Inequalities for Random Variables Over a Finite Interval

Neil S. Barnett

Pietro Cerone

Sever S. Dragomir

School of Computer Science & Mathematics, Victoria University, PO Box 14428, MC 8001, Melbourne, Victoria, Australia

E-mail address: {neil,pc,sever}@csm.vu.edu.au

URL: http://rgmia.vu.edu.au

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Preface

A chapter in the book "*Inequalities Involving Functions and Their Integrals and Derivatives*", Kluwer Academic Publishers, 1991, by Mitrinović, Pečarić and Fink is devoted to integral inequalities involving functions with bounded derivatives, or, Ostrowski type inequalities. This topic has now become a special domain in the Theory of Inequalities, there having been published many powerful results and a large number of applications in Numerical Integration, Probability Theory and Statistics, Information Theory and Integral Operator Theory.

The first monograph devoted to Ostrowski type inequalities and applications for quadrature rules was written by members of the Research Group in Mathematical Inequalities and Applications (RGMIA, see http://rgmia.vu.edu.au) in 2002. The book was entitled "*Ostrowski Type Inequalities and Applications in Numerical Integration*", edited by S.S. Dragomir & Th. M. Rassias, Kluwer Academic Publishers. The main aim of this monograph was to present some selected results of Ostrowski type inequalities for univariate and multivariate real functions and their natural application to the error analysis of numerical quadrature for both simple and multiple integrals as well as for the Riemann-Stieltjes integral. Due to space limitations, however, no attempt was made to present applications in other domains, more specifically, in Probability Theory.

It can be observed that Ostrowski type inequalities may also be successfully used to obtain various tight bounds for the expectation, variance and moments of continuous random variables defined over a finite interval. This had been noted in the late 1990's by many authors including members of the RGMIA located at Victoria University, Melbourne, Australia (see for instance the RGMIA Res. Rep. Coll., http://rgmia.vu.edu.au/reports.html for the years 1998-1999). The domain is now rich with results whose beneficial value will increase by being presented in a unified manner. This will then provide to all interested in Inequalities in Applied Probability Theory & Statistics, a primer of results and techniques that may well need further attention and polishing so as to obtain the best possible bounds and estimates.

It is from this view point that the current book is written and it is intended to be useful to both graduate students and established researchers working in Probability Theory & Statistics, Analytic Integral Inequalities and their applications in demography, economics, physics, biology, and other scientific areas.

The chapter outlines are given below and it is intended that they can be read independently if desired.

The first two chapters are concerned with natural applications to cumulative distribution functions (CDFs) and expectations for random variables (RVs) over a finite interval. The results use the latest Ostrowski type integral inequalities for functions that are of: bounded variation, convex, Hölder continuous, Lipschitzian or absolutely continuous. The tools used are both the Riemann-Stieltjes integral and the Lebesgue integral. Chapter 3 investigates the use of trapezoidal or corrected trapezoidal type inequalities developed recently in parallel with Ostrowski type inequalities for various classes of functions including the ones mentioned previously, but also for classes of much smoother functions whose second, third or fourth derivatives belong to the Lebesgue spaces L_p for $p = 1$. Chapter 4, deals with Grüss type or pre-Grüss type integral inequalities which provide error bounds for approximating the integral mean of a product (of two functions) in terms of the product of the integral means (for each individual function). Such inequalities are useful when the integral means of the individual functions are known or are more convenient to calculate. They also provide more accurate approximations, since the bounds are expressed in terms of the oscillation of a function rather than its sup norm that is usually not as tight. Utilising this type of estimate, various bounds for mathematical expressions incorporating the CDFs and the expectations are provided. Elementary and simple-looking bounds for the variance of continuous RVs are presented in Chapter 5. The tools used here are mostly Grüss and pre-Grüss type inequalities and some recent results obtained by the authors in connection with the problem of bounding the Čebyšev functional in its integral version over finite intervals in terms of various quantities and under certain assumptions for the involved integrable functions. Finally, in Chapter 6, by employing Taylor type expansions for n-time differentiable CDFs, various bounds involving the variance of a continuous random variable defined on a finite interval that are more accurate in terms of order of convergence, are outlined.

