UPPER AND LOWER BOUNDS FOR CSISZÁR $f-$DIVERGENCE IN TERMS OF HELLINGER DISCRIMINATION AND APPLICATIONS

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Abstract. In this paper we point out an upper and a lower bound for the Csiszár $f-$divergence of two discrete random variables in terms of the Hellinger discrimination. Some paricular cases for the Kullback-Leibler distance, triangular discrimination, χ^2-distance and the Rényi $\alpha-$entropy, etc.. are considered.

1. Introduction

Given a convex function $f:[0,\infty)\to\mathbb{R}$, the *$f-$divergence functional*

$$I_f(p,q)=\sum_{i=1}^{n}q_i f\left(\frac{p_i}{q_i}\right), \tag{1.1}$$

was introduced by Csiszár [1]-[2] as a generalized measure of information, a *"distance function"* on the set of probability distribution $\mathbb{P}^n$. The restriction here to discrete distributions is only for convenience, similar results hold for general distributions. As in Csiszár [1]-[2], we interpret undefined expressions by

$$f(0)=\lim_{t\to 0+}f(t),\quad 0f\left(\tfrac{0}{0}\right)=0,$$
$$0f\left(\tfrac{a}{0}\right)=\lim_{\varepsilon\to 0+}\varepsilon f\left(\tfrac{a}{\varepsilon}\right)=a\lim_{t\to\infty}\tfrac{f(t)}{t},\ a>0.$$

The following results (Theorems 1 and 2, and Corollary 1) were essentially given by Csiszár and Körner [3].

Theorem 1. ***(Joint Convexity)*** *If $f:[0,\infty)\to\mathbb{R}$ is convex, then $I_f(p,q)$ is jointly convex in p and q.*

Theorem 2. ***(Jensen's inequality)*** *Let $f:[0,\infty)\to\mathbb{R}$ be convex. Then for any $p,q\in\mathbb{R}^n_+$ with $P_n:=\sum_{i=1}^{n}p_i>0$, $Q_n:=\sum_{i=1}^{n}q_i>0$, we have the inequality*

$$I_f(p,q)\geq Q_n f\left(\frac{P_n}{Q_n}\right). \tag{1.2}$$

If f is strictly convex, equality holds in (1.2) iff

$$\frac{p_1}{q_1}=\frac{p_2}{q_2}=...=\frac{p_n}{q_n}. \tag{1.3}$$

It is natural to consider the following corollary.

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Corollary 1. ***(Nonnegativity)*** *Let $f:[0,\infty)\to\mathbb{R}$ be convex and normalised, i.e.,*

$$f(1)=0. \tag{1.4}$$

Then for any $p,q\in\mathbb{R}_+^n$ with $P_n=Q_n$, we have the inequality

$$I_f(p,q)\geq 0. \tag{1.5}$$

If f is strictly convex, equality holds in (1.5) iff

$$p_i=q_i \quad \text{for all} \quad i\in\{1,...,n\}. \tag{1.6}$$

In particular, if p,q are probability vectors, then Corollary 1 shows that, for strictly convex and normalized $f:[0,\infty)\to\mathbb{R}$ that

$$I_f(p,q)\geq 0 \ \text{ and } I_f(p,q)=0 \ \text{ iff } p=q. \tag{1.7}$$

We now give some more examples of divergence measures in Information Theory which are particular cases of Csiszár $f-$divergences.

