$ such that:*

$$f(x) = \begin{cases} f_1(x-ma) & for \quad x \in \left[a, \frac{a+b}{2}\right] \\ f_1\left(\frac{b+(1-2m)a}{2}\right) + f_2\left(\frac{b+(1-2m)a}{2}\right) & \\ -f_2(b+(1-m)x) & for \quad x \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

Proof. We can take:

$$f_1(t) = f(ma+t), \forall t \in [a(1-m),(b+(1-2m)a)/2]$$

and

$$f_2(t) = f((b+a)/2) + c - f(b+a(1-m)-t), \forall t \in [0,(b+(1-2m)a)/2],$$

where c is an arbitrary real number. ■

Using this lemma we can obtain the characterization and a method of construction of functions from $wS_m[a,b]$ and $wJ_m[a,b]$.

Theorem 4 *The function f belongs to:*

a) $wS_m[a,b]$ *if and only if*

$$f_1(t+a(1-m)) - mf_1(a(1-m)) \le f_2(t+a(1-m)) - f_2(0);$$

b) $wJ_m[a,b]$ *if and only if*

$$f_1(t+a(1-m)) + f_1\left(\frac{b+(1-2m)a}{2}\right) - \frac{1+m}{m} f_1\left(\frac{m(b-am)}{1+m}\right)$$

$$\ge f_2(t+a(1-m)) - f_2\left(\frac{b+(1-2m)a}{2}\right).$$

Corollary 5 *The function f belongs to $wS_m[a,b]$ if*

$$f_1(t) = f_2(t), \forall t \in [a(1-m),(b+(1-2m)a)/2]$$

and

$$f_1\left(\frac{b+(1-2m)a}{2}\right) \ge \frac{1+m}{2m} f_1\left(\frac{m(b-am)}{1+m}\right).$$

Corollary 6 *The function f belongs to $wJ_m[a,b]$ if*

$$f_1(t) = f_2(t), \forall t \in [a(1-m),(b+(1-2m)a)/2]$$

and

$$f_2(0) \le mf_1(a(1-m)).$$

Corollary 7 *The function f belongs to $wS_m[a,b] \cap wJ_m[a,b]$ if*

$$f_1(t) = f_2(t), \forall t \in [a(1-m), (b+(1-2m)a)/2]$$

$$f_2(0) \leq m f_1(a(1-m))$$

and

$$f_1\left(\frac{b+(1-2m)a}{2}\right) \geq \frac{1+m}{2m} f_1\left(\frac{m(b-am)}{1+m}\right).$$

Remark 8 *For $m = 1$ these results were proven in [11].*

4 Fejér's inequality

Let $L(\cdot, a, b) : C[a,b] \longrightarrow \mathbb{R}$ be an **isotonic linear functional**, that is, for $t, s \in \mathbb{R}, f, g \in C[a,b]$:

$$L(f; a, b) \geq 0 \quad if \quad f \geq 0$$

$$L(tf + sg; a, b) = tL(f; a, b) + sL(g; a, b).$$

If $f \in C[a,b]$ we denote by f_- the function defined by:

$$f_-(x) = f(a+b-x) \quad for \quad x \in [a,b].$$

Definition 9 *The functional $L(\cdot, a, b)$ is* ***symmetric*** *if:*

$$L(f_-; a, b) = L(f; a, b), \ \forall f \in C[a,b].$$

Theorem 10 *If $L(\cdot; a, b)$ is a symmetric isotonic linear functional, such that $L(1; a, b) = 1$, then:*

$$L(f; a, b) \leq [f(b+(1-m)a) + mf(a)]/2, \ \forall f \in wS_m[a,b]$$

and

$$L(f; a, b) \geq \frac{m+1}{2m} f\left(\frac{m(a+b)}{1+m}\right), \ \forall f \in wJ_m[a,b].$$