The book is self-contained in the sense that the reader needs only to be familiar with basic real analysis, integration theory and probability theory. All inequalities used in the text are explicitly stated and appropriately referenced. A comprehensive list of references on which the book is based is presented, complemented by other relevant literature that will allow the interested reader to be introduced to open problems, including the necessity to extend some of the obtained results to probability density functions defined on unbounded intervals.

Last, but no means least, the authors would like especially thank Professor George Anastassiou from Memphis University for his constant encouragement to write the book and whose numerous comments have been implemented in the final version.

The Authors
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CHAPTER 1

Ostrowski Type Inequalities for CDFs

1. An Inequality of the Ostrowski Type for CDFs

1.1. Inequalities. Let X be a random variable taking values in the finite interval $[a,b]$, with *cumulative distribution function* (CDF) $F(x) = \Pr(X \leq x)$.

The following theorem holds [**9**].

THEOREM 1. *Let X and F be as above, then we have the inequality*

$$\begin{aligned}(1.1)\quad & \left|\Pr(X \leq x) - \frac{b - E(X)}{b-a}\right| \\ & \leq \frac{1}{b-a}\left[[2x-(a+b)]\Pr(X \leq x) + \int_a^b \operatorname{sgn}(t-x)\,F(t)\,dt\right] \\ & \leq \frac{1}{b-a}\left[(b-x)\Pr(X \geq x) + (x-a)\Pr(X \leq x)\right] \\ & \leq \frac{1}{2} + \frac{\left|x - \frac{a+b}{2}\right|}{(b-a)}\end{aligned}$$

for all $x \in [a,b]$. All the inequalities in (1.1) *are sharp and the constant $\frac{1}{2}$ is the best possible.*

PROOF. Consider the kernel $p : [a,b]^2 \to \mathbb{R}$ given by

$$(1.2)\qquad p(x,t) := \begin{cases} t-a & \text{if } t \in [a,x] \\ t-b & \text{if } t \in (x,b] \end{cases}.$$

The Riemann-Stieltjes integral $\int_a^b p(x,t)\,dF(t)$ exists for any $x \in [a,b]$ and the formula of integration by parts for Riemann-Stieltjes integral gives:

$$\begin{aligned}(1.3)\quad & \int_a^b p(x,t)\,dF(t) \\ & = \int_a^x (t-a)\,dF(t) + \int_x^b (t-b)\,dF(t)\end{aligned}$$

$$= (t-a) F(t)\big|_a^x - \int_a^x F(t)\,dt + (t-b) F(t)\big|_x^b - \int_x^b F(t)\,dt$$
$$= (b-a) F(x) - \int_a^b F(t)\,dt.$$

On the other hand, the integration by parts formula for the Riemann-Stieltjes integral also gives:

$$E(X) := \int_a^b t dF(t) = tF(t)\big|_a^b - \int_a^b F(t)\,dt \tag{1.4}$$
$$= bF(b) - aF(a) - \int_a^b F(t)\,dt$$
$$= b - \int_a^b F(t)\,dt.$$

Now, using (1.3) and (1.4), we get the equality

$$(b-a) F(x) + E(X) - b = \int_a^b p(x,t)\,dF(t) \tag{1.5}$$

for all $x \in [a,b]$.

Now, assume that $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \to \infty$, where

$$\nu(\Delta_n) := \max\left\{x_{i+1}^{(n)} - x_i^{(n)} : i = 0, ..., n-1\right\}.$$

If $p : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and $\nu : [a,b] \to \mathbb{R}$ is monotonic nondecreasing, then the Riemann-Stieltjes integral $\int_a^b p(x)\,d\nu(x)$ exists and

$$\left|\int_a^b p(x)\,d\nu(x)\right| = \left|\lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right)\left[\nu\left(x_{i+1}^{(n)}\right) - \nu\left(x_i^{(n)}\right)\right]\right| \tag{1.6}$$
$$\leq \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left|p\left(\xi_i^{(n)}\right)\right| \left(\nu\left(x_{i+1}^{(n)}\right) - \nu\left(x_i^{(n)}\right)\right)$$
$$= \int_a^b |p(x)|\,d\nu(x).$$