(1) **Kullback-Leibler distance** ([12]). The *Kullback-Leibler distance* $D(\cdot,\cdot)$ is defined by

$$D(p,q):=\sum_{i=1}^{n}p_i\log\left(\frac{p_i}{q_i}\right). \tag{1.8}$$

If we choose $f(t)=t\ln t$, $t>0$, then obviously

$$I_f(p,q)=D(p,q). \tag{1.9}$$

(2) **Variational distance** (l_1-distance). The *variational distance* $V(\cdot,\cdot)$ is defined by

$$V(p,q):=\sum_{i=1}^{n}|p_i-q_i|. \tag{1.10}$$

If we choose $f(t)=|t-1|$, $t\in\mathbb{R}_+$, then we have

$$I_f(p,q)=V(p,q). \tag{1.11}$$

(3) **Hellinger discrimination** ([13]). The *Hellinger discrimination* $h^2(\cdot,\cdot)$ is defined by

$$h^2(p,q):=\frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{p_i}-\sqrt{q_i}\right)^2. \tag{1.12}$$

It is obvious that if $f(t)=\frac{1}{2}\left(\sqrt{t}-1\right)^2$, then

$$I_f(p,q)=h^2(p,q). \tag{1.13}$$

(4) **Triangular discrimination** ([24]). We define *triangular discrimination* between p and q by

$$\Delta(p,q)=\sum_{i=1}^{n}\frac{|p_i-q_i|^2}{p_i+q_i}. \tag{1.14}$$

It is obvious that if $f(t)=\frac{(t-1)^2}{t+1}$, $t\in(0,\infty)$, then

$$I_f(p,q)=\Delta(p,q). \tag{1.15}$$

(5) χ^2-**distance.** We define the χ^2-*distance* (chi-square distance) by

$$D_{\chi^2}(p,q) := \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i}. \tag{1.16}$$

It is clear that if $f(t) = (t-1)^2$, $t \in [0,\infty)$, then

$$I_f(p,q) = D_{\chi^2}(p,q) \tag{1.17}$$

(6) **Rényi** $\alpha-$**order entropy** ([14]). The $\alpha-$*order entropy* $(\alpha > 1)$ is defined by

$$R_\alpha(p,q) := \sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha}. \tag{1.18}$$

It is obvious that if $f(t) = t^\alpha \ (t \in (0,\infty))$, then

$$I_f(p,q) = R_\alpha(p,q). \tag{1.19}$$

For other examples of divergence measures, see the paper [22] by J. N. Kapur where further references are given.

2. Some Inequalities Between Csiszár $f-$Divergence and Hellinger Discrimination

In the recent paper [28], the author proved the following inequality for Csiszár $f-$divergence:

Theorem 3. *Let $\Phi : \mathbb{R}_+ \to \mathbb{R}$ be differentiable convex. Then for all $p, q \in \mathbb{R}_+^n$ we have the inequality:*

$$\Phi'(1)(P_n - Q_n) \le I_\Phi(p,q) - Q_n\Phi(1) \le I_{\Phi'}\left(\frac{p^2}{q}, p\right) - I_{\Phi'}(p,q), \tag{2.1}$$

where $P_n := \sum_{i=1}^{n} p_i > 0$, $Q_n := \sum_{i=1}^{n} q_i > 0$ and $\Phi' : (0,\infty) \to \mathbb{R}$ is the derivative of Φ.
If Φ is strictly convex and $p_i,\ q_i > 0\ (i = 1, ..., n)$, then the equality holds in (2.9) iff $p = q$,

If we assume that $P_n = Q_n$ and Φ is normalised, then we obtain the simpler

$$0 \le I_\Phi(p,q) \le I_{\Phi'}\left(\frac{p^2}{q}, p\right) - I_{\Phi'}(p,q). \tag{2.2}$$

Applications for particular divergences which are instances of Csiszár $f-$divergence were also given.

Asimilar result of the above theorem has been presented in antoher paper by the author [29].

Theorem 4. *Let Φ, p, q be as in Theorem 3. Then we have the inequality*

$$0 \le I_\Phi(p,q) - Q_n\Phi\left(\frac{P_n}{Q_n}\right) \le I_{\Phi'}\left(\frac{p^2}{q}, p\right) - \frac{P_n}{Q_n} I_{\Phi'}(p,q). \tag{2.3}$$

If Φ is strictly convex and $p_i,\ q_i > 0\ (i = 1, ..., n)$, then the equality holds in (2.3) iff $\frac{p_1}{q_1} = ... = \frac{p_n}{q_n}$.

