# SEVERAL INEQUALITIES OF HERMITE–HADAMARD, OSTROWSKI AND SIMPSON TYPE FOR $s$–CONVEX, QUASI–CONVEX AND $r$–CONVEX MAPPINGS AND APPLICATIONS

MOHAMMAD WAJEEH NAWAF ALOMARI

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FACULTY OF SCIENCE AND TECHNOLOGY
UNIVERSITI KEBANGSAAN MALAYSIA
BANGI

# PELBAGAI ANGGARAN RALAT BAGI BEBERAPA PERATURAN KUADRATUR DALAM SEBUTAN PALING BANYAK TERBITAN PERTAMA

MOHAMMAD WAJEEH NAWAF ALOMARI

TESIS YANG DIKEMUKAKAN UNTUK MEMPEROLEH IJAZAH
DOKTOR FALSAFAH

FAKULTI SAINS DAN TEKNOLOGI
UNIVERSITI KEBANGSAAN MALAYSIA
BANGI

## DECLARATION

I hereby declare that the work in this thesis is my own except for quotations and summaries which have been duly acknowledged.

MOHAMMAD WAJEEH NAWAF ALOMARI

## ACKNOWLEDGEMENT

*In the name of Allah, Most Gracious, Most Merciful.*

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*This thesis is especially dedicated to my beloved parents.*

## ABSTRACT

Inequalities play a significant role in almost all fields of mathematics. Several applications of inequalities are found in various areas of sciences such as, physical, natural and engineering sciences. In numerical analysis, inequalities play a main role in error estimations. A few years ago, a number of authors have considered an error analysis of some quadrature rules of Newton-Cotes type. In particular, the mid-point, trapezoid and Simpson's have been investigated more recently with the view of obtaining bounds for the quadrature rules in terms of at most second derivative. By using modern theory of inequalities and Peano kernel approach, this thesis is devoted to investigate several refinements inequalities for the Hermite–Hadamard's, Ostrowski's and Simpson's type and deduce explicit bounds for the mid-point, trapezoid and Simpson's quadrature rules in terms of a variety of quasi-convex, $s$-convex and $r$-convex mappings, at most second derivative. This approach allows us to investigate several quadrature rules that have restrictions on the behavior of the integrand and thus to deal with larger classes of functions. Several generalizations and improvements for a previous inequalities in the literature for function $f$ where $|f'|$ (or $|f'|^q, q \geq 1$) is convex (or other type of convexity) hold by applying the Hölder inequality and the power mean inequality. As applications, some error estimates for a proposed quadrature rules and for some special means are derived. A comparison between the presented results with the previous one is considered and discussed. In this way, this thesis provides a study of some of the most famous and fundamental inequalities originated by Hermite–Hadamard, Ostrowski and Simpson and shall gather interesting developments in this research area under a unified framework.

**PELBAGAI ANGGARAN BAGI BEBERAPA PERATURAN KUADRATUR DALAM SEBUTAN PALING BANYAK TERBITAN PERTAMA**

## ABSTRAK

Ketaksamaan memainkan peranan penting dalam hampir kesemua bidang matematik. Beberapa penggunaan ketaksamaan ditemui dalam pelbagai bidang sains termasuklah fizik, sains semulajadi dan kejuruteraan. Dalam analisis berangka, ketaksamaan memainkan peranan dalam anggaran ralat. Beberapa tahun yang lepas, sebilangan penulis menumpukan perhatian kepada analisis ralat bagi peraturan kuadratur jenis Newton-Cotes. Khususnya titik tengah, trapezoid dan Simpson yang terkini dikaji secara mendalam dengan membayangkan bahawa batas untuk peraturan kuadratur diperoleh dalam sebutan terbitan pertama paling banyak. Dengan menggunakan teori ketaksamaan moden dan kaedah inti Peano, tesis ini menekankan kepada pencarian beberapa penambahbaikan ketaksamaan untuk Hermite–Hadamard, Ostrowski dan Simpson. Lantas menghasilkan batas tersirat untuk titik tengah, trapezoid dan peraturan kuadratur Simpson dalam sebutan terbitan pertama paling banyak bagi pelbagai penjelmaan kuasi-cembung, $s$-cembung dan $r$-cembung. Kaedah ini membenarkan beberapa peraturan kuadratur yang mempunyai kekangan terhadap kelakuan pengamir dikaji dan seterusnya menerokai kelas fungsi yang lebih besar. Dalam kesusasteraan, beberapa pengitlakan dan penambahbaikan ketaksamaan terdahulu untuk fungsi $f$ cembung (atau jenis cembung lain) yang $|f'|$ (atau $|f'|^q, q \geq 1$) adalah benar dengan menggunakan ketaksamaan Hölder dan ketaksamaan min kuasa. Dari segi penggunaan, beberapa anggaran ralat untuk cadangan peraturan kuadratur dan beberapa min istimewa boleh diperoleh. Satu perbandingan diantara hasil terkini dengan yang lepas diberi perhatian dan dibincangkan. Secara am, hasil terkini yang diperoleh terbukti lebih berkesan dari yang sebelumnya. Secara tidak langsung, tesis ini mempersembahkan sorotan kajian awal beberapa ketaksamaan asas terkenal yang bermula dari Hermite–Hadamard, Ostrowski dan Simpson. Perkembangan yang menarik dikumpulkan dalam kajian ini di bawah kesepaduan ruang lingkup kerja.

## CONTENTS


| | | Page |
|---|---|---|
| **DECLARATION** | | iii |
| **ACKNOWLEDGMENT** | | iv |
| **ABSTRACT** | | v |
| **ABSTRAK** | | vi |
| **CONTENTS** | | vii |
| **LIST OF SYMBOLS** | | ix |
| **CHAPTER I** | **INTRODUCTION** | 1 |
| 1.1 | General Introduction | 1 |
| 1.2 | Problem Statement | 2 |
| 1.3 | Research Objectives | 2 |
| 1.4 | Research Methodology | 3 |
| 1.5 | Thesis Organization | 3 |
| **CHAPTER II** | **LITERATURE REVIEW AND BACKGROUND** | 5 |
| 2.1 | Introduction | 5 |
| 2.2 | Elementary Concepts | 5 |
| | 2.2.1 Convex functions | 5 |
| | 2.2.2 Quasi-convex functions | 8 |
| | 2.2.3 Mathematical means and $r$–convexity | 9 |
| | 2.2.4 $s$-Convex functions in the second sense | 12 |
| 2.3 | Inequalities Via Montgomery Identity and Peano Kernel | 15 |
| | 2.3.1 Hermite-Hadamard's type inequalities | 17 |
| | 2.3.2 Ostrowski's type inequalities | 23 |
| | 2.3.3 Simpson's type inequalities | 29 |
| **CHAPTER III** | **HERMITE-HADAMARD'S TYPE INEQUALITIES** | 33 |
| 3.1 | Introduction | 33 |
| 3.2 | Inequalities Via $s$-Convex Functions | 33 |
| 3.3 | Inequalities Via Quasi-Convex Functions | 58 |
| 3.4 | Inequalities Via $r$-Convex Functions | 63 |

| | | |
|---|---|---|
| 3.5 | Applications to Trapezoidal Formula | 67 |
| 3.6 | Summary and Conclusion | 68 |
| **CHAPTER IV** | **OSTROWSKI'S TYPE INEQUALITIES** | 70 |
| 4.1 | Introduction | 70 |
| 4.2 | On The Ostrowski's Inequality | 70 |
| 4.3 | Ostrowski's Type Inequalities Via Convex Functions | 78 |
| 4.4 | Ostrowski's Type Inequalities Via $s$-Convex Functions | 89 |
| 4.5 | Ostrowski's Type Inequalities Via Quasi-Convex Functions | 95 |
| 4.6 | Ostrowski's Type Inequalities Via $r$-Convex Functions | 100 |
| 4.7 | Applications to Midpoint Formula | 104 |
| 4.8 | Applications to Special Means | 106 |
| 4.9 | Summary and Conclusion | 109 |
| **CHAPTER V** | **SIMPSON'S TYPE INEQUALITIES** | 111 |
| 5.1 | Introduction | 111 |
| 5.2 | Simpson's Type Inequalities Via $s$-Convex Functions | 111 |
| 5.3 | Simpson's Type Inequalities Via Quasi-Convex Functions | 122 |
| 5.4 | Simpson's Type Inequalities Via $r$-Convex Functions | 131 |
| 5.5 | Applications to Simpson's Formula | 134 |
| 5.6 | Applications to Special Means | 137 |
| 5.7 | Summary and Conclusion | 139 |
| **CHAPTER VI** | **FURTHER RESEARCH** | 33 |
| 6.1 | Inequalities For Convex Mappings | 33 |
| 6.2 | Inequalities For two or More Variables | 33 |

# LIST OF SYMBOLS

| | |
|---|---|
| $\mathbb{R}$ | The real numbers |
| $[a, b]$ | Real interval |
| $L[a, b]$ | The space of integrable functions ON $[a, b]$ |
| $P$ | partition of $[a, b]$ |
| $\mathcal{P}[a, b]$ | The set of all possible partitions of $[a, b]$ |
| $T_n\,(f, P)$ | Trapezoidal rule |
| $M_n\,(f, P)$ | Midpoint rule |
| $S_n(f, P)$ | Simpson's rule |
| $\sup(\cdot, \cdot)$ | Supremum |
| $\max(\cdot, \cdot)$ | Maximum |
| $A(\cdot, \cdot)$ | The arithmetic mean |
| $G(\cdot, \cdot)$ | The geometric mean |
| $H(\cdot, \cdot)$ | The harmonic mean |
| $I(\cdot, \cdot)$ | The identric mean |
| $L(\cdot, \cdot)$ | The logarithmic mean |
| $L_r(\cdot, \cdot)$ | The generalized log-mean |

# CHAPTER I

# INTRODUCTION

## 1.1 GENERAL INTRODUCTION

In mathematics, the word ‘inequality’ means a disparity between two quantities, which is used to reflect the correlation between two objects. Simply, an ‘inequality’ means that two quantities are not equal. In the 19th century and with the emergence of calculus, the touch of the inequalities and its role increasingly became essential.

In modern mathematics, inequalities play a significant role in almost all fields of mathematics. Several applications of inequalities are found in various areas of sciences such as, physical and engineering sciences. In numerical analysis, the approximation of a definite integral of a real function $f(t)$ over an interval $[a, b]$ is a very interesting problem. Therefore, many methods appeared in literature to solve such problems, and one of the most popular examples of such methods are the Newton-Cotes formulas (e.g. midpoint, trapezoidal and Simpson formulas). Error bounds for these approximations involve higher order derivatives, i.e., for the midpoint and the trapezoidal formulas the error bound involves a second-order derivative. In particular, the error bound of Simpson’s method involves a fourth-order derivative, and therefore this method has many disadvantages, such as it requires a lot of differentiation (if we assume the derivatives exist), with bounded derivatives, that make this class of functions inefficient and inelastic to solve such problems. In recent years, modern theory of inequalities is used at large and many efforts devoted to establish several generalizations of the Midpoint, Trapezoid and Simpson’s inequalities for mappings of bounded variation and for Lipschitzian, monotonic, and absolutely continuous mappings as well as $n$-times differentiable via kernels to refine the error bounds of these inequalities.

The main concern in this thesis is to investigate, several refinements inequalities of Hermite-Hadamard's, Ostrowski's and Simpson's type via quasi-convex, $s$-convex and $r$-convex functions and therefore obtains explicit bounds through the use of a Peano kernel approach and the modern theory of inequalities. The error bounds derived involved only at most second derivative. A generalization of the obtained results are considered by applying Hölder and power mean inequalities. Improvement for the previous inequalities in the literature for function $f$ with $|f'|$ (or $|f'|^q, q \geq 1$) is convex is given. Finally, some error estimates for midpoint, trapezoid and Simpson's rules and for some special means are derived. In this way, this thesis provides a study of some famous and fundamental inequalities originated by Hermite-Hadamard, Ostrowski and Simpson via three types of convex functions. This shall bring latest developments in the research area under a unified framework.

## 1.2 PROBLEM STATEMENT

In this thesis, the main problem statement is devoted to introduce and discuss several inequalities of Hermite-Hadamard's, Ostrowski's and Simpson's type via three kinds of convexities, namely, $s$-convex, quasi-convex and $r$-convex mappings. These convexities are used to obtain several refinements of the above mentioned inequalities. In addition, the problems where the midpoint, trapezoid and Simpson's quadrature rules cannot not be applied will be discussed in the thesis.

## 1.3 RESEARCH OBJECTIVES

The objectives of the research are:

1. To improve the role of convexity in the theory of inequalities.

2. To introduce several new inequalities of Hermite-Hadamard's, Ostrowski's and Simpson's type via $s$-convex, quasi-convex and $r$-convex mappings.

3. To establish alternatives quadrature rules using first derivative.

4. To find several error inequalities for some quadrature rules and for some special means.

## 1.4 RESEARCH METHODOLOGY

Three types of convex mappings together with suitable Peano kernels and Montgomery identity, are used to establish variant inequalities of Hermite-Hadamard's, Ostrowski's and Simpson's type in terms of first derivative of a real function. Several generalizations, refinements and improvements for the corresponding version for powers of these inequalities are considered by applying the Hölder and power mean inequalities. Therefore, some new error estimate for some quadrature rules and for some special means are derived.

## 1.5 THESIS ORGANIZATION

In the following we give an outline of our thesis organization, Consists of six chapters defining the work contributed. The first chapter gives a general introduction of the research work where the motivation and objectives are defined.

In chapter II, some basic concepts of convex, quasi-convex, $r$-convex and $s$-convex functions including some of its properties are given. Some known inequalities of Hermite–Hadamard's, Ostrowski's and Simpson's type with some related refinements and generalizations are briefly introduced.

In chapter III, some refinements, improvement and new inequalities of Hermite-Hadamard's type via $s$-convex, quasi-convex and $r$-convex functions are introduced. As applications, an estimation of error bounds to trapezoidal formula and to some special means are given.

In chapter IV, some refinements, improvement and new inequalities of Ostrowski's type via $s$-convex, quasi-convex and $r$-convex functions are introduced. As applications, an estimation of error bounds to midpoint formula and to some special means are given.

In chapter V, some refinements, improvement and new inequalities of Simpson's type via $s$-convex, quasi-convex and $r$-convex functions are introduced. As applications, an estimation of error bounds to Simpson's formula and to some special means are given.

In chapter VI, some topics for further research are suggested.

Conclusion and final remarks of this work are presented in the end of each chapter.

# CHAPTER II

# LITERATURE REVIEW AND BACKGROUND

## 2.1 INTRODUCTION

A classification of functions of a real variable is concerned with various special properties, such as continuity, convexity, monotonicity and differentiability. It is know that, convexity plays a significant role in the development of several branches of mathematics. In this section, we begin with formal description of this concept followed by a precise definition. Next, we outline some basic terminologies associated with convex functions. Later, we give another types of convexities, including quasi-convex, $r$-convex and $s$-convex functions. By means of these convexities, we discuss here a number of properties and some results related to the functions. Finally, we discuss some inequalities of Hermite-Hadamard's, Ostrowski's and Simpson's type.

## 2.2 ELEMENTARY CONCEPTS

### 2.2.1 Convex Functions

Let $I$ be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and for all $\alpha \in [0, 1]$, the inequality

$$f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y) \qquad (2.2.1)$$

holds. If (2.2.1) is strictly for all $x \neq y$ and $\alpha \in (0, 1)$, then $f$ is said to be strictly convex. If the inequality in (2.2.1) is reversed, then $f$ is said to be concave. If (2.2.1) is strictly for all $x \neq y$ and $\alpha \in (0, 1)$, then $f$ is said to be strictly concave (see (Pečarić et al. 1992)).

A simple geometric interpretation of (2.2.1) is that the graph of $f$ lies below its chords, i.e, if $P, Q$ and $R$ are any three points on the graph of $f$ such that $Q$ lies between $P$ and $R$, then $Q$ is on or below the chord $PR$. Equivalently, for all distinct $x_1, x_2, x_3 \in I$, with $x_1 < x_2 < x_3$, the following inequality

$$f\left(x_2\right)\left(x_3 - x_1\right) \leq \left(x_3 - x_2\right) f\left(x_1\right) + \left(x_2 - x_1\right) f\left(x_3\right), \tag{2.2.2}$$

holds. Another way of writing (2.2.2) is instructive:

$$\frac{f\left(x_2\right) - f\left(x_1\right)}{x_2 - x_1} \leq \frac{f\left(x_3\right) - f\left(x_1\right)}{x_3 - x_1}. \tag{2.2.3}$$

Here, we note that, if $f$ is defined on $[a, b]$, convex (concave) on $[a, b]$ and differentiable at $x_0$, then for $x \in (a, b)$ we have

$$f\left(x\right) - f\left(x_0\right) \geq (\leq) f'\left(x_0\right)\left(x - x_0\right). \tag{2.2.4}$$

If $f$ is differentiable on $(a, b)$, the $f$ is convex (concave) iff (2.2.4) holds for all $x, x_0 \in (a, b)$. Also, one can characterize the convexity of $f$ through derivatives as follows:

**Theorem 2.2.1.** *(Pečarić et al. 1992) Suppose that $f''$ exists on $(a, b)$. Then $f$ is convex (strictly convex) if and only if $f''(x) \geq (>) 0$, for all $x \in (a, b)$.*

**Definition 2.2.2.** *(Apostol 1974) A function $f : I \to \mathbb{R}$ is said to satisfy a Lipschitz condition of order $\alpha$, $\alpha > 0$ if there exists a positive number $L$ such that*

$$\left|f\left(x\right) - f\left(c\right)\right| < L\left|x - c\right|^{\alpha}. \tag{2.2.5}$$

*Moreover, if $0 < \alpha \leq 1$, then $f$ is said to satisfy a Hölder condition.*

We recall that a function $f$ which satisfies a Lipschitz condition of order $\alpha$ is continuous at $c$ if $\alpha > 0$, and differentiable at $c$ if $\alpha > 1$.

**Definition 2.2.3.** *(Apostol 1974) A function $f : [a, b] \to \mathbb{R}$ is said to be absolutely continuous on $[a, b]$ if for $\epsilon > 0$, there is $\delta > 0$ such that for any collection $\{(a_i, b_i)\}_{i=1}^{n}$ of disjoint open subintervals of $[a, b]$ with $\sum_{i=1}^{n}\left(b_i - a_i\right) < \delta$, we have $\sum_{i=1}^{n}\left|f\left(b_i\right) - f\left(a_i\right)\right| < \epsilon$.*

The relation between convex, Lipschitz continuous and absolutely continuous functions, is obtained in the following theorem.

**Theorem 2.2.4.** *(Pečarić et al. 1992) If $f : I \to \mathbb{R}$ is convex, then $f$ satisfies a Lipschitz condition of order 1, on any closed interval $[a, b]$ contained in the interior $I^\circ$ of $I$. Consequently, $f$ is absolutely continuous on $[a, b]$ and continuous on $I^\circ$.*

Monotonicity is a significant property for real-valued function defined on a subset of $\mathbb{R}$ that corresponds to its graph being increasing or decreasing. A monotonic function or monotonically increasing (decreasing) is just a function $f$ which preserves the order, i.e., for $I \subseteq \mathbb{R}$ and $x, y \in I$ with $x \leq y$, we have $f(x) \leq (\geq) f(y)$. The following two theorems concerning the relation between the monotonicity and the derivatives of convex functions.

**Theorem 2.2.5.** *(Pečarić et al. 1992) If $f : I \to \mathbb{R}$ is convex (strictly convex), then $f'_-(x)$ and $f'_+(x)$ exists and are increasing (strictly increasing) on $I^\circ$.*

**Theorem 2.2.6.** *(Pečarić et al. 1992) Suppose that $f$ is differentiable on $(a, b)$. Then $f$ is convex (strictly convex) if and only if $f'$ increasing (strictly increasing).*

A functions of bounded variation is an interesting class of functions that is closely related to monotonic functions. Let us recall some facts about functions of bounded variation

**Definition 2.2.7.** *(Apostol 1974) If $[a, b]$ is a compact interval , a set of points $P := \{x_0, x_1, \cdots, x_n\}$, satisfying the inequalities*

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,$$

*is called a partition of $[a, b]$. The interval $[x_{k-1}, x_k]$ is called $k$th subinterval of $P$ and we write $\Delta x_k = x_k - x_{k-1}$, so that $\sum_{k=1}^{n} \Delta x_k = b - a$. The collection of all possible partitions of $[a, b]$ will be denoted by $\mathcal{P}[a, b]$.*

**Definition 2.2.8.** *(Apostol 1974) Let $f$ be defined on $[a, b]$. If $P := \{x_0, x_1, \cdots, x_n\}$ is a partition of $[a, b]$, write $\Delta f_k = f(x_k) - f(x_{k-1})$, for $k = 1, 2, \cdots, n$. If there exists* *a positive number $M$ such that $\sum_{k=1}^{n} |\Delta f_k| \leq M$ for all partition of $[a, b]$, then $f$ is said to be of bounded variation on $[a, b]$.*

**Definition 2.2.9.** *(Apostol 1974) Let $f$ be of bounded variation on $[a, b]$, and let $\sum(P)$ denote the sum $\sum_{k=1}^{n} |\Delta f_k|$ corresponding to the partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$. The number*

$$\bigvee_a^b (f) = \sup \left\{ \sum (P) : P \in \mathcal{P}[a, b] \right\},$$

*is called the total variation of $f$ on the interval $[a, b]$.*

We note that a continuous function need not be of bounded variation, for example consider $f(x) = x \cos(\frac{\pi}{2x})$ if $x \neq 0$, $f(0) = 0$. For further detailed properties for functions of bounded variation see Apostol (1974).

### 2.2.2 Quasi-Convex Functions

The notion of *quasi-convex functions* generalizes the notion of convex functions. More precisely,

**Definition 2.2.10.** *(Pečarić et al. 1992) A function $f : [a, b] \to \mathbb{R}$ is said quasi-convex on $[a, b]$ if*

$$f(\alpha x + (1 - \alpha) y) \leq \max \{f(x), f(y)\}, \tag{2.2.6}$$

*for any $x, y \in [a, b]$ and $\alpha \in [0, 1]$.*

Clearly, any convex function is a quasi-convex function. However, there do exist quasi-convex functions which are not convex.

**Example 2.2.11.** *(Ion 2007) The function $h : [-2, 2] \to \mathbb{R}$,*

$$h(x) = \begin{cases} 1, & t \in [-2, -1] \\ t^2, & t \in (-1, 2] \end{cases} \tag{2.2.7}$$

*is not a convex function on $[-2, 2]$ but it is a quasi-convex function on $[-2, 2]$.*

Also, a quasi-convex function may be neither convex nor continuous. For example, the *floor function* $f_{loor}(x) = \lfloor x \rfloor$, is the largest integer not greater than $x$, is an example of a monotonic increasing function which is quasi-convex but it is neither convex nor continuous. For more details, we refer the reader to Roberts & Varberg (1973).

It is convenient to note that, the quasi-convex mappings may be not of bounded variation, i.e., there exists quasi-convex functions which are not of bounded variation. For example, consider the function $f : [0, 2] \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} x \sin\left(\frac{\pi}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases},$$

therefore, $f$ is quasi-convex but not of bounded variation on [0,2].

### 2.2.3 Mathematical Means and $r$–Convexity

In the following we study certain generalizations of some notions for a positive-valued function of a positive variable.

**Definition 2.2.12.** *(Bullen 2003) A function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$, is called a Mean function if it has the following properties:*

1. *Homogeneity:* $M(ax, ay) = aM(x, y)$, *for all* $a > 0$,
2. *Symmetry :* $M(x, y) = M(y, x)$,
3. *Reflexivity :* $M(x, x) = x$,
4. *Monotonicity: If* $x \leq x'$ *and* $y \leq y'$, *then* $M(x, y) \leq M(x', y')$,
5. *Internality:* $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

We shall consider the means for arbitrary positive real numbers $\alpha, \beta$ $(\alpha \neq \beta)$, (see Bullen (2003) and Bullen et al. (1988) ). We take

1. The arithmetic mean :

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \quad \alpha, \beta \in \mathbb{R}_+.$$

2. The geometric mean :

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}, \quad \alpha, \beta \in \mathbb{R}_+$$

3. The harmonic mean :

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}, \quad \alpha, \beta \in \mathbb{R}_+ - \{0\}.$$

4. The power mean :

$$M_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1, \ \alpha, \beta \in \mathbb{R}_+$$

5. The identric mean:

$$I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{\frac{1}{\beta-\alpha}}, & \alpha \neq \beta \\ \alpha, & \alpha = \beta \end{cases}, \quad \alpha, \beta > 0$$

6. The logarithmic mean :

$$L := L(\alpha, \beta) = \frac{\alpha - \beta}{\ln|\alpha| - \ln|\beta|}, \ |\alpha| \neq |\beta|, \ \alpha, \beta \neq 0, \ \alpha, \beta \in \mathbb{R}_+.$$

7. The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}, \ p \in \mathbb{R} \backslash \{-1, 0\}, \ \alpha, \beta > 0.$$

It is well known that $L_p$ is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$.

A positive function $f$ is *log-convex* on a real interval $[a,b]$ if for all $x,y\in[a,b]$ and $\lambda\in[0,1]$ we have

$$f\left(\lambda x+(1-\lambda)\,y\right)\leq f^{\lambda}\left(x\right)f^{1-\lambda}\left(y\right). \tag{2.2.8}$$

If the reverse inequality holds, $f$ is said to be log-concave (see Pečarić et al. (1992)).

In 1997, Gill et al. used the power mean $M_r(x,y;\lambda)$ of order $r$ of positive numbers $x,y$, which is defined by

$$M_r\left(x,y;\lambda\right)=\begin{cases}\left(\lambda x^r+(1-\lambda)\,y^r\right)^{1/r}, & r\neq 0\\ x^{\lambda}y^{1-\lambda}, & r=0\end{cases} \tag{2.2.9}$$

to define the concept of $r$-convex mapping, as follows

**Definition 2.2.13.** *(Gill et al. 1997) A positive function $f:[a,b]\to\mathbb{R}_+$, is called $r$-convex function if for all $x,y\in[a,b]$ and $\lambda\in[0,1]$ we have*

$$f\left(\lambda x+(1-\lambda)\,y\right)\leq M_r\left(f\left(x\right),f\left(y\right);\lambda\right) \tag{2.2.10}$$

In the above definition, we have that 0-convex functions are simply log-convex functions and 1-convex functions are ordinary convex functions. Further generalization of the extension of Hadamard's inequality to $r$-convex functions and other related results are considered in and Pearce et al. (1998).

Also, Definition 2.2.13 of $r$-convexity can be expanded as the condition that

$$f^r\left(\lambda x+(1-\lambda)\,y\right)\leq\begin{cases}\lambda f^r\left(x\right)+(1-\lambda)\,f^r\left(y\right), & r\neq 0\\ f^{\lambda}\left(x\right)f^{1-\lambda}\left(y\right), & r=0\end{cases}$$

In 1998, Pearce et al., proved that for a nonnegative function $f$ that possesses a second derivative. If $r\geq 2$, then

$$\frac{d^2f^r}{dx^2}=r\left(r-1\right)f^{r-2}\left(f'\right)^2+rf^{r-1}f''$$

which is nonnegative if $f'' \geq 0$. Hence under the above restrictions, ordinary convexity implies $r$-convexity. The reverse implication is not the case, as is shown by the function, $f(x) = x^{1/2}$ for $x > 0$.

Another definition for $r$-convex mapping was known in the literature, which is quite different from Definition 2.2.13. Let us call Namely, a function $f : [a, b] \to \mathbb{R}$ is said to be $r$-convex ($r = 0, 1, 2, \cdots$), if for all choices of $x_0, x_1, \cdots, x_{r+1} \in [a, b]$ such that $x_0 < x_1 < ... < x_{r+1}$, the divided difference $[x_0, \cdots, x_{r+1}; f] \geq 0$, where,

$$[x_0; f] = f(x_0), \qquad [x_0, \cdots, x_r; f] = \frac{[x_0, \cdots, x_{r-1}; f] - [x_1, \cdots, x_r; f]}{x_0 - x_r},$$

see Pečarić et al. (1992). Equivalently, when the derivative $\frac{d^r}{dt^r} f(t)$ exists, $f$ is $r$-convex if and only if $\frac{d^r}{dt^r} f(t) \geq 0$. For example, Pearce et al. (1998), considered the function $f(x) = x(x^3 - x^2 + 1)$, $x \in (1/2, 1)$, and they showed that $\frac{d^2}{dt^2} < 0$ but $\frac{d^3}{dt^3} > 0$, so that $f$ is 3–convex but not convex. Also, the function $g = -f$ on the same domain is a function which is convex but not 3–convex.

### 2.2.4 $s$-Convex Functions in The Second Sense

Due to Hudzik and Maligranda (1994), two definitions of $s$-convexity $(0 < s \leq 1)$ of real-valued functions are known in the literature, and given below:

**Definition 2.2.14.** *(Orlicz 1961) A function $f : \mathbb{R}_+ \to \mathbb{R}$, where $\mathbb{R}_+ = [0, \infty)$, is said to be $s$-convex in the first sense if*

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y) \tag{2.2.11}$$

*for all $x, y \in [0, \infty)$, $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$ and for some fixed $s \in (0, 1]$. This class of functions is denoted by $K_s^1$.*

This definition of $s$-convexity, for so called $\varphi$-functions, was introduced by Orlicz in 1961 and was used in the theory of Orlicz spaces ( Matuszewska and Orlicz (1961), Musielak (1983), Rolewicz (1984)). A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a $\varphi$-function if $f(0) = 0$ and $f$ is nondecreasing and continuous. The symbol $\varphi$ stands for an Orlicz function,i.e., $\varphi$ is a convex, even, vanishing and continuous at zero function defined on the real line $\mathbb{R}$ and with values in $[0, +\infty]$.

In 2007, Pinheiro , claimed that this class of $K_s^1$ has many problems. Pinheiro studied this class of $s$-convex functions and explained why the first $s$-convexity sense was abandoned by the literature in the field. Pinheiro revised the class of $s$–convexity in the first sense and proposed a geometric interpretation for functions in $K_s^1$ with some related results. For further results concerning $s$-convexity in the first sense see Pinheiro (2008).

**Definition 2.2.15.** *(Breckner 1978) A function $f : \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+ = [0, \infty)$, is said to be $s$–convex in the second sense if*

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y) \tag{2.2.12}$$

*for all $x, y \in [0, \infty)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0, 1]$. This class of functions is denoted by $K_s^2$.*

This definition of $s$-convexity considered by Breckner, where the problem when the rational $s$-convex functions are $s$-convex was considered. Also, we note that, it can be easily seen that for $s = 1$, $s$-convexity (in both senses) reduces to the ordinary convexity of functions defined on $[0, \infty)$.

In 1994, Hudzik and Maligranda, realized the importance and undertook a systematic study of $s$-convex functions in both sense. They compared the notion of Breckner $s$-convexity with a similar one of Orlicz (1961). A function $f$ is Orlicz $s$-convex if the inequality (2.2.11) is satisfied for all $\alpha, \beta$ such that $\alpha^s + \beta^s = 1$. Hudzik & Maligranda, among others, gave an example of a non-continuous Orlicz $s$-convex function, which is not Breckner $s$-convex.

In the following, we shall consider some Hudzik and Maligranda results, that are connected with $s$-convex functions in the second sense.

**Theorem 2.2.16.** *(Hudzik & Maligranda 1994) If $f \in K_s^2$, then $f$ is non-negative on* $[0, \infty)$.

An example of $s$-convex functions of the second sense is introduced as follows:

**Example 2.2.17.** *(Hudzik & Maligranda 1994) Let $0 < s < 1$ and $a, b, c \in \mathbb{R}$. Defining for $u \in \mathbb{R}_+$,*

$$f(u) = \begin{cases} a, & u = 0, \\ \\ bu^s + c, & u > 0, \end{cases} \tag{2.2.13}$$

*we have the following*

*(i)* $b \geq 0$ *and* $0 \leq c \leq a$, *then* $f \in K_s^2$.

*(ii)* $b > 0$ *and* $c < 0$, *then* $f \notin K_s^2$.

As Hudzik and Maligranda pointed out, it is important to know where the condition $\alpha + \beta = 1$ in the definition of $K_s^2$ can be equivalently replaced by the condition $\alpha + \beta \leq 1$.

**Theorem 2.2.18.** *(Hudzik & Maligranda 1994) Let $f \in K_s^2$. Then the inequality (2.2.12) holds for all $u, v \in \mathbb{R}_+$ and $\alpha, \beta \geq 0$ with $\alpha + \beta \leq 1$ if and only if $f(0) = 0$.*

Some properties of $s$–convex mappings in the second sense are considered as follow:

**Theorem 2.2.19.** *(Hudzik & Maligranda 1994) Let $0 < s_1 \leq s_2 < 1$. If $f \in K_{s_2}^2$ and $f(0) = 0$, then $f \in K_{s_1}^2$.*

**Theorem 2.2.20.** *(Hudzik & Maligranda 1994) Let $f$ be a nondecreasing function in $K_s^2$ and $g$ be a nonnegative convex function on $[0, \infty)$. Then the composition $f \circ g$ of $f$ with $g$ belongs $K_s^2$.*

Recently, Pinheiro devoted her efforts to give a clear geometric definition for $s$-convexity in the second sense. In 2007, Pinheiro successfully proposed a geometric description for $s$-convex curve, as follows:

**Definition 2.2.21.** *(Pinheiro 2007) $f$ is called $s$–convex, $0 < s < 1$, $f \geq 0$, if the graph of $f$ lies below a 'bent chord' $L$ between any two points. That is, for every compact interval $J \subset I$, with boundary $\partial J$, it is true that*

$$\sup_J (L - f) \geq \sup_{\partial J} (L - f).$$

Indeed the geometric view for $s$-convex mapping of second sense is going through which Pinheiro called it '*limiting curve*', which is going to distinguish curves that are $s$-convex of second sense from those that are not. After that, Pinheiro obtained how the choice of '$s$' affects the limiting curve. In general a 'limiting curve' may be described by a *bent chord* joining $f(x)$ to $f(y)$-corresponding to the verification of the $s$-convexity property of the function $f$ in the interval $[x, y]$-forms representing the limiting height for the curve $f$ to be at, limit included, in case $f$ is $s$-convex. This curve is represented by $\lambda^s f(x) + (1 - \lambda)^s f(y)$, for each $0 < s < 1$.

## 2.3 INEQUALITIES VIA MONTGOMERY IDENTITY AND PEANO KERNEL

In recent years, a number of authors have considered an error analysis quadrature rules of Newton-Cotes type. In particular, the mid-point, trapezoid and Simpson's have been investigated more recently with the view of obtaining bounds on the quadrature rule in terms of a variety of convex mappings, at most first or second derivatives. This particular section will touch upon some background literature on some inequalities of Hermite-Hadamard's, Ostrowski's and Simpson's type via several real mappings.