Using (1.6) we have:

$$\begin{aligned}
&\left|\int_a^b p(x,t)\,dF(t)\right| \\
&= \left|\int_a^x (t-a)\,dF(t) + \int_x^b (t-b)\,dF(t)\right| \\
&\leq \left|\int_a^x (t-a)\,dF(t)\right| + \left|\int_x^b (t-b)\,dF(t)\right| \\
&\leq \int_a^x |t-a|\,dF(t) + \int_x^b |t-b|\,dF(t) \\
&= \int_a^x (t-a)\,dF(t) + \int_x^b (b-t)\,dF(t) \\
&= (t-a)\,F(t)\big|_a^x - \int_a^x F(t)\,dt \\
&\qquad - (b-t)\,F(t)\big|_x^b + \int_x^b F(t)\,dt \\
&= \left[[2x-(a+b)]\,F(x) - \int_a^x F(t)\,dt + \int_x^b F(t)\,dt\right] \\
&= [2x-(a+b)]\,F(x) + \int_a^b \operatorname{sgn}(t-x)\,F(t)\,dt.
\end{aligned} \tag{1.7}$$

Using the identity (1.5) and the inequality (1.7), we deduce the first part of (1.1).

We know that

$$\int_a^b \operatorname{sgn}(t-x)\,F(t)\,dt = -\int_a^x F(t)\,dt + \int_x^b F(t)\,dt.$$

As $F(\cdot)$ is monotonic nondecreasing on $[a,b]$, we can state that

$$\int_a^x F(t)\,dt \geq (x-a)\,F(a) = 0$$

and

$$\int_x^b F(t)\,dt \leq (b-x)\,F(b) = b-x$$

and then

$$\int_a^b \operatorname{sgn}(t-x)\,F(t)\,dt \leq b-x \qquad \text{for all } x \in [a,b].$$

Consequently, we have the inequality

$$\begin{aligned}
&[2x-(a+b)]\,F(x)+\int_a^b \operatorname{sgn}(t-x)\,F(t)\,dt \\
&\qquad \le [2x-(a+b)]\,F(x)+(b-x) \\
&\qquad = (b-x)\,(1-F(x))+(x-a)\,F(x) \\
&\qquad = (b-x)\Pr(X\ge x)+(x-a)\Pr(X\le x)
\end{aligned}$$

and the second part of (1.1) is proved.

Finally,

$$\begin{aligned}
&(b-x)\Pr(X\ge x)+(x-a)\Pr(X\le x) \\
&\qquad \le \max\{b-x,x-a\}\,[\Pr(X\ge x)+\Pr(X\le x)] \\
&\qquad = \frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|
\end{aligned}$$

and the last part of (1.1) is also proved.

We now assume that the inequality (1.1) holds for a constant $c>0$ instead of $\frac{1}{2}$, then,

$$\begin{aligned}
(1.8)\qquad &\left|\Pr(X\le x)-\frac{b-E(X)}{b-a}\right| \\
&\le \frac{1}{b-a}\left[[2x-(a+b)]\Pr(X\le x)+\int_a^b \operatorname{sgn}(t-x)\,F(t)\,dt\right] \\
&\le \frac{1}{b-a}\left[(b-x)\Pr(X\ge x)+(x-a)\Pr(X\le x)\right] \\
&\le c+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}
\end{aligned}$$

for all $x\in[a,b]$.

Choose the random variable X such that $F:[0,1]\to\mathbb{R}$,

$$F(x):=\begin{cases} 0 & \text{if } x=0 \\ \\ 1 & \text{if } x\in(0,1] \end{cases}.$$

We then have:

$$E(X)=0,\qquad \int_0^1 \operatorname{sgn}(t)\,F(t)\,dt=1$$

and by (1.8), for $x=0$, we get

$$1\le c+\frac{1}{2}$$

which shows that $c=\frac{1}{2}$ is the best possible value. ∎

REMARK 1. *Taking into account the fact that*

$$\Pr(X \geq x) = 1 - \Pr(X \leq x),$$

then from (1.1) *we get the equivalent inequality*

$$\begin{aligned}(1.9)\quad & \left|\Pr(X \geq x) - \frac{E(X) - a}{b-a}\right| \\ & \leq \frac{1}{b-a}\left[[2x-(a+b)]\Pr(X \leq x) + \int_a^b \operatorname{sgn}(t-x) F(t)\,dt\right] \\ & \leq \frac{1}{b-a}\left[(b-x)\Pr(X \geq x) + (x-a)\Pr(X \leq x)\right] \\ & \leq \frac{1}{2} + \frac{\left|x - \frac{a+b}{2}\right|}{b-a}\end{aligned}$$

for all $x \in [a,b]$.