Obviously, if $P_n = Q_n$ and Φ is normalised, then (2.3) becomes (2.2).

The following result concerning an upper and a lower bound for the Csiszár $f-$divergence in terms of the Kullback-Leibler distance $D(p,q)$ holds.

As in [30], we will say that the mapping $f : C \subset \mathbb{R} \to \mathbb{R}$, where C is an interval, (in [30], the definition was considered in general normed spaces) is

(i) $\alpha-$*lower convex on* C if $f - \frac{\alpha}{2} \cdot |\cdot|^2$ is convex on C;
(ii) $\beta-$*upper convex* on C if $\frac{\beta}{2} \cdot |\cdot|^2 - f$ is convex on C;
(iii) $(m,M)-$*convex on* C (with $m \le M$) if it is both $m-$lower convex and $M-$upper convex.

In [30], amongst others, the author has proved the following result for Csiszár $f-$divergence.

Theorem 5. *Let* $\Phi : \mathbb{R}_+ \to \mathbb{R}$ *and* $p, q \in \mathbb{R}_+^n$ *with* $P_n = Q_n$.

(i) *If* Φ *is* α*-lower convex on* $\mathbb{R}_+$*, then we have the inequality*

$$\frac{\alpha}{2} \cdot D_{\chi^2}(p,q) \le I_\Phi(p,q) - Q_n \Phi(1). \tag{2.4}$$

(ii) *If* Φ *is* $\beta-$*upper convex on* $\mathbb{R}_+$*, then we have the inequality*

$$I_\Phi(p,q) - Q_n \Phi(1) \le \frac{\beta}{2} \cdot D_{\chi^2}(p,q). \tag{2.5}$$

(iii) *If* Φ *is* $(m,M)-$*convex on* $\mathbb{R}_+$*, then we have the following sandwich inequality*

$$\frac{m}{2} \cdot D_{\chi^2}(p,q) \le I_\Phi(p,q) - Q_n \Phi(1) \le \frac{M}{2} \cdot D_{\chi^2}(p,q), \tag{2.6}$$

where $D_{\chi^2}(\cdot,\cdot)$ *is the* χ^2-*divergence.*

Of course, if Φ is normalised, i.e., $\Phi(1) = 0$ and p, q are probability distributions, then we get the simpler inequalities:

$$\frac{\alpha}{2} \cdot D_{\chi^2}(p,q) \le I_\Phi(p,q), \quad I_\Phi(p,q) \le \frac{\beta}{2} \cdot D_{\chi^2}(p,q) \tag{2.7}$$

and

$$\frac{m}{2} \cdot D_{\chi^2}(p,q) \le I_\Phi(p,q) \le \frac{M}{2} \cdot D_{\chi^2}(p,q). \tag{2.8}$$

In [30], some applications for particular instances of Csiszár $f-$divergences were also given.

The following result concerning an upper and a lower bound for the Csiszár $f-$divergence in terms of the Hellinger discrimination $h^2(p,q)$ holds. These results will complement, in a sense, the ones presented above.

Theorem 6. *Assume that the generating mapping* $f : (0,\infty) \to \mathbb{R}$ *is normalized, i.e.,* $f(1) = 0$ *and satisfies the assumptions*

(i) f *is twice differentiable on* (r,R)*, where* $0 \le r \le 1 \le R \le \infty$*,*
(ii) *there exists the real constants* m, M *such that*

$$m \le t^{\frac{3}{2}} f''(t) \le M \quad \text{for all } t \in (r,R). \tag{2.9}$$