Proof. Indeed in the first case we have

$$f(a+t) + f(b-t) = f(x) + f_-(x)$$

$$\leq f(b+(1-m)a) + mf(a), \forall x \in [a,b]$$

while in the second:

$$f(x) + f_-(x) \geq \frac{m+1}{m} f\left(\frac{m(a+b)}{1+m}\right), \forall x \in [a,b].$$

We need only to apply the functional $L(\cdot; a, b)$. ■

Corollary 11 *If $L(\cdot; a, b)$ is a symmetric isotonic linear functional, such that $L(1; a, b) = 1$, then:*

$$\frac{m+1}{2m} f\left(\frac{m(a+b)}{1+m}\right) \leq L(f; a, b) \leq \left[f(b + (1-m)a) + mf(a)\right]/2,$$

$$\forall f \in wS_m[a, b] \cap wJ_m[a, b].$$

Remark 12 *If $g \in C[a, b]$ is symmetric with respect to $\frac{a+b}{2}$, the functional defined by:*

$$L(f; a, b) = \int_a^b f(x)g(x)dx \;/\; \int_a^b g(x)dx$$

is a symmetric isotonic linear functional. As $K_m[a, b] \subset wS_m[a, b] \cap wJ_m[a, b]$ we obtained a generalization of the result of L. Fejér from [3] , thus also of the Hermite-Hadamard inequality. The generalization is effective even for $m = 1$ as was pointed out in [11]. Other generalizations of the Hermite-Hadamard inequality for m - convex functions were given in [2], [7], and [4].

5 Chebyshev-Andersson's inequality

In [10] we have shown that Chebyshev-Andersson's inequality is not only valid for convex functions but also for starshaped functions . A general result of this type was also proven in [12]. Let us now consider the case of $(\alpha, m)-$ starshaped functions. Denote by e the function defined by $e(x) = x$ and by c the constant function with value c.

Theorem 13 *If A and B are isotonic linear functionals, $f \in S^*_{m,\alpha}[a, b]$ and $g \in S^*_{n,\beta}[a, b]$ then the following inequality holds:*

$$A\left((e-ma)^{\alpha}(e-na)^{\beta}\right) B\left((f-mf(a))(g-ng(a))\right)$$

$$+B\left((e-ma)^{\alpha}(e-na)^{\beta}\right) A\left((f-mf(a))(g-ng(a))\right)$$

$$\leq A\left((e-ma)^{\alpha}(g-ng(a))\right) B\left((e-na)^{\beta}(f-mf(a))\right)$$

$$+B\left((e-ma)^{\alpha}(g-ng(a))\right) A\left((e-na)^{\beta}(f-mf(a))\right).$$

Proof. We have

$$\left[\frac{f(x)-mf(a)}{(x-ma)^{\alpha}} - \frac{f(z)-mf(a)}{(z-ma)^{\alpha}}\right](x-ma)^{\alpha}(z-ma)^{\alpha}$$

$$\cdot\left[\frac{g(x)-ng(a)}{(x-na)^{\beta}} - \frac{g(z)-ng(a)}{(z-na)^{\beta}}\right](x-na)^{\beta}(z-na)^{\beta} \geq 0,$$

or

$$
\begin{aligned}
& (z-ma)^{\alpha}(z-na)^{\beta}[f(x)-mf(a)][g(x)-ng(a)] \\
& -(z-ma)^{\alpha}[g(z)-ng(a)](x-na)^{\beta}[f(x)-mf(a)] \\
& -(z-na)^{\beta}[f(z)-mf(a)](x-ma)^{\alpha}[g(x)-ng(a)] \\
& +(x-ma)^{\alpha}(x-na)^{\beta}[f(z)-mf(a)][g(z)-ng(a)] \geq 0.
\end{aligned}
$$

If we now take the value of A for the functions of x and then the value of B for the functions of z, we obtain the announced inequality. ■

Remark 14 *Taking $A = B$ and/or $m = n, \alpha = \beta$, we deduce some consequences of the Chebyshev-Andersson type inequalities.*