REMARK 2. *The following particular cases are also interesting:*

$$(1.10)\quad \left|\Pr\left(X \leq \frac{a+b}{2}\right) - \frac{b - E(X)}{b-a}\right| \leq \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) F(t)\,dt \leq \frac{1}{2}$$

and

$$(1.11)\quad \left|\Pr\left(X \geq \frac{a+b}{2}\right) - \frac{E(X) - a}{b-a}\right| \leq \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) F(t)\,dt \leq \frac{1}{2}.$$

The following corollary can be useful in practice (see also [**9**]).

COROLLARY 1. *Under the above assumptions, we have*

$$(1.12)\quad \frac{1}{b-a}\left[\frac{a+b}{2} - E(X)\right] \leq \Pr\left(X \leq \frac{a+b}{2}\right) \leq \frac{1}{b-a}\left[\frac{a+b}{2} - E(X)\right] + 1.$$

PROOF. From the inequality (1.10), we get

$$-\frac{1}{2} + \frac{b - E(X)}{b-a} \leq \Pr\left(X \leq \frac{a+b}{2}\right) \leq \frac{1}{2} + \frac{b - E(X)}{b-a}.$$

But

$$-\frac{1}{2}+\frac{b-E(X)}{b-a} = \frac{-b+a+2b-2E(X)}{2(b-a)} = \frac{1}{b-a}\left[\frac{a+b}{2}-E(X)\right]$$

and

$$\begin{aligned}\frac{1}{2}+\frac{b-E(X)}{b-a} &= 1+\frac{b-E(X)}{b-a}-\frac{1}{2} \\ &= 1+\frac{2b-2E(X)-b+a}{2(b-a)} \\ &= 1+\frac{1}{b-a}\left[\frac{a+b}{2}-E(X)\right]\end{aligned}$$

and the inequality is proved. ∎

Remark 3. *Let* $0 \le \varepsilon \le 1$, *and assume that*

$$E(X) \ge \frac{a+b}{2}+(1-\varepsilon)(b-a), \tag{1.13}$$

then

$$\Pr\left(X \le \frac{a+b}{2}\right) \le \varepsilon. \tag{1.14}$$

Indeed, if (1.13) holds, then by the right-hand side of (1.12) we get

$$\begin{aligned}\Pr\left(X \le \frac{a+b}{2}\right) &\le \frac{1}{b-a}\left[\frac{a+b}{2}-E(X)\right]+1 \\ &\le \frac{(\varepsilon-1)(b-a)}{b-a}+1=\varepsilon.\end{aligned}$$

Remark 4. *Also, if*

$$E(X) \le \frac{a+b}{2}-\varepsilon(b-a) \tag{1.15}$$

then, by the right-hand side of (1.12),

$$\begin{aligned}\Pr\left(X \le \frac{a+b}{2}\right) &\ge \left[\frac{a+b}{2}-E(X)\right]\cdot\frac{1}{b-a} \\ &\ge \frac{\varepsilon(b-a)}{(b-a)}=\varepsilon\end{aligned}$$

and so

$$\Pr\left(X \le \frac{a+b}{2}\right) \ge \varepsilon, \qquad \varepsilon \in [0,1]. \tag{1.16}$$

The following corollary is also interesting (see also [**9**]):

COROLLARY 2. *Under the assumptions of Theorem 1,*

$$\frac{1}{b-x}\int_a^b\left[\frac{1+\operatorname{sgn}(t-x)}{2}\right]F(t)\,dt \geq \Pr(X\geq x) \geq \frac{1}{x-a}\int_a^b\left[\frac{1-\operatorname{sgn}(t-x)}{2}\right]F(t)\,dt \tag{1.17}$$

for all $x\in(a,b)$.