If p, q *are discrete probability distributions verifying the assumption*

$$r \le r_i := \frac{p_i}{q_i} \le R \quad \text{for all } i \in \{1,...,n\}, \tag{2.10}$$

then we have the inequality

$$4mh^2(p,q) \le I_f(p,q) \le 4Mh^2(p,q). \tag{2.11}$$

Proof. Define the mapping $H_m : (0,\infty) \to \mathbb{R}$, $H_m(t) = f(t) - 2m\left(\sqrt{t}-1\right)^2$. Then $H_m(\cdot)$ is normalised, twice differentiable and since

$$H_m''(t) = f''(t) - \frac{m}{t^{\frac{3}{2}}} = \frac{1}{t^{\frac{3}{2}}}\left(t^{\frac{3}{2}} f''(t) - m\right) \ge 0 \tag{2.12}$$

for all $t \in (a,b)$, implied by the first inequality in (2.9). Thus, the mapping $H_m(\cdot)$ is convex on (r,R).

Applying the nonnegativity property of the Csiszár $f-$divergence functional for $H_m(\cdot)$ and the linearity, we may state that

$$\begin{aligned} 0 &\le I_{H_m}(p,q) = I_f(p,q) - 2mI_{(\sqrt{\cdot}-1)^2}(p,q) \\ &= I_f(p,q) - 4mh^2(p,q), \end{aligned} \tag{2.13}$$

from where we get the first inequality in (2.11).

Define $H_M : (0,\infty) \to \mathbb{R}$, $H_M(t) = 2M\left(\sqrt{t}-1\right)^2 - f(t)$ which obviously is normalised, twice differentiable and, by (2.9), convex on (r,R).

Applying the nonnegativity property of Csiszár $f-$divergence for I_{H_M}, we obtain the second part of (2.11). □

The following theorem concerning the convexity property of the Csiszár $f-$divergence also holds.

Theorem 7. *Assume that f satisfies the assumptions (i) and (ii) from Theorem 6. If $p^{(j)}, q^{(j)}$ $(j = 1,2)$ are probability distributions satisfying (2.10), that is,*

$$r \le \frac{p_i^{(j)}}{q_i^{(j)}} \le R \quad \text{for all} \quad i \in \{1,...,n\} \quad \text{and} \quad j \in \{1,2\}, \tag{2.14}$$

then

$$r \le \frac{\lambda p_i^{(1)} + (1-\lambda)p_i^{(2)}}{\lambda q_i^{(1)} + (1-\lambda)q_i^{(2)}} \le R \quad \text{for all } i \in \{1,...,n\} \quad \text{and} \quad \lambda \in [0,1] \tag{2.15}$$

and

$$\begin{aligned} & 4m\big[h^2\left(\lambda p^{(1)} + (1-\lambda)p^{(2)}, \lambda q^{(1)} + (1-\lambda)q^{(2)}\right) \\ & -\lambda h^2\left(p^{(1)}, q^{(1)}\right) - (1-\lambda)h^2\left(p^{(2)}, q^{(2)}\right)\big] \\ \le\ & I_f\left(\lambda p^{(1)} + (1-\lambda)p^{(2)}, \lambda q^{(1)} + (1-\lambda)q^{(2)}\right) \\ & -\lambda I_f\left(p^{(1)}, q^{(1)}\right) - (1-\lambda)I_f\left(p^{(2)}, q^{(2)}\right) \\ \le\ & 4M\big[h^2\left(\lambda p^{(1)} + (1-\lambda)p^{(2)}, \lambda q^{(1)} + (1-\lambda)q^{(2)}\right) \\ & -\lambda h^2\left(p^{(1)}, q^{(1)}\right) - (1-\lambda)h^2\left(p^{(2)}, q^{(2)}\right)\big] \end{aligned} \tag{2.16}$$

for all $\lambda \in [0,1]$.

Proof. By (2.14), we have

$$r\lambda q_i^{(1)} \le \lambda p_i^{(1)} \le \lambda R q_i^{(1)} \quad \text{for all } i \in \{1,...,n\} \tag{2.17}$$

and

$$r(1-\lambda) q_i^{(2)} \le (1-\lambda) p_i^{(2)} \le R(1-\lambda) q_i^{(2)} \quad \text{for all } i \in \{1,...,n\}. \tag{2.18}$$

Summing the above (2.17) and (2.18), we obtain (2.15).