Before this, let us recall some famous results obtained in the literature. The following theorem contains the integral inequality which is known in the literature as Montgomery identity.

**Theorem 2.3.1.** *(Mitrinović et al. (1994)): Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ and $f' : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, then the following Montgomery identity holds*

$$f(x) = \frac{1}{b-a} \int_a^b f(t)\, dt + \int_a^b P(x, t) f'(t)\, dt \tag{2.3.1}$$

*where $P(x,t)$ is the Peano kernel,*

$$P(x,t)=\begin{cases}\frac{t-a}{b-a}, & a\le t\le x,\\ \frac{t-b}{b-a}, & x<t\le b.\end{cases}$$

Suppose now that $w:[a,b]\to[0,\infty)$ is some probability density function, i.e. is a positive integrable function satisfying $\int_a^b w(t)\,dt=1$, and

$$W(t)=\begin{cases}0, & t<a,\\ \int_a^t w(x)\,dx, & a\le t\le b,\\ 1, & b<t.\end{cases}$$

The following identity is a generalization of Montgomery's identity,

$$f(x)=\int_a^b w(t)\,f(t)\,dt+\int_a^b P_w(x,t)\,f'(t)\,dt \tag{2.3.2}$$

where the weighted Peano kernel is

$$P_w(x,t)=\begin{cases}W(t), & a\le t\le x,\\ W(t)-1, & x<t\le b.\end{cases}$$

This generalization of Montgomery's identity is considered by Pečarić (1980).

**Theorem 2.3.2.** *(Mitrinović et al. 1993) Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If $f$ and $g$ are real functions defined on $[a,b]$ and if $|f|^p$ and $|g|^q$ are integrable functions on $[a,b]$, then*

$$\int_a^b |f(x)\,g(x)|\,dx\le\left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b |g(x)|^q\,dx\right)^{\frac{1}{q}}. \tag{2.3.3}$$

**Theorem 2.3.3.** *(Mitrinović et al. 1993) Let $f$ be a convex function on the open $(a,b)$ and let $x(t):[c,d]\to\mathbb{R}$ be integrable with $a<x(t)<b$. If $\alpha(t):[c,d]\to\mathbb{R}$ is positive, $\int_c^d\alpha(t)\,dt=1$, and $(\alpha\cdot x)(t)$ is integrable on $[c,d]$, then*

$$f\left(\int_c^d \alpha(t)\,x(t)\,dt\right)\le\int_c^d \alpha(t)\,f(x(t))\,dt. \tag{2.3.4}$$

**Theorem 2.3.4.** *(Mitrinović et al. 1993) For real numbers $q, q_0$ with $q \geq q_0$ and positive real numbers $a_1, a_2, ..., a_n$, the following inequality holds:*

$$\left( \frac{\sum_{i=1}^{n} a_i^{q_0}}{n} \right)^{\frac{1}{q_0}} \leq \left( \frac{\sum_{i=1}^{n} a_i^{q}}{n} \right)^{\frac{1}{q}}. \tag{2.3.5}$$

Thus, the integral form may be written such as:

$$\left( \frac{\int_a^b p(x) f^{q_0}(x)\, dx}{\int_a^b p(x)\, dx} \right)^{\frac{1}{q_0}} \leq \left( \frac{\int_a^b p(x) f^{q}(x)\, dx}{\int_a^b p(x)\, dx} \right)^{\frac{1}{q}}. \tag{2.3.6}$$

For instance, if $q_0 = 1$, then we have

$$\frac{\int_a^b p(x) f(x)\, dx}{\int_a^b p(x)\, dx} \leq \left( \frac{\int_a^b p(x) f^{q}(x)\, dx}{\int_a^b p(x)\, dx} \right)^{\frac{1}{q}}. \tag{2.3.7}$$

### 2.3.1 Hermite-Hadamard's Type Inequalities

Let $f : [a, b] \to \mathbb{R}$, be a twice differentiable mapping such that $f''(x)$ exists on $(a, b)$ and $\|f''\|_\infty = \sup_{x \in (a,b)} |f''(x)| < \infty$. Then the midpoint inequality is known as:

$$\left| \int_a^b f(x)\, dx - (b-a) f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^3}{24} \|f''\|_\infty, \tag{2.3.8}$$

and, the trapezoid inequality

$$\left| \int_a^b f(x)\, dx - (b-a) \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)^3}{12} \|f''\|_\infty, \tag{2.3.9}$$

also hold. Therefore, we can approximate the integral $\int_a^b f(x)\, dx$ in terms of the midpoint and the trapezoidal rules, respectively such as:

$$\int_a^b f(x)\, dx \cong (b-a) f\left(\frac{a+b}{2}\right),$$

and

$$\int_a^b f(x)\, dx \cong (b-a) \frac{f(a) + f(b)}{2},$$

which are grouped in a very interesting and useful relationship, known as the Hermite-Hadamard's inequality. That is,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int\limits_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{2.3.10}$$

which hold for all convex functions $f:[a,b]\rightarrow\mathbb{R}$. As pointed out by Mitrinović and Lačković (1985) the inequalities (2.3.10) are due to Hermite who obtained it in 1893, ten years before Hadamard.

It is clear that if the mapping $f$ is not twice differentiable or the second derivative is not bounded on (a,b), then (2.3.8) and (2.3.9) cannot be applied. Prompting many authors to find alternative inequalities involving, at most the first derivative. A few years ago, several types of the Montgomery's identity and Peano kernel have been used to obtain various inequalities for several kinds of convex functions.

In 1998, Dragomir and Agarwal, obtained inequalities for differentiable convex mappings which are connected with the right-hand side of Hermite-Hadamard's (trapezoid) inequality and they used the following lemma to prove it.

**Lemma 2.3.5.** *(Dragomir & Agarwal 1998) Let $f: I\subset\mathbb{R}\rightarrow\mathbb{R}$ be a differentiable mapping on $I^{\circ}$ where $a,b\in I$ with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{b-a}{2}\int_0^1(1-2t)\,f'(ta+(1-t)\,b)\,dt. \tag{2.3.11}$$

Therefore, they proved the following result:

**Theorem 2.3.6.** *(Dragomir & Agarwal 1998) Let $f: I\subset\mathbb{R}\rightarrow\mathbb{R}$ be a differentiable mapping on $I^{\circ}$, where $a,b\in I$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{b-a}{8}\left[|f'(a)|+|f'(b)|\right]. \tag{2.3.12}$$

In 2000, Pearce and Pečarić generalized Theorem 2.3.6 and they proved the following inequalities:

**Theorem 2.3.7.** *(Pearce & Pečarić 2000) Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$, for some $q \geq 1$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq \frac{b-a}{4} \left[ \frac{|f(a)|^q + |f(b)|^q}{2} \right]^{\frac{1}{q}}, \tag{2.3.13}$$

*and*

$$\left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq \frac{b-a}{4} \left[ \frac{|f(a)|^q + |f(b)|^q}{2} \right]^{\frac{1}{q}}. \tag{2.3.14}$$

*If $|f|^q$ is concave on $[a, b]$ for some $q \geq 1$, then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq \frac{b-a}{4} \left| f'\left( \frac{a+b}{2} \right) \right| \tag{2.3.15}$$

*and*

$$\left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq \frac{b-a}{4} \left| f'\left( \frac{a+b}{2} \right) \right|. \tag{2.3.16}$$

In the same way as Dragomir and Agarwal approaches, inequalities for differentiable convex mappings which are connected with the left-hand side of Hermite-Hadamard's (midpoint) inequality was proved by Kirmaci in 2004 , using the following lemma:

**Lemma 2.3.8.** *(Kirmaci 2004) Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$ where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{1}{b-a} \int_a^b f(x)\, dx - f\left( \frac{a+b}{2} \right) = (b-a) \int_0^1 K(t)\, f'(ta + (1-t)b)\, dt \tag{2.3.17}$$

*where,*

$$K(t) = \begin{cases} t, & t \in \left[0, \frac{1}{2}\right], \\ \\ t-1, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Namely, Kirmaci proved the following result:

**Theorem 2.3.9.** *(Kirmaci 2004) Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} \left[ |f'(a)| + |f'(b)| \right]. \tag{2.3.18}$$

For more refinements, generalization and new results related to (2.3.12) and (2.3.18), are considered in Özdemir (2003), Kirmaci and Özdemir (2004a), Kirmaci and Özdemir (2004a), Kirmaci and Özdemir (2004b) and Kirmaci (2008).

In 2004, Yang obtained a very interesting inequalities for differentiable convex and concave mappings that are connected with the both sides of celebrated Hermite–Hadamard integral inequality as follow:

**Theorem 2.3.10.** *(Yang et al. 2004) Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}$ where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$, $q \geq 1$, then the following inequality holds:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq \frac{b-a}{12} \left[ |f'(a)|^q + \left| f'\left(\frac{a+b}{2}\right) \right|^q + |f'(b)|^q \right]^{1/q} \tag{2.3.19}$$

*and*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq \frac{b-a}{24} \left[ |f'(a)|^q + 4\left| f'\left(\frac{a+b}{2}\right) \right|^q + |f'(b)|^q \right]^{1/q}. \tag{2.3.20}$$

*If $|f|^q$ is concave on $[a, b]$ for some $q \geq 1$, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq \frac{b-a}{8} \left[ \left| f'\left(\frac{5a+b}{6}\right) \right| + \left| f'\left(\frac{a+5b}{6}\right) \right| \right] \tag{2.3.21}$$

*and*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq \frac{b-a}{8} \left[ \left| f'\left(\frac{2a+b}{3}\right) \right| + \left| f'\left(\frac{a+2b}{3}\right) \right| \right]. \tag{2.3.22}$$

For further results concerning the Hermite–Hadamard's inequality for monotone and Lipschitz mappings and functions of bounded variation, we refer the reader to Dragomir (1999c), Dragomir (2000a), Dragomir (2000b), Dragomir and Mabizela (2000), Cerone and Dragomir (2000a), Cerone and Dragomir (2000b), Cerone et al. (2000a), Dragomir (2001c), Dragomir (2001d), Barnett et al. (2002), (Barnett & Dragomir 2004), Dragomir (2004), Tseng et al. (2004), Dragomir et al. (2007), Tseng et al. (2007b) and Dragomir (2008).

Since there is a different types of convexities, many authors proved various inequalities of Hermite-Hadamard type. In Dragomir and Fitzpatrick (1999), proved a variant of Hadamard's inequality which holds for $s$-convex functions in the second sense, as follows:

**Theorem 2.3.11.** *(Dragomir & Fitzpatrick 1999) Suppose that $f : [0,\infty) \to [0,\infty)$ is an $s$–convex function in the second sense, where $s \in (0,1)$ and let $a,b \in [0,\infty)$, $a < b$. If $f \in L^1[a,b]$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{s+1}. \tag{2.3.23}$$

*The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (2.3.23). The above inequalities are sharp.*

Several inequalities of Hermite-Hadamard type for differentiable functions based on concavity and $s$-convexity established in Kirmaci et al. (2007), are presented below:

**Theorem 2.3.12.** *(Kirmaci et al. 2007) Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is $s$-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q \ge 1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\qquad \le \frac{b-a}{2}\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left[\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]^{\frac{1}{q}}\left(|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}}.\end{aligned} \tag{2.3.24}$$

**Theorem 2.3.13.** *(Kirmaci et al. 2007) Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is $s$-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \qquad (2.3.25) \\ &\le \frac{b-a}{2}\left[\frac{q-1}{2(2q-1)}\right]^{\frac{q-1}{q}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \\ &\quad \times \left[\left(|f'(a)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q\right)^{\frac{1}{q}}\right] \\ &\le \frac{b-a}{2}\left[\left(|f'(a)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q\right)^{\frac{1}{q}}\right]. \end{aligned}$$

**Theorem 2.3.14.** *(Kirmaci et al. 2007) Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is $s$-concave on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \qquad (2.3.26) \\ &\le \frac{b-a}{2}\left[\frac{q-1}{2(2q-1)}\right]^{\frac{q-1}{q}} 2^{\frac{s-1}{q}}\left(\left|f'\left(\frac{a+3b}{2}\right)\right| + \left|f'\left(\frac{3a+b}{2}\right)\right|\right) \\ &\le \frac{b-a}{2}\left(\left|f'\left(\frac{a+3b}{2}\right)\right| + \left|f'\left(\frac{3a+b}{2}\right)\right|\right). \end{aligned}$$

Recently, Ion (2007) obtained two inequalities of the right hand side of Hermite-Hadamard's type for functions whose derivatives in absolute values are quasi-convex functions, as follow:

**Theorem 2.3.15.** *(Ion 2007) Let $f : I^{\circ} \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on $I^{\circ}$, $a, b \in I^{\circ}$ with $a < b$. If $|f'|$ is quasi-convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{4}\max\{|f'(a)|, |f'(b)|\}. \qquad (2.3.27)$$

**Theorem 2.3.16.** *(Ion 2007) Let $f : I^{\circ} \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on $I^{\circ}$, $a, b \in I^{\circ}$ with $a < b$. If $|f'|^{p/(p-1)}$ is quasi-convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\qquad \le \frac{(b-a)}{2(p+1)^{1/p}}\left(\max\left\{|f'(a)|^{p/(p-1)}, |f'(b)|^{p/(p-1)}\right\}\right)^{(p-1)/p}. \qquad (2.3.28) \end{aligned}$$

In Gill et al. (1997), the authors developed some Hadamard-type inequalities for log-convex functions and more generally for $r$-convex functions, as follow:

**Theorem 2.3.17.** *(Gill et al. 1997) Let $f$ be a positive, log-convex function on $[a, b]$. Then*

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq L(f(a), f(b)). \tag{2.3.29}$$

*For $f$ a positive log-concave function, the inequality is reversed.*

**Theorem 2.3.18.** *(Gill et al. 1997) Let $f$ be a positive, $r$-convex function on $[a, b]$. Then*

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq L_r(f(a), f(b)). \tag{2.3.30}$$

*For $f$ a positive $r$-concave function, the inequality is reversed.*

Several inequalities are obtained for some $n$-time differentiable convex mappings that are connected with Hermite–Hadamard type inequalities for higher order derivatives in Cerone et al. (2000a), Cerone et al. (2000b), Cerone and Dragomir (2000a), Cerone and Dragomir (2000b), Cerone et al. (2000b), Barnett & Dragomir (2002a), Barnett and Dragomir (2002b), Hwang (2003), Pečarić and Vukelić (2003), Dedić et al. (2001b), (Dedić et al. 2005), Dedić et al. (2006), Barnett and Dragomir (2007), Ujević and Erceg (2007), and Dragomir and Sofo (2008). For a comprehensive list of refinements, counterparts, generalizations and new inequalities of Hermite–Hadamard's type see Mitrinović et al. (1993), Dragomir and Pearce (2000) and Pachpatte (2005a).

### 2.3.2 Ostrowski's Type Inequalities

In 1938, Ostrowski established a very interesting inequality for differentiable mappings with bounded derivatives, as follows:

**Theorem 2.3.19.** *(Ostrowski 1938) Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, the interior of the interval $I$, such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the following inequality,*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{2.3.31}$$

*holds for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.*

In 1992, Fink and earlier in 1976, Milovanović and Pečarić have obtained some interesting generalizations of (2.3.31) in the form

$$\left|\frac{1}{n}\left(f(x)+\sum_{k=1}^{n-1}F_k(x)\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le C(n,p,x)\left\|f^{(n)}\right\|_\infty \tag{2.3.32}$$

where,

$$F_k(x)=\frac{n-k}{n!}\frac{f^{(k-1)}(a)(x-a)^k-f^{(k-1)}(b)(x-b)^k}{b-a},$$

and, $\|\cdot\|_r$, $1 \le r \le \infty$ are the usual Lebesgue norms on $L_r[a,b]$, i.e.,

$$\|f\|_\infty := ess \sup_{t\in[a,b]} |f(t)|,$$

and

$$\|f\|_r := \left(\int_a^b |f(t)|^r\,dt\right)^{1/r}, \;\; 1 \le r < \infty.$$

In fact, Milovanović and Pečarić (see also Mitrinović et al. (1994)) have proved that

$$C(n,\infty,x)=\frac{(x-a)^{n+1}+(b-x)^{n+1}}{(b-a)\,n\,(n+1)!},$$

while Fink proved that the inequality (2.3.32) holds provided $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)} \in L_p[a,b]$, with

$$C(n,p,x)=\frac{\left[(x-a)^{nq+1}+(b-x)^{nq+1}\right]^{1/q}}{(b-a)\,n!}\beta^{1/q}\left((n-1)q+1,q+1\right),$$

for $1 < p \le \infty$, $\beta$ is the beta function, and

$$C(n,1,x)=\frac{(n-1)^{n-1}}{(b-a)\,n^n n!}\max\left\{(x-a)^n,(b-x)^n\right\}.$$

Recently, Pachpatte (2004b), Matić et al. (2002), Dedić et al. (2000) and Pearce and Pecaric (2000), have given some generalizations of Milovanović and Pečarić (1976) and Fink (1992) inequalities. For multivariate, univariate, higher order Ostrowski type inequalities over Euclidean domains and multivariate Fink type identity, we refer the reader to Anastassiou (1995), Anastassiou (1997), Anastassiou (2002), Pachpatte (2002) Anastassiou (2007) and Anastassiou and Goldstein (2007) and (2008).

In 2000, Dragomir introduced an Ostrowski type integral inequality for the Riemann-Stieltjes integral, as follows:

**Theorem 2.3.20.** *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation and $u : [a, b] \to \mathbb{R}$ a function of $r$-$H$-Hölder type, i.e.,*

$$|u(x) - u(y)| \le H |x - y|^r, \quad \forall x, y \in [a, b],$$

*where $r \in (0, 1]$ and $H > 0$ are given. Then, for any $x \in [a, b]$,*

$$\begin{aligned}
&\left| [u(b) - u(a)] f(x) - \int_a^b f(t)\, du(t) \right| \\
&\le H \left[ (x-a)^r \bigvee_a^x (f) + (b-x)^r \bigvee_x^b (f) \right] \qquad (2.3.33) \\
&\le H \times \begin{cases} [(x-a)^r + (b-x)^r] \left[ \frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right] \\ [(x-a)^{qr} + (b-x)^{qr}]^{1/q} \left[ \left( \bigvee_a^x (f) \right)^p + \left( \bigvee_x^b (f) \right)^p \right]^{1/p}, p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \left[ \frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^r \bigvee_a^b (f) \end{cases}
\end{aligned}$$

*where, $\bigvee_c^d (f)$ denotes the total variation of $f$ on the interval $[c, d]$.*

For other results concerning inequalities for Stieltjes integrals, see Liu (2004) and Cerone and Dragomir (2002). In 2007, Cerone et al. established some Ostrowski type inequalities for the Stieltjes integral where the integrand is absolutely continuous while the integrator is of bounded variation. Also, the case when $|f'|$ is convex was explored.

Dragomir and Wang (1997), (1998a) and (1998b), introduced the following results concerning inequalities of Ostrowski's type for absolutely continuous functions.

**Theorem 2.3.21.** *(Dragomir & Rassias 2002) Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f' \in L^q[a,b]$, then for all $x \in [a,b]$, we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left\|f'\right\|_\infty, & if\ f' \in L_\infty[a,b] \\ \frac{(b-a)^{1/p}\|f'\|_q}{(1+p)^{1/p}}\left[\left(\frac{x-a}{b-a}\right)^{p+1} + \left(\frac{b-x}{b-a}\right)^{p+1}\right]^{1/p}, & if\ f' \in L_q[a,b], \frac{1}{p}+\frac{1}{q}=1, p>1 \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left\|f'\right\|_1 \end{cases} \tag{2.3.34}$$

*where, $\|\cdot\|_r$, $1 \leq r \leq \infty$ are the usual Lebesgue norms on $L_r[a,b]$, i.e.,*

$$\|g\|_\infty := ess \sup_{t\in[a,b]} |g(t)|,$$

*and*

$$\|g\|_r := \left(\int_a^b |g(t)|^r\,dt\right)^{1/r}, \ \ 1 \leq r < \infty.$$

*The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{1/p}}$ and $\frac{1}{2}$ are the best possible in the sense that it cannot be replaced by a smaller constant.*

During the past few years, many researchers have given considerable attention to the Ostrowski's inequality. Further extended results to incorporate mappings of bounded variation, Lipschitzian mappings and monotonic mappings see Dragomir (1999f), Dragomir (2001b), Cerone et al. (2008) and Tseng et al. (2008),

An Ostrowski type inequality for convex functions was pointed out by Barnett et al., as follows:

**Theorem 2.3.22.** *(Barnett et al. 2003) Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous* *function on $[a,b]$ such that $|f'|$ is convex on $[a,b]$. Then for any $x\in[a,b]$ we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \begin{cases} \frac{1}{2}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[f'(x)+\|f'\|_\infty\right], & f'\in L_\infty[a,b]; \\ \frac{1}{2}\cdot\frac{1}{(q+1)^{1/q}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\,\||f'(x)|+|f'|\|_\infty & \\ \qquad f'\in L_p[a,b],\, p>1,\, \frac{1}{p}+\frac{1}{q}=1 & \\ \frac{1}{2}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[(b-a)f'(x)+\|f'\|_1\right]. & \end{cases} \tag{2.3.35}$$

*The constant $\frac{1}{2}$ in the first and second inequalities is sharp as is the first $\frac{1}{2}$ in the final.*

In the following, some Ostrowski type inequalities for absolutely continuous functions whose first derivative satisfies certain convexity assumptions are considered.

**Lemma 2.3.23.** *(Cerone & Dragomir 2004a) Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $I^\circ$ where $a,b\in I$ with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$f(x)-\frac{1}{b-a}\int_a^b f(u)\,du = \frac{(x-a)^2}{b-a}\int_0^1 tf'(tx+(1-t)a)\,dt - \frac{(b-x)^2}{b-a}\int_0^1 tf'(tx+(1-t)b)\,dt$$

*for each $x\in[a,b]$.*

**Theorem 2.3.24.** *(Cerone & Dragomir 2004a) Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $x\in[a,b]$. If $|f'|$ is convex on $[a,x]$ and $[x,b]$, then one has the inequality :*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{1}{6}\left[|f'(a)|\left(\frac{x-a}{b-a}\right)^2+|f'(b)|\left(\frac{b-x}{b-a}\right) + \left(1+2\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2|f'(x)|\right)(b-a)\right] \tag{2.3.36}$$

*The constant $\frac{1}{6}$ is best possible in the sense that cannot be replaced by a smaller value.*

The following Ostrowski type inequality for absolutely continuous functions for which $|f'|$ is quasi-convex holds.

**Theorem 2.3.25.** *(Cerone & Dragomir 2004a) Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $x \in [a,b]$.*

1. *If $|f'|$ is quasi-convex on $[a,x]$ and $[x,b]$, then one has the inequality :*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\,du \right| \\ &\quad \le \frac{1}{4} \left[ \left( \frac{x-a}{b-a} \right)^2 \left( |f'(a)| + |f'(x)| + ||f'(x)| - |f'(a)|| \right) \right. \\ &\qquad \left. + \left( \frac{b-x}{b-a} \right)^2 \left( |f'(a)| + |f'(x)| + ||f'(x)| - |f'(b)|| \right) \right]. \end{aligned} \tag{2.3.37}$$

2. *If $|f'|$ is log-convex on $[a,x]$ and $[x,b]$, then one has the inequality :*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\,du \right| \\ &\quad \le (b-a) \left[ \left( \frac{x-a}{b-a} \right)^2 |f'(a)| \frac{A \ln A + 1 - A}{(\ln A)^2} \right. \\ &\qquad \left. + \left( \frac{b-x}{b-a} \right)^2 |f'(b)| \frac{B \ln B + 1 - B}{(\ln B)^2} \right], \end{aligned} \tag{2.3.38}$$

*where,*

$$A := \frac{f'(x)}{f'(a)}, \quad B := \frac{f'(x)}{f'(b)}.$$

*The constant $\frac{1}{4}$ in (2.3.37) is best possible in the sense that cannot be replaced by a smaller value.*

In 2005, Pachpatte proved some inequalities of Ostrowski type involving two functions and their derivatives.

**Theorem 2.3.26.** *(Pachpatte 2005b) Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$, whose derivatives $f', g' : (a,b) \to \mathbb{R}$ are bounded, i.e., $\|f'\|_\infty =$*

$\sup\limits_{x\in(a,b)} |f'(x)| < \infty$, $\|g'\|_\infty = \sup\limits_{x\in(a,b)} |g'(x)| < \infty$. *Then*

$$\left|f(x)\,g(x) - \frac{1}{2(b-a)}\left[g(x)\int_a^b f(y)\,dy + f(x)\int_a^b g(y)\,dy\right]\right|$$
$$\leq \frac{b-a}{2}\left\{|g(x)|\,\|f'\|_\infty + |f(x)|\,\|g'\|_\infty\right\}\left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]. \quad (2.3.39)$$

For other inequalities of the type (2.3.39), see the book Mitrinović et al. (1994), where many other references are given. Further extended results was proved in Pachpatte (2006), where Pachpatte proved Ostrowski type inequalities involving product of two functions. The analysis used in the proofs is elementary and based on the use of the integral identity recently established in Dedić et al. (2003).

Several generalizations of (2.3.31) by considering $n$-times differentiable mappings via kernels, had been obtained in Anastassiou (1995), Dragomir (1999d), Dragomir (1999e), Dedić et al. (2000), Cerone et al. (1998), Dragomir and Barnett (1998), Sofo and Dragomir (1999), Cerone et al. (1999a), Cerone et al. (1999b), Cerone et al. (1999c), Cerone et al. (1999d), Dragomir and Sofo (2000), Pearce et al. (2000), Barnett et al. (2001), Dragomir (2001a), Sofo (2002), Ujević (2004b) and Pachpatte (2004a) . For recent comprehensive list of refinements, counterparts and generalizations of Ostrowski integral inequality see Dragomir and Rassias (2002) .

### 2.3.3 Simpson's Type Inequalities

The Simpson's inequality was known in the literature, as follows:

**Theorem 2.3.27.** *(Davis & Rabinowitz 1976) Suppose* $f : [a,b] \to \boldsymbol{R}$ *is four times continuously differentiable mapping on* $(a,b)$ *and* $\left\|f^{(4)}\right\|_\infty := \sup\limits_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. *The following inequality*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^4}{2880}\left\|f^{(4)}\right\|_\infty \quad (2.3.40)$$

*holds.*

Further extended results were considered by Ghizzetti & Ossicini (1970), Dragomir et al. (2000), Yang and Chu (2000), Ujević (2002), Ujević (2004a), Ujević (2004d), Ujević (2004c), Ujević (2005a), Ujević (2005b), Liu (2007a), Tseng et al. (2007a) and Ujević (2007).

In the recent years, a large progress concerning Simpson's inequality is appeared, where in 1998, Dragomir obtained Simpson's inequality for differentiable mappings whose derivatives belong to $L_p$ spaces.

**Theorem 2.3.28.** *(Dragomir 1998) Let* $f : [a,b] \to$ ***R*** *an absolutely continuous mapping on* $[a,b]$ *whose derivative belongs to* $L_p[a,b]$*. Then we have the inequality:*

$$\begin{aligned} &\left| \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\ \leq\ & \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{1/q} (b-a)^{1/q} \left\|f'\right\|_p \end{aligned} \tag{2.3.41}$$

*where,* $(1/p)+(1/q)=1$*,* $p>1$*.*

In 1999, Dragomir proved the Simpson's inequality for Lipschitzian mapping and functions of bounded variation, as follows:

**Theorem 2.3.29.** *(Dragomir 1999a) Let* $f : [a,b] \to \mathbb{R}$ *be an* $L$*-Lipschitzian mapping* $[a,b]$*. Then we have the inequality:*

$$\left| \int_a^b f(x)\,dx - \frac{(b-a)}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] \right| \leq \frac{5}{36}L(b-a)^2. \tag{2.3.42}$$

**Theorem 2.3.30.** *(Dragomir 1999b) Let* $f : [a,b] \to \mathbb{R}$ *be a mapping of bounded variation on* $[a,b]$*. Then we have the inequality:*

$$\left| \int_a^b f(x)\,dx - \frac{(b-a)}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] \right| \leq \frac{(b-a)}{3}\bigvee_a^b (f), \tag{2.3.43}$$

*where* $\bigvee_a^b(f)$ *denotes the total variation of* $f$ *on the interval* $[a,b]$*. The constant* $\frac{1}{3}$ *is the best possible.*

In 2000, Pečarić and Varošanec, obtained some inequalities of Simpson's type for functions whose $n$-th derivative, $n \in \{0, 1, 2, 3\}$ is of bounded variation, as follow:

**Theorem 2.3.31.** *(Pečarić & Varošanec 2000) Let $n \in \{0, 1, 2, 3\}$. Let $f$ be a real function on $[a, b]$ such that $f^{(n)}$ is function of bounded variation. Then*

$$\left| \int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \leq C_n (b-a)^{n+1} \bigvee_a^b \left(f^{(n)}\right), \quad (2.3.44)$$

*where,*

$$C_0 = \frac{1}{3},\ C_1 = \frac{1}{24},\ C_2 = \frac{1}{324},\ C_3 = \frac{1}{1152},$$

*and $\bigvee_a^b \left(f^{(n)}\right)$ is the total variation of $f^{(n)}$ on the interval $[a, b]$.*

Here we note that, Ghizzetti and Ossicini (1970), proved that if $f'''$ is an absolutely continuous mapping with total variation $\bigvee_a^b (f)$, then (2.3.44) holds with $n = 3$.

In 2001, Pečarić and Varošanec, generalized Dragomir result (2.3.42) for functions whose $n$-th derivative is Lipschitzian, as follow:

**Theorem 2.3.32.** *(Pečarić & Varošanec 2001a) Let $f$ be a real function on $[a, b]$ such that $f^{(n)}$ is $L_n$-Lipschitzian function. If $n = 1, 2, 3$, then*

$$\left| \int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \leq C_n (b-a)^{n+2} L_n, \quad (2.3.45)$$

*where,*

$$C_1 = \frac{1}{81},\ C_2 = \frac{1}{576},\ C_3 = \frac{1}{2880}.$$

*If $n \geq 4$*

$$\left| \int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] + \sum_{\substack{k=4 \\ k \text{ is even}}}^{n} (-1)^{+1} f^{(k)}\left(\frac{a+b}{2}\right) \frac{(b-a)^{k+1}}{2^k k!}\left(\frac{1}{k+1} - \frac{1}{3}\right) \right| \leq 2\left(\frac{b-a}{2}\right)^{n+2} \frac{1}{(n+1)!}\left(\frac{1}{3} - \frac{1}{n+1}\right) L_n. \quad (2.3.46)$$

In recent years, Pečarić and his group research consider another approach to obtain estimate of the error in several quadrature rules using $n$-convex mappings (the divided difference convexity) via Euler-type identities. The first main results in this way are considered by Dedić et al. (2000). In particular Dedić et al. (2001a) proved a number of inequalities of Euler-Simpson type, for functions whose derivatives are either functions of bounded variation or Lipschitzian functions or functions in $L_p$-spaces. For different approach, generalizations and new inequalities of Simpson type and other inequalities via Euler-type identities, we refer the reader to Dedić et al. (2001a), Pečarić and Varošanec (2001b), Pečarić and Vukelić (2003), Dedić et al. (2005), Pečarić and Franjić (2006), and Franjić et al. (2006)

# CHAPTER III

# HERMITE-HADAMARD'S TYPE INEQUALITIES

## 3.1 INTRODUCTION

This work brings together results for Hermite-Hadamard's inequalities type and thus giving explicit error bounds in the trapezoidal and midpoint rules, using Peano type kernels and results from the modern theory of inequalities. Although bounds through the use of Peano kernels have been obtained in some research papers on Hermite-Hadamard's inequality (see Chapter II), but these do not seem enough to perhaps the extent that they should be. In this chapter, we refine some inequalities of Hermite-Hadamard's type via $s$-convex, quasi-convex and $r$-convex functions. Some error estimates for the trapezoid and midpoint are obtained in terms of first derivative.

## 3.2 INEQUALITIES VIA $S$-CONVEX FUNCTIONS

Our aim in this section, is to give some improvements and further generalizations for Kirmaci inequalities (2.3.24)–(2.3.26), which are of trapezoid type via $s$-convex functions in the second sense. After that, we introduce some inequalities of midpoint type via $s$-convex functions. In order to prove our main result(s) we start with the following lemma:

**Lemma 3.2.1.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}$ where $a, b \in I$*

*with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\begin{aligned} & \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \\ = & \frac{b-a}{4}\left[\int_0^1 (-t)\, f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt + \int_0^1 t f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)dt\right]. \end{aligned}$$

*Proof.* It suffices to note that

$$\begin{aligned} I_1 &= \int_0^1 -tf'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt \\ &= -\frac{2}{a-b} f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)t\Big|_0^1 + \frac{2}{a-b}\int_0^1 f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt \\ &= -\frac{2}{a-b} f(a) + \frac{2}{a-b}\int_0^1 f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt. \end{aligned}$$

Setting $x = \frac{1+t}{2}a + \frac{1-t}{2}b$, and $dx = \frac{a-b}{2}dt$, which gives

$$I_1 = \frac{2}{b-a} f(a) - \frac{4}{(a-b)^2}\int_a^{\frac{a+b}{2}} f(x)\,dx.$$

Similarly, we can show that

$$\begin{aligned} I_2 &= \int_0^1 tf'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)dt \\ &= \frac{2}{b-a} f(b) - \frac{4}{(b-a)^2}\int_{\frac{a+b}{2}}^b f(x)\,dx. \end{aligned}$$

Thus,

$$\frac{b-a}{4}[I_1 + I_2] = \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx.$$

which is required. □

Next theorem gives an improvement of Kirmaci result (2.3.24) with $q = 1$.

**Theorem 3.2.2.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is an $s$-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{2^{s+1}} \cdot \frac{1+s2^s}{(s+1)(s+2)}\left[|f'(a)| + |f'(b)|\right]. \quad (3.2.1)$$

*Proof.* From Lemma 1 and since $|f'|$ is $s$-convex on $[a,b]$, then we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&= \frac{b-a}{4}\left|\int_0^1 -tf'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt + \int_0^1 tf'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)dt\right| \\
&\leq \frac{b-a}{4}\int_0^1 |-t|\left|f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|dt \\
&\quad + \frac{b-a}{4}\int_0^1 |t|\left|f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right|dt \\
&\leq \frac{b-a}{4}\int_0^1 t\left[\left(\frac{1+t}{2}\right)^s |f'(a)| + \left(\frac{1-t}{2}\right)^s |f'(b)|\right]dt \\
&\quad + \frac{b-a}{4}\int_0^1 t\left[\left(\frac{1+t}{2}\right)^s |f'(b)| + \left(\frac{1-t}{2}\right)^s |f'(a)|\right]dt \\
&= \frac{b-a}{4} \cdot \frac{1}{2^s}\int_0^1 \left[t(1+t)^s |f'(a)| + t(1-t)^s |f'(b)|\right]dt \\
&\quad + \frac{b-a}{4} \cdot \frac{1}{2^s}\int_0^1 \left[t(1+t)^s |f'(b)| + t(1-t)^s |f'(a)|\right]dt \\
&= \frac{(b-a)}{2^{s+1}} \cdot \frac{1+s2^s}{(s+1)(s+2)}\left[|f'(a)| + |f'(b)|\right],
\end{aligned}$$

where we have used the fact that

$$\int_0^1 t(1+t)^s\,dt = \frac{s2^{s+1}+1}{(s+1)(s+2)}$$

and

$$\int_0^1 t(1-t)^s\,dt = \frac{1}{(s+1)(s+2)}$$

which completes the proof. □

**Remark 3.2.3.** *If one choose* $s=1$ *in (3.2.1), then we refer to (2.3.12).*

A similar result may be embodied in the following theorem.