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Annals of the T. Popoviciu Seminar of Functional Equations, Approximation and Convexity ISSN 1584-4536, vol. 7, 2009, pp. 147-154

A hierarchy of logarithmic convexity of functions

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Abstract

In what follows, a hierarchy of logarithmic (h, m) - convexity is considered: we define logarithmic (h, m) - starshaped functions, logarithmic (h, m) - superadditive functions, Jensen logarithmic (h, m) - convex functions, Jensen logarithmic (h, m) - superadditive functions. Some inclusions between such classes of functions are established.

Mathematics subject classification (2000): 26A51

Key words and phrases: hierarchy of logarithmic (h, m) - convexity of functions.

1 Introduction

Let us consider the sets of continuous, convex, starshaped, and superadditive functions on $[a, b]$ given by:

$$C[a,b] = \{f : [a,b] \longrightarrow \mathbb{R}, f \quad \text{continuous}\},$$

$$K[a,b] = \{f \in C[a,b]; f(tx+(1-t)y) \leq tf(x)+(1-t)f(y), \forall x, y \in [a,b], t \in [0,1]\},$$

$$S^*[a,b] = \{f \in C[a,b]; f(tx+(1-t)a) \leq tf(x)+(1-t)f(a), \forall x \in [a,b], t \in [0,1]\},$$

and

$$S[a,b]=\{f\in C[a,b]; f(x)+f(y)\leq f(x+y-a)+f(a), \forall x,y,x+y-a\in[a,b]\},$$

respectively. For $a=0$ we denote by $C(b), K(b), S^*(b)$, and $S(b)$ respectively, the corresponding set of functions, restricted also under the condition $f(0)=0$. A.M. Bruckner and E. Ostrow have proven in [1] the strict inclusions:

$$K(b)\subset S^*(b)\subset S(b).$$

These inclusions, extended with some results of preservation of the above properties by the arithmetic integral mean, are collectively referred to in [2] as the **hierarchy of convexity**. Simple proofs and generalizations of the results of [1] may be found in [4].

Let us remark that we can also define a superadditive function by

$$f(x)+f(y)\leq f(x+y-a)+f(a), \forall x,y\in[a,b],$$

thus assuming $f\in C[a,2b-a]$. This is the preferred layout for superadditive functions in what follows.

In [5], one of the many generalizations on the convexity of functions - called m - convexity - was introduced. The set of m - convex functions is defined by:

$$K_m[a,b]=\{f\in C[a,b]; f(tx+m(1-t)y)\leq tf(x)+m(1-t)f(y),$$

$$\forall x,y\in[a,b], t\in[0,1]\},\ m\in[0,1].$$

A much larger generalization of convexity was given in [6] : the function $f:[a,b]\to\mathbb{R}$ belongs to $K_{g,h,\lambda,\mu}[a,b]$, or is $(g,h,\lambda,\mu)-$convex if

$$g(f(tx+(1-t)\lambda(y)))\leq h(t)g(f(x))+[1-h(t)]\mu(f(y)), \forall x,y\in[a,b], \forall t\in[0,1].$$

Here we define a hierarchy of convexity for it. The special case $\lambda(y)=\mu(y)=my$, $h(t)=t^\alpha$ and $g(x)=x$ was considered in [7]. Some other interesting special cases will be considered here.

