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Hierarhies

of

Convexity

of

Sequences

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Introduction

Let us consider the sets of continuous, convex, starshaped, or superadditive functions on $[0, b]$ given by:

$$C(b) = \{f : [0, b] \longrightarrow \mathbb{R}, f(0) = 0, f \quad \text{continuous}\},$$

$$K(b) = \{f \in C(b); f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \forall x, y \in [0, b], t \in [0, 1]\},$$

$$S^*(b) = \left\{ f \in C(b); \frac{f(x)}{x} \leq \frac{f(y)}{y},\ 0 < x < y \leq b \right\},$$

and

$$S(b) = \{f \in C(b); f(x) + f(y) \leq f(x + y), \forall x, y, x + y \in [0, b]\},$$

respectively. A.M. Bruckner and E. Ostrow have proven in [1] the strict inclusions:

$$K(b) \subset S^*(b) \subset S(b).$$

These inclusions were extended with some results of preservation of the above properties by the arithmetic integral mean

$$A\left(f\right)\left(x\right) = \frac{1}{x} \int_0^x f\left(t\right) dt.$$

A function f is said to have the property "P" in the mean if $A(f)$ has the property "P". Denoting by $MK(b), MS^*(b)$ and $MS(b)$ the sets of functions which are convex, starshaped, respectively superadditive in the mean, in [1] was proved that

$$K(b) \subset MK(b) \subset S^*(b) \subset S(b) \subset MS^*(b) \subset MS(b),$$

which was named in [2] as the **hierarchy of convexity of functions**.

In [3] was proved a first **hierarchy of convexity of sequences.** Let us consider the sets of convex, starshaped, or superadditive sequences given by:

$$K = \left\{(a_n)_{n=0}^{\infty}, a_{n+2} - 2a_{n+1} + a_n \geq 0, n \geq 0\right\},$$

$$S^* = \left\{(a_n)_{n=0}^{\infty}, \frac{a_n - a_0}{n} \leq \frac{a_{n+1} - a_0}{n+1}, n \geq 1\right\},$$

$$S = \left\{(a_n)_{n=0}^{\infty}, a_{n+m} + a_0 \geq a_n + a_m, n, m \in \mathbb{N}\right\}.$$

We say that the sequence $(a_n)_{n=0}^{\infty}$ has the property "P" in the mean, if the sequence $(A_n)_{n=0}^{\infty}$ has the property "P", where:

$$A_n = \frac{a_0 + \cdots + a_n}{n+1}.$$

Denoting by MK, MS^* and MS the sets of sequences which are convex, starshaped, respectively superadditive in the mean, in [3] was proved that

$$K \subset MK \subset S^* \subset S \subset MS^* \subset MS.$$

In [4] and [5] this hierarchy was generalized by proving that t*he sequence* $(A_n)_{n=0}^{\infty}$ *given by:*

$$A_n = \frac{p_0 a_0 + \cdots + p_n a_n}{p_0 + \cdots + p_n} n \geq 0$$

is convex(or starshaped) for any convex (respectively starshaped) *sequence* $(a_n)_{n=0}^{\infty}$ *if and only if:*

$$p_n = p_0 \binom{u+n-1}{n} n \geq 0$$

where $u > 0$ is arbitrary and:

$$\binom{v}{0} = 1, \quad \binom{v}{n} = \frac{1}{n!} \prod_{k=0}^{n-1} (v-k) k \geq 1, \ v \in \mathbb{R}.$$

In this case:

$$A_n = A_n^u = \frac{\sum_{k=0}^{n} \binom{u+k-1}{k} a_k}{\binom{u+n}{n}}.$$

We say that the sequence $(a_n)_{n=0}^{\infty}$ has the property "P" in u-mean if $(A_n^u)_{n=0}^{\infty}$ has the property P. Denoting by $M^u K, M^u S^*$ and $M^u S$ the sets of sequences which are convex, starshaped, respectively superadditive in u-mean, it was proved that if $0 < v < u$, then *hold the strict inclusions:*

$$K \subset M^v K \subset M^u K \subset S^* \subset M^v S^* \subset M^u S^*.$$

The inclusion in S or $M^u S$ is more complicated to study. Some results can be found in [6].

I am also the author or coauthor of other thirty papers with subject related to the hierarchy of convexity of sequeces. We have studied high order hierarchies, hierarchy of supermultiplicity of sequences in a semigroup, inequalities, or applications. Most of those papers were published with more than twenty years ago, in Romanian of other less known journals. As I got demands of copies of some of these papers, I decided to offer them with open access.

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THE REPRESENTATION OF n-CONVEX SEQUENCES

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1. The mathematical literature is quite rich in papers which treat problems of the following type: for two sets of sequences, K' and K'' construct a transformations A with the property that $A(K') \subseteq K''$. Usually, such a transformation is given by a triangular matrix:

$$\|p_{m,i}\|, \quad i = 1, \ldots, m \text{ for } m = 1, 2, \ldots$$

i.e., to the sequence $a = (a_m)_{m=1}^{\infty}$ is attached $A(a) = (A_m(a))_{m=1}^{\infty}$ where:

$$A_m(a) = \sum_{k=1}^{m} p_{m,k} a_k.$$

Many references to papers concerned to transformations that preserve the n-convexity may be found in [3]. A characterization of such transformations is contained in [2], while [1] presents a characterization of the transformations which map the set of p-monotone sequences in that of q-monotone sequences.

Our aim is to construct a transformation of the set of n-positive sequences, $\mathbf{R}_n^+ = \{(a_m)_{m=1}^{\infty} :\ a_m \geq 0 \text{ for } m > n\}$, in the set K_n of n-convex sequences. In fact, the transformation is a bijection, so that it gives a representation of n-convex sequences by means of n-positive sequences.

2. Let us to specify some notations and definitions used in what follows.

For a real sequence $(a_m)_{m=1}^{\infty}$, the n-th order difference is defined by:

$$\Delta^0 a_m = a_m,\ \Delta^n a_m = \Delta^{n-1} a_{m+1} - \Delta^{n-1} a_m. \tag{1}$$

Definition 1. A sequence $(a_m)_{m=1}^{\infty}$ is said to be convex of order n (or n-convex) if $\Delta^n a_m \geq 0$ for all m.

The set of all n-convex sequences is denoted by K_n.

Definition 2. A sequence $(c_m)_{m=1}^{\infty}$ is said to be n-positive if $c_m \geq 0$ for $m > n$.

3. Before giving the main result, which we have already announced, let us state some auxiliary lemmas, interesting by themselves. Although they are simple enough, we do not find them in the specialized literature.

Lemma 1. *If*

$$a_m = \sum_{i=1}^{m} b_i \textit{ for all } m,$$

then

$$\Delta^n a_m = \Delta^{n-1} b_{m+1} \textit{ for any } m \textit{ and any } n \geq 1. \tag{2}$$

The proof is easy to do by induction. As a direct consequence we have:

Lemma 2. *The sequence $(a_m)_{m=1}^{\infty}$ is n-convex, if and only if there is a sequence $(b_m)_{m=1}^{\infty}$ with the property that $(b_m)_{m=2}^{\infty}$ is convex of order $n-1$ and such that*

$$a_m = \sum_{i=1}^{m} b_i \textit{ for all } m \geq 1. \tag{3}$$

Because 0-convexity means positivity, we will prove by induction

Lemma 3. *The sequence* $(a_m)_{m=1}^{\infty}$ *is* n*-convex if and only if there is a* n*-positive sequence* $(c_m)_{m=1}^{\infty}$ *such that:*

$$a_m = \sum_{i=1}^{m} d_{i,m} c_i, \tag{4}$$

where the coefficients $d_{i,m}$ *do not depend on the two sequences.*

Proof. By the lemma 2, a sequence $(a_m^1)_{m=1}^{\infty}$ is 1-convex, if and only if there is a 1-positive sequence $(c_m)_{m=1}^{\infty}$, such that:

$$a_m^1 = \sum_{i=1}^{m} c_i \text{ for } m \geq 1. \tag{5}$$

Then, by the same lemma, the sequence $(a_m^2)_{m=1}^{\infty}$ is 2-convex if and only if:

$$a_m^2 = \sum_{i=1}^{m} a_i^1, \tag{6}$$

where $(a_m^1)_{m=2}^{\infty}$ is 1-convex. So, there is a 1-positive sequence $(c_m^1)_{m=1}^{\infty}$ such that:

$$a_{m+1}^1 = \sum_{i=1}^{m} c_i^1 \text{ for } m \geq 1. \tag{7}$$

Let us to denote $a_1^1 = c_1$ and $c_i^1 = c_{i+1}$ for $i \geq 1$. Then, the sequence $(c_m)_{m=1}^{\infty}$ is 2-positive and (7) becomes:

$$a_m^1 = \sum_{i=2}^{m} c_i \text{ for } m \geq 2 \tag{7'}$$

and, by (6), we have for $m \geq 2$:

$$a_m^2 = a_1^1 + \sum_{i=2}^{m} a_i^1 = c_1 + \sum_{i=2}^{m} \sum_{j=2}^{i} c_j = c_1 + \sum_{j=2}^{m} \sum_{i=j}^{m} c_j,$$

or:

$$a_m^2 = \begin{cases} c_1 & \text{for} \quad m = 1, \\ c_1 + \displaystyle\sum_{j=2}^{m} (m-j+1)c_j & \text{for} \quad m \geq 2. \end{cases} \tag{8}$$

Now suppose that a sequence is n-convex, if and only if there is a n-positive sequence $(c_m)_{m=1}^{\infty}$ such that:

$$a_m^n = \begin{cases} \displaystyle\sum_{i=1}^{m} p_{i,m}^n c_i & \text{for} \quad m < n, \\ \displaystyle\sum_{i=1}^{n-1} q_{i,m}^n c_i + \sum_{i=n}^{m} r_{i,m}^n c_i & \text{for} \quad m \geq n, \end{cases} \tag{9}$$

where the coefficients $p_{i,m}^n, q_{i,m}^n$ and $r_{i,m}^n$ are independent on the two sequences.

By the lemma 2, the sequence $(a_m^{n+1})_{m=1}^{\infty}$ is convex of order $n+1$ if and only if is a sequence $(a_m^n)_{m=1}^{\infty}$ such that $(a_m^n)_{m=2}^{\infty}$ is n-convex and:

$$a_m^{n+1} = \sum_{i=1}^{m} a_i^n \text{ for any } m \geq 1. \tag{10}$$

But then, as in (9), we must have a n-positive sequence $(c'_m)_{m=1}^{\infty}$ such that, for $m \geq 1$:

$$a_{m+1}^n = \begin{cases} \displaystyle\sum_{i=1}^{m} p_{i,m}^n c'_i & \text{for} \quad m < n, \\ \displaystyle\sum_{i=1}^{n-1} q_{i,m}^n c'_i + \sum_{i=n}^{m} r_{i,m}^n c'_i & \text{for} \quad m \geq n. \end{cases} \tag{11}$$

If we denote:

$$a_1^n = c_1 \text{ and } c'_i = c_{i+1} \text{ for } i \geq 1$$

the sequence $(c_m)_{m=1}^{\infty}$ is $n+1$-positive. Moreover, if we replace $i+1$ by i and, after that, $m+1$ by m, from (11) we get for $m \geq 2$:

$$a_m^n = \begin{cases} \sum_{i=2}^{m} p_{i-1,m-1}^n c_i & \text{for} \quad m-1 < n, \\ \sum_{i=2}^{n} q_{i-1,m-1}^n c_i + \sum_{i=n+1}^{m} r_{i-1,m-1}^n c_i & \text{for} \quad m-1 \geq n. \end{cases} \tag{12}$$

From (10) and (12) we have:

a) for $m = 1$:

$$a_1^{n+1} = a_1^n = c_1;$$

b) for $1 < m < n+1$:

$$\begin{aligned} a_m^{n+1} &= a_1^n + \sum_{i=2}^{m} a_i^n = c_1 + \sum_{i=2}^{m} \sum_{j=2}^{i} p_{j-1,i-1}^n c_j \\ &= c_1 + \sum_{j=2}^{m} \sum_{i=j}^{m} p_{j-1,i-1}^n c_j = \sum_{j=1}^{m} p_{j,m}^{n+1} c_j; \end{aligned}$$

c) for $m \geq n+1$:

$$a_m^{n+1} = a_1^n + \sum_{i=2}^{n} a_i^n + \sum_{i=n+i}^{m} a_i^n$$

$$= c_1 + \sum_{i=2}^{n} \sum_{j=2}^{i} p_{j-1,i-1}^n c_j + \sum_{i=n+1}^{m} \left[\sum_{j=2}^{n} q_{j-1,i-1}^n c_j + \sum_{j=n+1}^{m} r_{j-1,i-1}^n c_j \right]$$

$$= c_1 + \sum_{j=2}^{n} \sum_{i=j}^{n} p_{j-1,i-1}^n c_j + \sum_{j=2}^{n} \sum_{i=n+1}^{m} q_{j-1,i-1}^n c_j + \sum_{j=n+1}^{m} \sum_{i=j}^{m} r_{j-1,i-1}^n c_j$$

$$= \sum_{j=1}^{n} q_{j,m}^{n+1} c_j + \sum_{j=n+1}^{m} r_{j,m}^{n+1} c_j.$$

This completes the induction and, moreover, gives us the following recurrence relations:

$$p_{1,m}^{n+1} = 1, \; p_{j,m}^{n+1} = \sum_{i=j}^{m} p_{j-1,i-1}^n \text{ for } 2 \leq j \leq m < n+1; \tag{13}$$

$$q_{1,m}^{n+1} = q, \; q_{j,m}^{n+1} = \sum_{i=j}^{n} p_{j-1,i-1}^{n} + \sum_{i=n+1}^{m} q_{j-1,i-1}^{n} \text{ for } 2 \le j \le n; \tag{14}$$

$$r_{j,m}^{n+1} = \sum_{i=j}^{m} r_{j-1,i-1}^{n} \text{ for } j \ge n+1. \tag{15}$$

Using these relations, we may prove the following:

Theorem 1. *A sequence $(a_m)_{m=1}^{\infty}$ is n-convex, if and only if there is a n-positive sequence $(c_m)_{m=1}^{\infty}$, such that:*

$$a_m = \begin{cases} \displaystyle\sum_{i=1}^{m} \binom{m-1}{i-1} c_i & \textit{for} \quad m < n, \\ \displaystyle\sum_{i=1}^{n-1} \binom{n-1}{i-1} c_i + \sum_{i=n}^{m} \binom{m+n-i-1}{n-1} c_i & \textit{for} \quad m \ge n. \end{cases} \tag{16}$$

Proof. Let:

$$s_0(m) = m, \; s_j(m) = \sum_{i=1}^{m} s_{j-1}(i) \text{ for } j \ge 1. \tag{17}$$

From (13) we have successively:

$$p_{1,m}^{n} = p_{1,m} = 1,$$

$$p_{2,m}^{n} = \sum_{i=2}^{m} p_{1,i-1}^{n-1} = \sum_{i=2}^{m} p_{1,i-1} = p_{2,m} = m-1 = s_0(m-1),$$

$$p_{3,m}^{n} = \sum_{i=3}^{m} p_{2,i-1} = p_{3,m} = \sum_{i=3}^{m} s_0(i-2) = s_1(m-2).$$

Now suppose that for any n and m:

$$p_{j,m}^{n} = p_{j,m} = s_{j-2}(m-j+1). \tag{18}$$

Again by (13) we have then:

$$p_{j+1,m}^{n} = \sum_{i=j+1}^{m} p_{j,i-1}^{n-1} = \sum_{i=j+1}^{m} s_{j-2}(i-1-j+1) = p_{j+1,m}$$

$$= \sum_{i=1}^{m-j} s_{j-2}(i) = s_{j-1}(m-j),$$

that is (18) is true for any $j \geq 2$.

Similarly, from (14) we have:

$$q_{1,m}^n = q_{1,m} = 1 = p_{1,m},$$

$$q_{2,m}^n = \sum_{i=2}^{n} p_{1,i-1}^n + \sum_{i=n+1}^{m} q_{1,i-1}^n = \sum_{i=2}^{m} p_{1,i-1} = p_{2,m}.$$

Supposing:

$$q_{j,m}^n = p_{j,m}, \tag{19}$$

from (14) and (13) we have:

$$q_{j+1,m}^n = \sum_{i=j+1}^{n-1} p_{j,i-1}^{n-1} + \sum_{i=n}^{m} q_{j,i-1}^{n-1} = \sum_{i=j+1}^{n-1} p_{j,i-1} + \sum_{i=n}^{m} p_{j,i-1} = p_{j+1,m},$$

that is (19) holds.

As we saw in (8):

$$r_{j,m}^2 = m - j + 1 = s_0(m-j+1) \text{ for } 2 \leq j \leq m.$$

From (15) we have for $j \geq 3$:

$$r_{j,m}^3 = \sum_{i=j}^{m} r_{j-1,i-1}^2 = \sum_{i=j}^{m} s_0(i-j+1) = \sum_{i=1}^{m-j+1} s_0(i) = s_1(m-j+1).$$

Let us suppose that:

$$r_{j,m}^n = s_{n-2}(m-j+1). \tag{20}$$

Then

$$r_{j,m}^{n+1} = \sum_{i=j}^{m} r_{j-1,i-1}^n = \sum_{i=j}^{m} s_{n-2}(i-j+1) = \sum_{i=1}^{m-j+1} s_{n-2}(i),$$

that is, by induction for n, the assumption (20) is proved.

To finish the proof of the theorem, it is enough to determine the coefficients $s_k(m)$. We have:

$$s_1(m) = \sum_{i=1}^{m} s_0(i) = \sum_{i=1}^{m} i = \frac{m(m+1)}{2} = \binom{m+1}{2}.$$

Let us suppose that:

$$s_k(m) = \binom{m+k}{k+1}. \tag{21}$$

Then:

$$s_{k+1}(m) = \sum_{i=1}^{m} s_k(i) = \sum_{i=1}^{m} \binom{i+k}{k+1} = \binom{m+k+1}{k+2}.$$

From (9), (18), (19) and (21) we have (16).

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A HIERARCHY OF CONVEXITY FOR SEQUENCES

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An interesting property, called in [4] "hierarchy of convexity", was proved, for functions, by A.M. Bruckner and E. Ostrow in [3]. The main aim of this paper is to prove that this hierarchy is also valid in the case of sequences.

We begin by the definitions of sequence classes which we consider in what follows. We also prove representation theorems for some of this classes.

Definition 1. A sequence $(a_n)_{n=0}^{\infty}$ is called convex if its second order differences:

$$\Delta^2 a_n = a_{n+2} - 2a_{n+1} + a_n \tag{1}$$

are positive for any $n \geq 0$.

Although we have given in [7] a general representation theorem, for making a minor change in the formulation of the result, we prefer, in this particular case, to deduce it from the following:

Lemma 1. *If the sequence $(a_n)_{n=0}^{\infty}$ is given by:*

$$a_n = \sum_{k=0}^{n} (n-k+1) b_k \tag{2}$$

then:

$$\Delta^2 a_n = b_{n+2}. \tag{3}$$

The proof follows by a simple computation, hence it is omitted. Because the relation (2) is equivalent with:

$$b_0 = a_0,\ b_n = a_n - \sum_{k=0}^{n-1} (n-k+1) b_k \text{ for } n \geq 1 \tag{2'}$$

any sequence may be represented in this form and from lemma 1 we deduce:

Lemma 2. *The sequence $(a_n)_{n=0}^{\infty}$ is convex if and only if $b_n \geq 0$ for $n \geq 2$ in the representation (2).*

Definition 2. A sequence $(a_n)_{n=0}^{\infty}$ is called starshaped if it satisfies:

$$\frac{a_{n-1} - a_0}{n-1} \leq \frac{a_n - a_0}{n} \text{ for any } n \geq 2. \tag{4}$$

Remark 1. As it was proved by N. Ozeki (see [5]), a convex sequence $(a_n)_{n=0}^{\infty}$, with $a_0 = 0$, has the property:

$$\frac{a_{n-1}}{n-1} \leq \frac{a_n}{n}. \tag{4'}$$

Although this property may be easily put in connection with the similar property of functions, the definition of starshaped sequences we have not found neither in [5] nor elsewhere. We prefer the relation (4) instead of (4') to allow $a_0 \neq 0$.

Lemma 3. *The sequence $(a_n)_{n=0}^{\infty}$ is starshaped if and only if it may be represented by:*

$$a_n = n\sum_{k=1}^{n}\frac{c_k}{k} - (n-1)c_0 \tag{5}$$

with $c_k \geq 0$ for $k \geq 2$.

Proof. We denote $c_0 = a_0$ and $c_1 = a_1$. From (4), for $n = 2$, we have:

$$a_2 \geq 2a_1 - a_0 = 2c_1 - c_0$$

that is, there exists a number $c_2 \geq 0$ such that:

$$a_2 = 2c_1 - c_0 + c_2 = 2(c_1 + c_2/2) - c_0.$$

Suppose that (5) is valid for a natural n. From (4), for $n+1$, we have:

$$a_{n+1} \geq \frac{n+1}{n}a_n - \frac{1}{n}a_0$$

that is, for some $c_{n+1} \geq 0$:

$$a_{n+1} = c_{n+1} + \frac{n+1}{n}a_n - \frac{1}{n}a_0$$

$$= c_{n+1} + (n+1)\sum_{k=1}^{n}\frac{c_k}{k} - \left(\frac{n^2-1}{n} + \frac{1}{n}\right)c_0 = (n+1)\sum_{k=1}^{n+1}\frac{c_k}{k} - nc_0.$$

So, the lemma is proved by induction.

Lemma 4. *If the sequence $(a_n)_{n=0}^{\infty}$ is represented by (5), then:*

$$\Delta^2 a_n = c_{n+2} - \frac{n}{n+1}c_{n+1}. \tag{6}$$

Definition 3. The sequence $(a_n)_{n=0}^{\infty}$ is called superadditive if it verifies:

$$a_{n+m} + a_0 \geq a_n + a_m, \text{ for any } n, m \in \mathbb{N}. \tag{7}$$

Remark 2. As it is done in [2] for functions, we added the term a_0 in the first side of the relation (7) to avoid the restriction: $a_0 \leq 0$. As a matter of fact, this change is unimportant since from (7) follows that the sequence $(a'_n)_{n=0}^{\infty}$ given by $a'_n = a_n - a_0$, satisfies the usual relation:

$$a'_{n+m} \geq a'_n + a'_m. \tag{7'}$$

The following result, deduced from [6], is easily to check up:

Lemma 5. *The sequence $(a_n)_{n=0}^{\infty}$ is superadditive if it may be represented by:*

$$a_0 = d_0,\ \ a_n = d_0 + \sum_{k=1}^{n} \left[\frac{n}{k}\right] d_k,\ \ \textit{for } n \geq 1 \tag{8}$$

with $d_k \geq 0$ for $k \geq 1$, where $[x]$ denotes the integer part of x.

Remark 3. Any sequence $(a_n)_{n=0}^{\infty}$ may be represented by (8). It is superadditive if and only if every d_n verifies:

$$d_n \geq -\min_{p=1,\ldots,[n/2]} \sum_{k=2}^{n-1} \left(\left[\frac{n}{k}\right] - \left[\frac{p}{k}\right] - \left[\frac{n-p}{k}\right] \right) d_k \tag{9}$$

but (9) becomes $d_n \geq 0$ only for prime values of n.

Definition 4. The sequence $(a_n)_{n=0}^{\infty}$ has the property "P" in the mean, if the sequence $(A_n)_{n=0}^{\infty}$ has the property "P", where:

$$A_n = \frac{a_0 + \cdots + a_n}{n+1}. \tag{10}$$

Lemma 6. *The sequence $(a_n)_{n=0}^{\infty}$ is mean-convex if and only if it may be represented by:*

$$a_n = \sum_{k=0}^{n} (2n - k + 1) e_k \tag{11}$$

with $e_k \geq 0$ for $k \geq 2$.

Proof. By lemma 2, the sequence $(a_n)_{n=0}^{\infty}$ is mean-convex if and only if the sequence $(A_n)_{n=0}^{\infty}$ may be represented under the form:

$$A_n = \sum_{k=0}^{n}(n-k+1)e_k \tag{12}$$

with $e_k \geq 0$ for $k \geq 2$. From (10) we have:

$$a_0 = A_0, \; a_n = (n+1)A_n - nA_{n-1} \text{ for } n \geq 1. \tag{13}$$

Combining (12) and (13), by a simple calculation we get (11).

Lemma 7. *If the sequence $(a_n)_{n=0}^{\infty}$ is represented by means of (11), then:*

$$\Delta^2 a_n = (n+3)e_{n+2} - ne_{n+1}. \tag{14}$$

Lemma 8. *The sequence $(a_n)_{n=0}^{\infty}$ is mean-starshaped if and only if it may be represented by:*

$$a_n = (n+1)f_n + 2n\sum_{k=1}^{n-1}\frac{f_k}{k} - (2n-1)f_0 \tag{15}$$

where $f_k \geq 0$ for $k \geq 2$.

The proof is based, like that of lemma 6, on the relation (13), and uses for A_n the representation (5).

In what follows we denote by S_1, S_2, S_3, S_4, S_5 and S_6 the sets of convex, mean-convex, starshaped, superadditive, mean-starshaped, respectively mean-superadditive sequences. The main result, similar to that of [3], is given by the following:

Theorem. *The following inclusions:*

$$S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5 \subset S_6 \tag{16}$$

hold, each of them being strictly.

Proof. (i) Let us suppose that the sequence $(a_n)_{n=0}^{\infty}$ is represented as in (2) and also as in (11). Then, from (3) and (14) we deduce:

$$b_{n+2} = (n+3)e_{n+2} - ne_{n+1} \tag{17}$$

that is:

$$b_2 = 3e_2 \text{ and } (n+3)e_{n+2} = b_{n+2} + ne_{n+1}.$$

So, if $b_n \geq 0$ for $n \geq 2$, then $e_n \geq 0$ for $n \geq 2$. By lemmas 2 and 6, if the sequence $(a_n)_{n=0}^{\infty}$ is convex, it is mean-convex, i.e. $S_1 \subset S_2$. The inclusion is strictly because we have, for example, $b_3 = 4e_3 - e_2$, and so $e_2 = 1$ and $e_3 = 0$ give us $b_3 = -1 < 0$.

(ii) Let us represent the sequence $(a_n)_{n=0}^{\infty}$ under the forms (11) and (5). From (14) and (6) we have:

$$(n+3)e_{n+3} - ne_{n+1} = c_{n+2} - \frac{n}{n+1}c_{n+1} \tag{18}$$

that is:

$$c_2 = 3e_2, \quad c_3 = 4e_3 + 1/2 \cdot e_2$$

and:

$$c_n = (n+1)e_n + \frac{1}{n-1}\sum_{k=2}^{n-1}(k-1)e_k$$

what may be proved by induction. So, $e_n \geq 0$ for $n \geq 2$ implies $c_n \geq 0$ for $n \geq 2$, i.e. by lemmas 3 and 6, $S_2 \subset S_3$. On the other hand, for $c_2 = 3$ and $c_3 = 0$, we have $e_3 = -1/8 < 0$, that is the above inclusion is strictly.

(iii) Let us suppose that the sequence $(a_n)_{n=0}^{\infty}$ is in S_3. Then, on the basis of the representation given by the lemma 3:

$$a_{n+m} - a_0 - a_n - a_m = n\sum_{k=n+1}^{n+m}\frac{c_k}{k} + m\sum_{k=m+1}^{m+n}\frac{c_k}{k} \geq 0$$

that is $(a_n)_{n=0}^{\infty}$ is in S_4. The inclusion $S_3 \subset S_4$ is strictly because the sequence with the general term $a_n = [n/2]$ is, by lemma 5, in S_4 but:

$$\frac{a_3 - a_0}{3} - \frac{a_2 - a_0}{2} = -\frac{1}{6} < 0$$

so that it is not in S_3.

(iv) Let the sequence $(a_n)_{n=0}^{\infty}$ be in S_4. Then:

$$a_n + a_0 \geq a_k + a_{n-k} \text{ for } k = 1, \ldots, n-1$$

that is:

$$(n-1)(a_n + a_0) \geq 2 \sum_{k=1}^{n-1} a_k$$

or:

$$a_n \geq \frac{2}{n-1} \sum_{k=1}^{n-1} a_k - a_0.$$

So:

$$\frac{A_n - A_0}{n} = \frac{\sum_{k=1}^{n} a_k - n a_0}{n(n+1)} \geq \frac{\left(1 + \frac{2}{n-1}\right) \sum_{k=1}^{n-1} a_k - (n+1) a_0}{n(n+1)}$$

$$= \frac{\sum_{k=1}^{n-1} a_k - (n-1) a_0}{n(n-1)} = \frac{A_{n-1} - A_0}{n-1}$$

i.e. $(a_n)_{n=0}^{\infty}$ is in S_5. The inclusion $S_4 \subset S_5$ is, in his turn, strictly because if $(a_n)_{n=0}^{\infty}$ is represented through (5) we have:

$$a_4 + a_0 - a_3 - a_1 = 5c_4 - \frac{4}{3}c_3 + c_2 < 0$$

for $c_4 = c_2 = 0$, $c_3 = 1$.

(v) The inclusion $S_5 \subset S_6$ follows from (iii). His strictness also follows by taking $A_n = [n/2]$, that is:

$$a_n = (n+1)\left[\frac{n}{2}\right] - n\left[\frac{n-1}{2}\right]$$

which gives a sequence in S_6 but not in S_5.

Remark 4. As follows from [5], N. Ozeki has proved, by other means, the inclusion $S_1 \subset S_2$, and, in the case $a_0 = 0$, $S_1 \subset S_3$.

Remark 5. If we set the sequence $(a_n)_{n=0}^{\infty}$ in the form (15), we have:

$$a_{n+m} + a_0 - a_n - a_m = n(f_{n+m} - f_n) + m(f_{n+m} - f_m)$$

$$+f_{n+m} + f_n + f_m + 2n\sum_{k=n+1}^{n+m-1}\frac{f_k}{k} + 2m\sum_{k=m+1}^{m+n-1}\frac{f_k}{k}.$$

Taking into account the inclusion $S_4 \subset S_5$, this means that in order to get a superadditive sequence $(a_n)_{n=0}^{\infty}$ it is necessary to use in (15) a sequence $(f_n)_{n=0}^{\infty}$ with $f_n \geq 0$ for $n \geq 2$, and it is sufficiently that the sequence $(f_n)_{n=1}^{\infty}$ be increasing. In spite of this result and that given in the remark 3, we have unfortunately no satisfactory formula for the representation of superadditive sequences.

Remark 6. The theorem may be used to simplify some of the proofs from [3].

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α-CONVEX SEQUENCES

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In [3] we have shown that the "hierarchy of convexity" established for functions by A.M. Bruckner and E. Ostrow in [1], is also valid for sequences. In [4] we have extended this hierarchy, inserting between the set of convex sequences and that of starshaped sequences an infinity of sets of sequences, namely sequences with convex weighted arithmetic means. In this paper we also insert an infinity of sets of sequences between the set of convex sequences and that of starshaped sequences, by defining α-convex sequences. Such a definition was given for analytic functions by P.T. Mocanu in [2].

Let us recall the following:

Definition 1. A sequence $(a_n)_{n=0}^{\infty}$ is called:

a) **convex**, if:

(1) $$\Delta^2 a_n = a_{n+2} - 2a_{n+1} + a_n \geq 0, \text{ for } n \geq 0;$$

b) **starshaped**, if:

(2) $$\frac{a_{n+1} - a_0}{n+1} \geq \frac{a_n - a_0}{n}, \text{ for } n \geq 1.$$

Now for any $\alpha \in [0,1]$ we give the following:

Definition 2. A sequence $(a_n)_{n=0}^{\infty}$ is called α**-convex** if the sequence:

$$\left(\alpha(a_{n+1}-a_n)+(1-\alpha)\frac{a_n-a_0}{n}\right)_{n=1}^{\infty}$$

is increasing.

Obviously we have:

Lemma 1. *The sequence $(a_n)_{n=0}^{\infty}$ is α-convex, if and only if:*

$$\alpha\Delta^2 a_n+(1-\alpha)\left(\frac{a_{n+1}-a_0}{n+1}-\frac{a_n-a_0}{n}\right)\geq 0,\ \textit{for } n\geq 1. \tag{3}$$

Lemma 2. *The sequence $(a_n)_{n=0}^{\infty}$ is α-convex if and only if the sequence:*

$$(a_n-a_0+\alpha[n(a_{n+1}-a_n)-(a_n-a_0)])_{n=0}^{\infty}$$

is starshaped.

It can be also proved:

Lemma 3. *Any α-convex sequence is starshaped.*

Lemma 4. *The sequence $(a_n)_{n=0}^{\infty}$ is α-convex (for $\alpha>0$) if and only if it may be represented by:*

$$a_n=n\sum_{k=1}^{n}\frac{b_k}{k}-(n-1)b_0,\ \textit{for } n\geq 0 \tag{4}$$

with

$$b_{n+2}\geq\left[1-\frac{1}{\alpha(n+1)}\right]b_{n+1},\ \textit{for } n\geq 0,\ b_2\geq 0. \tag{5}$$

Theorem. *If a sequence if α-convex, then it is also β-convex for any $0\leq\beta\leq\alpha$.*

Remark. Obviously 1-convexity means convexity and 0-convexity means starshapedness. So, the theorem gives an infinite chain of sequence classes between the set of convex sequences and that of starshaped sequences. Every inclusion is proper because of (5).

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A TRANSFORMATION THAT PRESERVES SOME SEQUENCE CLASSES

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Abstract. Se demonstrează că dacă transformarea (2) păstrează clasa de şiruri definită de (9), atunci ponderile p_n sunt de forma (4). Aplicăm rezultatul la şirurile α-convexe.

In [2] are exposed more results on convex sequences, that is sequences $(x_n)_{n=0}^{\infty}$ with the property:

$$\Delta^2 x_n = x_{n+2} - 2x_{n+1} + x_n \geq 0 \text{ for } n \geq 0. \tag{1}$$

Some of these results deal with the preservation of convexity by matrix transformations and particularly by weighted arithmetic means. So is the following characterization theorem from [7], which we present in the form given in [1] because in [7] the sequences are indexed starting from 1 not from 0 as we do. The difference seems minor but it allows a simplification of the results.

Theorem 0. *The sequence $(X_n)_{n=0}^{\infty}$ given by:*

$$X_n = \frac{p_0 x_0 + \cdots + p_n x_n}{p_0 + \cdots + p_n} \textit{ for } n \geq 0 \tag{2}$$

is convex for any convex sequence $(x_n)_{n=0}^{\infty}$ *if and only if:*

$$p_n = \frac{\prod_{k=0}^{n-1}(kp_0 + p_1)}{n!p_0^{n-1}} \text{ for } n \geq 2 \tag{3}$$

with $p_0 > 0$ *and* $p_1 > 0$ *arbitrary.*

In [4] we put (3) in the form:

$$p_n = p_0\binom{u+n-1}{n} \text{ for } n \geq 0 \tag{4}$$

where $u > 0$ is arbitrary (in fact it is p_1/p_0) and:

$$\binom{v}{0} = 1, \quad \binom{v}{n} = \frac{1}{n!}\prod_{k=0}^{n-1}(v-k) \text{ for } k \geq 1, \ v \in \mathbb{R}. \tag{5}$$

In this form, the proof of Theorem 0 is simpler and (2) becomes:

$$X_n^u = \frac{\sum_{k=0}^{n}\binom{u+n-1}{n}x_k}{\binom{u+n}{n}}. \tag{6}$$

In [5] we proved that (4) characterizes also the transformations that preserve starshaped sequences, i.e. sequence $(x_n)_{n=0}^{\infty}$ with the property:

$$D^1 x_n = \frac{x_{n+1} - x_0}{n+1} - \frac{x_n - x_0}{n} \geq 0 \text{ for } n \geq 1. \tag{7}$$

In what follows we generalize the necessity part of the result to some other sequence classes.

For some fixed functions $a, b, c, d : \mathbb{N} \to \mathbb{R}$, let us denote:

$$Tx_n = a(n)x_{n+2} + b(n)x_{n+1} + c(n)x_n + d(n)x_0 \tag{8}$$

and consider the set:

$$S = \{(x_n)_{n=0}^{\infty};\ Tx_n \geq 0,\ \forall\ n \geq 0\}. \tag{9}$$

Theorem 1. *If $(kn)_{n=0}^{\infty} \in S$ for any $k \in \mathbb{R}$ and for $(x_n)_{n=0}^{\infty} \in S$, (2) gives an $(X_n)_{n=0}^{\infty}$ in S, then there is an $u > 0$ such that the weights p_n be given by (4).*

Proof. We first observe that $(kn)_{n=0}^{\infty} \in S$ for any real k, if and only if:

$$a(n)(n+2) + b(n)(n+1) + c(n)n = 0. \tag{10}$$

Then, denoting by:

$$X_n = \frac{k \sum_{i=0}^{n} i p_i}{\sum_{i=0}^{n} p_i}$$

from the hypothesis we deduce that $(X_n)_{n=0}^{\infty}$ is in S. Taking $u = p_1/p_0$, we have:

$$p_1 = p_0 u = p_0 \binom{u}{1}.$$

As for $n = 0$, (10) becomes:

$$2a(2) + b(2) = 0$$

and

$$X_0 = 0, \quad X_1 = \frac{ku}{1+u}, \quad X_2 = k\frac{up_0 + 2p_2}{p_0(1+u) + p_2}$$

we have:

$$\begin{aligned} TX_0 &= k\left[a(2)\frac{up_0 + 2p_2}{p_0(1+u) + p_2} + b(2)\frac{u}{1+u}\right] \\ &= ka(2)\left[\frac{up_0 + 2p_2}{p_0(1+u) + p_2} - \frac{2u}{1+u}\right]. \end{aligned}$$

So $TX_0 \geq 0$ for any k, if and only if:

$$(up_0 + 2p_2)(1+u) - 2u[p_0(1+u) + p_2] = 0$$

or:

$$p_2 = \frac{u(1+u)}{2} p_0 = p_0 \binom{u+1}{2}.$$

Assume (4) valid for $n \leq m+1$. Then:

$$X_m = \frac{k \sum_{i=0}^{m} i \binom{u+i-1}{i}}{\binom{u+m}{m}} = k \frac{u \sum_{i=1}^{m} \binom{u+i-1}{i-1}}{\binom{u+m}{m}} = \frac{kum}{u+1}.$$

In the same manner:

$$X_{m+1} = \frac{ku(m+1)}{u+1}$$

and

$$X_{m+2} = k \frac{up_0 \binom{u+m+1}{m} + (m+2)p_{m+2}}{p_0 \binom{u+m+1}{m+1} + p_{m+2}}.$$

Hence, by (10):

$$TX_m = ka(m) \left[\frac{up_0 \binom{u+m+1}{m} + (m+2)p_{m+2}}{p_0 \binom{u+m+1}{m+1} + p_{m+2}} - \frac{u(m+2)}{u+1} \right]$$

and so $TX_n \geq 0$ for any k if and only if it is zero, that is:

$$\begin{aligned} p_{m+2} &= p_0 \frac{u}{m+2} \left[(m+2) \binom{u+m+1}{m+1} - (m+1) \binom{u+m+1}{m+1} \right] \\ &= p_0 \binom{u+m+1}{m+2}. \end{aligned}$$

So, by induction, (4) is valid for any n.

In [6] we have given the following:

Definition. The sequence $(x_n)_{n=0}^{\infty}$ is called α-convex (with $\alpha \geq 0$) if:

(11) $$\alpha \Delta^2 x_n + (1-\alpha) D^1 x_n \geq 0 \text{ for } n \geq 0.$$

Consequence. If $(X_n)_{n=0}^{\infty}$ given by (2) is α-convex for any α-convex sequence $(x_n)_{n=0}^{\infty}$, then there is an $u > 0$ such that p_n be given by (4).

Proof. For $x_n = kn$, we have:

$$\alpha \Delta^2 x_n + (1-\alpha) D^1 x_n = 0$$

so that it is α-convex and we may apply Theorem 1.

In [3] we have considered, beside convex and starshaped sequences, also superadditive sequences, that is sequences $(x_n)_{n=0}^{\infty}$ that satisfy:

$$x_{n+m} + x_0 \geq x_n + x_m \text{ for any } n, m \geq 0. \tag{12}$$

Although their definition is not of the form (8) and (9), may be proved:

Theorem 2. *If $(X_n)_{n=0}^{\infty}$ given by (2) is superadditive for any superadditive sequence $(x_n)_{n=0}^{\infty}$, then p_n are of the form (4) for some $u > 0$.*

As we stated above, for convex and starshaped sequences we proved in [4] and [5] a converse of Theorem 1. In the case of superadditivity we can prove only the weaker result:

Lemma. *If $(x_n)_{n=0}^{\infty}$ satisfies:*

$$x_{n+1} + x_0 \geq x_n + x_1 \tag{13}$$

and $(X_n)_{n=0}^{\infty}$ is given by (6), for some $u > 0$, then this also verifies the relation (13).

Proof. From (13) we have:

$$x_{n+1} - x_k \geq (n-k+1)(x_1 - x_0)$$

and so:

$$X_{n+1}+X_0-X_n-X_1=\frac{\binom{u+n}{n+1}\sum_{k=0}^{n}\binom{u+k-1}{k}(x_{n+1}-x_k)}{\binom{u+n}{n}\binom{u+n+1}{n+1}}+u\frac{x_0-x_1}{u+1}$$

$$\geq\frac{u(x_1-x_0)}{(u+1)\binom{u+n+1}{n}}\left[(n+1)\binom{u+n}{n}-u\binom{u+n}{n-1}\right]+\frac{u(x_0-x_1)}{u+1}=0.$$

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SOME GENERALIZATIONS OF CONVEXITY FOR SEQUENCES

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In [1] it is defined the mean-convexity and it is proved that this generalizes the convexity.

A sequence $(a_n)_{n=0}^{\infty}$ is called mean-convex if the sequence $(C_n)_{n=0}^{\infty}$ of Cesaro means

(1) $$C_n = \frac{a_0 + \cdots + a_n}{n+1}$$

is convex, that is:

(2) $$\Delta^2 C_n = C_{n+1} - 2C_{n+1} + C_n \geq 0, \ \forall \ n \geq 0.$$

The first generalization that we propose in this paper consists in the substitution of (1) by a simpler mean:

(3) $$A_n = \frac{a_n + a_{n+1}}{2}$$

or, more generally by:

(4) $$A_n^p = \frac{a_n + pa_{n+1}}{1+p},$$

where p is a given nonnegative real number.

It is easy to prove:

Theorem 1. *If the sequence* $(a_n)_{n=0}^{\infty}$ *is convex then* $(A_n^p)_{n=0}^{\infty}$ *is also convex, for any* $p \geq 0$.