**Theorem 3.2.4.** *Let* $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ *be a differentiable mapping on* $I^\circ$ *such that* $f' \in L[a,b]$*, where* $a, b \in I$ *with* $a<b$*. If* $|f'|^q$ *is an* $s$*-convex on* $[a,b]$*, for some fixed* $s \in (0,1]$ *and* $q>1$*, then the following inequality holds:*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &\le \left(\frac{b-a}{4}\right)\left(\frac{q-1}{2q-1}\right)^{(q-1)/q} \cdot \frac{1}{(s+1)^{1/q}\,2^{s/q}} \\ &\times \left\{\left[\left(2^{s+1}-1\right)|f'(a)|^q + |f'(b)|^q\right]^{1/q}\right. \\ &\left. + \left[|f'(a)|^q + \left(2^{s+1}-1\right)|f'(b)|^q\right]^{1/q}\right\}.\end{aligned} \tag{3.2.2}$$

*Proof.* Suppose that $q>1$. From Lemma 3.2.1 and using the Hölder's inequality, then we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &= \frac{b-a}{4}\left|\int_0^1 -tf'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt + \int_0^1 tf'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)dt\right| \\ &\le \frac{b-a}{4}\int_0^1 |-t|\left|f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|dt \\ &\quad + \frac{b-a}{4}\int_0^1 |t|\left|f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right|dt\end{aligned}$$

$$\leq \frac{b-a}{4}\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1 \left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{1/q}$$
$$+\frac{b-a}{4}\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1 \left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{1/q}.$$

Because $|f'|^q$ is $s$-convex, we have

$$\begin{aligned}
&\int_0^1 \left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt \\
\leq\ & \int_0^1 \left[\left(\frac{1+t}{2}\right)^s |f'(a)|^q+\left(\frac{1-t}{2}\right)^s |f'(b)|^q\right] dt \\
=\ & \frac{1}{2^s}|f'(a)|^q\int_0^1 (1+t)^s dt+\frac{1}{2^s}|f'(b)|^q\int_0^1 (1-t)^s dt, \qquad (3.2.3)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt \\
\leq\ & \int_0^1 \left[\left(\frac{1+t}{2}\right)^s |f'(b)|^q+\left(\frac{1-t}{2}\right)^s |f'(a)|^q\right] dt \\
=\ & \frac{1}{2^s}|f'(b)|^q\int_0^1 (1+t)^s dt+\frac{1}{2^s}|f'(a)|^q\int_0^1 (1-t)^s dt. \qquad (3.2.4)
\end{aligned}$$

A combination between (3.2.3)–(3.2.4), gives the following

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{(b-a)}{4(p+1)^{1/p}}\cdot\frac{1}{(s+1)^{1/q}2^{s/q}}\cdot\left\{\left[\left(2^{s+1}-1\right)|f'(a)|^q+|f'(b)|^q\right]^{1/q}\right. \\
&\left.+\left[|f'(a)|^q+\left(2^{s+1}-1\right)|f'(b)|^q\right]^{1/q}\right\}.
\end{aligned}$$

A simple calculations give the required result (3.2.2), where $\frac{1}{p}+\frac{1}{q}=1$. □

Next result gives a new refinement for the inequality (2.3.25).

**Theorem 3.2.5.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is an $s$-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \left(\frac{b-a}{4}\right)\left(\frac{q-1}{2q-1}\right)^{(q-1)/q} \cdot \frac{1}{(s+1)^{1/q}} \cdot \left\{\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(a)|^q\right]^{1/q}\right. \\
&\left. + \left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q\right]^{1/q}\right\} \qquad (3.2.5) \\
\leq\ & \frac{(b-a)}{4} \cdot \left\{\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(a)|^q\right]^{1/q}\right. \\
&\left. + \left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q\right]^{1/q}\right\}. \qquad (3.2.6)
\end{aligned}$$

*Proof.* We proceed similarly as in the proof of Theorem 3.2.4, but using inequality (2.3.23) for $|f'|^q$ $s$-convex mapping, we have

$$\int_0^1 \left|f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|^q dt \leq \frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(a)|^q}{s+1},$$

and

$$\int_0^1 \left|f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right|^q dt \leq \frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q}{s+1}.$$

So that,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \left(\frac{b-a}{4}\right)\left(\frac{q-1}{2q-1}\right)^{(q-1)/q} \cdot \frac{1}{(s+1)^{1/q}} \cdot \left\{\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(a)|^q\right]^{1/q}\right. \\
&\left. + \left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q\right]^{1/q}\right\}.
\end{aligned}$$

Also, since $\left(\frac{q-1}{2q-1}\right)^{(q-1)/q} \leq 1$ and $\left(\frac{1}{s+1}\right)^{1/q} < 1$, $s \in (0,1)$, $q \in (1,\infty)$, then we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \left(\frac{b-a}{4}\right)\left(\frac{q-1}{2q-1}\right)^{(q-1)/q} \cdot \frac{1}{(s+1)^{1/q}} \cdot \left\{\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(a)|^q\right]^{1/q}\right. \\
&\left. + \left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q\right]^{1/q}\right\} \\
\leq\ & \frac{(b-a)}{4} \cdot \left\{\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(a)|^q\right]^{1/q}\right. \\
&\left. + \left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q\right]^{1/q}\right\},
\end{aligned}$$

which completes the proof. □

A generalization for the inequality (3.2.5), may be given as follows:

**Theorem 3.2.6.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is an $s$-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \left(\frac{b-a}{4}\right)\left(\frac{q-1}{2q-1}\right)^{(q-1)/q} \cdot \frac{1}{(s+1)^{1/q}\,2^{s/q}} \\
&\times \left[\left(2^{s+1}-1\right)^{1/q} + 1\right](|f'(a)| + |f'(b)|) \qquad (3.2.7) \\
\leq\ & \frac{(b-a)}{4}(|f'(a)| + |f'(b)|). \qquad (3.2.8)
\end{aligned}$$

*Proof.* We consider the inequality (3.2.2), i.e.,

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq\ & \left(\frac{b-a}{4}\right)\left(\frac{q-1}{2q-1}\right)^{(q-1)/q} \cdot \frac{1}{(s+1)^{1/q}\,2^{s/q}} \\
&\times \left\{\left[\left(2^{s+1}-1\right)|f'(a)|^q + |f'(b)|^q\right]^{1/q}\right. \\
&\left. + \left[|f'(a)|^q + \left(2^{s+1}-1\right)|f'(b)|^q\right]^{1/q}\right\}.
\end{aligned}$$

Let $a_1 = (2^{s+1} - 1)\,|f'(a)|^q$, $b_1 = |f'(b)|^q$, $a_2 = |f'(a)|^q$ and $b_2 = (2^{s+1} - 1)\,|f'(b)|^q$.

Here, $0 < 1/q < 1$, for $q > 1$. Using the fact

$$\sum_{i=1}^{n} (a_i + b_i)^r \leq \sum_{i=1}^{n} a_i^r + \sum_{i=1}^{n} b_i^r,$$

for $0 \leq r < 1$, $a_1, a_2, ..., a_n \geq 0$ and $b_1, b_2, ..., b_n \geq 0$, we obtain

$$\begin{aligned} \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| &\leq \left(\frac{b-a}{4}\right)\left(\frac{q-1}{2q-1}\right)^{(q-1)/q} \cdot \frac{1}{(s+1)^{1/q}\,2^{s/q}} \\ &\quad \times \left\{ \left[\left(2^{s+1}-1\right)|f'(a)|^q + |f'(b)|^q\right]^{1/q} \right. \\ &\quad \left. + \left[|f'(a)|^q + \left(2^{s+1}-1\right)|f'(b)|^q\right]^{1/q} \right\}. \\ &\leq \left(\frac{b-a}{4}\right)\left(\frac{q-1}{2q-1}\right)^{(q-1)/q} \cdot \frac{1}{(s+1)^{1/q}\,2^{s/q}} \\ &\quad \times \left[\left(2^{s+1}-1\right)^{1/q} + 1\right]\left(|f'(a)| + |f'(b)|\right), \end{aligned}$$

which is required. □

Another new bound for the trapezoid inequality may be stated as follows:

**Theorem 3.2.7.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is an $s$-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q \geq 1$, then the following inequality holds:*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\ \leq\ & \frac{(b-a)}{8}\left(\frac{2^{1-s}}{(s+1)(s+2)}\right)^{1/q}\left[\left(\left(1+s2^{s+1}\right)|f'(a)|^q + |f'(b)|^q\right)^{1/q}\right. \\ &\left. + \left(|f'(a)|^q + \left(1+s2^{s+1}\right)|f'(b)|^q\right)^{1/q}\right]. \qquad (3.2.9) \end{aligned}$$

*Proof.* Suppose that $q > 1$. From Lemma 3.2.1 and using the power mean inequality, then we have

$$\begin{aligned}
& \left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
= & \frac{b-a}{4}\left|\int_0^1 -tf'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt+\int_0^1 tf'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt\right| \\
\leq & \frac{b-a}{4}\int_0^1 |-t|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt \\
& +\frac{b-a}{4}\int_0^1 |t|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt \\
\leq & \frac{b-a}{4}\left(\int_0^1 tdt\right)^{1-1/q}\left(\int_0^1 t\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{1/q} \\
& +\frac{b-a}{4}\left(\int_0^1 tdt\right)^{1-1/q}\left(\int_0^1 t\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{1/q}.
\end{aligned}$$

Because $|f'|$ is $s$-convex, we have

$$\begin{aligned}
& \int_0^1 t\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt \\
\leq & \int_0^1 t\left[\left(\frac{1+t}{2}\right)^s|f'(a)|^q+\left(\frac{1-t}{2}\right)^s|f'(b)|^q\right]dt \\
= & \frac{1}{2^s}|f'(a)|^q\int_0^1 t(1+t)^s\,dt+\frac{1}{2^s}|f'(b)|^q\int_0^1 t(1-t)^s\,dt, \qquad (3.2.10)
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 t\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt \\
\leq & \int_0^1 t\left[\left(\frac{1+t}{2}\right)^s|f'(b)|^q+\left(\frac{1-t}{2}\right)^s|f'(a)|^q\right]dt \\
= & \frac{1}{2^s}|f'(b)|^q\int_0^1 t(1+t)^s\,dt+\frac{1}{2^s}|f'(a)|^q\int_0^1 t(1-t)^s\,dt. \qquad (3.2.11)
\end{aligned}$$

A combination between (3.2.10)–(3.2.11) gives the following

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\left(\frac{1}{2^s(s+1)(s+2)}\right)^{1/q}\left[\left(\left(1+s2^{s+1}\right)|f'(a)|^q+|f'(b)|^q\right)^{1/q}\right.$$
$$\left.+\left(|f'(a)|^q+\left(1+s2^{s+1}\right)|f'(b)|^q\right)^{1/q}\right],$$

simple calculations give the required result (3.2.9). $\square$

Another approach leads to the following result.

**Theorem 3.2.8.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is an $s$-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{8}\left(\frac{2^{1-s}}{(s+1)(s+2)}\right)^{1/q}\left(1+\left(1+s2^{s+1}\right)^{1/q}\right)(|f'(a)|+|f'(b)|$$

*Proof.* We consider the inequality (3.2.9), i.e.,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\left(\frac{1}{2^s(s+1)(s+2)}\right)^{1/q}\left[\left(\left(1+s2^{s+1}\right)|f'(a)|^q+|f'(b)|^q\right)^{1/q}\right.$$
$$\left.+\left(|f'(a)|^q+\left(1+s2^{s+1}\right)|f'(b)|^q\right)^{1/q}\right],$$

Let $a_1 = (1+s2^{s+1})|f'(a)|^q$, $b_1 = |f'(b)|^q$, $a_2 = |f'(a)|^q$ and $b_2 = (1+s2^{s+1})|f'(b)|^q$.

Here, $0 < 1/q < 1$, for $q \geq 1$. Using the fact

$$\sum_{i=1}^n (a_i+b_i)^r \leq \sum_{i=1}^n a_i^r + \sum_{i=1}^n b_i^r,$$

for $0 \leq r < 1$, $a_1, a_2, ..., a_n \geq 0$ and $b_1, b_2, ..., b_n \geq 0$, we obtain

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \\
\leq \; & \frac{(b-a)}{8} \left( \frac{2^{1-s}}{(s+1)(s+2)} \right)^{1/q} \left[ \left( 2^{s+1} s \left| f'(a) \right|^q + \left| f'(b) \right|^q \right)^{1/q} \right. \\
& \left. + \left( \left| f'(a) \right|^q + 2^{s+1} s \left| f'(b) \right|^q \right)^{1/q} \right]. \\
\leq \; & \frac{(b-a)}{8} \left( \frac{2^{1-s}}{(s+1)(s+2)} \right)^{1/q} \left( 1 + \left( 1 + s2^{s+1} \right)^{1/q} \right) \left( \left| f'(a) \right| + \left| f'(b) \right| \right),
\end{aligned}$$

which is required. □

In the following, we obtain some inequalities of Hermite–Hadamard type for $s$-concave mappings. We begin with following result, which is different from (2.3.26).

**Theorem 3.2.9.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ interior of $I$, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is $s$-concave on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality holds:*

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \\
\leq \; & \frac{(b-a)}{4(p+1)^{1/p}} \left[ \left| f'\left( \frac{3a+b}{4} \right) \right| + \left| f'\left( \frac{a+3b}{4} \right) \right| \right].
\end{aligned} \tag{3.2.12}$$

*Proof.* From Lemma 3.2.1 and using the Hölder inequality for $q > 1$, and $p = \frac{q}{q-1}$, we obtain

$$\begin{aligned}
\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq \frac{b-a}{4} & \left[ \int_0^1 t \left| f'\left( \frac{1+t}{2}a + \frac{1-t}{2}b \right) \right| dt \right. \\
& \left. + \int_0^1 t \left| f'\left( \frac{1+t}{2}b + \frac{1-t}{2}a \right) \right| dt \right]
\end{aligned}$$

$$\leq \frac{b-a}{4}\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1 \left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{1/q}$$
$$+\frac{b-a}{4}\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1 \left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{1/q},$$

where, $p$ is the conjugate of $q$.

We note that, since $|f'|^q$ is concave on $[a,b]$, and using the power mean inequality, we have

$$\begin{aligned}
\left|f'\left(\lambda x+(1-\lambda)y\right)\right|^q &\geq \lambda\left|f'(x)\right|^q+(1-\lambda)\left|f'(y)\right|^q \\
&\geq \left(\lambda\left|f'(x)\right|+(1-\lambda)\left|f'(y)\right|\right)^q, \quad \forall x,y\in[a,b].
\end{aligned}$$

Hence,

$$\left|f'\left(\lambda x+(1-\lambda)y\right)\right| \geq \lambda\left|f'(x)\right|+(1-\lambda)\left|f'(y)\right|,$$

so $|f'|$ is also concave.

By the Jensen integral inequality, we have

$$\begin{aligned}
\int_0^1 \left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt &\leq \left(\int_0^1 t^0 dt\right)\left|f'\left(\frac{\int_0^1 \left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt}{\int_0^1 t^0 dt}\right)\right|^q \\
&\leq \left|f'\left(\frac{3a+b}{4}\right)\right|^q,
\end{aligned}$$

and analogously,

$$\int_0^1 \left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt \leq \left|f'\left(\frac{a+3b}{4}\right)\right|^q.$$

Combining all obtained inequalities, we get the required result. $\square$

Next result gives a new inequality which of trapezoid type for $s$–concave mappings.

**Theorem 3.2.10.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ interior of $I$, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is $s$-concave on $[a,b]$, for some fixed $s \in (0,1]$ and $q \geq 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq\ & \frac{(b-a)}{8}\left[\left|f'\left(\frac{3a+b}{2}\right)\right|^q + \left|f'\left(\frac{a+3b}{2}\right)\right|^q\right]^{\frac{1}{q}}, \end{aligned} \tag{3.2.13}$$

*where, $q \geq 1$.*

*Proof.* For $q = 1$. From Lemma 3.2.1 and using the Jensen's integral inequality, we obtain

$$\begin{aligned} \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &\leq \frac{b-a}{4}\left[\int_0^1 t\left|f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right| dt \right. \\ &\quad \left. + \int_0^1 t\left|f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right| dt\right] \\ &\leq \frac{b-a}{4}\left(\int_0^1 t dt\right)\left|f'\left(\frac{\int_0^1 \left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt}{\int_0^1 t dt}\right)\right| \\ &\quad + \frac{b-a}{4}\left(\int_0^1 t dt\right)\left|f'\left(\frac{\int_0^1 \left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) dt}{\int_0^1 t dt}\right)\right| \\ &\leq \frac{b-a}{8}\left[\left|f'\left(\frac{3a+b}{2}\right)\right| + \left|f'\left(\frac{a+3b}{2}\right)\right|\right], \end{aligned}$$

which proves this case.

Now, for $q > 1$, using the Hölder inequality for $q > 1$, and then the Jensen's integral inequality, we obtain

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$
\begin{aligned}
&\leq \frac{b-a}{4}\left[\int_0^1 t\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right.\\
&\qquad\left.+\int_0^1 t\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right]\\
&=\frac{b-a}{4}\int_0^1 t^{1-\frac{1}{q}}\cdot t^{\frac{1}{q}}\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\\
&\qquad+\frac{b-a}{4}\int_0^1 t^{1-\frac{1}{q}}\cdot t^{\frac{1}{q}}\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\\
&\leq \frac{b-a}{4}\left(\int_0^1 tdt\right)^{\frac{q-1}{q}}\left(\int_0^1 t\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\\
&\qquad+\frac{b-a}{4}\left(\int_0^1 tdt\right)^{\frac{q-1}{q}}\left(\int_0^1 t\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\\
&\leq \frac{b-a}{4}\left(\int_0^1 tdt\right)^{\frac{q-1}{q}}\left[\left(\int_0^1 tdt\right)\left|f'\left(\frac{\int_0^1\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt}{\int_0^1 tdt}\right)\right|^q\right.\\
&\qquad\left.+\left(\int_0^1 tdt\right)\left|f'\left(\frac{\int_0^1\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt}{\int_0^1 tdt}\right)\right|^q\right]^{\frac{1}{q}}\\
&=\frac{(b-a)}{8}\left[\left|f'\left(\frac{3a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+3b}{2}\right)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}
$$

which completes the proof. □

Note that we can apply the estimates in (2.3.9) only if the second derivative $f''$ exists and bounded. It means that we cannot use (2.3.9) to estimate directly the error when approximating the integral of such a well-behaved function as $f(t) = \sqrt{t^3}$ on $[0,1]$, since $f''(t) = 3/4\sqrt{t}$ is unbounded on $[0,1]$. Also, we can apply the estimates in (2.3.12) only if the first derivative $f'$ exists and $|f'|$ is convex, so that we cannot use (2.3.12) to estimate the error in case we have $f$ is $s$-convex ( $0 < s \lneqq 1$). In fact, it is not easy to construct an example for which this case holds, i.e., $f$ is $s$-convex ( $0 < s \lneqq$

1) (and not convex) in one variable, however, the case will be easy if we consider $f$ in two variables, for more details see Dragomir & Fitzpatrick (1997), where the $s$-Orlicz convex mappings in linear spaces are discussed.

Next, we consider several refinements for the left Hermite-Hadamard's (midpoint) inequality via $s$-convex functions and in order to prove our main result(s) we consider the following lemma:

**Lemma 3.2.11.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \qquad (3.2.14) \\ = \ & \frac{b-a}{4}\left[\int_0^1 tf'\left(t\frac{a+b}{2} + (1-t)\,a\right)dt + \int_0^1 (t-1)\,f'\left(tb + (1-t)\,\frac{a+b}{2}\right)dt\right]. \end{aligned}$$

*Proof.* We note that

$$\begin{aligned} I_1 &= \int_0^1 tf'\left(t\frac{a+b}{2} + (1-t)\,a\right)dt \\ &= \frac{2}{b-a} tf\left(t\frac{a+b}{2} + (1-t)\,a\right)\Bigg|_0^1 - \frac{2}{b-a}\int_0^1 f\left(t\frac{a+b}{2} + (1-t)\,a\right)dt \\ &= \frac{2}{b-a} f\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_0^1 f\left(t\frac{a+b}{2} + (1-t)\,a\right)dt \end{aligned}$$

Setting $x = t\frac{a+b}{2} + (1-t)\,a$, and $dx = \frac{b-a}{2}dt$, which gives

$$I_1 = \frac{2}{b-a} f\left(\frac{a+b}{2}\right) - \frac{4}{(b-a)^2}\int_a^{\frac{a+b}{2}} f(x)\,dx.$$

Similarly, we can show that

$$\begin{aligned} I_2 &= \int_0^1 (t-1)\,f'\left(tb + (1-t)\,\frac{a+b}{2}\right)dt \\ &= \frac{2}{b-a} f\left(\frac{a+b}{2}\right) - \frac{4}{(b-a)^2}\int_{\frac{a+b}{2}}^b f(x)\,dx. \end{aligned}$$

and therefore,

$$\begin{aligned} I &= \frac{b-a}{4}\left[I_1+I_2\right] \\ &= \frac{b-a}{4}\left[\frac{4}{b-a}f\left(\frac{a+b}{2}\right)-\frac{4}{(b-a)^2}\int\limits_a^b f(x)\,dx\right] \\ &= f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(x)\,dx. \end{aligned}$$

which completes the proof. □

Next theorem refines the inequalities (2.3.14) and (2.3.20) for $s$-convex mappings.

**Theorem 3.2.12.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|$ is an $s$-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:*

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(x)\,dx\right| \\ \le\ & \frac{(b-a)}{4(s+1)(s+2)}\left[\left|f'(a)\right|+2(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'(b)\right|\right] \qquad (3.2.15) \\ \le\ & \frac{\left(2^{2-s}+1\right)(b-a)}{4(s+1)(s+2)}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \end{aligned}$$

*Proof.* From Lemma 3.2.11, we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{b-a}{4}\left[\int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)\,a\right)\right|dt\right. \\
&\left.+\int_0^1 |t-1|\left|f'\left(tb+(1-t)\,\frac{a+b}{2}\right)\right|dt\right] \\
\leq\ & \frac{b-a}{4}\int_0^1 t\left[t^s\left|f'\left(\frac{a+b}{2}\right)\right|+(1-t)^s\,|f'(a)|\right]dt \\
&+\frac{b-a}{4}\int_0^1 (1-t)\left[t^s\,|f'(b)|+(1-t)^s\left|f'\left(\frac{a+b}{2}\right)\right|\right]dt \\
=\ & \frac{b-a}{4}\left[\frac{1}{s+2}\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{1}{(s+1)(s+2)}\,|f'(a)|\right] \\
&+\frac{b-a}{4}\left[\frac{1}{(s+1)(s+2)}\,|f'(b)|+\frac{1}{s+2}\left|f'\left(\frac{a+b}{2}\right)\right|\right] \\
=\ & \frac{(b-a)}{4(s+1)(s+2)}\left[|f'(a)|+2(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(b)|\right] \qquad (3.2.16)
\end{aligned}$$

which proves the first inequality in (3.2.15). To prove the second inequality in (3.2.15), since $|f'|$ is $s$-convex on $[a,b]$, for any $t\in[0,1]$, then by (2.3.23) we have

$$2^{s-1}\left|f'\left(\frac{a+b}{2}\right)\right| \leq \frac{|f'(a)|+|f'(b)|}{s+1}. \qquad (3.2.17)$$

A combination of (3.2.16) and (3.2.17), we get

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{(b-a)}{4(s+1)(s+2)}\left[|f'(a)|+2(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(b)|\right] \\
\leq\ & \frac{(b-a)}{4(s+1)(s+2)}\left[|f'(a)|+2(s+1)\,2^{1-s}\frac{|f'(a)|+|f'(b)|}{s+1}+|f'(b)|\right] \\
=\ & \frac{(2^{2-s}+1)(b-a)}{4(s+1)(s+2)}\left[|f'(a)|+|f'(b)|\right]
\end{aligned}$$

which proves the second inequality in (3.2.15), where we have used the fact that

$$\int_0^1 t^{s+1}dt=\int_0^1 (1-t)^{s+1}\,dt=\frac{1}{s+2}$$

and

$$\int_0^1 t^s (1-t)\, dt = \int_0^1 t (1-t)^s\, dt = \frac{1}{(s+1)(s+2)}$$

which completes the proof. $\square$

**Remark 3.2.13.** *We note that, the first inequality in (3.2.15) with $s = 1$ reduces to the inequality (2.3.20), and the second inequality in (3.2.15) with $s = 1$ reduces to the bound of (2.3.18).*

Simply, we can state the following result.

**Corollary 3.2.14.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\, dx \right| \\ \leq\; & \frac{b-a}{24}\left[ |f'(a)| + 4\left| f'\left(\frac{a+b}{2}\right)\right| + |f'(b)| \right] \qquad (3.2.18) \\ \leq\; & \frac{(b-a)}{8}\left( |f'(a)| + |f'(b)| \right). \end{aligned}$$

*Proof.* Put $s = 1$ in Theorem 3.2.12, we get the required result. $\square$

Next theorem gives a new upper bound of the left Hermite–Hadamard's inequality for $s$-convex mappings.

**Theorem 3.2.15.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is an $s$-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $p > 1$, then the following inequality holds:*

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\, dx \right| \\ \leq\; & \left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{2/q} \\ & \left[ \left( \left(2^{1-s}+s+1\right)|f'(a)|^q + 2^{1-s}|f'(b)|^q \right)^{1/q} \right. \\ & \left. + \left( 2^{1-s}|f'(a)|^q + \left(2^{1-s}+s+1\right)|f'(b)|^q \right)^{1/q} \right], \qquad (3.2.19) \end{aligned}$$

*where* $q = p/(p-1)$.

*Proof.* Suppose that $p > 1$. From Lemma 3.2.11 and using the Hölder inequality, we have

$$
\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\
\leq\ & \frac{b-a}{4}\left[\int_0^1 t\left|f'\left(t\frac{a+b}{2} + (1-t)\,a\right)\right| dt \right. \\
&\left. + \int_0^1 (1-t)\left|f'\left(tb + (1-t)\,\frac{a+b}{2}\right)\right| dt\right] \\
\leq\ & \frac{b-a}{4}\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1 \left|f'\left(t\frac{a+b}{2} + (1-t)\,a\right)\right|^q dt\right)^{1/q} \\
& + \frac{b-a}{4}\left(\int_0^1 (1-t)^p\,dt\right)^{1/p}\left(\int_0^1 \left|f'\left(tb + (1-t)\,\frac{a+b}{2}\right)\right|^q dt\right)^{1/q}
\end{aligned}
$$

Because $|f'|^q$ is $s$-convex, we have

$$
\begin{aligned}
\int_0^1 \left|f'\left(t\frac{a+b}{2} + (1-t)\,a\right)\right|^q dt &\leq \int_0^1 \left[t^s\left|f'\left(\frac{a+b}{2}\right)\right|^q + (1-t)^s\left|f'(a)\right|^q\right] dt \\
&= \frac{1}{s+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \frac{1}{s+1}\left|f'(a)\right|^q
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^1 \left|f'\left(tb + (1-t)\,\frac{a+b}{2}\right)\right|^q dt &\leq \int_0^1 \left[t^s\left|f'(b)\right|^q + (1-t)^s\left|f'\left(\frac{a+b}{2}\right)\right|^q\right] dt \\
&= \frac{1}{s+1}\left|f'(b)\right|^q + \frac{1}{s+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{1/q}\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q + \left|f'(a)\right|^q\right)^{1/q}\right. \\
&\left. + \left(\left|f'(b)\right|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right]. \qquad (3.2.20)
\end{aligned}
$$

Now, since $|f'|^q$ is $s$-convex on $[a,b]$, for any $t\in[0,1]$, then by (2.3.23) we have

$$2^{s-1}\left|f'\left(\frac{a+b}{2}\right)\right|\leq\frac{|f'(a)|+|f'(b)|}{s+1}. \tag{3.2.21}$$

A combination of (3.2.20) and (3.2.21), we get

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| &\leq \left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{1/q}\\ &\times\left[\left(\frac{2^{1-s}}{s+1}\left(|f'(a)|^q+|f'(b)|^q\right)+|f'(a)|^q\right)^{1/q}\right.\\ &\left.+\left(|f'(b)|^q+\frac{2^{1-s}}{s+1}\left(|f'(a)|^q+|f'(b)|^q\right)\right)^{1/q}\right]\\ &\leq\left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{2/q}\\ &\left[\left(\left(2^{1-s}+s+1\right)|f'(a)|^q+2^{1-s}|f'(b)|^q\right)^{1/q}\right.\\ &\left.+\left(2^{1-s}|f'(a)|^q+\left(2^{1-s}+s+1\right)|f'(b)|^q\right)^{1/q}\right]\end{aligned}$$

where $1/p+1/q=1$, which is required. $\square$

Therefore, Theorem 3.2.15 may be extended to be as follows:

**Corollary 3.2.16.** *Let $f:I\subseteq\mathbb{R}_+\to\mathbb{R}_+$ be a differentiable mapping on $I^\circ$ such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^{p/(p-1)}$ is an $s$-convex on $[a,b]$, for some fixed $s\in(0,1]$ and $p>1$, then the following inequality holds:*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\ &\left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{2/q}\\ &\times\left\{2^{(1-s)/q}+\left(2^{1-s}+s+1\right)^{1/q}\right\}\left(|f'(a)|+|f'(b)|\right)\end{aligned} \tag{3.2.22}$$

*where $q=p/(p-1)$.*

*Proof.* We consider the inequality (3.2.19), for $p > 1$, $q = p/(p-1)$

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\
\leq\ & \left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{2/q} \\
&\left[\left(\left(2^{1-s}+s+1\right)|f'(a)|^q + 2^{1-s}|f'(b)|^q\right)^{1/q}\right. \\
&\left. + \left(2^{1-s}|f'(a)|^q + \left(2^{1-s}+s+1\right)|f'(b)|^q\right)^{1/q}\right].
\end{aligned}$$

Let $a_1 = (2^{1-s}+s+1)|f'(a)|^q$, $b_1 = 2^{1-s}|f'(b)|^q$, $a_2 = 2^{1-s}|f'(a)|^q$ and $b_2 = (2^{1-s}+s+1)|f'(b)|^q$. Here, $0 < 1/q < 1$, for $q > 1$. Using the fact

$$\sum_{i=1}^{n}(a_i+b_i)^r \leq \sum_{i=1}^{n} a_i^r + \sum_{i=1}^{n} b_i^r,$$

for $0 < r < 1$, $a_1, a_2, ..., a_n \geq 0$ and $b_1, b_2, ..., b_n \geq 0$, we obtain

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\
\leq\ & \left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{2/q} \\
&\left[\left(\left(2^{1-s}+s+1\right)|f'(a)|^q + 2^{1-s}|f'(b)|^q\right)^{1/q}\right. \\
&\left. + \left(2^{1-s}|f'(a)|^q + \left(2^{1-s}+s+1\right)|f'(b)|^q\right)^{1/q}\right] \\
\leq\ & \left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{2/q} \\
&\times\left\{2^{(1-s)/q} + \left(2^{1-s}+s+1\right)^{1/q}\right\}\left(|f'(a)| + |f'(b)|\right),
\end{aligned}$$

which completes the proof. ☐

Another result is given in the following theorem.