2 A hierarchy of $(h,\lambda,\mu)-$convexity

The set of $(h,\lambda,\mu)-$ **convex functions** is defined by

$$K_{h,\lambda,\mu}[a,b]=\{f\in C[a,b]; f(tx+(1-t)\lambda(y))\leq h(t)f(x)+(1-h(t))\mu(f(y)),$$

$$\forall x,y\in[a,b], t\in[0,1]\}.$$

Note that for $t=0$ and $y=a$, if $h(0)=0$, we have the condition $f(\lambda(a))\leq\mu(f(a))$ meaning that the function must be defined on $\lambda(a)$. In fact, to assure that all the definitions and results that follow are valid we will assume that

$\lambda(a) \leq a$ and the functions are defined on $[\lambda(a), 2b-\lambda(a)]$. We define also the following sets of functions:

$$S^{*}_{h,\lambda,\mu}[a,b] = \{f \in C[a,b]; f(tx+(1-t)\lambda(a)) \leq h(t)f(x) + (1-h(t))\mu(f(a)),$$
$$\forall x \in [a,b], t \in [0,1]\},$$

called $(h,\lambda,\mu)-$ **starshaped functions**;

$$S_{h,\lambda,\mu}[a,b] = \left\{ f \in C[a,b]; \frac{[f(x+y-\lambda(a)) - \mu(f(a))]\,(x+y-2\lambda(a))}{h\,(x+y-2\lambda(a))} \geq \right.$$
$$\left. \frac{[f(x)-\mu(f(a))]\,(x-\lambda(a))}{h(x-\lambda(a))} + \frac{[f(y)-\mu(f(a))]\,(y-\lambda(a))}{h(y-\lambda(a))}, \forall x,y \in [a,b] \right\},$$

called $(h,\lambda,\mu)-$ **superadditive functions**; and

$$J^{*}_{h,\lambda,\mu}[a,b] = \left\{ f \in C[a,b]; \frac{f(2x-\lambda(a)) - \mu(f(a))}{h\,(2x-2\lambda(a))} \geq \frac{f(x)-\mu(f(a))}{h(x-\lambda(a))}, \forall x \in [a,b] \right\},$$

called **Jensen** $(h,\lambda,\mu)-$ **starshaped functions**.

For these sets, we have the following main results.

Theorem 1 *If the function h is supermultiplicative, that is it has the property*

$$h(t \cdot s) \geq h(t) \cdot h(s), \forall t,s \geq 0,$$

then the following inclusions

$$K_{h,\lambda,\mu}[a,b] \subseteq S^{*}_{h,\lambda,\mu}[a,b] \subseteq S_{h,\lambda,\mu}[a,b] \subseteq J^{*}_{h,\lambda,\mu}[a,b],$$

hold.

Proof. a) For $f \in K_{h,\lambda,\mu}[a,b]$ and $y = a$ we obtain $f \in S^{*}_{h,\lambda,\mu}[a,b]$. b) Assuming that $f \in S^{*}_{h,\lambda,\mu}[a,b]$ we have

$$f(xt+(1-t)\lambda(a)) - \mu(f(a)) \leq h(t)\,[f(x)-\mu(f(a))]\,.$$

Denoting $xt+(1-t)\lambda(a) = z$ we deduce that

$$\frac{f(x)-\mu(f(a))}{h\,(x-\lambda(a))} \geq \frac{f(z)-\mu(f(a))}{h(t)h\,(x-\lambda(a))} \geq \frac{f(z)-\mu(f(a))}{h(t\,(x-\lambda(a)))},$$

thus

$$\frac{f(x)-\mu(f(a))}{h\,(x-\lambda(a))} \geq \frac{f(z)-\mu(f(a))}{h\,(z-\lambda(a))}, \text{for } \lambda(a) < z < x \leq b.$$

So

$$\frac{[f(x+y-\lambda(a))-\mu(f(a))]\,(x+y-2\lambda(a))}{h\,(x+y-2\lambda(a))} =$$
$$\frac{f(x+y-\lambda(a))-\mu(f(a))}{h\,(x+y-2\lambda(a))}\,(x-\lambda(a)) + \frac{f(x+y-\lambda(a))-\mu(f(a))}{h\,(x+y-2\lambda(a))}\cdot$$
$$(y-\lambda(a)) \geq \frac{f(x)-\mu(f(a))}{h\,(x-\lambda(a))}(x-\lambda(a)) + \frac{f(y)-\mu(f(a))}{h\,(y-\lambda(a))}(y-\lambda(a)),$$

thus $f \in S_{h,\lambda,\mu}[a,b]$. c) For $f \in S_{h,\lambda,\mu}[a,b]$ if we take $x = y$ we obtain that $f \in J^{*}_{h,\lambda,\mu}[a,b]$. ■