Remark 1. If we denote by S the set of convex sequences and by

$$S^p = \{(a_n)_{n=0}^{\infty} : \ (A_n^p)_{n=0}^{\infty} \in S\},$$

the Theorem 1 means:

$$S \subset S^p, \ \forall \ p \geq 0. \tag{5}$$

It is also easy to prove the following representation theorem:

Theorem 2. *The sequence* $(a_n)_{n=0}^{\infty}$ *belongs to* S^p *if and only if*

$$a_n = \sum_{k=0}^{n} b_k,$$

where the sequence $(b_n + pb_{n+1})_{n=1}^{\infty}$ *is nondecrasing.*

Remark 2. As it was proved in [1], the sequence $(A_n^p)_{n=0}^{\infty}$ is convex if and only if:

$$A_n^p = \sum_{k=0}^{n}(n-k+1)c_k, \quad c_k \geq 0, \text{ for } k \geq 2. \tag{6}$$

Using this result we may find simpler representation theorems. So we have:

Theorem 3. *The sequence* $(a_n)_{n=0}^{\infty}$ *belongs to* S^1 *if and only if:*

$$a_n = 2\sum_{k=0}^{n-1}\left[\frac{n+1-k}{2}\right]c_k + (-1)^n c_0, \quad c_k \geq 0 \text{ for } k \geq 2 \tag{7}$$

([ ] denotes the integer part).

Remark 3. It is easy to see that there is no inclusion relation between S^p and S^q if $p \neq q$. Also there is no relation between a set S^p with $p > 0$ and the set of starshaped sequences, or that of superadditive sequences (see [3]).

From this point of view, the mean (1) is better than (3), because for the mean (1) such relations between S^p and S^q are valid (see [1]).

Remark 4. The mean (3) may be also generalized by iteration, or by taking:

$$A_n = \frac{1}{k} \sum_{i=0}^{k-1} a_{n+i}. \tag{4'}$$

Another generalization of convexity is given by the following:

Definition 1. The sequence $(a_n)_{n=0}^{\infty}$ is called weakly quasi-convex if

$$a_n \leq \max\{a_{n-1}, a_{n+1}\}, \ \forall\, n \geq 1. \tag{8}$$

If the equality is excluded from (8) the sequence $(a_n)_{n=0}^{\infty}$ is called strictly quasi-convex.

We have directly:

Theorem 4. *The sequence* $(a_n)_{n=0}^{\infty}$ *is*

a) weakly qausi-convex if and only if:

$$a_n = \sum_{k=0}^{n} b_k, \tag{9}$$

where $b_k > 0$ *implies* $b_{k+1} \geq 0$*, for* $k > 1$*;*

b) strictly quasi-convex if and only if in (9), $b_k > 0$ *implies* $b_{k+1} > 0$*, for* $k > 1$*, and at most one* b_k *is zero.*

Definition 2. The sequence $(a_n)_{n=0}^{\infty}$ is unimodal (strictly unimodal) if $(a_k)_{k=0}^{n}$ is nonincreasing (decreasing) and $(a_k)_{k=n+1}^{\infty}$ is nondecreasing (respectively, increasing), for some $n \geq 0$.

Consequence 1. The sequence $(a_n)_{n=0}^{\infty}$ is strictly quasi-convex if and only if it is strictly unimodal.

Remark 5. An unimodal sequence is weakly quasi-convex but the converse assertion is false. For instance, the sequence: 2,1,2,2,1,2,2,1,... is weakly quasi-convex but it is not unimodal. To amend this situation, we consider also the following:

Definition 3. The sequence $(a_n)_{n=0}^{\infty}$ is called quasi-convex if for any $0 \leq n < m < p$, we have:

$$a_m \leq \max\{a_n, a_p\}. \tag{10}$$

Theorem 5. *The sequence $(a_n)_{n=0}^{\infty}$ is quasi-convex if and only if in the representation (9), $b_k > 0$ implies $b_{k+i} \geq 0$, for any $k, i > 1$.*

Consequence 2. The sequence $(a_n)_{n=0}^{\infty}$ is quasi-convex if and only if it is unimodal.

Theorem 6. *The following implications hold:*

a) strictly convex $\Rightarrow$ strictly quasi-convex;

b) convex $\Rightarrow$ quasi-convex;

c) strictly quasi-convex $\Rightarrow$ quasi-convex;

d) quasi-convex $\Rightarrow$ weakly quasi-convex.

Theorem 7. *If the sequence $(a_n)_{n=0}^{\infty}$ is quasi-convex (weakly quasi-convex, respectively strictly quasi-convex) then so is also the sequence $(A_n^p)_{n=0}^{\infty}$, for any $p \geq 0$.*

It can be proved as in the case of the functions (see [2]) the following:

Theorem 8. *If $(a_n)_{n=0}^{\infty}$ is a positive convex sequence and $(b_n)_{n=0}^{\infty}$ is a strictly positive, concave sequence, then the sequence $(a_n/b_n)_{n=0}^{\infty}$ is quasi-convex.*

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ON SOME INEQUALITIES INVOLVING CONVEX SEQUENCES

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Several inequalities connected with convex sequences are known. Let us mention those of Nanson [4], Steinig [6] and Ozeki (see [2]). In what follows, we shall use a simple method which allows the substitution of the conditions

$$\Delta^2 a_n = a_{n+2} - 2a_{n+1} + a_n \geq 0, \text{ for } n \geq 1, \tag{1}$$

that characterize convex sequences, by

$$m \leq \Delta^2 a_n \leq M, \text{ for } n \geq 1. \tag{2}$$

The obtained inequalities are not only more general, but, as we shall see on some examples, they strengthen the initial inequalities. The same method was used for functions in [5] and [1].

Before presenting the main results, let us give a representation theorem of sequences that satisfy (2). A more general result is given in [7], but we sketch here the proof which is simple in this particular case.

Theorem 1. *The real sequence* $(a_n)_{n\geq 1}$ *satisfies (2) if and only if there is a sequence* $(b_n)_{n\geq 1}$ *which verifies*

$$m \leq b_n \leq M, \ \textit{for } n > 2 \tag{3}$$

such that

$$a_n = \sum_{k=1}^{n} (n-k+1) b_k, \ \textit{for any } n \geq 1. \tag{4}$$

Proof. Any sequence $(a_n)_{n\geq 1}$ may be written as (4) by taking

$$b_1 = a_1, \ b_n = a_n - \sum_{k=1}^{n-1} (n-k+1) b_k, \ \text{for } n \geq 2.$$

Because (4) implies

$$\Delta^2 a_n = b_{n+2} \tag{5}$$

the conditions (2) and (3) are equivalent.

The method which gives the results what follow is based on a simple remark: if the sequence $(a_n)_{n\geq 1}$ satisfies (2), then the sequences $(c_n)_{n\geq 1}$ and $(d_n)_{n\geq 1}$ given by

$$c_n = a_n - m\frac{n^2}{2}, \quad d_n = M\frac{n^2}{2} - a_n \tag{6}$$

are convex. So, we can apply to these the results valid for convex sequences. To complete the proofs one requires only some simple calculations which we omit. As a matter of fact, we content ourselves to present for exemplification only two results: the first obtained from the inequality of Nanson [4], the second from that of Steinig [6].

Theorem 2. *If the sequence $(a_n)_{n\geq 1}$ satisfies (2), then for any $n \geq 1$ hold*

$$(7)\quad \frac{2n+1}{6}m \leq \frac{a_1+a_3+\cdots+a_{2n+1}}{n+1} - \frac{a_2+a_4+\cdots+a_{2n}}{n} \leq \frac{2n+1}{6}M$$

and

$$(8)\quad \frac{n(2n+1)}{6}m \leq a_1-a_2+a_3-a_4+\cdots+a_{2n+1} - \frac{a_1+a_3+\cdots+a_{2n+1}}{n+1} \leq \frac{n(2n+1)}{6}M$$

Applications. Let $a_n = a^{n-1}$. Then $\Delta^2 a_n = a^{n-1}(a-1)^2$. If $a > 1$, then $m = (a-1)^2$ and (7) gives us

$$(9)\quad \frac{1+a^2+\cdots+a^{2n}}{n+1} - \frac{a+a^3+\cdots+a^{2n-1}}{n} \geq \frac{2n+1}{6}(a-1)^2$$

This is an improvement of an inequality of Wilson (see [3]).

From (8) it follows

$$(10)\quad 1-a+a^2-a^3+\cdots+a^{2n} - \frac{1+a^2+\cdots+a^{2n}}{n+1} \geq \frac{n(2n+1)}{6}(a-1)^2$$

which is an improvement of an inequality of Steinig [6]. If $0 < a < 1$, then $m = 0$ and $M = (a-1)^2$, so that (7) and (8) give

$$(11)\quad 0 \leq \frac{1+a^2+\cdots+a^{2n}}{n+1} - \frac{a+a^3+\cdots+a^{2n-1}}{n} \leq \frac{2n+1}{6}(a-1)^2,$$

respectively

$$(12)\quad 0 \leq 1-a+a^2-a^3+\cdots+a^{2n} - \frac{1+a^2+\cdots+a^{2n}}{n+1} \leq \frac{n(2n+1)}{6}(a-1)^2.$$

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GENERALIZED CONVEX SEQUENCES

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The notion of convexity was generalized in many ways. Some of these generalizations are based on the geometric interpretation of convexity and resort to an alteration of the finite differences. In this case it was impossible to transpose them for high order convexities. In this paper, we propose another generalization of convexity based on the notion of finite differences. For the moment we study the convexity of sequences of elements of an abelian group.

Let $(X,+)$ be an abelian group and $(x_m)_{m=1}^{\infty}$ a sequence of elements of X. With usual notations, we define the finite differences by the relations:

$$\Delta^0 x_m = x_m, \quad \Delta^{n+1} x_m = \Delta^n x_{m+1} - \Delta^n x_m \text{ for } n \geq 0. \tag{1}$$

One proves by induction, as in the classical case, the validity of the following relation:

$$\Delta^n x_m = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} x_{m+i} \tag{2}$$

where the second member must be interpreted in the natural way by means of the operation of the group. In fact, finite differences defined for sequences of elements of a commutative field was considered previously by M.D. Torres in [10], but obviously a group structure is enough for our purpose.

Let P be an arbitrary proper subset of X.

Definition 1. The sequence $(x_m)_{m=1}^{\infty}$ is said to be $P - n$-convex if $\Delta^n x_m \in P$ for any m.

Before passing to the study of the notion just introduces, let us give some examples.

Example 1. For the group $(\mathbb{R}, +)$ with $P = \mathbb{R}_+$ we obtain the usual n-convexity (see [5] for more references).

Example 2. In the same group, for $P = \{0\}$ we obtain n-polynomial sequences (met especially in the case of functions). Particularly, for $n = 2$ one get the arithmetical progressions.

Example 3. Ju.N. Subbotin has considered in [8] the set of the sequences with the property: $|\Delta^n x_m| \leq 1$ for any m. This may be obtained by choosing $P = [-1, 1]$.

Example 4. The case $(\mathbb{R} - \{0\}, \cdot)$ with $P = [1, \infty)$ corresponds to logarithmic n-convexity. In fact, one obtains a generalization of this because the sequences so defined need not to be positive.

Example 5. In the same group, but with $P = \{1\}$ we obtain sequences which can call logarithmic n-polynomial. Particularly, for $n = 2$ one get geometrical progressions.

Example 6. In the group $(Q - \{0\}, \cdot)$ with $P = \mathbb{N}$ we obtain sequences which we name n-divisible.

Remark 1. Although the way which we have chosen to arrive at the definition 1 is suitable, the following method may be regarded more natural. Let $(P, +)$ be a semigroup and $(x_m)_{m=1}^{\infty}$ be a sequence of elements of P. We say that x_m has finite difference of first order if there is $d \in P$ such that $x_{m+1} = x_m + d$. In this case we denote d by $\Delta^1 x_m$. Similarly may exist the differences of higher order. A sequence is named n-convex if all his elements have differences of order n. This method is suggested by example 6.

Remark 2. Analogously, we may define the convexity of a function with values in a group (in which we have fixed certain subset P).

Although the definition 1 seems to be too general, we may transpose for it all the results which we obtained in [9] concerning the representation of n-convex sequences. We begin with the following useful result which is easy to prove by induction:

Lemma 1. *If the sequences $(x_m)_{m=1}^{\infty}$ and $(y_m)_{m=1}^{\infty}$ are related by:*

$$x_m = \sum_{i=1}^{m} y_i \text{ for } m \geq 1 \tag{3}$$

then:

$$\Delta^n x_m = \Delta^{n-1} y_{m+1} \text{ for } n \geq 1. \tag{4}$$

As a direct consequence, we have:

Lemma 2. *The sequence $(x_m)_{m=1}^{\infty}$ is P-n-convex if and only if there is a sequence $(y_m)_{m=1}^{\infty}$ such that holds (3) and $(y_m)_{m=2}^{\infty}$ be $P-n-1$-convex.*

To formulate the following result (which may be obtained by successive application of lemma 2) we need the following:

Definition 2. The sequence $(y_m)_{m=1}^{\infty}$ is a $n-P$ sequence if $y_m \in P$ for $m > n$.

Lemma 3. *There are the natural numbers $p^n_{m,i}$ (for any n, m and $i \leq m$) such that a sequence $(x_m)_{m=1}^{\infty}$ is $P-n$-convex if and only if it may be represented by:*

$$x_m = \sum_{i=1}^{m} p^n_{m,i} y_i \text{ for any } m \tag{5}$$

with a $n-P$ sequence $(y_m)_{m=1}^{\infty}$.

In [9] we have determined the numbers $p^n_{m,i}$ for the usual case of n-convex sequences (example 1). We shall see that they are generally valid. We prove first:

Lemma 4. *For an arbitrary sequence $(y_m)_{m=1}^{\infty}$, define the sequence $(x_m)_{m=1}^{\infty}$ by:*

$$x_m = \begin{cases} \displaystyle\sum_{i=1}^{m} \binom{m-1}{i-1} y_i & \text{for } \; m < n \\ \displaystyle\sum_{i=1}^{n-1} \binom{n-1}{i-1} y_i + \sum_{i=n}^{m} \binom{n+m-i-1}{n-1} y_i & \text{for } \; m \geq n. \end{cases} \tag{6}$$

Then:

$$\Delta^n x_m = y_{n+m} \text{ for any } m \geq 1 \tag{7}$$

and

$$\Delta^k x_1 = y_{k+1} \text{ for } k < n. \tag{7'}$$

Proof. We shall prove only the relations (7) for $m < n$. The other cases may be proved analogously. From (2) and (6) we have:

$$\Delta^n x_m = \sum_{j=0}^{n-m-1} (-1)^{n-j} \binom{n}{j} \sum_{i=1}^{m+j} \binom{m+j-1}{i-1} y_i$$

$$+ \sum_{j=n-m}^{n} (-1)^{n-j} \binom{n}{j} \sum_{i=1}^{n-1} \binom{m+j-1}{i-1} y_i + \sum_{i=n}^{m+j} \binom{m+n+j-i-1}{n-1} y_i$$

or, changing the order of addition:

$$\Delta^n x_m = \sum_{i=1}^{m-1} y_i \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \binom{m+j-1}{i-1}$$

$$+ \sum_{i=m}^{n-1} y_i \sum_{j=i-m}^{n} (-1)^{n-j} \binom{n}{j} \binom{m+j-1}{i-1}$$

$$+ \sum_{i=n}^{n+m} y_i \sum_{j=i-m}^{n} (-1)^{n-j} \binom{n}{j} \binom{m+n+j-i-1}{n-1}.$$

the first sum missing for $m = 1$. Because, for any m and n we have:

$$(8) \qquad \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \binom{n+j}{k} = 0, \text{ if } k < m \text{ and } k \le n$$

(as is proved, for example, in [7] p.48), the first sum is zero. Making in the other two sums the changement of variable: $j = i - m + k$, by (8), we get (7).

So we get the coefficients $p_{m,j}^n$ from lemma 3, that is:

Theorem 1. *A sequence $(x_m)_{m=1}^{\infty}$ is $P - n$-convex if and only if there is a $n - P$ sequence $(y_m)_{m=1}^{\infty}$ such that (6) holds.*

Remark 3. In the usual case of n-convex sequences, in [9] we found the representation (6) by induction from lemmas 2 and 3. Taking in account the lemma 4, we can obtain a similar representation by solving the system of equations (7) and (7') (see [1]). Prof. A. Lupaş pointed out to me "the fundamental formula of transformation of divided differences" given by T. Popoviciu in [6], from which (6) may be also deduced if we make the notations (7) and (7'). In [2] and [3] may be also found some formulas related to Popoviciu's formula.

Remark 4. As is usually done in defining the logarithmic-convexity (see [4] for αm-convexity also), we may assume that the transformed sequence, by some fixed function, is convex. That is, given the set M, the group $(X,+)$, the set $P \subset X$, and the function $f : M \to X$, we may define the $f - F - n$-convexity of a sequence $(x_m)_{m=1}^{\infty}$ from M, taking $\Delta^0 x_m = f(x_m)$. For example, for $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined by $f(x) = 1/x$ and the addition on $\mathbb{R}$, we obtain "harmonic progressions" for $P = \{0\}$ and a related convexity for $P = [0, \infty)$. If f is injective, we may obtain also the representation of such sequences using $f^{-1} : f(M) \to M$.

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STARSHAPED SEQUENCES

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In [4] we have shown that the "hierarchy convexity" established, for the functions by A.M. Bruckner and E. Ostrow in [1], is also valid for sequences. Starting from a property proved in [7] and generalized in [2], we have extended in [6] this hierarchy, inserting between the set of convex sequences and that of starshaped sequences an infinity of sets of sequences. In this paper we prove similar results for starshaped sequences.

Let us recall some definitions and some results from [4] and [6] which we need in what follows.

Definition 1. A sequence $(a_n)_{n=0}^{\infty}$ is called:

a) convex, if:

$$\Delta^2 a_n = a_{n+2} - 2a_{n+1} + a_n \geq 0, \text{ for } n \geq 0; \tag{1}$$

b) starshaped, if it satisfies:

$$D^1 a_n = \frac{a_{n+1} - a_0}{n+1} - \frac{a_n - a_0}{n} \geq 0, \text{ for } n \geq 1; \tag{2}$$

c) superadditive, if:

$$a_{n+m} + a_0 \geq a_n + a_m, \text{ for any } n \text{ and } m. \tag{3}$$

Definition 2. The sequence $(a_n)_{n=0}^{\infty}$ has the property "P" in the mean, if the sequence $(A_n)_{n=0}^{\infty}$ has the property "P", where:

$$A_n = \frac{a_0 + \cdots + a_n}{n+1}, \text{ for } n \geq 0. \tag{4}$$

Remark 1. Denoting by S_1, S_2, S_3, S_4, S_5 and S_6 the sets of convex, mean-convex, starshaped, superadditive, mean-starshaped, respectively mean-superadditive sequences, we have proved in [4] the proper inclusions:

$$S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5 \subset S_6. \tag{5}$$

As it is shown in [3], N. Ozeki has proved that $S_1 \subset S_2$ and, if $a_0 = 0$, $S_1 \subset S_3$. As a matter of fact, we attached a_0 in (2) and (3) just to allow $a_0 \neq 0$ in (5).

Remark 2. Instead of the arithmetic mean (4), in [7] is considered the weighted mean:

$$\overline{A}_n = \frac{p_0 a_0 + \cdots + p_n a_n}{p_0 + \cdots + p_n} \tag{6}$$

with $p_n > 0$ for $n \geq 0$. The result from [7] was put in [6] in the following simpler form: the sequence $(\overline{A}_n)_{n=0}^{\infty}$ is convex for any convex $(a_n)_{n=0}^{\infty}$ if and only if there is an $u > 0$ such that:

$$p_n = p_0 \binom{u+n-1}{n}, \text{ for } n \geq 1 \tag{7}$$

where:

$$\binom{v}{0} = 1, \ \binom{v}{n} = \frac{1}{n!}\prod_{k=0}^{n-1}(v-k), \text{ for } n \geq 1, \ v \in \mathbb{R}. \tag{8}$$

In this case:

$$\overline{A}_n = A_n^u = \frac{\sum_{k=0}^{n} \binom{u+k-1}{k} a_k}{\binom{u+n}{n}}. \tag{9}$$

Definition 3. The sequence $(a_n)_{n=0}^{\infty}$ is u-mean-convex if $(A_n^u)_{n=0}^{\infty}$ is convex. The set of all u-mean-convex sequences is denoted by S_2^u.

In [6] we have proved:

Theorem 1. *If $0 < v < u$, then hold the strict inclusions:*

$$S_1 \subset S_2^u \subset S_2^v \subset S_3. \tag{10}$$

For some fixed functions $f, g, h, k : \mathbb{N} \to \mathbb{R}$, let us denote by:

$$Ta_n = f(n)a_{n+2} + g(n)a_{n+1} + h(n)a_n + k(n)a_0 \tag{11}$$

and consider the set:

$$S = \{(a_n)_{n=0}^{\infty} : \ Ta_n \geq 0, \ \forall \ n \geq 0\}. \tag{12}$$

We have proved in [5] the following general result:

Lemma 1. *If $(cn)_{n=0}^{\infty} \in S$ for any real c and if (6) gives an $(\overline{A}_n)_{n=0}^{\infty}$ in S for any $(a_n)_{n=0}^{\infty}$ from S, then there is an $u > 0$ such that the weights p_n be given by (7).*

Consequence. If $(\overline{A}_n)_{n=0}^{\infty}$ given by (6) is starshaped for any starshaped $(a_n)_{n=0}^{\infty}$ then the weights p_n are given by (7) with $u > 0$ adequate chosen.

Definition 4. The sequence $(a_n)_{n=0}^{\infty}$ is u-mean-starshaped if $(A_n^u)_{n=0}^{\infty}$, given by (9) is starshaped. The set of u-mean-starshaped sequences is denoted by S_5^u.

Lemma 2. *The sequence $(a_n)_{n=0}^{\infty}$ is u-mean-starshaped if and only if it may be represented by:*

$$a_0 = c_0,\ a_1 = (1+1/u)c_1 - c_0/u, \tag{13}$$

$$a_n = \left(1+\frac{n}{u}\right)c_n + n\left(1+\frac{1}{u}\right)\sum_{k=1}^{n-1}\frac{c_k}{k} - \left(n-1+\frac{n}{u}\right)c_0,\ \textit{for } n \geq 2$$

where $c_k \geq 0$ for $k \geq 2$.

Proof. As it is proved in [4], a sequence $(a_n)_{n=0}^{\infty}$ is starshaped if and only if it may be represented by:

$$a_n = n\sum_{k=1}^{n}\frac{b_k}{k} - (n-1)b_0 \tag{14}$$

with $b_k \geq 0$ for $n \geq 2$. So, $(a_n)_{n=0}^{\infty}$ is u-mean-starshaped if and only if $(A_n^u)_{n=0}^{\infty}$ may be represented by:

$$A_n^u = n\sum_{k=1}^{n}\frac{c_k}{k} - (n-1)c_0 \tag{14'}$$

with $c_k \geq 0$ for $k \geq 2$. But, from (9), we have:

$$\binom{u+n-1}{n}a_n = \binom{u+n}{n}A_n^u - \binom{u+n-1}{n-1}A_{n-1}^u$$

that is:

$$a_n = \left(1+\frac{n}{u}\right)A_n^u - \frac{n}{u}A_{n-1}^u. \tag{15}$$

From (14') and (15) we get (13).

Remark 3. It was proved in [4] that if the sequence $(a_n)_{n=1}^{\infty}$ is represented by (14), then:

$$\Delta^2 a_n = b_{n+2} - \frac{n}{n+1} b_{n+1}. \tag{16}$$

It is easy to check also the following results:

Lemma 3. *If the sequence $(a_n)_{n=0}^{\infty}$ is represented by (13), then:*

$$\Delta^2 a_0 = \left(1 + \frac{2}{u}\right) c_2, \tag{17}$$

$$\Delta^2 a_n = \left(1 + \frac{n+2}{u}\right) c_{n+2} - \frac{n}{n+1} \left(1 + \frac{2n+3}{u}\right)$$
$$c_{n+1} + \frac{n-1}{u} c_n, \ \textit{for } n > 0.$$

Theorem 2. *If $0 < v < u$, then the strict inclusions hold:*

$$S_3 \subset S_5^u \subset S_5^v. \tag{18}$$

Proof. (i) Let us suppose that the sequence $(a_n)_{n=0}^{\infty}$ is represented as in (13) and also as in (14). This may be done for every sequence. Then, from (16) and (17) we deduce:

$$b_2 = \left(1 + \frac{2}{u}\right) c_2$$

and

$$b_{n+2} - \frac{n}{n+1} b_{n+1} = \left(1 + \frac{n+2}{u}\right) c_{n+2} - \frac{n}{n+1} \left(1 + \frac{2n+3}{u}\right) c_{n+1}$$
$$+ \frac{n-1}{u} c_n$$

for $n \geq 1$, which give, by induction:

$$c_n = \frac{u}{u+n} b_n + \frac{n(n-2)}{(u+n)(n-1)} c_{n-1}, \text{ for } n \geq 2.$$

So, if $b_n \geq 0$ for $n \geq 2$, we obtain, step by step, $c_n \geq 0$ for $n \geq 2$. That is $S_3 \subset S_5^u$. The inclusion is proper because we have:

$$b_3 = \left(1 + \frac{3}{u}\right) c_3 - \frac{2}{3u} c_2$$

which yields $b_3 < 0$ for $c_2 = 1$ and $c_3 = 0$.

(ii) Now suppose that the sequence $(a_n)_{n=0}^{\infty}$ is represented by (13) and by:

$$(13') \qquad a_0 = d_0, \quad a_1 = (1 + 1/v)d_1 - d_0/v,$$

$$a_n = (1 + n/v)d_n + n\left(1 + \frac{1}{v}\right) \sum_{k=1}^{n-1} \frac{d_k}{k} - \left(n - 1 + \frac{n}{v}\right) d_0, \text{ for } n \geq 2.$$

From (17) we have:

$$(1 + 2/u)c_2 = (1 + 2/v)d_2$$

and for $n \geq 1$:

$$\left(1 + \frac{n+2}{u}\right) c_{n+2} - \frac{n}{n+1} \left(1 + \frac{2n+3}{u}\right) c_{n+1} + \frac{n-1}{u} c_n$$

$$= \left(1 + \frac{n+2}{v}\right) d_{n+2} - \frac{n}{n+1} \left(1 + \frac{2n+3}{v}\right) d_{n+1} + \frac{n-1}{v} d_n.$$

So, again by induction:

$$d_n = \frac{v(u+n)}{u(v+n)} c_n + \frac{v(u-v)}{u} \frac{n!}{n-1} \sum_{k=2}^{n-1} \frac{(k-1)c_k}{k!(v+k)\dots(v+n)}$$

Since $u > v$, if $c_k \geq 0$ for $n \geq 2$, then $d_n \geq 0$ for $n \geq 2$. Hence, by Lemma 2, $S_5^u \subset S_5^v$. The inclusion is proper because:

$$c_3 = \frac{u(v+3)}{v(u+3)} d_3 + \frac{3}{2} \frac{u(v-u)}{v(u+2)(u+3)} d_2$$

and $d_3 = 0$, $d_2 > 0$ give $c_3 < 0$.

Remark 4. For $u = 1$, $S_5^u = S_5$, so that $S_4 \subset S_5^1$. By Theorem 2, we have also:

$$S_4 \subset S_5^u, \text{ for } 0 < u < 1. \tag{19}$$

As we shall prove by the following examples, there is no inclusion between S_4 and S_5^u for $u > 1$.

Example 1. For $a_n = [n/2]$, where $[x]$ denotes the integer part of x, we have:

$$D^1 A_2^u = \frac{u(1-u)}{6(u+2)(u+3)}$$

so that $(a_n)_{n=0}^{\infty}$ is in S_4 (see [4]) but not in S_5^u for $u > 1$.

Example 2. For an arbitrary sequence of the form (13) we have:

$$a_n + a_0 - a_{n-1} - a_1 = \left(1 + \frac{n}{u}\right) c_n + \frac{u - n^2 + 3n - 1}{u(n-1)} c_{n-1}$$

$$+ \left(1 + \frac{1}{u}\right) \sum_{k=2}^{n-2} \frac{c_k}{k}.$$

For any $u > 0$ there is an n_0 such that $n_0^2 - 3n + 1 > u$. Hence, if we take $c_k = 0$ for $k \neq n_0 - 1$ and $c_{n_0 - 1} = 1$, we get a sequence $(a_n)_{n=0}^{\infty}$ in S_5^u but not in S_4.

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ON p, q-CONVEX SEQUENCES

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In this paper we give the representation and some properties of p, q-convex, of p, q-starshaped and of p, q-superadditive sequences. We also prove some relations among their classes.

The p, q-differences are natural generalizations of finite differences (of order two). They are defined for any real sequence $(a_n)_{n\geq 0}$ by:

$$L_{pq}(a_n) = a_{n+2} - (p+q)a_{n+1} + pqa_n. \tag{1}$$

Definition 1. A real sequence $(a_n)_{n\geq 0}$ is called p, q-convex if:

$$L_{pq}(a_n) \geq 0, \ \forall\ n \geq 0. \tag{2}$$

For some problems about p, q-differences and p, q-convex sequences one can consult the papers [1], [2] and [3].

An example of p, q-convex sequence, which plays an important place in what follows is given by the following result that is easy to verify.

Lemma 1. *The sequence:*

$$w_n = \begin{cases} \dfrac{p^n - q^n}{p - q}, & \text{if} \quad p \neq q \\ np^{n-1}, & \text{if} \quad p = q \end{cases} \tag{3}$$

satisfies the relation:

$$L_{pq}(w_n) = 0, \ \forall \ n \geq 0. \tag{4}$$

Theorem 1. *The sequence $(a_n)_{n\geq 0}$ is p, q-convex if and only if it may be represented by:*

$$a_n = \sum_{i=0}^{n} w_{n-i+1} b_i, \quad n \geq 0 \tag{5}$$

where w_i is given by (3) and

$$b_i \geq 0 \text{ for } i \geq 2. \tag{6}$$

Proof. Any sequence $(a_n)_{n\geq 0}$ may be represented by (5) with a invariant sequence $(b_n)_{n\geq 0}$. Because from (5) and (4) we get:

$$L_{pq}(a_n) = b_{n+1}, \quad n \geq 0 \tag{7}$$

the conditions (7) and (6) and equivalent.

Corollary 1. *A sequence $(a_n)_{n\geq 0}$ satisfies the relation:*

$$L_{pq}(a_n) = 0, \ \forall \ n \geq 0 \tag{7'}$$

if and only if it may be represented by:

$$a_n = w_{n+1} b_0 + w_n b_1 \tag{7''}$$

where b_0 and b_1 are arbitrary real numbers.

Lemma 2. *A sequence* $(a_n)_{n\geq 0}$ *verifies:*

$$\frac{a_{n+1}-y_{n+1}a_0}{w_{n+1}} \geq \frac{a_n - y_n a_0}{w_n}, \ \forall\ n \geq 1 \tag{8}$$

where $(y_n)_{n\geq 1}$ *is a given real sequence, if and only if* $(a_n)_{n\geq 0}$ *can be represented by:*

$$a_n = w_n \sum_{i=1}^{n} \frac{c_i}{u_i} + (y_n - w_n y_1)c_0 \tag{9}$$

where $y_0 = 1$ *and*

$$a_i \geq 0 \text{ for } i \geq 2. \tag{10}$$

The proof may be done by induction. It is also easy to verify:

Corollary 2. *If* $(a_n)_{n\geq 0}$ *is represented by (9), then:*

$$L_{pq}(a_n) = c_{n+2} - pq\frac{w_n}{w_{n+1}}c_{n+1} + L_{pq}(y_n)c_0. \tag{11}$$

Lemma 3. *The relation (8) is verified by any pq-convex sequence* $(a_n)_{n\geq 0}$ *if and only if:*

$$L_{pq}(y_n) = 0, \ \forall\ n \geq 0. \tag{12}$$

Proof. If the sequence $(a_n)_{n\geq 0}$ is pq-convex and verifies (8), by (7) and (11) we have:

$$b_{n+2} = c_{n+2} - pq\frac{w_n}{w_{n+1}}b_{n+1} + L_{pq}(v_n)c_0. \tag{13}$$

Then (6) implies (10) if and only if holds (12).

Remark 1. So $(y_n)_{n\geq 0}$ is given by (5'). To get $y_n = 1, \forall\, n$ if $p = q = 1$, we choose $b_0 = 1$ and $b_1 = -(p+q)/2$, that is:

$$y_n = \frac{p^n + q^n}{2}. \tag{14}$$

Definition 2. The sequence $(a_n)_{n\geq 0}$ is called pq-starshaped if it verifies (8), where $(y_n)_{n\geq 0}$ is given by (14).

Remark 2. So, a pq-starshaped sequence is represented by (9) and (10). If we denote by K_{pq} the set of all pq-convex sequences and by S^*_{pq} the set of all pq-starshaped sequences, from Lemma 3, we deduce:

Theorem 2. *Holds the strict inclusion:*

$$K_{pq} \subset S^*_{pq}. \tag{15}$$

Definition 3. The sequence $(a_n)_{n\geq 0}$ is said to be pq-superadditive if it satisfies the relation:

$$a_{n+m} - y_n a_m - y_m a_n + (2y_n y_m - y_{n+m})a_0 \geq 0 \tag{16}$$

for any $n, m \geq 0$.

Let us denote by S_{pq} the set of all pq-superadditive sequences.

Theorem 3. *Holds the inclusion:*

$$S^*_{pq} \subset S_{pq}. \tag{17}$$

Proof. For $n = 0$, (16) is verified for any sequence. Suppose $0 < n < m$. As:

$$y_n w_m + y_m w_n = w_{n+m} \tag{18}$$

if $(a_n)_{n\geq 0}$ is given by (9), then:

$$a_{n+m} - y_n a_m - y_n a_n + (2y_n y_m - y_{n+m})a_0 = y_m w_n \sum_{i=n+1}^{n+m} \frac{c_i}{w_i}$$

which is positive because of (10).

Remark 3. For $p = q = 1$, all the result of this paper were proved in [4].

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STARSHAPEDNESS AND SUPERADDITIVITY OF HIGH ORDER OF SEQUENCES

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1. The convexity of high order of sequences is well known (see [2]). In [6] we have also defined a starshapedness (of order two) which may be extended to an arbitrary order.

In what follows, by (a_n) we mean a real sequence for $n = 0, 1, \ldots$. The finite differences of the sequence (a_n) are defined inductively by:

$$\Delta^0 a_n = a_n,\ \Delta^{k+1} a_n = \Delta^k a_{n+1} - \Delta^k a_n, \quad k \geq 0,\ n \geq 0. \tag{1}$$

Definition 1. A sequence (a_n) is said to be convex of order r if $\Delta^r a_n \geq 0$ for any $n \geq 0$.

Let us denote by K_r the set of all convex of order r sequences. In [7] we have proved the following:

Lemma 1. *A sequence (a_n) is in K_r if and only if it may be represented by:*

$$a_n = \sum_{k=0}^{n} \binom{n+r-k-1}{r-1} b_k \tag{2}$$

where $b_k \geq 0$ for $k \geq r$.

Consequence 1. If (a_n) is represented by (2), then:

$$\Delta^r a_n = b_{n+r}. \tag{3}$$

Definition 2. The sequence (a_n) is said to be starshaped of order r if

$$\left(\frac{a_{n+1} - a_0}{n+1}\right) \in K_{r-1}.$$

If we denote by S_r^* the set of all starshaped of order r sequences, from Lemma 1 we obtain:

Lemma 2. *A sequence $(a_n) \in S_r^*$ if and only if it may represented by:*

$$a_n = n \sum_{k=1}^{n} \binom{n+r-k-2}{r-2} c_k + c_0, \quad n \geq 0 \tag{4}$$

with $c_k \geq 0$ for $k > r$.

Consequence 2. If the sequence (a_n) is represented by (4) then:

$$\Delta^r a_n = (n+r)c_{n+r} - nc_{n+r-1}, \quad n \geq 0. \tag{5}$$

Proof. Applying (3) to (4) we get:

$$\Delta^{r-1} \frac{a_{n+1} - a_0}{n+1} = c_{n+r}. \tag{6}$$

Putting $x_0 = a_0$ and $x_{n+1} = (a_{n+1} - a_0)/(n+1)$, we have:

$$a_n = nx_n + a_0, \quad n \geq 0$$

and so:

$$\Delta^r a_n = n\Delta^r x_n + r\Delta^{r-1} x_{n+1} = n\Delta c_{n+r-1} + rc_{n+r}$$

which gives (5).

In [7] we have also proved the following:

Theorem 1. *The sequence* (A_n) *given by:*

$$A_n = \frac{p_0 a_0 + \cdots + p_n a_n}{p_0 + \cdots + p_n}, \quad p_i > 0 \text{ for } i \geq 0 \tag{7}$$

is convex of order r *for any* $(a_n) \in K_r$ *if and only if there is an* $n > 0$ *such that:*

$$p_n = p_0 \binom{u+n-1}{n}, \text{ for } n \geq 1 \tag{8}$$

where:

$$\binom{v}{0} = 1, \quad \binom{v}{n} = \frac{v(v-1)\dots(v-n+1)}{n!}, \quad n > 0, \; v \in \mathbb{R}. \tag{9}$$

If it is so, then:

$$A_n = A_n^u = \sum_{k=0}^{n} \binom{u+k-1}{k} a_k \Big/ \binom{u+n}{n}. \tag{7'}$$

If we denote by $M^u K_r$ the set of all sequences (a_n) such that (7') gives a sequence (A_n^u) in K_r, in [7] was proved:

Lemma 3. *A sequence* (a_n) *is in* $M^u K_r$ *if and only if it may be represented by:*

$$a_n = \sum_{k=0}^{n} \binom{n+r-k-2}{r-2} \left(\frac{n+r-k-1}{r-1} + \frac{n}{u} \right) d_k, \quad n \geq 0 \tag{10}$$

where $d_k \geq 0$ *for* $k \geq 0$.

Consequence 3. If (a_n) is given by (10) then:

$$\Delta^r a_n = \frac{n+r+u}{u} d_{n+r} - \frac{n}{u} d_{n+r-1}, \quad n \geq 0. \tag{11}$$

Theorem 2. *For any* $u > 0$ *hold the following strict inclusions:*

$$K_r \subset M^u K_r \subset S_r^*. \tag{12}$$

Proof. The first inclusion was proved in [7]. To prove the second, suppose (a_n) be represented by (10) and by (4). This is possible for any sequence. Then (11) and (5) gives:

$$\frac{n+r+u}{u}d_{n+r}-\frac{n}{u}d_{n+r-1}=(n+r)c_{n+r}-nc_{n+r-1},\quad n\geq 0. \tag{13}$$

Thus $c_r=[(r+u)/(ru)]d_r$ and generally, by induction:

$$c_{n+r}=\frac{n+r+u}{u(n+r)}d_{n+r}+n!\sum_{k=0}^{n-1}\frac{d_{r+k}}{k!(r+k)}.$$

So, if $d_n\geq 0$ it follows also $c_n\geq 0$ for $n\geq r$, thus the inclusion is proved. That it is strictly also follows from (13) by choosing $c_r=1/r+1/u$, $c_{r+1}=0$, because we get $d_{r+1}=-1/r$.

Remark 1. For sequences with $a_0=0$, the inclusion $K_r\subset S_r^*$, was proved in [1], although the definition of starshapedness of order r is missing there.

2. In fact, the starshapedness is a convexity with one point fixed. So that we can consider also other types of starshapedness, fixing more points, as was done for functions in [3]. Starting from the following formula for divided differences:

$$[x_0,x_1,\ldots,x_n;f]=[x_0,\ldots,x_p;[x_{p+1},\ldots,x_n,x;f]]$$

we have:

$$[0,1,\ldots,p,n+p+1,\ldots,n+r;f]=[n+p+1,\ldots,n+r;[0,1,\ldots,p,x;f]].$$

As:

$$[0,1,\ldots,p,x;f]=\frac{f(x)}{x(x-1)\ldots(x-p)}+\frac{(-1)^{p+1}}{p!}\sum_{i=0}^{p}(-1)^i\binom{p}{i}\frac{f(i)}{x-i}$$

$$= \frac{(-1)^p}{p!} \sum_{i=0}^{p} (-1)^i \binom{p}{i} \frac{f(x) - f(i)}{x - i}$$

we give the following:

Definition 3. The sequence (a_n) is called $(p+1)$-starshaped of order r (with $p+1 < r$) if the sequence:

$$\left(\frac{(-1)^p}{p!} \sum_{i=0}^{p} (-1)^i \binom{p}{i} \frac{a_{n+p+1} - a_i}{n+p-i+1} \right)$$

belong to K_{r-p-1}.

If we denote by $S_r^{(p+1)*}$ the set of all $(p+1)$-starshaped of order r sequences, we have:

Lemma 4. *The sequence (a_n) belongs to $S_r^{(p+1)*}$ if and only if it may be represented by:*

$$a_n = \frac{n!}{(n-p-1)!} \frac{(-1)^p}{p!} \sum_{i=0}^{p} (-1)^i \binom{p}{i} \frac{c_{p,i}}{n-i}$$

$$+ \sum_{i=p+1}^{n} \binom{n+r-p-i-2}{r-p-2} c_{p,i}$$

where $c_{p,i} \geq 0$ for $i \geq r$.

Proof. By definition, the sequence $(a_n) \in S_r^{(p+1)*}$ if and only if $(a_{p,n}) \in K_{r-p-1}$, where:

$$a_{p,n} = \frac{(-1)^p}{p!} \sum_{i=0}^{p} (-1)^i \binom{p}{i} \frac{a_{n+p+1} - a_i}{n+p-i+1} = \frac{n! a_{n+p+1}}{(n+p+1)!} \tag{15}$$

$$+ \frac{(-1)^{p+1}}{p!} \sum_{i=0}^{p} (-1)^i \binom{p}{i} \frac{a_i}{n+p-i+1}.$$

From Lemma 1, we have thus:

$$a_{p,n} = \sum_{k=0}^{n} \binom{n+r-p-k-2}{r-p-2} c'_k \tag{16}$$

with $c'_k \geq 0$ for $k \geq r - p - 1$. Putting $c_{p,i} = a_i$ for $i = 0, \ldots, n$ and $c_{p,p+i+1} = c'_i$ for $i = 0, \ldots, n$, from (15) and (16) we get (14).