**Theorem 3.2.17.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is an $s$-convex on $[a,b]$, for some fixed*

$s \in (0,1]$ *and* $q \geq 1$*, then the following inequality holds:*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\
\leq\ & \left(\frac{b-a}{8}\right)\left(\frac{2}{(s+1)(s+2)}\right)^{1/q} \Big[\left\{\left(2^{1-s}+1\right)|f'(a)|^q + 2^{1-s}|f'(b)|^q\right\}^{1/q} \\
&+ \left\{\left(2^{1-s}+1\right)|f'(b)|^q + 2^{1-s}|f'(a)|^q\right\}^{1/q}\Big] \qquad (3.2.23)
\end{aligned}$$

*Proof.* Suppose that $p \geq 1$. From Lemma 3.2.11 and using the power mean inequality, we have

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\
\leq\ & \frac{b-a}{4}\left[\int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right. \\
&\left.+\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right] \\
\leq\ & \frac{b-a}{4}\left(\int_0^1 t\,dt\right)^{1-1/q}\left(\int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{1/q} \\
&+\frac{b-a}{4}\left(\int_0^1 (1-t)\,dt\right)^{1-1/q}\left(\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{1/q}
\end{aligned}$$

Because $|f'|^q$ is $s$-convex, we have

$$\begin{aligned}
&\int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt \\
\leq\ & \int_0^1 \left[t^{s+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q + t(1-t)^s|f'(a)|^q\right]dt \\
=\ & \frac{1}{s+2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \frac{1}{(s+1)(s+2)}|f'(a)|^q
\end{aligned}$$

and

$$
\begin{aligned}
& \int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt \\
\leq & \int_0^1 \left[(1-t)\,t^s\,|f'(b)|^q + (1-t)^{s+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right] dt \\
= & \frac{1}{s+2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \frac{1}{(s+1)(s+2)}\,|f'(b)|^q
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
& \left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq & \left(\frac{b-a}{8}\right)\left(\frac{2}{(s+1)(s+2)}\right)^{1/q}\left[\left((s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(a)|^q\right)^{1/q}\right. \\
& \left. + \left(|f'(b)|^q + (s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right]. \qquad (3.2.24)
\end{aligned}
$$

Now, since $|f'|^q$ is $s$-convex on $[a,b]$, for any $t \in [0,1]$, then by (2.3.23) we have

$$
2^{s-1}\left|f'\left(\frac{a+b}{2}\right)\right|^q \leq \frac{|f'(a)|^q + |f'(b)|^q}{s+1}. \qquad (3.2.25)
$$

A combination of (3.2.24) and (3.2.25), we get

$$
\begin{aligned}
& \left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq & \left(\frac{b-a}{8}\right)\left(\frac{2}{(s+1)(s+2)}\right)^{1/q}\left[\left((s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(a)|^q\right)^{1/q}\right. \\
& \left. + \left(|f'(b)|^q + (s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right] \\
\leq & \left(\frac{b-a}{8}\right)\left(\frac{2}{(s+1)(s+2)}\right)^{1/q}\left[\left(2^{1-s}\left(|f'(a)|^q + |f'(b)|^q\right) + |f'(a)|^q\right)^{1/q}\right. \\
& \left. + \left(|f'(b)|^q + 2^{1-s}\left(|f'(a)|^q + |f'(b)|^q\right)\right)^{1/q}\right] \\
= & \left(\frac{b-a}{8}\right)\left(\frac{2}{(s+1)(s+2)}\right)^{1/q}\left[\left\{\left(2^{1-s}+1\right)|f'(a)|^q + 2^{1-s}\,|f'(b)|^q\right\}^{1/q}\right. \\
& \left. + \left\{\left(2^{1-s}+1\right)|f'(b)|^q + 2^{1-s}\,|f'(a)|^q\right\}^{1/q}\right].
\end{aligned}
$$

which is required, and the proof is complete. $\square$

**Corollary 3.2.18.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is an $s$-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q \geq 1$, then the following inequality holds:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \leq \left(\frac{b-a}{8}\right)\left(\frac{2}{(s+1)(s+2)}\right)^{1/q} \times \left\{2^{(1-s)/q} + \left(2^{1-s}+1\right)^{1/q}\right\}\left(|f'(a)| + |f'(b)|\right). \tag{3.2.26}$$

*Proof.* We consider the inequality (3.2.23), i.e.,

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\ \leq\ &\left(\frac{b-a}{8}\right)\left(\frac{2}{(s+1)(s+2)}\right)^{1/q}\left[\left\{\left(2^{1-s}+1\right)|f'(a)|^q + 2^{1-s}|f'(b)|^q\right\}^{1/q}\right. \\ &\left. + \left\{\left(2^{1-s}+1\right)|f'(b)|^q + 2^{1-s}|f'(a)|^q\right\}^{1/q}\right] \end{aligned}$$

Let $a_1 = (2^{1-s}+1)|f'(a)|^q$, $b_1 = 2^{1-s}|f'(b)|^q$, $a_2 = 2^{1-s}|f'(a)|^q$ and $b_2 = (2^{1-s}+1)|f'(b)|^q$. Here, $0 < 1/q \leq 1$, for $q \geq 1$. Using the fact

$$\sum_{i=1}^{n}(a_i+b_i)^r \leq \sum_{i=1}^{n} a_i^r + \sum_{i=1}^{n} b_i^r,$$

for $0 < r \leq 1$, $a_1, a_2, ..., a_n \geq 0$ and $b_1, b_2, ..., b_n \geq 0$, we obtain

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| &\leq \left(\frac{b-a}{8}\right)\left(\frac{2}{(s+1)(s+2)}\right)^{1/q} \\ &\quad \times \left[\left(\left(2^{1-s}+1\right)|f'(a)|^q + 2^{1-s}|f'(b)|^q\right)^{1/q}\right. \\ &\quad \left. + \left(2^{1-s}|f'(a)|^q + \left(2^{1-s}+1\right)|f'(b)|^q\right)^{1/q}\right] \\ &\leq \left(\frac{b-a}{8}\right)\left(\frac{2}{(s+1)(s+2)}\right)^{1/q} \\ &\quad \times \left\{2^{(1-s)/q} + \left(2^{1-s}+1\right)^{1/q}\right\}\left(|f'(a)| + |f'(b)|\right), \end{aligned}$$

which gives the required result. □

Now, we give the following midpoint type inequality for concave mappings.

**Theorem 3.2.19.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is an $s$-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality holds:*

$$\begin{aligned}&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\ \leq\ & \left(\frac{b-a}{4}\right)\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[\left|f'\left(\frac{3a+b}{4}\right)\right| + \left|f'\left(\frac{a+3b}{4}\right)\right|\right]. \quad (3.2.27)\end{aligned}$$

*Proof.* From Lemma 3.2.11 and using the Hölder inequality for $q > 1$ and $p = \frac{q}{q-1}$, we obtain

$$\begin{aligned}&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\ \leq\ & \frac{b-a}{4}\left[\int_0^1 t\left|f'\left(t\frac{a+b}{2} + (1-t)\,a\right)\right| dt \right. \\ &\left. + \int_0^1 (1-t)\left|f'\left(tb + (1-t)\,\frac{a+b}{2}\right)\right| dt\right]\end{aligned}$$

$$\begin{aligned}\leq\ & \frac{b-a}{4}\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1 \left|f'\left(t\frac{a+b}{2} + (1-t)\,a\right)\right|^q dt\right)^{1/q} \\ & + \frac{b-a}{4}\left(\int_0^1 (1-t)^p\,dt\right)^{1/p}\left(\int_0^1 \left|f'\left(tb + (1-t)\,\frac{a+b}{2}\right)\right|^q dt\right)^{1/q}\end{aligned}$$

It can be easily checked that

$$\int_0^1 t^{\frac{q}{q-1}}dt = \int_0^1 (1-t)^{\frac{q}{q-1}}\,dt = \frac{q-1}{2q-1}.$$

We note that, since $|f|^q$ is concave on $[a,b]$, and using the power mean inequality, we have

$$\begin{aligned}|f'(\lambda x + (1-t)\,y)|^q &\geq \lambda\,|f'(x)|^q + (1-t)\,|f'(y)|^q \\ &\geq (\lambda\,|f'(x)| + (1-t)\,|f'(y)|)^q\,, \quad \forall x,y \in [a,b].\end{aligned}$$

Hence,

$$\left|f'\left(\lambda x+\left(1-t\right)y\right)\right|\geq\lambda\left|f'\left(x\right)\right|+\left(1-t\right)\left|f'\left(y\right)\right|,$$

so $|f'|$ is also concave.

By the Jensen integral inequality, we have

$$\begin{aligned}\int_0^1\left|f'\left(t\frac{a+b}{2}+\left(1-t\right)a\right)\right|^q dt &\leq \left(\int_0^1 t^0 dt\right)\left|f'\left(\frac{\int_0^1\left(t\frac{a+b}{2}+\left(1-t\right)a\right)dt}{\int_0^1 t^0 dt}\right)\right|^q\\ &\leq \left|f'\left(\frac{3a+b}{4}\right)\right|^q,\end{aligned}$$

and analogously,

$$\int_0^1\left|f'\left(tb+\left(1-t\right)\frac{a+b}{2}\right)\right|^q dt\leq\left|f'\left(\frac{a+3b}{4}\right)\right|^q.$$

Combining all obtained inequalities, we get the required result. □

### 3.3 INEQUALITIES VIA QUASI-CONVEX FUNCTIONS

In the following theorem we shall propose new upper bound for the right-hand side of Hadamard's inequality via quasi-convex mappings which gives new result different from Ion's result (2.3.27).

**Theorem 3.3.1.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|$ is a quasi-convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \qquad (3.3.1)\\ \leq\ &\frac{b-a}{8}\left[\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'\left(a\right)\right|\right\}+\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'\left(b\right)\right|\right\}\right].\end{aligned}$$

*Proof.* From Lemma 3.2.1, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$=\frac{b-a}{4}\left|\int_0^1(-t)\,f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt+\int_0^1 tf'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt\right|.$$

Since $|f'|$ is quasi-convex on $[a,b]$, for any $t\in[0,1]$ we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{b-a}{4}\left[\int_0^1|(-t)|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right.$$
$$\left.+\int_0^1|t|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right]$$
$$\leq\frac{b-a}{4}\left[\int_0^1 t\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,|f'(a)|\right\}dt\right.$$
$$\left.+\int_0^1 t\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,|f'(b)|\right\}dt\right]$$
$$=\frac{b-a}{8}\left[\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,|f'(a)|\right\}+\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,|f'(b)|\right\}\right],$$

which completes the proof. □

**Corollary 3.3.2.** *Let $f$ as in Theorem 3.3.1, if in addition*

1. *$|f'|$ is increasing, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{b-a}{8}\left[|f'(b)|+\left|f'\left(\frac{a+b}{2}\right)\right|\right].\qquad(3.3.2)$$

*2. $|f'|$ is decreasing, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{8}\left[|f'(a)|+\left|f'\left(\frac{a+b}{2}\right)\right|\right]. \quad (3.3.3)$$

*Proof.* It follows directly by Theorem 3.3.1 □

Another similar result may be extended in the following theorem.

**Theorem 3.3.3.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is an quasi-convex on $[a,b]$, for $p > 1$, then the following inequality holds:*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| &\le \frac{(b-a)}{4(p+1)^{1/p}}\left[\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{p/(p-1)}, |f'(b)|^{p/(p-1)}\right\}\right)^{(p-1)/p}\right.\\ &\quad \left.+\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{p/(p-1)}, |f'(a)|^{p/(p-1)}\right\}\right)^{(p-1)/p}\right]. \quad (3.3.4)\end{aligned}$$

*Proof.* From Lemma 3.2.1 and using the well known Hölder integral inequality, we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| &\le \frac{b-a}{4}\left[\int_0^1 |(-t)|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right.\\ &\quad \left.+\int_0^1 |t|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right]\end{aligned}$$

$$\leq \frac{b-a}{4}\left[\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1 \left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{1/q}\right.$$
$$\left.+\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1 \left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{1/q}\right]$$

$$\leq \frac{b-a}{4\left(p+1\right)^{1/p}}\left[\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'\left(a\right)\right|^q\right\}\right)^{1/q}\right.$$
$$\left.+\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'\left(b\right)\right|^q\right\}\right)^{1/q}\right],$$

where $1/p+1/q=1$, which completes the proof. □

**Corollary 3.3.4.** *Let $f$ as in Theorem 3.3.3, if in addition*

1. $|f'|^{p/(p-1)}$ *is increasing, then we have*

$$\left|\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right|$$
$$\leq \frac{\left(b-a\right)}{4\left(p+1\right)^{1/p}}\left[\left|f'\left(b\right)\right|+\left|f'\left(\frac{a+b}{2}\right)\right|\right]. \quad (3.3.5)$$

2. $|f'|^{p/(p-1)}$ *is decreasing, then we have*

$$\left|\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right|$$
$$\leq \frac{\left(b-a\right)}{4\left(p+1\right)^{1/p}}\left[\left|f'\left(a\right)\right|+\left|f'\left(\frac{a+b}{2}\right)\right|\right]. \quad (3.3.6)$$

**Remark 3.3.5.** *We note that the inequalities (3.3.1) and (3.3.4) are two new refinements for the trapezoid inequality for the quasi-convex functions.*

A generalization of Theorem 3.3.1 is given in the following result.

**Theorem 3.3.6.** *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}$, $a, b \in I^{\circ}$ with $a < b$. If $|f'|^q$ is an quasi-convex on $[a, b]$, $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \\
&\qquad \leq \frac{b-a}{8} \left[ \left( \max\left\{ \left| f'\left( \frac{a+b}{2} \right) \right|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \right. \\
&\qquad\qquad \left. + \left( \max\left\{ \left| f'\left( \frac{a+b}{2} \right) \right|^q, |f'(a)|^q \right\} \right)^{\frac{1}{q}} \right]. \qquad (3.3.7)
\end{aligned}$$

*Proof.* From Lemma 3.2.1 and using the well known power mean inequality, we have

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \\
&\leq \frac{b-a}{4} \left[ \int_0^1 |(-t)| \left| f'\left( \frac{1+t}{2} a + \frac{1-t}{2} b \right) \right| dt \right. \\
&\qquad \left. + \int_0^1 |t| \left| f'\left( \frac{1+t}{2} b + \frac{1-t}{2} a \right) \right| dt \right] \\
&\leq \frac{b-a}{4} \left[ \left( \int_0^1 t dt \right)^{1-1/q} \left( \int_0^1 t \left| f'\left( \frac{1+t}{2} a + \frac{1-t}{2} b \right) \right|^q dt \right)^{1/q} \right. \\
&\qquad \left. + \left( \int_0^1 t dt \right)^{1-1/q} \left( \int_0^1 t \left| f'\left( \frac{1+t}{2} b + \frac{1-t}{2} a \right) \right|^q dt \right)^{1/q} \right] \\
&\leq \frac{b-a}{8} \left[ \left( \max\left\{ \left| f'\left( \frac{a+b}{2} \right) \right|^q, |f'(a)|^q \right\} \right)^{1/q} \right. \\
&\qquad \left. + \left( \max\left\{ \left| f'\left( \frac{a+b}{2} \right) \right|^q, |f'(b)|^q \right\} \right)^{1/q} \right],
\end{aligned}$$

which completes the proof. $\square$

**Remark 3.3.7.** *For $q = 1$ this reduces to Theorem 3.3.1. For $q = p/(p-1)$ $(p > 1)$ we have an improvement of the constants in Theorem 3.3.3, since $4^p > p + 1$ if $p > 1$ and accordingly*

$$\frac{1}{8} < \frac{1}{4(p+1)^{1/p}}.$$

**Corollary 3.3.8.** *Let $f$ as in Theorem 3.3.6, if in addition*

1. $|f'|$ *is increasing, then (3.3.2) holds.*

2. $|f'|$ *is decreasing, then (3.3.3) holds.*

**Remark 3.3.9.** *One can use Lemma 3.2.11 to obtain several inequalities of midpoint type via quasi-convex mappings.*

## 3.4 INEQUALITIES VIA $R$-CONVEX FUNCTIONS

We begin with the following theorem.

**Theorem 3.4.1.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is $r$-convex on $[a,b]$, for $p > 1$, then the following inequality holds:*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \\ &\quad \leq \frac{(b-a)}{4(p+1)^{1/p}} \left[ \left( L_r \left\{ \left| f'\left( \frac{a+b}{2} \right) \right|^{p/(p-1)}, |f'(b)|^{p/(p-1)} \right\} \right)^{(p-1)/p} \right. \\ &\qquad \left. + \left( L_r \left\{ \left| f'\left( \frac{a+b}{2} \right) \right|^{p/(p-1)}, |f'(a)|^{p/(p-1)} \right\} \right)^{(p-1)/p} \right]. \qquad (3.4.1) \end{aligned}$$

*where, $L_r(\cdot,\cdot)$ is the generalized logarithmic mean.*

*Proof.* From Lemma 3.2.1 and using the well known Hölder integral inequality, we have

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq \frac{b-a}{4} \left[ \int_0^1 |(-t)| \left| f'\left( \frac{1+t}{2}a + \frac{1-t}{2}b \right) \right| dt \right.$$

$$+\int_0^1 |t| \left| f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right| dt\Bigg]$$

$$\leq \frac{b-a}{4}\left[\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1 \left| f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{1/q}\right.$$

$$\left.+\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1 \left| f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{1/q}\right].$$

Since $|f'|^{p/(p-1)}$ is $r$-convex, then by (2.3.30), we have

$$\int_0^1 \left| f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt \leq L_r\left\{\left| f'\left(\frac{a+b}{2}\right)\right|^q, |f'(a)|^q\right\}$$

and

$$\int_0^1 \left| f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt \leq L_r\left\{|f'(b)|^q, \left| f'\left(\frac{a+b}{2}\right)\right|^q\right\}.$$

Therefore,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{4(p+1)^{1/p}}\left[\left(L_r\left\{\left| f'\left(\frac{a+b}{2}\right)\right|^q, |f'(a)|^q\right\}\right)^{1/q}\right.$$

$$\left.+\left(L_r\left\{\left| f'\left(\frac{a+b}{2}\right)\right|^q, |f'(b)|^q\right\}\right)^{1/q}\right],$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, which completes the proof. □

**Corollary 3.4.2.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|^{p/(p-1)}$ is a log-convex on $[a,b]$, for $p>1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq \frac{(b-a)}{4(p+1)^{1/p}}\left[\left(L\left\{\left| f'\left(\frac{a+b}{2}\right)\right|^{p/(p-1)}, |f'(b)|^{p/(p-1)}\right\}\right)^{(p-1)/p}\right.$$

$$\left.+\left(L\left\{\left| f'\left(\frac{a+b}{2}\right)\right|^{p/(p-1)}, |f'(a)|^{p/(p-1)}\right\}\right)^{(p-1)/p}\right], \quad (3.4.2)$$

*where, $p>1$, and $L(\cdot,\cdot)$ is the log-mean.*

**Theorem 3.4.3.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is $r$-convex $(r \geq 1)$ on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq \frac{(b-a)\cdot r}{4(1+3r+2r^2)}\left[\left(r2^{-1/r}+2\right)|f'(a)| + r2^{1-\frac{1}{r}}\left|f'\left(\frac{a+b}{2}\right)\right| + \left(r2^{-1/r}+2\right)|f'(b)|\right] \tag{3.4.3}$$

$$\leq (b-a)\frac{r\cdot\left(r2^{-\frac{1}{r}}+r2^{1-\frac{2}{r}}+2\right)}{4(1+3r+2r^2)}\left[|f'(a)|+|f'(b)|\right] \tag{3.4.4}$$

*Proof.* From Lemma 3.2.1 and since $|f'|$ is $r$-convex, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq \frac{b-a}{4}\left[\int_0^1 |(-t)|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt + \int_0^1 |t|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right]$$

$$\leq \frac{b-a}{4}\left\{\int_0^1 t\cdot\left[\left(\frac{1+t}{2}\right)|f'(a)|^r + \left(\frac{1-t}{2}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^r\right]^{1/r}dt + \int_0^1 t\cdot\left[\left(\frac{1+t}{2}\right)|f'(b)|^r + \left(\frac{1-t}{2}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^r\right]^{1/r}dt\right\}.$$

Using the fact that $\sum_{i=1}^{n}(a_i+b_i)^k \leq \sum_{i=1}^{n}a_i^k + \sum_{i=1}^{n}b_i^k$, for $0 < k < 1$, $a_1, a_2, ..., a_n \geq 0$ and $b_1, b_2, ..., b_n \geq 0$, we obtain

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{3.4.5}$$

$$\leq \frac{b-a}{4}\left\{\int_0^1 t\cdot\left[\left(\frac{1+t}{2}\right)^{1/r}|f'(a)| + \left(\frac{1-t}{2}\right)^{1/r}\left|f'\left(\frac{a+b}{2}\right)\right|\right]dt + \int_0^1 t\cdot\left[\left(\frac{1+t}{2}\right)^{1/r}|f'(b)| + \left(\frac{1-t}{2}\right)^{1/r}\left|f'\left(\frac{a+b}{2}\right)\right|\right]dt\right\}$$

$$= \frac{(b-a)\cdot r}{4(1+3r+2r^2)}\left[\left(r2^{-1/r}+2\right)|f'(a)| + r2^{1-\frac{1}{r}}\left|f'\left(\frac{a+b}{2}\right)\right| + \left(r2^{-1/r}+2\right)|f'(b)|\right],$$

which proves (3.4.3). To prove (3.4.4) and since $|f'|$ is $r$-convex, $r \geq 1$, then we have

$$\left|f'\left(\frac{a+b}{2}\right)\right| \leq \left(\frac{1}{2}\left|f'(a)\right|^r + \frac{1}{2}\left|f'(b)\right|^r\right)^{1/r} \leq \frac{|f'(a)| + |f'(b)|}{2^{\frac{1}{r}}}. \tag{3.4.6}$$

Thus, substitute (3.4.6) in (3.4.5), we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{(b-a)\cdot r}{4(1+3r+2r^2)}\left[\left(r2^{-1/r}+2\right)|f'(a)| + r2^{1-\frac{2}{r}}\left|f'\left(\frac{a+b}{2}\right)\right| + \left(r2^{-1/r}+2\right)|f'(b)|\right]. \\
&\leq \frac{(b-a)\cdot r}{4(1+3r+2r^2)}\left(r2^{-\frac{1}{r}} + r2^{1-\frac{2}{r}} + 2\right)\left[|f'(a)| + |f'(b)|\right]
\end{aligned}$$

which completes the proof. $\square$

Therefore, we deduce the following trapezoid inequality.

**Corollary 3.4.4.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &\leq \frac{(b-a)}{48}\left[5|f'(a)| + \left|f'\left(\frac{a+b}{2}\right)\right| + 5|f'(b)|\right] \\
&\leq \frac{(b-a)}{8}\left(|f'(a)| + |f'(b)|\right)
\end{aligned} \tag{3.4.7}$$

Also, we may state the following trapezoid inequality :

**Corollary 3.4.5.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$, $a, b \in I^\circ$ with $a < b$, which satisfies $f'(a) = f'(b) = 0$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{48}\left|f'\left(\frac{a+b}{2}\right)\right| \tag{3.4.8}$$

**Remark 3.4.6.** *One can use Lemma 3.2.11 to obtain several inequalities of midpoint type via $r$-convex mappings.*

## 3.5 APPLICATIONS TO TRAPEZOIDAL FORMULA

In the classical Trapezoid rule (2.3.9), it is clear that if the mapping $f$ is not twice differentiable or the second derivative is not bounded on $(a,b)$, then (2.3.9) cannot be applied. In this section, we choose two results in the sections 3.3 and 3.4 to derive some new error estimates for the trapezoidal rule in terms of first derivative, similarly one can deduce several error estimates by using different inequalities.

**Proposition 3.5.1.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. Assume that $|f'|$ is a quasi-convex on $[a,b]$. If $P := a = x_0 < x_1 < \cdots < x_n = b$ is a partition of the interval $[a,b]$, $h_i = x_{i+1} - x_i$, for $i = 0,1,2,\cdots,n-1$ and*

$$T_n(f,P) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \cdot h_i,$$

*then*

$$\begin{aligned}\left|E_n^T(f,P)\right| &= \left|\int_a^b f(x)\,dx - T_n(f,P)\right| \\ &\le \frac{1}{8}\sum_{i=0}^{n-1} h_i^2 \cdot \left[\max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, |f'(x_{i+1})|\right\}\right. \\ &\qquad\left. + \max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, |f'(x_i)|\right\}\right].\end{aligned}$$

*Proof.* Applying Theorem 3.3.1 on the subintervals $[x_i, x_{i+1}]$, for $i = 0,1,...,n-1$ of the division $P$, we get

$$\begin{aligned}\left|E_n^T(f,P)\right| &= \left|\int_a^b f(x)\,dx - T_n(f,P)\right| \\ &= \left|\int_{x_i}^{x_{i+1}} f(x)\,dx - h_i \frac{f(x_i)+f(x_{i+1})}{2}\right| \\ &\le h_i^2 \left[\max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, |f'(x_{i+1})|\right\}\right. \\ &\qquad\left. + \max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, |f'(x_i)|\right\}\right].\end{aligned}$$

Summing over $i$ from $0$ to $n-1$ and taking into account that $|f'|$ is quasi-convex, we deduce that

$$\left|E_n^T(f,P)\right| \le \frac{1}{8}\sum_{i=0}^{n-1} h_i^2 \left[\max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, |f'(x_{i+1})|\right\}\right.$$
$$\left. + \max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, |f'(x_i)|\right\}\right],$$

which completes the proof. □

**Proposition 3.5.2.** *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}_+$ *be a differentiable mapping on* $I^\circ$ *such that* $f' \in L[a,b]$*, where* $a, b \in I$ *with* $a < b$*. Assume that* $|f'|$ *is convex on* $[a,b]$ *and* $f'(x_i) = f'(x_{i+1}) = 0$*. If* $P := a = x_0 < x_1 < \cdots < x_n = b$ *is a partition of the interval* $[a,b]$*,* $h_i = x_{i+1} - x_i$*, for* $i = 0, 1, 2, \cdots, n-1$ *and*

$$T_n(f,P) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \cdot h_i,$$

*then*

$$\left|E_n^T(f,P)\right| = \left|\int_a^b f(x)\,dx - T_n(f,P)\right| \le \frac{1}{48}\sum_{i=0}^{n-1} h_i^2 \cdot \left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|$$

*Proof.* The proof can be done similar to that of Proposition 3.5.1 and using Corollary 3.4.4. □

## 3.6 SUMMARY AND CONCLUSION

In the presented chapter, inequalities for differentiable $s$-convex (concave), quasi-convex, $r$-convex and log-convex mappings that are connected with the both sides of celebrated Hermite–Hadamard integral inequality are established. The idea of these are results summed and manifested by writing the differences

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx, \qquad f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx$$

in terms of $\int_0^1 p(t)\, f'(ta + (1-t)b)\,dt$, where $p(t)$ is a suitable Peano kernel, after that using the convexity condition of $|f'|$ we obtain the desirable results. Several generalizations, refinements and improvements for the corresponding version for powers of these inequalities are considered by applying the Hölder and the power mean inequalities.

In this way, we highlight the role of convexity to obtain several refinements for the Hermite–Hadamard's inequality and thus for the midpoint and the trapezoid inequalities. More precisely, the obtained trapezoid type inequalities (3.2.1)–(3.2.12) via $s$-convex functions, refine and improve Kirmaci results (2.3.24)–(2.3.26), where the obtaining constants in our results are better than Kirmaci results. Similarly, the presented midpoint type inequalities (3.2.15)–(3.2.27) are new for $s$-convex functions. For quasi-convex functions, the presented inequalities (3.3.1)–(3.3.7) are new and different from (2.3.27)–(2.3.28). In the same sense, for $r$-convex functions, the inequalities (3.4.1)–(3.4.4) and (3.4.7)–(3.4.8) are new. In general, along the presented chapter our results are new and in some cases is better than the old results.

# CHAPTER IV

# OSTROWSKI'S TYPE INEQUALITIES

## 4.1 INTRODUCTION

In this section, the classical Ostrowski's inequality holds with weaker conditions. Several inequalities of Ostrowski's type via concave, $s$-convex, quasi-convex and $r$-convex functions are introduced. Some bounds for the difference between the integral mean of a function $f$ defined on the interval $[a, b]$ and it is value in the midpoint $\frac{a+b}{2}$ are provided. Therefore, the inequalities are related to the left hand side of Hadamard inequality. In this way, a generalizations and improvement for a previous inequalities in the literature for functions $f$ with $|f|$ (or $|f|^q, q \geq 1$) convex. The proofs follow from standard arguments and a Montgomery-type equality. Finally, some inequalities between some special means are derived.

## 4.2 ON THE OSTROWSKI'S INEQUALITY

We start by giving another proof for the well known Ostrowski's inequality:

**Theorem 4.2.1.** *Let $f : I \to \mathbb{R}$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$ and $f'$ is bounded i.e., $\|f'\|_\infty = \sup\limits_{y \in (a,b)} |f'(y)| < \infty$. If $f$ is concave on $I$, then we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right| \leq (b-a) \|f'\|_\infty \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right], \qquad (4.2.1)$$

*for all $x \in I^\circ$.*

*Proof.* Since $f$ is differentiable on $I^\circ$ and concave on $I$, then for any $x, t \in I^\circ$

$$\frac{f(x) - f(t)}{x - t} \leq f'(t).$$

It follows that

$$f(x) \leq f(t) + (x - t) f'(t).$$

Integrating both sides over $[a, b]$, with respect to $t$, we get

$$(b - a) f(x) \leq \int_a^b f(t)\, dt + \int_a^b (x - t) f'(t)\, dt,$$

which is equivalent to write

$$f(x) - \frac{1}{b - a}\int_a^b f(t)\, dt \leq \frac{1}{b - a}\int_a^b (x - t) f'(t)\, dt.$$

Therefore, since $f'$ is bounded, then we have

$$\begin{aligned}
\left| f(x) - \frac{1}{b - a}\int_a^b f(t)\, dt \right| &\leq \left| \frac{1}{b - a}\int_a^b (x - t) f'(t)\, dx \right| \\
&\leq \frac{1}{b - a}\int_a^b |x - t|\, |f'(t)|\, dt \\
&\leq \frac{1}{b - a}\sup_{t \in (a,b)} |f'(t)| \int_a^b |x - t|\, dt \\
&\leq \frac{\|f'\|_\infty}{b - a}\left[ \int_a^x (x - t)\, dt + \int_x^b (t - x)\, dt \right] \\
&\leq \frac{\|f'\|_\infty}{b - a}\left[ \frac{1}{2}(x - a)^2 + \frac{1}{2}(b - x)^2 \right] \\
&= (b - a)\|f'\|_\infty \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b - a)^2} \right].
\end{aligned}$$

which completes the proof. $\square$

Using the same technique, we may state the following result for Riemann-Stieltjes integral.

**Theorem 4.2.2.** *Let $f$ as in Theorem 4.2.1. Let $g' : I^\circ \to \mathbb{R}_+$ be defined on $I^\circ$ such that $g' \in L[a, b]$. If $g'$ is bounded i.e., $\|g'\|_\infty = \sup_{x \in (a,b)} |g'(x)| < \infty$, then we have*

$$\left| f(x)\left[g(b) - g(a)\right] - \int_a^b f(t)\, dg \right| \leq \|f'\|_\infty \|g'\|_\infty \left[ \frac{(x - a)^2 + (b - x)^2}{2} \right], \quad (4.2.2)$$

*for each $x \in I$.*

*Proof.* Since $f$ is concave function on $I$, then for any $x, t \in [a, b]$

$$f(x) \leq f(t) + (x - t) f'(t).$$

Since $g' > 0$, multiplying both side by $g'(t)$, we get

$$f(x) g'(t) \leq f(t) g'(t) + (x - t) f'(t) g'(t).$$

Integrating both sides over $[a, b]$, with respect to $t$, we get

$$f(x)[g(b) - g(a)] \leq \int_a^b f(t) g'(t) \, dt + f'(t) \int_a^b (x - t) g'(t) \, dt,$$

which is equivalent to write

$$f(x)[g(b) - g(a)] - \int_a^b f(t) \, dg \leq \int_a^b (x - t) f'(t) g'(t) \, dt.$$

Therefore, since $f', g'$ are bounded, then we have

$$\begin{aligned}
\left| f(x)[g(b) - g(a)] - \int_a^b f(t) \, dg \right| &\leq \left| \int_a^b (x - t) f'(t) g'(t) \, dt \right| \\
&\leq \int_a^b |x - t| \, |f'(t)| \, |g'(t)| \, dt \\
&\leq \sup_{t \in (a,b)} |f'(t)| \cdot \sup_{t \in (a,b)} |g'(t)| \int_a^b |x - t| \, dt \\
&= \|f'\|_\infty \|g'\|_\infty \left[ \frac{(x - a)^2 + (b - x)^2}{2} \right],
\end{aligned}$$

which is required. □

A new Ostrowski's type inequality, which gives the weighted difference between the integrands of a function $f$ and it is first derivative, is considered as follows:

**Theorem 4.2.3.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on $I^\circ$ where $a, b \in I$ with $a < b$. Assume that $f$ and $f'$ are concave on $(a, b)$. If $\|f'\|_\infty = \sup_{t \in (a,b)} |f'(t)| < \infty$ and $\|f''\|_\infty = \sup_{t \in (a,b)} |f''(t)| < \infty$ then the following inequality holds:*

$$\begin{aligned}
\left| f'(y) - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \left| \frac{f(b) - f(a)}{b - a} - f(x) \right| &+ \|f'\|_\infty \left[ \frac{(x - a)^2 + (b - x)^2}{2(b - a)} \right] \\
&+ \|f''\|_\infty \left[ \frac{(y - a)^2 + (b - y)^2}{2(b - a)} \right], \quad (4.2.3)
\end{aligned}$$

*where $x, y \in (a, b)$.*

*Proof.* Since $f$ is concave on $[a,b]$ and $f'$ is concave on $(a,b)$ then for any $s,t\in(a,b)$

$$f(x)\le f(t)+(x-t)f'(t), \tag{4.2.4}$$

and

$$f'(y)\le f'(s)+(y-s)f''(s), \tag{4.2.5}$$

Integrating both sides of (4.2.4) over $[a,b]$, with respect to $t$, and (4.2.5) over $[a,b]$, with respect to $s$, we get

$$(b-a)f(x)\le\int_a^b f(t)\,dt+\int_a^b(x-t)f'(t)\,dt, \tag{4.2.6}$$

and

$$\int_a^b f'(y)\,ds\le\int_a^b f'(s)\,ds+\int_a^b(y-s)f''(s)\,ds \tag{4.2.7}$$

Adding (4.2.6), (4.2.7), we get

$$\begin{aligned}&(b-a)f'(y)-\int_a^b f(t)\,dt\\&\le f(b)-f(a)-(b-a)f(x)+\int_a^b(x-t)f'(t)\,dt+\int_a^b(y-s)f''(s)\,ds\end{aligned} \tag{4.2.8}$$

Therefore, since $\|f'\|_\infty=\sup\limits_{x\in(a,b)}|f'(t)|<\infty$ and $\|f''\|_\infty=\sup\limits_{x\in(a,b)}|f''(t)|<\infty$, then we have

$$\begin{aligned}\left|f'(y)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|&\le\left|\frac{f(b)-f(a)}{b-a}-f(x)\right|+\frac{1}{b-a}\int_a^b|x-t|\,|f'(t)|\,dt\\&\quad+\frac{1}{b-a}\int_a^b|y-s|\,|f''(s)|\,ds\\&\le\left|\frac{f(b)-f(a)}{b-a}-f(x)\right|+\frac{\|f'\|_\infty}{b-a}\int_a^b|x-t|\,dt\\&\quad+\frac{\|f''\|_\infty}{b-a}\int_a^b|y-s|\,ds\\&\le\left|\frac{f(b)-f(a)}{b-a}-f(x)\right|+\|f'\|_\infty\left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right]\\&\quad+\|f''\|_\infty\left[\frac{(y-a)^2+(b-y)^2}{2(b-a)}\right],\end{aligned}$$

for $x,y\in(a,b)$, which completes the proof. □

**Remark 4.2.4.** *In the inequality (4.2.3), one can see that when $t, y \to b^-$, then we have*

$$\left| f'(b) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\ \leq \left| \frac{f(b)-f(a)}{b-a} - f(b) \right| + \frac{(b-a)}{2}\left[\|f'\|_\infty + \|f''\|_\infty\right], \quad (4.2.9)$$

*similarly, when $t, y \to a^+$, then we have*

$$\left| f'(a) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\ \leq \left| \frac{f(b)-f(a)}{b-a} - f(a) \right| + \frac{(b-a)}{2}\left[\|f'\|_\infty + \|f''\|_\infty\right]. \quad (4.2.10)$$

*Also, for $t = y = \frac{a+b}{2}$, we have*

$$\left| f'\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\ \leq \left| \frac{f(b)-f(a)}{b-a} - f\left(\frac{a+b}{2}\right) \right| + \frac{(b-a)}{4}\left[\|f'\|_\infty + \|f''\|_\infty\right] \quad (4.2.11)$$

In the following result we propose an error estimation for the first derivative in terms of convexity.

**Theorem 4.2.5.** *Consider the assumptions in Theorem 4.2.3. Then the following inequality holds:*

$$\left| f'(y) - \frac{f(b)-f(a)}{b-a} \right| \leq \|f'\|_\infty \frac{(b-a)^2}{3} + \|f''\|_\infty \left[ \frac{(y-a)^2 + (b-y)^2}{2(b-a)} \right], \quad (4.2.12)$$

$\forall y \in (a, b)$.