PROOF. From the inequality (1.1) we have

$$\Pr(X\leq x)-\frac{b-E(X)}{b-a} \leq \frac{1}{b-a}\left[[2x-(a+b)]\Pr(X\leq x)+\int_a^b\operatorname{sgn}(t-x)F(t)\,dt\right]$$

which is equivalent to

$$(b-a)\Pr(X\leq x)-[2x-(a+b)]\Pr(X\leq x) \leq b-E(X)+\int_a^b\operatorname{sgn}(t-x)F(t)\,dt,$$

that is,

$$2(b-x)\Pr(X\leq x)\leq b-E(X)+\int_a^b\operatorname{sgn}(t-x)F(t)\,dt.$$

As

$$b-E(X)=\int_a^b F(t)\,dt$$

then from the above inequality we deduce the first part of (1.17).

The second part follows by a similar argument from

$$\Pr(X\leq x)-\frac{b-E(X)}{b-a} \geq -\frac{1}{b-a}\left[[2x-(a+b)]\Pr(X\leq x)+\int_a^b\operatorname{sgn}(t-x)F(t)\,dt\right].$$

The details are omitted. ∎

REMARK 5. *If we put* $x = \frac{a+b}{2}$ *in* (1.17), *then we get*

$$\begin{aligned}(1.18)\quad & \frac{1}{b-a}\int_a^b \left[1+\operatorname{sgn}\left(t-\frac{a+b}{2}\right)\right] F(t)\,dt \\ & \geq \Pr\left(X \geq \frac{a+b}{2}\right) \\ & \geq \frac{1}{b-a}\int_a^b \left[1-\operatorname{sgn}\left(t-\frac{a+b}{2}\right)\right] F(t)\,dt.\end{aligned}$$

1.2. Applications for a Beta Random Variable. A Beta random variable X with parameters (p,q) has the probability density function

$$f(x;p,q) := \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}; \qquad 0<x<1,$$

where $\Omega = \{(p,q) : p,q>0\}$ and $B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,dt$.

We have, further, that

$$E(X) = \frac{1}{B(p,q)}\int_0^1 x\cdot x^{p-1}(1-x)^{q-1}\,dx = \frac{B(p+1,q)}{B(p,q)},$$

and so

$$E(X) = \frac{p}{p+q}.$$

Let X be a Beta random variable with parameters (p,q). Then we have:

$$\left|\Pr(X\leq x) - \frac{q}{p+q}\right| \leq \frac{1}{2} + \left|x-\frac{1}{2}\right|$$

and

$$\left|\Pr(X\geq x) - \frac{p}{p+q}\right| \leq \frac{1}{2} + \left|x-\frac{1}{2}\right|$$

for all $x\in[0,1]$ and particularly

$$\left|\Pr\left(X\leq\frac{1}{2}\right) - \frac{q}{p+q}\right| \leq \frac{1}{2}$$

and

$$\left|\Pr\left(X\geq\frac{1}{2}\right) - \frac{p}{p+q}\right| \leq \frac{1}{2}$$

respectively.

The proof follows by application of Theorem 1.

2. Random Variables whose PDFs Belong to $L_{\infty}[a,b]$

2.1. Inequalities. Let X be a random variable with the probability density function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}_{+}$ and with cumulative distribution function $F(x)=\Pr(X\leq x)$.

The following theorem holds [**13**].

THEOREM 2. *Let* $f\in L_{\infty}[a,b]$ *and put* $\|f\|_{\infty}=\sup_{t\in[a,b]}f(t)<\infty$. *Then we have the inequality:*

$$\left|\Pr(X\leq x)-\frac{b-E(X)}{b-a}\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^{2}}{(b-a)^{2}}\right](b-a)\|f\|_{\infty} \tag{1.19}$$

or equivalently,

$$\left|\Pr(X\geq x)-\frac{E(X)-a}{b-a}\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^{2}}{(b-a)^{2}}\right](b-a)\|f\|_{\infty} \tag{1.20}$$

for all $x\in[a,b]$. *The constant* $\frac{1}{4}$ *in* (1.19) *and* (1.20) *is sharp.*

PROOF. Let x, $y\in[a,b]$, then

$$|F(x)-F(y)|=\left|\int_{x}^{y}f(t)\,dt\right|\leq|x-y|\,\|f\|_{\infty}$$

which shows that F is $\|f\|_{\infty}$ $-$Lipschitzian on $[a,b]$.