3 A hierarchy of $(h,m)-$convexity

Taking $\lambda(y)=\mu(y)=my, m\in[0,1]$, we have the set of functions:

$$K_{h,m}[a,b]=\{f\in C[a,b]; f(tx+(1-t)my)\leq h(t)f(x)+(1-h(t))mf(y),$$
$$\forall x,y\in[a,b], t\in[0,1]\},$$
$$S^{*}_{h,m}[a,b]=\{f\in C[a,b]; f(tx+(1-t)ma)\leq h(t)f(x)+(1-h(t))mf(a),$$
$$\forall x\in[a,b], t\in[0,1]\},$$
$$S_{h,m}[a,b]=\left\{f\in C[a,b]; \frac{[f(x+y-ma)-mf(a)]\,(x+y-2ma)}{h\,(x+y-2ma)}\geq\right.$$
$$\left.\frac{[f(x)-mf(a)]\,(x-ma)}{h(x-ma)}+\frac{[f(y)-mf(a)]\,(y-ma)}{h(y-ma)}, \forall x,y\in[a,b]\right\},$$
$$J^{*}_{h,m}[a,b]=\left\{f\in C[a,b]; \frac{f(2x-ma)-mf(a)}{h\,(2x-2ma)}\geq\frac{f(x)-mf(a)}{h(x-ma)}, \forall x\in[a,b]\right\}.$$

In this case we can define some new sets of functions:

$$J_{h,m}[a,b]=\left\{f\in C[a,b]; f\left(\frac{m(x+y)}{1+m}\right)\right.$$
$$\left.\leq h\left(\frac{m}{1+m}\right)f(x)+m\left[1-h\left(\frac{m}{1+m}\right)\right]f(y), \forall x,y\in[a,b]\right\},$$

called $(h,m)-$ **Jensen convex functions**;

$$H_{h,m}[a,b]=\left\{f\in C[a,b]; f(tx)\leq\left[m+(1-m)h\left(\frac{t-m}{1-m}\right)\right]f(x),\right.$$
$$\left. a\leq x\leq b, m\leq t\leq 1\right\},$$

called (h,m) $-$ **subhomogenous functions**; and

$$H^{*}_{h,m}[a,b]=\left\{f\in C[a,b]; f\left(\frac{2mx}{1+m}\right)\right.$$
$$\left.\leq\left[m+(1-m)\,h\left(\frac{m}{1+m}\right)\right]f(x),\ a\leq x\leq b\right\},$$

called **Jensen** $(h,m)-$ **subhomogenous functions.**

For these sets, we have the following main results.

Theorem 2 *If the function h is supermultiplicative, then the following inclusions*

$$K_{h,m}[a,b]\subseteq S^{*}_{h,m}[a,b]\subseteq S_{h,m}[a,b]\subseteq J^{*}_{h,m}[a,b],$$

and

$$H^{*}_{h,m}[a,b]\supseteq H_{h,m}[a,b]\supseteq K_{h,m}[a,b]\subseteq J_{h,m}[a,b]\subseteq H^{*}_{h,m}[a,b]$$

hold.