*Proof.* Integrating the both sides of (4.2.8) over $[a, b]$ with respect to $x$, we get

$$(b-a)^2 f'(y) - (b-a)\int_a^b f(t)\,dt \\ \leq (b-a)(f(b)-f(a)) - (b-a)\int_a^b f(x)\,dx \\ - \int_a^b \int_a^b (x-t) f'(t)\,dxdt + (b-a)\int_a^b (y-s) f''(s)\,ds,$$

and we write

$$
\begin{aligned}
f'(y) - \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(b)-f(a)}{b-a} &- \frac{1}{b-a}\int_a^b f(x)\,dx \\
&- \int_a^b\int_a^b (x-t)\,f'(t)\,dxdt \\
&+ \frac{1}{b-a}\int_a^b (y-s)\,f''(s)\,ds,
\end{aligned}
$$

which follows that

$$
f'(y) - \frac{f(b)-f(a)}{b-a} \le \frac{1}{b-a}\int_a^b (y-s)\,f''(s)\,ds - \int_a^b\int_a^b (x-t)\,f'(t)\,dxdt.
$$

Since $f'$ and $f''$ are bounded, then we have

$$
\begin{aligned}
\left| f'(y) - \frac{f(b)-f(a)}{b-a} \right| &\le \frac{1}{b-a}\int_a^b |y-s|\,|f''(s)|\,ds + \int_a^b\int_a^b |x-t|\,|f'(t)|\,dxdt \\
&\le \frac{\|f''\|_\infty}{b-a}\int_a^b |y-s|\,ds + \frac{\|f'\|_\infty}{b-a}\int_a^b\int_a^b |x-t|\,dxdt \\
&\le \|f''\|_\infty \left[ \frac{(y-a)^2+(b-y)^2}{2(b-a)} \right] \\
&\quad + \|f'\|_\infty \int_a^b \frac{(t-a)^2+(b-t)^2}{2(b-a)}dt \\
&= \|f''\|_\infty \left[ \frac{(y-a)^2+(b-y)^2}{2(b-a)} \right] + \|f'\|_\infty \frac{(b-a)^2}{3},
\end{aligned}
$$

$\forall y \in (a,b)$, which completes the proof. □

**Remark 4.2.6.** *In the inequality (4.2.12), we have*

$$
\left| f'(b) - \frac{f(b)-f(a)}{b-a} \right| \le \|f'\|_\infty \frac{(b-a)^2}{3} + \|f''\|_\infty \frac{(b-a)}{2}, \tag{4.2.13}
$$

$$
\left| f'(a) - \frac{f(b)-f(a)}{b-a} \right| \le \|f'\|_\infty \frac{(b-a)^2}{3} + \|f''\|_\infty \frac{(b-a)}{2}, \tag{4.2.14}
$$

*and*

$$
\left| f'\left(\frac{a+b}{2}\right) - \frac{f(b)-f(a)}{b-a} \right| \le \|f'\|_\infty \frac{(b-a)^2}{3} + \|f''\|_\infty \frac{(b-a)}{4} \tag{4.2.15}
$$

Using the same technique in the proof of Theorem 4.2.3, we can generalize the inequality (4.2.3) for $n$-times differentiable mappings as follows:

**Corollary 4.2.7.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be $n$-times differentiable mapping on $I^\circ$ where $a, b \in I$ with $a < b$. Assume that $f$ and $f^{(n-1)}$ are concave, $n \geq 2$ on $(a, b)$. If $\|f'\|_\infty = \sup\limits_{x\in(a,b)} |f'(t)| < \infty$ and $\left\|f^{(n)}\right\|_\infty = \sup\limits_{x\in(a,b)} \left|f^{(n)}(t)\right| < \infty$, then the following inequality holds:*

$$\left|f^{(n-1)}(y) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \left|\frac{f^{(n-2)}(b) - f^{(n-2)}(a)}{b-a} - f(t)\right| + \|f'\|_\infty \left[\frac{(t-a)^2 + (b-t)^2}{2(b-a)}\right] + \left\|f^{(n)}\right\|_\infty \left[\frac{(y-a)^2 + (b-y)^2}{2(b-a)}\right], \quad (4.2.16)$$

*for $n \geq 2$ and $t, y \in (a, b)$.*

*Proof.* The proof goes likewise the proof of Theorem 4.2.3, we omit the details. ☐

Next result gives an Ostrowski type inequality involving product of two functions, which is different from (2.3.39).

**Theorem 4.2.8.** *Let $f, g : I \to \mathbb{R}_+$ be two bounded differentiable mapping on $I^\circ$ such that $f', g' \in L[a, b]$, where $a, b \in I$ with $a < b$ whose derivatives $f', g'$ are bounded. If $f$ is concave and $M = \max\limits_{x\in(a,b)} \{|f(x)|, |f'(x)|, |g(x)|, |g'(x)|\}$, then we have*

$$\left|\int_a^b f(x)\,g(x)\,dx - (b-a)\,f(t)\,g(s)\right| \leq M\left[\frac{(t-a)^2 + (b-t)^2}{2} + \frac{(s-a)^2 + (b-s)^2}{2} + \frac{b^3 - a^3}{3} - \frac{b^2 - a^2}{2}(t+s) + ts(b-a) + \begin{cases} \frac{(t-s)^3}{3}, & s < t \\ \frac{(s-t)^3}{3}, & t \leq s \end{cases}\right], \quad (4.2.17)$$

*where $t, s \in [a, b]$.*

*Proof.* Since $f$ and $g$ are concave function on $I$, then for any $x, t \in (a, b)$

$$f(x) \leq f(t) + (x-t)\,f'(t)$$

and

$$g(x) \leq g(s) + (x-s)\,g'(s)$$

Multiplying the above inequalities, we get

$$\begin{aligned} f(x)\,g(x) &\leq [f(t) + (x-t)\,f'(t)]\,[g(s) + (x-s)\,g'(s)] \\ &= [f(t)\,g(s) + (x-t)\,f'(t)\,g(s)] \\ &\quad + [f(t)\,(x-s)\,g'(s) + (x-t)\,f'(t)\,(x-s)\,g'(s)] \end{aligned}$$

Integrating both sides over $[a,b]$, with respect to $x$, we get

$$\begin{aligned} \int_a^b f(x)\,g(x)\,dx &\leq (b-a)\,f(t)\,g(s) + f'(t)\,g(s)\int_a^b (x-t)\,dx \\ &\quad + f(t)\,g'(s)\int_a^b (x-s)\,dx + f'(t)\,g'(s)\int_a^b (x-t)\,(x-s)\,dx \end{aligned}$$

which is equivalent to write

$$\begin{aligned} &\int_a^b f(x)\,g(x)\,dx - (b-a)\,f(t)\,g(s) \\ &\leq f'(t)\,g(s)\int_a^b (x-t)\,dx + f(t)\,g'(s)\int_a^b (x-s)\,dx \\ &\quad + f'(t)\,g'(s)\int_a^b (x-t)\,(x-s)\,dx \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\left|\int_a^b f(x)\,g(x)\,dx - (b-a)\,f(t)\,g(s)\right| \\ &\leq M\left[\int_a^b |x-t|\,dx + \int_a^b |x-s|\,dx + \int_a^b |x-t|\,|x-s|\,dx\right] \\ &= M\left[\frac{(t-a)^2+(b-t)^2}{2} + \frac{(s-a)^2+(b-s)^2}{2}\right. \\ &\quad + \frac{t^3-s^3}{3} - 2\left(\begin{cases} 0, & s<t \\ 1, & t\leq s \end{cases}\right)\cdot ts^2 + \frac{b^3-a^3}{3} - \frac{b^2-a^2}{2}t \\ &\quad - \frac{b^2-a^2}{2}s - t^2s + ts^2 + tsb - tsa + 2\left(\begin{cases} 0, & s<t \\ 1, & t\leq s \end{cases}\right)\cdot t^2s \\ &\quad \left. -\frac{2}{3}\left(\begin{cases} 0, & s<t \\ 1, & t\leq s \end{cases}\right)\cdot t^3 + \frac{2}{3}\left(\begin{cases} 0, & s<t \\ 1, & t\leq s \end{cases}\right)\cdot s^3\right] \end{aligned}$$

$$= M\left[\frac{(t-a)^2+(b-t)^2}{2}+\frac{(s-a)^2+(b-s)^2}{2}+\frac{b^3-a^3}{3}\right.$$

$$\left.-\frac{b^2-a^2}{2}(t+s)+ts(b-a)+\begin{cases}\frac{(t-s)^3}{3}, & s<t\\ \frac{(s-t)^3}{3}, & t\le s\end{cases}\right]$$

which is required. □

## 4.3 OSTROWSKI'S TYPE INEQUALITIES VIA CONVEX FUNCTIONS

In the following, we introduce some inequalities of Ostrowski's type via $s$-convex function (in the second sense). We note that the functions appearing in the main results are in terms of the first derivatives which is $s$-convex in the second sense. As we note in Chapter II, the $s$-convexity is a weaker condition than the usual convexity and coincides in the case $s = 1$.

**Lemma 4.3.1.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$ where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$f(x)-\frac{1}{b-a}\int_a^b f(u)\,du=(b-a)\int_0^1 p(t)\,f'(ta+(1-t)b)\,dt \tag{4.3.1}$$

*for each $t \in [0, 1]$, where*

$$p(t)=\begin{cases} t, & t\in\left[0,\frac{b-x}{b-a}\right]\\ t-1, & t\in\left(\frac{b-x}{b-a},1\right]\end{cases},$$

*for all $x \in [a, b]$.*

*Proof.* Integrating by parts

$$I=\int_0^1 p(t)\,f'(ta+(1-t)b)\,dt$$
$$=\int_0^{\frac{b-x}{b-a}} tf'(ta+(1-t)b)\,dt+\int_{\frac{b-x}{b-a}}^1 (t-1)\,f'(ta+(1-t)b)\,dt$$
$$=t\frac{f(ta+(1-t)b)}{a-b}\Big|_0^{\frac{b-x}{b-a}}-\int_0^{\frac{b-x}{b-a}}\frac{f(ta+(1-t)b)}{a-b}dt$$
$$+(t-1)\frac{f(ta+(1-t)b)}{a-b}\Big|_{\frac{b-x}{b-a}}^1-\int_{\frac{b-x}{b-a}}^1\frac{f(ta+(1-t)b)}{a-b}dt$$

$$= \frac{b-x}{(b-a)^2} f(x) - \int_0^{\frac{b-x}{b-a}} \frac{f(ta+(1-t)b)}{a-b} dt + \frac{x-a}{(b-a)^2} f(x)$$
$$- \int_{\frac{b-x}{b-a}}^1 \frac{f(ta+(1-t)b)}{a-b} dt$$
$$= \frac{1}{b-a} f(x) - \frac{1}{(b-a)^2} \int_a^b f(u)\, du.$$

Thus, $(b-a) \cdot I$, gives the desired representation (4.3.1). $\square$

An Ostrowski-like inequality may be stated as follows:

**Theorem 4.3.2.** *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* $I^\circ$ *such that* $f' \in L[a,b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|$ *is convex on* $[a,b]$, *then the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right|$$
$$\leq \frac{b-a}{6} \left[ \left( 4\left(\frac{b-x}{b-a}\right)^3 - 3\left(\frac{b-x}{b-a}\right)^2 + 1 \right) |f'(a)| \right.$$
$$\left. + \left( 9\left(\frac{b-x}{b-a}\right)^2 - 4\left(\frac{b-x}{b-a}\right)^3 - 6\left(\frac{b-x}{b-a}\right) + 2 \right) |f'(b)| \right], \tag{4.3.2}$$

*for each* $x \in [a,b]$. *The constant* $\frac{1}{6}$ *is best possible in the sense that it cannot be replaced by a smaller value.*

*Proof.* Using triangle inequality in Lemma 4.3.1 and since $|f'|$ is convex, then we have

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right|$$
$$\leq (b-a) \int_0^{\frac{b-x}{b-a}} t\, |f'(ta+(1-t)b)|\, dt$$
$$+ (b-a) \int_{\frac{b-x}{b-a}}^1 (1-t)\, |f'(ta+(1-t)b)|\, dt$$

$$
\begin{aligned}
&\le (b-a)\int_0^{\frac{b-x}{b-a}} t\left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt \\
&\qquad + (b-a)\int_{\frac{b-x}{b-a}}^{1}(1-t)\left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt \\
&= \frac{(b-x)^3}{3(b-a)^3}\left|f'(a)\right| - \frac{(b-x)^3}{3(b-a)^3}\left|f'(b)\right| + \frac{(b-x)^2}{2(b-a)^2}\left|f'(b)\right| \\
&\quad + \frac{1}{3}\left|f'(b)\right|\left(1-\frac{(b-x)^3}{(b-a)^3}\right) - \frac{1}{3}\left|f'(a)\right|\left(1-\frac{(b-x)^3}{(b-a)^3}\right) \\
&\quad + \frac{1}{2}\left|f'(a)\right|\left(1-\frac{(b-x)^2}{(b-a)^2}\right) - \left|f'(b)\right|\left(1-\frac{(b-x)^2}{(b-a)^2}\right) \\
&\quad + \left|f'(b)\right|\left(1-\frac{b-x}{b-a}\right),
\end{aligned}
$$

which follows that

$$
\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\le (b-a)\left[\frac{2}{3}\left(\frac{b-x}{b-a}\right)^3-\frac{1}{2}\left(\frac{b-x}{b-a}\right)^2+\frac{1}{6}\right]\cdot\left|f'(a)\right| \\
&\qquad + (b-a)\left[-\frac{2}{3}\left(\frac{b-x}{b-a}\right)^3+\frac{3}{2}\left(\frac{b-x}{b-a}\right)^2-\frac{b-x}{b-a}+\frac{1}{3}\right]\cdot\left|f'(b)\right|,
\end{aligned}
$$

and we can write

$$
\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\qquad\le \frac{b-a}{6}\left[\left(4\left(\frac{b-x}{b-a}\right)^3-3\left(\frac{b-x}{b-a}\right)^2+1\right)\left|f'(a)\right|\right. \\
&\qquad\qquad \left.+\left(9\left(\frac{b-x}{b-a}\right)^2-4\left(\frac{b-x}{b-a}\right)^3-6\left(\frac{b-x}{b-a}\right)+2\right)\left|f'(b)\right|\right].
\end{aligned}
$$

To prove that the constant $1/6$ is best possible, let us assume that (4.3.2) holds with constant $C>0$, i.e.,

$$
\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\qquad\le C(b-a)\left[\left(4\left(\frac{b-x}{b-a}\right)^3-3\left(\frac{b-x}{b-a}\right)^2+1\right)\left|f'(a)\right|\right. \\
&\qquad\qquad \left.+\left(9\left(\frac{b-x}{b-a}\right)^2-4\left(\frac{b-x}{b-a}\right)^3-6\left(\frac{b-x}{b-a}\right)+2\right)\left|f'(b)\right|\right].
\end{aligned}
$$

Let $f(x) = x$, and then set $x = a$, we get

$$\left|a - \frac{a+b}{2}\right| \le C\,(b-a)\,[(4-3+1)\cdot 1 + (9-4-6+2)\cdot 1]$$

therefore

$$\frac{b-a}{2} \le 3C\,(b-a)\,,$$

which gives $C \ge \frac{1}{6}$, and the inequality (4.3.2) is proved. □

One can deduce an Ostrowski like inequality for functions whose derivative are bounded, as follows:

**Corollary 4.3.3.** *In Theorem 4.3.2. Additionally, if* $|f'(x)| \le M$*,* $M > 0$*, then inequality*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \le M\,(b-a)\left[\left(\frac{b-x}{b-a}\right)^2 - \left(\frac{b-x}{b-a}\right) + \frac{1}{2}\right], \qquad (4.3.3)$$

*holds. The constant* $\frac{1}{2}$ *is best possible in the sense that it cannot be replaced by a smaller constant.*

*Proof.* In the proof of Theorem , assume that $|f'(x)| \le M$ we get the required result. To prove the sharpness we use the identity function. □

The corresponding version for powers of the absolute value of the first derivative is incorporated in the following result:

**Theorem 4.3.4.** *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* $I^\circ$ *such that* $f' \in L[a,b]$*, where* $a, b \in I$ *with* $a < b$*. If* $|f'|^{p/(p-1)}$ *is convex on* $[a,b]$*, then the following inequality holds:*

$$\begin{aligned}\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \le \frac{2^{-1/q}}{[(b-a)(p+1)]^{1/p}}\Big[&(b-x)^{\frac{p+1}{p}}\left(|f'(x)|^q + |f'(b)|^q\right)^{1/q} \\ &+ (x-a)^{\frac{p+1}{p}}\left(|f'(a)|^q + |f'(x)|^q\right)^{1/q}\Big]\end{aligned} \qquad (4.3.4)$$

*for each* $x \in [a,b]$*, where* $\frac{1}{p} + \frac{1}{q} = 1$.

*Proof.* Suppose that $p > 1$. From Lemma 4.3.1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
&\le (b-a)\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|dt \\
&\qquad + (b-a)\int_{\frac{b-x}{b-a}}^1 |t-1|\left|f'(ta+(1-t)b)\right|dt \\
&\le (b-a)\left(\int_0^{\frac{b-x}{b-a}} t^p dt\right)^{1/p}\left(\int_0^{\frac{b-x}{b-a}} \left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q} \\
&\qquad + (b-a)\left(\int_{\frac{b-x}{b-a}}^1 (1-t)^p\,dt\right)^{1/p}\left(\int_{\frac{b-x}{b-a}}^1 \left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q}.
\end{aligned}$$

Since $|f'|$ is convex, by Hermite-Hadamard inequality (2.3.10), we have,

$$\int_0^{\frac{b-x}{b-a}} \left|f'(ta+(1-t)b)\right|dt \le \frac{|f'(x)|+|f'(b)|}{2},$$

and

$$\int_{\frac{b-x}{b-a}}^1 \left|f'(ta+(1-t)b)\right|dt \le \frac{|f'(a)|+|f'(x)|}{2}.$$

Therefore,

$$\begin{aligned}
\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \le \frac{2^{-1/q}}{\left[(b-a)(p+1)\right]^{1/p}} &\left[(b-x)^{\frac{p+1}{p}}\left(|f'(x)|^q+|f'(b)|^q\right)^{1/q}\right. \\
&\left. + (x-a)^{\frac{p+1}{p}}\left(|f'(a)|^q+|f'(x)|^q\right)^{1/q}\right],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. This completes the proof. □

**Corollary 4.3.5.** *In Theorem 4.3.4. Additionally, if* $|f'(x)| \le M$, $M > 0$, *then inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \le M \cdot \frac{(b-x)^{\frac{p+1}{p}} + (x-a)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}(b-a)^{\frac{1}{p}}} \tag{4.3.5}$$

*holds, where* $\frac{1}{p}+\frac{1}{q}=1$.

**Corollary 4.3.6.** *In Theorem 4.3.4, choose* $x = \frac{a+b}{2}$*, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{(b-a)}{4(p+1)^{1/p}}\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right)^{1/q} + \left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right]. \quad (4.3.6)$$

**Theorem 4.3.7.** *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* $I^\circ$ *such that* $f' \in L[a,b]$*, where* $a, b \in I$ *with* $a < b$*. If* $|f'|^{p/(p-1)}$ *is concave on* $[a,b]$*, then the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{(b-a)}{(p+1)^{1/p}}\left[\left(\frac{b-x}{b-a}\right)^{(p+1)/p}\left|f'\left(\frac{b+x}{2}\right)\right| + \left(\frac{x-a}{b-a}\right)^{(p+1)/p}\left|f'\left(\frac{a+x}{2}\right)\right|\right], \quad (4.3.7)$$

*for each* $x \in [a,b]$*, where* $p > 1$.

*Proof.* Suppose that $p > 1$. From Lemma 4.3.1 and using the Hölder inequality, we have

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\ &\leq (b-a)\int_0^{\frac{b-x}{b-a}} t\,|f'(ta+(1-t)b)|\,dt \\ &\quad + (b-a)\int_{\frac{b-x}{b-a}}^1 |t-1|\,|f'(ta+(1-t)b)|\,dt\end{aligned}$$

$$\begin{aligned}&\leq (b-a)\left(\int_0^{\frac{b-x}{b-a}} t^p dt\right)^{1/p}\left(\int_0^{\frac{b-x}{b-a}} |f'(ta+(1-t)b)|^q\,dt\right)^{1/q} \\ &\quad + (b-a)\left(\int_{\frac{b-x}{b-a}}^1 (1-t)^p\,dt\right)^{1/p}\left(\int_{\frac{b-x}{b-a}}^1 |f'(ta+(1-t)b)|^q\,dt\right)^{1/q}.\end{aligned}$$

Since $|f'|^q$ is concave on $[a,b]$, by Hermite-Hadamard's inequality (2.3.10), we get

$$\int_0^{\frac{b-x}{b-a}} |f'(ta+(1-t)b)|^q\,dt \leq \left|f'\left(\frac{b+x}{2}\right)\right|^q$$

and

$$\int_{\frac{b-x}{b-a}}^{1} |f'(ta+(1-t)b)|^q \, dt \leq \left| f'\left(\frac{a+x}{2}\right) \right|^q$$

Therefore,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \frac{(b-a)}{(p+1)^{1/p}} \left[ \left(\frac{b-x}{b-a}\right)^{(p+1)/p} \left| f'\left(\frac{b+x}{2}\right) \right| + \left(\frac{x-a}{b-a}\right)^{(p+1)/p} \left| f'\left(\frac{a+x}{2}\right) \right| \right]$$

This completes the proof. $\square$

**Corollary 4.3.8.** *In Theorem 4.3.7, choose* $x = \frac{a+b}{2}$*, then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \frac{(b-a)}{\left(2^{p+1}(p+1)\right)^{1/p}} \left[ \left| f'\left(\frac{a+3b}{4}\right) \right| + \left| f'\left(\frac{3a+b}{4}\right) \right| \right], \quad (4.3.8)$$

*for each* $x \in [a,b]$*, where* $p > 1$.

The following result refines the above inequality (4.3.7).

**Theorem 4.3.9.** *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* $I^\circ$ *such that* $f' \in L[a,b]$*, where* $a, b \in I$ *with* $a < b$*. If* $|f'|^{p/(p-1)}$ *is concave on* $[a,b]$*, then the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \frac{(b-x)^2}{(b-a)(p+1)^{1/p}} \left| f'\left(\frac{b+x}{2}\right) \right| + \frac{(x-a)^2}{(b-a)(p+1)^{1/p}} \left| f'\left(\frac{a+x}{2}\right) \right| \quad (4.3.9)$$

*for each* $x \in [a,b]$*, where* $p > 1$.

*Proof.* Suppose that $p > 1$. From Lemma 4.3.1 and using the Hölder inequality, we

have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\leq (b-a)\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|dt \\
&\qquad + (b-a)\int_{\frac{b-x}{b-a}}^1 |t-1|\left|f'(ta+(1-t)b)\right|dt \\
&\leq (b-a)\left(\int_0^{\frac{b-x}{b-a}} t^p dt\right)^{1/p}\left(\int_0^{\frac{b-x}{b-a}} \left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q} \\
&\qquad + (b-a)\left(\int_{\frac{b-x}{b-a}}^1 (1-t)^p\,dt\right)^{1/p}\left(\int_{\frac{b-x}{b-a}}^1 \left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q}.
\end{aligned}$$

Since $|f'|^q$ is concave on $[a,b]$, we can use the Jensen's integral inequality to obtain

$$\begin{aligned}
\int_0^{\frac{b-x}{b-a}} \left|f'(ta+(1-t)b)\right|^q dt &\leq \int_0^{\frac{b-x}{b-a}} t^0\left|f'(ta+(1-t)b)\right|^q dt \\
&\leq \left(\int_0^{\frac{b-x}{b-a}} t^0 dt\right)\left|f'\left(\frac{1}{\int_0^{\frac{b-x}{b-a}} t^0 dt}\int_0^{\frac{b-x}{b-a}} (ta+(1-t)b)\,dt\right)\right|^q \\
&= \frac{b-x}{b-a}\left|f'\left(\frac{b+x}{2}\right)\right|^q
\end{aligned}$$

and

$$\begin{aligned}
\int_{\frac{b-x}{b-a}}^1 \left|f'(ta+(1-t)b)\right|^q dt &\leq \int_{\frac{b-x}{b-a}}^1 t^0\left|f'(ta+(1-t)b)\right|^q dt \\
&\leq \left(\int_{\frac{b-x}{b-a}}^1 t^0 dt\right)\left|f'\left(\frac{1}{\int_{\frac{b-x}{b-a}}^1 t^0 dt}\int_{\frac{b-x}{b-a}}^1 (ta+(1-t)b)\,dt\right)\right|^q \\
&= \frac{x-a}{b-a}\left|f'\left(\frac{a+x}{2}\right)\right|^q
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| &\leq \frac{(b-x)^2}{(b-a)(p+1)^{1/p}}\left|f'\left(\frac{b+x}{2}\right)\right| \\
&\qquad + \frac{(x-a)^2}{(b-a)(p+1)^{1/p}}\left|f'\left(\frac{a+x}{2}\right)\right|
\end{aligned}$$

This completes the proof. □

**Corollary 4.3.10.** *In Theorem 4.3.9, choose* $x = \frac{a+b}{2}$*, then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \frac{(b-a)}{4(p+1)^{1/p}}\left[\left|f'\left(\frac{a+3b}{4}\right)\right| + \left|f'\left(\frac{3a+b}{4}\right)\right|\right], \quad (4.3.10)$$

*for each* $x \in [a,b]$*, where* $p > 1$.

A different approach for powers of the absolute value of the first derivative leads to the following result:

**Theorem 4.3.11.** *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* $I^\circ$ *such that* $f' \in L[a,b]$*, where* $a, b \in I$ *with* $a < b$*. If* $|f'|^q$ *is convex on* $[a,b]$*,* $q \geq 1$*, and* $|f'(x)| \leq M$*,* $x \in [a,b]$*, then the following inequality holds:*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\ &\leq (b-a)\left(\frac{1}{2}\frac{(b-x)^2}{(b-a)^2}\right)^{1-\frac{1}{q}}\left\{\left(\frac{1}{2}\frac{(b-x)^2}{(b-a)^2} - \frac{1}{3}\frac{(b-x)^3}{(b-a)^3}\right)|f'(b)|^q \right. \\ &\qquad \left. +\frac{1}{3}\frac{(b-x)^3}{(b-a)^3}|f'(a)|^q\right\}^{\frac{1}{q}} + (b-a)\left(\frac{1}{2} - \frac{(b-x)}{(b-a)} + \frac{1}{2}\frac{(b-x)^2}{(b-a)^2}\right)^{1-\frac{1}{q}} \\ &\qquad \times\left\{\left[\frac{1}{3}\left(1 - \frac{(b-x)^3}{(b-a)^3}\right) - \left(1 - \frac{(b-x)^2}{(b-a)^2}\right) + \left(1 - \frac{(b-x)}{(b-a)}\right)\right]|f'(b)|^q \right. \\ &\qquad\quad \left. + \left[\frac{1}{2}\left(1 - \frac{(b-x)^2}{(b-a)^2}\right) - \frac{1}{3}\left(1 - \frac{(b-x)^3}{(b-a)^3}\right)\right]|f'(a)|^q\right\}^{\frac{1}{q}} \quad (4.3.11) \end{aligned}$$

*for each* $x \in [a,b]$.

*Proof.* Suppose that $q \geq 1$. From Lemma 4.3.1 and using the well known power mean inequality, we have

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\ &\leq (b-a)\int_0^{\frac{b-x}{b-a}} t\,|f'(ta+(1-t)b)|\,dt \\ &\qquad + (b-a)\int_{\frac{b-x}{b-a}}^1 |t-1|\,|f'(ta+(1-t)b)|\,dt \end{aligned}$$

$$\leq (b-a)\left(\int_0^{\frac{b-x}{b-a}} tdt\right)^{1-1/q}\left(\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q}$$
$$+(b-a)\left(\int_{\frac{b-x}{b-a}}^1 (1-t)\,dt\right)^{1-1/q}\left(\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q}.$$

Since $|f'|^q$ is convex, we have

$$\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|^q dt \leq \int_0^{\frac{b-x}{b-a}} t\cdot\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt$$
$$=\left(\frac{1}{2}\frac{(b-x)^2}{(b-a)^2}-\frac{1}{3}\frac{(b-x)^3}{(b-a)^3}\right)\left|f'(b)\right|^q+\frac{1}{3}\frac{(b-x)^3}{(b-a)^3}\left|f'(a)\right|^q$$

and

$$\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'(ta+(1-t)b)\right|^q dt$$
$$\leq \int_{\frac{b-x}{b-a}}^1 (1-t)\cdot\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt$$
$$=\left[\frac{1}{3}\left(1-\frac{(b-x)^3}{(b-a)^3}\right)-\left(1-\frac{(b-x)^2}{(b-a)^2}\right)+\left(1-\frac{(b-x)}{(b-a)}\right)\right]\left|f'(b)\right|^q$$
$$+\left[\frac{1}{2}\left(1-\frac{(b-x)^2}{(b-a)^2}\right)-\frac{1}{3}\left(1-\frac{(b-x)^3}{(b-a)^3}\right)\right]\left|f'(a)\right|^q$$

Therefore, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq (b-a)\left(\frac{1}{2}\frac{(b-x)^2}{(b-a)^2}\right)^{1-\frac{1}{q}}\left\{\left(\frac{1}{2}\frac{(b-x)^2}{(b-a)^2}-\frac{1}{3}\frac{(b-x)^3}{(b-a)^3}\right)\left|f'(b)\right|^q\right.$$
$$\left.+\frac{1}{3}\frac{(b-x)^3}{(b-a)^3}\left|f'(a)\right|^q\right\}^{\frac{1}{q}}+(b-a)\left(\frac{1}{2}-\frac{(b-x)}{(b-a)}+\frac{1}{2}\frac{(b-x)^2}{(b-a)^2}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left[\frac{1}{3}\left(1-\frac{(b-x)^3}{(b-a)^3}\right)-\left(1-\frac{(b-x)^2}{(b-a)^2}\right)+\left(1-\frac{(b-x)}{(b-a)}\right)\right]\left|f'(b)\right|^q\right.$$
$$\left.+\left[\frac{1}{2}\left(1-\frac{(b-x)^2}{(b-a)^2}\right)-\frac{1}{3}\left(1-\frac{(b-x)^3}{(b-a)^3}\right)\right]\left|f'(a)\right|^q\right\}^{\frac{1}{q}}$$

which is required. □

**Corollary 4.3.12.** *In Theorem 4.3.11, choose* $x = \frac{a+b}{2}$*, then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq (b-a)\frac{8^{\frac{1}{q}}}{192}\left[\left(|f'(a)|^q + 2|f'(b)|^q\right)^{\frac{1}{q}} + \left(2|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}}\right] \quad (4.3.12)$$

*For instance, if* $q = 1$*, then (4.3.12) becomes*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \frac{(b-a)}{8}\left(|f'(a)| + |f'(b)|\right).$$

In the following, we may refine the inequalities (4.3.7) and (4.3.9):

**Theorem 4.3.13.** *Let* $f : I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* $I^\circ$ *such that* $f' \in L[a,b]$*, where* $a, b \in I$ *with* $a < b$*. If* $|f'|^q$ *is concave on* $[a,b]$*,* $q \geq 1$*, then the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq 2^{-1/q}(b-a)\left[\left(\frac{b-x}{b-a}\right)^2 \left|f'\left(\frac{b+2x}{3}\right)\right| + \left(\frac{x-a}{b-a}\right)^2 \left|f'\left(\frac{a+2x}{3}\right)\right|\right], \quad (4.3.13)$$

*for each* $x \in [a,b]$.

*Proof.* First, we note that by concavity of $|f'|^q$ and the power-mean inequality, we have

$$|f'(\alpha x + (1-\alpha)y)|^q \geq \alpha |f'(x)|^q + (1-\alpha)|f'(y)|^q.$$

Hence,

$$|f'(\alpha x + (1-\alpha)y)| \geq \alpha |f'(x)| + (1-\alpha)|f'(y)|.$$

so, $|f'|$ is also concave.

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\leq (b-a)\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|dt \\
&\qquad + (b-a)\int_{\frac{b-x}{b-a}}^1 |t-1|\left|f'(ta+(1-t)b)\right|dt \\
&\leq (b-a)\left(\int_0^{\frac{b-x}{b-a}} t dt\right)^{1-1/q}\left(\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q} \\
&\qquad + (b-a)\left(\int_{\frac{b-x}{b-a}}^1 (1-t)\,dt\right)^{1-1/q}\left(\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q}.
\end{aligned}$$

Accordingly, by Lemma 4.3.1 and the Jensen integral inequality, we have

$$\begin{aligned}
\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|^q dt &\leq \left(\int_0^{\frac{b-x}{b-a}} t dt\right)\left|f'\left(\frac{\int_0^{\frac{b-x}{b-a}} t\,(ta+(1-t)b)\,dt}{\int_0^{\frac{b-x}{b-a}} t dt}\right)\right|^q \\
&= \frac{1}{2}\left(\frac{b-x}{b-a}\right)^2\left|f'\left(\frac{b+2x}{3}\right)\right|^q
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'(ta+(1-t)b)\right|^q dt \\
&\qquad\leq \left(\int_{\frac{b-x}{b-a}}^1 (1-t)\,dt\right)\left|f'\left(\frac{\int_{\frac{b-x}{b-a}}^1 (1-t)\,(ta+(1-t)b)\,dt}{\int_{\frac{b-x}{b-a}}^1 (1-t)\,dt}\right)\right|^q \\
&\qquad= \frac{1}{2}\left(\frac{x-a}{b-a}\right)^2\left|f'\left(\frac{a+2x}{3}\right)\right|^q
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq 2^{-1/q}(b-a)\Bigg[&\left(\frac{b-x}{b-a}\right)^2\left|f'\left(\frac{b+2x}{3}\right)\right| \\
&+\left(\frac{x-a}{b-a}\right)^2\left|f'\left(\frac{a+2x}{3}\right)\right|\Bigg],
\end{aligned}$$

which completes the proof. □

Finally, a midpoint type inequalities may be deduced as follow:

**Corollary 4.3.14.** *In Theorem 4.3.13, choose* $x = \frac{a+b}{2}$*, we get*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \frac{2^{-1/q}}{4}(b-a)\left[\left|f'\left(\frac{a+2b}{3}\right)\right| + \left|f'\left(\frac{2a+b}{3}\right)\right|\right]. \quad (4.3.14)$$

*For instance if* $q = 1$*, then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \frac{(b-a)}{8}\left[\left|f'\left(\frac{a+2b}{3}\right)\right| + \left|f'\left(\frac{2a+b}{3}\right)\right|\right]. \quad (4.3.15)$$

## 4.4 OSTROWSKI'S TYPE INEQUALITIES VIA $S$-CONVEX FUNCTIONS

In this section, we consider some inequalities of Ostrowski's type for $s$-convex (concave) functions. We start with the following result:

**Theorem 4.4.1.** *Let* $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ *be a differentiable mapping on* $I^\circ$ *such that* $f' \in L[a,b]$*, where* $a, b \in I$ *with* $a < b$*. If* $|f'|$ *is* $s$*-convex in the second sense on* $[a,b]$ *for some fixed* $s \in (0,1]$ *and* $|f'(x)| \leq M$*,* $x \in [a,b]$*, then the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \frac{M}{b-a}\cdot\left[\frac{(x-a)^2 + (b-x)^2}{s+1}\right], \quad (4.4.1)$$

*for each* $x \in [a,b]$.