Consider the kernel $p:[a,b]^{2}\to\mathbb{R}$ given by (1.2). The Riemann-Stieltjes integral $\int_{a}^{b}p(x,t)\,dF(t)$ exists for any $x\in[a,b]$ and the formula of integration by parts for Riemann-Stieltjes integral gives:

$$\int_{a}^{b}p(x,t)\,dF(t)=(b-a)F(x)-\int_{a}^{b}F(t)\,dt. \tag{1.21}$$

The integration by parts formula also gives

$$E(X)=b-\int_{a}^{b}F(t)\,dt. \tag{1.22}$$

Now, using (1.21) and (1.22), we get the equality

$$(b-a)F(x)+E(X)-b=\int_{a}^{b}p(x,t)\,dF(t), \tag{1.23}$$

for all $x\in[a,b]$.

Now, assume that

$$\Delta_{n}:a=x_{0}^{(n)}<x_{1}^{(n)}<...<x_{n-1}^{(n)}<x_{n}^{(n)}=b$$

is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \to \infty$, where

$$\nu(\Delta_n) := \max\left\{x_{i+1}^{(n)} - x_i^{(n)} : i = 0, ..., n-1\right\}.$$

If $p : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ and $\nu : [a,b] \to \mathbb{R}$ is L-Lipschitzian (Lipschitzian with the constant L), then we have

$$\begin{aligned}
(1.24) \quad & \left|\int_a^b p(x)\,d\nu(x)\right| \\
&= \left|\lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right)\left[\nu\left(x_{i+1}^{(n)}\right) - \nu\left(x_i^{(n)}\right)\right]\right| \\
&\le \lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1}\left|p\left(\xi_i^{(n)}\right)\right|\left(x_{i+1}^{(n)} - x_i^{(n)}\right)\left|\frac{\nu\left(x_{i+1}^{(n)}\right) - \nu\left(x_i^{(n)}\right)}{x_{i+1}^{(n)} - x_i^{(n)}}\right| \\
&\le L\lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1}\left|p\left(\xi_i^{(n)}\right)\right|\left(x_{i+1}^{(n)} - x_i^{(n)}\right) = L\int_a^b |p(x)|\,dx.
\end{aligned}$$

Applying the inequality (1.24) for the mappings $p(x,\cdot)$ and $F(\cdot)$,we get

$$\begin{aligned}
\left|\int_a^b p(x,t)\,dF(t)\right| &\le \|f\|_\infty \int_a^b |p(x,t)|\,dt \\
&= \|f\|_\infty\left[\int_a^x (t-a)\,dt + \int_x^b (b-t)\,dt\right] \\
&= \|f\|_\infty\left[\frac{(x-a)^2 + (b-x)^2}{2}\right] \\
&= \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right]\|f\|_\infty
\end{aligned}$$

for all $x \in [a,b]$.

Finally, by the identity (1.23) we deduce that for all $x \in [a,b]$,

$$\left|F(x) - \frac{b - E(X)}{b-a}\right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f\|_\infty$$

which proves (1.19).

Now, taking into account the fact that

$$\Pr(X \ge x) = 1 - \Pr(X \le x),$$

the inequality (1.20) is also obtained.

To prove that the constant $\frac{1}{4}$ is best, assume that the inequality (1.19) holds with a constant $c>0$, that is,

$$\left|\Pr(X\le x)-\frac{b-E(X)}{b-a}\right|\le\left[c+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f\|_\infty \tag{1.25}$$

for all $x\in[a,b]$.

Assume that X_0 is a random variable having the probability density function $f_0:[0,1]\to\mathbb{R}$ given by $f_0(t)=1$, then we find that

$$\Pr(X_0\ge x)=x,\ \ (x\in[0,1]),\ \ E(X_0)=\frac{1}{2},\ \text{and}\ \ \|f_0\|_\infty=1.$$

Consequently, (1.25) becomes

$$\left|x-\frac{1}{2}\right|\le c+\left(x-\frac{1}{2}\right)^2 \qquad \text{for all } x\in[0,1].$$

Choosing $x=0$, we get $c\ge\frac{1}{4}$ and the result is proved. ∎

The above theorem has some interesting corollaries for the expectation of X (see also [**13**]).