Proof. The first three inclusions were proved in the first theorem. a) If $f \in K_{h,m}[a,b]$ for $t = m/(1+m)$ we deduce that $f \in J_{h,m}[a,b]$. b) For $f \in J_{h,m}[a,b]$ if we take $x = y$ we obtain that $f \in H^*_{h,m}[a,b]$. g) If $f \in K_{h,m}[a,b]$ for $x = y$ we obtain

$$f(x(m+t(1-m)) \leq [h(t) + m(1-h(t))]\, f(x)$$

and denoting $m+t(1-m) = s$ we deduce that $f \in H_{h,m}[a,b]$. h) If $f \in H_{h,m}[a,b]$, for $t = 2m/(1+m)$ it follows that $f \in H^*_{h,m}[a,b]$. ■

Remark 3 *The special case when the function h is multiplicative, thus $h(x) = x^{\alpha}$, was treated in [7].*

4 A hierarchy of logarithmic (h,m) −convexity

It is easy to see that $f \in K_{g,h,\lambda,\mu}[a,b]$ if and only if $g \circ f \in K_{h,\lambda,\mu}[a,b]$. So we can consider a hierarchy of (g,h,λ,μ) −convexity. Let us illustrate this by defining a hierarchy of $(\ln, h, m, m)$ −convexity, which we call a hierarchy of logarithmic (h,m) −convexity.

We denote the following sets of functions:

$$LK_{h,m}[a,b] = \{f \in C[a,b]; f(tx+(1-t)my) \leq [f(x)]^{h(t)} \cdot [f(y)]^{(1-h(t))m},$$

$$\forall x,y \in [a,b], t \in [0,1]\},$$

$$LS^*_{h,m}[a,b] = \{f \in C[a,b]; f(tx+(1-t)ma) \leq [f(x)]^{h(t)} \cdot [f(a)]^{(1-h(t))m},$$

$$\forall x \in [a,b], t \in [0,1]\},$$

$$LS_{h,m}[a,b] = \{f \in C[a,b]; \left[\frac{f(x+y-ma)}{f(a)^m}\right]^{\frac{(x+y-2ma)}{h(x+y-2ma)}} \geq$$

$$\left[\frac{f(x)}{f(a)^m}\right]^{\frac{(x-ma)}{h(x-ma)}} \cdot \left[\frac{f(y)}{f(a)^m}\right]^{\frac{(y-ma)}{h(y-ma)}}, \forall x,y \in [a,b]\},$$

$$LJ^*_{h,m}[a,b] = \left\{f \in C[a,b]; \left[\frac{f(2x-ma)}{f(a)^m}\right]^{\frac{1}{h(2x-2ma)}} \geq\right.$$

$$\left.\left[\frac{f(x)}{f(a)^m}\right]^{\frac{1}{h(x-ma)}}, \forall x \in [a,b]\right\},$$

$$LJ_{h,m}[a,b] = \left\{f \in C[a,b]; f\left(\frac{m(x+y)}{1+m}\right)\right.$$

$$\left.\leq [f(x)]^{h\left(\frac{m}{1+m}\right)} \cdot [f(y)]^{m\left[1-h\left(\frac{m}{1+m}\right)\right]}, \forall x,y \in [a,b]\right\},$$

$$LH_{h,m}[a,b] = \left\{f \in C[a,b]; f(tx) \leq [f(x)]^{\left[m+(1-m)h\left(\frac{t-m}{1-m}\right)\right]},\right.$$

$$a \leq x \leq b, m \leq t \leq 1\},$$

and

$$LH^*_{h,m}[a,b] = \left\{ f \in C[a,b]; f\left(\frac{2mx}{1+m}\right) \right.$$

$$\left. \leq [f(x)]^{\left[m+(1-m)h\left(\frac{m}{1+m}\right)\right]},\ a \leq x \leq b \right\}.$$

For these sets, we have the following main results.

Theorem 4 *If the function h is supermultiplicative, then the following inclusions*

$$LK_{h,m}[a,b] \subseteq LS^*_{h,m}[a,b] \subseteq LS_{h,m}[a,b] \subseteq LJ^*_{h,m}[a,b],$$

and

$$LH^*_{h,m}[a,b] \supseteq LH_{h,m}[a,b] \supseteq LK_{h,m}[a,b] \subseteq LJ_{h,m}[a,b] \subseteq LH^*_{h,m}[a,b]$$

hold.

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