Consequence 4. If (a_n) is represented by (14) then:

$$\Delta^r a_n = (p+1)! \sum_{j=0}^{\min\{n,p+1\}} (-1)^i \binom{n}{j} \binom{r+n-1}{p+1-i} c_{p,r+n-j}. \tag{17}$$

Proof. If we denote by:

$$(a)_0 = 1, \quad (a)_k = a(a+1)\ldots(a+k-1), \tag{18}$$

and also by:

$$x_n = a_n, \quad n = 0, \ldots, p, \quad r_{n+p+1} = r_{p,n}, \quad n = 0, \ldots$$

from (15) we have:

$$a_n = (n-p)_{p+1} x_n + \frac{(-1)^p}{p!} \sum_{i=0}^{p} (-1)^i \binom{p}{i} a_i \frac{(n-p)_{p+i}}{n-1}, \quad n \geq 0.$$

As $\Delta^i (n-p)_{p+1} = (p-i+2)_i (n-p+i)_{p-i+1}$ for $i \leq p+1$ (and zero for $i > p+1$), and from (16) we have $\Delta^{r-p-1} a_{p,n} = c'_{n+r-p-1}$, that is $\Delta^{r-p-1} x_{n+p+1} = c_{p,n+r}$, we get:

$$\Delta^r a_n = \sum_{i=0}^{p+1} \binom{r}{i} (p-i+2)_i (n-p+i)_{p-i+1} \Delta^{p-i+1} c_{p,n+r-p+i-1}. \tag{19}$$

Of course, for $n \le p$, $(n-p+i)_{p-i+1} = 0$ if $i = 0, \ldots, p-n$, thus:

$$\begin{aligned}
\Delta^r a_n &= \sum_{i=p-n+1}^{n+1} \binom{r}{i} \frac{(p+1)!}{(p-i+1)!} \frac{n!}{(n-p+i-1)!} \Delta^{p-i+1} c_{p,n+r-p+i-1} \\
&= \sum_{j=0}^{n} \binom{r}{p-n+j+1} (p+1)! \binom{n}{j} \Delta^{n-j} c_{p,r+j} \\
&= (p+1)! \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{r}{p-n+j+1} \binom{n}{j} (-1)^{n-i} \binom{n-j}{i-j} c_{p,r+i} \\
&= (p+1)! \sum_{i=0}^{n} (-1)^{n-i} c_{p,r+i} \binom{n}{i} \sum_{j=0}^{i} \binom{r}{p-n+j+1} \binom{i}{i-j} \\
&= (p+1)! \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \binom{r+i}{p-n+i+1} c_{p,r+i}
\end{aligned}$$

In the last equality we have used Vandermonde's relation (see [5]). Putting $n-i=j$, we get (17), because $\min\{n, p+1\} = n$ in this case. If $n > p$, (19) leads again to (17), on the same way with $\min\{n, p+1\} = p+1$.

Remark 2. For $p = 0$, we have $S_r^{1*} = S_r^*$ and the relations (14) and (17) one reduces to (4) and (5).

Theorem 3. *For any $1 \le p \le r-2$, we have:*

$$S_r^{p*} \subset S_r^{(p+1)*}. \tag{20}$$

Proof. Consider a sequence (a_n) represented by (14) for p and $p+1$. Then (17) gives:

$$\sum_{j=0}^{\min\{n,p\}} (-1)^j \binom{n}{j} \binom{r+n-j}{p-j} c_{p-1,r+n-j} \tag{21}$$

$$= (p+1) \sum_{j=0}^{\min\{n,p+1\}} (-1)^j \binom{n}{j} \binom{r+n-j}{p+1-j} c_{p,r+n-j}.$$

Denote:

$$c_{p,r+n} = \sum_{k=0}^{n} x_{r+n}^{r+k} c_{p-1,r+k}, \quad n \geq 0. \tag{22}$$

For $n \leq p$, we have:

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{r+n-j}{p-j} c_{p-1,r+n-j}$$

$$= (p+1) \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{r+n-j}{p+1-j} \sum_{k=0}^{p-j} x_{r+n-j}^{r+k} c_{p-1,r+k}$$

or

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{r+k}{p-n+k} c_{p-1,r+k}$$

$$= (p+1) \sum_{k=0}^{n} c_{p-1,r+k} \sum_{j=0}^{n-k} (-1)^j \binom{n}{j} \binom{r+n-j}{p+1-j} x_{r+n-j}^{r+k}.$$

Thus, for $k = 0, \ldots, n$ we must have:

$$\sum_{j=0}^{n-k} (-1)^j \binom{n}{j} \binom{r+n-j}{p+1-j} x_{r+n-j}^{r+k} = \frac{(-1)^{n-k}}{p+1} \binom{n}{k} \binom{r+k}{p-n+k}. \tag{23}$$

So, for $k = n$ we get $x_{r+n}^{r+n} = 1/(r-p+n)$, and generally:

$$x_{r+n}^{r+n-k} = \frac{1}{k+1} \binom{n}{k} \Big/ \binom{r-p+n}{k+1}. \tag{24}$$

Analogously, for $n > p$, from (21) we get:

$$\sum_{j=0}^{p} (-1)^j \binom{n}{j} \binom{r+n-j}{p-j} c_{p-1,r+n-j}$$

$$= (p+1) \sum_{j=0}^{p+1} (-1)^j \binom{n}{j} \binom{r+n-j}{p+1-j} \sum_{k=0}^{n-j} x_{r+n-j}^{r+k} c_{p-1,r+k}$$

or

$$\sum_{k=n-p}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{p-n+k}c_{p-1,r+k}$$

$$=(p+1)\sum_{k=0}^{n}c_{p-1,r+k}\sum_{j=0}^{\min\{n-k,p+1\}}(-1)^j\binom{n}{j}\binom{r+n-j}{p+1-j}x_{r+n-j}^{r+k}.$$

Thus:

$$\sum_{j=0}^{p+1}(-1)^j\binom{n}{j}\binom{r+n-j}{p+1-j}x_{r+n-j}^{r+k}=0,\ \text{ for } k=0,\dots,n-p-1 \tag{25}$$

and

$$\sum_{j=0}^{n-k}(-1)^j\binom{n}{j}\binom{r+n-j}{p+1-j}x_{x+n-j}^{r+k} \tag{26}$$

$$=\frac{(-1)^{n-k}}{p+1}\binom{n}{k}\binom{r+k}{p-n+k},\qquad k=n-p,\dots,n.$$

As above, we get $x_{r+n}^{r+k}\geq 0$, thus $c_{p,n}\geq 0$ if $c_{p-1,n}\geq 0$ for $n\geq r$ and so the desired inclusion.

3. In the hierarchy of convexity from [6] appears also the superadditivity. To extend it at an arbitrary order r, we have as model the inequalities proved for functions (convex of order r) by T. Popoviciu in [4] and much later by P. M. Vasić in [9]. So we give the following:

Definition 4. The sequence (a_n) is superadditive of order r if for any indices $n_1,\dots,n_r>0$ holds:

$$\sum_{k=0}^{r}(-1)^{r-k}\sum_{(i_1,\dots,i_k)}a_{n_{i_1}+n_{i_k}}\geq 0 \tag{27}$$

where the second sum is extended to $\binom{r}{k}$ possible choices of indices $i_1,\dots,i_k$ from $1,\dots,r$ and reduces at a_0 for $k=0$.

Remark 3. Let us denote by S_r the set of all superadditive of order r sequences. For $r = 2$ we have proved in [6] that $K_2 \subset S_2^* \subset S_2$. Also, for $r = 3$, we have proved in [8] that:

$$K_3 \subset S_3^* \subset S_3 \subset S_3^{2*} \tag{29}$$

The properties proved in [4] and [9] for functions means that: $K_r \subset S_r$. From Theorem 2 and Theorem 3 we have:

$$K_r \subset S_r^{1*} \subset S_r^{2*} \subset \cdots \subset S_r^{(r-1)*}. \tag{29}$$

From (28) we suppose that hold also the inclusion:

$$S_r^i \subset S_r \subset S_r^{(i+1)*} \tag{30}$$

but we can't yet prove it.

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ON A GENERAL INEQUALITY FOR CONVEX SEQUENCES

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The finite differences of a real sequence $a = (a_1, a_2, \dots,)$ are defined by:

$$\Delta^1 a_n = a_{n+1} - a_n,\ \Delta^m a_n = \Delta^1(\Delta^{m-1} a_n),\ \nabla^m a_n = (-1)^m \Delta^m a_n.$$

The sequence a is said to be convex of order m if $\Delta^m a_n \geq 0$ for $n \geq 1$.

In [2] J.E. Pečarić proved the following:

Theorem 1. *Let $p = (p_1, \dots, p_n)$ be a real n-tuple $(n > m)$. The inequality:*

$$\sum_{i=1}^{n} p_i a_i \geq 0 \tag{1}$$

holds for every sequence a convex of order m, if and only if:

$$\sum_{i=1}^{n} (i-1)^{(k)} p_i = 0, \quad k = 0, 1, \dots, m-1 \tag{2}$$

and

$$(3) \qquad \sum_{i=k}^{n}(i-k+m-1)^{(m-1)}p_i \geq 0, \quad k=m+1,\dots,n$$

where:

$$(4) \qquad x^{(0)}=1, \quad x^{(k)}=x(x-1)\dots(x-k+1), \; k\geq 1.$$

Remark. An analogous result was proved for functions (convex of order m) by T. Popoviciu (see [3] and [1]). Using a method from [4], we can give the representation of the n-tuple p.

Theorem 2. *The inequality (1) holds for any sequence a, convex of order m, if and only if:*

$$(5) \qquad p_k=\nabla^m q_k, \quad k=1,\dots,n$$

where:

$$(6) \qquad q_k=0, \; \textit{for } k=1,\dots,m \textit{ and } k=n+1,\dots,n+m$$

and

$$(7) \qquad q_k\geq 0, \; \textit{for } k=m+1,\dots,n.$$

Proof. If we put:

$$(8) \quad \sum_{i=k}^{n}(i-k+m-1)^{(m-1)}p_i=(m-1)^{(m-1)}q_k, \quad k=m+1,\dots,n$$

then (3) is equivalent to (7) and (2) gives the first part of (6). To get (5) from (8) for $k=n-m+1,\dots,n$, we must add the second part of (6).

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ON THE HIERARCHY OF CONVEXITY OF ORDER THREE OF SEQUENCES

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1. In the paper [4] we have proved a hierarchy of convexity for sequences, analogous with that known for functions. In this paper we prove similar properties in the case of the convexity of order three. This case underlines more clear the way for further generalizations.

Let $a = (a_n)$ $(n = 0, 1, \dots)$ be a real sequence. The finite differences of the sequence (a_n) are defined inductively by:

$$(1) \qquad \Delta^0 a_n = a_n,\ \Delta^{k+1} a_n = \Delta^k a_{n+1} - \Delta^k a_n \quad (k \geq 0,\ n \geq 0).$$

Definition 1. A sequence (a_n) is said to be convex of order 3 if $\Delta^3 a_n \geq 0$, for any $n \geq 0$.

Simplifying a result from [5], we have:

Lemma 1. *If the sequence* (a_n) *is given by:*

$$a_n = \sum_{k=0}^{n} \binom{n-k+2}{2} b_k \tag{2}$$

then:

$$\Delta^3 a_n = b_{n+3}, \quad n \geq 0. \tag{3}$$

Consequence 1. The sequence (a_n) is convex of order 3 if and only if in his representation (2) it has $b_n \geq 0$ for $n \geq 3$.

In [4] we have introduced the following notion:

Definition 2'. The sequence (a_n) is called starshaped if it satisfies

$$\frac{a_{n+1} - a_0}{n+1} \geq \frac{a_n - a_0}{n}, \text{ for } n \geq 1. \tag{4}$$

Remark 1. As (4) means that the sequence $((a_{n+1} - a_0)/(n+1))$ is increasing, that is convex of order 1, we extend this definition to the following:

Definition 2. The sequence (a_n) is called starshaped of order 3 if the sequence $((a_{n+1} - a_0)/(n+1))$ is convex (of order 2).

Lemma 2. *The sequence* (a_n) *is starshaped of order 3 if and only if it may be represented by:*

$$a_n = n \sum_{k=1}^{n} (n-k+1) c_k + c_0, \textit{ for } n \geq 0 \tag{5}$$

where $c_k \geq 0$ *for* $k \geq 3$.

Consequence 2. If the sequence (a_n) is represented by (5), then:

$$\Delta^3 a_n = (n+3) c_{n+3} - n c_{n+2}, \quad n \geq 0. \tag{6}$$

Proof. From (5) we have:

$$\Delta^1 a_n = (n+1)c_{n+1} + \sum_{k=1}^{n}(2n-k+2)c_k$$

then:

$$\Delta^2 a_n = (n+2)c_{n+2} + 2\sum_{k=1}^{n+1} c_k$$

and so (6).

Remark 2. In the hierarchy of convexity occurs also the superadditivity: a sequence (a_n) is called superadditive if:

$$(7') \qquad a_{n+m} + a_0 \geq a_n + a_m, \text{ for any } n, m \geq 0.$$

We introduce here an analogous notion for order three:

Definition 3. The sequence (a_n) is called superadditive of order 3 if it satisfies the relation:

$$(7) \quad p^3_{n,m,p}(a) = a_{n+m+p} - a_{n+m} - a_{m+p} - a_{p+n} + a_n + a_m + a_p - a_0 \geq 0$$

for any $n, m, p \geq 0$.

Remark 3. The relation (7) suggests Elawka's inequality (see [2]).

Definition 4. The sequence (a_n) is said to be 2-starshaped of order 3 if it satisfies the relation:

$$(8) \qquad \frac{a_{n+3} - a_0}{n+3} \geq \frac{a_{n+2} - a_1}{n+1}, \text{ for any } n \geq 0.$$

Lemma 3. *The sequence (a_n) is 2-starshaped of order 3 if and only if it may be represented by:*

$$(9) \qquad a_n = n(n-1)\sum_{k=2}^{n} d_k + nd_1 - (n-1)d_0, \quad n \geq 0$$

where $d_k \geq 0$ for $k \geq 3$.

Proof. Any sequence may be represented by (9). Then:

$$\frac{a_{n+3}-a_0}{n+3}-\frac{a_{n+2}-a_1}{n+1}=(n+2)d_{n+3}$$

which shows that (8) is equivalent with $d_n \geq 0$ for $n \geq 3$.

Consequence 3. If a sequence (a_n) is represented by (9) then:

$$\Delta^3 a_n=(n+2)(n+3)d_{n+3}-2n(n+2)d_{n+2}+n(n-1)d_{n+1}. \tag{10}$$

Let us denote by K_3, S_3^*, S_3 and S_3^{2*} the sets of sequences convex of order 3, starshaped of order 3, superadditive of order 3, respectively 2-starshaped of order 3.

Theorem 1. *Hold the following inclusions:*

$$K_3 \subset S_3^* \subset S_3 \subset S_3^{2*}. \tag{11}$$

Proof. a) Any sequence (a_n) may be represented by (2) and by (5). Then from (3) and (6) we get:

$$b_{n+3}=(n+3)c_{n+3}-nc_{n+2}, \quad n \geq 0.$$

If (a_n) is in K_3, then $b_n \geq 0$ for $n \geq 3$, thus $c_3=b_3/3 \geq 0$ and by mathematical induction $c_n \geq 0$ for $n > 3$. So (a_n) is also in S_3^*.

b) If (a_n) is represented by (5) and $c_n \geq 0$ for $n \geq 3$ then:

$$D^3_{n,m,p}(a)=$$

$$=n\left[n\sum_{k=n+1}^{n+p}c_k+\sum_{k=n+p+1}^{n+p+m}(n+m+p-k+1)c_k-\sum_{k=n+1}^{n+m}(n+m-k+1)c_k\right]$$

$$+m\left[p\sum_{k=m+1}^{m+n}c_k+\sum_{k=n+m+1}^{n+m+p}(n+m+p-k+1)c_k-\sum_{k=m+1}^{m+p}(m+p-k+1)c_k\right]$$

$$+p\left[n\sum_{k=p+1}^{p+m}c_k+\sum_{k=m+p+1}^{m+p+n}(n+m+p-k+1)c_k-\sum_{k=p+1}^{p+n}(n+p-k+1)c_k\right]$$

Supposing $n<m<p<m+n$:

$$p^3_{n,m,p}(a)=\sum_{k=n+1}^{m}n(k-n-1)c_k+\sum_{k=m+1}^{p}[n(k-n+1)+m(k-m-1)]c_k$$

$$+\sum_{k=p+1}^{n+m}[n(k-n-1)+m(k-m-1)+p(k-p-1)]c_k$$

$$+\sum_{k=n+m+1}^{n+p}[2nm+p(k-p-1)]c_k+\sum_{k=n+p+1}^{m+p}n(n+2p+2m-k+1)c_k$$

$$+\sum_{k=m+p+1}^{m+p+n}(n+m+p)(n+m+p-k+1)c_k$$

As $n\geq 1$ and $c_k\geq 0$ for $k\geq 3$, we get $D^3_{n,m,p}(a)\geq 0$. Similarly may be proved the case $n<m<n+m<p$. If we have an equality between the parameters, one or more sums will not appear in the expression of $D^3_{n,m,p}(a)$. Thus $S_3^*\subset S_3$.

c) From (7) we have:

$$a_n-a_{n-1}-a_{k+1}-a_{n-k}+a_k+a_{n-k-1}+a_1-a_0\geq 0$$

for $k=1,\ldots,n-2$. By addition we get:

$$(n-2)(a_n-a_{n-1}+a_1-a_0)+2(a_1-a_{n-1})\geq 0$$

which is (8).

Remark 4. If $a_0=0$, the inclusion $K_3\subset S_3^*$ was proved in [4], by other means, even in the case of the convexity of order r. For functions, the inclusion $K_r\subset S_r$ was also proved by T. Popoviciu in [3] and much later by P.M. Vasić in [7].

2. The hierarchy of convexity from [4] was enlarged in [6]. Let us to complete also (11) in the same manner. For this, to any sequence (a_n) and any real $u > 0$, we attach the sequence (A_n^u) given by:

$$A_n^u = \sum_{k=0}^{n} \binom{u+k-1}{k} a_k \Big/ \binom{u+n}{n} \tag{12}$$

where:

$$\binom{v}{0} = 1, \ \binom{v}{n} = v(v-1)\dots(v-n+1)/n!, \quad n \geq 1, \ \forall\, v \in \mathbb{R}. \tag{13}$$

Definition 5. The sequence (a_n) has the property P in the u-mean if the sequence (A_n^u) has the property P.

Let us to denote by $M^u K_3$, $M^u S_3^*$, $M^u S_3$ and $M^u S_3^{2*}$ the sets of sequences which are convex, starshaped, superadditive respectively 2-starshaped of order 3 in the u-mean.

Lemma 4. *The sequence (a_n) is in $M^u K_3$ if and only if it may be represented by:*

$$a_n = \sum_{k=0}^{n} (n+1-k) \left(\frac{n}{u} + \frac{n-k}{2} + 1 \right) e_k \tag{14}$$

with $e_k \geq 0$ for $k \geq 3$.

Proof. The sequence (a_n) is in $M^u K_3$ if and only if (A_n^u) is in K_3, that is:

$$A_n^u = \sum_{k=0}^{n} \binom{n-k+2}{2} e_k, \quad e_k \geq 0, \text{ for } k \geq 3. \tag{15}$$

But (12) gives:

$$a_n = A_n^u + \frac{n}{u}(A_n^u - A_{n-1}^u). \tag{16}$$

From (15) and (16) we get (14).

Similarly we can prove:

Lemma 5. *A sequence (a_n) is in $M^u S_3^*$ if and only if it may be represented by:*

$$a_n = n \sum_{k=1}^{n} \left(n - k + 1 + \frac{2n-k}{n} \right) f_k + f_0 \tag{17}$$

with $f_k \geq 0$ for $k \geq 3$.

Lemma 6. *A sequence (a_n) is in $M^u s_3^{2*}$ if and only if it may be represented by:*

$$a_n = n(n-1) \left(1 + \frac{n}{u} \right) g_n \tag{18}$$

$$+n(n-1) \left(1 + \frac{2}{u} \right) \sum_{k=2}^{n-1} g_k + (1 + 1/u) n g_1 - (n - 1 + n/u) g_0$$

where $g_k \geq 0$ for $k \geq 3$.

Consequence 4. If the sequence (a_n) is given:

a) by (14), then:

$$\Delta^3 a_n = \left(\frac{n+3}{u} + 1 \right) e_{n+3} - \frac{n}{u} e_{n+2}, \text{ for } n \geq 0; \tag{19}$$

b) by (17), then:

$$\Delta^3 a_n = (n+3) \left(1 + \frac{n+3}{u} \right) f_{n+3} \tag{20}$$

$$-n \left(1 + \frac{2n+5}{u} \right) f_{n+2} + \frac{n(n-1)}{u} f_{n+1}, \quad n \geq 0;$$

c) by (18), then:

$$\Delta^3 a_n = (n+2)(n+3) \left(1 + \frac{n+3}{u} \right) g_{n+3} \tag{21}$$

$$-n(n+2) \left(2 + \frac{3n+7}{u} \right) g_{n+2}$$

$$+n(n-1)\left(1+\frac{3n+5}{u}\right)g_{n+1}-\frac{n}{u}(n-1)(n-2)g_n, \quad n\geq 0.$$

Theorem 2. *For any $u>0$, hold the following inclusions:*

$$(22)\qquad \begin{array}{ccccccc} K_3 & \subset & S_3^* & \subset & S_3 & \subset & S_3^{2*} \\ \cap & & \cap & & & & \cap \\ M^u K_3 & \subset & M^u S_3^* & \subset & M^u S_3 & \subset & M^u S_3^{2*} \end{array}$$

Proof. a) Let (a_n) be represented by (2) and by (14). Then (3) and (19) give:

$$(23)\qquad b_{n+3}=\left(\frac{n+3}{u}+1\right)e_{n+3}-\frac{n}{u}e_{n+2}, \quad n\geq 0.$$

That is, if $b_k\geq 0$ for $k\geq 3$, it follows that $e_k\geq 0$ for $k\geq 3$. By Consequence 1 and Lemma 3, it results the inclusion $K_3\subset M^uK_3$.

b) If (a_n) is represented by (5) and by (17), from (6) and (20) follows:

$$(24)\qquad (n+3)c_{n+3}-nc_{n+2}=(n+3)\left(1+\frac{n+3}{u}\right)f_{n+3}$$

$$-n\left(1+\frac{2n+5}{u}\right)f_{n+2}+n(n-1)f_{n+1}/u.$$

We get $f_3=u/(u+3)c_3$ and generally, by induction:

$$f_{n+3}=\frac{u}{u+n+3}c_{n+3}+\sum_{k=3}^{n+2}\frac{un(n-1)\dots(k-2)}{(u+n+3)\dots(u+k)}c_k$$

so that $c_k\geq 0$ implies $f_k\geq 0$, for $k\geq 3$. By Lemma 2 and Lemma 5, we have $S_3^*\subset M^uS_3^*$.

c) Suppose (a_n) represented by (9) and by (18). From (10) and (21) we have:

$$(25)\qquad (n+2)(n+3)d_{n+3}-2n(n+2)d_{n+2}+n(n-1)d_{n+1}=$$

$$= (n+2)(n+3)g_{n+3} - n(n+2)\left(2 + \frac{3n+7}{u}\right) g_{n+2}$$

$$+ n(n-1)\left(1 + \frac{3n+5}{u}\right) g_{n+1} - n(n-1)(n-2)/u g_n.$$

That is $g_3 = u/(u+3)d_3$ and generally, supposing $g_n = \sum_{k=3}^{n} x_n^k d_k$, we get, by induction:

$$g_{n+3} = \frac{u}{u+n+3} d_{n+3} + \sum_{k=3}^{n+2} \frac{un(n-1)\dots(k-2)}{(u+n+3)\dots(u+k)} d_k.$$

So, if $d_k \geq 0$ for $k \geq 3$, we have also $g_k \geq 0$ for $k \geq 3$. By Lemma 3 and Lemma 6, $S_3^{2*} \subset M^u S_3^{2*}$. The other inclusions from (22) were proved in Theorem 1, or are its direct consequences.

Remark 5. It is natural to suppose also the inclusion $S_3 \subset M^u S_3$ but we cannot yet prove it. In exchange we give the following additional results:

Theorem 3. *Hold the following inclusions:*

$$M^u K_3 \subset S_3^* \tag{26}$$

and

$$M^u S_3^* \subset S_3^{2*}. \tag{27}$$

Proof. a) Let (a_n) be represented by (14) and by (5). Then (19) and (6) give:

$$(u+n+3)/u e_{n+3} - n/u e_{n+2} = (n+3)c_{n+3} - nc_{n+2}, \ n \geq 0 \tag{28}$$

that is $c_3 = (u+3)/3ue_3$ and generally:

$$c_{n+3} = \frac{u+n+3}{u(n+3)} e_{n+3} + \sum_{k=3}^{n+2} \frac{n(n-1)\dots(k-2)}{(n+3)(n+2)\dots k} e_k.$$

So, if $e_n \geq 0$ for $n \geq 3$, we have also $c_n \geq 0$ for $n \geq 3$. By Lemma 3 and Lemma 2 we get (26).

b) If (a_n) is given by (17) and (9), the relations (20) and (10) give:

$$(29)\quad (n+3)(u+n+3)/uf_{n+3} - n(u+2n+5)/uf_{n+2} + n(n-1)/uf_{n+3}$$

$$= (n+2)(n+3)d_{n+3} - 2n(n+2)d_{n+2} + n(n-1)d_{n+1}, \quad n \geq 0.$$

Thus:

$$d_{n+3} = \frac{u+n+3}{u(n+2)} f_{n+3} + \frac{u+1}{u(n+1)(n+2)} \sum_{k=3}^{n+2} (k-2) f_k.$$

So, again, $f_k \geq 0$ implies $d_k \geq 0$ for $k \geq 3$ and by Lemma 5 and Lemma 3 we have (27).

Theorem 4. *If $0 < u < v$ then:*

$$(30)\quad \begin{array}{ccccccc} M^v K_3 & \subset & M^v S_3^* & \subset & M^v S_3 & \subset & M^v S_3^{2*} \\ \cap & & \cap & & & & \cap \\ M^u K_3 & \subset & M^u S_3^* & \subset & M^u S_3 & \subset & M^u S_3^{2*} \end{array}$$

Proof. a) Let (a_n) be represented by (14) and by:

$$(14')\qquad a_n = \sum_{k=0}^{n} (n+1-k) \left(\frac{n}{v} + \frac{n-k}{2} + 1 \right) e'_k.$$

Then, by (19) we have:

$$(u+n+3)/ue_{n+3} - n/ue_{n+2} = (v+n+3)/ve'_{n+3} - n/ve'_{n+2},$$

and so:

$$e_{n+3} = \frac{u(v+n+3)}{v(u+n+3)} e'_{n+3} + (v-u)u/v \sum_{k=3}^{n+2} \frac{n(n-1)\dots(k-2)}{(u+n+3)\dots(u+k)} e'_k.$$

As $v > u$, if $e'_n \geq 0$ for $n \geq 3$, we have also $e_n \geq 0$. By Lemma 4, $M^v K_3 \subset M^u K_3$.

b) Now let (a_n) be represented by (17) and also by:

$$a_n = n \sum_{k=1}^{n} \left(n - k + 1 + \frac{2n-k}{v} \right) f'_k + f'_0. \tag{17'}$$

From (20) we have:

$$(n+3)(u+n+3)/u f_{n+3} - n(u+2n+5)/u f_{n+2} + n(n-1)/u f_{n+1}$$

$$= (n+3)(v+n+3)/v f'_{n+3} - n(v+2n+5)/v f'_{n+2} + n(n-1) v f'_{n+1}.$$

As above we get:

$$f_{n+3} = \frac{u(v+n+3)}{v(u+n+3)} f'_{n+3} + (v-u)u/v \sum_{k=3}^{n+2} \frac{n(n-1)\dots(k-2)}{(u+n+3)\dots(u+k)} f'_k$$

thus, by Lemma 5: $M^v S_3^* \subset M^u S_3^*$.

c) If (a_n) is represented by (18) and by:

$$a_n = n(n-1)\left(1 + \frac{n}{v}\right) g'_n + n(n-1)\left(1 + \frac{2}{v}\right) \sum_{k=2}^{n-1} g'_k \tag{18'}$$

$$+ n\left(1 + \frac{1}{v}\right) g'_1 - (n - 1 + n/v) g'_0$$

from (21) we get:

$$(n+2)(n+3)(u+n+3) : u g_{n+3} - n(n+2)(2u+3n+7) : u g_{n+2}$$

$$+ n(n-1)(u+3n+5) : u g_{n+1} - n(n-1)(n-2) : u g_n$$

$$= (n+2)(n+3)(v+n+3) : v g'_{n+3} - n(n+2)(2v+3n+7) : v g'_{n+2}$$

$$+n(n-1)(v+3n+5) : vg'_{n+1} - n(n-1)(n-2) : vg'_n.$$

Thus:

$$g_{n+3} = \frac{u(v+n+3)}{v(u+n+3)} g'_{n+3} + (v-u)u : v \sum_{k=3}^{n+2} \frac{n(n-1)\dots(k-2)}{(u+n+3)\dots(u+k)} g'_k$$

and, by Lemma 6, $M^v S_3^{2*} \subset M^u S_3^{2*}$.

Remark 6. The method of demonstration can be used to prove the strictness of all the inclusions.

These inclusions (together with those left yet unproved) show a large hierarchy for the convexity of order three. In what follows we show that it is the largest possible of this type.

Theorem 5. *If the sequence* (A_n) *given by:*

(31) $$A_n = \frac{p_0 a_0 + \cdots + p_n a_n}{p_0 + \cdots + p_n}, \quad n \geq 0$$

(where $p_k > 0$ *for* $n \geq 0$*) is convex (starshaped, superadditive respectively 2-starshaped) of order 3 for any sequence* (a_n) *with the same property, then the sequence* (p_n) *must be of the form:*

(32) $$p_n = p_0 \binom{u+n-1}{n}, \quad n \geq 0$$

where $u = p_1 : p_0$*.*

Proof. The sequence (a'_n) given by:

(33) $$a'_n = cn(n-1)$$

is in K_3 (thus it is also in S_3^*, in S_3 and in S_3^{2*}) so that (A'_n) given by:

(31′) $$A'_n = \frac{c \sum_{i=0}^{n} i(i-1)p_i}{\sum_{i=0}^{n} p_i}$$

must be in S_3^{2*} for any $c \in \mathbb{R}$. But this happens if and only if:

$$(34) \qquad \frac{\sum_{i=2}^{n+3} i(i-1)p_i}{(n+3)\sum_{i=0}^{p+3} p_i} = \frac{\sum_{i=2}^{n+2} i(i-1)p_i}{(n+1)\sum_{i=0}^{n+2} p_i}, \text{ for any } n \geq 0.$$

For $n = 0$ we have thus the condition:

$$(35) \qquad p_3 = \frac{2p_2(p_0 + p_1 + p_2)}{3(p_0 + p_1)}.$$

Analogously:

$$(33') \qquad a''_n = cn$$

gives a sequence in K_3, so that:

$$(31'') \qquad A''_n = \frac{c\sum_{i=0}^{n} ip_i}{\sum_{i=0}^{n} p_i}$$

defines a sequence in S_3^{2*} for any c, which implies:

$$(34') \qquad \frac{\sum_{i=1}^{n+3} ip_i}{(n+3)\sum_{i=0}^{p+3} p_i} = \frac{1}{n+1}\left[\frac{\sum_{i=1}^{n+2} ip_i}{\sum_{i=0}^{n+2} p_i} - \frac{p_1}{p_0 + p_1}\right], \quad n \geq 0.$$

For $n = 0$ this gives:

$$(35') \qquad p_3 = \frac{(p_0 + p_1 + p_2)[p_2(4p_0 + p_1) - p_1(p_0 + p_1)]}{3[(p_0 + p_1)^2 - p_0p_2]}$$

From (35) and (35') we have successively:

$$2p_2[(p_0 + p_1)^2 - p_0p_2] = (p_0 + p_1)[p_2(4p_0 + p_1) - p_1(p_0 + p_1)]$$

and

$$2p_0p_2^2 - p_2(p_0+p_1)(p_1-2p_0) - p_1(p_0+p_1)^2 = 0$$

that is:

$$p_2 = \frac{p_1(p_0+p_1)}{2p_0}.$$

Putting $u = p_1 : p_0$, we get (32) for $n = 0, 1, 2$ and 3 (the last one from (35)). Supposing (32) valid for $n \le m+2$ we obtain:

$$A'_n = cn(n-1)u : (u+2), \text{ for } n \le m+2$$

and

$$A'_{m+3} = c\left[(m+2)(m+3)p_{m+3} + p_0u(u+1)\binom{u+m+2}{m}\right] /$$

$$/ \left[p_{m+3} + p_0\binom{u+m+2}{m+2}\right]$$

because:

(36) $$\sum_{k=0}^{n}\binom{v+k}{k} = \binom{v+n+1}{n}.$$

That is (34) becomes:

$$\frac{(m+2)(m+3)p_{m+3} + p_0u(u+1)\binom{u+m+2}{m}}{(m+3)\left[p_{m+3} + p_0\binom{u+m+2}{m+2}\right]} = \frac{u(m+2)}{u+2}$$

which gives for p_{m+3} the same representation (32).

Remark 7. If p_n is given by (32), then (31) becomes (12) because of (36).

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A GENERAL HIERARCHY OF CONVEXITY OF SEQUENCES

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1. Introduction

In this paper we define some measures: of the convexity, of the starshapedness and of the superadditivity of a sequence. For sets, a measure of nonconvexity was given by J. Eisenfeld and V. Kalshmikantham in [2]. But our definitions are like that given by Gr.S. Sălăgean in [3] for complex functions. That is, they offer the possibility of evaluation of the deviation from a given property, but also of his strengthening. So we use them to introduce more classes of sequences and then we prove a hierarchy among them. This generalizes the hierarchy proved in [4] (which is similar with that given for functions by A.M. Bruckner and E. Ostrow in [1]) and even that from [5].

2. Notations and definitions

For a real sequence $a = (a_k)_{k\geq 0}$ we consider the following differences:

$$C_i(a) = a_{i+2} - 2a_{i+1} + a_i$$

$$D_i(a) = \frac{a_{i+2} - a_0}{i+2} - \frac{a_{i+1} - a_0}{i+1}$$

and

$$P_{ij}(a) = a_{i+j} - a_i - a_j + a_0.$$

With their help, we can define the well known classes of sequences:

$$K = \{a : C_i(a) \geq 0,\ \forall\ i \geq 0\}$$

$$S^* = \{a : D_i(a) \geq 0,\ \forall\ i \geq 0\}$$

and

$$S = \{a : P_{ij}(a) \geq 0,\ \forall\ i, j > 0\}$$

that is the sets of convex, starshaped, respectively superadditive sequences. We shall consider also the class:

$$W = \{a : P_{i1}(a) \geq 0,\ \forall\ i > 0\}$$

of weak superadditive sequences.

They suggest also the definition of the following measures:

a) of convexity, given by:

$$k_n(a) = \inf\{C_i(a) : 0 \leq i \leq n-2\}$$

b) of starshapedness, given by:

$$s_n^*(a) = \inf\{D_i(a) : 0 \leq i \leq n-2\}$$

c) of superadditivity, given by:

$$s_n(a) = \inf\{P_{ij}(a)/ij : 0 \le i, j, i+j \le n\};$$

d) of weak superadditivity, given by:

$$w_n(a) = \inf\{P_{i1}(a)/i : 0 < i < n\}.$$

These measures permit the consideration of the following classes of sequences:

$$K_{np} = \{a : k_n(a) \ge p\}$$

$$S^*_{np} = \{a : s^*_n(a) \ge p\}$$

$$S_{np} = \{a : s_n(a) \ge p\}$$

and

$$W_{np} = \{a : w_n(a) \ge p\}.$$

For $p = 0$ and n arbitrary we get the previous classes.

3. Main results

We begin by indicating a method for the determination of the above measures for a given sequence.

3.1. Lemma. *a) If the sequence a is represented by:*

$$a_i = \sum_{j=0}^{i} (i-j+1)b_j, \quad 0 \le i \le n \tag{1}$$

then:

$$k_n(a) = \inf\{b_i : 2 \le i \le n\};$$

b) If a is given by:

$$a_i = i\sum_{j=1}^{i} c_j + c_0, \quad 0 \le i \le n, \quad \sum_{j=1}^{i} c_j = 0,\ i < 1 \tag{2}$$

then:

$$s_n^*(a) = \inf\{c_i : 2 \le i \le n\};$$

c) If a is represented by:

$$a_i = \sum_{j=2}^{i} f_j + if_1 - (i-1)f_0, \quad 0 \le i \le n \tag{3}$$

then:

$$w_n(a) = \inf\{f_i/(i-1) : 2 \le i \le n\}.$$

Proof. From (1) we have: $C_i(a) = b_{i+2}$. From (2) also: $D_i(a) = c_{i+2}$, and from (3): $P_{i1}(a) = f_{i+1}$. All these give the desired conclusions.

3.2. Theorem. *For any sequence a we have:*

$$k_n(a) \le 2s_n^*(a) \le s_n(a) \le w_n(a). \tag{4}$$

Proof. Any sequence a may be represented by (1) and so, for $i \le n-2$:

$$D_i(a) = \frac{1}{(i+1)(i+2)}\sum_{j=2}^{i+2}(j-1)b_j \ge \frac{k_n(a)}{(i+1)(i+2)}\sum_{j=2}^{i+2}(j-1) = \frac{1}{2}k_n(a)$$

which gives the first part of (4). Similarly, using (2), we have, for $i+j \le n$:

$$P_{ij}(a) = i\sum_{l=i+1}^{i+j} c_l + j\sum_{l=j+1}^{j+i} c_l \ge 2ijs_n^*(a)$$

which gives the second part of (4). The last inequality from (4) is obviously.

3.3. Corollary. *For every natural n and every real p hold the following inclusions:*

$$K_{np} \subset S^*_{n,p/2} \subset S_{np} \subset W_{np}.$$

3.4. Remark. This result is more eloquent than a similar one from [4].

4. Weighted arithmetic means

In [6] are characterized the weight sequences $p = (p_i)_{i\geq 0}$ with the property that the sequence $A = (A_i)_{i\geq 0}$ given by:

$$A_i = (p_0 a_) + \cdots + p_i a_i)/(p_0 + \cdots + p_i) \tag{5}$$

is convex for any sequence $a = (a_i)_{i\geq 0}$. In [5], we have specified and extended this result. Now we want to generalize the result to the classes defined here, giving also a simpler proof.

4.1. Theorem. *If the sequence A given by (5) is in K_{n0}, S^*_{n0}, S_{n0} respectively W_{n0} for any sequence a from the same set, then there is an $u > 0$ such that:*

$$p_i = p_0 \binom{u+i-1}{i}, \quad 0 \leq i \leq n \tag{6}$$

where:

$$\binom{v}{0} = 1, \quad \binom{v}{i} = \frac{v(v-1)\dots(v-i+1)}{i!}, \quad i \geq 1.$$

Proof. For any $c \in \mathbb{R}$, the sequence a given by $a_i = ci$, $i \geq 0$ is in K_{n0}, S^*_{n0}, S_{n0} and W_{n0}. By the hypothesis and Corollary 3.3, the sequence

A^0 given by:

$$A_i^0 = c\left(\sum_{j=0}^{i} jp_j\right) / \left(\sum_{j=0}^{i} p_j\right)$$

is in W_{n0}. But c being of arbitrary sign, this implies that, for $i < n$:

$$\left(\sum_{j=1}^{i+1} jp_j\right) / \left(\sum_{j=0}^{i+1} p_j\right) - \left(\sum_{j=1}^{i} jp_j\right) / \left(\sum_{j=0}^{i} p_j\right) - \frac{p_1}{p_0 + p_1} = 0. \tag{7}$$

For $i = 1$ we get:

$$p_2 = p_1(p_0 + p_1)/2p_0.$$

Putting $p_1/p_0 = u$, we have (6) for $i \leq 2$. Supposing it valid for $i \leq m$ and using:

$$\sum_{j=0}^{i} \binom{v+j}{j} = \binom{v+i+1}{i}$$

the relation (7) becomes for $i = m$:

$$\left[p_0 u\binom{u+m}{m-1} + (m+1)p_{m+1}\right] / \left[p_0\binom{u+m}{m} + p_{m+1}\right] =$$

$$= u\binom{u+m}{m-1} / \binom{u+m}{m} + \frac{u}{u+1}$$

which gives p_{m+1} of the form (6).