It is already known that the mappings H_m, H_M as defined in Theorem 6 are convex and normalised.

Applying the "Joint Convexity Principle" for $I_{H_m}(\cdot,\cdot)$, i.e.,

$$\begin{aligned} & I_{H_m}\left(\lambda\left(p^{(1)},q^{(1)}\right)+(1-\lambda)\left(p^{(2)},q^{(2)}\right)\right) \\ \le\ & \lambda I_{H_m}\left(p^{(1)},q^{(1)}\right)+(1-\lambda) I_{H_m}\left(p^{(2)},q^{(2)}\right) \end{aligned} \tag{2.19}$$

and rearranging the terms, we end up with the first inequality in (2.16).

The second inequality follows likewise if we apply the same property to the Csiszár $f-$divergence $I_{H_M}(\cdot,\cdot)$.

We omit the details. □

Remark 1. *If $m>0$ in (2.9), then the inequality (2.11) is a better result than the positivity property of Csiszár divergence. The same will apply for the joint convexity of Csiszár divergence if $m>0$.*

Using the inequality (2.2) which holds for Φ, a differentiable convex and normalised function, for p,q probability distributions, we can state the following theorem as well.

Theorem 8. *Let $f:[0,\infty)\to\mathbb{R}$ be a normalised mapping. That is, $f(1)=0$ and satisfies the assumption*

- *(i) f is twice differentiable on (r,R), where $0\le r\le 1\le R\le\infty$;*
- *(ii) there exists the constants m,M such that*

$$m \le t^{\frac{3}{2}} f''(t) \le M \quad \text{for all } t\in(r,R). \tag{2.20}$$

If p,q are discrete probability distributions verifying the assumption

$$r \le r_i := \frac{p_i}{q_i} \le R \quad \text{for all } i\in\{1,...,n\}, \tag{2.21}$$

then we have the inequality

$$\begin{aligned} & I_{f'}\left(\frac{p^2}{q},p\right) - I_{f'}(p,q) - 2MC(p,q) + 4Mh^2(p,q) \\ \le\ & I_f(p,q) \\ \le\ & I_{f'}\left(\frac{p^2}{q},p\right) - I_{f'}(p,q) - 2mC(p,q) + 4mh^2(p,q), \end{aligned} \tag{2.22}$$

where $C(p,q) := \sum_{i=1}^{n}(q_i-p_i)\sqrt{\frac{q_i}{p_i}}$.

Proof. We know (see the proof of Theorem 6), that the mapping $H_m:[0,\infty)\to\mathbb{R}$, $H_m(t) := f(t) - 2m\left(\sqrt{t}-1\right)^2$ is normalised, twice differentiable and convex on (r,R).

If we apply the second inequality from (2.2) for H_m, we may write

$$I_{H_m}(p,q) \le I_{H'_m}\left(\frac{p^2}{q},p\right) - I_{H'_m}(-p,q). \tag{2.23}$$

However,

$$\begin{aligned}
I_{H_m}(p,q) &= I_f(p,q) - 4mh^2(p,q), \\
I_{H'_m}\left(\frac{p^2}{q},p\right) &= I_{f'(\cdot)-4m\left(\frac{1}{2}-\frac{1}{2\sqrt{\cdot}}\right)}\left(\frac{p^2}{q},p\right) \\
&= I_{f'}\left(\frac{p^2}{q},p\right) - 2m + 2mI_{\frac{1}{\sqrt{\cdot}}}\left(\frac{p^2}{q},p\right) \\
&= I_{f'}\left(\frac{p^2}{q},p\right) - 2m + 2m\sum_{i=1}^{n} p_i\left(\frac{1}{\sqrt{\frac{p_i^2}{q_i}\cdot\frac{1}{p_i}}}\right) \\
&= I_{f'}\left(\frac{p^2}{q},p\right) - 2m + 2m\sum_{i=1}^{n} p_i\sqrt{\frac{q_i}{p_i}} \\
&= I_{f'}\left(\frac{p^2}{q},p\right) - 2m + 2m\sum_{i=1}^{n} \sqrt{p_iq_i}
\end{aligned}$$