*Proof.* By Lemma 2.3.23 and since $|f'|$ is $s$-convex, then we have

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\ &\leq \frac{(x-a)^2}{b-a}\int_0^1 t\,|f'(tx+(1-t)a)|\,dt \\ &\qquad + \frac{(b-x)^2}{b-a}\int_0^1 t\,|f'(tx+(1-t)b)|\,dt \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(x-a)^2}{b-a}\int_0^1 t\left[t^s\left|f'(x)\right| + (1-t)^s\left|f'(a)\right|\right]dt \\
&\qquad + \frac{(b-x)^2}{b-a}\int_0^1 t\left[t^s\left|f'(x)\right| + (1-t)^s\left|f'(b)\right|\right]dt \\
&\leq \frac{M(x-a)^2}{b-a}\left(\frac{1}{s+2}+\frac{1}{(s+1)(s+2)}\right) \\
&\qquad + \frac{M(b-x)^2}{b-a}\left(\frac{1}{s+2}+\frac{1}{(s+1)(s+2)}\right) \\
&= \frac{M}{b-a}\cdot\left[\frac{(x-a)^2+(b-x)^2}{s+1}\right],
\end{aligned}
$$

where we have used the fact that

$$\int_0^1 t^{s+1}dt = \frac{1}{s+2} \qquad \text{and} \qquad \int_0^1 t(1-t)^s\,dt = \frac{1}{(s+1)(s+2)}.$$

This completes the proof. □

The corresponding version for powers of the absolute value of the first derivative is incorporated in the following result:

**Theorem 4.4.2.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is $s$-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, $p, q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$ and $|f'(x)| \leq M$, $x \in [a,b]$, then the following inequality holds:*

$$
\begin{aligned}
&\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\qquad\qquad \leq \frac{M}{(1+p)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right], \qquad (4.4.2)
\end{aligned}
$$

*for each $x \in [a,b]$.*

*Proof.* Suppose that $p > 1$. From Lemma 2.3.23 and using the Hölder inequality, we

have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\le \frac{(x-a)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)a)\right|dt \\
&\qquad+\frac{(b-x)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)b)\right|dt \\
&\le \frac{(x-a)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&\qquad+\frac{(b-x)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is $s$-convex in the second sense and $|f'(x)| \le M$, then we have

$$\begin{aligned}
\int_0^1 \left|f'(tx+(1-t)a)\right|^q dt &\le \int_0^1 \left[t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(a)\right|^q\right]dt \\
&= \frac{\left|f'(x)\right|^q+\left|f'(a)\right|^q}{s+1} \le \frac{2M^q}{s+1}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 \left|f'(tx+(1-t)b)\right|^q dt &\le \int_0^1 \left[t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt \\
&= \frac{\left|f'(x)\right|^q+\left|f'(b)\right|^q}{s+1} \le \frac{2M^q}{s+1}.
\end{aligned}$$

Therefore, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \le \frac{M}{(1+p)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right],$$

where $1/p+1/q=1$, which is required. $\square$

The previous observation can be formulated in case that $f$ is convex as follows:

**Corollary 4.4.3.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|^{p/(p-1)}$ is convex on $[a,b]$, $p>1$, and $|f'(x)| \le M$, $x \in [a,b]$, then the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \le \frac{M}{b-a}\left[\frac{(x-a)^2+(b-x)^2}{(1+p)^{\frac{1}{p}}}\right] \tag{4.4.3}$$

*for each $x \in [a,b]$.*

A different approach for powers of the absolute value of the first derivative is obtained in the following result:

**Theorem 4.4.4.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is $s$-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$ and $q \geq 1$, and $|f'(x)| \leq M$, $x \in [a,b]$, then the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right| \leq M \left( \frac{2}{s+1} \right)^{\frac{1}{q}} \left[ \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right] \tag{4.4.4}$$

*for each $x \in [a,b]$.*

*Proof.* Suppose that $q \geq 1$. From Lemma 2.3.23 and using the well known power mean inequality, we have

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right| \\ &\leq \frac{(x-a)^2}{b-a} \int_0^1 t \left| f'(tx + (1-t)a) \right| dt \\ &\qquad + \frac{(b-x)^2}{b-a} \int_0^1 t \left| f'(tx + (1-t)b) \right| dt \end{aligned}$$

$$\begin{aligned} &\leq \frac{(x-a)^2}{b-a} \left( \int_0^1 t dt \right)^{1-\frac{1}{q}} \left( \int_0^1 t \left| f'(tx + (1-t)a) \right|^q dt \right)^{\frac{1}{q}} \\ &\qquad + \frac{(b-x)^2}{b-a} \left( \int_0^1 t dt \right)^{1-\frac{1}{q}} \left( \int_0^1 t \left| f'(tx + (1-t)b) \right|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f'|^q$ is $s$-convex, we have

$$\begin{aligned} \int_0^1 t \left| f'(tx + (1-t)a) \right|^q dt &\leq \int_0^1 \left[ t^{s+1} \left| f'(x) \right|^q + t(1-t)^s \left| f'(a) \right|^q \right] dt \\ &= \frac{\left| f'(x) \right|^q + (s+1) \left| f'(a) \right|^q}{(s+1)(s+2)} \leq \frac{M^q}{s+1}, \end{aligned}$$

and

$$\begin{aligned} \int_0^1 t \left| f'(tx + (1-t)b) \right|^q dt &\leq \int_0^1 \left[ t^{s+1} \left| f'(x) \right|^q + t(1-t)^s \left| f'(b) \right|^q \right] dt \\ &= \frac{\left| f'(x) \right|^q + (s+1) \left| f'(b) \right|^q}{(s+1)(s+2)} \leq \frac{M^q}{s+1}. \end{aligned}$$

Therefore, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\le M\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right],$$

which is required. □

**Remark 4.4.5.** *Since* $(1+p)^{\frac{1}{p}}<2$ *for any* $p>1$, *then we observe that the inequality (4.4.4) is better than the inequality (4.4.2) meaning that the approach via power mean inequality is a better approach than the one through Hölder's inequality.*

A midpoint type inequality for functions whose derivatives in absolute value are $s$-convex in the second sense may be obtained from the previous results as follows:

**Corollary 4.4.6.** *If in (4.4.4) we choose* $x=\frac{a+b}{2}$, *then we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\le\frac{M(b-a)}{4}\left(\frac{2}{s+1}\right)^{\frac{1}{q}},\quad q\ge 1,\tag{4.4.5}$$

*where* $s\in(0,1]$ *and* $|f'|^q$ *is* $s$*-convex in the second sense on* $[a,b]$, $q\ge 1$.

Now, we obtain an Ostrowski's type inequality for the following result holds for $s$-concave mapping.

**Theorem 4.4.7.** *Let* $f:I\subseteq\mathbb{R}_+\to\mathbb{R}_+$ *be a differentiable mapping on* $I^\circ$ *such that* $f'\in L[a,b]$, *where* $a,b\in I$ *with* $a<b$. *If* $|f'|^q$ *is* $s$*-concave on* $[a,b]$ $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\le\frac{2^{(s-1)/q}}{(1+p)^{1/p}(b-a)}\left[(x-a)^2\left|f'\left(\frac{x+a}{2}\right)\right|+(b-x)^2\left|f'\left(\frac{b+x}{2}\right)\right|\right],\tag{4.4.6}$$

*for each* $x\in[a,b]$.

*Proof.* Suppose that $q>1$. From Lemma 2.3.23 and using the Hölder inequality, we

have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\
&\leq \frac{(x-a)^2}{b-a}\int_0^1 t\,|f'(tx+(1-t)a)|\,dt \\
&\qquad + \frac{(b-x)^2}{b-a}\int_0^1 t\,|f'(tx+(1-t)b)|\,dt \\
&\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1 |f'(tx+(1-t)a)|^q\,dt\right)^{1/q} \\
&\qquad + \frac{(b-x)^2}{b-a}\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1 |f'(tx+(1-t)b)|^q\,dt\right)^{1/q}.
\end{aligned}$$

But since $|f'|^q$ is concave, using the inequality (2.3.23), we have

$$\int_0^1 |f'(tx+(1-t)a)|^q\,dt \leq 2^{s-1}\left| f'\left(\frac{x+a}{2}\right)\right|^q,$$

and

$$\int_0^1 |f'(tx+(1-t)b)|^q\,dt \leq 2^{s-1}\left| f'\left(\frac{b+x}{2}\right)\right|^q.$$

A combination of the above numbered inequalities, we get

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
&\leq \frac{2^{(s-1)/q}}{(1+p)^{1/p}(b-a)}\left[(x-a)^2\left| f'\left(\frac{x+a}{2}\right)\right| + (b-x)^2\left| f'\left(\frac{b+x}{2}\right)\right|\right]
\end{aligned}$$

This completes the proof. □

Therefore, we can deduce the following midpoint type inequality for functions whose derivatives in absolute value are $s$-concave in the second sense:

**Corollary 4.4.8.** *If in (4.4.6) we choose $s = 1$ and $x = \frac{a+b}{2}$, then we have*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
&\qquad\qquad \leq \frac{(b-a)}{4(1+p)^{1/p}}\left[\left| f'\left(\frac{3a+b}{4}\right)\right| + \left| f'\left(\frac{a+3b}{4}\right)\right|\right], \quad (4.4.7)
\end{aligned}$$

*where, $|f'|^q$ is concave on $[a,b]$, $p > 1$.*

## 4.5 OSTROWSKI'S TYPE INEQUALITIES FOR QUASI-CONVEX FUNCTIONS

In this section, we follows the same techniques by obtaining several Ostrowski's type inequalities for quasi-convex functions which are different from (2.3.37). Let begin with the following result:

**Theorem 4.5.1.** *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* $I^{\circ}$ *such that* $f' \in L[a,b]$*, where* $a,b \in I$ *with* $a < b$*. If* $|f'|$ *is quasi-convex on* $[a,b]$*, then the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \le \frac{(b-x)^2}{2(b-a)} \max\{|f'(x)|, |f'(b)|\} + \frac{(x-a)^2}{2(b-a)} \max\{|f'(x)|, |f'(a)|\}, \quad (4.5.1)$$

*for each* $x \in [a,b]$.

*Proof.* By Lemma 2.3.23 and since $|f'|$ is quasi-convex, then we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
&\le (b-a)\int_0^{\frac{b-x}{b-a}} t\,|f'(ta+(1-t)b)|\,dt \\
&\quad + (b-a)\int_{\frac{b-x}{b-a}}^1 |t-1|\,|f'(ta+(1-t)b)|\,dt \\
&\le (b-a)\int_0^{\frac{b-x}{b-a}} t\cdot\max\{|f'(x)|, |f'(b)|\}\,dt \\
&\quad + \int_{\frac{b-x}{b-a}}^1 (1-t)\cdot\max\{|f'(x)|, |f'(a)|\}\,dt
\end{aligned}$$

$$\begin{aligned}
&= (b-a)\max\{|f'(x)|, |f'(b)|\}\int_0^{\frac{b-x}{b-a}} t\,dt \\
&\quad + (b-a)\max\{|f'(x)|, |f'(a)|\}\int_{\frac{b-x}{b-a}}^1 (1-t)\,dt \\
&= \frac{(b-x)^2}{2(b-a)}\max\{|f'(x)|, |f'(b)|\} + \frac{(x-a)^2}{2(b-a)}\max\{|f'(x)|, |f'(a)|\}.
\end{aligned}$$

This completes the proof. $\square$

**Corollary 4.5.2.** *In Theorem 4.5.1. Additionally, if $f'$ is bounded on $[a,b]$, i.e., there exists $M>0$ such that $|f'(x)|\leq M$, $x\in[a,b]$, then inequality (2.3.31) holds.*

**Corollary 4.5.3.** *In Theorem 4.5.1, Additionally, if*

1. *$|f'|$ is increasing, then we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\frac{(b-x)^2}{2(b-a)}|f'(b)|+\frac{(x-a)^2}{2(b-a)}|f'(x)|. \quad (4.5.2)$$

2. *$|f'|$ is decreasing, then we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\frac{(b-x)^2}{2(b-a)}|f'(x)|+\frac{(x-a)^2}{2(b-a)}|f'(a)|. \quad (4.5.3)$$

**Corollary 4.5.4.** *In Theorem 4.5.1, choose $x=\frac{a+b}{2}$, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\\&\leq\frac{(b-a)}{8}\left[\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,|f'(b)|\right\}+\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,|f'(a)|\right\}\right].\end{aligned} \quad (4.5.4)$$

*Therefore, we have*

1. *If $|f'|$ is increasing, then we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\frac{b-a}{8}\left[|f'(b)|+\left|f'\left(\frac{a+b}{2}\right)\right|\right]. \quad (4.5.5)$$

2. *If $|f'|$ is decreasing, then we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\frac{b-a}{8}\left[|f'(a)|+\left|f'\left(\frac{a+b}{2}\right)\right|\right]. \quad (4.5.6)$$

The corresponding version for powers via quasi-convex mapping is incorporated in the following result:

**Theorem 4.5.5.** *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $I^\circ$ such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|$ is quasi-convex on $[a,b]$, then the*

*following inequality holds:*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\qquad\leq\left(\frac{(b-x)^{p+1}}{(b-a)(p+1)}\right)^{\frac{1}{p}}\left[\max\left\{|f'(x)|^q,|f'(b)|^q\right\}\right]^{\frac{1}{q}} \\
&\qquad\quad+\left(\frac{(x-a)^{p+1}}{(b-a)(p+1)}\right)^{\frac{1}{p}}\left[\max\left\{|f'(x)|^q,|f'(a)|^q\right\}\right]^{\frac{1}{q}}, \qquad (4.5.7)
\end{aligned}$$

*for each* $x\in[a,b]$*, where* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* Suppose that $p>1$. From Lemma 4.3.1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\leq(b-a)\int_0^{\frac{b-x}{b-a}}t\left|f'(ta+(1-t)b)\right|dt \\
&\qquad+(b-a)\int_{\frac{b-x}{b-a}}^1|t-1|\left|f'(ta+(1-t)b)\right|dt \\
&\leq(b-a)\left(\int_0^{\frac{b-x}{b-a}}t^p dt\right)^{1/p}\left(\int_0^{\frac{b-x}{b-a}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q} \\
&\qquad+(b-a)\left(\int_{\frac{b-x}{b-a}}^1(1-t)^p\,dt\right)^{1/p}\left(\int_{\frac{b-x}{b-a}}^1\left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q} \\
&=\frac{(b-x)^{\frac{p+1}{p}}}{(b-a)^{\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[\max\left\{|f'(x)|^q,|f'(b)|^q\right\}\right]^{1/q} \\
&\qquad+\frac{(x-a)^{\frac{p+1}{p}}}{(b-a)^{\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[\max\left\{|f'(x)|^q,|f'(a)|^q\right\}\right]^{1/q}.
\end{aligned}$$

This completes the proof. □

**Corollary 4.5.6.** *In Theorem 4.5.5, Additionally, if*

1. $|f'|$ *is increasing, then we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{1}{(b-a)^{\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[(b-x)^{\frac{p+1}{p}}\left|f'(b)\right|+(x-a)^{\frac{p+1}{p}}\left|f'(x)\right|\right], \quad (4.5.8)$$

2. $|f'|$ *is decreasing, then we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{1}{(b-a)^{\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[(b-x)^{\frac{p+1}{p}}\left|f'(x)\right|+(x-a)^{\frac{p+1}{p}}\left|f'(a)\right|\right]. \quad (4.5.9)$$

**Corollary 4.5.7.** *In Corollary 4.5.6, choose $x=\frac{a+b}{2}$, then we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{(b-a)}{2^{1/p}(p+1)^{1/p}}\left[\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(b)\right|^q\right\}^{\frac{1}{q}} + \max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(a)\right|^q\right\}^{\frac{1}{q}}\right]. \quad (4.5.10)$$

*Therefore, we have*

1. $|f'|$ *is increasing, then we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{2^{1/p}(p+1)^{1/p}}\left[\left|f'(b)\right|+\left|f'\left(\frac{a+b}{2}\right)\right|\right]. \quad (4.5.11)$$

2. $|f'|$ *is decreasing, then we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{2^{1/p}(p+1)^{1/p}}\left[\left|f'(a)\right|+\left|f'\left(\frac{a+b}{2}\right)\right|\right]. \quad (4.5.12)$$

A different approach leads to the following result:

**Theorem 4.5.8.** *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* $I^{\circ}$ *such that* $f' \in L[a,b]$*, where* $a, b \in I$ *with* $a < b$*. If* $|f'|^q$ *is quasi-convex on* $[a,b]$*,* $q \geq 1$*, and* $|f'(x)| \leq M$*,* $x \in [a,b]$ *, then the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \frac{(x-a)^2}{2(b-a)}\left(\max\left\{|f'(x)|^q, |f'(a)|^q\right\}\right)^{\frac{1}{q}} + \frac{(b-x)^2}{2(b-a)}\left(\max\left\{|f'(x)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}} \quad (4.5.13)$$

*for each* $x \in [a,b]$ .

*Proof.* Suppose that $q \geq 1$. From Lemma 4.3.1 and using the well known power mean inequality, we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
&\leq (b-a)\int_0^{\frac{b-x}{b-a}} t\,|f'(ta+(1-t)b)|\,dt \\
&\qquad + (b-a)\int_{\frac{b-x}{b-a}}^1 |t-1|\,|f'(ta+(1-t)b)|\,dt \\
&\leq (b-a)\left(\int_0^{\frac{b-x}{b-a}} t dt\right)^{1-1/q}\left(\int_0^{\frac{b-x}{b-a}} t\,|f'(ta+(1-t)b)|^q\,dt\right)^{1/q} \\
&\qquad + (b-a)\left(\int_{\frac{b-x}{b-a}}^1 (1-t)\,dt\right)^{1-1/q}\left(\int_{\frac{b-x}{b-a}}^1 (1-t)\,|f'(ta+(1-t)b)|^q\,dt\right)^{1/q}.
\end{aligned}$$

Since $|f'|^q$ is quasi-convex, we have

$$\begin{aligned}
\int_0^{\frac{b-x}{b-a}} t\,|f'(ta+(1-t)b)|^q\,dt &\leq \int_0^{\frac{b-x}{b-a}} t\cdot\max\left\{|f'(x)|^q, |f'(b)|^q\right\}dt \\
&= \frac{(b-x)^2}{2(b-a)^2}\cdot\max\left\{|f'(x)|^q, |f'(b)|^q\right\}
\end{aligned}$$

and

$$\begin{aligned}
\int_{\frac{b-x}{b-a}}^1 (1-t)\,|f'(ta+(1-t)b)|^q\,dt &\leq \int_{\frac{b-x}{b-a}}^1 (1-t)\cdot\max\left\{|f'(a)|^q, |f'(x)|^q\right\}dt \\
&= \frac{(x-a)^2}{2(b-a)^2}\cdot\max\left\{|f'(a)|^q, |f'(x)|^q\right\}
\end{aligned}$$

Therefore, we have

$$\left|f\left(x\right)-\frac{1}{b-a}\int_{a}^{b}f\left(u\right)du\right|\leq\frac{\left(x-a\right)^{2}}{2\left(b-a\right)}\left(\max\left\{\left|f'\left(x\right)\right|^{q},\left|f'\left(a\right)\right|^{q}\right\}\right)^{\frac{1}{q}} + \frac{\left(b-x\right)^{2}}{2\left(b-a\right)}\left(\max\left\{\left|f'\left(x\right)\right|^{q},\left|f'\left(b\right)\right|^{q}\right\}\right)^{\frac{1}{q}},$$

which is required. □

**Corollary 4.5.9.** *In Theorem 4.5.8, Additionally, if*

1. $|f'|$ *is increasing, then (4.5.2) holds.*

2. $|f'|$ *is decreasing, then (4.5.3) holds.*

**Corollary 4.5.10.** *In Theorem 4.5.8, choose* $x=\frac{a+b}{2}$*, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f\left(u\right)du\right| \leq\frac{\left(b-a\right)}{8}\left[\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{q},\left|f'\left(b\right)\right|^{q}\right\}\right)^{1/q} + \left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{q},\left|f'\left(a\right)\right|^{q}\right\}\right)^{1/q}\right]. \quad (4.5.14)$$

*Therefore, we have*

1. *If* $|f'|$ *is increasing, then (4.5.5) holds.*

2. *If* $|f'|$ *is decreasing, then (4.5.6) holds.*

## 4.6 OSTROWSKI'S TYPE INEQUALITIES FOR $R$-CONVEX FUNCTIONS

In this section we consider some inequalities of Ostrowski's type via $r$-convex functions. The type of these results are obtained for the first time.

We begin with the following theorem.

**Theorem 4.6.1.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}_+$ be a positive differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$, $p > 1$ is $r$-convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \;& \frac{1}{(b-a)(p+1)^{1/p}}\left[(x-a)^2 L_r^{1/q}\left(|f'(x)|^q, |f'(a)|^q\right)\right. \\ & \left.+ (b-x)^2 L_r^{1/q}\left(|f'(x)|^q, |f'(b)|^q\right)\right], \end{aligned} \tag{4.6.1}$$

*for each $x \in [a,b]$, where, $p > 1$, and $L_r(\cdot,\cdot)$ is the generalized log-mean.*

*Proof.* By Lemma 2.3.23 and since $|f'|^q$ is $r$-convex, then we have

$$\begin{aligned}&\left|f(x) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\leq \frac{(x-a)^2}{b-a}\int_0^1 t\,|f'(tx+(1-t)a)|\,dt \\ &\qquad + \frac{(b-x)^2}{b-a}\int_0^1 t\,|f'(tx+(1-t)b)|\,dt \\ &\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1 |f'(tx+(1-t)a)|^q\,dt\right)^{1/q} \\ &\qquad + \frac{(b-x)^2}{b-a}\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1 |f'(tx+(1-t)b)|^q\,dt\right)^{1/q}.\end{aligned}$$

Since $|f'|^q$ is $r$-convex, by (2.3.30), we have

$$\int_0^1 |f'(tx+(1-t)a)|^q\,dt \leq L_r\left(|f'(x)|^q, |f'(a)|^q\right)$$

and

$$\int_0^1 |f'(tx+(1-t)b)|^q\,dt \leq L_r\left(|f'(x)|^q, |f'(b)|^q\right).$$

Combining all above inequalities, we get

$$\begin{aligned}&\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\qquad \leq \frac{1}{(b-a)(p+1)^{1/p}}\left[(x-a)^2 L_r^{1/q}\left(|f'(x)|^q, |f'(a)|^q\right)\right. \\ &\qquad\qquad \left.+ (b-x)^2 L_r^{1/q}\left(|f'(x)|^q, |f'(b)|^q\right)\right],\end{aligned}$$

where $1/p + 1/q = 1$, which completes the proof. □

In the following we obtain an inequality of Ostrowski's type for log-convex mappings which is different from (2.3.38) as follows:

**Corollary 4.6.2.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}_+$ be a positive differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$, $p > 1$ is log-convex on $[a,b]$, then the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \frac{1}{(b-a)(p+1)^{1/p}} \left[ (x-a)^2 L^{1/q} \left( |f'(x)|^q, |f'(a)|^q \right) + (b-x)^2 L^{1/q} \left( |f'(x)|^q, |f'(b)|^q \right) \right], \tag{4.6.2}$$

*for each $x \in [a,b]$, where, $p > 1$, and $L(\cdot,\cdot)$ is the log-mean.*

*Proof.* The proof goes likewise the proof of Theorem 4.6.1, and using (2.3.29). □

**Corollary 4.6.3.** *If in Theorem 4.6.1, choose $x = \frac{a+b}{2}$*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(u)\,du \right| \leq \frac{(b-a)}{4(1+p)^{1/p}} \left[ L_r^{1/q} \left( |f'(a)|^q, \left| f'\left(\frac{a+b}{2}\right) \right|^q \right) \right. \left. + L_r^{1/q} \left( \left| f'\left(\frac{a+b}{2}\right) \right|^q, |f'(b)|^q \right) \right]. \tag{4.6.3}$$

*For instance, for $r = 0$, the result holds for log-convex functions, and*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(u)\,du \right| \leq \frac{(b-a)}{4(1+p)^{1/p}} \left[ L^{1/q} \left( |f'(a)|^q, \left| f'\left(\frac{a+b}{2}\right) \right|^q \right) \right. \left. + L^{1/q} \left( \left| f'\left(\frac{a+b}{2}\right) \right|^q, |f'(b)|^q \right) \right]. \tag{4.6.4}$$

**Theorem 4.6.4.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}_+$ be a positive differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is $r$-convex ($r \geq 1$), on $[a,b]$, then the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq \frac{(x-a)^2}{b-a} \left[ |f'(x)| \frac{r}{1+2r} + |f'(a)| \frac{r^2}{(1+r)(1+2r)} \right] + \frac{(b-x)^2}{b-a} \left[ |f'(x)| \frac{r}{1+2r} + |f'(b)| \frac{r^2}{(1+r)(1+2r)} \right], \tag{4.6.5}$$

*for each $x \in [a,b]$.*

*Proof.* By Lemma 2.3.23 and since $|f'|$ is $r$-convex, then we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{(x-a)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)a)\right|dt \\
&\qquad + \frac{(b-x)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)b)\right|dt \\
&\leq \frac{(x-a)^2}{b-a}\int_0^1 t\left[t\left|f'(x)\right|^r+(1-t)\left|f'(a)\right|^r\right]^{1/r}dt \\
&\qquad + \frac{(b-x)^2}{b-a}\int_0^1 t\left[t\left|f'(x)\right|^r+(1-t)\left|f'(b)\right|^r\right]^{1/r}dt
\end{aligned}$$

Using the fact that $\sum_{i=1}^{n}(a_i+b_i)^k \leq \sum_{i=1}^{n} a_i^k + \sum_{i=1}^{n} b_i^k$, for $0<k<1$, $a_1, a_2, ..., a_n \geq 0$ and $b_1, b_2, ..., b_n \geq 0$, we obtain

$$\begin{aligned}
\left|f(x)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| &\leq \frac{(x-a)^2}{b-a}\int_0^1\left(t^{1+\frac{1}{r}}\left|f'(x)\right|+t(1-t)^{\frac{1}{r}}\left|f'(a)\right|\right)dt \\
&\qquad + \frac{(b-x)^2}{b-a}\int_0^1\left(t^{1+\frac{1}{r}}\left|f'(x)\right|+t(1-t)^{\frac{1}{r}}\left|f'(b)\right|\right)dt \\
&= \frac{(x-a)^2}{b-a}\left[\left|f'(x)\right|\frac{r}{1+2r}+\left|f'(a)\right|\frac{r^2}{(1+r)(1+2r)}\right] \\
&\qquad + \frac{(b-x)^2}{b-a}\left[\left|f'(x)\right|\frac{r}{1+2r}+\left|f'(b)\right|\frac{r^2}{(1+r)(1+2r)}\right],
\end{aligned}$$

which completes the proof. □

**Corollary 4.6.5.** *If in Theorem 4.6.4, there is $M>0$ such that $|f'(x)| \leq M$, for all $x \in [a,b]$, then the inequality (4.6.5), becomes*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{M}{(b-a)}\left(\frac{r}{r+1}\right)\left[(x-a)^2+(b-x)^2\right], \qquad (4.6.6)$$

*for $r \geq 1$.*

**Corollary 4.6.6.** *If in Theorem 4.6.4, choose $x=\frac{a+b}{2}$, then we have*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\leq \frac{(b-a)}{4(1+r)(1+2r)}\left[r^2\left|f'(a)\right|+2r(r+1)\left|f'\left(\frac{a+b}{2}\right)\right|+r^2\left|f'(b)\right|\right], \qquad (4.6.7)
\end{aligned}$$

*for $r \geq 1$. For instance, for $r=1$, we have*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\qquad \leq \frac{(b-a)}{24}\left[\left|f'(a)\right|+4\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'(b)\right|\right]. \qquad (4.6.8)
\end{aligned}$$

We note that, Yang et al. (2004) proved that the inequality (4.6.8) holds for convex mappings (see Theorem 2.3.10). In Corollary 4.6.6 we generalize Yang et al. (2004) result (2.3.20) for $r$-convex mapping ($r \geq 1$), which is weaker than the usual convexity.

## 4.7 APPLICATIONS TO MIDPOINT FORMULA

In the classical Midpoint rule (2.3.8), it is clear that if the mapping $f$ is not twice differentiable or the second derivative is not bounded on $(a, b)$, then (2.3.8) cannot be applied. In this section, we derive some new error estimates for the midpoint rule in terms of first derivative.

**Proposition 4.7.1.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}_+$ be a positive differentiable mapping on $I^\circ$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. Assume that $|f'|$ is a $r$-convex ($r \geq 1$) on $[a, b]$. If $P := a = x_0 < x_1 < \cdots < x_n = b$ is a partition of the interval $[a, b]$, $h_i = x_{i+1} - x_i$, for $i = 0, 1, 2, \cdots, n-1$ and*

$$M_n(f, P) = \sum_{i=0}^{n-1} h_i \cdot f\left(\frac{x_i + x_{i+1}}{n}\right),$$

*then*

$$\begin{aligned}\left|E_n^M(f, P)\right| &= \left|\int_a^b f(x)\,dx - M_n(f, P)\right| \\ &\leq \frac{1}{4(1+r)(1+2r)} \sum_{i=0}^{n-1} h_i^2 \cdot \left[r^2 \left|f'(x_i)\right| + 2r(r+1)\left|f'\left(\frac{x_i + x_{i+1}}{2}\right)\right| \right. \\ &\qquad \left. + r^2 \left|f'(x_{i+1})\right|\right].\end{aligned}$$

*Proof.* Applying Corollary 4.6.6 on the subintervals $[x_i, x_{i+1}]$, for $i = 0, 1, ..., n-1$ of the division $P$, we get

$$\begin{aligned}
\left|E_n^M(f,P)\right| &= \left|\int_a^b f(x)\,dx - M_n(f,P)\right| \\
&= \left|\int_{x_i}^{x_{i+1}} f(x)\,dx - h_i^2 \cdot f\left(\frac{x_i+x_{i+1}}{2}\right)\right| \\
&\leq \frac{h_i^2}{4(1+r)(1+2r)}\left[r^2\left|f'(x_i)\right| + 2r(r+1)\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right. \\
&\qquad \left. + r^2\left|f'(x_{i+1})\right|\right].
\end{aligned}$$

Summing over $i$ from $0$ to $n-1$ and taking into account that $|f'|$ is $r$-convex ($r \geq 1$), we deduce that

$$\begin{aligned}
\left|E_n^M(f,P)\right| \leq \frac{1}{4(1+r)(1+2r)}\sum_{i=0}^{n-1} h_i^2 \cdot &\left[r^2\left|f'(x_i)\right| + 2r(r+1)\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right. \\
&\left. + r^2\left|f'(x_{i+1})\right|\right],
\end{aligned}$$

which completes the proof. □

**Corollary 4.7.2.** *In Proposition 4.7.1, if $f$ is convex on $[a,b]$, then we have*

$$\begin{aligned}
\left|E_n^M(f,P)\right| &= \left|\int_a^b f(x)\,dx - M_n(f,P)\right| \\
&\leq \frac{1}{24}\sum_{i=0}^{n-1} h_i^2 \cdot \left[\left|f'(x_i)\right| + 4\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right| + \left|f'(x_{i+1})\right|\right].
\end{aligned}$$

*Proof.* In the proof of Proposition 4.7.1 setting $r$=1, we get the required result. □

**Proposition 4.7.3.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. Assume that $|f'|^q$ ($q \geq 1$) is concave on $[a,b]$. If $P := a = x_0 < x_1 < \cdots < x_n = b$ is a partition of the interval $[a,b]$, $h_i = x_{i+1} - x_i$, for $i = 0,1,2,\cdots,n-1$ and*

$$M_n(f,P) = \sum_{i=0}^{n-1} h_i \cdot f\left(\frac{x_i+x_{i+1}}{n}\right),$$

*then*

$$\begin{aligned}
\left|E_n^M(f,P)\right| &= \left|\int_a^b f(x)\,dx - M_n(f,P)\right| \\
&\leq \frac{2^{-1/q}}{4}\sum_{i=0}^{n-1} h_i^2 \cdot \left[\left|f'\left(\frac{2x_i+x_{i+1}}{3}\right)\right| + \left|f'\left(\frac{x_i+2x_{i+1}}{3}\right)\right|\right].
\end{aligned}$$

*Proof.* The proof can be done similar to that of Proposition 4.7.1 and using Corollary 4.3.14. □

**Proposition 4.7.4.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. Assume that $|f'|$ is a quasi-convex on $[a,b]$. If $P := a = x_0 < x_1 < \cdots < x_n = b$ is a partition of the interval $[a,b]$, $h_i = x_{i+1} - x_i$, for $i = 0, 1, 2, \cdots, n-1$ and*

$$M_n(f,P) = \sum_{i=0}^{n-1} h_i \cdot f\left(\frac{x_i + x_{i+1}}{n}\right),$$

*then*

$$\begin{aligned}\left|E_n^T(f,P)\right| &= \left|\int_a^b f(x)\,dx - M_n(f,P)\right| \\ &\le \frac{1}{8}\sum_{i=0}^{n-1} h_i^2 \cdot \left[\max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, \left|f'(x_{i+1})\right|\right\}\right. \\ &\qquad\left. + \max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, \left|f'(x_i)\right|\right\}\right].\end{aligned}$$

## 4.8 APPLICATIONS TO SPECIAL MEANS

In this section, we obtain some error estimates for some special means of real numbers.