4.2. Remark. Taking p_m of the form (6), A_i becomes:

$$A_i = A_i^u = \sum_{j=0}^{i} \binom{u+j-1}{j} a_j / \binom{u+i}{i}. \tag{8}$$

If we denote by $A^u = (A_i^u)_{i \geq 0}$, we can consider also the following measures (of u-mean):

$$k_n^u(a) = k_n(A^u),\ s_n^{*u}(a) = s_n^*(A^u),\ s_n^u(a) = s_n(A^u),\ w_n^u(a) = w_n(A^u).$$

4.3. Theorem. *For any sequence $a = (a_i)_{i\geq 0}$ and any $0 < v < u$, we have the following relations:*

$$k_n(a) \leq (1+2/u)k_n^u(a) \leq (1+2/v)k_n^v(a) \leq 2s_n^*(a) \tag{9}$$

$$s_n^*(a) \leq (1+2/u)s_n^{*u}(a) \leq (1+2/v)s_n^{*v}(a) \tag{10}$$

and

$$w_n(a) \leq (1+2/u)w_n^u(a) \leq (1+2/v)w_n^v(a). \tag{11}$$

Proof. i) Let a be given by (1) and A^u by:

$$A_i^u = \sum_{j=0}^{i}(i-j+1)d_j^u, \quad i \geq 0. \tag{12}$$

Then we have:

$$a_i = (1+i/u)A_i^u - (i/u)A_{i-1}^u = \sum_{j=0}^{i}[i(1+1/u)-j+1]d_j^u \tag{13}$$

and so:

$$C_i(a) = b_{i+2} = [1+(i+2)/u]d_{i+2}^u - (i/u)d_{i+1}^u, \quad i \geq 0 \tag{14}$$

which gives, by mathematical induction:

$$d_i^u = \frac{u}{u+i}b_i + u\sum_{j=2}^{i-1}\frac{(i-2)\dots(j-1)}{(u+i)\dots(u+j)}b_j.$$

Hence, for $i \leq n$:

$$d_i^u \geq \left[\frac{u}{u+i} + \frac{u(i-2)!}{(u+i)\dots(u+2)}\sum_{j=2}^{i-1}\binom{u+j-1}{j-2}\right]k_n(a) = \frac{u}{u+2}k_n(a)$$

and Lemma 3.1 gives the first inequality from (9).

ii) Taking (12) for u and v, (14) gives:

$$\left(1+\frac{i+2}{u}\right)d_{i+2}^{u}-\frac{i}{u}d_{i+1}^{u}=\left(1+\frac{i+2}{v}\right)d_{i+2}^{v}-\frac{i}{v}d_{i+1}^{v}$$

and by induction

$$d_{i+2}^{v}=\frac{v(u+i+2)}{u(v+i+2)}d_{i+2}^{u}+(u-v)\frac{v}{u}\sum_{j=2}^{i+1}\frac{i\ldots(j-1)}{(v+i+2)\ldots(v+j)}d_{j}^{u}.$$

So, for $i\le n-2$:

$$d_{i+2}^{v}\ge\frac{v}{u}\left[\frac{u+i+2}{v+i+2}+\frac{(u-v)i!}{(v+i+2)\ldots(v+2)}\sum_{j=2}^{i+1}\binom{v+j-1}{j-2}\right]k_{n}^{u}(a)$$

$$=\frac{v(u+2)}{u(v+2)}k_{n}^{u}(a)$$

and applying again Lemma 3.1 we have the second inequality from (9).

iii) Taking v instead u in (13), we have for $i\le n$:

$$D_i(a)=d_{i+2}^{v}/v+1/[(i+1)(i+2)]\sum_{j=2}^{i+2}(j-1)d_{j}^{v}\ge(1/v+1/2)k_{n}^{v}(a)$$

that is the third inequality from (9).

iv) If a is given by (2), then:

$$A_{i}^{u}=\frac{ui}{u+1}\sum_{j=1}^{i}c_j-u/\binom{u+i}{i}\sum_{j=2}^{i+1}\binom{u+j-1}{j-2}c_j+c_0$$

and thus:

$$D_i(A^u)=\frac{uc_{i+2}}{u+i+2}+\frac{u}{(u+2)\binom{u+i+2}{i}}\sum_{j=2}^{i+1}\binom{u+j-1}{j-2}c_j.$$

Hence, for $i \leq n$:

$$D_i(A^u) \geq \left[\frac{u}{u+i+2} + \frac{u}{(u+2)\binom{u+i+2}{i}} \sum_{j=2}^{i+1} \binom{u+j-1}{j-2} \right] s_n^*(a)$$

$$= \frac{u}{u+2} s_n^*(a)$$

which gives the first inequality from (10).

v) Let A^u be given by:

$$A_i^u = i \sum_{j=1}^{i} e_j^u + e_0^u \tag{15}$$

Then, as in (13):

$$a_i = i(1 + i/u)e_i^u + i(1 + 1/u) \sum_{j=1}^{i-1} e_j^u + e_0^u \tag{16}$$

and so:

$$C_i(a) = (i+2)\left(1 + \frac{i+2}{u}\right) e_{i+2}^u - i\left(1 + \frac{2i+3}{u}\right) e_{i+1}^u - \frac{i(i-1)}{u} e_i^u. \tag{17}$$

Taking (15), (16) and (17) for u and v, we get:

$$(i+2)\left(1 + \frac{i+2}{u}\right) e_{i+2}^u - i\left(1 + \frac{2i+3}{u}\right) e_{i+1}^u + \frac{i(i-1)}{u} e_i^n =$$

$$= (i+2)\left(1 + \frac{i+2}{v}\right) e_{i+2}^v - i\left(1 + \frac{2i+3}{v}\right) e_{i+1}^v + \frac{i(i-1)}{v} e_i^v$$

hence, by mathematical induction:

$$e_{i+2}^v = \frac{v(u+i+2)}{u(v+i+2)} e_{i+2}^u + (u-v)\frac{v}{u} \sum_{j=2}^{i+1} \frac{i \ldots (j-1)}{(v+i+2)\ldots(v+j)} e_j^u$$

which gives, as in ii), the second inequality from (10).

(vi) If a is given by (3) and A^u by:

$$A_i^u = \sum_{j=2}^{i} g_j^u + i g_1^u - (i-1) g_0^u \tag{18}$$

we have, as in (13):

$$a_i = (1+i/u)g_i^u + \sum_{j=2}^{i-1} g_j^u + i(1+1/u)g_1^u - [i(1+1/u)-1]g_0^u$$

thus:

$$a_{i+1} - a_i = (1+(i+1)/u)g_{i+1}^u - (i/u)g_i^u + (1+1/u)(g_1^u - g_0^u)$$

and taking into account (3):

$$f_{i+1} = (1+(i+1)/u)g_{i+1}^u - (i/u)g_i^u, \quad i \geq 2. \tag{19}$$

As $g_2^u = (u/(u+2))f_2$, we get for $i \leq n$:

$$g_i^u = \frac{u}{u+i} f_i + u \sum_{j=2}^{i-1} \frac{(i-1)\dots j}{(u+i)\dots(u+j)} f_j$$

$$\geq \left[\frac{u(i-1)}{u+i} + u \sum_{j=2}^{i-1} \frac{(i-1)\dots j}{(u+i)\dots(u+j)} (j-1) \right] w_n(a) = u w_n(a) \frac{i-1}{u+2}$$

which gives the first inequality from (11).

vii) Considering (18) for u and v, we have from (19):

$$\left(1 + \frac{i+1}{u}\right) g_{i+1}^u - \frac{i}{u} g_i^u = \left(1 + \frac{i+1}{v}\right) g_{i+1}^v - \frac{i}{v} g_i^v, \quad i \geq 2$$

and

$$g_2^v = \frac{v(u+2)}{u(v+2)} g_2^u.$$

So, for $i \leq n$:

$$g_i^v = \frac{v(u+i)}{u(v+i)} g_i^u + (u-v) \frac{v}{u} \sum_{j=2}^{i-1} \frac{(i-1)\dots j}{(v+i)\dots(v+j)} g_j^u$$

$$\geq \frac{v}{u}\left[\frac{u+i}{v+i}(i-1)+(u-v)\sum_{j=2}^{i-1}\frac{(i-1)\dots j}{(v+i)\dots(v+j)}(j-1)\right]w_n^u(a)$$
$$=(i-1)\frac{v(u+2)}{u(v+2)}w_n^u(a)$$

thus the second inequality from (11).

If we denote by $M^uK_{np}, M^uS^*_{np}, M^uS_{np}$ and M^uW_{np} the sets of the sequences a with the property that the sequence A^u, associated by (8), belongs to K_{np}, S^*_{np}, S_{np} respectively W_{np}, we have the following:

4.4. Corollary. *For any* $0 < v < u$, $n \in \mathbb{N}$ *and* $p \in \mathbb{R}$, *denoting by* $p * u = pu/(u+2)$, *hold the following inclusions:*

$$\begin{array}{ccccc} K_{np} \subset M^uK_{n,pu} \subset M^vK_{n,pv} \subset S^*_{n,p/2} & \subset & M^uS^*_{n,(p/2)*u} & \subset & M^vS^*_{n,(p/2)*v} \\ \cap & & \cap & & \cap \\ S_{np} & & M^uS_{n,p*u} & & M^vS_{n,p*v} \\ \cap & & \cap & & \cap \\ W_{np} & \subset & M^uW_{n,p*u} & \subset & M^vW_{n,p*v} \end{array}$$

4.5. Remark. Among these sets may exist also other inclusions. For example, in [4] it was proved that for $u = 1$ (which corresponds to the usual arithmetic mean in (8)) and $p = 0$:

$$K_{n0} \subset M^1K_{n0} \subset S^*_{n0} \subset S_{n0} \subset M^1S^*_{n0} \subset M^1S_{n0}.$$

Hence $S_{n0} \subset M^uS^*_{n0}$ for $u < 1$. In [5] it is proved that there is no relation between S_{n0} and $M^uS^*_{n0}$ for $u > 1$.

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ON THE REPRESENTATION OF CONVEX SEQUENCES

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Let $(a_m)_{n\geq 1}$ be a sequence of real numbers and the operator Δ^k defined as usual by:

$$\Delta^0 a_m = a_m, \quad \Delta^{k+1} a_m = \Delta^k a_{m+1} - \Delta^k a_m, \quad k \geq 0.$$

The sequence $(a_m)_{n\geq 1}$ is said to be convex of order n (or n-convex) if $\Delta^n a_m \geq 0$ for $m \geq 1$.

In [1] it was proved the following identity:

$$\sum_{i=1}^{m} p_i a_i = \sum_{k=1}^{n} \sum_{i=1}^{m} \binom{i-1}{k-1} p_i \Delta^{k-1} a_1$$

$$+ \sum_{k=n+1}^{m} \sum_{i=1}^{m} \binom{i-k+n-1}{n-1} p_i \Delta^n a_{k-m}.$$

For $p_1 = \cdots = p_{m-1} = 0$ and $p_m = 1$, we obtain the Taylor's formula for sequences:

$$a_m = \sum_{k=1}^{n} \binom{m-1}{k-1} \Delta^{k-1} a_1 \tag{1}$$

$$+\sum_{k=n+1}^{m}\binom{m+n-k-1}{n-1}\Delta^n a_{k-n},\ \text{for } m>n$$

and

$$a_m=\sum_{k=1}^{m}\binom{m-1}{k-1}\Delta^{k-1}a_1,\ \text{for } m\le n. \tag{2}$$

In [4] it is given the following definition: a sequence $(c_m)_{n\geq 1}$ is said to be n-positive if $c_m\geq 0$ for $m>n$. The formulas (1) and (2) suggest the following representation of n-convex sequences:

Theorem 1. *A sequence $(a_m)_{n\geq 1}$ is n-convex if and only if there is a n-positive sequence $(c_m)_{n\geq 1}$ such that:*

$$a_m=\begin{cases}\displaystyle\sum_{i=1}^{m}\binom{m-1}{i-1}c_i, & \textit{for}\quad m\le n\\[2ex] \displaystyle\sum_{i=1}^{n}\binom{m-1}{i-1}c_i+\sum_{i=n+1}^{m}\binom{m+n-i-1}{n-1}c_i, & \textit{for}\quad m>n.\end{cases} \tag{3}$$

Lemma 1. *If the sequence $(a_m)_{m\geq 1}$ is represented by (3), then for $k=1,\dots,n-1$:*

$$\Delta^k a_m=\begin{cases}\displaystyle\sum_{i=k+1}^{m+k}\binom{m-1}{i-k-1}c_i, & \textit{for}\quad m<n-k+2\\[2ex] \displaystyle\sum_{i=k+1}^{n}\binom{m-1}{i-k-1}c_i\\ \displaystyle\quad+\sum_{i=n+1}^{m+k}\binom{m+n-i-1}{n-k-1}c_i, & \textit{for}\quad m\geq n-k+2\end{cases} \tag{4}$$

and

$$\Delta^k a_m=c_{m+k},\ \textit{for } m\geq 1,\ k=n. \tag{5}$$

Proof. The relation (4) can be proved by induction. As:

$$\Delta^{n-1} a_m = \sum_{i=n}^{m+n-1} c_i$$

we have also (5).

From (1), (2) and Lemma 1 it results not only Theorem 1 but also the following generalization of it:

Theorem 2. *A sequence $(a_m)_{m\geq 1}$ is convex of order $p, p+1, \ldots, n$ if and only if it may be represented by (3) with a p-positive sequence $(c_m)_{m\geq 1}$.*

In [4] and [6] are given two other representation theorems but they cannot be generalized as was (3) in the theorem 2 (or it is more difficult to do it).

In [6] it is given the following definition: a sequence $(a_m)_{m\geq 1}$ is said to be starshaped of order n if $((a_{m+1} - a_1)/m)_{m\geq 1}$ is convex of order $n-1$. From Theorem 2 we obtain:

Theorem 3. *A sequence $(a_m)_{m\geq 1}$ is starshaped of orders $p, p+1, \ldots, n$ if and only if there is a p-positive sequence $(d_m)_{m\geq 1}$ such that:*

$$(6) \qquad a_m = \begin{cases} (m-1)\displaystyle\sum_{j=2}^{m} \binom{m-2}{j-2} d_j + d_1, & \textit{for} \quad m \leq n \\[2em] (m-1)\left[\displaystyle\sum_{j=2}^{n} \binom{m-2}{j-2} d_j \right. \\ \left. + \displaystyle\sum_{j=n+1}^{m} \binom{m+n-j-2}{n-2} d_j \right] + d_1, & \textit{for} \quad m > n. \end{cases}$$

If $(a_m)_{m\geq 1}$ is given by (3) then, for $N > m$:

$$\sum_{m=1}^{N} q_m a_m = \sum_{i=1}^{n} c_i \sum_{m=i}^{N} \binom{m-1}{i-1} q_m + \sum_{i=n+1}^{n} c_i \sum_{m=i}^{N} \binom{m+n-i-1}{n-1} q_m.$$

From Theorem 2, we obtain so a result from [3]:

Theorem 4. *The inequality:*

$$\sum_{m=1}^{N} q_m a_m \geq 0 \tag{7}$$

holds for every sequence $(a_m)_{m\geq 1}$ convex of orders $p, p+1, \dots, n$ if and only if $(q_m)_{m\geq 1}^{N}$ satisfies:

$$\sum_{m=i}^{N} \binom{m-1}{i-1} q_m = 0 \textit{ for } i = 1, \dots, p$$

$$\sum_{m=i}^{N} \binom{m-1}{i-1} q_m \geq 0 \textit{ for } i = p+1, \dots, n$$

$$\sum_{m=i}^{N} \binom{m+n-i-1}{n-1} q_m \geq 0, \textit{ for } i = n+1, \dots, N.$$

Similarly, from Theorem 3 we get a result from [2]:

Theorem 5. *The inequality (7) holds for every sequences $(a_m)_{m\geq 1}$ starshaped of orders $p, \dots, n$ if and only if $(q_m)_{m=1}^{N}$ satisfies:*

$$\sum_{m=1}^{N} q_m = 0, \quad \sum_{m=j}^{N} (m-1) \binom{m-2}{j-2} q_m = 0 \textit{ for } j = 2, \dots, p,$$

$$\sum_{m=j}^{N} (m-1) \binom{m-2}{j-2} q_m \geq 0 \textit{ for } j = p+1, \dots, n,$$

$$\sum_{m=j}^{N}(m-1)\binom{m+n-j-2}{n-2}q_m \geq 0 \textit{ for } j=n+1,\ldots,N.$$

Finally we remark that all these results may be generalized following the ideas from [5]. Also the operator Δ^k may be replaced by ∇^k.

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THE RESOLUTION OF SOME INEQUATIONS WITH FINITE DIFFERENCES

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Let us consider the linear equation with finite differences:

$$L_p(x_n) = \sum_{i=0}^{p} c_i \Delta^i x_n = \sum_{j=0}^{p} d_j x_{n+j} = 0, \quad n \geq 0, \tag{1}$$

where d_p and d_0 does not vanish. As one knows (see [1]), the resolution of this equation is related to the solutions of the algebraic equation:

$$L_p(t^n)/t^n = \sum_{i=0}^{p} d_i t^i = d_p \prod_{i=1}^{p} (t - t_i). \tag{2}$$

In what follows, we shall deal with the set of convex sequences in respect to the operator L_p, that is:

$$K_m(t_1, \ldots, t_p) = \{(x_n)_{n=0}^{m} : \ L_p(x_n) \geq 0, \ 0 \leq n \leq m - p\}$$

or:

$$K(t_1, \ldots, t_p) = \{(x_n)_{n \geq 0} : \ L_p(x_n) \geq 0, \ n \geq 0\}.$$

The case $t_1 = \cdots = t_p = 1$, corresponds to the usual convexity of order p (that is $L_p = \Delta^p$). In [8] we have proved that a sequence $x = (x_n)_{n \geq 0}$

is convex of order p if and only if it may be represented by:

$$x_n = \sum_{i=0}^{n} \binom{n+p-i-1}{p-1} y_i, \text{ with } y_i \geq 0 \text{ for } i \geq p.$$

Such representations were also given in [10], in the case $p = 2$, for any t_1 and t_2. We want to extend this result to the general case.

A leading part will be played by the sequence $(u_n)_{n\geq 0}$ defined by:

$$L_p(u_n) = 0, \ \forall\ n \geq 0; \ u_0 = \cdots = u_{p-2} = 0, \ u_{p-1} = 1/d_p.$$

For example, if $t_1 = \cdots = t_p$, then $u_n = (t_1^n/d_p)\binom{n}{p-1}$ and if $t_i \neq t_j$ for $i \neq j$, then

$$u_n = (1/d_p) \sum_{k=1}^{p} \left[t_k^n / \prod_{\substack{i=1 \\ i\neq k}}^{p} (t_k - t_i) \right].$$

Lemma 1. *If:*

$$x_n = \sum_{i=0}^{n} u_{n+p-i-1} y_i \tag{2}$$

then:

$$L_p(x_n) = y_{n+p}.$$

Proof. From (1) and (2) we have:

$$L_p(x_n) = \sum_{j=0}^{p} d_j \sum_{i=0}^{n+j} u_{n+j+p-i-1} y_i$$

$$= \sum_{i=0}^{n} \left(\sum_{j=0}^{p} d_j u_{n+j+p-i-1} \right) y_i + \sum_{i=n+1}^{n+p} \left(\sum_{j=n+1}^{p} d_j u_{n+j+p-i-1} \right) y_i$$

$$= \sum_{i=0}^{n} L_p(u_{n+p-i-1}) y_i - \sum_{k=1}^{p-1} \left(\sum_{j=0}^{k-1} d_j u_{p+j-k-1} \right) y_{n+k} + d_p u_{p-1} y_{n+p} = y_{n+p}.$$

Remark 1. As from (2) we obtain:

$$y_n = d_p \left[x_n - x_{n-1} - \sum_{i=0}^{n-1} (u_{n+p-i-1} - u_{n+p-i-2}) y_i \right]$$

it results the following:

Lemma 2. *Let $P \subset \mathbb{R}$. We have $L_p(x_n) \in P$ for every $n \geq 0$, if and only if $(x_n)_{n\geq 0}$ is represented by (2) with $y_i \in P$ for $i \geq p$.*

Lemma 3. *The sequence $(x_n)_{n\geq 0}$ verifies the equation:*

$$L_p(x_n) = z_n, \quad n \geq 0$$

if and only if it is represented by (2) with $y_i = z_{i-p}$ for $i \geq p$.

Theorem 1. *The sequence $(x_n)_{n\geq 0}^m$ belongs to $K_m(t_1, \ldots, t_p)$ if and only if it may be represented by (2) with $y_i \geq 0$ for $p \leq i \leq m - p$.*

Remark 2. Some other sequences can also be represented using (2). For example, in [9] we have given the following definition: the sequence $x = (x_n)_{n\geq 0}$ is starshaped of order p if $\Delta^{p-1}((x_{n+1} - x_0)/(n+1)) \geq 0$, for $n \geq 0$. So, the sequence x is starshaped of order p if and only if it may be represented by:

$$x_n = y_0 + n \sum_{k=1}^{n} \binom{n+p-k-2}{p-2} y_k, \text{ with } y_k \geq 0 \text{ for } k \geq p.$$

Remark 3. In what follows, we are interested in the determination of the dual cone of $K_m(t_1, \ldots, t_p)$, i.e.

$$K_m^*(t_1, \ldots, t_p) = \left\{ (a_n)_{n=0}^m : \sum_{n=0}^{m} a_n x_n \geq 0, \ \forall \ x \in K_m(t_1, \ldots, t_p) \right\}.$$

As it is stated even in [2], results of this nature were obtained for the first time by T. Popoviciu (see [7]).

Theorem 2. *The sequence* $(a_n)_{n=0}^m$ *belongs to* $K_m^*(t_1, \ldots, t_p)$ *if and only if it satisfies the relations:*

$$\sum_{n=k}^{m} a_n u_{n+p-k-1} = 0 \text{ for } k = 0, \ldots, p-1 \tag{3}$$

and

$$\sum_{n=k}^{m} a_n u_{n+p-k-1} \geq 0 \text{ for } k = p, \ldots, m. \tag{4}$$

Proof. From (2) we have:

$$\sum_{n=0}^{m} a_n x_n = \sum_{n=0}^{m} a_n \sum_{k=0}^{n} u_{n+p-k-1} y_k = \sum_{k=0}^{m} y_k \sum_{n=k}^{m} u_{n+p-k-1} a_n \geq 0. \tag{5}$$

As y_k is of arbitrary sign for $k = 0, \ldots, p-1$, but it is nonnegative for $k = p$, the relation (5) is equivalent with (3) and (4).

Remark 4. For $L_p = \Delta^p$ the result may be find in [2] (in the special case $p = 2$) and in [6] (in the general case). In [10] we have put the result in a more convenient form. We want to do the same thing for the general case. For this we need the operator:

$$L_p^*(x_n) = \sum_{j=0}^{p} d_{p-j} x_{n+j}.$$

Theorem 3. *The sequence* $(a_n)_{n=0}^m$ *belongs to* $K_m^*(t_1, \ldots, t_p)$ *if and only if it may be represented by:*

$$a_n = L_p^*(b_n), \text{ for } n = 0, \ldots, m \tag{6}$$

with

$$b_n \geq 0 \text{ for } p \leq n \leq m; \quad b_n = 0 \text{ for } n \leq p-1 \text{ or } n > m. \tag{7}$$

Proof. If we put:

$$\sum_{n=k}^{m} u_{n+p-k-1} a_n = b_k \tag{8}$$

from (3) and (4) we have (7). But (8) may be written as:

$$\begin{bmatrix} u_{p-1} & u_p & u_{p+1} & \dots & u_{p+m-1} \\ 0 & u_{p-1} & u_p & \dots & u_{p+m-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & u_{p-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \dots \\ a_m \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \dots \\ b_m \end{bmatrix}$$

which gives:

$$\begin{bmatrix} a_0 \\ a_1 \\ \dots \\ a_m \end{bmatrix} = \begin{bmatrix} d_p & d_{p-1} & d_{p-2} & \dots & 0 \\ 0 & d_p & d_{p-1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_p \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \dots \\ b_m \end{bmatrix}$$

that is (6).

Remark 5. If $L_p = \Delta^p$, then $L_p^* = \nabla^p = (-1)^p \Delta^p$, and we get the result from [10]. The transition from the conditions (3) and (4) to (6) and (7) remind the Minkowski-Farkas Lemma [11], but it does not represent a simple consequence of it, we needing the conditions $b_n = 0$ for $n = 0, \dots, p-1$.

Remark 6. Let the triangular matrix $Q = (q_{n,k})_{\substack{n=0,1,\dots \\ k=0,\dots,n}}$. It defines a transformation in the set of sequences: to any sequence $x = (x_n)_{n\geq 0}$ corresponds the sequence $X = Q(x) = (X_n)_{n\geq 0}$ given by:

$$X_n = \sum_{k=0}^{n} q_{n,k} x_k. \tag{9}$$

We have the following problem: what are the matrices Q with the property that $x \in K_m(t_1, \dots, t_p)$ implies $Q(x) \in K_m(t_1, \dots, t_p)$. For this we need:

$$L_p(X_n) = \sum_{i=0}^{p} d_i \sum_{k=0}^{n+i} q_{n+i,k} x_k$$

$$= \sum_{k=0}^{n} \left(\sum_{i=0}^{p} d_i q_{n+i,k} \right) x_k + \sum_{k=n+1}^{n+p} \left(\sum_{i=k-n}^{p} d_i q_{n_i,k} \right) x_k \geq 0$$

for any $0 \leq n \leq m - p$ if $x \in K_m(t_1, \ldots, t_p)$. This means that the sequences $a^n = (a_k^n)_{k=0}^{n+p}$ given by:

$$a_k^n = \sum_{i=j}^{p} d_i q_{n+i,k}, \text{ for } k = 0, \ldots, n+p, \; j = \max\{0, k-n\}$$

belong to $K_{n+p}^*(t_1, \ldots, t_p)$. From (3) and (4) we have the following:

Theorem 4. *The sequence X given by (9) is in $K_m(t_1, \ldots, t_p)$ for any $x \in K_m(t_1, \ldots, t_p)$ if and only if:*

$$\sum_{i=0}^{p} d_i \sum_{k=1}^{n+i} u_{k+p-l-1} q_{n+i,k} = 0, \quad l = 0, \ldots, p-1$$

$$\sum_{i=j}^{p} d_i \sum_{k=1}^{n+i} u_{k+p-l-1} q_{n+i,k} \geq 0, \quad l = p, \ldots, n+p, \; j = \max\{0, 1-n\}$$

for every $0 \leq n \leq m - p$.

Remark 7. For $L_p = \Delta^p$ such results may be found in [3] and [4] and for L_2 arbitrary in [5]. We want to put the result in another form, using the theorem 3.

Theorem 5. *The matrix Q has the property $Q(x) \in K_m(t_1, \ldots, t_p)$ for any $x \in K_m(t_1, \ldots, t_p)$ if and only if, for every $0 \leq n \leq m-p$, there is a nonnegative sequence $v^n = (v_k^n)_{k \geq 0}$ such that $v_k^n = 0$ for $k < p$ and for $k > n + p$, with the property that:*

$$\sum_{i=j}^{p} d_i q_{n+i,k} = L_p^*(v_k^n), \text{ for } k = 0, \ldots, n+p, \; j = \max\{0, k-n\}.$$

Remark 8. So $q_{i,j}$ may be chosen arbitrarily for $i = 0, \dots, p-1$ and $j = 0, \dots, i$ and then, taking v^n as it is requested by the theorem 5, we can build, step by step, $q_{n,k}$ for $n = p, p+1, \dots$ and $k = 0, \dots, n$.

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ON SOME PROPERTIES OF CONVEX SEQUENCES

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In [4] we have shown that the "hierarchy of convexity" established for the functions by A.M. Bruckner and E. Ostrow in [1], is also valid for sequences. Particularly we have proved that every convex sequence is mean-convex and every mean-convex sequence is starshaped. Starting from a property proved in [5], in this paper we shall extend this hierarchy, inserting between the set of convex sequences and that of starshaped sequences an infinity of sets of sequences.

Let us recall some definitions and some results.

Definition 1. A sequence $(a_n)_{n=0}^{\infty}$ is called:

a) convex, if:

$$\Delta^2 a_n = a_{n+2} - 2a_{n+1} + a_n \geq 0, \text{ for } n \geq 0; \tag{1}$$

b) mean-convex, if the sequence $(A_n)_{n=0}^{\infty}$ is convex, where:

$$A_n = \frac{a_0 + \cdots + a_n}{n+1}, \text{ for } n \geq 0; \tag{2}$$

c) starshaped, if it satisfies:

$$\frac{a_{n-1} - a_0}{n-1} \leq \frac{a_n - a_0}{n}, \text{ for } n \geq 2. \tag{3}$$

Remark 1. We adopted (3) instead of:

$$\frac{a_{n-1}}{n-1} \leq \frac{a_n}{n} \tag{3'}$$

to allow $a_0 \neq 0$.

We have proved in [4] representation formulas for these sequence classes. They are contained in the following:

Theorem 1. *A sequence $(a_n)_{n=0}^{\infty}$ is convex, mean-convex, respectively starshaped if and only if it may be represented by:*

$$a_n = \sum_{k=0}^{n} (n-k+1) b_k, \tag{4}$$

$$a_n = \sum_{k=0}^{n} (2n-k+1) c_k \tag{5}$$

respectively:

$$a_n = n \sum_{k=1}^{n} \frac{d_k}{k} - (n-1) d_0 \tag{6}$$

with $b_k \geq 0$, $c_k \geq 0$ and $d_k \geq 0$ for $k \geq 2$.

Remark 2. If the sequence $(a_n)_{n=0}^{\infty}$ is represented by (4), (5) or (6) then:

$$\Delta^2 a_n = b_{n+2}, \tag{7}$$

$$\Delta^2 a_n = (n+3) c_{n+2} - n c_{n+1} \tag{8}$$

respectively:

$$\Delta^2 a_n = d_{n+2} - \frac{n}{n+1} d_{n+1}. \tag{9}$$

Using these relations we have proved in [4] the validity of the strict inclusions:

$$S_1 \subset S_2 \subset S_3 \tag{10}$$

where S_1, S_2 and S_3 denote the sets of convex, mean-convex, respectively starshaped sequences. By other means N. Ozeki has proved (see [3]) that $S_1 \subset S_2$, and, if $a_0 = 0$, $S_1 \subset S_3$.

Remark 3. We want now to recall the result from [5] mentioned before, but it is more convenient for us to present it in the form given in [2] because in [5] the sequence is indexed starting from 1 not from 0 as we do. The result is given by the following:

Theorem A. *The sequence* $(\overline{A}_n)_{n=0}^{\infty}$ *given by*

$$\overline{A}_n = \frac{p_0 a_0 + \cdots + p_n a_n}{p_0 + \cdots + p_n}, \ \textit{for } n \geq 0 \tag{11}$$

is convex for any convex sequence $(a_n)_{n=0}^{\infty}$ *if and only if:*

$$p_n = \frac{\prod_{k=0}^{n-1}(kp_0 + p_1)}{n! p_0^{n-1}}, \ \textit{for } n \geq 2 \tag{12}$$

with $p_0 > 0$ *and* $p_1 > 0$ *arbitrary.*

We want to put (12) in a more natural form and also to simplify the proof of Theorem A given in [5]. We begin with:

Lemma 1. *In order that the sequence* $(A_n)_{n=0}^{\infty}$ *given by (11) be convex for any convex sequence* $(a_n)_{n=0}^{\infty}$ *it is necessary that:*

$$p_n = p_0 \binom{u+n-1}{n}, \ \textit{for } n \geq 0 \tag{13}$$

where $u > 0$ is arbitrary, and:

(14) $$\binom{v}{0} = 1, \quad \binom{v}{n} = \frac{1}{n!}\prod_{k=0}^{n-1}(v-k), \ \textit{for } n \geq 1, \ v \in \mathbb{R}.$$

In this case:

$$\overline{A}_n = A_n^u = \frac{\sum_{k=0}^{n}\binom{u+k-1}{k}a_k}{\binom{u+n}{n}}. \tag{15}$$

Proof. We proceed by induction. If we put $p_1/p_0 = u$, then $p_1 = p_0 u = p_0\binom{u}{1}$. Suppose (13) is valid for $n = 0, 1, \ldots, m, m+1$. Since the sequence $(a_n)_{n=0}^{\infty}$ given by $a_n = cn$ is convex for any real c, the attached sequence $(\overline{A}_n)_{n=0}^{\infty}$ must also be convex. We have:

$$\overline{A}_m = \frac{\sum_{k=0}^{m}\binom{u+k-1}{k}a_k}{\sum_{k=0}^{m}\binom{u+k-1}{k}} = \frac{\sum_{k=0}^{m}\binom{u+k-1}{k}ck}{\binom{u+m}{m}}$$

$$= \frac{cu\sum_{k=1}^{m}\binom{u+k-1}{k-1}}{\binom{u+m}{m}} = \frac{cu\binom{u+m}{m-1}}{\binom{u+m}{m}} = \frac{cum}{u+1}$$

because:

(16) $$\sum_{k=0}^{m}\binom{v+k}{k} = \binom{v+m+1}{m}.$$

By the same way:

$$\overline{A}_{m+1} = \frac{cu(m+1)}{u+1}$$

and

$$\overline{A}_{m+2} = \frac{cup_0\binom{u+m+1}{m} + c(m+2)p_{m+2}}{p_0\binom{u+m+1}{m+1} + p_{m+2}}$$

that is:

$$\Delta^2 A_m = \frac{c}{p_0\binom{u+m+1}{m+1} + p_{m+2}}\left[\frac{m+2}{u+1}p_{m+2} - \frac{p_0 u}{m+1}\binom{u+m+1}{m}\right].$$

Since c is arbitrary, we have $\Delta^2 A_m \geq 0$ if and only if the expression in brackets vanishes, that is:

$$p_{m+2} = p_0\frac{u(u+1)}{(m+1)(m+2)}\binom{u+m+1}{m} = p_0\binom{u+m+1}{m+2}.$$

We will prove the sufficiency of the condition (13) in a more general context (Theorem 3).

Definition 2. A sequence $(a_n)_{n=0}^{\infty}$ is called u-mean-convex if the sequence $(A_n^u)_{n=0}^{\infty}$ given by (15) is convex. The set of all u-mean-convex sequences is denoted by S_2^u.

Theorem 2. *A sequence $(a_n)_{n=0}^{\infty}$ is u-mean-convex if and only if it may be represented by:*

$$a_n = \sum_{k=0}^{n}\left[n\left(1+\frac{1}{u}\right) - k + 1\right]c_k \tag{17}$$

where $c_k \geq 0$ for $k \geq 2$.

Proof. From (15) we have:

$$\binom{u+n-1}{n}a_n = \binom{u+n}{n}A_n^u - \binom{u+n-1}{n-1}A_{n-1}^u$$

that is:

$$a_n = \left(1+\frac{n}{u}\right)A_n^u - \frac{n}{u}A_{n-1}^u. \tag{18}$$

If $(A_n^u)_{n=0}^{\infty} \in S_1$, by Theorem 1, it may be represented by:

$$A_n^u = \sum_{k=0}^{n} (n-k+1)c_k, \quad c_k \geq 0 \text{ for } n \geq 2.$$

Hence we obtain from (18);

$$a_n = \left(1+\frac{n}{u}\right)c_n + \sum_{k=0}^{n-1}\left[\left(1+\frac{n}{u}\right)(n-k+1) - \frac{n}{u}(n-k)\right]c_k$$

$$= \left(1+\frac{n}{u}\right)c_k + \sum_{k=0}^{n-1}\left(n-k+1+\frac{n}{u}\right)c_k$$

which is (17).

Remark 4. For $u = 1$, $S_2^u = S_2$, and (17) reduces to (5).

Lemma 2. *If the sequence* $(a_n)_{n=0}^{\infty}$ *is represented by (17), then:*

(19) $$\Delta^2 a_n = \left(1+\frac{n+2}{u}\right)c_{n+2} - \frac{n}{u}c_{n+1}, \text{ for } n \geq 0.$$

Proof. we have directly from (17):

$$a_{n+1} - a_n = \left(1+\frac{n+1}{u}\right)c_{n+1} + \sum_{k=0}^{n}\left(1+\frac{1}{u}\right)c_k$$

and then:

$$\Delta^2 a_n = (a_{n+2} - a_{n+1}) - (a_{n+1} - a_n)$$
$$= \left(1+\frac{n+2}{u}\right)c_{n+2} + \left(1+\frac{1}{u}\right)c_{n+1} - \left(1+\frac{n+1}{u}\right)c_{n+1}$$

that is (19).

Theorem 3. *If* $0 < v < u$, *then we have the strict inclusions:*

(20) $$S_1 \subset S_2^u \subset S_2^v \subset S_3.$$

Proof. (i) Let us suppose that the sequence $(a_n)_{n=0}^{\infty}$ is represented as in (4) and also as in (17). This may be done for every sequence. Then from (7) and (19) we deduce:

$$b_{n+2} = \left(1 + \frac{n+2}{u}\right) c_{n+2} - \frac{n}{u} c_{n+1}, \text{ for } n \geq 0 \tag{21}$$

that is:

$$\left(1 + \frac{2}{u}\right) c_2 = b_2 \text{ and } \left(1 + \frac{n+2}{u}\right) c_{n+2} = b_{n+2} + \frac{n}{u} c_{n+1}, \quad n \geq 1.$$

So, if $b_n \geq 0$ for $n \geq 2$, then $c_n \geq 0$ for $n \geq 2$. By the theorems 1 and 2, if the sequence $(a_n)_{n=0}^{\infty}$ is convex, it is u-mean-convex, i.e. $S_1 \subset S_2^u$ for any $u > 0$. The inclusion is proper because we have, for example:

$$b_3 = \left(1 + \frac{3}{u}\right) c_3 - \frac{1}{u} c_2$$

which with $c_2 = 1$ and $c_3 = 0$ yields $b_3 = -1/u < 0$.

(ii) Now suppose that the sequence $(a_n)_{n=0}^{\infty}$ is represented by (17) and by:

$$a_n = \sum_{k=0}^{n} \left[n \left(1 + \frac{1}{v}\right) - k + 1 \right] e_k. \tag{17'}$$

From (19) we have:

$$\left(1 + \frac{n+2}{u}\right) c_{n+2} - \frac{n}{u} c_{n+1} = \left(1 + \frac{n+2}{v}\right) e_{n+2} - \frac{n}{v} e_{n+1} \tag{22}$$

for $n \geq 0$, which gives successively:

$$e_2 = \frac{v(u+2)}{u(v+2)} c_2,$$

$$e_3 = \frac{v(u+3)}{u(v+3)} c_3 + \frac{v(u-v)}{u(v+2)(v+3)} c_2$$

and supposing:

$$e_{n+1} = \frac{v(u+n+1)}{u(v+n+1)} c_{n+1} + \sum_{k=2}^{n} x_k^{n+1} c_k \tag{23}$$

one obtains also:

$$e_{n+2} = \frac{v(u+n+2)}{u(v+n+2)} c_{n+2} + \frac{n}{v+n+2} e_{n+1} - \frac{vn}{u(v+n+2)} c_{n+1}$$

$$= \frac{v(u+n+2)}{u(v+n+2)} c_{n+2} + \frac{vn(u-v)}{u(v+n+2)(v+n+1)} c_{n+1} + \sum_{k=2}^{n} \frac{n}{v+n+2} x_k^{n+1} c_k.$$

That is, by induction, if $u > v$ all the coefficients x_k^{n+1} in (23) are positive and so if $c_n \geq 0$ for $n \geq 2$ then $e_n \geq 0$ for $n \geq 2$. By Theorem 2 this means $S_2^u \subset S_2^v$. The inclusion is proper because:

$$c_3 = \frac{u(v+3)}{v(u+3)} e_3 + \frac{u(v-u)}{v(u+2)(u+3)} e_2$$

and $e_3 = 0$, $e_2 > 0$, for example, give $c_3 < 0$.

(iii) If $(a_n)_{n=0}^{\infty}$ is represented by (17') and by (6), from (9) and (19) it results:

$$\left(1 + \frac{n+2}{v}\right) e_{n+2} - \frac{n}{v} e_{n+1} = d_{n+2} - \frac{n}{n+1} d_{n+1} \tag{24}$$

for $n \geq 0$, or:

$$d_2 = \left(1 + \frac{2}{v}\right) e_2,$$

$$d_3 = \left(1 + \frac{3}{v}\right) e_3 + \frac{1}{2} e_2$$

and generally:

$$d_n = \left(1 + \frac{n}{v}\right) e_n + \frac{1}{n-1} \sum_{k=1}^{n-1} (k-1) e_k,$$

which may be proved by induction. As above, $e_n \geq 0$ for $n \geq 2$ implies $d_n \geq 0$ for $n \geq 2$, that is, by the theorem 1 and 2, $S_2^v \subset S_3$. Since:

$$\left(1 + \frac{3}{v}\right) e_3 = d_3 - \frac{v}{2(v+2)} d_2,$$

$d_3 = 0$ and $d_2 > 0$ give $e_3 < 0$, that is the above inclusion is proper.

Remark 5. The first inclusion, $S_1 \subset S_2^u$, gives implicitely the sufficiency part of the theorem A.

We finish by giving another result which combines the convexity and the starshapedness.

Theorem 4. *If the sequence $(a_n)_{n=0}^{\infty}$ is convex, then for any integers $n \geq 2$ and $0 \leq q \leq n-2$ and any real $s \geq 0$ we have:*

$$\frac{a_n - a_{q+1} + s(a_{q+1} - a_q)}{n - q - 1 + s} \geq \frac{a_{n-1} - a_{q+1} + s(a_{q+1} - a_q)}{n - q - 2 + s}. \tag{25}$$

Proof. Since for any $k \geq 0$ we have: $a_{k+2} - 2a_{k+1} + a_k \geq 0$, then for any $p_k \geq 0$ and any $0 \leq q \leq n-2$:

$$\sum_{k=q}^{n-2} p_{k+2}(a_{k+2} - 2a_{k+1} + a_k) \geq 0 \tag{26}$$

or:

$$\sum_{k=q+2}^{n} p_k s_k - 2 \sum_{k=q+1}^{n-1} p_{k+1} a_k + \sum_{k=q}^{n-2} p_{k+2} a_k \geq 0$$

that is, if $q \leq n-4$:

$$p_n a_n + (p_{n-1} - 2p_n) a_{n-1} \tag{27}$$

$$+ \sum_{k=q+2}^{n-2} (p_k - 2p_{k+1} + p_{k+2}) a_k + (p_{q+3} - 2p_{q+2}) a_{q+1} + p_{q+2} a_q \geq 0.$$

Choose the sequence $(p_k)_{k=q+2}^{n}$ such that:

$$p_k - 2p_{k+1} + p_{k+2} = 0, \text{ for } k = q+2, \ldots, n-2$$

that is:

$$p_k = p_{q+2} + (k-q-2)(p_{q+3} - p_{q+2})$$

or denoting: $p_{q+2} = p \geq 0$, $p_{q+3} - p_{q+2} = r > 0$, we have from (27):

$$[p + (n-q-2)r]a_n - [p + (n-q-1)r]a_{n-1} + (r-p)a_{q+1} + pa_q \geq 0$$

or:

$$[p + (n-q-2)r]\left[a_n - a_{q+1} + \frac{p}{r}(a_{q+1} - a_q)\right]$$

$$\geq [p + (n-q-1)r]\left[a_{n-1} - a_{q+1} + \frac{p}{r}(a_{q+1} - a_q)\right]$$

and hence we have (25) for $s = p/r$. For $q = n-2$ and $q = n-3$, (25) may be put in the form:

$$a_n - 2a_{n-1} + a_{n-2} \geq 0$$

respectively:

$$s[(a_n - a_{n-1}) - (a_{n-2} - a_{n-3})] + (a_n - 2a_{n-1} + a_{n-2}) \geq 0$$

that is (25) is valid also for them.

Taking $s = 1$ we obtain the following:

Corollary. *If the sequence $(a_n)_{n=0}^{\infty}$ is convex and $0 \leq q \leq n-2$, then:*

$$\frac{a_n - a_q}{n - q} \geq \frac{a_{n-1} - a_q}{n - 1 - q}. \tag{25'}$$

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ON AN INEQUALITY OF NANSON

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In this paper we give a generalization of the following inequality of Nanson [1] (see also [2]) for the case of p, q-convex sequences:

Theorem 1. *If the real sequence $a = (a_1, \dots, a_{2n+1})$ is convex, then*

$$\frac{a_1 + a_3 + \cdots + a_{2n+1}}{n+1} \geq \frac{a_2 + a_4 + \cdots + a_{2n}}{n} \tag{1}$$

with equality if and only if a represents an arithmetic progression.