Corollary 3. *Under the above assumptions, we have the double inequality*

$$b-\frac{1}{2}(b-a)^2\|f\|_\infty\le E(X)\le a+\frac{1}{2}(b-a)^2\|f\|_\infty. \tag{1.26}$$

Proof. We know that

$$a\le E(X)\le b.$$

Now, choose $x=a$ in (1.19) to obtain

$$\left|\frac{b-E(X)}{b-a}\right|\le\frac{1}{2}(b-a)\|f\|_\infty$$

that is,

$$b-E(X)\le\frac{1}{2}(b-a)^2\|f\|_\infty,$$

which is equivalent to the first inequality in (1.26).

Also, choose $x=b$ in (1.19) to get

$$\left|1-\frac{b-E(X)}{b-a}\right|\le\frac{1}{2}(b-a)\|f\|_\infty$$

which reduces to:

$$E(X)-a\le\frac{1}{2}(b-a)^2\|f\|_\infty,$$

proving the second inequality (1.26). ∎

Remark 6. *We know that*

$$1 = \int_a^b f(x)\,dx \leq (b-a)\,\|f\|_\infty$$

which gives

$$\|f\|_\infty \geq \frac{1}{b-a}.$$

If we assume that $\|f\|_\infty$ is not too large, say,

$$\|f\|_\infty \leq \frac{2}{b-a}, \tag{1.27}$$

then

$$a + \frac{1}{2}(b-a)^2\,\|f\|_\infty \leq b$$

and

$$b - \frac{1}{2}(b-a)^2\,\|f\|_\infty \geq a$$

which shows that the inequality (1.26) is a tighter inequality than $a \leq E(X) \leq b$ when (1.27) holds.

Another equivalent inequality to (1.26), which can be more useful in practice, is the one following (see also [**13**]).

Corollary 4. *With the above assumptions,*

$$\left|E(X) - \frac{a+b}{2}\right| \leq \frac{(b-a)^2}{2}\left(\|f\|_\infty - \frac{1}{b-a}\right). \tag{1.28}$$

Proof. From the inequality (1.26),

$$\begin{aligned} b - \frac{a+b}{2} - \frac{1}{2}(b-a)^2\,\|f\|_\infty &\leq E(X) - \frac{a+b}{2} \\ &\leq a - \frac{a+b}{2} + \frac{1}{2}(b-a)^2\,\|f\|_\infty, \end{aligned}$$

giving,

$$\begin{aligned} -\frac{(b-a)^2}{2}\left(\|f\|_\infty - \frac{1}{b-a}\right) &\leq E(X) - \frac{a+b}{2} \\ &\leq \frac{(b-a)^2}{2}\left(\|f\|_\infty - \frac{1}{b-a}\right), \end{aligned}$$

which is exactly (1.28). ∎

This corollary provides the mechanism for finding a sufficient condition, in terms of $\|f\|_\infty$, for the expectation $E(X)$ to be close to the midpoint of the interval, $\frac{a+b}{2}$ (see also [**13**]).

COROLLARY 5. *Let X and f be as above and $\varepsilon > 0$. If*

$$\|f\|_{\infty} \le \frac{1}{b-a} + \frac{2\varepsilon}{(b-a)^2} \tag{1.29}$$

then,

$$\left| E(X) - \frac{a+b}{2} \right| \le \varepsilon.$$

The proof is obvious and hence the details are omitted.

The following corollary of Theorem 2 also holds (see also [**13**]).

COROLLARY 6. *Let X and f be as above, then:*

$$\begin{aligned} &\left| \Pr\left(X \le \frac{a+b}{2}\right) - \frac{1}{2} \right| \\ &\le \frac{1}{4}(b-a)\|f\|_{\infty} + \frac{1}{b-a}\left| E(X) - \frac{a+b}{2} \right| \\ &\le \frac{3}{4}(b-a)\|f\|_{\infty} - \frac{1}{2}. \end{aligned} \tag{1.30}$$