1. Consider $f : [a,b] \to \mathbb{R}$, $(0 < a < b)$, $f(x) = x^r$, $r \in \mathbb{R}\setminus\{-1, 0\}$. Then,

$$\frac{1}{b-a}\int_a^b f(x)\,dx = L_r^r(a,b),$$

   (a) Using the inequality (4.3.3), we get

$$|x^r - L_r^r| \le (b-a)\,\mu_r(a,b)\left[\frac{1}{3} + \frac{(b-x)^3 + (x-a)^3}{3(b-a)^3}\right],$$

   where,

$$\mu_r(a,b) = \begin{cases} rb^{r-1}, & r \ge 1 \\ |p|\,a^{r-1}, & r \in (-\infty, 0) \cup (0,1) \setminus \{-1\} \end{cases}$$

   For instance, if we choose

i. $x = A$, then we get

$$|A^r - L_r^r| \leq \frac{5(b-a)}{12}\mu_r(a,b),$$

ii. $x = G$, then we get

$$|G^r - L_r^r| \leq (b-a)\,\mu_r(a,b)\left[\frac{1}{3} + \frac{(b-G)^3 + (G-a)^3}{3(b-a)^3}\right],$$

iii. $x = H$, then we get

$$|H^r - L_r^r| \leq (b-a)\,\mu_r(a,b)\left[\frac{1}{3} + \frac{(b-H)^3 + (H-a)^3}{3(b-a)^3}\right],$$

iv. $x = I$, then we get

$$|I^r - L_r^r| \leq (b-a)\,\mu_r(a,b)\left[\frac{1}{3} + \frac{(b-I)^3 + (I-a)^3}{3(b-a)^3}\right],$$

v. $x = L$, then we get

$$|L^r - L_r^r| \leq (b-a)\,\mu_r(a,b)\left[\frac{1}{3} + \frac{(b-L)^3 + (L-a)^3}{3(b-a)^3}\right].$$

(b) Using the inequality (4.3.5), we get

$$|x^r - L_r^r| \leq \mu_r(a,b)\,\frac{(b-x)^{\frac{p+1}{p}} + (x-a)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}(b-a)^{\frac{1}{p}}},$$

where, $p > 1$. For instance, if we choose

i. $x = A$, then we get

$$|A^r - L_r^r| \leq \mu_r(a,b)\,\frac{(b-A)^{\frac{p+1}{p}} + (A-a)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}(b-a)^{\frac{1}{p}}},$$

ii. $x = G$, then we get

$$|G^r - L_r^r| \leq \mu_r(a,b)\,\frac{(b-G)^{\frac{p+1}{p}} + (G-a)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}(b-a)^{\frac{1}{p}}},$$

iii. $x = H$, then we get

$$|H^r - L_r^r| \leq \mu_r(a,b)\,\frac{(b-H)^{\frac{p+1}{p}} + (H-a)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}(b-a)^{\frac{1}{p}}},$$

iv. $x = I$, then we get

$$\left|I^r - L_r^r\right| \leq \mu_r(a,b) \frac{(b-I)^{\frac{p+1}{p}} + (I-a)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}} (b-a)^{\frac{1}{p}}},$$

v. $x = L$, then we get

$$\left|L^r - L_r^r\right| \leq \mu_r(a,b) \frac{(b-L)^{\frac{p+1}{p}} + (L-a)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}} (b-a)^{\frac{1}{p}}}.$$

2. Consider $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$, $(0 < a < b)$, $f(x) = \ln x$, then,

$$\frac{1}{b-a} \int_a^b f(x)\, dx = \ln I(a,b) := \ln I,$$

(a) Using the inequality (4.3.7), we get

$$|\ln x - \ln I| \leq \frac{2}{(b-a)^{1/p} (p+1)^{1/p}} \left[ \frac{(b-x)^{(p+1)/p}}{b+x} + \frac{(x-a)^{(p+1)/p}}{x+a} \right].$$

where, $x \neq I$ and $p > 1$. For instance, if we choose

i. $x = A$, then we get

$$|\ln A - \ln I| \leq \frac{2}{(b-a)^{1/p} (p+1)^{1/p}} \left[ \frac{(b-A)^{(p+1)/p}}{b+A} + \frac{(A-a)^{(p+1)/p}}{A+a} \right],$$

ii. $x = G$, then we get

$$|\ln G - \ln I| \leq \frac{2}{(b-a)^{1/p} (p+1)^{1/p}} \left[ \frac{(b-G)^{(p+1)/p}}{b+G} + \frac{(G-a)^{(p+1)/p}}{G+a} \right],$$

iii. $x = H$, then we get

$$|\ln H - \ln I| \leq \frac{2}{(b-a)^{1/p} (p+1)^{1/p}} \left[ \frac{(b-H)^{(p+1)/p}}{b+H} + \frac{(H-a)^{(p+1)/p}}{H+a} \right],$$

iv. $x = L$, then we get

$$|\ln L - \ln I| \leq \frac{2}{(b-a)^{1/p} (p+1)^{1/p}} \left[ \frac{(b-L)^{(p+1)/p}}{b+L} + \frac{(L-a)^{(p+1)/p}}{L+a} \right].$$

(b) Using the inequality (4.3.13), we get

$$|\ln x - \ln I| \leq \frac{3 \cdot 2^{-1/q}}{(b-a)} \left[ \frac{(b-x)^2}{b+2x} + \frac{(x-a)^2}{a+2x} \right],$$

where, $x \neq I$ and $q \geq 1$. For instance, if we choose

i. $x = A$, then we get

$$|\ln A - \ln I| \leq \frac{3 \cdot 2^{-1/q}}{(b-a)} \left[ \frac{(b-A)^2}{b+2A} + \frac{(A-a)^2}{a+2A} \right],$$

ii. $x = G$, then we get

$$|\ln G - \ln I| \leq \frac{3 \cdot 2^{-1/q}}{(b-a)} \left[ \frac{(b-G)^2}{b+2G} + \frac{(G-a)^2}{a+2G} \right],$$

iii. $x = H$, then we get

$$|\ln H - \ln I| \leq \frac{3 \cdot 2^{-1/q}}{(b-a)} \left[ \frac{(b-H)^2}{b+2H} + \frac{(H-a)^2}{a+2H} \right],$$

iv. $x = L$, then we get

$$|\ln L - \ln I| \leq \frac{3 \cdot 2^{-1/q}}{(b-a)} \left[ \frac{(b-L)^2}{b+2L} + \frac{(L-a)^2}{a+2L} \right].$$

## 4.9 SUMMARY AND CONCLUSION

In the presented chapter, Ostrowski's type inequalities for differentiable convex, concave, $s$-convex (concave), quasi-convex, $r$-convex and log-convex mappings are established. In section 4.2, for differentiable concave mapping, the well known inequality

$$f(x) - f(x_0) \leq f'(x_0)(x - x_0).$$

is used to obtain new inequalities of Ostrowski's type. Indeed, the type of the inequalities (4.2.3) and (4.2.12) are presented for the first time. Additionally, the inequality (4.2.2) is a new refinement for Ostrowski inequality for Riemann-Stieltjes integral, which is different than (2.3.33). In the sections 4.3–4.6, new inequalities of Ostrowski's type via convex, $s$-convex, quasi-convex and $r$-convex functions are considered. In fact, the new Montgomery-type identity (4.3.1) and Lemma 2.3.23 are used to obtain the required results. Several generalizations, refinements and improvements for concave functions and for the corresponding version for powers of these inequalities are considered by applying the Hölder and the power mean inequalities. Choosing $x = \frac{a+b}{2}$ in the obtained inequalities, several midpoint type inequalities are deduced. In this way, we highlight the role of convexity in the Ostrowski's inequality.

# CHAPTER V

# SIMPSON'S TYPE INEQUALITIES

## 5.1 INTRODUCTION

In this chapter, we obtain several inequalities of Simpson's type and thus giving explicit error bounds in the Simpson's rules and deduced various inequalities for some special means, using Peano type kernels and results from the modern theory of inequalities. Although bounds through the use of Peano kernels have been obtained in some research papers on Simpson's inequality (see Chapter II), however the approach presented here using $s$-convex, quasi-convex and $r$-convex functions in terms of at most second derivatives are obtained for the first time.

## 5.2 INEQUALITIES OF SIMPSON'S TYPE FOR $S$–CONVEX

In order to prove our main theorems regarding Simpson's inequality via $s$-convex functions, we need the following lemma:

**Lemma 5.2.1.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping on $I^\circ$ where $a, b \in I$ with $a < b$. Then the following equality holds:*

$$\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx = (b-a)\int_0^1 p(t)\, f'(tb + (1-t)a)\,dt \quad (5.2.1)$$

*where,*

$$p(t) = \begin{cases} t - \frac{1}{6}, & t \in \left[0, \frac{1}{2}\right) \\ t - \frac{5}{6}, & t \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

*Proof.* We note that

$$\begin{aligned} I = \int_0^1 p(t) f'(tb + (1-t)a)\, dt &= \int_0^{1/2} \left(t - \frac{1}{6}\right) f'(tb + (1-t)a)\, dt \\ &+ \int_{1/2}^1 \left(t - \frac{5}{6}\right) f'(tb + (1-t)a)\, dt. \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} I &= \left(t - \frac{1}{6}\right) \frac{f(tb + (1-t)a)}{b-a}\Bigg|_0^{1/2} - \int_0^{1/2} \frac{f(tb + (1-t)a)}{b-a} dt \\ &+ \left(t - \frac{5}{6}\right) \frac{f(tb + (1-t)a)}{b-a}\Bigg|_{1/2}^{1} - \int_{1/2}^1 \frac{f(tb + (1-t)a)}{b-a} dt \\ &= \frac{1}{6(b-a)}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \int_0^1 \frac{f(tb + (1-t)a)}{b-a} dt. \end{aligned}$$

Setting $x = tb + (1-t)a$, and $dx = (b-a)dt$, gives

$$(b-a) \cdot I = \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x) dt$$

which gives the desired representation (5.2.1). □

The next theorem gives a new refinement of the Simpson inequality via $s$-convex functions.

**Theorem 5.2.2.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable function on $I^\circ$ and $a, b \in I$ with $a < b$. If $|f'|$ is $s$-convex on $[a, b]$, for some fixed $s \in (0, 1]$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\, dx\right| \\ &\qquad \le (b-a) \frac{6^{-s} - 9(2)^{-s} + (5)^{s+2} 6^{-s} + 3s - 12}{18(s^2 + 3s + 2)} \left[|f'(a)| + |f'(b)|\right]. \end{aligned} \tag{5.2.2}$$

*Proof.* From Lemma 5.2.1, and since $f$ is $s$-convex, we have

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\, dx\right| \\ \le\ & (b-a)\left|\int_0^1 s(t) f'(tb + (1-t)a)\, dt\right| \\ \le\ & (b-a)\int_0^{1/2} \left|\left(t - \frac{1}{6}\right)\right| |f'(tb + (1-t)a)|\, dt \end{aligned}$$

$$
\begin{aligned}
& +(b-a)\int_{1/2}^{1}\left|\left(t-\frac{5}{6}\right)\right|\left|f'(tb+(1-t)a)\right|dt \\
\leq\ & (b-a)\int_{0}^{1/2}\left|\left(t-\frac{1}{6}\right)\right|\left(t^{s}\left|f'(b)\right|+(1-t)^{s}\left|f'(a)\right|\right)dt \\
& +(b-a)\int_{1/2}^{1}\left|\left(t-\frac{5}{6}\right)\right|\left(t^{s}\left|f'(b)\right|+(1-t)^{s}\left|f'(a)\right|\right)dt \\
=\ & (b-a)\int_{0}^{1/6}\left(\frac{1}{6}-t\right)\left(t^{s}\left|f'(b)\right|+(1-t)^{s}\left|f'(a)\right|\right)dt \\
& +(b-a)\int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)\left(t^{s}\left|f'(b)\right|+(1-t)^{s}\left|f'(a)\right|\right)dt \\
& +(b-a)\int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)\left(t^{s}\left|f'(b)\right|+(1-t)^{s}\left|f'(a)\right|\right)dt \\
& +(b-a)\int_{5/6}^{1}\left(t-\frac{5}{6}\right)\left(t^{s}\left|f'(b)\right|+(1-t)^{s}\left|f'(a)\right|\right)dt \\
=\ & (b-a)\,\frac{6^{-s}-9\,(2)^{-s}+(5)^{s+2}\,6^{-s}+3s-12}{18\,(s^{2}+3s+2)}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]
\end{aligned}
$$

which completes the proof. □

Therefore, we can deduce the following result.

**Corollary 5.2.3.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable function on $I^\circ$ and $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$
\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right| \leq \frac{5\,(b-a)}{72}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \quad (5.2.3)
$$

Applying Hölder inequality on the previous theorem, a similar result is embodied in the following theorem:

**Theorem 5.2.4.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable function on $I^\circ$ and $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is $s$-convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $p > 1$, then* *the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq (b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\frac{1}{(s+1)^{1/q}}\left[\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right.$$
$$\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right)^{1/q}\right]. \quad (5.2.4)$$

*where,* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* From Lemma 5.2.1, and since $f$ is $s$-convex, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & (b-a)\left|\int_0^1 s(t)\,f'(tb+(1-t)a)\,dt\right| \\
\leq\ & (b-a)\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right||f'(tb+(1-t)a)|\,dt \\
&+(b-a)\int_{1/2}^{1}\left|\left(t-\frac{5}{6}\right)\right||f'(tb+(1-t)a)|\,dt \\
\leq\ & (b-a)\left(\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right|^p dt\right)^{1/p}\left(\int_0^{1/2}|f'(tb+(1-t)a)|^q\,dt\right)^{1/q} \\
&+(b-a)\left(\int_{1/2}^{1}\left|\left(t-\frac{5}{6}\right)\right|^p dt\right)^{1/p}\left(\int_{1/2}^{1}|f'(tb+(1-t)a)|^q\,dt\right)^{1/q} \\
=\ & (b-a)\left(\int_0^{1/6}\left(\frac{1}{6}-t\right)^p dt+\int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)^p dt\right)^{1/p} \\
&\times\left(\int_0^{1/2}|f'(tb+(1-t)a)|^q\,dt\right)^{1/q} \\
&+(b-a)\left(\int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)^p dt+\int_{5/6}^{1}\left(t-\frac{5}{6}\right)^p dt\right)^{1/p} \\
&\times\left(\int_{1/2}^{1}|f'(tb+(1-t)a)|^q\,dt\right)^{1/q}.
\end{aligned}$$

Since $f$ is $s$–convex by (2.3.23), we have

$$\int_0^{1/2}|f'(tb+(1-t)a)|^q\,dt\leq\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}, \quad (5.2.5)$$

and

$$\int_{1/2}^{1} |f'(tb+(1-t)a)|^q\,dt \le \frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q}{s+1}. \tag{5.2.6}$$

Therefore, by (5.2.5) and (5.2.6), we get

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\le\ &(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\frac{1}{(s+1)^{1/q}}\left[\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right. \\
&\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right)^{1/q}\right]
\end{aligned}$$

which completes the proof. $\square$

**Corollary 5.2.5.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping on $I^\circ$ and $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is convex on $[a,b]$, for some fixed $p > 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\qquad\le 2^{-1/q}(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\left[\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right. \\
&\qquad\qquad\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right)^{1/q}\right],
\end{aligned} \tag{5.2.7}$$

*where,* $\frac{1}{p}+\frac{1}{q}=1$.

Our next result gives another refinement for the Simpson's inequality:

**Theorem 5.2.6.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable mapping on $I^\circ$ and $a, b \in I$ with $a < b$. If $|f'|^q$ is $s$-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q \ge 1$, then the*

*following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{[216(s^2+3s+2)]^{1/q}}\left(\frac{5}{72}\right)^{1-1/q}$$
$$\times\left\{\left(\left[\left(3^{-s}\right)\left(2^{1-s}\right)+3s\left(2^{1-s}\right)+3\left(2^{-s}\right)\right]|f'(b)|^q\right.\right.$$
$$\left.+\left[5^{s+2}3^{-s}2^{1-s}-6s\left(2^{-s}\right)-21\left(2^{-s}\right)+6s-24\right]|f'(a)|^q\right)^{1/q}$$
$$+\left(\left[\left(3^{-s}\right)\left(2^{1-s}\right)+3s\left(2^{1-s}\right)+3\left(2^{-s}\right)\right]|f'(a)|^q\right.$$
$$\left.\left.+\left[5^{s+2}3^{-s}2^{1-s}-6s\left(2^{-s}\right)-21\left(2^{-s}\right)+6s-24\right]|f'(b)|^q\right)^{1/q}\right\}. \quad (5.2.8)$$

*Proof.* From Lemma 5.2.1, and since $f$ is $s$-convex, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq (b-a)\left|\int_0^1 s(t)\,f'(tb+(1-t)a)\,dt\right|$$
$$\leq (b-a)\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right||f'(tb+(1-t)a)|\,dt$$
$$+(b-a)\int_{1/2}^1\left|\left(t-\frac{5}{6}\right)\right||f'(tb+(1-t)a)|\,dt$$
$$\leq (b-a)\left(\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right|dt\right)^{1-1/q}\left(\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right||f'(tb+(1-t)a)|^q\,dt\right)^{1/q}$$
$$+(b-a)\left(\int_{1/2}^1\left|\left(t-\frac{5}{6}\right)\right|dt\right)^{1-1/q}\left(\int_{1/2}^1\left|\left(t-\frac{5}{6}\right)\right||f'(tb+(1-t)a)|^q\,dt\right)^{1/q}.$$

Since $|f'|^q$ is $s$-convex therefore, we have

$$\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right||f'(tb+(1-t)a)|^q\,dt$$
$$\leq \int_0^{1/6}\left(\frac{1}{6}-t\right)\left(t^s|f'(b)|^q+(1-t)^s|f'(a)|^q\right)dt$$
$$+\int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)\left(t^s|f'(b)|^q+(1-t)^s|f'(a)|^q\right)dt$$
$$=\frac{(3^{-s})(2^{1-s})+3s(2^{1-s})+3(2^{-s})}{36(s^2+3s+2)}|f'(b)|^q$$
$$+\frac{5^{s+2}3^{-s}2^{1-s}-6s(2^{-s})-21(2^{-s})+6s-24}{36(s^2+3s+2)}|f'(a)|^q \quad (5.2.9)$$

and

$$\int_{1/2}^{1}\left|\left(t-\frac{5}{6}\right)\right|\left|f'\left(tb+\left(1-t\right)a\right)\right|^{q}dt$$
$$\le \int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)\left(t^{s}\left|f'\left(b\right)\right|^{q}+\left(1-t\right)^{s}\left|f'\left(a\right)\right|^{q}\right)dt$$
$$+\int_{5/6}^{1}\left(t-\frac{5}{6}\right)\left(t^{s}\left|f'\left(b\right)\right|^{q}+\left(1-t\right)^{s}\left|f'\left(a\right)\right|^{q}\right)dt$$
$$=\frac{\left(3^{-s}\right)\left(2^{1-s}\right)+3s\left(2^{1-s}\right)+3\left(2^{-s}\right)}{36\left(s^{2}+3s+2\right)}\left|f'\left(a\right)\right|^{q}$$
$$+\frac{5^{s+2}3^{-s}2^{1-s}-6s\left(2^{-s}\right)-21\left(2^{-s}\right)+6s-24}{36\left(s^{2}+3s+2\right)}\left|f'\left(b\right)\right|^{q} \quad (5.2.10)$$

Also, we note that

$$\int_{0}^{1/2}\left|\left(t-\frac{1}{6}\right)\right|dt=\int_{1/2}^{1}\left|\left(t-\frac{5}{6}\right)\right|dt=\frac{5}{72}. \quad (5.2.11)$$

Combination of (5.2.9), (5.2.10) and (5.2.11), gives the required result which completes the proof. □

**Corollary 5.2.7.** *Let $f$ be as in Theorem 5.2.6, let $s=1$, therefore the inequality holds for convex functions:*

$$\left|\frac{1}{6}\left[f\left(a\right)+4f\left(\frac{a+b}{2}\right)+f\left(b\right)\right]-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\right|$$
$$\le\frac{\left(b-a\right)}{\left(1296\right)^{1/q}}\left(\frac{5}{72}\right)^{1-1/q}\left[\left(29\left|f'\left(b\right)\right|^{q}+61\left|f'\left(a\right)\right|^{q}\right)^{1/q}\right.$$
$$\left.+\left(61\left|f'\left(b\right)\right|^{q}+29\left|f'\left(a\right)\right|^{q}\right)^{1/q}\right] \quad (5.2.12)$$

*Moreover, if $\left|f'\left(x\right)\right|\le M, \forall x\in I$, then we have*

$$\left|\frac{1}{6}\left[f\left(a\right)+4f\left(\frac{a+b}{2}\right)+f\left(b\right)\right]-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\right|\le\frac{5\left(b-a\right)}{36}\cdot M. \quad (5.2.13)$$

**Remark 5.2.8.** *We note that, the inequality (5.2.13) with $s=1$ gives an improvement for the inequality (2.3.42).*

Therefore, we can give the following refinements for (5.2.8), as follows:

**Corollary 5.2.9.** *Let $f$ as in Theorem 5.2.6, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{(b-a)}{\left[216\left(s^2+3s+2\right)\right]^{1/q}}\left(\frac{5}{72}\right)^{1-1/q}\left(\left|f'(a)\right|+\left|f'(b)\right|\right) \\
&\qquad \left\{\left[\left(3^{-s}\right)\left(2^{1-s}\right)+3s\left(2^{1-s}\right)+3\left(2^{-s}\right)\right]^{1/q}\right. \\
&\qquad \left.+\left[5^{s+2}3^{-s}2^{1-s}-6s\left(2^{-s}\right)-21\left(2^{-s}\right)+6s-24\right]^{1/q}\right\}
\end{aligned} \tag{5.2.14}$$

*Moreover, if $s=1$, we have*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\qquad\qquad \leq \frac{5\,(1668)^{1/q}}{72\,(6480)^{1/q}}(b-a)\left(\left|f'(a)\right|+\left|f'(b)\right|\right)
\end{aligned} \tag{5.2.15}$$

*Proof.* We consider the inequality (5.2.8), for $p>1$, $q=p/(p-1)$. Let

$$a_1=\left[\left(3^{-s}\right)\left(2^{1-s}\right)+3s\left(2^{1-s}\right)+3\left(2^{-s}\right)\right]\left|f'(b)\right|^q,$$

$$b_1=\left[5^{s+2}3^{-s}2^{1-s}-6s\left(2^{-s}\right)-21\left(2^{-s}\right)+6s-24\right]\left|f'(a)\right|^q,$$

$$a_2=\left[\left(3^{-s}\right)\left(2^{1-s}\right)+3s\left(2^{1-s}\right)+3\left(2^{-s}\right)\right]\left|f'(a)\right|^q,$$

and $b_2=\left[5^{s+2}3^{-s}2^{1-s}-6s\left(2^{-s}\right)-21\left(2^{-s}\right)+6s-24\right]\left|f'(b)\right|^q$.

Here, $0<1/q<1$, for $q>1$. Using the fact

$$\sum_{i=1}^{n}\left(a_i+b_i\right)^r \leq \sum_{i=1}^{n}a_i^r+\sum_{i=1}^{n}b_i^r,$$

for $0<r<1$, $a_1,a_2,...,a_n\geq 0$ and $b_1,b_2,...,b_n\geq 0$, (in our case $n=2$), we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{(b-a)}{\left[216\left(s^2+3s+2\right)\right]^{1/q}}\left(\frac{5}{72}\right)^{1-1/q}\left\{\left[\left(3^{-s}\right)\left(2^{1-s}\right)+3s\left(2^{1-s}\right)+3\left(2^{-s}\right)\right]^{1/q}\right. \\
&\qquad\qquad \left.+\left[5^{s+2}3^{-s}2^{1-s}-6s\left(2^{-s}\right)-21\left(2^{-s}\right)+6s-24\right]^{1/q}\right\} \\
&\qquad\qquad \times\left(\left|f'(a)\right|+\left|f'(b)\right|\right)
\end{aligned}$$

which completes the proof. $\square$

The next result, gives an inequality of Simpson's type for concave functions.

**Theorem 5.2.10.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable function on $I^\circ$ and $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$, for some fixed $q \geq 1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{5(b-a)}{72}\left[\left|f'\left(\frac{29b+61a}{90}\right)\right|+\left|f'\left(\frac{61b+29a}{90}\right)\right|\right]. \quad (5.2.16)$$

*Proof.* From Lemma 5.2.1, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq (b-a)\left|\int_0^1 p(t)\,f'(tb+(1-t)a)\,dt\right| \\
&\leq (b-a)\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right|\left|f'(tb+(1-t)a)\right|dt \\
&\qquad + (b-a)\int_{1/2}^{1}\left|\left(t-\frac{5}{6}\right)\right|\left|f'(tb+(1-t)a)\right|dt.
\end{aligned}$$

We note that by concavity of $|f'|^q$ and the power-mean inequality, we have

$$|f'(\alpha x+(1-\alpha)y)|^q \geq \alpha|f'(x)|^q+(1-\alpha)|f'(y)|^q.$$

Hence,

$$|f'(\alpha x+(1-\alpha)y)| \geq \alpha|f'(x)|+(1-\alpha)|f'(y)|.$$

so, $|f'|$ is also concave.

Accordingly, by the Jensen integral inequality, we have

$$\begin{aligned}
&\int_0^{1/2}\left|t-\frac{1}{6}\right| f'(tb+(1-t)a)\,dt \\
&\leq \left(\int_0^{1/2}\left|t-\frac{1}{6}\right|dt\right)\left|f'\left(\frac{\int_0^{1/2}\left|t-\frac{1}{6}\right|[tb+(1-t)a]\,dt}{\int_0^{1/2}\left|t-\frac{1}{6}\right|dt}\right)\right| \\
&= \frac{5}{72}\left|f'\left(\frac{29b+61a}{90}\right)\right|, \quad (5.2.17)
\end{aligned}$$

and

$$\int_{1/2}^{1}\left|t-\frac{5}{6}\right| f'\left(tb+(1-t)a\right)dt$$
$$\leq\left(\int_{1/2}^{1}\left|t-\frac{5}{6}\right| dt\right)\left|f'\left(\frac{\int_{1/2}^{1}\left|t-\frac{5}{6}\right|\left[tb+(1-t)a\right]dt}{\int_{1/2}^{1}\left|t-\frac{5}{6}\right| dt}\right)\right|$$
$$=\frac{5}{72}\left|f'\left(\frac{61b+29a}{90}\right)\right|. \quad (5.2.18)$$

A combination of the above numbered inequalities gives the result, that is

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{5(b-a)}{72}\left[\left|f'\left(\frac{29b+61a}{90}\right)\right|+\left|f'\left(\frac{61b+29a}{90}\right)\right|\right].$$

which completes the proof. □

Another result is considered as follows:

**Theorem 5.2.11.** *Let $f: I\subseteq\mathbb{R}_+\to\mathbb{R}_+$ be differentiable function on $I^\circ$ and $a,b\in I$ with $a<b$. If $|f'|^q$ is concave on $[a,b]$, for some fixed $q>1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq(b-a)\left(\frac{q-1}{2q-1}\right)\left(2^{\frac{2q-1}{q-1}}+1\right)\left[\left|f'\left(\frac{3b+a}{4}\right)\right|+\left|f'\left(\frac{b+3a}{4}\right)\right|\right]. \quad (5.2.19)$$

*Proof.* From Lemma 5.2.1, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq (b-a)\int_0^{1/2}\left|t-\frac{1}{6}\right|\left|f'\left(tb+(1-t)a\right)\right|dt$$
$$+(b-a)\int_{1/2}^{1}\left|t-\frac{5}{6}\right|\left|f'\left(tb+(1-t)a\right)\right|dt$$

Using Hölder inequality, for $q>1$ and $p=\frac{q}{q-1}$, we obtain

$$(b-a)\int_0^{1/2}\left|t-\frac{1}{6}\right|\left|f'\left(tb+(1-t)a\right)\right|dt$$
$$\leq (b-a)\left(\int_0^{1/2}\left|t-\frac{1}{6}\right|^{\frac{q}{q-1}}dt\right)^{\frac{q}{q-1}}\left(\int_0^{1/2}\left|f'\left(tb+(1-t)a\right)\right|^q dt\right)^{1/q},$$

and

$$
\begin{aligned}
& (b-a)\int_{1/2}^{1}\left|t-\frac{5}{6}\right|\left|f'\left(tb+(1-t)a\right)\right|dt \\
\leq\ & (b-a)\left(\int_{1/2}^{1}\left|t-\frac{5}{6}\right|^{\frac{q}{q-1}}dt\right)^{\frac{q}{q-1}}\left(\int_{1/2}^{1}\left|f'\left(tb+(1-t)a\right)\right|^{q}dt\right)^{1/q}.
\end{aligned}
$$

It is easy to check that

$$
\int_{0}^{1/2}\left|t-\frac{1}{6}\right|^{\frac{q}{q-1}}dt=\int_{1/2}^{1}\left|t-\frac{5}{6}\right|^{\frac{q}{q-1}}dt=\frac{1}{6^{\frac{2q-1}{q-1}}}\left(\frac{q-1}{2q-1}\right)\left(2^{\frac{2q-1}{q-1}}+1\right).
$$

Since $|f'|^q$ is concave on $[a,b]$ we can use the integral Jensen's inequality to obtain

$$
\begin{aligned}
\int_{0}^{1/2}\left|f'\left(tb+(1-t)a\right)\right|^{q}dt &= \int_{0}^{1/2}t^{0}\left|f'\left(tb+(1-t)a\right)\right|^{q}dt \\
&\leq \left(\int_{0}^{1/2}t^{0}dt\right)\left|f'\left(\frac{\int_{0}^{1/2}\left(tb+(1-t)a\right)dt}{\int_{0}^{1/2}t^{0}dt}\right)\right|^{q} \\
&= \frac{1}{2}\left|f'\left(2\int_{0}^{1/2}\left(tb+(1-t)a\right)dt\right)\right|^{q} \\
&= \frac{1}{2}\left|f'\left(\frac{b+3a}{4}\right)\right|^{q}.
\end{aligned}
$$

Analogously,

$$
\int_{1/2}^{1}\left|f'\left(tb+(1-t)a\right)\right|^{q}dt\leq\frac{1}{2}\left|f'\left(\frac{3b+a}{4}\right)\right|^{q}.
$$

Combining all obtained inequalities we get

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right| \\
&\leq \frac{(b-a)}{6^{\frac{2q-1}{q-1}}}\left(\frac{q-1}{2q-1}\right)\left(2^{\frac{2q-1}{q-1}}+1\right)\left(\frac{1}{2}\right)^{q}\left[\left|f'\left(\frac{3b+a}{4}\right)\right|+\left|f'\left(\frac{b+3a}{4}\right)\right|\right] \\
&\leq (b-a)\left(\frac{q-1}{2q-1}\right)\left(2^{\frac{2q-1}{q-1}}+1\right)\left[\left|f'\left(\frac{3b+a}{4}\right)\right|+\left|f'\left(\frac{b+3a}{4}\right)\right|\right],
\end{aligned}
$$

which completes the proof. $\square$

**Theorem 5.2.12.** *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be differentiable function on $I^{\circ}$ and $a,b \in I$ with $a<b$. If $|f'|^q$ is $s$–concave on $[a,b]$, for some fixed $s \in (0,1]$ and $q>1$, then the*

*following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\le (b-a)\,2^{(s-1)/q}\frac{1}{6^{\frac{2q-1}{q-1}}}\left(\frac{q-1}{2q-1}\right)\left(2^{\frac{2q-1}{q-1}}+1\right)$$
$$\times\left[\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{a+3b}{4}\right)\right|\right]. \tag{5.2.20}$$

*Proof.* We proceed similarly as in the proof of Theorem 5.2.11, by using (2.3.23) instead of the Jensen's integral inequality for concave functions. For $|f'|^q$ $s$–concave, we have

$$\int_0^{1/2}|f'(tb+(1-t)a)|^q\,dt\le 2^{s-1}\left|f'\left(\frac{3a+b}{4}\right)\right|^q,$$

and

$$\int_{1/2}^{1}|f'(tb+(1-t)a)|^q\,dt\le 2^{s-1}\left|f'\left(\frac{a+3b}{4}\right)\right|^q,$$

so that,

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\le (b-a)\,2^{(s-1)/q}\frac{1}{6^{\frac{2q-1}{q-1}}}\left(\frac{q-1}{2q-1}\right)\left(2^{\frac{2q-1}{q-1}}+1\right)\left[\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{a+3b}{4}\right)\right|\right],$$

which completes the proof. □

## 5.3 INEQUALITIES OF SIMPSON'S TYPE FOR QUASI-CONVEX FUNCTIONS

In order to prove our main theorems, we need the following lemma:

**Lemma 5.3.1.** *Let $f':I\subseteq\mathbb{R}\to\mathbb{R}$ be a absolutely continuous mapping on $I^\circ$ where $a,b\in I$ with $a<b$. Then the following equality holds:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$
$$=(b-a)^2\int_0^1 p(t)\,f''(tb+(1-t)a)\,dt \tag{5.3.1}$$

*where,*

$$p(t)=\begin{cases}\frac{1}{6}t(3t-1), & t\in\left[0,\frac{1}{2}\right]\\[2mm] \frac{1}{6}(t-1)(3t-2), & t\in\left(\frac{1}{2},1\right]\end{cases}$$

*Proof.* We note that

$$I = \int_0^1 p(t)\, f''(tb + (1-t)a)\, dt = \frac{1}{6}\int_0^{1/2} t(3t-1)\, f''(tb+(1-t)a)\, dt + \frac{1}{6}\int_{1/2}^1 (t-1)(3t-2)\, f''(tb+(1-t)a)\, dt.$$

Integrating by parts, we get

$$\begin{aligned}
I &= \frac{1}{6}t(3t-1)\frac{f'(tb+(1-t)a)}{b-a}\Bigg|_0^{1/2} - \left[\frac{1}{2}t + \frac{1}{6}(3t-1)\right]\frac{f(tb+(1-t)a)}{(b-a)^2}\Bigg|_0^{1/2} \\
&\quad + \int_0^{1/2}\frac{f(tb+(1-t)a)}{(b-a)^2}dt + \frac{1}{6}(t-1)(3t-2)\frac{f'(tb+(1-t)a)}{b-a}\Bigg|_{1/2}^1 \\
&\quad - \left[\frac{1}{2}(t-1) + \frac{1}{6}(3t-2)\right]\frac{f(tb+(1-t)a)}{(b-a)^2}\Bigg|_{1/2}^1 + \int_{1/2}^1 \frac{f(tb+(1-t)a)}{(b-a)^2}dt \\
&= \frac{1}{24}\frac{f'\left(\frac{a+b}{2}\right)}{b-a} - \frac{1}{3}\frac{f\left(\frac{a+b}{2}\right)}{(b-a)^2} - \frac{1}{6}\frac{f(a)}{(b-a)^2} + \int_0^{1/2}\frac{f(tb+(1-t)a)}{(b-a)^2}dt \\
&\quad - \frac{1}{6}\frac{f(b)}{(b-a)^2} - \frac{1}{24}\frac{f'\left(\frac{a+b}{2}\right)}{b-a} - \frac{1}{3}\frac{f\left(\frac{a+b}{2}\right)}{(b-a)^2} + \int_{1/2}^1\frac{f(tb+(1-t)a)}{(b-a)^2}dt \\
&= \frac{1}{(b-a)^2}\int_0^1 f(tb+(1-t)a)\, dt - \frac{1}{6(b-a)^2}\left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right)\right].
\end{aligned}$$

Setting $x = tb + (1-t)a$, and $dx = (b-a)dt$, gives

$$(b-a)^2 \cdot I = \frac{1}{(b-a)}\int_a^b f(x)\, dx - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right],$$

which gives the desired representation (5.3.1). □

The next theorem gives a new refinement of the Simpson's inequality for quasi-convex functions.