Definition. A real sequence $a = (a_1, a_2, \dots)$ is p, q-convex $(p, q > 0)$ if $L_{pq}(a_n) \geq 0$ for $n \geq 1$, where

$$L_{pq}(a_n) = a_{n+2} - (p+q)a_{n+1} + pqa_n. \tag{2}$$

Remark 1. For $p = q = 1$ we get the usual notion of convexity. In [4] we have shown that a remarkable place in the theory of p, q-convex sequences is played by the sequence $w = (w_1, w_2, \dots)$ given by

$$w_n = \begin{cases} \dfrac{p^n - q^n}{p - q} & \text{if} \quad p \neq q \\ np^{n-1} & \text{if} \quad p = q. \end{cases} \tag{3}$$

For example, we have proved:

Lemma 1. *The sequence a satisfies the relation*

$$L_{pq}(a_n) = 0, \quad n = 1, \dots, n \tag{4}$$

if and only if

$$a_n = uw_n + vw_{n+1} \tag{5}$$

where u and v are arbitrary real numbers.

Theorem 2. *If the real sequence $a = (a_1, a_2, \dots, a_{2n+1})$ is p, q-convex, then:*

$$\begin{aligned} &\frac{(pq)^n a_1 + (pq)^{n-1} a_3 + \cdots + a_{2n+1}}{w_{n+1}} \\ &\geq \frac{(pq)^{n-1} a_2 + (pq)^{n-2} a_4 + \cdots + a_{2n}}{w_n} \end{aligned} \tag{6}$$

with equality if and only if a satisfies (4).

Proof. Since

$$a_{2k+1} - (p+q)a_{2k} + pqa_{2k-1} \geq 0, \quad k = 1, \dots, n$$

and

$$a_{2k+2} - (p+q)a_{2k+1} + pqa_{2k} \geq 0, \quad k = 1, \dots, n-1,$$

we have the inequalities:

$$\sum_{k=1}^{n} \frac{w_k w_{n-k+1}}{(pq)^{k-1}} (a_{2k+1} - (p+q)a_{2k} + pqa_{2k-1}) \geq 0 \tag{7}$$

and

$$\sum_{k=1}^{n-1} \frac{w_k w_{n-k}}{(pq)^{k-1}}(a_{2k+2} - (p+q)a_{2k+1} + pqa_{2k}) \geq 0. \tag{8}$$

Because:

$$w_{k+1}w_{n-k} - pqw_k w_{n-k-1} = w_n \tag{9}$$

adding (7) and (8) we obtain:

$$pqw_n \sum_{k=1}^{n+1} \frac{a_{2k-1}}{(pq)^{k-1}} - w_{n+1} \sum_{k=1}^{n} \frac{a_{2k}}{(pq)^{k-1}} \geq 0$$

which is (6).

Remark 2. For

$$s_n = w_1 + \cdots + w_n \tag{10}$$

we have

$$L_{pq}(s_n) = 1. \tag{11}$$

So we obtain the following:

Lemma 2. *If the real sequence $a = (a_1, a_2, \dots)$ satisfies:*

$$m \leq L_{pq}(a_n) \leq M, \quad n = 1, 2, \dots \tag{12}$$

then the sequences (b_n) and (c_n), $n = 1, 2, \dots,$ given by

$$b_n = Ms_n - a_n, \quad c_n = a_n - ms_n, \tag{13}$$

are p, q-convex.

Theorem 3. *If the sequence a satisfies (12), then*

$$mz_n \leq \frac{(pq)^n a_1 + \cdots + a_{2n+1}}{w_{n+1}} - \frac{(pq)^{n-1}a_2 + \cdots + a_{2n}}{w_n} \leq Mz_n \tag{14}$$

where

$$(15) \qquad z_n = \frac{w_n + \sum_{k=1}^{n-1} (pq)^{n-k} w_k (w_{n-k+1} + w_{n-k})}{w_n w_{n+k}}$$

Proof. Because (b_n) and (c_n), $n = 1, 2, \dots$, given by (13), are p, q-convex, we may apply (6). Taking into account (11), we can make with (s_k), $k = 1, 2, \dots$, the same operations as in the proof of Theorem 2 and we get (14) and (15).

Corollary. *Since for* $p = q = 1$, $w_n = n$, *(6) becomes (1) and (14) an inequality proved in [3], where* $z_n = \dfrac{2n+1}{6}$.

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CONVEX SEQUENCES AND FOURIER SERIES

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1. Introduction

In their book [11], A.W. Roberts and D.E. Varberg have proposed, for an independent study project, the convex sequences and have given as basic references two books on trigonometric series: [1] of N.K. Bary and [19] of A. Zygmund. In what follows, we use the results given in these books and also in that of R. Edwards [4], regarding the properties of convex sequences useful in Fourier series. We remind also some recent papers which define other classes of sequences. For the first two of these classes we give representation theorems and study some of their properties.

2. Notations and definitions

Given the sequence $(a_n)_{n\geq 0}$ we consider the first and the second finite differences:

$$\Delta a_n = a_{n+1} - a_n$$

and

$$\Delta^2 a_n = a_{n+2} - 2a_{n+1} + a_n.$$

In what follows we shall refer at a lot of sequence classes. We give now some of them, which are well known: the set of all real sequences, that of bounded sequences, of convergent sequences, of null-sequences, of decreasing sequences, of sequences with bounded variation, of convex sequences and of quasiconvex sequences, respectively:

$$S = \{a : \ a = (a_n)_{n\geq 0}, \ a_n \in \mathbb{R}\}$$

$$B = \{a \in S : \ \exists \ m, \ |a_n| \leq m, \forall \ n \geq 0\}$$

$$C = \{a \in B : \ a_n \to l, \text{ for } n \to \infty\}$$

$$C_0 = \{a \in B : \ a_n \to 0, \text{ for } n \to \infty\}$$

$$D = \{a \in S : \ a_{n+1} \leq a_n, \text{ for } n \geq 0\}$$

$$BV = \left\{a \in S : \ \sum_{n\geq 0} |\Delta a_n| < \infty\right\}$$

$$K = \{a \in S : \ \Delta^2 a_n \geq 0, \text{ for } n \geq 0\}$$

$$Q = \left\{a \in S : \ \sum_{n\geq 0} (n+1)|\Delta^2 a_n| < \infty\right\}.$$

Let us also denote by $A_0 = A \cap C_0$, where A is any class of sequences. Before ending, we remark that quasiconvexity is also used in another sense, analogues with that given for functions (see [16]).

3. Trigonometric series

For a given sequence $a \in S$, we consider the trigonometric series:

$$a_0/2 + \sum_{n\geq 1} a_n \cos nx, \quad \sum_{n\geq 1} a_n \sin nx$$

which are usually called cosine respectively sine series.

For their convergence it is necessary that the sequence $a \in C_0$. If $\sum_{n\geq 0} |a_n| < \infty$ the series converge absolutely, but this condition is too strongly. Using Abel's lemma, in [1] it is proved that if $a \in BV_0$ the series converge at least for $x \neq 2k\pi$, $k \in \mathbb{Z}$. This happens for example for sequences from D_0.

But, even if the series are convergent, they are not necessarily Fourier series. For the sine series in [1] it is proved that it is a Fourier series if and only if: $\sum_{n\geq 1} a_n/n < \infty$. If we pass to the cosine series, this condition is only sufficient (to be a Fourier series) and other sufficient conditions (other classes of sequences) may be also find. The first one was K_0 given by W.H. Young in [18]. The second was Q_0 finded by A.N. Kolmogorov in [10]. The third example is the class T_0 of S.A. Telyakovskii [14]: the sequence $a = (a_n)_{n\geq 0}$ belongs to T if there is a sequence $b = (b_n)_{n\geq 0} \in D_0$ such that $|\Delta a_n| \leq b_n$ and $\sum_{n\geq 0} b_n < \infty$. Other classes were given in [3], [5], [6], [7], [8], [9], [13] and their relationships are analysed in [17].

For some of these classes it is also proved that the convergence of the Fourier series to the limit function holds even in the L^1 norm if and only if: $\lim_{n\to\infty} a_n \ln n = 0$ (see [1] and []2]).

4. Bounded convex sequences

In [4] it is proved that if the sequence a is convex and bounded, then it is decreasing, $\lim\limits_{n\to\infty} n\Delta_n = 0$ and $\sum\limits_{n\geq 0}(n+1)\Delta^2 a_n = a_0 - \lim\limits_{n\to\infty} a_n$.

Taking into account the result from [15] we obtain the following representation theorem for such sequences.

4.1. Theorem. *The sequence $a = (a_n)_{n\geq 0}$ is in $K \cap B$ if and only if there is a sequence $b = (b_n)_{n\geq 0}$ such that:*

$$a_n = \sum_{k=0}^{n}(n-k+1)b_k, \quad n \geq 0 \tag{1}$$

$$b_k \geq 0 \textit{ for } k \geq 2 \tag{2}$$

$$n\sum_{k=0}^{n} b_k \to 0, \quad n \to \infty \tag{3}$$

and

$$\sum_{k\geq 0}(k+1)b_{k+2} < \infty. \tag{4}$$

Proof. Any sequence a may be represented by (1). As this gives:

$$\Delta a_n = \sum_{k=0}^{n+1} b_k \text{ and } \Delta^2 a_n = b_{n+2}$$

the sequence is convex if and only if holds (2). From $\Delta^2 a_n \geq 0$ for $n \geq 0$ we get:

$$\Delta a_0 \leq \Delta a_1 \leq \cdots \leq \Delta a_n \leq \dots$$

If $\Delta a_p = c > 0$ for some $p \geq 0$, it results that $\Delta a_n \geq c$ for $n \geq p$, thus:

$$a_n = a_0 + \sum_{k=0}^{n-1} \Delta a_k \geq a_0 + \sum_{k=0}^{p-1} \Delta a_k + (n-p)c, \quad n > p \tag{5}$$

hence $a_n \to \infty$ for $n \to \infty$.

So $\Delta a_n \leq 0$ for $n \geq 0$ and hence $\Delta a_n \to l \leq 0$ $(n \to \infty)$. If $l < 0$, we have $\Delta a_n \leq l < 0$, for every $n \geq 0$, thus $a_n \leq a_0 + nl$ and so $a_n \to -\infty$ $(n \to \infty)$. Hence if $a \in B \cap K$, the sequence $(\Delta a_n)_{n \geq 0}$ increases at zero. Thus $a \in D \cap B \subset C$. From (5) we have:

$$\sum_{k=0} (-\Delta a_k) < \infty, \quad (-\Delta a_k)_{k \geq 0} \in D_0$$

and by the theorem of Olivier (see [12]), we have $n\Delta a_n \to 0$, that is (3). Applying to (5) Abel's summation formula, we get:

$$a_n - a_0 = \sum_{k=0}^{n-1} \Delta a_k = n\Delta a_{n-1} - \sum_{k=0}^{n-2} (k+1)\Delta^2 a_k \tag{6}$$

and from (3) we have:

$$\sum_{k=0} (k+1)\Delta^2 a_k = a_0 - \lim_{n \to \infty} a_n$$

which gives (4).

Conversely, if $(a_n)_{n \geq 0}$ is given by (1), then (2) guarantees that it is from K, while (3), (4) and (6) give:

$$\lim_{n \to \infty} a_n = a_0 - \sum_{k \geq 0} (k+1) b_{k+2}$$

that is $a \in C \subset B$.

4.2. Corollary. *The following inclusion holds:*

$$K \cap B \subset C \cap D \cap BV \cap Q.$$

4.3. Corollary. *The sequence a belongs to K_0 if and only if there is a sequence b such that (1), (2), (3) hold and*

$$(4') \qquad \sum_{k\geq 0}(k+1)b_{k+2} = a_0.$$

4.4. Theorem. *The sequence a is quasiconvex if and only if there is a sequence b such that hold: (1) and*

$$(7) \qquad \sum_{n\geq 0}(n+1)|b_{n+2}| < \infty.$$

Proof. Any sequence may be represented by (1) and it implies $\Delta^2 a_n = b_{n+2}$, which gives the equivalence.

4.5. Theorem. *If the sequence a is quasiconvex, then the following properties are equivalent:*

(i) $a \in BV$

(ii) $a \in B$

(iii) $(n\Delta a_n)_{n\geq 0} \in B$.

Proof. The implication (i) $\Rightarrow$ (ii) holds for any sequence a from S as:

$$|a_n| = \left| a_0 + \sum_{k=0}^{n-1} \Delta a_k \right| \leq |a_0| + \sum_{k=0}^{n-1} |\Delta a_k|, \quad n \geq 0.$$

(ii) $\Rightarrow$ (iii) Suppose a given by (1) and:

$$|a_n| \leq M, \forall\ n \geq 0; \quad \sum_{n\geq 0}(n+1)|b_{n+2}| \leq L.$$

Then:

$$|n\Delta a_n| = \left| n \sum_{k=0}^{n+1} b_k \right|$$

$$= \left| \sum_{k=0}^{n+1} (n-k+2) b_k - (2b_0 + b_1) + \sum_{k=2}^{n+1} (k-1) b_k - \sum_{k=2}^{n+1} b_k \right|$$

$$\le |a_{n+1}| + |a_1| + \sum_{k=2}^{n+1} (k-1)|b_k| + \sum_{k=2}^{n+1} |b_k| \le 2(M+L).$$

(iii) $\Rightarrow$ (i) Suppose:

$$\sum_{n\ge 0} (n+1)|\Delta^2 a_n| \le L; \quad |n\Delta a_n| \le M, \ \forall\ n \ge 0.$$

From Abel's summation formula, in [4] it is proved that for any $a \in S$:

$$\sum_{k=0}^{n} |\Delta a_k| \le (n+1)|\Delta a_n| + \sum_{k=0}^{n} k|\Delta^2 a_k|$$

and so:

$$\sum_{k\ge 0} |\Delta a_k| \le 2M + L.$$

In [4] it is proved the "only if" part of the following:

4.6. Theorem. *The sequence $a \in Q$ if convergent if and only if:*

$$(n\Delta a_n)_{n\ge 0} \in C_0.$$

Proof. As $a \in Q$, the series: $\sum_{n\ge 0} (n+1)\Delta^2 a_n$ is absolutely convergent and so the affirmation follows from (6). Moreover, we have:

$$\lim_{n\to\infty} a_n = a_0 - \sum_{n\ge 0} (n+1)\Delta^2 a_n.$$

4.7. Corollary. *The sequence a belongs to Q_0 if and only if there is a sequence b such that hold (1), (7), (3) and (4').*

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JESSEN'S INEQUALITY FOR SEQUENCES

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ABSTRACT. În lucrare se demonstrează că o funcţională liniară şi continuă A este pozitivă pentru orice şir convex de ordin n dacă şi numai dacă ea verifică relaţiile (5) şi (6) pentru şirurile "test" ce au componentele date de (1).

A very large extension of the well known inequality of Jensen for convex function was given by B. Jessen in [2]. Some generalizations of Jessen's inequality are analysed in [1]. In [8] we have succeeded in transposing the result to convexity of higher order. We have proved that a function f is convex of order n if and only if:

$$f(A(c)) \leq A(f) \quad \text{(where } e(x) = x\text{)}$$

for every continuous linear functional A with the properties:

$$p(A(e)) = A(p), \ \forall \ p \text{ polynomial of degree } n-1$$

and

$$w_c^n(A(e)) \leq A(w_c^n), \ \forall \ c \in (a,b)$$

where

$$w_c^n(x) = \begin{cases} 0 & \text{for} \quad x < c \\ (x-c)^{n-1} & \text{for} \quad x \geq c. \end{cases}$$

To pass to sequences, we must note that the inequality of Jessen has generally no meaning in this case. But, as we have remarked in [8], there is a bijection between the functionals which satisfy Jessen's inequality and the positive functionals on the set of convex functions. For sequences we follow this second way, which was initiated by T. Popoviciu (see [6]). For "finite" sequences some results are given in [9]. A possibility for passing to infinite sequences was remarked in [3] and used in [4] and [5].

Let K_n be the space of all n-convex sequences, that is of sequences $(x_m)_{m\geq 0}$ with the property that $\Delta^n x_m \geq 0$ for every m, where:

$$\Delta^0 x_m = x_m, \quad \Delta^n x_m = \Delta^{n-1} x_{m+1} - \Delta^{n-1} x_m, \quad n \geq 1.$$

In [7] we have given the following result:

Theorem 1. *A sequence $(x_m)_{m\geq 0}$ is in K_n if and only if it may be represented by:*

$$x_m = \sum_{k=0}^{m} \binom{m+n-k-1}{n-1} y_k$$

where $y_k \geq 0$ for $k \geq n$.

This result may be put in another form. In the vector space S of all sequences, it is considered the metric d defined by:

$$d(x,y) = \sum_{k=0}^{\infty} 2^{-k} \frac{|x_k - y_k|}{1 + |x_k - y_k|}$$

for $x = (x_k)_{k\geq 0}$ and $y = (y_k)_{k\geq 0}$.

If we consider the sequences $e_m^0 = (e_{m,k}^0)_{k\geq 0}$ given by:

$$e_{m,m}^0 = 1, \quad e_{m,k}^0 = 0 \text{ for } k \neq m$$

we have, for any sequence $x = (x_k)_{k \geq 0}$:

$$x = \lim_{p \to \infty} \sum_{k=0}^{p} x_k e_k^0 = \sum_{k=0}^{\infty} x_k e_k^0$$

where the limit is taken in the metric d. For n-convex sequences we look for other "basic" sequences.

For an $n > 0$, let us consider the sequences $e_m^n = (e_{m,k}^n)_{k \geq 0}$ with the components:

$$e_{m,k}^n = \binom{n-m+k-1}{n-1} \tag{1}$$

where $\binom{m}{k} = 0$ if $m < k$. It can be proved that, for every fixed n, the sequences $e_0^n, e_1^n \dots$ give a base for S (as it was for $n = 0$).

We may use Δ as an operator, $\Delta : S \to S$, given by:

$$\Delta x = (\Delta x_k)_{k \geq 0}, \text{ for } x = (x_k)_{k \geq 0}.$$

We have: $\Delta e_m^n = e_{m-1}^{n-1}$ and so:

$$\Delta^p e_m^n = e_{m-p}^{n-p}, \quad 1 \leq p \leq n.$$

If we denote by:

$$S_+ = \{x = (x_k)_{k \geq 0} : \; x_k \geq 0, \; \forall \; k \geq 0\}$$

the set of n-convex sequences is given by:

$$K_n = \{x \in S : \; \Delta^n x \in S_+\}.$$

For the sequence:

$$x_m = y_0 e_0^n + \cdots + y_m e_m^n, \quad m \geq n$$

we have:

$$\Delta^n x_m = y_n e_0^0 + \cdots + y_m e_{m-n}^0 = (y_n, \dots, y_m, 0, \dots)$$

and so we get the following:

Theorem 2. *The sequence x is in K_n if and only if:*

$$x = \lim_{m\to\infty} (y_0 e_0^n + \cdots + y_m e_m^n), \quad y_m \geq 0 \text{ for } m \geq n. \tag{2}$$

Using the representation theorem, we can prove the following result:

Theorem 3. *Let the functional $A : S \to \mathbb{R}$ be superadditive, positively homogeneous and upper semicontinuous. In order that $A(x) \geq 0$ for every $x \in K_n$ it is necessary and sufficient that:*

$$A(e_m^n) \geq 0 \text{ for } m \geq 0 \tag{3}$$

and

$$A(-e_m^n) \geq 0 \text{ for } 0 \leq m < n. \tag{4}$$

Proof. From the theorem 2 we have that $e_m^n \in K_n$, $\forall\ m \geq 0$ and $-e_m^n \in K_n$ for $0 \leq m < n$, so that the conditions (3) and (4) are necessary. They are also sufficient. Indeed, let us take an $x \in K_n$. By the theorem 2 we have (2), which gives:

$$A(y_0 e_0^n + \cdots + y_m e_m^n) \geq A(y_0 e_0^n) + \cdots + A(y_m e_m^n)$$

$$= |y_0| A((\mathrm{sgn} y_0) e_0^n) + \cdots + |y_{n-1}| A((\mathrm{sgn} y_{n-1}) e_{n-1}^n)$$

$$+ y_n A(e_n^n) + \cdots + y_m A(e_m^n) \geq 0.$$

As A is upper semicontinuous, it follows that $A(x) \geq 0$.

Consequence. Let $A : S \to \mathbb{R}$ be a continuous linear functional. In order that $A(x) \geq 0$ for every $x \in K_n$ it is necessary and sufficient that:

$$A(e_m^n) \geq 0 \text{ for } m \geq n \tag{5}$$

and

$$A(e^n_m) = 0 \text{ for } 0 \leq m < n. \tag{6}$$

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DISCRETE CONVEXITY CONES

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1. Let us consider the linear recurrence of order p:

(1) $$L_p(x_n) = \sum_{j=0}^{p} d_j x_{n+j} = 0, \quad n \geq 0$$

where $d_p = 1$ and $d_0 \neq 0$. As it is known (see [2]), the representation of the sequences which satisfy this relation is related to the solutions of the algebraic equation:

(2) $$L_p(t^n)/t^n = \sum_{j=0}^{p} d_j t^j = \prod_{j=1}^{p} (t - t_j).$$

For example, we shall use the sequence $(u_n)_{n\geq 0}$ defined by:

(3) $$L_p(u_n) = 0,\ \forall\ n \geq 0,\ u_0 = \cdots = u_{p-2} = 0,\ u_{p-1} = 1.$$

If the roots of (2) are s_i multiple of order q_i, for $i = 1, \dots, r$ (with $q_1 + \dots + q_r = p$), then:

$$u_n = \sum_{i=1}^{r} P_i(n) s_i^n$$

where P_i is a polynomial of degree q_i and

$$\sum_{i=1}^{r} P_i(j) s_i^j = u_j \text{ for } j = 0, \dots, p-1.$$

So, if $r = 1$, that is $t_1 = \dots = t_p = s$, then:

$$u_n = s^n \binom{n}{p-1}$$

and if $r = p$, that is $t_i \neq t_j$ for $i \neq j$, then:

$$u_n = \sum_{j=1}^{p} \left[t_j^n / \prod_{\substack{i=1 \\ i \neq j}}^{p} (t_j - t_i) \right].$$

References to other methods of representation of recurrent sequences may be found in [1].

Our basic method of study is furnished by the following result which may be proved by simple computation (see [16]):

Lemma 1. *If the sequence* $(x_n)_{n \geq 0}$ *is represented by:*

$$x_n = \sum_{i=0}^{n} u_{n+p-i-1} y_i \tag{4}$$

where $(u_n)_{n \geq 0}$ *is given by (3), then:*

$$L_p(x_n) = y_{n+p}.$$

If $(x_n)_{n \geq 0}$ is given, then $(y_n)_{n \geq 0}$ may be found, step by step, from (4), so that we get:

Lemma 2. *Let $P \subset \mathbb{R}$. In order that $L_p(x_n) \in P$ for every $n \geq 0$ it is necessary and sufficient that $(x_n)_{n\geq 0}$ be represented by (4) with $y_i \in P$ for $i \geq p$.*

Corollary 1. *The sequence $(x_n)_{n\geq 0}$ verifies the relations:*

$$L_p(x_n) = z_n, \quad n \geq 0$$

if and only if it is represented by (4) with $y_i = z_{i-p}$ for $i \geq p$.

Corollary 2. *The sequence $(x_n)_{n\geq 0}$ verifies the relation (1) if and only if it is represented by:*

$$x_n = \sum_{i=0}^{p} u_{n+p-i-1} y_i. \tag{5}$$

On the vector space S of all sequences, let us consider the shift operator E defined for any $x = (x_n)_{n\geq 0}$ by:

$$Ex = x' = (x'_n)_{n\geq 0}, \quad x'_0 = 0, \quad x'_n = x_{n-1}, \quad n \geq 1.$$

If we define the sequence:

$$u^p = (u_{p-1+n})_{n\geq 0} \tag{6}$$

the relation (5) may be written as:

$$x = \sum_{i=0}^{p-1} y_i E^i u^p$$

where $E^0 x = x$ and E^i is obtained by the composition of i exemplars of E. Thus we have:

Corollary 3. *The sequences:*

$$u^p, Eu^p, \ldots, E^{p-1}u^p$$

form a basis for the subspace of sequences which verify (1).

2. In what follows, we shall deal with the cone of convex sequences in respect to the operator L_p, that is:

$$K_m(L_p) = \{(x_n)_{n=0}^m : \ L_p(x_n) \geq 0, \ 0 \leq n \leq m-p\}$$

or

$$K(L_p) = \{(x_n)_{n\geq 0} : \ L_p(x_n) \geq 0, \ n \geq 0\}.$$

The case $t_1 = \cdots = t_p = 1$ corresponds to the usual convexity of order p as $L_p = \Delta^p$ (see [12]). We have given the representation of these (ordinary) convex sequences in [15], for the case $p = 2$ (and L_2 arbitrary) in [9] and for the general case in [16]. This follows from Lemma 1.

Theorem 1. *a) The sequence $(x_n)_{n=0}^m$ belongs to $K_m(L_p)$ if and only if it may be represented by (4), with $y_i \geq 0$ for $p \leq i \leq m-p$.*

b) The sequence $(x_n)_{n\geq 0}$ belongs to $K(L_p)$ if and only if it may be represented by (4) with $y_i \geq 0$ for $i \geq p$.

The result from part b) may be reformulated if we consider (as it was done in [5] and then in [10], [11] and [17]) the metric d on S, defined by:

$$d(x, y) = \sum_{n=0}^{\infty} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

for $x = (x_n)_{n\geq 0}$ and $y = (y_n)_{n\geq 0}$. Let us also put:

$$L_p(x) = (L_p(x_n))_{n\geq 0}.$$

We have at once:

Lemma 3. *If u^p is given by (6) then:*

$$L_p(E^k u^p) = 0 \text{ for } 0 \leq k \leq p-1$$

and

$$L_p(E^k u^p) = (\delta_{n,k-p})_{n\geq 0} \text{ for } k \geq p$$

where $\delta_{n,k}$ is Kronecker's symbol.

Theorem 2. *The sequence x belongs to $K(L_p)$ if and only if:*

$$x = \lim_{n\to\infty} x^n = \sum_{n=0}^{\infty} x^n \tag{7}$$

where

$$x^n = \sum_{k=0}^{n} y_k E^k u^p, \text{ with } y_k \geq 0 \text{ for } k \geq p$$

and the limit is taken in respect to the metric d.

Proof. As $E^n u^p$ has the first n components zero, any sequence x is the limit of such a linear combination (in fact, x and x^n have the same first $n+1$ components). But

$$L_p(x^n) = (y_p, \dots, y_n, 0, 0, \dots) \to L_p(x)$$

so that x is in $K(L_p)$ if and only if $y_n \geq 0$ for $n \geq p$.

3. In [16] we have also characterized the elements of the dual cone of $K_m(L_p)$ that is:

$$K_m^*(L_p) = \left\{ (a_n)_{n=0}^m : \sum_{k=0}^{m} a_k x_k \geq 0, \ \forall\ (x_k)_{k=0}^m \in K_m(L_p) \right\}.$$

As it is stated in [3], such results were obtained for the first time for convex functions by T. Popoviciu (see [14] for more references).

They were transposed for convex sequences by J.E. Pečarić in [13]. A constructive characterization is given in [20]. The representation for $p = 2$ is given in [8]. The general case follows easy from Theorem 1.

Theorem 3. *The sequence* $(a_n)_{n=0}^m$ *belongs to* $K_m^*(L_p)$ *if and only if it satisfies the relations:*

$$\sum_{n=k}^{m} a_n u_{n+p-k-1} = 0 \textit{ for } 0 \leq k \leq p-1$$

and

$$\sum_{n=k}^{m} a_n u_{n+p-k-1} \geq 0 \textit{ for } p \leq k \leq m.$$

Using Theorem 2 we can transpose the result for the case of m infinite. But, as in [17] we want to deal with a more general case. We remind some definitions. The functional $A : S \to \mathbb{R}$ is said to be:

a) *superadditive*, if:

$$A(x+y) \geq A(x) + A(y), \ \forall \ x, y \in S;$$

b) *positively superhomogeneous*, if:

$$A(ax) \geq aA(x), \ \forall \ x \in S, \ a \geq 0;$$

c) *upper semicontinuous*, if:

(8) $$\limsup_{n\to\infty} A(x^n) \leq A\left(\lim_{n\to\infty} x^n\right).$$

Theorem 4. *Let* $A : S \to \mathbb{R}$ *be a superadditive, positively superhomogenous, upper semicontinuous functional. In order that* $A(x) \geq 0$ *for every* $x \in K(L_p)$ *it is necessary and sufficient that:*

(9) $$A(E^k u^p) \geq 0 \textit{ for } k \geq 0$$

and

(10) $$A(-E^k u^p) \geq 0 \textit{ for } 0 \leq k < p.$$

Proof. From the theorem 2, we have $E^k u^p \in K(L_p)$ for $k \geq 0$ and also $-E^k u_p \in K(L_p)$ for $0 \leq k < p$, so that the conditions (9) and (10) are necessary. They are also sufficient. For an $x \in K(L_p)$ we have (7) and so, for $n > p$:

$$A(x^n) = A(y_0 u^p + y_1 E u^p + \cdots + y_n E^n u^p)$$

$$\geq A(y_0 u^p) + A(y_1 E u^p) + \cdots + A(y_n E^n u^p)$$

$$\geq |y_0| A((\mathrm{sgn} y_0) u^p) + \cdots + |y_{p-1}| A((\mathrm{sgn} y_{p-1}) E^{p-1} u^p)$$

$$+ y_p A(E^p u^p) + \cdots + y_n A(E^n u^p) \geq 0$$

thus, from (8), $A(x) \geq 0$.

Corollary 4. *Let $A : S \to \mathbb{R}$ be a linear and continuous functional. In order that $A(x) \geq 0$ for every $x \in K(L_p)$ it is necessary and sufficient that:*

$$A(E^k u^p) = 0 \textit{ for } 0 \leq k < p$$

and

$$A(E^k u^p) \geq 0 \textit{ for } k \geq p.$$

We remark that in this corollary $\mathbb{R}$ can be replaced by an arbitrary linear topological space with a "positive" cone.

If we don't work with divergent series, Corollary 4 takes the following form. Let us denote:

$$K^*(L_p) = \Big\{ a = (a_n)_{n \geq 0},\ \exists\, n_0 :\ a_n = 0 \text{ if } n > n_0$$

$$\text{and } ax = \sum_{n=0}^{\infty} a_n x_n \geq 0,\ \forall\, x = (x_n)_{n \geq 0} \in K(L_p) \Big\}.$$

Corollary 5. *The finally null sequence a belongs to* $K^*(L_p)$ *if and only if:*

$$aE^k u^p = 0 \text{ for } 0 \leq k < p$$

and

$$aE^k u^p \geq 0 \text{ for } k \geq p.$$

We point out that these results generalize the corresponding theorems from [5] and [17].

4. We can further generalize these results as follows. Let $A : S \to S$ be a continuous linear operator on S and L_p, L'_q two linear recurrences of the form (1). The problem is when holds:

$$A(K(L_p)) \subset K(L'_q). \tag{11}$$

Theorem 5. *If* $A : S \to S$ *is a linear continuous operator, then (11) holds if and only if:*

$$L'_q(A(E^k u^p)) = 0 \text{ for } 0 \leq k < p$$

and

$$L'_q(A(E^k u^p)) \geq 0 \text{ for } k \geq p.$$

Proof. As $E^k u^p \in K(L_p)$ for $k \geq 0$ and $-E^k u^p \in K(L_p)$ for $0 \leq k < p$, the conditions are necessary. They are also sufficiently. Indeed, let $x \in K(L_p)$. By (7), $x = \lim_{n\to\infty} x^n$, where $x^n = \sum_{k=0}^{n} y_k E^k u^p$ and $y_k \geq 0$ for $k \geq p$. So:

$$L'_q(A(x)) = \lim_{n\to\infty} L'_q(A(x_r))$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} y_k \cdot L'_q(A(E^k u^p)) = \lim_{n\to\infty} \sum_{k=p}^{n} y_k \cdot L'_q(A(E^k u^p)) \geq 0.$$

We remark that A is usually given by a double infinite matrix $A = (a_{nk})_{n,k\geq 0}$ with the property that for any $n \geq 0$ there is a k_n such that $a_{nk} = 0$ for $k > k_n$. If $x = (x_k)_{k\geq 0}$ then

$$A(x) = \left(\sum_{k=0}^{\infty} a_{nk} x_k \right)_{n\geq 0}.$$

The case of triangular matrices, that is $k_n = n$, was studied, for $L_p = L'_q = \Delta$ in [4] and [7]. His special case of generalized arithmetic means is effectively solved: the case $p = 2$ in [21] and in an improved form in [18], while the general case was initiated in [6] and accomplished in [19]. We shall give this result in the next paragraph. Also, the case $L_2 = L'_2$ is studied in [8].

5. Let $q = (q_n)_{n\geq 0}$ be a sequence of positive numbers. It defines an operator $Q : S \to S$ by: if $x = (x_n)_{n\geq 0}$ then $Q(x) = X = (X_n)_{n\geq 0}$ is given by:

$$X_n = (q_0 x_0 + \cdots + q_n x_n)/(q_0 + \cdots + q_n).$$

We denote by $L_p = K(\Delta^p)$ the set of (ordinary) p-convex sequences. In [19] we have proved that $Q(K_p) \subset K_p$ if and only if:

$$q_n = q_0 \binom{v+n-1}{n}, \quad n \geq 1 \tag{12}$$

with $v = q_1/q_0$, where:

$$\binom{w}{0} = 1, \quad \binom{w}{n} = \frac{w(w-1)\dots(w-n+1)}{n!} \text{ for } n \geq 1.$$

Let us denote by $M^v K_p$ the set of sequences x with the property that $Q(x) \in K_p$, where q is given by (12). In [19] it is proved that $x \in M^v K_p$

if and only if:

$$x_n = \sum_{k=0}^{n} \binom{n+p-k-2}{p-2} \left(\frac{n+p-k-1}{p-1} + \frac{n}{v} \right) z_k, \quad z_k \geq 0 \text{ for } k \geq p.$$

This may be transcript as follows:

Lemma 4. *The sequence x belongs to $M^v K_p$ if and only if:*

$$x = \sum_{k=0}^{\infty} \left[\left(1 + \frac{p-1}{v} \right) E^k u^p + \frac{k-p+1}{v} E^k u^{p-1} \right] z_k, \quad z_k \geq 0 \textit{ for } k \geq p.$$

As in the other cases this gives:

Theorem 6. *The linear continuous functional $A : S \to \mathbb{R}$ verifies the condition $A(x) \geq 0$ for every $x \in M^v K_p$ if and only if:*

$$(v+p-1)A(E^k u^p) + (k-p+1)A(E^k u^{p-1}) = 0 \textit{ for } 0 \leq k < p$$

and

$$(v+p-1)A(E^k u^p) + (k-p+1)A(E^k u^{p-1}) \geq 0 \textit{ for } k \geq p.$$

In the special case $p = 2$, $v = 1$ this result is given in [11].

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ON THE CONVEXITY OF HIGH ORDER OF SEQUENCES

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ABSTRACT. We improve some results of Lacković and Simić [2] concerning the weighted arithmetic means that preserve the convexity of high order of sequences.

In [1] and [3] a characterization is given for triangular matrices which define transformations in the set of sequences preserving convexity of order r. In the particular case of weighted arithmetic means, explicit expressions were given before in Lacković and Simić [2]. In this paper we improve the results from [2] generalizing some of the properties that we proved in [7] for the convexity of order two.

At the beginning, let us specify some notation and definitions which will be used throughout the paper.

Let $a = (a_n)$ $(n = 0, 1, \dots)$ be a real sequence. The r-th order difference of the sequence a is defined by:

$$\Delta^0 a_n = a_n, \quad \Delta^r a_n = \Delta^{r-1} a_{n+1} - \Delta^{r-1} a_n \tag{1}$$

$(r = 1, 2, \dots,\ n = 0, 1, \dots)$.

Definition 1. A sequence $a = (a_n)$ is said to be convex of order r if $\Delta^r a_n \geq 0$ for all $n \in \mathbb{N}$.

Let $p = (p_n)$ be a sequence of positive numbers. If defines a transformation P in the set of sequences: any sequence $a = (a_n)$ is transformed into the sequence $P(a) = A = (A_n)$ given by:

$$A_n = \frac{p_0 a_0 + \cdots + p_n a_n}{p_0 + \cdots + p_n} \quad (n = 0, 1, \dots) \tag{2}$$

Definition 2. The transformation P is said to be r-convex if the sequence $A = P(a)$ is convex of order r for any sequence a convex of order r.

In [2] the following theorem is given:

Theorem 0. *The transformation P is r-convex if and only if the sequence $p = (p_n)$ is given by:*

$$p_n = \frac{(r-1)! p_{r-1}}{n!(p_0 + \cdots + p_{r-2})^{n-r+1}} \prod_{k=r-2}^{n-2} (k+1)(p_0 + \cdots + p_{r-2}) + (r-1)p_{r-1}$$

for $n \geq r$, with $p_0, \dots, p_{r-1}$ arbitrary positive numbers.

Remark 1. For $a_0 = 0$ and $a_n = (3 + 6n - 2n^2)/3$ if $n \geq 1$, we have $\Delta^3 a_0 = 1$ and $\Delta^3 a_n = 0$ if $n \geq 1$, so that the sequence (a_n) is convex of order 3. Let us choose for $r = 3$: $p_0 = 6$, $p_1 = 1$ and $p_2 = 7/2$. From (3) we get $p_3 = 7/2$ and so from (2), we have $A_0 = 0$, $A_1 = 1/3$, $A_2 = 1$ and $A_3 = 1$, that is $\Delta^3 A_0 = -1$. Hence the result from Theorem 0 is not valid in this form. To amend it, we begin by putting (3) in a simpler shape. For this we use the following notation:

$$\binom{u}{0} = 1, \quad \binom{u}{n} = \frac{u(u-1)\dots(u-n+1)}{n!}, \text{ for } n \geq 1 \tag{4}$$

where u is an arbitrary real number.

Lemma 1. *If the transformation P is r-convex, then the sequence (p_n) must be given by:*

$$p_n = p_{r-1}\binom{u+n-1}{n-r+1}\Big/\binom{n}{r-1}, \quad \textit{for } n \geq r \tag{5}$$

where:

$$u = \frac{(r-1)p_{r-1}}{p_0 + \cdots + p_{r-2}}, \qquad p_k > 0 \textit{ for } k = 0, \ldots, r-1. \tag{6}$$

Proof. Because (5) is only a transcription of (3) using (4) and (6), the result was proved in [2]. However we sketch here another proof by mathematical induction. As in [2] we use the sequence $a_n = cn(n-1)\ldots(n-r+2)$ for which we have $\Delta^r a_n = 0$ for any n. Hence it is convex of order r for any real c, and so must be (A_n) too. But this happens if and only if for $c = 1$ we have $\Delta^r A_n = 0$ for any n. For $n = 0$ we get $p_r = p_{r-1}(u+r-1)/r$ which is (5) for $n = r$. Suppose (5) is valid for $n \leq m$. To obtain A_n for $r \leq n \leq m$, we must calculate:

$$\sum_{k=0}^{n} p_k = \sum_{k=0}^{r-2} p_k + p_{r-1} + \sum_{k=r}^{n} p_k$$

$$= p_{r-1}\left[\frac{r-1}{u} + 1 + \sum_{i=0}^{n-r}\binom{u+r+i-1}{i+1}\Big/\binom{r+i}{i+1}\right].$$

From this it can be shown, by mathematical induction, that:

$$\sum_{k=0}^{n} p_k = p_{r-1}\frac{n-r+2}{u}\binom{u+n}{n-r+2}\Big/\binom{n}{n-r+1}. \tag{7}$$

So:

$$A_n = \frac{u(r-1)!}{u+r-1}\binom{n}{r-1}, \qquad n \leq m$$

and

$$A_{m+1} = \left[p_{r-1}(r-1)!\binom{u+m}{m-r+1} + p_{m+1}(r-1)!\binom{m+1}{r-1}\right]$$

$$/\left[p_{r-1}\frac{m-r+2}{u}\binom{u+m}{m-r+2}\Big/\binom{m}{m-r+1} + p_{m+1}\right].$$

From $\Delta^r A_{m-r+1} = 0$, we obtain (5) for $m+1$, and so for every n.

Lemma 2. *If the sequence (a_n) is given by:*

$$a_n = \sum_{k=0}^{n} \binom{n+r-k-1}{r-1} b_k, \tag{8}$$

then

$$\Delta^r a_n = b_{n+r} \quad (n = 0, 1, \dots) \tag{9}$$

Remark 2. This result is connected with some relations from [1] and [6]. Because any sequence may be put in the form (8), we obtain a representation theorem simpler than that given in [6]:

Corollary 1. *The sequence (a_n) is convex of order r if and only if in its representation (8), it has $b_n \geq 0$ for $n \geq r$.*

Lemma 3. *If the transformation P is r-convex, then for every $n \leq r$:*

$$\sum_{k=0}^{n-1} p_k = np_n/u. \tag{10}$$

Proof. Let (A_n) be represented by:

$$A_n = \sum_{k=0}^{n} \binom{n+r-k-1}{r-1} c_k. \tag{11}$$

Then:

$$a_n = \left(A_n \sum_{i=0}^{n} p_i - A_{n-1} \sum_{i=0}^{n-1} p_i \right) / p_n.$$

If

$$q_n = \frac{1}{p_n} \sum_{k=0}^{n-1} p_k$$

then:

$$a_n = A_n + q_n(A_n - A_{n-1}) = \sum_{k=0}^{n} \left[\binom{n+r-i-1}{r-1} + q_n \binom{n+r-i-2}{r-2} \right] c_i$$

for $n \geq 1$ and $a_0 = A_0 = c_0$. So:

$$\Delta^r a_0 = \sum_{j=0}^{r} (-1)^j \binom{r}{j} a_{r-j}$$

$$= \sum_{j=0}^{r-1} \left\{ \sum_{i=0}^{r-1} \left[\binom{2r-j-i-1}{r-1} + q_{r-j} \binom{2r-j-i-2}{r-2} \right] c_i \right\} (-1)^j \binom{r}{j}$$
$$+ (-1)^r c_0$$

$$= \sum_{i=0}^{r} \left\{ \sum_{j=0}^{r-i} \left[\binom{2r-j-i-1}{r-1} + q_{r-j} \binom{2r-j-i-2}{r-2} \right] (-1)^j \binom{r}{j} \right\} c_i$$
$$+ \left\{ \sum_{k=0}^{r-1} \left[\binom{2r-j-1}{r-1} + q_{r-j} \binom{2r-j-2}{r-2} \right] (-1)^j \binom{r}{j} + (-1)^r \right\} c_0.$$

But, as it is proved in [5]:

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} j^p = 0 \text{ for } p < n$$

and hence:

$$\sum_{j=0}^{n} (-1)^j \binom{r}{j} Q(j) = 0$$

for any polynomial Q of degree less than n. So:

$$\sum_{j=0}^{m} (-1)^j \binom{r}{j} \binom{m+r-j-1}{r-1} = 0 \text{ for } m = 1, \ldots, r \tag{12}$$

because:

$$\sum_{j=0}^{m}(-1)^j \frac{r!}{j!(r-j)!}\cdot\frac{(m+r-j-1)!}{(r-1)!(m-j)!}=\frac{r}{m}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\binom{m+r-j-1}{m-1}$$

and $\binom{m+r-j-1}{m-1}$ is a polynomial of degree $m-1$ in j. Hence:

$$\Delta^r a_0 = c_r + \sum_{i=0}^{r}\left[\sum_{j=0}^{r-i}(-1)^j\binom{r}{j}\binom{2r-j-i-2}{r-2}q_{r-j}\right]c_i$$

$$+\sum_{j=0}^{r-1}(-1)^j\binom{r}{j}\binom{2r-j-2}{r-2}q_{r-j}c_0.$$

As the coefficient of c_r is $1+q_r>0$, $\Delta^r a_0 \geq 0$ implies $\Delta^r A_0 = c_r \geq 0$ if and only if the coefficients of c_i are zero for $i=0,\ldots,r-1$. For $i=r-1$ we have: $(r-1)q_r - rq_{r-1}=0$ and as (6) means $q_{r-1}=(r-1)/u$, we also have $q_r = r/u$. Assuming (10) valid for $r-j$ ($j=0,\ldots,m-1;\ m<r-1$) it may be deduced for $r-m$, because we have:

$$\sum_{j=0}^{m}(-1)^j\binom{r}{j}\binom{m+r-j-2}{r-2}(r-j)=0,\ \text{for } m<r-1$$

and

$$\sum_{j=0}^{r-1}(-1)^j\binom{r}{j}\binom{2r-j-2}{r-2}(r-j)=0$$

which may be verified as in (12).