and

$$\begin{aligned}
I_{H'_m}(p,q) &= I_{f'}(p,q) - 2m + 2mI_{\frac{1}{\sqrt{\cdot}}}(p,q) \\
&= I_{f'}(p,q) - 2m + 2m\sum_{i=1}^{n} q_i\left(\frac{1}{\sqrt{\frac{p_i}{q_i}}}\right) \\
&= I_{f'}(p,q) - 2m + 2m\sum_{i=1}^{n} q_i\sqrt{\frac{q_i}{p_i}}
\end{aligned}$$

and then, by (2.23), we obtain

$$\begin{aligned}
&I_f(p,q) - 4mh^2(p,q) \\
\le\ & I_{f'}\left(\frac{p^2}{q},p\right) - 2m + 2m\sum_{i=1}^{n} p_i\sqrt{\frac{q_i}{p_i}} - I_{f'}(p,q) + 2m - 2m\sum_{i=1}^{n} q_i\sqrt{\frac{q_i}{p_i}},
\end{aligned}$$

which is equivalent to the second inequality in (2.22).

If we consider $H_M(t) := 2M\left(\sqrt{t}-1\right)^2 - f(t)$, $t \ge 0$, then we observe that $H_M(\cdot)$ is normalised, twice differentiable and convex on (r,R). Applying the second inequality from (2.2), we deduce the first part of (2.22). □

The above results have natural applications when the Hellinger distance is compared with a number of other divergence measures arising in Information Theory.

3. Some Particular Cases

Using Theorem 6, we shall be able to point out the following particular cases which are of interest in Information Theory.

Proposition 1. *Let p,q be two probability distributions with the property that*

$$0<r\leq\frac{p_i}{q_i}=:q_i\leq R<\infty \quad \textit{for all} \quad i\in\{1,...,n\}. \tag{3.1}$$

Then we have the inequality

$$4\sqrt{r}h^2(p,q)\leq D(p,q)\leq 4\sqrt{R}h^2(p,q). \tag{3.2}$$

Proof. Consider the mapping $f:(0,\infty)\to\mathbb{R}$, $f(t)=t\ln t$. Then

$$f''(t)=\frac{1}{t},\ t\in(0,\infty).$$

Consider the mapping $g:[r,R]\to\mathbb{R}$, $g(t)=t^{\frac{3}{2}}\cdot\frac{1}{t}=t^{\frac{1}{2}}$. Then

$$\inf_{t\in[r,R]}g(t)=\sqrt{r},\quad \sup_{t\in[r,R]}g(t)=\sqrt{R}.$$

Therefore, applying the inequality (2.11) with $m=\sqrt{r}$, $M=\sqrt{R}$, we obtain (3.2). □

Remark 2. *The following inequality is well known in the literature (see for example Dacunha-Castelle* [25]*):*

$$D(p,q)\geq 2h^2(p,q) \tag{3.3}$$

for any p,q probability distributions.

From the first inequality in (3.2) we have the inequality

$$D(p,q)\geq 4\sqrt{r}h^2(p,q) \tag{3.4}$$

We remark that if $4\sqrt{r}\geq 2$, i.e., $r\geq\frac{1}{4}$, then the inequality (3.4) is better than (3.3).

The following proposition also holds.