**Theorem 5.3.2.** *Let $f' : I \subseteq \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on $I^\circ$ and $a, b \in I$ with $a < b$. If $|f''|$ is quasi-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\, dx - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\
&\qquad \le \frac{(b-a)^2}{162}\cdot\left[\max\left\{|f''(a)|, \left|f''\left(\frac{a+b}{2}\right)\right|\right\}\right. \\
&\qquad\qquad \left. + \max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|, |f''(b)|\right\}\right] \qquad (5.3.2)
\end{aligned}$$

*Proof.* By Lemma 5.3.1 and since $|f''|$ is quasi-convex, then we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{(b-a)^2}{6}\int_0^{\frac{1}{2}} t\,|3t-1|\,|f''(tb+(1-t)a)|\,dt \\
&\qquad + \frac{(b-a)^2}{6}\int_{\frac{1}{2}}^1 |t-1|\,|3t-2|\,|f''(tb+(1-t)a)|\,dt \\
&\leq \frac{(b-a)^2}{6}\cdot\max\left\{|f''(a)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right\}\left(\int_0^{\frac{1}{3}} t(1-3t)\,dt+\int_{\frac{1}{3}}^{\frac{1}{2}} t(3t-1)\,dt\right) \\
&\qquad + \frac{(b-a)^2}{6}\cdot\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|,|f''(b)|\right\}\left(\int_{\frac{1}{2}}^{\frac{2}{3}}(1-t)(2-3t)\,dt\right. \\
&\qquad\qquad \left.+\int_{\frac{\frac{2}{3}}{2}}^{1}(1-t)(3t-2)\,dt\right) \\
&\leq \frac{(b-a)^2}{162}\cdot\left[\max\left\{|f''(a)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right\}+\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|,|f''(b)|\right\}\right],
\end{aligned}$$

which completes the proof. □

**Corollary 5.3.3.** *In Theorem 5.3.2, Additionally, if*

1. $|f''|$ *is increasing, then we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{162}\cdot\left[\left|f''\left(\frac{a+b}{2}\right)\right|+|f''(b)|\right]; \quad (5.3.3)$$

2. $|f''|$ *is decreasing, then we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{162}\cdot\left[|f''(a)|+\left|f''\left(\frac{a+b}{2}\right)\right|\right]. \quad (5.3.4)$$

As a special case, we refine the following midpoint type inequality for quasi-convex functions:

**Corollary 5.3.4.** *In Theorem 5.3.2, Additionally, if* $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$*, then we have,*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right|\\ &\le \frac{(b-a)^2}{162}\cdot\left[\max\left\{|f''(a)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right\}+\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|,|f''(b)|\right\}\right].\end{aligned} \tag{5.3.5}$$

*For instance, for* $M > 0$*, if* $|f''(x)| < M$*, for all* $x \in [a,b]$*, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2}{81}M. \tag{5.3.6}$$

The corresponding version for powers of the absolute value of the second derivative is incorporated in the following result:

**Theorem 5.3.5.** *Let* $f' : I \subseteq \mathbb{R} \to \mathbb{R}$ *be an absolutely continuous function on* $I^\circ$ *and* $a, b \in I$ *with* $a < b$*. If* $|f'|^{p/(p-1)}$ *is quasi-convex on* $[a,b]$*, for some fixed* $p > 1$*, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\qquad\le \frac{(b-a)^2}{6}\cdot\left(3^{-p-1}\beta(p+1,p+1)+\frac{4(3)^{-p}+3(2)^{-p}(p-1)}{12(2+3p+p^2)}\right)^{\frac{1}{p}}\\ &\qquad\qquad\left[\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^{p/(p-1)},|f''(b)|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}\right.\\ &\qquad\qquad\qquad\left.+\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^{p/(p-1)},|f''(a)|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}\right].\end{aligned} \tag{5.3.7}$$

*for* $p > 1$*, where, where* $\beta(x,y)$ *is the Beta function of Euler type.*

*Proof.* From Lemma 5.3.1, and since $f$ is quasi-convex, we have

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\le \frac{(b-a)^2}{6}\int_0^{\frac{1}{2}} t\,|3t-1|\,|f''(tb+(1-t)a)|\,dt\\ &\qquad+\frac{(b-a)^2}{6}\int_{\frac{1}{2}}^1 |t-1|\,|3t-2|\,|f''(tb+(1-t)a)|\,dt\end{aligned}$$

$$\leq \frac{(b-a)^2}{6}\left(\int_0^{\frac{1}{2}}(t\,|3t-1|)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)^2}{6}\left(\int_{\frac{1}{2}}^1(|t-1|\,|3t-2|)^p\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1|f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}$$

$$=\frac{(b-a)^2}{6}\left(\int_0^{\frac{1}{3}}t^p(1-3t)^p\,dt+\int_{\frac{1}{3}}^{\frac{1}{2}}t^p(3t-1)^p\,dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^{\frac{1}{2}}|f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)^2}{6}\left(\int_{\frac{1}{2}}^{\frac{2}{3}}(1-t)^p(2-3t)^p\,dt+\int_{\frac{2}{3}}^1(1-t)^p(3t-2)^p\,dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_{\frac{1}{2}}^1|f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}$$

Since $f$ is quasi-convex, we have

$$\int_0^{1/2}|f''(tb+(1-t)a)|^q\,dt\leq\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(a)|^q\right\},\tag{5.3.8}$$

and

$$\int_{1/2}^1|f''(tb+(1-t)a)|^q\,dt\leq\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(b)|^q\right\}.\tag{5.3.9}$$

Therefore,

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq\frac{(b-a)^2}{6}\cdot\left(3^{-p-1}\beta(p+1,p+1)+\frac{4(3)^{-p}+3(2)^{-p}(p-1)}{12(2+3p+p^2)}\right)^{\frac{1}{p}}$$

$$\left[\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^{p/(p-1)},|f''(b)|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}\right.$$

$$\left.+\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^{p/(p-1)},|f''(a)|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}\right],$$

for $p>1$, where we have used the fact that

$$\int_0^{\frac{1}{3}}t^p(1-3t)^p\,dt=\int_{\frac{2}{3}}^1(1-t)^p(3t-2)^p\,dt=3^{-p-1}\beta(p+1,p+1),$$

and

$$\int_{\frac{1}{3}}^{\frac{1}{2}} t^p (3t-1)^p \, dt = \int_{\frac{1}{2}}^{\frac{2}{3}} (1-t)^p (2-3t)^p \, dt = \frac{4(3)^{-p} + 3(2)^{-p}(p-1)}{12(2+3p+p^2)},$$

for details see Gradshteyn and Ryzhik (2007), which completes the proof. □

**Corollary 5.3.6.** *Let $f$ be as in Theorem 5.3.5. Additionally, if*

1. $|f'|$ *is increasing, then we have*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\quad\le \frac{(b-a)^2}{6}\cdot\left(3^{-p-1}\beta(p+1,p+1) + \frac{4(3)^{-p} + 3(2)^{-p}(p-1)}{12(2+3p+p^2)}\right)^{\frac{1}{p}} \\ &\qquad\times\left(\left|f'\left(\frac{a+b}{2}\right)\right| + |f'(b)|\right). \quad (5.3.10)\end{aligned}$$

2. $|f'|$ *is decreasing, then we have*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\quad\le \frac{(b-a)^2}{6}\cdot\left(3^{-p-1}\beta(p+1,p+1) + \frac{4(3)^{-p} + 3(2)^{-p}(p-1)}{12(2+3p+p^2)}\right)^{\frac{1}{p}} \\ &\qquad\times\left(|f'(a)| + \left|f'\left(\frac{a+b}{2}\right)\right|\right). \quad (5.3.11)\end{aligned}$$

Another refinement for the Midpoint inequality via quasi-convex functions may be stated as follows:

**Corollary 5.3.7.** *In Theorem 5.3.5, Additionally, if $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$, then we have,*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \\ &\quad\le \frac{(b-a)^2}{6}\cdot\left(3^{-p-1}\beta(p+1,p+1) + \frac{4(3)^{-p} + 3(2)^{-p}(p-1)}{12(2+3p+p^2)}\right)^{\frac{1}{p}} \\ &\qquad\left[\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^{p/(p-1)}, |f''(b)|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}\right. \\ &\qquad\left.+\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^{p/(p-1)}, |f''(a)|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}\right]. \quad (5.3.12)\end{aligned}$$

A generalization of (5.3.2) is given in the following theorem:

**Theorem 5.3.8.** *Let $f' : I \subseteq \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on $I^\circ$ and $a, b \in I$ with $a < b$. If $|f'|^q$ is quasi-convex on $[a, b]$, $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\qquad\leq \frac{(b-a)^2}{162}\left[\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}}\right. \\
&\qquad\qquad\left. + \left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, |f''(a)|^q\right\}\right)^{\frac{1}{q}}\right]. \quad (5.3.13)
\end{aligned}$$

*Proof.* From Lemma 5.3.1, and since $f$ is quasi-convex, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{(b-a)^2}{6}\int_0^{\frac{1}{2}} t\,|3t-1|\,|f''(tb + (1-t)a)|\,dt \\
&\qquad + \frac{(b-a)^2}{6}\int_{\frac{1}{2}}^1 |t-1|\,|3t-2|\,|f''(tb + (1-t)a)|\,dt \\
&\leq \frac{(b-a)^2}{6}\left(\int_0^{\frac{1}{2}} t\,|3t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t\,|3t-1|\,|f''(tb + (1-t)a)|^q\,dt\right)^{\frac{1}{q}} \\
&\qquad + \frac{(b-a)^2}{6}\left(\int_{\frac{1}{2}}^1 |t-1|\,|3t-1|\,dt\right)^{1-\frac{1}{q}} \\
&\qquad\qquad \times\left(\int_{\frac{1}{2}}^1 |t-1|\,|3t-1|\,|f''(tb + (1-t)a)|^q\,dt\right)^{\frac{1}{q}} \\
&= \frac{(b-a)^2}{6}\left(\int_0^{\frac{1}{3}} t(1-3t)\,dt + \int_{\frac{1}{3}}^{\frac{1}{2}} t(3t-1)\,dt\right)^{1-\frac{1}{q}} \\
&\qquad\qquad \times\left(\int_0^{\frac{1}{2}} t\,|3t-1|\,|f''(tb + (1-t)a)|^q\,dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\frac{(b-a)^2}{6}\left(\int_{\frac{1}{2}}^{\frac{2}{3}}(1-t)(2-3t)\,dt+\int_{\frac{2}{3}}^{1}(1-t)(3t-2)\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_{\frac{1}{2}}^{1}|t-1|\,|3t-2|\,|f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}$$

Since $f$ is quasi-convex, we have

$$\int_0^{\frac{1}{2}} t\,|3t-1|\,|f''(tb+(1-t)a)|^q\,dt$$
$$=\frac{1}{27}\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,|f'(a)|^q\right\} \quad (5.3.14)$$

and

$$\int_{1/2}^{1}|t-1|\,|3t-2|\,|f'(tb+(1-t)a)|^q\,dt$$
$$=\frac{1}{27}\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,|f'(b)|^q\right\} \quad (5.3.15)$$

where, we used the fact

$$\int_0^{1/2} t\,|3t-1|\,dt=\int_{1/2}^{1}|t-1|\,|3t-2|\,dt=\frac{1}{27}. \quad (5.3.16)$$

Combination of (5.3.14), (5.3.15) and (5.3.16), gives the required result which completes the proof. □

**Corollary 5.3.9.** *Let $f$ be as in Theorem 5.3.8. Additionally, if*

1. *$|f'|$ is increasing, then (5.3.3) holds.*
2. *$|f'|$ is decreasing, then (5.3.4) holds.*

*Proof.* It follows directly by Theorem 5.3.8. □

**Corollary 5.3.10.** *In Theorem 5.3.8, Additionally, if $f(a)=f\left(\frac{a+b}{2}\right)=f(b)$, then we have,*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|$$
$$\le\frac{(b-a)^2}{162}\cdot\left[\max\left\{|f''(a)|^q,\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}\right. \quad (5.3.17)$$
$$\left.+\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(b)|^q\right\}\right].$$

**Remark 5.3.11.** *For*

$$h(p)=\left(3^{-p-1}\beta\left(p+1,p+1\right)+\frac{4\left(3\right)^{-p}+3\left(2\right)^{-p}\left(p-1\right)}{12\left(2+3p+p^{2}\right)}\right)^{\frac{1}{p}},\quad p>1,$$

*we have*

$$\lim_{p\to1^{+}}h\left(p\right)=\frac{1}{27},$$

*using the fact that*

$$\sum_{i=1}^{n}\left(a_{i}+b_{i}\right)^{r}\leq\sum_{i=1}^{n}a_{i}^{r}+\sum_{i=1}^{n}b_{i}^{r},$$

*for* $0<r<1$, $a_{1},a_{2},...,a_{n}\geq0$ *and* $b_{1},b_{2},...,b_{n}\geq0$*, we obtain*

$$\begin{aligned}\lim_{p\to\infty}h\left(p\right)&\leq\lim_{p\to\infty}3^{-1-\frac{1}{p}}\beta^{\frac{1}{p}}\left(p+1,p+1\right)+\lim_{p\to\infty}\frac{4^{\frac{1}{p}}\left(3\right)^{-1}+3^{\frac{1}{p}}\left(2\right)^{-1}\left(p-1\right)^{\frac{1}{p}}}{\left(12\right)^{\frac{1}{p}}\left(2+3p+p^{2}\right)^{\frac{1}{p}}}\\&=\frac{1}{3}\lim_{p\to\infty}\beta^{\frac{1}{p}}\left(p+1,p+1\right)+1,\end{aligned}$$

*also, Stirling's approximation gives the asymptotic formula*

$$\beta\left(x,y\right)\simeq\sqrt{2\pi}\frac{x^{x-\frac{1}{2}}y^{y-\frac{1}{2}}}{\left(x+y\right)^{x+y-\frac{1}{2}}},$$

$$\lim_{p\to\infty}\beta^{\frac{1}{p}}\left(p+1,p+1\right)\cong\sqrt{2\pi}\lim_{p\to\infty}\frac{\left(p+1\right)^{2p+1}}{\left(2p+2\right)^{2p+\frac{3}{2}}}=\lim_{p\to\infty}\frac{\sqrt{2\pi}}{\left(2\right)^{2p+\frac{3}{2}}}\frac{1}{\left(p+1\right)^{\frac{1}{2}}}\to0,$$

*so that,* $\lim\limits_{p\to\infty}h\left(p\right)\to1$*, therefore* $h(p)$ *satisfies*

$$\frac{1}{27}\leq h\left(p\right)\leq1.$$

*Therefore, since*

$$h(p)\leq162,\quad\forall p>1,$$

*then we observe that the inequality (5.3.13) is better than the inequality (5.3.7) meaning that the approach via power mean inequality is a better approach than the one through Hölder's inequality.*

## 5.4 INEQUALITIES OF SIMPSON'S TYPE FOR $R$-CONVEX FUNCTIONS

The next theorem gives a new refinement for the Simpson's inequality via $r$-convex functions.

**Theorem 5.4.1.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}_+$ be differentiable function on $I^\circ$ and $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is $r$-convex on $[a, b]$, for some fixed $p > 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\le (b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\left[\left(L_r\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{p/(p-1)},|f'(b)|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}\right. \\
&\qquad\left.+\left(L_r\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{p/(p-1)},|f'(a)|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}\right]. \qquad (5.4.1)
\end{aligned}$$

*where, $p > 1$, and $L_r$ is the generalized log-mean.*

*Proof.* From Lemma 5.2.1, and since $f$ is $r$-convex, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\le\ & (b-a)\left|\int_0^1 p(t)\,f'(tb+(1-t)a)\,dt\right| \\
\le\ & (b-a)\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right||f'(tb+(1-t)a)|\,dt \\
&+(b-a)\int_{1/2}^1\left|\left(t-\frac{5}{6}\right)\right||f'(tb+(1-t)a)|\,dt \\
\le\ & (b-a)\left(\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right|^p dt\right)^{1/p}\left(\int_0^{1/2}|f'(tb+(1-t)a)|^q\,dt\right)^{1/q} \\
&+(b-a)\left(\int_{1/2}^1\left|\left(t-\frac{5}{6}\right)\right|^p dt\right)^{1/p}\left(\int_{1/2}^1|f'(tb+(1-t)a)|^q\,dt\right)^{1/q} \\
=\ & (b-a)\left(\int_0^{1/6}\left(\frac{1}{6}-t\right)^p dt+\int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)^p dt\right)^{1/p}
\end{aligned}$$

$$\times\left(\int_0^{1/2}\left|f'\left(tb+(1-t)a\right)\right|^q dt\right)^{1/q}$$
$$+(b-a)\left(\int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)^p dt+\int_{5/6}^{1}\left(t-\frac{5}{6}\right)^p dt\right)^{1/p}$$
$$\times\left(\int_{1/2}^{1}\left|f'\left(tb+(1-t)a\right)\right|^q dt\right)^{1/q}.$$

Since $f$ is $r$-convex by (2.3.30), we have

$$\int_0^{1/2}\left|f'\left(tb+(1-t)a\right)\right|^q dt \le L_r\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(a)\right|^q\right\}, \tag{5.4.2}$$

and

$$\int_{1/2}^{1}\left|f'\left(tb+(1-t)a\right)\right|^q dt \le L_r\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(b)\right|^q\right\}. \tag{5.4.3}$$

Therefore,

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\le (b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\left[\left(L_r\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{p/(p-1)},\left|f'(b)\right|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}\right.$$
$$\left.+\left(L_r\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{p/(p-1)},\left|f'(a)\right|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}\right].$$

where, $L_r(\cdot,\cdot)$ is the generalized log-mean, which completes the proof. □

**Corollary 5.4.2.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}_+$ be differentiable function on $I^\circ$ and $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is log-convex on $[a, b]$, for some fixed $p > 1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\le (b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\left[\left(L\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{p/(p-1)},\left|f'(b)\right|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}\right.$$
$$\left.+\left(L\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{p/(p-1)},\left|f'(a)\right|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}\right].$$

*where, $p > 1$ and $L(\cdot,\cdot)$ is the log-mean.*

*Proof.* In the proof of Theorem 4.5.1, using (2.3.29) instead of (2.3.30), therefore, the result holds. □

**Theorem 5.4.3.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}_+$ be differentiable function on $I^\circ$ and $a, b \in I$ with $a < b$. If $|f'|$ is $r$-convex on $[a, b]$, $r \geq 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\leq \frac{(b-a)}{108}\left\{\frac{r\left(r(6)^{-\frac{1}{r}} - 27(2)^{-\frac{1}{r}}r - 54r + 125r(5)^{\frac{1}{r}}(6)^{-\frac{1}{r}} + 18\right)}{(6r^2+5r+1)}|f'(b)|\right. \\ &\quad + \left[\frac{r^2\left(-162r + 18 + 325r(5)^{\frac{1}{r}}(6)^{-\frac{1}{r}} + 25(5)^{\frac{1}{r}}(6)^{-\frac{1}{r}}\right)}{(6r^3+11r^2+6r+1)}\right. \\ &\quad \left.\left.+\frac{r^2\left(17r(6)^{-\frac{1}{r}} + 5(6)^{-\frac{1}{r}} - 135r(2)^{-\frac{1}{r}} - 27(2)^{-\frac{1}{r}}\right)}{(6r^3+11r^2+6r+1)}\right]|f'(a)|\right\} \end{aligned} \tag{5.4.4}$$

*Proof.* From Lemma 5.2.1, and since $f$ is $r$-convex ($r \geq 1$), we have

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\leq (b-a)\left|\int_0^1 p(t)\,f'(tb+(1-t)a)\,dt\right| \\ &\leq (b-a)\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right| |f'(tb+(1-t)a)|\,dt \\ &\qquad + (b-a)\int_{1/2}^1 \left|\left(t-\frac{5}{6}\right)\right| |f'(tb+(1-t)a)|\,dt \\ &\leq (b-a)\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right| \left(t|f'(b)|^r + (1-t)|f'(a)|^r\right)^{1/r} dt \\ &\qquad + (b-a)\int_{1/2}^1 \left|\left(t-\frac{5}{6}\right)\right| \left(t|f'(b)|^r + (1-t)|f'(a)|^r\right)^{1/r} dt. \end{aligned}$$

Using the fact that $\sum_{i=1}^{n}(a_i+b_i)^k \leq \sum_{i=1}^{n} a_i^k + \sum_{i=1}^{n} b_i^k$, for $0 < k < 1$, $a_1, a_2, ..., a_n \geq 0$ and

$b_1, b_2, ..., b_n \geq 0$, we obtain

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq (b-a)\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right|\left(t\left|f'(b)\right|^r+(1-t)\left|f'(a)\right|^r\right)^{1/r}dt$$
$$+(b-a)\int_{1/2}^{1}\left|\left(t-\frac{5}{6}\right)\right|\left(t\left|f'(b)\right|^r+(1-t)\left|f'(a)\right|^r\right)^{1/r}dt$$
$$\leq (b-a)\int_0^{1/2}\left|\left(t-\frac{1}{6}\right)\right|\left(t^{\frac{1}{r}+1}\left|f'(b)\right|+t(1-t)^{\frac{1}{r}}\left|f'(a)\right|\right)dt$$
$$+(b-a)\int_{1/2}^{1}\left|\left(t-\frac{5}{6}\right)\right|\left(t^{\frac{1}{r}+1}\left|f'(b)\right|+t(1-t)^{\frac{1}{r}}\left|f'(a)\right|\right)dt$$
$$=\frac{(b-a)}{108}\left\{\frac{r\left(r(6)^{-\frac{1}{r}}-27(2)^{-\frac{1}{r}}r-54r+125r(5)^{\frac{1}{r}}(6)^{-\frac{1}{r}}+18\right)}{(6r^2+5r+1)}\left|f'(b)\right|\right.$$
$$+\left[\frac{r^2\left(-162r+18+325r(5)^{\frac{1}{r}}(6)^{-\frac{1}{r}}+25(5)^{\frac{1}{r}}(6)^{-\frac{1}{r}}\right)}{(6r^3+11r^2+6r+1)}\right.$$
$$\left.\left.+\frac{r^2\left(17r(6)^{-\frac{1}{r}}+5(6)^{-\frac{1}{r}}-135r(2)^{-\frac{1}{r}}-27(2)^{-\frac{1}{r}}\right)}{(6r^3+11r^2+6r+1)}\right]\left|f'(a)\right|\right\},$$

which gives the required result and the proof is complete. □

**Corollary 5.4.4.** *In Theorem 5.4.3, choose $r = 1$, the result holds for convex functions, i.e.,*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{7776}\left(211\left|f'(a)\right|+329\left|f'(b)\right|\right).$$

## 5.5 APPLICATIONS TO SIMPSON'S FORMULA

Let $P$ be a division or partition of the interval $[a, b]$, i.e., $d : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$, $h_i = (x_{i+1} - x_i)/2$ and consider the Simpson's formula

$$S_n(f, P) = \sum_{i=0}^{n-1} \frac{f(x_i) + 4f(x_i + h_i) + f(x_{i+1})}{6}(x_{i+1} - x_i). \tag{5.5.1}$$

It is well known that if the mapping $f : [a, b] \to \mathbf{R}$, is differentiable such that $f^{(4)}(x)$ exists on $(a, b)$ and $K = \max_{x \in (a,b)} \left| f^{(4)}(x) \right| < \infty$, then

$$I = \int_a^b f(x)\, dx = S_n(f, P) + E_n^S(f, P), \tag{5.5.2}$$

where the approximation error $E_n^S(f, P)$ of the integral $I$ by the Simpson's formula $S_n(f, P)$ satisfies

$$\left| E_n^S(f, P) \right| \leq \frac{K}{90} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^5. \tag{5.5.3}$$

In the classical Simpson's rule (5.5.2), It is clear that if the mapping $f$ is not fourth differentiable or the fourth derivative is not bounded on $(a, b)$, then (5.5.2) cannot be applied. In this section, we derive some new error estimates for the Simpson's rule in terms of first and second derivatives.

**Proposition 5.5.1.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping on $I^\circ$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then in (5.5.2), for every division $d$ of $[a, b]$, the following holds:*

$$\left| E_n^S(f, P) \right| \leq \frac{5}{72} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left[ |f'(x_i)| + |f'(x_{i+1})| \right].$$

*Proof.* Applying Corollary 5.2.3 on the subintervals $[x_i, x_{i+1}]$, $(i = 0, 1, ..., n-1)$ of the division $P$, we get

$$\begin{aligned} & \left| \frac{(x_{i+1} - x_i)}{3} \left( f(x_i) + 4f\left( \frac{x_i + x_{i+1}}{2} \right) + f(x_{i+1}) \right) - \int_{x_i}^{x_{i+1}} f(x)\, dx \right| \\ \leq & \frac{5(x_{i+1} - x_i)^3}{72} \left[ |f'(x_i)| + |f'(x_{i+1})| \right] \end{aligned}$$

Summing over $i$ from $0$ to $n-1$ and taking into account that $|f'|$ is $s$–convex, we deduce, by the triangle inequality, that

$$\left| S_n(f, P) - \int_a^b f(x)\, dx \right| \leq \frac{5}{72} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3 \left[ |f'(x_i)| + |f'(x_{i+1})| \right].$$

which completes the proof. $\square$

**Proposition 5.5.2.** *Let $f' : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping on $I^{\circ}$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$, for some fixed $q \geq 1$, then in (5.5.2), for every division $P$ of $[a,b]$, the following holds:*

$$\left|E_n^S(f,d)\right| \leq \frac{5}{72}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\left[\left|f'\left(\frac{29x_{i+1}+61x_i}{90}\right)\right| + \left|f'\left(\frac{61x_{i+1}+29x_i}{90}\right)\right|\right].$$

*Proof.* The proof can be done similar to that of Proposition 5.5.1 and using the proof of Theorem 5.2.10. □

**Proposition 5.5.3.** *Let $f' : I \subseteq \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on $I^{\circ}$ such that $f'' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f''|$ is quasi-convex on $[a,b]$, then in (5.5.2), for every division $P$ of $[a,b]$, the following holds:*

$$\begin{aligned}\left|E_n^S(f,P)\right| &= \left|S_n(f,d) - \int_a^b f(x)\,dx\right| \\ &\leq \frac{1}{162}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3 \cdot \left[\max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, |f'(x_{i+1})|\right\}\right. \\ &\quad \left. + \max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, |f'(x_i)|\right\}\right].\end{aligned}$$

*Proof.* The proof can be done similar to that of Proposition 5.5.1 and using the proof of Theorem 5.3.2. □

**Proposition 5.5.4.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping on $I^{\circ}$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then in (5.5.2), for every division $P$ of $[a,b]$, the following holds:*

$$\left|E_n^S(f,P)\right| \leq \frac{1}{7776}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\left[211\,|f'(x_i)| + 239\,|f'(x_{i+1})|\right].$$

*Proof.* The proof can be done similar to that of Proposition 5.5.1 and using the proof of Theorem 5.4.4. □

## 5.6 APPLICATIONS TO SPECIAL MEANS

Let $s \in (0,1]$ and $u, v, w \in \mathbb{R}$. We define a function $f : [0,\infty) \to [0,\infty)$ as

$$f(t) = \begin{cases} u, & t = 0 \\ vt^s + w, & t > 0. \end{cases}$$

If $v \geq 0$ and $0 \leq w \leq u$, then $f \in K_s^2$ (see Example (2.2.17)). Hence, for $u = w = 0$, $v = 1$, we have $f : [a,b] \to [0,\infty)$, $f(t) = t^s$, $f \in K_s^2$.

In the following some new inequalities are derived for the above means.

1. Consider $f : [a,b] \to \mathbb{R}$, $(0 < a < b)$, $f(x) = x^s$, $s \in (0,1]$. Then,

$$\frac{1}{b-a}\int_a^b f(x)\,dx = L_s^s(a,b),$$

$$\frac{f(a)+f(b)}{2} = A(a^s, b^s),$$

$$f\left(\frac{a+b}{2}\right) = A^s(a,b).$$

(a) Using the inequality (5.2.2), we get

$$\left|\frac{1}{3}A(a^s,b^s) + \frac{2}{3}A^s(a,b) - L_s^s(a,b)\right|$$
$$\leq s(b-a)\frac{6^{-s} - 9(2)^{-s} + (5)^{s+2}6^{-s} + 3s - 12}{18(s^2+3s+2)}\left[|a|^{s-1} + |b|^{s-1}\right].$$

For instance, if $s = 1$ then we get

$$|A(a,b) - L(a,b)| \leq \frac{5}{72}(b-a).$$

(b) Using the inequality (5.2.4), we get

$$\left|\frac{1}{3}A(a^s,b^s) + \frac{2}{3}A^s(a,b) - L_s^s(a,b)\right|$$
$$\leq (b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\frac{s}{(s+1)^{1/q}}\left[\left(\left|a^{s-1}\right|^q + \left|A^{s-1}(a,b)\right|^q\right)^{1/q}\right.$$
$$\left. + \left(\left|A^{s-1}(a,b)\right|^q + \left|b^{s-1}\right|^q\right)^{1/q}\right].$$

where, $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. For instance, if $s=1$ then we get

$$\left|A(a,b)-L(a,b)\right|\leq 2^{-1/q}(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p},\quad p>1.$$

2. Consider $f:[a,b]\subseteq(0,\infty)\rightarrow\mathbb{R}$, $(0<a<b)$, $f(x)=\frac{1}{x^s}\in K_s^2$ (by Theorem 2.2.20), $s\in(0,1]$. Then,

$$\frac{1}{b-a}\int_a^b f(x)\,dx = L_{-s}^s(a,b),$$

$$\frac{f(a)+f(b)}{2}=A\left(a^{-s},b^{-s}\right),$$

$$f\left(\frac{a+b}{2}\right)=A^{-s}(a,b).$$

(a) Using the inequality (5.2.2), we get

$$\left|\frac{1}{3}A\left(a^{-s},b^{-s}\right)+\frac{2}{3}A^{-s}(a,b)-L_{-s}^s(a,b)\right|$$
$$\leq s(b-a)\frac{6^{-s}-9(2)^{-s}+(5)^{s+2}6^{-s}+3s-12}{18(s^2+3s+2)}\left[|a|^{-s-1}+|b|^{-s-1}\right].$$

For instance, if $s=1$ then we get

$$\left|\frac{1}{3}A\left(a^{-1},b^{-1}\right)+\frac{2}{3}A^{-1}(a,b)-L_{-1}(a,b)\right|\leq\frac{5}{72}(b-a)\left[|a|^{-2}+|b|^{-2}\right].$$

(b) Using the inequality (5.2.4), we get

$$\left|\frac{1}{3}A\left(a^{-s},b^{-s}\right)+\frac{2}{3}A^{-s}(a,b)-L_{-s}^s(a,b)\right|$$
$$\leq(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\frac{s}{(s+1)^{1/q}}\left[\left(\left|a^{-s-1}\right|^q+\left|A^{-s-1}(a,b)\right|^q\right)^{1/q}\right.$$
$$\left.+\left(\left|A^{-s-1}(a,b)\right|^q+\left|b^{-s-1}\right|^q\right)^{1/q}\right],$$

where, $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. For instance, if $s=1$ then we get

$$\left|\frac{1}{3}A\left(a^{-1},b^{-1}\right)+\frac{2}{3}A^{-1}(a,b)-L_{-1}(a,b)\right|$$
$$\leq(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\frac{1}{2^{1/q}}\left[\left(\left|a^{-2}\right|^q+\left|A^{-2}(a,b)\right|^q\right)^{1/q}\right.$$
$$\left.+\left(\left|A^{-2}(a,b)\right|^q+\left|b^{-2}\right|^q\right)^{1/q}\right],\quad p>1.$$

## 5.7 SUMMARY AND CONCLUSION

In the presented chapter, Simpson's type inequalities for convex, concave, $s$-convex (concave), quasi-convex, $r$-convex and log-convex mappings in terms of at most second derivative are established. In fact, the approaches considered here using convexities are considered for the first time. In the sections 5.2–5.4, the approaches that used are summed by writing the difference

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

in terms of $\int\limits_0^1 p(t)\, f'(ta+(1-t)\,b)\,dt$, where $p(t)$ is a suitable Peano kernel, after that using the convexity condition of $|f'|$ we obtain the desirable results. Several generalizations, refinements and improvements for the corresponding version for powers of these inequalities are considered by applying the Hölder and the power mean inequalities.

# CHAPTER VI

# FURTHER RESEARCH

## 6.1 INEQUALITIES FOR CONVEX MAPPINGS

In 1976, a generalization of Ostrowski's inequality for $n$–times differentiable mappings was proved by Mitrinović et al. (1994) (see Chapter II). In special case, they proved the following inequality regarding a twice differentiable mappings.

$$\left|\frac{1}{2}\left[f(x)+\frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{\|f''\|_\infty}{4}(b-a)^2\left[\frac{1}{12}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right] \quad (6.1.1)$$

for all $x \in [a,b]$, such that $f'' : (a,b) \to \mathbb{R}$ is bounded, i.e., $\|f''\|_\infty = \sup_{t\in[a,b]} |f''(t)| < \infty$.

In Cheng (2001), considered this inequality for differentiable mappings. Recently, a generalizations of Ostrowski type inequality for functions of Lipschitzian type are established in Liu (2007b). In future, we will study this inequality for convex mappings.

In 2002, Guessab and Schmeisser, studied the companion of Ostrowski inequality. Indeed, they have proved among others, the following companion of Ostrowski's inequality:

$$\left|\frac{f(x)+f(a+b-x)}{2}-\int_a^b f(t)\,dt\right| \le \left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)M \quad (6.1.2)$$

for any $x \in [a, \frac{a+b}{2}]$, where, $f$ is assumed to satisfies the Lipschitz condition on [a,b], i.e., $|f(t) - f(s)| \leq M|t - s|$, for all $t, s \in [a, b]$.

In Dragomir (2005), some companions of Ostrowski's integral inequality for absolutely continuous functions. Also, in Dragomir (2002), some inequalities for the this companion for mappings of bounded variation. Recently, Liu (2009), introduced some companions of an Ostrowski type integral inequality for functions whose derivatives are absolutely continuous. In future research, we will continue our study to consider this inequality for convex type mappings.

## 6.2 INEQUALITIES FOR TWO OR MORE VARIABLES

Although important for applications, numerical integration in two or more dimensions is still a much less developed area than its one-dimensional counterpart, which has been worked on intensively. In the recent study Hanna (2009), we find the author introduce some important inequalities of Ostrowski's type and used it to study some cubature rules from a generalized Taylor perspective. On the other hand, for Ostrowski, Hermite–Hadamard and Simpson inequalities the mappings of two or more variables which is of bounded variation, Lipschitzian, absolutely continuous and etc, have not been discussed yet. For further researches we refer the reader to Anastassiou (1997), Anastassiou (2002), Anastassiou and Goldstein (2007), Anastassiou (2007) and Hanna (2009)

Finally, we recommend other researchers to go inside this area to develop these and other types of inequalities and so that to reach a very interesting applications in the numerical integrations.

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# APPENDIX A

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