Theorem 1. *The transformation P is r-convex if and only if the sequence (p_n) is given by:*

$$p_n = p_0\binom{u+n-1}{n},\ \textit{for } n\geq 1,\ \textit{with } u=p_1/p_0. \tag{13}$$

Proof. Necessity. Lemma 1 and Lemma 3 give the necessary conditions (5) and (10). From (10) we have: $u = p_1/p_0$ for $n = 1$, and $p_2 = u(p_0 + p_1)/2 = p_0\binom{u+1}{2}$; supposing (13) valid for $n \le m < r-1$, (10) gives:

$$p_{m+1} = \frac{up_0}{m+1}\sum_{k=0}^{m}\binom{u+k-1}{k} = p_0\frac{u}{m+1}\binom{u+m}{m} = p_0\binom{u+m}{m+1}$$

that is, (13) holds for $n \le r-1$. Hence, from (5) we also get:

$$p_n = p_0\binom{u+r-2}{r-1}\binom{u+n-1}{n-r+1}\Big/\binom{n}{r-1} = p_0\binom{u+n-1}{n}$$

for $n \ge r$.

Sufficiency. With (13), the sequence (2) becomes:

$$A_n = \left[\sum_{k=0}^{n}\binom{u+k-1}{k}a_k\right]\Big/\binom{u+n}{n} \tag{14}$$

and so we have the relation:

$$a_n = A_n + n(A_n - A_{n-1})/u, \ \text{ for } n > 0. \tag{15}$$

Taking A_n of the form (11), from (15) we obtain:

$$a_n = \sum_{k=0}^{n}\binom{n+r-k-2}{r-2}\left(\frac{n+r-k-1}{r-1} + \frac{n}{u}\right)c_k. \tag{16}$$

Because $\Delta^r A_n = c_{n+r}$, applying to (15) the known relation (see [4]):

$$\Delta^r(a_n b_n) = \sum_{i=0}^{r}\binom{r}{i}\Delta^i a_n \Delta^{r-i} b_{n+i}$$

we obtain:

$$\Delta^r a_n = (n+r+u)u^{-1}c_{n+r} - nu^{-1}c_{n+r-1}, \quad n \ge 1. \tag{17}$$

From the proof of Lemma 3 we have: $\Delta^r a_0 = c_r(r+u)/u$, that is (17) is valid for $n = 0$ too. Assuming (a_n) given by (8), (9) is valid; thus:

$$b_r = (r+u)/u, \quad b_{n+r} = (n+r+u)/uc_{n+r} - n/uc_{n+r-1}. \tag{18}$$

Hence, if $b_n \geq 0$ for $n \geq r$, then also $c_n \geq 0$ for $n \geq r$; that is, if (a_n) is convex of order r, so is (A_n) too.

Remark 3. The sufficiency part of Theorem 1 was also proved in [1]. In what follows we improve also this result. Let us denote by K_r the set of all sequences convex of order r and by K_r^u the set of all sequence (a_n) with the property that (14) gives a sequence (A_n) in K_r.

Theorem 2. *If $0 < v < u$ then the following strict inclusions hold:*

$$K_r \subset K_r^u \subset K_r^v.$$

Proof. The first inclusion was proved in Theorem 1. Its strictness follows from (18): the positivity of c_n $(n \geq r)$ does not imply that of b_n. Now suppose (a_n) given by (16) and also by:

$$a_n = \sum_{k=0}^{n} \binom{n+r-k-2}{r-2} \left(\frac{n+r-k-1}{r-1} + \frac{n}{v} \right) d_k.$$

So (17) holds and $\Delta^r a_n = (n+r+v)v^{-1}d_{n+r} - nv^{-1}d_{n+r-1}$ that is:

$$(n+r+v)/vd_{n+r} - nv^{-1}d_{n+r-1} = (n+r+u)u^{-1}c_{n+r} - nu^{-1}c_{n+r-1}$$

Hence $d_r = \dfrac{v(r+u)}{u(r+v)} c_r$ and generally, by mathematical induction:

$$d_{r+n} = \frac{u+r+n}{v+r+n} \cdot \frac{c_{r+n}}{uv} + \frac{u-v}{uv} \sum_{i=0}^{n-1} \frac{c_{r+i}}{n-i+1} \binom{n}{i} \Big/ \binom{v+r+n}{n-i+1}; \tag{19}$$

that is, $c_n \geq 0$ for $n \geq r$ implies $d_n \geq 0$ for $n \geq r$ and so, if (a_n) is in K_r^u, it is also in K_r^v. That the inclusion $K_r^u \subset K_r^v$ is strict follows also from (19) as above.

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PROPERTIES OF BOUNDED CONVEX SEQUENCES

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Abstract. Some properties of the bounded convex sequences of order $m \geqq 2$ are considered in this paper. As it is shown, these properties are in direct connection with the convergence of a certain class of series.

1. Introduction

When studying convergence conditions, summability and other properties of series the knowledge of sequence properties is of a decisive importance. Due to their specific nature, different classes of sequences, for example classes of bounded, convergent, convex, starshaped and other sequences (see for example [1]-[16]) are studied hardly. In this paper some properties of bounded convex sequences with order of convexity $m(\geqq 2)$ and their relation with a certain class of real series is considered.

Let us first introduce some notation and definitions. Denote by S_m the class of real sequences (a_n), $n \in N_0$, with the following properties

$$(1.1)\qquad M_2 \leqq a \leqq M_1, \quad n = 0, 1, \dots \ (M_1, M_2 = \text{const.}),$$

and

$$(1.2)\qquad \nabla^r a_n \geqq 0, \quad r = 2, \dots, m, \ (m \geqq 2), \ n = 0, 1, \dots$$

where

$$\nabla^r a_n = (-1)^r \Delta^r a_n, \quad (\Delta a_n = a_{n+1} - a_n, \quad \Delta^r a_n = \Delta(\Delta^{r-1} a_n)).$$

Let $(s)_p = s(s+1)\dots(s+p-1)$ and $V_n^r = \binom{n+r}{r}$ for each $p = 1, 2, \dots,$ $s = 0, 1, \dots,$ $n = 0, 1, \dots$. We shall also quote some results, known in the literature which are in relation to those obtained in this paper.

For sequences belonging to S_2 class, the following result is proved in paper [13]:

Theorem A. *The sequence (a_n), $n \in N_0$, belongs to S_2 class if and only if there is a sequence (b_n), $n \in N_0$, such that*

$$a_n = \sum_{k=0}^{n} (n-k+1) b_k, \quad n \geqq 0,$$

$$b_k \geq 0 \textit{ for } k \geqq 2,$$

$$n \sum_{k=0}^{n} b_k \to 0, \quad n \to \infty,$$

$$\sum_{k=0}^{\infty} (k+1) b_{k+1} < +\infty.$$

The following results can be found, for example in the paper [10]:

Theorem B. *Let the sequence* (a_n), $n \in N_0$, *satisfy the following properties:* $\lim\limits_{n\to\infty} a_n = 0$ *and* $\nabla^2 a_n \geqq 0$ *for* $n = 0, 1, \ldots$ *. Then* $\nabla a_n \geqq 0$,

$$\lim_{n\to\infty} n\nabla a_n = 0 \text{ and } \sum_{n=0}^{\infty}(n+1)\nabla^2 a_n = \sum_{n=0}^{\infty}\nabla a_n = a_0.$$

Theorem C. *For any convergent series* $\sum\limits_{n=1}^{\infty} a_n$, *we have* $na_n \to 0$ *in the sense of Cesàro.*

The following result is given in monograph [3]:

Theorem D. *If a sequence* (a_n), $n \in N_0$, *belongs to* S_2 *class, then:*

(a) the sequence (a_n), $n \in N_0$ *is decreasing;*

(b) $\lim\limits_{n\to\infty} n\nabla a_n = 0$*;*

(c) the series $\sum\limits_{n=0}^{\infty}(n+1)\nabla^2 a_n$ *is a convergent one and its sum is* $a_0 - \lim\limits_{n\to\infty} a_n$.

2. Main results

Before presenting the main results of this paper, we shall prove several lemmas giving specific properties of the sequences belonging to S_m class.

Lemma 1. *If a sequence* (a_n), $n \in N_0$, *belongs to* S_m $(m \geq 2)$ *class then the following inequality*

$$0 \leqq \nabla^r a_n \leqq (M_1 - M_2)\frac{V_n^r}{V_n^{2r}} = (M_1 - M_2)\frac{(2r)!}{r!(n+r+1)_r}, \tag{2.1}$$

holds for $r = 1, \ldots, m-1$.

Proof. As $\nabla^r a_n \geqq 0$, for $r = 2, \ldots, m$ the inequality

$$\nabla^{r-1} a_n \geqq \nabla^{r-1} a_{n+1} \quad (n = 0, 1, \ldots) \tag{2.2}$$

is valid. According to Theorem D we conclude that the following implication

$$(a_n) \in S_2 \Rightarrow \nabla a_n \geqq 0 \tag{2.3}$$

holds.

As (a_n), $n \in N_0$, belongs to S_m class $\nabla^r a_n \geqq 0$ holds for $r = 2, \dots, m$, then, taking into account (2.3), we prove the left inequality in (2.1). The right side of the inequality (2.1) shall be proved by means of mathematical induction. For $r = 1$ we have

$$(M_1 - M_2)V_n^1 = (M_1 - M_2)(n+1)$$

$$= M_1(n+1) - M_2(n+1) \geqq \sum_{k=0}^{n} a_k - M_2(n+1)$$

$$= \sum_{k=0}^{n}(k+1)\nabla a_k + (n+1)a_{n+1} - M_2(n+1) \geqq \nabla a_n \sum_{k=0}^{n}(k+1) = V_n^2 \nabla a_n,$$

i.e.

$$0 \leqq \nabla a_n \leqq (M_1 - M_2)\frac{V_n^1}{V_n^2}.$$

Let us suppose that (2.1) is valid for some $r = p$ $(1 \leqq p \leqq m-2)$, i.e.

$$0 \leqq \nabla^p a_n \leqq (M_1 - M_2)\frac{V_n^p}{V_n^{2p}}. \tag{2.4}$$

According to equality $\sum_{i=0}^{k} V_i^s = V_k^{s+1}$ and the inductive assumption (2.4) we have

$$(M_1 - M_2)V_n^{p+1} = (M_1 - M_2)\sum_{k=0}^{n} V_k^p \geqq \sum_{k=0}^{n} V_k^{2p}\nabla^p a_k$$

$$= \sum_{k=0}^{n} V_k^{2p+1}\nabla^{p+1} a_k + V_n^{2p+1}\nabla^p a_{n+1} \geqq \nabla^{p+1} a_n V_n^{2p+2},$$

i.e.,

$$\nabla^{p+1} a_n \geqq (M_1 - M_2)\frac{V_n^{p+1}}{V_n^{2p+2}},$$

which had to be proved.

Using Abel's lemma we directly obtain the following result:

Lemma 2. *For each sequence of real numbers* (a_n), $n \in N_0$, *the equality*

$$a_n = a_0 - \sum_{k=1}^{n} V_{n-k}^{k} \nabla^k a_{n-k} - \sum_{k=0}^{n-p-1} V_k^p \nabla^{p+1} a_k, \quad \left(\sum_{k=1}^{0} = 0\right), \tag{2.5}$$

holds, where $p < n$.

Lemma 3. *Let the sequence* (a_n), $n \in N_0$, *belong to the class* S_m $(m \geqq 2)$. *Then, equalities*

$$\lim_{n\to\infty} V_{n-k}^{k} \nabla^k a_{n-k} = 0, \tag{2.6}$$

for $k = 1, \ldots, m-1$, *and*

$$\lim_{n\to\infty} \sum_{k=0}^{n} V_k^{m-1} \nabla^m a_k = a_0 - \lim_{n\to\infty} a_n \tag{2.7}$$

hold.

Proof. For $m = 2$ Lemma 3 is proved in Theorem A (i.e. Theorem D). Assume that Lemma 3 holds for $k = 1, \ldots, m-2$. According to Lemma 2 we have:

$$a_n = a_0 - \sum_{k=1}^{m-2} V_{n-k}^{k} \nabla^k a_{n-k} - \sum_{k=0}^{n-m+1} V_k^{m-2} \nabla^{m-1} a_k,$$

and by the inductive assumption

$$\lim_{n\to\infty} V_{n-k}^{k} \nabla^k a_{n-k} = 0$$

for $k = 1, \ldots, m-2$. On this basis, we conclude that the series $\sum_{k=0}^{\infty} V_k^{m-2} \nabla^{m-1} a_k$ is a convergent one. According to Theorem C we obtain a sequence $(nV_n^{m-2}\nabla^{m-1}a_n)$, i.e. $(V_n^{m-1}\nabla^{m-1}a_n)$ which tends to zero in Cesàro sense. In other words the equality

$$\lim_{n\to\infty} \frac{V_0^{m-1}\nabla^{m-1}a_0 + \cdots + V_n^{m-1}\nabla^{m-1}a_n}{n+1} = 0, \tag{2.8}$$

holds. Let us prove that the sequence $(V_n^{m-1}\nabla^{m-1}a_n)$ tends toward zero. Assume that it is not true. Then, there would be a constant $C(\geqq 0)$ such that beginning from some index n the inequality $V_n^{m-1}\nabla^{m-1}a_n \geqq C$ holds. According to (2.2) for each $k \leqq n$ the following inequality

$$\Delta^{m-1}a_k \geq \nabla^{m-1}a_n \geqq \frac{C}{V_n^{m-1}}$$

holds. On the other hand, the relation

$$\frac{V_0^{m-1}\nabla^{m-1}a_0 + \cdots + V_n^{m-1}\nabla^{m-1}a_n}{n+1} \geqq C\frac{V_n^m}{(n+1)V_n^{m-1}} \underset{n\to\infty}{\longrightarrow} \frac{C}{m} \neq 0,$$

is in opposition with equality (2.8). It contradicts the assumption that the sequence $(V_n^m\nabla^{m-1}a_n)$ does not tend to zero. It also means that (2.6) holds even for $k = m-1$. If we substitute $p = m-1$ in (2.5), the assumption (2.7) is directly obtained from the equality

$$\sum_{k=0}^{n-m} V_k^{m-1}\nabla^m a_k = a_0 - \sum_{k=1}^{m-1} V_{n-k}^k \nabla^k a_{n-k} - a_n.$$

From paper [14] we directly obtain the following result.

Lemma 4. *The sequence of real numbers (a_n), $n \in N_0$, has the property $\nabla^r a_n \geqq 0$, if and only if there is a sequence (b_n), $n \in N_0$, so that $b_n \geqq 0$*

for $n \geqq r$ for which equality

$$a_n = (-1)^r \sum_{k=0}^{n} V_{n-k}^{r-1} b_k \tag{2.9}$$

holds.

According to the given lemmas we immediately obtain the following result:

Theorem 1. *The sequence (a_n), $n \in N_0$, belongs to S_m class if and only if there is a sequence (b_n), $n \in N_0$, such that:*

$$a_n = (-1)^m \sum_{k=0}^{n} V_{n-k}^{m-1} b_k \tag{2.10}$$

$$(-1)^{m+j} \sum_{k=0}^{n} V_{n-k}^{m-j-1} b_k \geqq 0 \text{ for } j = 2, \ldots, m-1, \ n \geq j \tag{2.11}$$

$$b_k \geqq 0 \text{ for } k \geqq m, \tag{2.12}$$

$$V_{n-k}^{k} \sum_{i=0}^{n} V_{n-i}^{n-k-1} b_i \overset{n\to\infty}{\longrightarrow} 0, \text{ for } k = 1, \ldots, m-1, \tag{2.13}$$

and

$$\sum_{k=0}^{\infty} V_k^{m-1} b_{k+m} \text{ is convergent.} \tag{2.14}$$

Corollary 1. *The sequence of real numbers (a_n), $n \in N_0$, has the following properties $\nabla^r a_n \geqq 0$, for $r = 2, \ldots, m$ and $\lim_{n\to\infty} a_n = 0$, if and only if there is a sequence (b_n), $n \in N_0$, such that properties (2.10)-(2.13) and $\sum_{k=0}^{\infty} V_k^{m-1} b_{k+m} = a_0$ are satisfied.*

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A HIERARCHY OF SUPERMULTIPLICITY OF SEQUENCES IN A SEMIGROUP

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ABSTRACT. In this paper are defined the classes of convex, starshaped and supermultiplicative sequences in a semigroup. Also some relations among them are proved.

1. Introduction

In [1] it was proved that all the convex functions are starshaped and these are all superadditive. This property was named in [4] hierarchy of convexity of functions. In [7] we have proved a similar property for real sequences and now we want to transpose it to sequences from a semigroup. But it seems to me to be more adequate to call the property "hierarchy of supermultiplicity" because all the sequences are supermultiplicative.

In the nest paragraph we shall given the notions relative to semigroups which we need in what follows. Some of them are taken from [3] but the others can be new because we couldn't find them in the accessible literature. Then we define the classes of convex, starshaped and supermultiplicative sequences in a semigroup and prove some relations among them.

2. Semigroups

By a **semigroup** $(G, \cdot)$ we shall mean a non-empty set G on which is defined an associative binary operation. We suppose that the semigroup is commutative and has an identity e, thus:

$$ex = x,\ \forall\ x \in G.$$

A semigroup can have a zero, that is an element z with the property:

$$zx = z,\ \forall\ x \in G.$$

If $xx = x$, the element x is called idempotent.

A basic relation which we need in what follows is the divisibility:

$$a|b \Leftrightarrow \exists\ x \in G,\ b = ax.$$

Also, we shall consider semigroups in which some kinds of reductions are valid.

Definition 1. The semigroup $(G, \cdot)$ is **cancellative** if:

$$xa = xb \Rightarrow a = b. \tag{1}$$

We remind that for the product of n elements, each equal to x, it is used the notation x^n.

Definition 2. The semigroup $(G, \cdot)$ has **radical** if:

$$x^n = y^n \Rightarrow x = y. \tag{2}$$

Definition 3. The semigroup $(G, \cdot)$ **preserves the divisibility** if:

$$x^n | y^n \Rightarrow x | y. \tag{3}$$

Remark 1. Some results are immediate. If $(G, \cdot)$ is a group then it is cancellative and preserves the divisibility. If every element of $(G, \cdot)$ is idempotent, the semigroup has radicals and preserves the divisibility but is non cancallative. Also, if $(G, \cdot)$ has a zero element it is non cancellative.

We show by examples some relations between the three definitions.

Example 1. The ensemble of subsets of a non-empty set X with respect to intersectuon is a non cancellative semigroup but which has radicals and preserves the divisibility (any element is idempotent).

Example 2. The set of classes $\widehat{k}$ of integers congruent modulo 4 with respect to addition represents a cancellative semigroup which preserves the divisibility (in fact it is a group) but which has no radical as:

$$\widehat{2} + \widehat{2} = \widehat{0} + \widehat{0} = \widehat{0}.$$

Example 3. In the semigroup generated by the transformation:

$$t = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 3 \end{pmatrix}$$

in respect to composition, we have: $(t^2)^2 t = t^2$ and $(t^2)^3 = t^3$, but $t^2 t^k \neq t$ for any $k \geq 0$, thus the semigroup is non cancellative, has no radical and do not preserves the divisibility.

To give examples of semigroups which satisfy all the conditions (1)-(3), we consider also the next notion:

Definition 4. The semigroup $(G, \cdot)$ has the **base** B if it is satisfies the following condition:

$$(4) \qquad \forall\ x \in G - \{e\},\ \exists\ !n \geq 1,\ \exists\ !p_1, \ldots, p_n \in B,$$

$$\exists\ !k_1,\ldots,k_n \geq 1:\ x = \prod_{i=1}^{n} p_i^{k_i}.$$

Remark 2. As in the case of a vectorial space, a base of a semigroup generates him. From the unicity of the representation in (4), it follows also that the base is an independent set in the sense that if $p, q \in B$ then $p \nmid q^n$ for no $n \geq 1$. We must suppose $p^0 = e,\ \forall\ p \in B$. So we can represent any two elements of G by the same elements of the base B but with non-negative powers:

$$x = \prod_{i=1}^{n} p_i^{k_i}, \quad y = \prod_{i=1}^{n} p_i^{h_i}, \quad k_i, h_i \geq 0.$$

It remains true that $x = y$ if and only if $k_i = h_i$, $i = 1, \ldots, n$. We have also some consequences:

Lemma 1. *If the semigroup $(G, \cdot)$ has a base, then hold the following implications for every $x, y, z \in G$, $m, n \geq 1$:*

a) $x^n = x^m \Rightarrow n = m$;

b) $xz = yz \Rightarrow x = y$;

c) $x^n = y^n \Rightarrow x = y$;

d) $x^n | y^n \Rightarrow x | y$.

Proof. All the affirmations follow from the unicity of the representation in the base B. For example, for the last implication, we suppose that $y^n = x^n z$ where:

$$x = \prod_{i=1}^{n} p_i^{k_i}, \quad y = \prod_{i=1}^{n} p_i^{h_i}, \quad z = \prod_{i=1}^{n} p_i^{j_i}.$$

Then $nh_i = nk_i + j_i$, for $i = 1, \ldots, n$. So $j_i = ng_i$, where $g_i = h_i - k_i$, and putting:

$$w = \prod_{i=1}^{n} p_i^{g_i}$$

we have $y = xw$, thus $x|y$.

Remark 3. Thus, if the semigroup has a base it is cancellative, has radicals and preserves the divisibility. But, because of the property a), it cannot be finite. Examples of semigroups with base are $(N, +)$ with $B = \{1\}$ and $(N, \cdot)$ with the base consisting of all prime numbers.

3. Sequences in a semigroup

Let $(x_n)_{n\geq 1}$ be a sequence of elements of the semigroup $(G, \cdot)$.

Definition 5. The sequence $(x_n)_{n\geq 1}$ is **convex** if it verifies the relation:

$$x_n^2 | x_{n+1}x_{n-1}, \ \forall\, n \geq 1. \tag{5}$$

Lemma 2. *a) If:*

$$x_n = \prod_{i=1}^{n} y_i^{n-i+1}, \quad n \geq 1 \tag{6}$$

where $(y_n)_{n\geq 1}$ is arbitrary, then the sequence $(x_n)_{n\geq 1}$ is convex.

b) If $(G, \cdot)$ is cancellative, then every convex sequence may be represented by (6) with adequate $(y_n)_{n\geq 1}$.

Proof. a) From (6) we deduce:

$$x_{n+1}x_{n-1} = x_n^2 y_{n+1}$$

that is (7).

b) For $n = 1$, (6) is $x_1 = y_1$, which we consider. Suppose (6) valid for $n \le m$. Then (5) gives an y_{m+1} such that:

$$\prod_{i=1}^{m} y_i^{2(m-i+1)} y_{m+1} = x_{m+1} \prod_{i=1}^{m-1} y_i^{m-i}$$

and by cancellation we get (6) for $n = m + 1$.

Definition 6. A sequence $(x_n)_{n\ge1}$ is called **starshaped** if:

$$x_n^{n+1} | x_{n+1}^n, \ \forall\, n \ge 1. \tag{7}$$

Lemma 3. *a) If the sequence $(x_n)_{n\ge1}$ is starshaped, then it may be represented by:*

$$x_n^{(n-1)!} = z_1^{n!} \prod_{i=2}^{n} z_i^{n!/i(i-1)} \tag{8}$$

with an adequate sequence $(z_n)_{n\ge1}$.

b) If $(G, \cdot)$ has radicals and the sequence $(x_n)_{n\ge1}$ is represented by (8) then it is starshaped.

Proof. a) We take $z_1 = x_1$. Suppose (8) be valid for $m - 1$. Then (7) gives a z_{m+1} such that:

$$x_{m+1}^m = x_m^{m+1} z_{m+1}.$$

So:

$$x_{m+1}^{m!} = z_1^{(m+1)!} \prod_{i=2}^{m} z_i^{(m+1)!/i(i-1)} z_{m+1}^{(m-1)!}$$

that is (8).

b) From (8) we have:

$$x_{n+1}^{n!} = x_n^{(n+1)(n-1)!} z_n^{(n-1)!}$$

and taking the $(n-1)!$-th radical we get (7).

Definition 7. A sequence $(x_n)_{n\geq 1}$ is called **supermultiplicative** if it verifies the relation:

$$x_n x_m | x_{n+m}, \ \forall \ n, m \geq 1. \tag{9}$$

Remark 4. We can consider also the sequences $(x_n)_{n\geq 0}$ but then the relation (7) must be replaced by:

$$x_n^{n+1} | x_{n+1}^n x_0, \ \forall \ n \geq 0 \tag{7'}$$

and (9) by:

$$x_n x_m | x_{n+m} x_0, \ \forall \ n, m \geq 0. \tag{9'}$$

Otherwise we must suppose $x_0 = e$.

Lemma 4. *If the sequence $(x_n)_{n\geq 1}$ is given by:*

$$x_n = \prod_{i=1}^{n} w_i^{[n/i]}, \quad n \geq 1 \tag{10}$$

where $(w_n)_{n\geq 1}$ is an arbitrary sequence and $[x]$ represents the integer part of x, then it is supermultiplicative.

Proof. We can write:

$$x_n x_m = \prod_{i=1}^{n+m} w_i^{[n/i]+[m/i]}$$

because $[n/i] = 0$ for $i > n$. As:

$$[(n+m)/i] \geq [n/i] + [m/i]$$

it follows (9).

Remark 5. In [6] we have stated that for $(N, \cdot)$ the representation (10) is also necessary for supermultiplicity. The problem was also posed in [5] and a result analogous with that from (6) was "proved" in [2]. But as we have remarked in [7], a condition like (10) is not necessary even in the case of the semigroup $(\mathbb{R},+)$. The affirmation is valid in all the cases.

Example 4. For a fixed $p \in G$ and the sequence of integers $(c_n)_{n\geq 1}$ we consider:

$$x_n = p^{\sum\limits_{i=1}^{n} c_i[n/i]}.$$

If $c_i \geq 0,\ \forall\ i$, it follows that it is represented by (10) with:

$$w_i = p^{c_i}.$$

But the sequence $(x_n)_{n\geq 1}$ can be supermultiplicative also for some negative values of c_i and then it cannot be represented by (10). For example we can take $c_1 = c_2 = c_3 = 1 = -c_4$ and then:

$$c_k \geq - \min_{p=1,\dots,n-1} \prod_{k=2}^{n-1} \left(\left[\frac{n}{k}\right] - \left[\frac{p}{k}\right] - \left[\frac{n-p}{k}\right] \right) c_k$$

to get a supermultiplicative sequence.

4. A hierarchy of supermultiplicity of sequences

Let us denote by K, S^* and S the set of convex, starshaped respective supermultiplicative sequences from $(G,\cdot)$. Let also denote by K' and $S^{*'}$ the set of sequences from $(G,\cdot)$ which may be represented by (6) respectively by (8).

Theorem. *a) For every semigroup $(G,\cdot)$ hold the inclusions:*

$$K' \subset S^* \subset S^{*'}. \tag{11}$$

b) If $(G,\cdot)$ preserves the divisibility, then holds also:

$$S^{*'} \subset S. \tag{12}$$

Proof. a) If the sequence $(x_n)_{n\geq 1}$ is in K', it may be represented by (6) and so:

$$x_n^{n-1} = x_{n-1}^n \prod_{k=2}^{n} y_k^{k-1}$$

thus it belongs to S^*. The second inclusion follows from Lemma 3.

b) From (8) we get:

$$x_{m+n}^{(m+n-1)!} = x_m^{(m+n-1)!} \cdot x_n^{(m+n-1)!} \prod_{k=m+1}^{m+n-1} z_k^{m(m+n-1)!/k(k-1)}$$

$$\cdot \prod_{k=n+1}^{n+m-1} z_k^{n(m+n-1)!/k(k-1)} z_{n+m}^{(m+n-2)!}$$

and as the semigroup preserves the divisibility, we deduce that the sequence $(x_n)_{n\geq 1}$ is supermultiplicative, that is we get (12).

Corollary. *If the semigroup $(G, \cdot)$ is cancellative and preserves the divisibility then hold the inclusions:*

$$K \subset S^* \subset S.$$

Proof. Indeed, then $K = K'$ and (11) and (12) are valid.

Remark 6. In the case $(\mathbb{R}, +)$ more results may be found in [7].

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INVARIANT TRANSFORMATIONS OF p, q-CONVEX SEQUENCES

GH. TOADER

1. Introduction

A sequence $a = (a_n)_{n\geq 0}$ is called p, q-convex if:

$$L_{pq}(a_n) = a_{n+2} - (p+q)a_{n+1} + pqa_n \geq 0,\ \forall\ n \geq 0.$$

The set of p, q-convex sequences is denoted by K_{pq}. For $p = q$ we have proved in [4] the following:

1.1. Theorem. *If the sequence $P(a) = (A_n)_{n\geq 0}$ given by:*

$$A_n = (p_0 p^n a_0 + p_1 p^{n-1} a_1 + \cdots + p_n a_n)/(p_0 + \cdots + p_n) \tag{1}$$

is in K_{pp} for every $a = (a_n)_{n\geq 0}$ from K_{pp}, then there is an $u > 0$ such that:

$$p_n = p_0 \binom{u+n-1}{n},\ \forall\ n \geq 0, \tag{2}$$

where:

$$\binom{v}{0} = 1, \quad \binom{v}{n} = \frac{v}{n}\binom{v-1}{n-1}, \quad n \geq 1.$$

Are also known more general transformations of following type: given an infinite triangular positive matrix $P = (p_{nk})_{0 \leq k \leq n}$ and a sequence $a = (a_n)_{n \geq 0}$, one defines the transformed sequence $P(a) = (A_n)_{n \geq 0}$ by:

$$A_n = \sum_{k=0}^{n} p_{nk} a_k. \tag{3}$$

One knows more characterizations of matrices p which give invariant transformation of K_{pq} that is with the property:

$$a \in K_{pq} \Rightarrow P(a) \in K_{pq} \tag{4}$$

(see for example [1], [2] or [3]). But non of them offers concret examples like (2) for $p \neq q$. As we have shown in [4] there is no example similar with (1) in this case. It is the aim of this paper to give an example of invariant transformation of K_{pq} for $p \neq q$ which has a form more general than (1) but more special than (3).

2. Results

We need a notation and some results which are well known or may be found in [4]. Let us denote:

$$K_{pq}^0 = \{w = (w_n)_{n \geq 0} : \ L_{pq}(w_n) = 0, \ \forall \ n \geq 0\}.$$

2.1. Lemma. *The sequence $w = (w_n)_{n \geq 0}$ belongs to K_{pq}^0 for $p \neq q$ if and only if there are the numbers A and B such that:*

$$w_n = Ap^n + Bq^n, \ \forall \ n \geq 0.$$

2.2. Lemma. *If the transformation defined by (3) satisfies (4) then it verifies also:*

$$w \in K^0_{pq} \Rightarrow P(w) \in K^0_{pq}. \tag{5}$$

2.3. Theorem. *For $p \neq q$ there are sequences $(p_k)_{k\geq 0}$ and $(z_n)_{n\geq 0}$ such that the transformation $P(a) = (A_n)_{n\geq 0}$ given by*

$$A_n = \sum_{k=0}^{n} p_k a_{n-k}/z_n, \quad n \geq 0 \tag{6}$$

has the property (5).

Proof. Taking into account Lemma 1, we must find A, B, C, D such that:

$$\sum_{k=0}^{n} p_k p^{n-k}/z_n = Ap^n + Bq^n \tag{7}$$

and

$$\sum_{k=0}^{n} p_k q^{n-k}/z_n = Cp^n + Dq^n. \tag{8}$$

Letting $p_0 = z_0 = 1$, for $n = 0$ and $n = 1$ we find from (7) and (8) that:

$$A = (p + p_1 - qz_1)/z_1(p - q),$$

$$B = (pz_1 - p - p_1)/z_1(p - q),$$

$$C = (q + p_1 - qz_1)/z_1(p - q),$$

$$D = (pz_1 - q - p_1)/z_1(p - q).$$

Subtracting (8) from (7) we get:

$$z_n = \sum_{k=0}^{n-1} p_k(p^{n-k} - q^{n-k})/((A-C)p^n + (B-D)q^n)$$

$$= z_1 \left(p \sum_{k=0}^{n-1} p_k p^{n-1-k} - q \sum_{k=0}^{n-1} p_k q^{n-1-k} \right) / (p^n - q^n)$$

$$= z_1(pz_{n-1}(Ap^{n-1} + Bq^{n-1}) - qz_{n-1}(Cp^{n-1} + Dq^{n-1}))/(p^n - q^n)$$

$$= z_1 z_{n-1}(p^{n-1}(Ap - Cq) + q^{n-1}(Bp - Dq))/(p^n - q^n).$$

Thus:

$$z_n = z_{n-1}((p+q+p_1)(p^{n-1} - q^{n-1}) - pqz_1(p^{n-2} - q^{n-2}))/(p^n - q^n).$$

For $n = 2$ we get:

$$z_2 = z_1(p+q+p_1)/(p+q).$$

If we put:

$$p_1 = r(p+q)$$

it follows:

$$z_2 = z_1(r+1).$$

Then:

$$z_3 = z_2((p^2+q^2)(r+1) + pq(2r+2-z_1))/(p^2+pq+r^2).$$

If we choose:

$$z_1 = r+1$$

we have:

$$z_2 = (r+1)^2, \quad z_3 = (r+1)^3$$

and generally:

$$z_n = z_{n-1}(r+1) = (r+1)^n. \tag{9}$$

From (7) we have:

$$
\begin{aligned}
p_n &= (Ap^n + Bq^n)z_n - p\sum_{k=0}^{n-1} p_k p^{n-1-k} \\
&= (Ap^n + Bq^n)z_n - p(Ap^{n-1} + Bq^{n-1})z_{n-1} \\
&= (1+r)^{n-1}((Ap^n + Bq^n)(r+1) - p(Ap^{n-1} + Bq^{n-1})) \\
&= (r+1)^{n-1}(rAp^n + Bq^{n-1}(q + rq - p))
\end{aligned}
$$

thus

$$p_n = (r+1)^{n-2}(p^n + q^n + r(p^{n+1} - q^{n+1})/(p-q)).$$

So, with $(p_n)_{n\geq 0}$ given by (10) and $(z_n)_{n\geq 0}$ by (9), r arbitrary, the transformation defined by (6) verifies (5).

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LOGARITMICALLY CONVEX SEQUENCES

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1. Introduction

Let $(a_n)_{n\geq 0}$ be a real sequence of strictly positive numbers. It is called convex if:

$$a_{n+1} - 2a_n + a_{n-1} \geq 0,\ \forall\ n \geq 1$$

and logarithmically convex (see [1]) if:

$$a_n^2 \leq a_{n-1}a_{n+1},\ \forall\ n \geq 1.$$

As it is easy to see, the sequence $(a_n)_{n\geq 0}$ is logarithmically convex if and only if $(\log a_n)_{n\geq 0}$ is convex so that it is also called log-convex.

We can remark that the sequence $(a_n)_{n\geq 0}$ is convex if and only if:

$$a_n \leq A(a_{n-1}, a_{n+1}),\ \forall\ n \geq 1$$

and it is log-convex if and only if:

$$a_n \leq G(a_{n-1}, a_{n+1}),\ \forall\ n \geq 1$$

where A and G denote the arithmetic mean respectively the geometric mean.

In what follows we define the convexity in respect to an arbitrary mean, generalizing the above convexities. Also we transpose for this case some results proved in [2] for the usual convexity. By example we define starshapedness and superadditivity and establish some relations between them.

2. Means

We begin by giving some definitions and results related to means. By a mean we understand a function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$ with the property:

$$\min(a, b) \leq M(a, b) \leq \max(a, b),\ \forall\ a, b > 0.$$

For example, the power means are defined by:

$$P_r(a, b) = ((a^r + b^r)/2)^{1/r},\ \text{for } r \neq 0$$

and

$$P_0(a, b) = G(a, b) = (ab)^{1/2}.$$

We have also:

$$P_1 = A \text{ and } P_{-1} = H \quad \text{(the harmonic mean)}.$$

More generally, for a strictly monotonic function $f : \mathbb{R}_+ \to \mathbb{R}_+$ we get a quasi-arithmetic mean (see [1]) defined by:

$$A_f(a, b) = f^{-1}((f(a) + f(b))/2).$$

For $f = f_r$, where $f_r(x) = x^r$ if $r \neq 0$ and $f_0(x) = \log x$, we have $A_{f_r} = P_r$.

Given two means M and N we write $M < N$ if $M(a,b) < N(a,b)$ for $a \neq b$. For example (see again [1]) we have $A_f < A_g$ if and only if f is increasing and fg^{-1} is Jensen concave or f is decreasing and fg^{-1} is Jensen convex. As a consequence we have:

$$P_r < P_s \text{ iff } r < s.$$

3. M-convex sequences

Let M be a mean. We consider the following:

3.1. Definition. The sequence $(a_n)_{n\geq 0}$ is called M-convex if:

$$a_n \leq M(a_{n-1}, a_{n+1}),\ \forall\ n \geq 1.$$

Of course, A-convexity means convexity and G-convexity is log-convexity. Also for $M = \max$ we get the quasi-convexity which we have studied in [4]. For $M = \min$ we obtain only constant sequences.

We denote by KM the set of M-convex sequences. Directly from the definition we get the following result.

3.2. Lemma. *If the means M and N are in the relation $M < N$ then it is valid the inclusion:*

$$KM \subset KN.$$

3.3. Consequence. We have the inclusions:

$$KH \subset KG \subset KA$$

and more generally:

$$KA_f \subset KA_g$$

if f is increasing and fg^{-1} is concave or f is decreasing and fg^{-1} is convex.

Also, as we did in [4] in the case of quasi-convexity, we can define a stronger variant of convexity.

3.4. Definition. The sequence $(a_n)_{n\geq 0}$ is strongly M-convex if:

$$a_n \leq M(a_m, a_p),\ \text{for } 0 < m < n < p.$$

We denote by sKM the set of strongly M-convex sequences. Of course, for every mean M we have:

$$sKM \subset KM$$

and generally the inclusion is proper. For example, the sequence given by:

$$a_n = n,\ n \geq 0$$

belongs to KA but not to sKA. So a convex sequence doesn't satisfy:

$$a_n \leq (a_m + a_p)/2,\ \text{for } 0 < m < n < p \text{ arbitrary}$$

but, as we have proved in [3] it verifies the relations:

$$a_n \leq ((p-n)a_m + (n-m)a_p)/(p-m), \quad 0 < m < n < p.$$

Analogously, a log-convex sequence has the property:

$$a_n^{p-m} \leq a_m^{p-n} a_p^{n-m} \text{ for } 0 < m < n < p.$$

4. Starshapedness and superadditivity

In [2] we have considered together with the set of convex sequences $K = KA$, the set of starshaped sequences:

$$S^* = \{(a_n)_{n\geq 0} : \ (a_n - a_0)/n \leq (a_{n+1} - a_0)/(n+1), \ n \geq 1\}$$

and that of superadditive sequences:

$$S = \{(a_n)_{n\geq 0} : \ a_n + a_m \geq a_{n+m} + a_0, \ \forall \ n, m \geq 1\}.$$

Also we have proved the inclusions:

$$K \subset S^* \subset S. \tag{1}$$

Let f be strictly increasing. We remark that the sequence $(a_n)_{n\geq 0}$ is A_f-convex if:

$$f(a_n) \leq (f(a_{n-1}) + f(a_{n+1}))/2. \tag{2}$$

We can give also the following:

4.1. Definition. The sequence $(a_n)_{n\geq 1}$ is said to be A_f-starshaped if:

$$f(a_n)/n \leq f(a_{n+1})/(n+1), \ \forall \ n \geq 1 \tag{3}$$

and it is called A_f-superadditive if it satisfies:

$$f(a_n) + f(a_m) \leq f(a_{n-m}), \ \forall \ n, m \geq 1. \tag{4}$$

For f decreasing we take the converse inequalities in (2), (3) and (4).