Proposition 2. *Let p,q be two probability distributions with the property (3.1). Then we have the inequality:*

$$\frac{4}{\sqrt{R}}h^2(p,q)\leq D(q,p)\leq\frac{4}{\sqrt{r}}h^2(p,q). \tag{3.5}$$

Proof. Consider the mapping $f:[r,R]\to\mathbb{R}$, $f(t)=-\ln t$. Define $g(t)=t^{\frac{3}{2}}f''(t)=\frac{1}{\sqrt{t}}$. Then obviously

$$\sup_{t\in[r,R]}g(t)=\frac{1}{\sqrt{r}},\quad \inf_{t\in[r,R]}g(t)=\frac{1}{\sqrt{R}}.$$

In addition,

$$I_f(p,q)=-\sum_{i=1}^n q_i\ln\left(\frac{p_i}{q_i}\right)=\sum_{i=1}^n q_i\ln\left(\frac{q_i}{p_i}\right)=D(q,p).$$

Now, using the inequality (3.2), we get the desired inequality (3.5). □

The following result for the χ^2-distance also holds.

Proposition 3. *Let p,q be two probability distributions satisfying the condition (3.1). Then we have the inequality:*

$$8r^{\frac{3}{2}}h^2(p,q)\leq D_{\chi^2}(p,q)\leq 8R^{\frac{3}{2}}h^2(p,q). \tag{3.6}$$

Proof. Consider the mapping $f:(0,\infty)\to\mathbb{R}$, $f(t)=(t-1)^2$. Define $g:[r,R]\to\mathbb{R}$, $g(t)=t^{\frac{3}{2}}f''(t)=2t^{\frac{3}{2}}$. Obviously,

$$\sup_{t\in[r,R]}g(t)=2R^{\frac{3}{2}}\quad\text{and}\quad\inf_{t\in[r,R]}g(t)=2r^{\frac{3}{2}}.$$

Since

$$I_f(p,q)=D_{\chi^2}(p,q),$$

then, applying the inequality (2.11) with $m=2r^{\frac{3}{2}}$, $M=2R^{\frac{3}{2}}$, we get the desired inequality (3.6). □

Now, let us consider the $J-$*divergence* [26]

$$J(p,q)=\sum_{i=1}^{n}(p_i-q_i)\ln\left(\frac{p_i}{q_i}\right).$$

The following proposition holds.

Proposition 4. *Let p,q be two probability distributions. Then we have the inequality*

$$8h^2(p,q)\le J(p,q). \tag{3.7}$$

Proof. Consider the mapping $f:(0,\infty)\to\mathbb{R}$, $f(t)=(t-1)\ln t$. Define $g:[r,R]\to\mathbb{R}$,

$$g(t)=t^{\frac{3}{2}}f''(t)=t^{\frac{1}{2}}+\frac{1}{t^{\frac{1}{2}}}\ge 2,$$

which shows that

$$\inf_{t\in(0,\infty)}g(t)=2.$$

Since

$$I_f(p,q)=J(p,q),$$

then, applying the inequality (2.11) with $m=2$, we get the desired inequality. □

If we know more about $r_i:=\frac{p_i}{q_i}$ $(i=1,...,n)$, i.e., the condition (3.1) holds, then we can point out an upper bound for the $J(\cdot,\cdot)$ as follows:

Proposition 5. *If $0<r\le r_i\le R<\infty$ for all $i\in\{1,...,n\}$, then we have the inequality*

$$J(p,q)\le 4\max\left\{\sqrt{r}+\frac{1}{\sqrt{r}},\sqrt{R}+\frac{1}{\sqrt{R}}\right\}h^2(p,q). \tag{3.8}$$

Proof. As above, we have

$$g(t)=t^{\frac{1}{2}}+\frac{1}{t^{\frac{1}{2}}}.$$

For the mapping $h(u)=u+\frac{1}{u}$, we have

$$h'(u)=\frac{u^2-1}{u^2},$$

which shows that the mapping is strictly decreasing on $(0,1)$ and strictly increasing on $(1,\infty)$ Therefore

$$\sup_{t\in[r,R]}g(t)=\max[g(r),g(R)]=\max\left\{\sqrt{r}+\frac{1}{\sqrt{r}},\sqrt{R}+\frac{1}{\sqrt{R}}\right\}.$$