Let us denote by S^*A_f the set of A_f-starshaped sequences and by SA_f the set of A_f-superadditive sequences. From (1) we deduce also the inclusions:

$$KA_f \subset S^*A_f \subset SA_f.$$

For example if $f = \log$ we get the implications:

$$a_n^2 \leq a_{n-1}a_{n+1},\ \forall\ n \geq 1 \Rightarrow a_n^{1/n} \leq a_{n+1}^{1/(n+1)},\ \forall\ n \geq 1$$

$$\Rightarrow a_n a_m \leq a_{n+m},\ \forall\ n, m \geq 1.$$

Given two strictly monotonic functions f and g, we have in Consequence 3.3 conditions for the validity of the inclusion:

$$KA_f \subset KA_g.$$

Similarly we have:

4.2. Lemma. *If f and g are strictly increasing and gf^{-1} is positively subhomogeneous, then:*

$$S^*A_f \subset S^*A_g.$$

Proof. If the sequence $(a_n)_{n\geq 1}$ belong to S^*A_f then:

$$f(a_n) \leq f(a_{n+1})n/(n+1).$$

As gf^{-1} is increasing and positively subhomogeneous, we get:

$$g(a_n) \leq gf^{-1}((n/(n+1))f(a_{n+1})) \leq (n/(n+1))g(a_{n+1})$$

thus $(a_n)_{n\geq 1}$ is also in S^*A_g.

4.3. Lemma. *If f and g are increasing and gf^{-1} is superadditive, then:*

$$SA_f \subset SA_g.$$

Proof. If

$$f(a_{n+m}) \geq f(a_n) + f(a_m)$$

as gf^{-1} is increasing and superadditive, we have:

$$g(a_{n+m}) \geq gf^{-1}(f(a_n) + f(a_m)) \geq g(a_n) + g(a_m).$$

Also we have some variants in the case when one or two of the functions f and g are decreasing.

For example we have the inclusions:

$$\begin{array}{ccccc} KH & \subset & S^*H & \subset & SH \\ \cap & & \cup & & \cup \\ K & \subset & S^* & \subset & S \end{array}$$

but no relation between S^* and S^*G or S^*G and S^*H.

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PROBLEMS AND SOLUTIONS

This section publishes problems and solutions believed to be new and interesting. Problems are designated by P1,P2,..., and solutions by P1S1,P2S2,..., and remarks by P1R1,P2R2,.... Correspondence regarding this section should be sent to the Problems Editor, Professor S.A. Vanstone, Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada, N2L3G1. In case several similar solutions are received, solutions may be edited by credits given to the individual contributors.

P279S2 - Gh. Toader (Cluj-Napoca, Romania)

In answering a problem of T. Popoviciu, we proved in [3] the following result. A sequence of positive integers $(a_n : n \geq 1)$ has the property

$$a_n a_m | a_{n+m}, \quad n, m \geq 1 \tag{1}$$

if and only if there exists a sequence of natural numbers $(b_n : n \geq 1)$ such that

$$a_n = \prod_{i=1}^{n} b_i^{[n/i]}, \quad n \geq 1. \tag{2}$$

We have also proposed that the same problem be studied in an arbitrary semigroup. As we remarked in [4], for the case of the additive semigroup of positive real numbers, a representation similar to (2) is sufficient but not necessary for the corresponding relation (1).

At the 24-th International Symposium on Functional Equations, R. D. Snow raised the same question (see [2]) and a solution was published in [1].

The sequence $(a_n : n \geq 1)$ has the property (1) if and only if it can be represented by

$$a_n = \prod_i p_i^{\sum_j c_j(p_i)[n/j]} \tag{3}$$

where $(p_i : j \geq 1)$ is the sequence of prime numbers, $(c_j : j \geq 1)$ is a sequence of natural numbers and the product over i is finite.

As is easy to verify, the representations (2) and (3) are equivalent. They are sufficient but not necessary for the validity of (1). The last assertion is proved by a simple counter-example. If we take

$$a_1 = 2, \quad a_2 = 2^3, \quad a_3 = 2^5, \quad a_4 = 2^6$$

we can continue to get a sequence which satisfies (1) but in the representation given by (3) we have

$$c_1 = c_2 = c_3 = -c_4 = 1.$$

That is, $c_4 < 0$.

Instead of these inexact results we can prove only the following less effective characterization.

Theorem. *The sequence* $(a_n : n \geq 1)$ *has the property (1) if and only if in its representation*

$$a_n = \prod_i p_i^{q_{in}} \tag{4}$$

the sequences $(q_{in} : n \geq 1)$ *are superadditive for every fixed value of* i.

Remark. The sequence $(q_{in} : n \geq 1)$ can be represented by

$$q_{in} = \sum_{j=1}^{n} c_{ij}[n/j].$$

As a consequence of (4) above, it is superadditive if and only if

$$c_{in} = -\min_{p=1,2,\ldots,[n/2]} \left(\sum_{k=2}^{n-1} ([n/k] - [p/k] - [(n-p)/k])c_{ik} \right).$$

Of course, this last inequality is true if $c_{in} \geq 0$ for all n, i, but this is necessary only for prime values of n.

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ON REPRESENTATION OF A LINEAR OPERATOR ON THE SET OF MEAN-CONVEX SEQUENCES

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1. Introduction

Let S be the set of all real sequences $a = (a_0, a_1, \dots)$, S_1 the set of all convex sequences ($S_1 \subset S$) and S_2 the set of all mean-convex sequences, i.e. $A = (A_0, A_1, \dots) \in S_1$, where

$$A_n = \frac{a_0 + \cdots + a_n}{n+1} \quad (n = 0, 1, \dots).$$

The set S of all sequences becomes a vector space if it is supplied by addition and multiplication by scalars in the usual way. Namely, for $\lambda \in \mathbb{R}$ and arbitrary $x, y \in S$, where $x = (x_0, x_1, \dots)$ and $y = (y_0, y_1, \dots)$ we put $\lambda x = (\lambda x_0, \lambda x_1, \dots)$, $x + y = (x_0 + y_0, x_1 + y_1, \dots)$. Denote by $e_n = (e_{n0}, e_{n1}, \dots)$ where

$$e_{nk} = \begin{cases} 0, & n \neq k \\ 1, & n = k \end{cases}$$

the basic sequences in S. (For example $e_0 = (1, 0, \dots)$, $e_1 = (0, 1, \dots)$). Metric in S is introduced as follows

$$d(x, y) = \sum_{k=0}^{+\infty} 2^{-k} \frac{|x_k - y_k|}{1 + |x_k - y_k|}. \tag{1.1}$$

It is easy to see that every sequence $u = (u_0, u_1, \dots)$ from the metric space (S, d) is the limit of sequences

$$u^{(n)} = \sum_{k=0}^{n} u_k e_k \tag{1.2}$$

in the sense of metric (1.1), i.e. $\lim d(u^{(n)}, u) = 0$.

Therefore, every $u \in S$ can be represented in the form

$$u = \sum_{k=0}^{+\infty} u_k e_k. \tag{1.3}$$

Let $E_0 = \sum_{n=0}^{+\infty} e_n$, $E_1 = \sum_{n=0}^{+\infty} n e_n$ and $W_n = \sum_{k=n}^{+\infty} (2k - n + 1) e_k$ for $n = 2, 3, \dots$ Let L be linear operator defined on S with values in $F(D)$ - the set of all real functions $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}$. We also suppose that L is continuous, i.e. for every $a^{(n)} \to a$ $(n \to +\infty)$, it holds that $L(a^{(n)}) \to L(a)$ $(n \to +\infty)$.

The purpose of this work is to determine the necessary and sufficient conditions for a real sequence $p = (p_0, p_1, \dots)$ such that the inequality

$$\sum_{k=0}^{n} p_k a_k \geq 0, \tag{1.4}$$

holds for all sequences $a = (a_0, a_1, \dots) \in S_2$. Besides, we shall state the necessary and sufficient conditions for linear operator L, defined on S_2, to be positive.

2. Main result

Theorem 1. *Let $p = (p_0, p_1, \dots) \in S$ be given arbitrary. Inequality (1.4) holds for every sequence a from S_2 if and only if the following conditions*

$$\sum_{k=0}^{n} p_k = 0, \tag{2.1}$$

$$\sum_{k=1}^{n} kp_k = 0, \tag{2.2}$$

$$\sum_{j=k}^{n} (2j-k+1)p_j \geq 0 \text{ for } k = 2, 3, \dots, n, \tag{2.3}$$

are fulfilled.

Proof. Suppose that (1.4) holds. The sequences $a = (c, c, \dots)$ and $-a = (-c, -c, \dots)$, $c =$ const., belong to S_2, so necessity of (2.1) follows. Further, $a = (0, 1, \dots)$ and $-a = (0, -1, \dots)$ also belongs to S_2 so, from (1.4) we get that (2.2) is necessary. Finally, the sequences $a = (a_0, a_1, \dots)$ where $a_0 = \dots = a_k = 0$, $a_j = 2j-k+1$, $j = k+1, \dots, n$ and $k = 1, \dots, n$ belong to S_2 so, conditions (2.3) are necessary too.

The sufficiency of the conditions (2.1), (2.2), (2.3) is a consequence of the following identity:

$$\sum_{k=0}^{n} p_k a_k = a_0 \sum_{k=0}^{n} p_k + 2(\Delta A_0) \sum_{k=0}^{n} kp_k$$

$$+ \sum_{k=2}^{n} \left(\sum_{j=k}^{n} (2j-k+1)p_j \right) \Delta^2 A_{k-2}. \quad \square$$

Remark 1. Results, analogous to those explained in Theorem 1, but for different classes of sequences are proved in [1-10].

Theorem 2. *a) Every sequence $a^{(n)} = (a_0^{(n)}, a_1^{(n)}, \dots)$ of the form*

$$a^{(n)} = \alpha^{(n)} E_0 + \beta^{(n)} E_1 + \sum_{k=2}^{n} \gamma_k^{(n)} W_k \quad (n = 0, 1, \dots) \tag{2.4}$$

where $\alpha^{(n)}, \beta^{(n)} \in \mathbb{R}$, $\gamma_k^{(n)} \geq 0$ $(k = 0, 1, \dots)$ for fixed n, is mean-convex.

b) Every sequence $a \in S_2$ is a limit (in d-metric) of sequences $u^{(n)}$ given by (2.4).

c) Let $L : S \to F(D)$ be a continuous linear operator. Then, for every a $(a \in S)$ the implication

$$a \in S_2 \Rightarrow L(a) \geq 0 \tag{2.5}$$

holds, if and only if

$$L(E_0) = L(E_1) = 0 \text{ and } L(W_n) \geq 0 \text{ for } n = 2, 3, \dots \tag{2.6}$$

Proof. a) This assertion follows from (2.4) directly.

b) In virtue of representation

$$a_n = a_0 + 2n(\Delta A_0) + \sum_{k=0}^{n-1} (2n - k - 1)\Delta^2 A_k,$$

(see [11]), we get

$$a = a_0 e_0 + a_1 e_1 + \sum_{n=2}^{+\infty} \left(a_0 + 2n\Delta A_0 + \sum_{k=0}^{n-2} (2n - k - 1)\Delta^2 A_k \right) e_n$$

$$= a_0 \sum_{n=0}^{+\infty} e_n + 2\Delta A_0 \sum_{n=0}^{+\infty} n e_n + \sum_{n=2}^{+\infty} \left(\sum_{k=n}^{+\infty} (2k - n + 1) e_k \right) \Delta^2 A_{n-2}.$$

Using this identity we can get, that every sequence $a \in S_2$, is a limit of sequences $a^{(n)}$, where

$$a^{(n)} = a_0 E_0 + 2E_1 \Delta A_0 + \sum_{k=2}^{n} W_k \Delta^2 A_{k-2}$$

in metric space (S_2, d).

c) Suppose that (2.5) holds. By the fact that the sequences $+E_0$, $-E_0$, $+E_1$, $-E_1$, W_n, for $n = 2, 3, \ldots$, belong to S_2, we get that the conditions (2.6) are necessary. If we suppose that the conditions (2.6) are fulfilled, then

$$\begin{aligned} L(a) &= L\left(\lim_{n\to+\infty} a^{(n)}\right) = \lim_{n\to+\infty} L(a^{(n)}) \\ &= \lim_{n\to+\infty}\left(\alpha^{(n)} L(E_0) + \beta^{(n)} L(E_1) + \sum_{k=2}^{n} \gamma_k^{(n)} L(W_k)\right) \geq 0 \end{aligned}$$

from which we see that the conditions (2.6) are sufficient. $\Box$

Remark 2. Results, analogous to those explained in Theorem 2, but for different classes of sequences are proved in [2], [5].

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A MEASURE OF CONVEXITY OF SEQUENCES

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1. Introduction

In [1] a hierarchy of convexity of functions is proved which we have transposed in [4] for convexity of sequences and in [3] for p, q-convexity of sequences. But this hierarchy is also related to some linear transformations that preserves the convexity. Though there are some characterizations of such transformations (see [2] and [6]) there is no concrete example in the case of p, q-convexity. We shall give here such examples in the case $p = q$. We have generalized the result of [4] in [5] with the help of a measure of convexity. We want to transpose it now to p, p-convexity which we call here only p-convexity. In fact it can be deduced from ordinary convexity by some transformations. But we give here direct proofs.

2. A hierarchy of p-convexity of sequences

For a real sequence $x = (x_i)_{i\geq 0}$ we consider the p-differences (of order two) defined by:

$$c_{pi}(x) = x_{i+2} - 2px_{i+1} + p^2 x_i.$$

The sequence x is called p-convex if $c_{pi}(x) \geq 0, \ \forall \ i \geq 0$. This is a generalization of the convexity which corresponds to $p = 1$. In [3] we have also defined generalizations of starshapedness and of superadditivity: the sequence x is said to be p-starshaped if:

$$d_{pi}(x) = (x_{i+2}/p^{i+2} - x_0)/(i+2) - (x_{i+1}/p^{i+1} - x_0)/(i+1) \geq 0, \ \forall \ i \geq 0$$

or p-superadditive if:

$$a_{pij}(x) = x_{i+j} - p^i x_j - p^j x_i + p^{i+j} x_0 \geq 0, \ \forall \ i, j \geq 0.$$

We shall denote by K_p, S_p^* and S_p the sets of p-convex, p-starshaped respectively p-supperadditive sequences. Let us consider also the set of weakly p-superadditive sequences:

$$W_p = \{x; \ a_{pi1}(x) \geq 0, \ \forall \ i \geq 0\}.$$

The first form of the hierarchy of p-convexity is represented by the following chain of inclusions:

$$K_p \subset S_p^* \subset S_p \subset W_p. \tag{1}$$

We don't prove it now because we shall give stronger results in what follows.

3. A measure of p-convexity

As we have done in [5] for the case $p = 1$, we define the following measures:

(a) of p-convexity, by:

$$k_{pn}(x) = \min\{(c_{pi}x)/p^{i+2}, \ 0 \leq i \leq n-2\}$$

(b) of p-starshapedness, by:

$$s_{pn}^*(x) = \min\{2d_{pi}(x), \ 0 \leq i \leq n-2\}$$

(c) of p-superadditivity, by:

$$s_{pn}(x) = \min\{a_{pij}(x)/ijp^{i+j},\ 0 < i, j, i+j \leq n\}$$

(d) of weakly p-superadditivity, by:

$$w_{pn}(x) = \min\{a_{pi1}(x)/ip^{i+1},\ 0 < i < n\}.$$

Lemma 1. *(a) If the sequence x is represented by:*

$$x_i = \sum_{j=0}^{i} (i-j+1)p^{i-j}b_j \tag{2}$$

then:

$$k_{pn}(x) = \min\{b_i/p^i,\ 2 \leq i \leq n\}.$$

(b) If x is given by:

$$x_i = ip^i \sum_{j=1}^{i} b_j - (i-1)p^i b_0$$

then:

$$s^*_{pn}(x) = \min\{2b_i,\ 2 \leq i \leq n\}.$$

(c) If x is given by:

$$x_i = \sum_{j=2}^{i} p^{i-j}b_j + ip^{i-1}b_1 - (i-1)p^i b_0 \tag{4}$$

then:

$$w_{pn}(x) = \min\{b_{i+1}/ip^{i+1},\ 1 \leq i < n\}.$$

Proof. From (2) we have:

$$c_{pi}(x) = b_{i+2}$$

from (3):

$$d_{pi}(x) = b_{i+2}$$

and from (4) also:

$$a_{pi1}(x) = b_{i+1}.$$

Lemma 2. *For every sequence x, every $p > 0$ and $n \geq 2$, we have:*

$$k_{pn}(x) \leq s^*_{pn}(x) \leq s_{pn}(x) \leq w_{pn}(x). \tag{5}$$

Proof. Every sequence x may be represented by (2) and so, for $i \leq n-2$:

$$d_{pi}(x) = \frac{1}{(i+1)(i+2)} \sum_{j=2}^{i+2} (j-1) \frac{b_j}{p^j} \geq \frac{k_{pn}(x)}{(i+1)(i+2)} \sum_{j=2}^{i+2} (j-1)$$

which gives the first part of (5). But the sequence x may be also represented by (3) and so:

$$a_{pij}(x) = p^{i+j} \left[i \sum_{k=i+1}^{i+j} b_k + j \sum_{k=j+1}^{j+i} b_k \right] \geq p^{i+j} ij s^*_{pn}(x)$$

which gives the second inequality from (5). The last one is obvious.

Remark 1. The defined measures permit the consideration of the following classes of sequences:

$$K_{pan} = \{x;\ k_{pn}(x) \geq a\}$$

$$S^*_{pan} = \{x;\ s^*_{pn}(x) \geq a\}$$

$$S_{pan} = \{x;\ s_{pn}(x) \geq a\}$$

$$W_{pan} = \{x;\ w_{pn}(x) \geq a\}.$$

If the corresponding conditions are fulfilled for any n we renounce at this index getting the sets: K_{pa}, S^*_{pa}, S_{pa} and W_{pa}. For $a = 0$ we find also the sets from (1). But from Lemma 2 we have the following generalization of (1).

Theorem 1. *For every $p > 0$, $n \geq 2$ and a real, hold the following inclusions:*

$$K_{pan} \subset S^*_{pan} \subset S_{pan} \subset W_{pan}. \tag{6}$$

Remark 2. Let us consider also the following classes of sequences:

$$K_p^0 = \{x;\ c_{pi}(x) = 0,\ \forall\ i \geq 0\}$$

$$S_p^{*0} = \{x;\ d_{pi}(x) = 0,\ \forall\ i \geq 0\}$$

$$S_p^0 = \{x;\ a_{pij}(x) = 0,\ \forall\ i, j \geq 0\}$$

$$W_p^0 = \{x;\ a_{pi1}(x) = 0,\ \forall\ i \geq 0\}$$

$$Z_p = \{x;\ \exists\ a, b \in \mathbb{R},\ x_i = p^i(ai + b),\ \forall\ i \geq 0\}.$$

From Lemma 1 we deduce that $K_p^0 = S_p^{*0} = Z_p$. Also $Z_p \subset S_p^0 \subset W_p^0$ and from:

$$c_{pi}(x) = a_{i,i+1,1}(x) - pa_{p,i,1}(x)$$

we deduce $W_p^0 \subset K_p^0$, thus:

$$K_p^0 = S_p^{*0} = S_p^0 = W_p^0 = Z_p.$$

4. Invariant transformations

In [5] are indicated all the weight sequences $a = (a_i)_{i \geq 0}$ which define a transformation T_a of sequence by $T_a(x) = (X_i)_{i \geq 0}$, where:

$$X_i = (a_0 x_0 + \cdots + a_i x_i)/(a_0 + \cdots + a_i) \tag{7}$$

with the property that it preserves the classes K_1, S_1^*, S_1 or W_1. In [2] and [6] one can found characterization of such transformations (even of more general type) which preserves the p-convexity, but no example is known. One reason may be that there is no transformation of type (7).

A more general transformation may be given by a triangular matrix $A = (a_{ij})_{0 \le j \le i}$ putting $T_A(x) = (X_i)_{i \ge 0}$ where:

$$X_i = a_{i0}x_0 + \cdots + a_{ii}x_i.$$

Lemma 3. *If the transformation T_A preserves one of the sets K_p, S_p^*, S_p or W_p then it preserves also the set Z_p.*

Proof. If, for example, T_A preserves K_p, then for every $x \in Z_p \subset K_p$ we have:

$$c_{pi}(T_A(\pm x)) = \pm c_{pi}(T_A(x)) \ge 0, \ \forall \ i \ge 0$$

that is $T_A(x) \in K_p^0 = Z_p$.

Lemma 4. *If the transformation T_a given by $T_a(x) = (X_i)_{i \ge 0}$, where:*

$$X_i = (a_i x_0 + a_{i-1}x_1 + \cdots + a_0 x_i)/(i+1)$$

preserves the set Z_p then:

$$a_i = a_0 p^i.$$

Proof. If $T_a(Z_p) \subset Z_p$, there are the real numbers A, B, C and D such that:

$$ia_0p^i + \cdots + a_{i-1}p = (i+1)p^i(Ai+B) \tag{8}$$

and

$$a_0p^i + \cdots + a_{i-1}p + a_i = (i+1)p^i(Ci+D). \tag{9}$$

For $i = 0$ and $i = 1$ it follows:

$$A = a_0/2, \quad B = 0, \quad C = (a_1 - a_0p)/2p, \quad D = a_0$$

and for $i = 2$:

$$a_1 = a_0p, \quad a_2 = a_0p^2.$$

Subtracting (8) from (9) we get:

$$a_i = a_0 p^i (i-1) + a_1 p^{i-1}(i-2) + \cdots + a_{i-2}p^2 - a_0(i+1)(i-2)p^i/2$$

which gives, by mathematical induction:

$$a_i = a_0 p^i.$$

This result suggests to consider a more general case.

Theorem 2. *If the transformation T_a given by $T_a(x) = (X_i)_{i\geq 0}$, with:*

$$X_i = (a_0 p^i x_0 + a_1 p^{i-1} x_1 + \cdots + a_i x_i)/(a_0 + \cdots + a_i) \tag{10}$$

preserves the set Z_p, then there is a $v > 0$ such that:

$$a_i = a_0 \binom{v+i-1}{i}, \ \forall\ i \geq 0 \tag{11}$$

where

$$\binom{v}{0} = 1, \quad \binom{v}{i} = \frac{v}{i}\binom{v-1}{i-1}, \quad i \geq 1.$$

Proof. We must find the numbers A and B such that:

$$(ia_i + (i-1)a_{i-1} + \cdots + a_1)p^i = (a_0 + \cdots + a_i)(Ai + B)p^i. \tag{12}$$

For $i = 0$ we have $B = 0$ and for $i = 1$ we get also $A = a_1/(a_0 + a_1)$. Then $i = 2$ gives:

$$a_2 = a_1(a_0 + a_1)/2a_0$$

and putting $a_1 = va_0$ we have (11) for $i \leq 2$. From (12) we deduce:

$$a_i = \sum_{k=0}^{i-1}(iv - k(v+1))a_k/i$$

which gives relation (11) for every i. For this one used the mathematical induction and the relation:

$$\sum_{j=0}^{i} \binom{v+j}{j} = \binom{v+i+1}{i}.$$

Remark 3. Taking in (10) a_i as given by (11), it becomes:

$$X_i = X_i^v = \sum_{j=0}^{i} \binom{v+j-1}{j} p^{i-j} x_j \Big/ \binom{v+i}{i}. \tag{13}$$

Writing $X^v = (X_i^v)_{i\geq 0} = A_v(x)$ we can consider the following measures (in v-mean) of sequences:

$$k_{pn}^v(x) = k_{pn}(X^v), \quad s_{pn}^{*v}(x) = s_{pn}^*(X^v),$$

$$s_{pn}^v(x) = s_{pn}(X^v), \quad w_{pn}^v(x) = w_{pn}(X^v).$$

Theorem 3. *For any sequence $x = (x_i)_{i\geq 0}$ and any $0 < v < u$ we have the following relations:*

$$k_{pn}(x) \leq (1+2/u)k_{pn}^u(x) \leq (1+2/v)k_{pn}^v(x) \leq s_{pn}^*(x)/p^2 \tag{14}$$

$$s_{pn}^*(x) \leq (1+2/u)s_{pn}^{*u}(x) \leq (1+2/v)s_{pn}^{*v}(x) \tag{15}$$

and

$$w_{pn}(x) \leq (1+2/u)w_{pn}^u(x) \leq (1+2/v)w_{pn}^v(x). \tag{16}$$

Proof. (i) Let x be given by (2) and X^u by:

$$X_i^u = \sum_{j=0}^{i} (i-j+1)p^{i-j} b_j^u, \quad i \geq 0. \tag{17}$$

Then from (13) we have also:

$$x_i = \frac{u+i}{u} X_i^u - p\frac{i}{u} X_{i-1}^u = \sum_{j=0}^{i} (i(i+1/u) - j + 1) p^{i-j} b_j^u \tag{18}$$

and so:

$$c_{pi}(x) = b_{i+2} = (1 + (i+2)/u) b_{i+2}^u - ipb_{i+1}^u/u. \tag{19}$$

This gives, step by step:

$$\frac{b_i^u}{p^i} = \frac{u}{u+i}\frac{b_i}{i} + u\sum_{j=2}^{i-1} \frac{(i-2)\dots(j-1)}{(u+i)\dots(u+j)} \frac{b_j}{p^j}$$

thus, for $i \le n$:

$$\frac{b_i^u}{p^{i+2}} \ge \left(\frac{u}{u+i} + u\sum_{j=2}^{i-1} \frac{(i-2)\dots(j-1)}{(u+i)\dots(u+j)} \right) k_{pn}(x)$$

$$= \left(\frac{u}{u+i} + \frac{u(i-2)!}{(u+i)\dots(u+2)} \sum_{j=2}^{i-1} \binom{u+j-1}{j-2} \right) k_{pn}(x) = \frac{u}{u+2} k_{pn}(x)$$

and hence, by Lemma 1, we have the first inequality from (14).

(ii) Taking (17) for v and u, (19) gives:

$$\left(1 + \frac{i+2}{u}\right) b_{i+2}^u - \frac{ip}{u} b_{i+1}^u = \left(1 + \frac{i+2}{v}\right) b_{i+2}^v - \frac{ip}{v} b_{i+1}^v$$

and so, by mathematical induction:

$$\frac{b_{i+2}^v}{p^{i+2}} = \frac{v(u+i+2)}{u(v+i+2)} \frac{b_{i+2}^u}{p^{i+2}} + (u-v)\frac{v}{u} \sum_{j=2}^{i+1} \frac{i\dots(j-1)}{(v+i+2)\dots(v+j)} \frac{b_j^u}{p^j}.$$

Hence, for $i \le n-2$:

$$\frac{b_{i+2}^v}{p^{i+4}} \ge \frac{v}{u} \left(\frac{u+i+2}{v+i+2} + \frac{(u-v)i!}{(v+i+2)\dots(v+2)} \sum_{j=2}^{i+1} \binom{v+j-1}{j-2} \right) k_{pn}^u(x)$$

$$= \frac{v(u+2)}{u(v+2)} k_{pn}^u(x)$$

thus obtaining the second inequality from (14).

(iii) Taking v instead of u in (18), we have for $i \leq n$:

$$d_{pi}(x) = (1/v)(b^v_{i+2}/p^{i+2}) + \left(\sum_{j=2}^{i+2}(j-1)b^v_j/p^j\right)/((i+1)(i+2)).$$

Hence:

$$d_{pi}(x)/p^2 \geq \left(1/v + \sum_{j=2}^{i+2}(j-1)/((i+1)(i+2))\right) k^v_{pn}(x)$$

$$= (1/v + 1/2)k^v_{pn}(x)$$

that is the last inequality from (14).

(iv) If x is given by (3), then (13) gives:

$$X^u_i/p^i = \frac{ui}{u+1}\sum_{j=1}^{i} b_j - \left(u/\binom{u+i}{i}\right)\sum_{j=2}^{i}\binom{u+j-1}{j-2}b_j - b_0\left(\frac{ui}{u+1}-1\right)$$

thus:

$$d_{pi}(X^u) = \frac{ub_{i+2}}{u+i+2} + \frac{u}{(u+2)\binom{u+i+2}{i}}\sum_{j=2}^{i+1}\binom{u+j-1}{j-2}b_j$$

and so, for $i \leq n-2$:

$$2d_{pi}(X^u) \geq \left(\frac{u}{u+i+2} + \frac{u}{(u+2)\binom{u+i+2}{i}}\sum_{k=0}^{i-1}\binom{u+k+1}{k}\right)s^*_{pn}(x)$$

$$= \frac{u}{u+2}s^*_{pn}(x)$$

which gives the first inequality from (15).

(v) Let X^u be given as in (3) by:

$$X^u_i = ip^i\sum_{j=1}^{i} b^u_j - (i-1)p^i b^u_0. \tag{20}$$

Then as in (18) we have:

$$x_i/p^i = i(1+i/u)b_i^u + i(1+1/u)\sum_{j=1}^{i-1} b_j^u + (1-i(1+1/u))b_0^u$$

and so:

$$c_{pi}(x)/p^{i+2} = (i+2)(1+(i+2)/u)b_{i+2}^u - i(1+(2i+3)/u)b_{i+1}^u + i(i-1)b_i^u/u.$$

Taking it for $0 < v < u$, we get:

$$(i+2)(1+(i+2)/u)b_{i+2}^u - i(1+(2i+3)/u)b_{i+1}^u + i(i-1)b_i^u/u$$

$$= (i+2)(1+(i+2)/v)b_{i+2}^v - i(1+(2i+3)/v)b_{i+1}^v + i(i-1)b_i^v/v$$

thus, by mathematical induction:

$$b_{i+2}^v = \frac{v(u+i+2)}{u(v+i+2)}b_{i+2}^u + (u-v)\frac{v}{u}\sum_{j=2}^{i+1}\frac{i\dots(j-1)}{(v+i+2)\dots(v+j)}b_j^u$$

which gives as in (ii) the second inequality from (15).

(vi) If x is given by (4) and X^u by:

$$x_i^u = \sum_{j=2}^{i} p^{i-j}b_j^u + ip^{i-1}b_1^u - (i-1)b_0^u p^i$$

we have as in (18):

$$x_i = \left(1+\frac{i}{u}\right)b_i^u + \sum_{j=2}^{i-1} b_j^u p^{i-j} + p^{i-1}i\left(1+\frac{1}{u}\right)b_1^u + \left(1-i\left(1+\frac{1}{u}\right)\right)b_0^u p^i$$

and so, for $i > 1$:

(21) $$a_{pil}(x) = b_{i+1} = ((1+(i+1)/u)b_{i+1}^u - (pi/u)b_i^u$$

and

$$a_{p11}(x) = b_2 = (1+2/u)b_2^u.$$

By mathematical induction it follows that:

$$b_i^u = \frac{u}{u+i} b_i + u \sum_{j=2}^{i-1} \frac{(i-1)\dots j p^{i-j}}{(u+i)\dots(u+j)} b_j$$

thus, using Lemma 1:

$$\frac{b_i^u}{(i-1)p^i} = \frac{u}{u+i} \frac{b_i}{(i-1)p^i} + u \sum_{j=2}^{i-1} \frac{(i-2)\dots(j-1)}{(u+i)\dots(u+j)} \frac{b_j}{(j-1)p^i}$$

$$\geq \left(\frac{u}{u+i} + \frac{u(i-2)!}{(u+i)\dots(u+2)} \sum_{j=2}^{i-1} \binom{u+j-1}{j-2} \right) w_{pn}(x) = \frac{u}{u+2} w_{p^n}(x)$$

which gives the first inequality from (16).

(vii) Taking (20) for u and v, we have from (21):

$$(1+(i+1)/u) b_{i+1}^u - (pi/u) b_i^u = (1+(i+1)/v) b_{i+1}^v - (pi/v) b_i^v, \quad i \geq 2$$

and

$$b_2^v = \frac{v(u+2)}{u(v+2)} b_2^u.$$

So, step by step, we get for $i \leq n$:

$$b_i^v = \frac{v(u+i)}{u(v+i)} b_i^u + \frac{v(u-v)}{u} \sum_{j=2}^{i-1} \frac{(i-1)\dots j p^{i-j}}{(v+i)\dots(v+j)} b_j^u$$

or, using again Lemma 1:

$$\frac{b_i^v}{(i-1)p^i} = \frac{v(u+i)}{u(v+i)} \frac{b_i^u}{(i-1)p^i} + \frac{v(u-v)}{u} \sum_{j=2}^{i-1} \frac{(i-2)\dots(j-1)}{(v+i)\dots(v+j)} \frac{b_j^u}{(j-1)p^j}$$

$$\geq \left(\frac{v(u+i)}{u(v+i)} + \frac{v(u-v)(i-2)!}{u(v+i)\dots(v+2)} \sum_{j=2}^{i-1} \binom{v+j-1}{j-2} \right) w_{pn}^u(x)$$

$$= \frac{v(u+2)}{u(v+2)} w_{pn}^u(x)$$

getting the last inequality from (16).

Remark 4. Let us denote by $M^v K_{pan}$, $M^v S^*_{pan}$, $M^v S_{pan}$ and $M^v W_{pan}$ the sets of sequences x with the property that the sequence X^v given by (13) belongs to K_{pan}, S^*_{pan}, S_{pan}, W_{pan}, respectively. For $a = 0$ and n unbounded, we denote them by $M^v K_p$, $M^v S^*_p$, $M^v S_p$, $M^v W_p$, respectively. From Theorem 3 we have the following:

Corollary 1. *For every $p > 0$, $0 < v < u$, $n \geq 2$ and a real, we have the following inclusions:*

$$\begin{array}{lllll}
K_{p^{an}} \subset M^u K_{p,af(u),n} \subset & M^v K_{p,a(v),n} & & & \\
 & \cap & & & \\
 & S^*_{p,ap^2,n} & \subset M^u S^*_{p,ap^2 f(u),n} & \subset M^v S^*_{p,ap^2 f(v),n} \\
 & \cap & \cap & \cap \\
 & S_{p,ap^2,n} & M^u S_{p,ap^2 f(u),n} & M^v S_{p,ap^2 f(v),n} \\
 & \cap & \cap & \cap \\
 & W_{p,ap^2,n} & \subset M^u W_{p,ap^2 f(u),n} & \subset M^v W_{p,ap^2 f(v),n}
\end{array}$$

where $f(n) = n/(n+2)$.

Corollary 2. *For every $p > 0$ and $0 < v < u$ we have the inclusions:*

$$\begin{array}{llll}
K_p \subset M^u K_p \subset M^v K_p \subset & S^*_p & \subset M^u S^*_p & \subset M^v S^*_p \\
 & \cap & \cap & \cap \\
 & S_p & M^u S_p & M^v S_p \\
 & \cap & \cap & \cap \\
 & W_p & \subset M^u W_p & \subset M^v W_p.
\end{array}$$

Remark 5. Among these sets other inclusions may also exist. For example in [4] it is proved that for $p = 1$ and $u = 1$ (which corresponds to the arithmetic mean):

$$K_1 \subset M^1 K_1 \subset S^*_1 \subset S_1 \subset M^1 S^*_1 \subset M^1 S_1.$$

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ALEXANDER'S OPERATOR FOR SEQUENCES

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Abstract. In this paper we define two weak types of starshaped sequences. One of them shows a close connection between starshaped and superadditive sequences, while the other one is used for the determination of linear operators which conserve some sequence classes. We obtain so a discrete operator of Alexander type.

1. Introduction

Sequences with some special properties can occur in many unexpected branches. For example, if the positive sequence $(a_n)_{n\geq 1}$ has the property:

$$(k+1)a_{k+1} \leq ka_k, \ \forall \ k \geq 1,$$

then the complex function f defined by $f(z) = z + a_2z^2 + \dots \ (a_1 = 1)$ is close-to-convex and a similar condition implies that f is a starlike function (see [6]). Also convex, quasiconvex and other sequences are used in the theory of Fourier series (see [3] for many references), giving conditions for summability.

In this paper we deal with more classes of sequences. The following sets are well known (see for example [4]): the set of convex sequences:

$$K = \{(x_n)_{n\geq 0} : \ x_{n+2} - 2x_{n+1} + x_n \geq 0, \ \forall \ n \geq 0\}$$

and also that of superadditive sequences:

$$S = \{(x_n)_{n\geq 0} : \ x_{n+m} + x_0 \geq x_n + x_m, \ \forall \ n, m \geq 1\}.$$

In [7] we have considered the set of starshaped sequences:

$$S^* = \{(x_n)_{n\geq 0} : \ (x_n - x_0)/n \leq (x_{n+1} - x_0)/(n+1), \ \forall \ n \geq 1\}$$

proving also that:

$$K \subset S^* \subset S. \tag{1}$$

Then we have used in [9] a weaker form of superadditivity introducing the set:

$$W = \{(x_n)_{n\geq 0} : \ x_{n+1} + x_0 \geq x_n + x_1, \ \forall \ n \geq 1\}.$$

Here we define also two weaker kinds of starshapedness and establish their relations with the previous notions.

In [10] there are characterized the weighted arithmetic means that preserve the convexity. We have obtained a simpler characterization in [8] and then we have proved that it is also valid for the preservation of the starshapedness or the superadditivity (see [9]). In what follows we want to determinate all the linear positive operators of another special type which conserve one of the above properties. Thus we get a discrete operator which resembles Alexander's integral operator used in the theory of complex functions (see for example [5]).

2. Weakly starshaped sequences

Define the following two sets of sequences:

$$J^* = \{(x_n)_{n\geq 0} : \; x_{nm} - x_0 \geq n(x_m - x_0), \; \forall \; n, m \geq 1\}$$

and

$$V^* = \{(x_n)_{n\geq 0} : \; x_n - x_0 \geq n(x_1 - x_0), \; \forall \; n \geq 1\}.$$

The first of them can be considered as a Jensen starshapedness and the second as a very weak kind of starshapedness. We have obviously:

$$S^* \subset J^* \subset V^*$$

but we want to combine it with (1). Before doing this we add also the set of linear (or zero) sequences:

$$Z = \{(x_n)_{n\geq 0} : \; \exists \; a, b \in \mathbb{R}, \; x_n = an + b, \; \forall \; n \geq 0\}.$$

Lemma 1. *The following inclusions:*

$$\begin{array}{ccccccccc} Z & \subset & K & \subset & S^* & \subset & S & \subset & W \\ & & & & & & \cap & & \cap \\ & & & & & & J^* & \subset & V^* \end{array} \tag{2}$$

hold.

Proof. The inclusions $S \subset J^*$ and $W \subset V^*$ can be proved by mathematical induction. The other relations are in (1) or are obvious.

Remark 1. The inclusions:

$$S^* \subset S \subset J^*$$

show a close connection between starshapedness and superadditivity. In fact a superadditivity sequence verifies even a stronger inequality than that used in the definition of J^*, namely:

$$x_n - x_0 \geq [n/m](x_m - x_0), \quad n \geq m,$$

where $[x]$ denotes the integer part of x.

We have given in [7] a representation formula for sequences from K: a sequence $(x_n)_{n\geq 0}$ belongs to K is and only if

$$x_n = \sum_{k=0}^{n}(n-k+1)y_k, \text{ with } y_k \geq 0 \text{ for } k \geq 2.$$

Also we have used representation formulas for sequences from S^* and in [9] for those from W. We add here such formulas for sequences from S, J^* and V^*. Each of them is easy to verify.

Lemma 2. *Every sequence $(x_n)_{n\geq 0}$ can be represented by:*

$$x_n = n\sum_{k=1}^{n} z_k - (n-1)z_0, \qquad \textit{for } n \geq 0. \tag{3}$$

It belongs to:

i) S^* if and only if

$$z_k \geq 0 \textit{ for } k \geq 2; \tag{4}$$

ii) S if and only if

$$n\sum_{k=n+1}^{n+m} z_k + m\sum_{k=m+1}^{m+n} z_k \geq 0, \textit{ for } n, m \geq 1;$$

iii) J^* if and only if

$$\sum_{k=n+1}^{nm} z_k \geq 0, \textit{ for } n \geq 1,\ m \geq 2; \tag{5}$$

iv) W if and only if

$$nz_{n+1} + \sum_{k=2}^{n+1} z_k \geq 0, \ \textit{for } n \geq 1;$$

v) V^* if and only if

$$\sum_{k=n+1}^{2n} z_k \geq 0, \ \textit{for } n \geq 1.$$

Remark 2. For sequences from W and V^* we have also simpler representations:

$$x_n = \sum_{k=2}^{n} w_k + nx_1 - (n-1)x_0, \ \text{with } w_k \geq 0, \ \text{for } k \geq 2, \tag{6}$$

respectively:

$$x_n = v_n + nx_1 - (n-1)x_0, \ \text{with } v_n \geq 0, \ \text{for } n \geq 2, \tag{7}$$

both valid for $n \geq 2$.

3. Linear operators

Let $Q = (q_{nm})_{0 \leq m \leq n}$ be a strictly positive triangular matrix. For a sequence $x = (x_n)_{n \geq 0}$ consider the associated sequence $L^Q(x)$ defined by:

$$L_n^Q(x) = \sum_{k=0}^{n} q_{nk} x_k, \ \forall \ n \geq 0.$$

We get so a linear operator L^Q defined on the space of all real sequences with values in the same space. It is also isotonic, that is $L^Q(x)$ is positive if x is positive. Given a set X of sequences, an usual problem is to characterize the matrices Q with the property that X is invariant under L^Q, that is $L^Q(X) \subset X$. We have such characterizations for the set K of convex sequences (see [1] and [2]). We have also the following general result:

Lemma 3. *If one of the sets K, S^*, S, W, J^* or V^* is invariant under L^Q, then Z is also invariant with respect to it.*

Proof. Let x be an arbitrary sequence from Z. If the set

$$X \in \{K, S^*, S, W, J^*, V^*\}$$

is invariant under L^Q, as $x \in X$, we have $L^Q(x) \in X$. By (2) we get $L^Q(x) \in V^*$. But $-x$ also belongs to Z, which gives $L^Q(x) \in Z$.

In what follows we want to give explicitly the matrices Q with the property that Z is invariant under L^Q, supposing Q of some special types. We begin with the case of weighted arithmetic means studied in [8] and [10]. Let $p = (p_n)_{n\geq 0}$ be a strictly positive sequence. For any sequence $x = (x_n)_{n\geq 0}$ we define the sequence $A^p(x)$ of weighted arithmetic means of x by:

$$A_n^p(x) = \frac{\sum_{k=0}^{n} p_k x_k}{\sum_{k=0}^{n} p_k}, \ \forall\, n \geq 0. \tag{8}$$

We note that one can define a matrix Q using p by:

$$q_{nm} = \frac{p_m}{\sum_{k=0}^{n} p_k}, \quad 0 \leq m \leq n.$$

Lemma 4. *The inclusion:*

$$A^p(Z) \subset Z \tag{9}$$

is valid if and only if there is an $u \geq 0$ *such that:*

$$p_n = p_0 \binom{u+n-1}{n}, \ \forall\, n \geq 0, \tag{10}$$

where

$$\binom{v}{0} = 1, \quad \binom{v}{n} = \frac{v(v-1)\dots(v-n+1)}{n!}, \quad n \geq 1.$$

Proof. If (9) holds, we must have $a, b \in \mathbb{R}$ such that:

$$\frac{\sum_{k=0}^{n} k p_k}{\sum_{k=0}^{n} p_k} = an + b, \ \forall\, n \geq 0. \tag{11}$$

For $n = 0$ we get $b = 0$ and $n = 1$ gives $a = p_1/(p_0 + p_1)$ so that (11) becomes:

$$\sum_{k=0}^{n} k p_k = \frac{n p_1 \sum_{k=0}^{n} p_k}{p_0 + p_1}, \ \forall\, n \geq 0. \tag{12}$$

Thus, by subtraction, we have:

$$(n+1) p_{n+1} = (p_1/p_0) \sum_{k=0}^{n} p_k.$$

Denoting $p_1/p_0 = u$, we obtain, again by subtraction:

$$p_{n+1} = \frac{p_n(u+n)}{n+1}$$

which gives (10).