Applyig Theorem 6 we deduce the desired result. □

Remark 3. *We observe that*

$$\sqrt{R}+\frac{1}{\sqrt{R}}-\sqrt{r}-\frac{1}{\sqrt{r}}=\frac{\left(\sqrt{R}-\sqrt{r}\right)\left(\sqrt{rR}-1\right)}{\sqrt{rR}}$$

and then the inequality (3.8) can be rewritten in the equivalent form as

$$J\left(p,q\right)\leq 4h^{2}\left(p,q\right)\times\begin{cases}\sqrt{R}+\frac{1}{\sqrt{R}} & \text{if} \quad R\geq\frac{1}{r}\\ \\ \sqrt{r}+\frac{1}{\sqrt{r}} & \text{if} \quad 1\leq R<\frac{1}{r}\end{cases}. \tag{3.9}$$

Let us now consider the *harmonic distance*

$$M\left(p,q\right):=\sum_{i=0}^{n}\frac{2p_{i}q_{i}}{p_{i}+q_{i}}.$$

The following proposition holds.

Proposition 6. *Let p,q be two probability distributions. Then we have the inequality*

$$0\leq 1-M\left(p,q\right)\leq\frac{1}{2}h^{2}\left(p,q\right). \tag{3.10}$$

Proof. Consider the function $f:(0,\infty)\to\mathbb{R}$, $f\left(t\right)=1-\frac{2t}{t+1}$. Then

$$f'\left(t\right)=-\frac{2}{\left(1+t\right)^{2}},\ f''\left(t\right)=\frac{4}{\left(t+1\right)^{3}}.$$

Define the mapping

$$g\left(t\right)=t^{\frac{3}{2}}f''\left(t\right)=\frac{4t^{\frac{3}{2}}}{\left(t+1\right)^{3}}.$$

A simple calculation shows that

$$g'\left(t\right)=\frac{6\sqrt{t}\left(1-t\right)}{\left(t+1\right)^{4}}.$$

Consequently, the mapping g is increasing on the interval $(0,1)$ and decreasing on $(1,\infty)$. Moreover,

$$\sup_{t\in(0,\infty)}=g\left(1\right)=\frac{1}{2}\ \text{ and }\ I_{f}\left(p,q\right)=1-M\left(p,q\right).$$

Applying the inequality (2.11) for $M=\frac{1}{2}$, we deduce (3.10). □

If we know more about $r_i:=\frac{p_i}{q_i}$, that is, the condition (3.1) holds, then we can improve the first inequality in (3.10) as follows.

Proposition 7. *Assume that the probability distributions p,q satisfy (3.1). Then we have the inequality*

$$16\min\left\{\frac{r^{\frac{3}{2}}}{\left(r+1\right)^{3}},\frac{R^{\frac{3}{2}}}{\left(R+1\right)^{3}}\right\}h^{2}\left(p,q\right)\leq 1-M\left(p,q\right). \tag{3.11}$$

Proof. Taking into account that the mapping $g\left(t\right)=\frac{4t^{\frac{3}{2}}}{\left(t+1\right)^{3}}$ is monotonic increasing on $(0,1)$ and decreasing on $(1,\infty)$, we may assert that

$$\inf_{t\in[r,R]}g\left(t\right)=\min\left\{g\left(r\right),g\left(R\right)\right\}=4\min\left\{\frac{r^{\frac{3}{2}}}{\left(r+1\right)^{3}},\frac{R^{\frac{3}{2}}}{\left(R+1\right)^{3}}\right\}.$$

Using the inequality (2.11), we deduce the desired lower bound (3.11). □

Remark 4. *Similar results can be stated by Applying Theorem 8, but we omit the details.*

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