Conversely, if p_n is given by (10) then (9) is valid, because (12) means $A^p(z) = (u/(u+1))z$, where $z = (n)_{n \geq 0}$.

The second case which we study is obtained by putting $q_{nk} = p_k$. Thus we have the linear operator B^p defined by:

$$B^p_n(x) = \sum_{k=0}^{n} p_k x_k, \ \forall \ n \geq 0.$$

We denote by:

$$Z_0 = \{(x_n)_{n\geq 0} : \ \exists \ a, x_n = an, \ \forall \ n \geq 0\}.$$

Lemma 5. *i) There is no sequence p with property:*

$$B^p(Z) \subset Z; \tag{13}$$

ii) The operator B^p satisfies the inclusion:

$$B^p(Z_0) \subset Z_0$$

if and only is

$$p_n = p_1/n, \ \forall \ n \geq 1.$$

Proof. To obtain (13) it is necessary and sufficient that for arbitrary a and b there exist constants c, d, e and f such that:

$$\sum_{k=0}^{n} akp_k = cn + d, \quad \sum_{k=0}^{n} bp_k = en + f, \quad n \geq 0.$$

For $n = 0$ we have $d = 0$, $bp_0 = f$ and for $n \geq 1$: $anp_n = c$, $bp_n = e$. Since $b \neq 0$ leads to a contradiction, we must have $b = 0$, $e = 0$, $c = ap_1$ and $p_n = p_1/n$.

Remark 3. Taking $p_1 = 1$, we get an operator which we denote simply by B, thus:

$$B_n(x) = \sum_{k=1}^{n} x_k/x, \ \forall \ n \geq 1. \tag{14}$$

As we pointed out in the introduction, this operator resembles Alexander's integral operator.

4. A hierarchy of starshapedness

In what follows we want to investigate the sufficiency of the previous conditions. First, we denote:

$$M^u T = \{x : A^p(x) \in T\},$$

where T is an arbitrary set of sequences and A^p is given by (8) with p taken as in (10). We have proved in [8] and [9] that:

$$\begin{array}{ccccccccc} K & \subset & M^u K & \subset & S^* & \subset & S & \subset & W \\ & & & & \cap & & & & \cap \\ & & & & M^u S^* & \subset & M^u S & \subset & M^u W \end{array} \tag{15}$$

that is, the condition is sufficient for the sets K, S^* and W. We try to extend this result by taking into account (2). But, as in the case of the set S, we are not able to prove the inclusion $J^* \subset M^u J^*$ because the representation given by (3) and (5) for the sequences of J^* is too complicated.

Lemma 6. *For every $u \geq 0$ the inclusion:*

$$V^* \subset M^u V^*$$

is valid.

Proof. Let $x = (x_n)_{n\geq 0}$ be an arbitrary sequence of V^*. It may be represented as in (7) by:

$$x_n = v_n + nx_1 - (n-1)x_0$$

with $v_0 = v_1 = 0$ and $v_n \geq 0$ for $n \geq 2$. So:

$$
\begin{aligned}
A_n^u(x) &= \sum_{k=0}^{n} \frac{\binom{u+k-1}{k} x_k}{\binom{u+n}{n}} \\
&= \sum_{k=0}^{n} \frac{\binom{u+k-1}{n} v_k}{\binom{u+n}{n}} + (x_1 - x_0)nu/(u+1) + x_0 \\
&= w_n + nA_1^u(x) - (n-1)A_0^u(x),
\end{aligned}
$$

where $w_0 = w_1 = 0$ and $w_n \geq 0$ for $n \geq 2$, that is $A^u(x) \in V^*$.

To use the operator B given by (14), taking into account Lemma 5, we must use only sequences which have the first item zero. So, for a given set T of sequences, we denote by:

$$T_0 = \{x = (x_n)_{n\geq 0},\ x \in T,\ x_0 = 0\}$$

its subset with desired property. Also we denote:

$$M^0 T_0 = \{x : B(x) \in T_0\}.$$

We get the following characterizations:

Lemma 7. *The sequence $(x_n)_{n\geq 1}$ belongs to:*

i) $M^0 K_0$ iff $x_{n+1}/(n+1) \geq x_n/n$, for $n \geq 2$;

ii) $M^0 S_0^*$ iff $\sum_{k=1}^{n}(x_n/n - x_k/k) \geq 0$ for $n \geq 2$;

iii) $M^0 W_0$, iff $x_n/n \geq x_1$ for $n \geq 2$;

iv) $M^0 V_0^*$, iff $\sum_{k=2}^{n}(x_k/k - x_1) \geq 0$ for $n \geq 2$.

Proof.

We have only to compute:

i) $B_{n+2}(x) - 2B_{n+1}(x) + B_n(x) = x_{n+2}/(n+2) - x_{n+1}/(n+1)$;

ii) $B_{n+1}(x)/(n+1) - B_n(x)/n = \dfrac{nx_{n+1}/(n+1) - \sum_{k=1}^{n} x_k/k}{n(n+1)}$;

iii) $B_{n+1}(x) - B_n(x) - B_1(x) = x_{n+1}/(n+1) - x_1$;

iv) $B_n(x) - nB_1(x) = \sum_{k=1}^{n} x_k/k - nx_1 = \sum_{k=1}^{n}(x_k/k - x_1)$.

Lemma 8. *The inclusions*

$$\begin{array}{ccccccccc} K_0 & \subset & S_0^* & \subset & S_0 & \subset & W_0 & \subset & V_0^* \\ & & \cap & & & & & & \cap \\ M^0K_0 & \subset & M^0S_0^* & \subset & M^0S_0 & \subset & M^0W_0 & \subset & M^0V_0^* \end{array} \tag{16}$$

hold.

Proof. The inclusions from the first and the second lines follow from (2), while $S_0^* \subset M^0K_0$ and $V_0^* \subset M^0W_0$ are proved in assertions i) respectively iii) of Lemma 7.

Remark 4. It is easy to see that a superadditive sequence satisfies the condition ii) of Lemma 7 for $n = 2, 3, 4, 5$, so we conjecture that:

$$S_0 \subset M^0S_0^*.$$

To the contrary, $W_0 \not\subset M^0S_0$. For example, the sequence x given by $x_1 = 0$, $x_n = 1$ for $n \geq 2$, belongs to W_0, but not to M^0S_0.

Also we can combine the diagrams (15) and (16).

Lemma 9. *For every $u > 0$ the inclusions:*

$$\begin{array}{ccccccc} M^uS_0^* & \subset & M^uS_0 & \subset & M^uW_0 & \subset & M^uV_0^* \\ \cap & & & & \cap & & \cap \\ M^0S_0^* & \subset & M^0S_0 & \subset & M^0W_0 & \subset & M^0V_0^* \end{array}$$

hold.

Proof. i) If $x=(x_n)_{n\geq 0}\in M^uS_0^*$, the using (3) and (4) we can represent $A^u(x)$ by:

$$A_n^u(x)=n\sum_{k=1}^n z_k,\ \text{with } z_k\geq 0 \text{ for } k\geq 2.$$

But then, as in [9], we have:

$$x_n=\left(1+\frac{n}{u}\right)A_n^u(x)-\frac{n}{u}A_{n-1}^u(x), \tag{17}$$

hence

$$x_n=n\left((n-1)z_n+(u+1)\sum_{k=1}^n z_k\right)/u.$$

We deduce that:

$$\frac{x_n+1}{n+1}-\frac{1}{n}\sum_{k=1}^n\frac{x_k}{k}=\left(1+\frac{n+1}{u}\right)z_{n+1}+\sum_{k=2}^n\frac{k-1}{u}z_k\geq 0,$$

that is $x\in M^0S_0^*$.

ii) If $x\in M^uW_0$, we have by (6):

$$A_n^u(x)=\sum_{k=2}^n w_k+nw_1,\ \text{with } w_k\geq 0 \text{ for } k\geq 2$$

if $n\geq 2$ and $A_1^u(x)=w_1$. Then $x_1=(1+1/u)w_1$ and for $n\geq 2$, from (17):

$$x_n=nw_1+\sum_{k=2}^n w_k+\frac{n}{u}(w_1+w_n)\geq n(1+1/u)w_1=nx_1,$$

that is $x\in M^0W_0$.

iii) For $x\in M^uV_0^*$ we have from (7) $A_1^u(x)=v_1$ and:

$$A_n^u(x)=v_n+nv_1,\ \text{with } v_n\geq 0 \text{ for } n\geq 2.$$

So $x_1 = (1 + 1/u)v_1$ and from (17):

$$x_n = \left(1 + \frac{n}{u}\right) v_n - \frac{n}{u} v_{n-1} + n\left(1 + \frac{1}{u}\right) v_1.$$

Thus:

$$\sum_{k=2}^{n} (x_k/k - x_1) = v_n/u + \sum_{k=2}^{n} v_k/k \geq 0$$

and by Lemma 7, $x \in M^0 V_0^*$.

We summarize the above results in the following:

Theorem. *For arbitrary $u \geq 0$ we have the inclusions:*

$$\begin{array}{ccccccccccc}
K_0 & \subset & S_0^* & \subset & S_0 & \subset & W_0 & \subset & V_0^* & & \\
 & & \cap & & & & & & \cap & & \\
\cap & & M^0K_0 & \subset & M^0S_0^* & \subset & M^0S_0 & \subset & M^0W_0 & \subset & M^0V_0^* \\
 & & & & \cup & & & & \cup & & \cup \\
M^uK_0 & & \subset & & M^uS_0^* & \subset & M^uS_0 & \subset & M^uW_0 & & M^uV_0^*
\end{array}$$

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SUPERMULTIPLICATIVE SEQUENCES IN SEMIGROUPS

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1. Real sequences

In [1], solving the problem [2], the following result is proved:

Let $(a_n)_{n\geq 1}$ be a sequence of real numbers satisfying the relation $a_{n+m} \leq a_n + a_m,\ \forall\ n, m \geq 1$. Then

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \frac{a_n}{n} \sum_{k=1}^{n} \frac{1}{k}. \tag{1}$$

We can analyse this result by taking into account some definitions and results from [4]. There we have considered the sets of starshaped and of superadditive sequences defined by

$$S^* = \left\{ (a_n)_{n\geq 1} :\ \frac{a_n}{n} \leq \frac{a_{n+1}}{n+1},\ \forall\ n \geq 1 \right\}$$

respectively:

$$S = \{(a_n)_{n\geq 1} :\ a_{n+m} \geq a_n + a_m,\ \forall\ n, m \geq 1\}$$

and we have proved the proper inclusion

$$S^* \subset S. \tag{2}$$

Now, multiplying inequality (1) by -1, we get

$$\sum_{k=1}^{n} \frac{1}{k} \left(\frac{a_n}{n} - \frac{a_k}{k} \right) \geq 0$$

which is obviously valid for any sequence from S^*, but the result of [1] means that it holds even for all sequences from S.

Starting from this remark, in [8] we have posed the problem of determination of positive weight sequences $(p_k)_{k\geq 1}$ with the property that:

$$\sum_{k=1}^{n} p_k \left(\frac{a_n}{n} - \frac{a_k}{k} \right) \geq 0, \ \forall \ n \geq 1 \tag{3}$$

for every sequence $(a_n)_{n\geq 1} \in S$. In what follows we want to generalize the results from [8] for the case of sequences in a semigroup which we have considered in [5].

2. Sequences in a semigroup

Remarking that the relation $\geq$ can be interpreted as a relation of divisibility in the additive semigroup of the positive reals, we have transposed in [5] some of the results of [4] for semigroups.

Let $(G, \cdot)$ be a semigroup, that is the binary operation $\cdot : G \times G \to G$ is associative. We suppose also that the semigroup is commutative and has an identity

We consider the usual divisibility relation

$$a|b \Leftrightarrow \exists \ c \in G, \ b = ac.$$

Let $(x_n)_{n\geq 1}$ be a sequence of elements of $(G, \cdot)$. In [5], we have called this sequence:

a) starshaped if

$$x_n^{n+1} | x_{n+1}^n, \ \forall \ n \geq 1;$$

b) supermultiplicative if

$$x_n x_m | x_{n+m}, \ \forall \ n, m \geq 1.$$

In what follows we replace the definition of starshapedness by a stronger one:

$$x_n^m | x_m^n, \ \forall \ n < m$$

and denote by S_G^* and S_G the set of starshaped, supermultiplicative respectively, sequences from $(G, \cdot)$.

If the semigroup has some properties, a relation like (2) can be valid.

For example, in [5] we have proved that if $(G, \cdot)$ preserves the divisibility, that is

$$x^n | y^n \Rightarrow x | y$$

then

$$S_G^* \subset S_G$$

holds.

By analogy with relation (3), for every sequence $(x_n)_{n\geq 1} \in S_G^*$ and every sequence of natural numbers $(q_n)_{n\geq 1}$ we have

$$\left(\prod_{k=1}^{n} x_k^{q_k} \right)^n | x_n^{\sum\limits_{k=1}^{n} k q_k} . \tag{4}$$

We denote by W_G the set of sequence $(q_n)_{n\geq 1}$ of natural numbers with the property that (4) is valid for every sequence $(x_n)_{n\geq 1}$ from S_G. We remark that W_G is an "integer" cone, that is, it is closed with respect to addition and multiplication by positive integer numbers.

Lemma 1. *The constant sequence given by*

$$q_n = 2, \ \forall \ n \geq 1$$

belongs to W_G.

Proof. For every sequence $(x_n)_{n\geq 1}$ from S_G we have

$$x_k x_{n-k} | x_n, \quad 1 \leq k \leq n$$

thus

$$\prod_{k=1}^{n} x_k^2 | x_n^{n+1}$$

or

$$\left(\prod_{k=1}^{n} x_k^2\right) | x_n^{\sum\limits_{k=1}^{n} 2k}.$$

Remark 1. In the case of usual superadditive sequences, we have proved a similar result in [8].

Remark 2. For noninteger sequences $(q_k)_{k\geq 1}$, we must find other types of formulations. So, for the sequence defined by $q_k = 1/k$, we have the following:

Conjecture. If the sequence $(x_n)_{n\geq 1}$ belongs to S_G, then, for every $n \geq 1$, we have

$$\prod_{k=1}^{n} x_k^{\frac{n!}{k}} | x_n^{n!}. \tag{5}$$

We can prove it for small values of n (say $n \leq 10$). For example, for $n = 5$ we use

$$x_4 x_1 | x_5, \ x_2 x_3 | x_5, \ x_1 x_2^2 | x_5 \text{ and } x_1^5 | x_5$$

at the power 30, 40, 10 respectively 16 and then multiplied. Also we can verify (5) for the sequences $(x_n)_{n\geq 1}$ of the subset T_G of S_G defined as

$$T_G = \left\{ (x_n)_{n\geq 1} : \ x_n = \prod_{i=1}^{n} w_i^{\left[\frac{n}{i}\right]}, \ w_i \in G, \ \forall \ i, n \geq 1 \right\},$$

where $[x]$ denotes the integer part of x. We have proved in [5] that T_G is a proper subset of S_G.

Lemma 2. *Every sequence* $(x_n)_{n\geq 1}$ *of* T_G *verifies (5).*

Proof. We have

$$\prod_{k=1}^{n} x_k^{\frac{n!}{k}} = \prod_{k=1}^{n} \left(\prod_{i=1}^{n} w_i^{\left[\frac{k}{i}\right]} \right)^{\frac{n!}{k}} = \prod_{i=1}^{n} w_i^{\sum_{k=1}^{n} \left[\frac{k}{i}\right] \frac{n!}{k}}$$

thus (5) is fulfilled because

$$\sum_{k=1}^{n} \frac{1}{k} \left[\frac{k}{i}\right] \leq \left[\frac{n}{i}\right], \ \forall\, n, i \geq 1$$

(see [3]).

Remark 3. For the case of superadditive sequences we made this conjecture in [7] and the corresponding special case of Lemma 2 we have proved in [8].

Remark 4. In [6] another kind of starshapedness and superadditivity related to the logarithmic convexity is defined. So, if the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, we say that the sequence $(a_n)_{n\geq 1}$ is

i) f-starshaped if

$$\frac{f(a_n)}{n} \leq \frac{f(a_{n+1})}{n+1}, \ \forall\, n \geq 1;$$

ii) f-superadditive if

$$f(a_n) + f(a_m) \leq f(a_{n+m}), \ \forall\, n, m \geq 1.$$

For example, log-starshapedness means

$$a_n^{1/n} \leq a_{n+1}^{1/(n+1)}, \ \forall\, n \geq 1$$

and it implies log-superadditivity, i.e.

$$a_n a_m \leq a_{n+m}, \ \forall\, n, m \geq 1.$$

By Lemma 1, this last relation implies

$$\prod_{k=1}^{n} a_k \leq a_n^{(n+1)/2}, \ \forall\, n \geq 1.$$

The conjecture means, for this case, that every log-superadditive sequence verifies

$$\prod_{k=1}^{n} a_k^{1/k} \leq a_n, \ \forall\, n \geq 1,$$

which is obviously true for log-starshaped sequences.

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ON CHEBYSHEV'S INEQUALITY FOR SEQUENCES

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Abstract. We give improved versions of Chebyshev's inequality for starshaped sequences, valid also for convex sequences.

1. Introduction

The classical Chebyshev's inequality for sequences, as it may be found in [4], asserts that is $a = \{a_j | 1 \leq j \leq n\}$ and $b = \{b_j | 1 \leq j \leq n\}$ are increasing sequences, then the following inequality is true:

$$\sum_{j=1}^{n} a_j \sum_{k=1}^{n} b_k \leq n \sum_{j=1}^{n} a_j b_j. \tag{1}$$

For convex sequences a and b, Pečarić proved in [5] the analogous inequality

$$n \sum_{j=1}^{n} a_j b_j - \sum_{j=1}^{n} a_j \sum_{k=1}^{n} b_k \geq \frac{12}{n^2 - 1} \sum_{k=1}^{n} \left(k - \frac{n+1}{2}\right) a_k \sum_{j=1}^{n} \left(j - \frac{n+1}{2}\right) b_j. \tag{2}$$

In [7] Seymour and Welsh give the following generalization of Chebyshev's inequality: if $p = \{p_j | 1 \leq j \leq n\}$ is log convex and a and b are increasing sequences, then

$$\sum_{j=1}^{n} p_j a_j \sum_{k=1}^{n} p_k b_k \leq \sum_{j=1}^{n} p_j \sum_{k=1}^{n} p_k a_k b_k. \tag{3}$$

In [1] Beck and Krogdahl proved a 2-dimensional version of this result.

But it is known (see [3]) that the inequality (3) is also valid for all positive sequences p, and not only for those log convex. Moreover, in [2] the relation (3) is proved under weaker conditions on a and b: replace monotonicity by monotonicity in p-mean (to be explained below). Also the classical Chebyshev's inequaloty (1) was generalized in [6] in a similar sense, but the result is not comparable with that of [2].

In this paper we want to improve (3) in the case of convexity, but of the sequences a and b and not of p. In fact, the result is proved even for starshaped sequences and not only for convex ones. We also generalize in the same direction some results from [2], [6].

2. Definitions and auxiliary results

A sequence $a = \{a_j | 1 \leq j \leq n\}$ is said to be

i) convex, if

$$a_{j+1} \leq \frac{a_j + a_{j+2}}{2} \text{ for } j = 1, \ldots, n-2,$$

ii) log convex, if it is positive and

$$a_{j+1}^2 \leq a_j a_{j+2} \text{ for } j = 1, \ldots, n-2,$$

iii) starshaped, if

$$\frac{a_j}{j} \leq \frac{a_{j+1}}{j+1} \text{ for } j = 1, \ldots, n-1.$$

We recall that a log convex sequence is also convex (as follows from the inequality between the arithmetic and the geometric mean). Also we have the following result.

Lemma 1. *If the sequence* $a = \{a_j | 1 \le j \le n\}$ *is convex and* $a_1 \le a_2/2$ *then it is starshaped.*

The proof can be done by induction. Usually, one supposes that the sequence a starts, not with a_1, but with $a_0 = 0$ (or $a_0 \le 0$). This is only to get the condition $a_1 \le a_2/2$.

Let $p = \{p_j | 1 \le j \le n\}$ be a fixed positive sequence. The sequence $a = \{a_j | 1 \le j \le n\}$ is said to be increasing in p-mean if the sequence

$$\left(\sum_{k=1}^{j} p_k a_k / \sum_{k=1}^{j} p_k \right)_{j=1}^{n}$$

is increasing. If $p_j = 1$ for all j, we say that the sequence is increasing in 1-mean or simply increasing in mean. Of course, an increasing sequence is increasing in p-mean.

The result of [2] cited above, may be formulated as follows.

Theorem A. *If the sequences a and b are increasing in p-mean then the inequality (3) is valid. If one of the sequences is increasing in p-mean and the other is decreasing in p-mean then the reverse inequality holds.*

To obtain an improved variant of this result for some sequences, we give the following definition: the sequence a is said to be starshaped in p-mean if the sequence $\{a_j/j | 1 \le j \le n\}$ is increasing in p-mean. Obviously, we have the following result.

Proposition. *All starshaped sequences are starshaped in p-mean.*

We need also the following results.

Lemma 2. *The sequence* $a = \{a_j | 1 \le j \le n\}$ *is increasing in p-mean if and only if*

$$a_{j+1} \ge \sum_{k=1}^{j} p_k a_k / \sum_{h=1}^{j} p_h \text{ for } j = 1, \ldots, n-1. \tag{4}$$

Proof. We have

$$\sum_{k=1}^{j+1} p_k a_k / \sum_{h=1}^{j+1} p_h - \sum_{k=1}^{j} p_k a_k / \sum_{h=1}^{j} p_j$$

$$= \left(a_{j+1} - \sum_{k=1}^{j} p_k a_k / \sum_{h=1}^{j} p_h \right) p_{j+1} / \sum_{k=1}^{j+1} p_k,$$

which gives the desired result as the sequence p is positive.

Lemma 3. *If the sequence* $a = \{a_j | 1 \le j \le n\}$ *is increasing in p-mean and the positive sequence* $q = \{q_j | 1 \le j \le n\}$ *is decreasing in p-mean, then the sequence*

$$\left(\sum_{k=1}^{j} p_k q_k a_k / \sum_{h=1}^{j} p_h q_h \right)_{j=1}^{n}$$

is increasing.

Proof. We have to use only (4) and Theorem A.

Corollary 1. *If the sequence a is increasing (starshaped) in p-mean, then it is also increasing (respectively starshaped) in pq-mean for every positive decreasing sequence q, where* $pq = \{p_j q_j | 1 \le j \le n\}$.

We also need the result of [6] given by:

Theorem B. *If the sequence a verifies the conditions*

$$\frac{1}{m} \sum_{j=1}^{m} a_j \le \frac{1}{n} \sum_{j=1}^{n} a_j \text{ for } m < n \tag{5}$$

and the sequence b is increasing, then the inequality (1) is valid.

We see that (5) is a condition weaker than the 1-mean monotonicity, because we compare all the means only with the last mean (n is fixed).

3. Main results

We want to prove (3), and even a stronger inequality, for starshaped sequences.

Theorem 1. *If the sequence p is positive and the sequences a and b are starshaped then the following inequality holds:*

$$\sum_{i=1}^{n} i^2 p_i \sum_{j=1}^{n} p_j a_j \sum_{k=1}^{n} p_k b_k \leq \left(\sum_{j=1}^{n} j p_j \right)^2 \sum_{k=1}^{n} p_k a_k b_k. \tag{6}$$

If $a_j = b_j = 1$ for $1 \leq j \leq n$ we have the inequality in (6).

Proof. Applying (3) for the weight sequence $\{j^2 p_j | 1 \leq j \leq n\}$ and the increasing sequences $\{a_j/j | 1 \leq j \leq n\}$ and $\{b_j/j | 1 \leq j \leq n\}$ we get

$$\sum_{j=1}^{n} j p_j a_j \sum_{k=1}^{n} k p_k b_k \leq \sum_{j=1}^{n} j^2 p_j \sum_{k=1}^{n} p_k a_k b_k. \tag{7}$$

Again (3) for the sequences $\{j p_j | 1 \leq j \leq n\}$, $\{a_j/j | 1 \leq j \leq n\}$ and $\{j | 1 \leq j \leq n\}$ gives

$$\sum_{j=1}^{n} p_j a_j \sum_{k=1}^{n} k^2 p_k \leq \sum_{j=1}^{n} j p_j \sum_{k=1}^{n} k p_k a_k. \tag{8}$$

Using (7) and (8), for a and b, we have (6).

Remark. From (3) we also deduce

$$\sum_{j=1}^{n} p_j \sum_{k=1}^{n} k^2 p_k \geq \left(\sum_{j=1}^{n} j p_j \right)^2,$$

which means that (6) is really stronger than (3).

Remark 2. By Lemma 1, the inequality (6) is also valid for (log) convex sequences with the property that $a_1 \le a_2/2$.

Corollary 2. *If the sequences a and b are starshaped, then*

$$\sum_{j=1}^{n} a_j \sum_{k=1}^{n} b_k \le \frac{3n(n+1)}{2(2n+1)} \sum_{j=1}^{n} a_j b_j. \tag{9}$$

Of course, (9) is obtained from (6) by taking $p_j = 1$, for all j. On the other hand, as $(n+1)/(2n+1) \le \dfrac{3}{5}$ for $n \ge 2$, from (9) we get an improvement of (1) for starshaped (or convex) sequences:

Corollary 3. *If the sequences a and b are starshaped then, for $n > 1$, we have the inequality*

$$\sum_{j=1}^{n} a_j \sum_{k=1}^{n} b_k \le \frac{9}{10} n \sum_{j=1}^{n} a_j b_j. \tag{10}$$

We can replace 0.9 be a smaller constant c_q if we desire the inequality (10) only for $n > q$, and of course $c_q \to 0.75$ as $q \to \infty$.

Let us consider the sequences $e^i = \{j^i | 1 \le j \le n\}$, for $i \ge 1$.

Theorem 2. *If the sequence p is positive and the sequences a and b are starshaped in e^2p-mean, then the inequality (6) is valid.*

Proof. Using Theorem A for the sequences $\{a_j/j | 1 \le j \le n\}$ and $\{b_j/j | 1 \le j \le n\}$ and the weight sequence e^2p, we get (7). Then, from Corollary 1 we deduce that the sequences a and b are also starshaped in e^1p-mean. So, we can use again Theorem A for the sequences $\{a_j/j | 1 \le j \le n\}$ and e^1 and the weight sequence e^1p obtaining (8). Employing it, and a similar inequality for b, in (7) we have (6).

Also we can generalize Theorem B by introducing a weight sequence.

Theorem 3. *If the sequence p is such that*

$$\sum_{j=1}^{m} p_j > 0 \text{ for all } m \le n,$$

and a verifies the condition

$$\sum_{j=1}^{m} p_j a_j / \sum_{k=1}^{m} p_k \le \sum_{j=1}^{n} p_j a_j / \sum_{k=1}^{n} p_k \text{ for all } m < n, \tag{11}$$

then the inequality (3) is valid for all increasing sequences b.

Proof. If we define P_m and A_m by

$$P_m = \sum_{j=1}^{m} p_j \text{ and } A_m = \sum_{j=1}^{m} p_j a_j,$$

condition (11) becomes $A_m \le (P_m a_n)/P_n$ and then we get

$$\begin{aligned} \sum_{j=1}^{n} p_j a_j b_j &= A_n b_n - \sum_{m=1}^{n-1} A_m (b_{m+1} - b_m) \\ &\ge b_n A_n - \sum_{m-1}^{n-1} P_m (b_{m+1} - b_m) A_n / P_n \\ &= (A_n / P_n) \sum_{m=1}^{n} p_m b_m \end{aligned}$$

which is (3).

Remark 3. There is also a version of this theorem for starshaped sequences. On the other hand, in all the results of this paper the increasing monotonicity can be replaced by decreasing monotonicity. In fact, it can be generalized even by a property of synchronism. The sequences a and b are synchronous if

$$(a_i - a_j)(b_i - b_j) \ge 0 \text{ for } 1 \le j < j \le n.$$

If the above inequality is reversed so are the inequalities (1) and (3).

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ON THE DUAL CONE OF A CONE OF SEQUENCES

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Abstract. In the paper there are generalized or adapted some results from the paper [1] to the case of a cone of sequences.

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1. Introduction

Let S_n denote the set of finite real sequences $(a_k)_{k=0}^n$. As it is known (see [2]), a subset $K_n \subseteq S_n$ is called a cone if:

$$(pa_k)_{k=0}^n \in K_n, \text{ for every } p \geq 0, \quad (a_k)_{k=0}^n \in K_n.$$

The cone K_n is convex if:

$$(a_k + b_k)_{k=0}^n \in K_n \text{ for all } (a_k)_{k=0}^n, \ (b_k)_{k=0}^n \in K_n.$$

The dual cone K_n^* of K_n is defined by:

$$K_n^* = \left\{ (p_k)_{k=0}^n \in S_n : \ \sum_{k=0}^n p_k a_k \geq 0, \ \forall \ (a_k)_{k=0}^n \in K_n \right\}.$$

As it is stated in [2], characterizations of the elements of a dual cone were obtained for the first time for convex functions by T. Popoviciu (see [5] for more references). They were transposed for convex sequences by J.E. Pečarić in [4]. In [6] we can find more results and references to papers in which it is determined the dual cone of some cones of sequences. Essentially the cones K_n which are considered in these papers consist in different kinds of convex or of starshaped sequences.

In this paper we want to generalize and to transpose to above context the results from [1].

2. Some results of M.P. Drazin

The finite differences Δ^k are defined for any sequence $(a_i)_{i\geq 0}$ recurrently by:

$$\Delta^0 a_i = a_i,\ \Delta^1 a_i = a_{i+1} - a_i,\ \Delta^k a_i = \Delta^1(\Delta^{k-1} a_i),\ k \geq 2,\ i \geq 0.$$

M.P. Drazin proved in [1] the following results:

i) For any sequence $(a_i)_{i\geq 0}$ and any y holds:

$$\sum_{i=0}^{n} \binom{n}{i} y^i a_i = (-1)^n \sum_{j=0}^{n} \binom{n}{j} (-1-y)^j \Delta^{n-j} a_j, \quad n \geq 0. \tag{1}$$

ii) If:

$$(-1)^{n-j} \Delta^{n-j} a_j \geq 0,\ \text{for } 0 \leq j \leq n,$$

with at least one inequality, then:

$$\sum_{i=0}^{n} \binom{n}{i} y^i a_i > 0,\ \text{for } y > -1. \tag{2}$$

iii) If:

$$\Delta^{n-j} a_j \geq 0,\ \text{for } 0 \leq j \leq n,$$

with at least one inequality, then:

$$(-1)^n \sum_{i=0}^{n} \binom{n}{i} y^i a_i > 0, \text{ for } y < -1. \tag{3}$$

3. A new proof

Let us remind the following notation:

$$i^{(k)} = i(i-1)\dots(i-k+1).$$

In [2] and [3] are given two identities which we can use for proving and generalizing (1):

$$\sum_{i=0}^{n} p_i a_i = \sum_{k=0}^{n} \left(\frac{1}{k!} \sum_{i=k}^{n} i^{(k)} p_i \right) \Delta^k a_0 \tag{1}$$

and

$$\sum_{i=0}^{n} p_i a_i = \sum_{k=0}^{n} \left(\frac{1}{(n-k)!} \sum_{i=0}^{k} (n-i)^{(n-k)} p_i \right) (-1)^{(n-k)} \Delta^{n-k} a_k. \tag{5}$$

For $p_i = \binom{n}{i} y^i$, (5) becomes:

$$\begin{aligned} \sum_{i=0}^{n} \binom{n}{i} y^i a_i &= (-1)^n \sum_{k=0}^{n} (-1)^k \left(\sum_{i=0}^{k} \binom{n-i}{n-k} \binom{n}{i} y^i \right) \Delta^{n-k} a_k \\ &= (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1-y)^k \Delta^{n-k} a_k, \end{aligned}$$

i.e. (1).

Similarly, for $p_i = \binom{n}{i} y^{n-i}$, (4) gives:

$$\sum_{i=0}^{n} \binom{n}{i} y^{n-i} a_i = \sum_{k=0}^{n} \binom{n}{k} (1+y)^{n-k} \Delta^k a_0.$$

4. The main results

Let $r = (r_k)_{k=0}^n$, $r_k \in \{0,1\}$ be a given sequence. We define:

$$K_{n,r} = \{(a_k)_{k=0}^n : \ (-1)^{r_k}\Delta^k a_0 \geq 0, \ \text{for } 0 \leq k \leq n\},$$

and

$$L_{n,r} = \{(a_k)_{k=0}^n : \ (-1)^{r_k}\Delta^{n-k} a_k \geq 0, \ \text{for } 0 \leq k \leq n\}.$$

Obviously $K_{n,r}$ and $L_{n,r}$ are convex cones. Using (4) we have the following result:

Theorem 1. *If p_k $(k = 0, 1, \ldots, n)$ are real numbers such that:*

$$(-1)^{r_k} \sum_{i=k}^{n} i^{(k)} p_i \geq 0, \ \textit{for } 0 \leq k \leq n,$$

then $(p_k)_{k=0}^n \in K_{n,k}^*$.

Also, using (5) we get:

Theorem 2. *If the real numbers p_k $(k = 0, 1, \ldots, n)$ are such that:*

$$(-1)^{r_k+n-k} \sum_{i=0}^{k} (n-i)^{(n-k)} p_i \geq 0, \ \textit{for } 0 \leq k \leq n,$$

then $(p_k)_{k=0}^n \in L_{n,r}^*$.

Consequence. *Let f be a function continuous on $[0,n]$, differentiable n times in $(0,n)$, positive for $x = n$ and*

$$(-1)^k f^{(k)}(x) \geq 0, \ \textit{for } 1 \leq k \leq n, \ n-k < x < n.$$

If the real numbers p_k $(k = 0, 1, \ldots, n)$ are such that:

$$\sum_{i=0}^{k} (n-i)^{(n-k)} p_i \geq 0, \ \textit{for } k = 0, 1, \ldots, n \tag{6}$$

then:

$$\sum_{i=0}^{n} p_i f(i) \geq 0.$$

Proof. From (5) we have:

$$\sum_{i=0}^{n} p_i f(i) = \sum_{k=0}^{n} \frac{1}{(n-k)!} \left(\sum_{i=0}^{k} (n-i)^{(n-k)} p_i \right) (-1)^{n-k} \Delta^{n-k} f(k).$$

As it is proved in [1], for $k = 0, 1, \ldots, n-1$, there is a $x_k \in (k, n)$ such that:

$$\Delta^{n-k} f(k) = f^{(n-k)}(x_k).$$

We can consider also $x_n = n$ and from (6) we get the result. □

As an application, we can prove the following generalization of a proposition from [1]:

Theorem 3. *Let $f_1, \ldots, f_q$ be given continuous functions on the interval $0 \leq x \leq n$, each differentiable n times in the open interval and positive for $x = n$; suppose also that:*

$$(-1)^k f_j^{(k)}(x) \geq 0, \;\; j = 1, \ldots, q; \;\; k = 1, \ldots, n; \;\; n - k \leq x \leq n.$$

If p_k $(k = 0, 1, \ldots, n)$ are real numbers verifying (6) then:

$$\sum_{i=0}^{n} p_i \left(\prod_{j=1}^{q} f_j(i) \right) \geq 0.$$

As in [1], this result can be illustrated by the set of functions:

$$f_j(x) = (1 + a_j x)^{-b_j}, \quad j = 1, \ldots, q$$

where a_j and b_j can be any positive numbers.

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LALESCU SEQUENCES

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The convergence of some sequences related to the Lalescu sequences is studied.

The Romanian mathematical journal Gazeta Mathematică (Bukarest) appears monthly since 1895. In one of the first volumes, more exactly in a number from 1900 (see [**6**]), T.Lalescu has proposed, as problem 579, the study of a sequence with the general term

$$L_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}.$$

It is called now Lalescu sequence, at least by Romanian mathematicians, and many variants of it have appeared during this century in the same journal. The first one was considered as the problem 2042 (see [**5**]) and has the general term

$$I_n = (n+1)\sqrt[n+1]{(n+1)} - n\sqrt[n]{n}.$$

We relate to them also a third sequence given by

$$J_n = \frac{(n+1)^n}{n^{n-1}} - \frac{n^{n-1}}{(n-1)^{n-2}}$$

which appears as the problem 4600 (see [**7**]). At the end of this paper we will give some other sequences which have appeared in the last years.

We begin by indicating a general result giving the limit of a sequence which looks like the sequences mentioned above. The method of proof is that used in the first published solution for Lalescu's problem. This was forget and many other more sophisticated solutions were considered (see [**1**] for more information).

We study sequences with the general term

$$x_n = y_n - z_n$$

where

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = \infty, \qquad \lim_{n\to\infty} \frac{y_n}{z_n} = 1.$$

Theorem 1. *If there exist the positive constants b and c such that*

$$\lim_{n\to\infty} \frac{z_n}{n^c} = z > 0, \qquad \lim_{n\to\infty} \left(\frac{y_n}{z_n}\right)^{n^b} = y > 0,$$

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then

$$\lim_{n\to\infty}(y_n - z_n) = \begin{cases} 0 & (c < b), \\ z\ln y & (c = b), \\ \infty & (c > b, y > 1), \\ -\infty & (c > b, y < 1). \end{cases}$$

Proof. We write

$$y_n - z_n = \frac{\dfrac{y_n}{z_n} - 1}{\ln\left(\dfrac{y_n}{z_n}\right)} \,\frac{z_n}{n^c}\, n^{c-b}\, \ln\left(\frac{y_n}{z_n}\right)^{n^b}$$

and use the hypotheses and the well known result $\lim\limits_{n\to\infty}\dfrac{t-1}{\ln t} = 1$

Example 1. The sequence

$$x_n = \frac{(n+p)^n}{n^{n-k}} - \frac{n^{n-h}}{(n-p)^{n-h-k}}$$

has a finite limit if and only if $k = 1$ and in this case the limit is $pe^p(h+k-p)$. Indeed, in this case

$$y_n = \frac{(n+p)^n}{n^{n-k}}, \qquad z_n = \frac{n^{n-h}}{(n-p)^{n-h-k}}$$

and so

$$\lim_{n\to\infty}\frac{y_n}{n^k} = e^p$$

while

$$\lim_{n\to\infty}\left(\frac{y_n}{z_n}\right)^n = \lim_{n\to\infty}\left(\left(\frac{n+p}{n}\right)^{h+k}\left(\frac{n^2-p^2}{n^2}\right)^{n-h-k}\right)^n q = e^{p(h+k-p)}.$$

Taking $p = k = h = 1$ we get the sequence (J_n) with limit e.

To study the sequences (L_n) or (I_n) we want to apply Theorem 1 to a sequence with the general term

$$x_n = \sqrt[n+1]{p_{n+1}} - \sqrt[n]{q_n}.$$

Theorem 2. *If the positive sequences (p_n) and (q_n) have the infinite limits and for some $c > 0$ satisfy*

$$\lim_{n\to\infty}\frac{p_{n+1}}{n^c p_n} = p > 0, \qquad \frac{q_n}{p_n} = q > 0$$

then

$$\lim_{n\to\infty}\left(\sqrt[n+1]{p_{n+1}} - \sqrt[n]{q_n}\right) = \begin{cases} 0 & (c < 1), \\ \dfrac{p}{e}\ln\dfrac{e}{q} & (c = 1), \\ \infty & (c > 1, q < e^c), \\ -\infty & (c > 1, q > e^c). \end{cases}$$

Proof. We take $y_n = \sqrt[n+1]{p_{n+1}}, \quad z_n = \sqrt[n]{q_n}$. Of course

$$\lim_{n\to\infty} \frac{q_{n+1}}{n^c q_n} = \lim_{n\to\infty} \frac{q_{n+1}}{p_{n+1}} \frac{p_{n+1}}{n^c p_n} \frac{p_n}{q_n} = p$$

and so

$$\lim_{n\to\infty} \frac{z_n}{n^c} = \lim_{n\to\infty} \sqrt[n]{\frac{q_n}{n^{nc}}} = \lim_{n\to\infty} \frac{q_{n+1}}{(n+1)^{(n+1)c}} \frac{n^{nc}}{q_n}$$
$$= \lim_{n\to\infty} \frac{q_{n+1}}{n^c q_n} \left(\frac{n}{n+1}\right)^{(n+1)c} = \frac{p}{e^c}\,.$$

We have used the well known implication

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1 \;\Rightarrow\; \lim_{n\to\infty} \sqrt[n]{x_n} = 1.$$

Also

$$\lim_{n\to\infty} \left(\frac{y_n}{z_n}\right)^n = \lim_{n\to\infty} \frac{p_{n+1}}{n^c p_n} \frac{n^c}{\sqrt[n+1]{p_{n+1}}} \frac{p_n}{q_n} = \frac{e^c}{q}.$$

thus we can apply Theorem 1 with $b = 1$.

We use in what follows only a special case of this result.

Consequence. *If the positive sequence* (p_n) *is such that*

$$\lim_{n\to\infty} \frac{p_{n+1}}{np_n} = p > 0$$

then

$$\lim_{n\to\infty} \left(\sqrt[n+1]{p_{n+1}} - \sqrt[n]{p_n} \right) = \frac{p}{e}\,.$$

With its help we can find the limit of some sequences given in the above mentioned journal Gazeta Matematica. First of all, for $p_n = n!$ we get the sequence (L_n) with limit $1/e$ and for $p_n = n^{n+1}$ we get (I_n) with limit 1. If $p_n = n^{2n}/n!$ we have a sequence given in [**2**] with limit e. Taking $p_n = \Gamma\left(\frac{n+1}{2}\right)$ we get the sequence from [**4**] having limit $1/e$. Also for $p_n = \sqrt[3]{n!n^{2n}}$ we have the sequence given in [**3**] with limit $1/\sqrt[3]{e}$ and the list of examples can be continued.

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