PROOF. If we choose $x = \frac{a+b}{2}$ in (1.19), then we have:

$$\left| \Pr\left(X \le \frac{a+b}{2}\right) - \frac{b - E(X)}{b-a} \right| \le \frac{1}{4}(b-a)\|f\|_{\infty},$$

which is clearly equivalent to

$$\left| \Pr\left(X \le \frac{a+b}{2}\right) - \frac{1}{2} + \frac{1}{b-a}\left(E(X) - \frac{a+b}{2}\right) \right| \le \frac{1}{4}(b-a)\|f\|_{\infty}.$$

Using the triangle inequality, we get

$$\begin{aligned} &\left| \Pr\left(X \le \frac{a+b}{2}\right) - \frac{1}{2} \right| \\ &= \left| \Pr\left(X \le \frac{a+b}{2}\right) - \frac{1}{2} + \frac{1}{b-a}\left(E(X) - \frac{a+b}{2}\right) \right. \\ &\qquad \left. - \frac{1}{b-a}\left(E(X) - \frac{a+b}{2}\right) \right| \end{aligned}$$

$$\leq \left|\Pr\left(X \leq \frac{a+b}{2}\right) - \frac{1}{2} + \frac{1}{b-a}\left(E(X) - \frac{a+b}{2}\right)\right| + \left|\frac{1}{b-a}\left(E(X) - \frac{a+b}{2}\right)\right|$$
$$\leq \frac{1}{4}(b-a)\|f\|_{\infty} + \frac{1}{b-a}\left|E(X) - \frac{a+b}{2}\right|$$
$$\leq \frac{3}{4}(b-a)\|f\|_{\infty} - \frac{1}{2}$$

and the desired inequality is obtained. ∎

Remark 7. *A similar result applies for*

$$\Pr\left(X \geq \frac{a+b}{2}\right),$$

and the details are omitted.

Finally, the following result holds (see also [**13**]).

Corollary 7. *Let X and f be as above, then:*

$$\left|E(X) - \frac{a+b}{2}\right| \leq \frac{1}{4}(b-a)^2\|f\|_{\infty} + (b-a)\left|\Pr\left(X \leq \frac{a+b}{2}\right) - \frac{1}{2}\right|. \tag{1.31}$$

Proof. As in the above Corollary 6, we have

$$\frac{1}{b-a}\left|E(X) - \frac{a+b}{2}\right|$$
$$\leq \left|\Pr\left(X \leq \frac{a+b}{2}\right) - \frac{1}{2} + \frac{1}{b-a}\left(E(X) - \frac{a+b}{2}\right)\right| + \left|\Pr\left(X \leq \frac{a+b}{2}\right) - \frac{1}{2}\right|$$
$$\leq \frac{1}{4}(b-a)\|f\|_{\infty} + \left|\Pr\left(X \leq \frac{a+b}{2}\right) - \frac{1}{2}\right|,$$

from which we get (1.31). ∎

Remark 8. *If we assume that f is continuous on $[a,b]$, then F is differentiable on (a,b) and we get, in view of Ostrowski's inequality, (see for instance [**86**]),*

$$\left|F(x) - \frac{1}{b-a}\int_a^b F(t)\,dt\right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f\|_{\infty}$$

for all $x \in [a, b]$.

Using the identity (1.22) we recapture the inequality (1.19) and (1.20) for random variables whose probability density functions are continuous on $[a, b]$.

2.2. Application for a Beta Random Variable. Assume that X is a Beta random variable with parameters (p, q) as defined in (1.2). We observe that for $0 < p < 1$,

$$\|f(\cdot, p, q)\|_\infty = \sup_{x\in(0,1)} \left[\frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}\right] = \infty.$$

Assume that $p,\ q \geq 1$, then we find that

$$\begin{aligned}
&\frac{df(x,p,q)}{dx} \\
&= \frac{1}{B(p,q)}\left[(p-1)x^{p-2}(1-x)^{q-1} - (q-1)x^{p-1}(1-x)^{q-2}\right] \\
&= \frac{x^{p-2}(1-x)^{q-2}}{B(p,q)}\left[(p-1)(1-x) - (q-1)x\right] \\
&= \frac{x^{p-2}(1-x)^{q-2}}{B(p,q)}\left[-(p+q-2)x + (p-1)\right].
\end{aligned}$$

We observe that for $p,\ q > 1$, $\frac{df(x,p,q)}{dx} = 0$ if and only if $x_0 = \frac{p-1}{p+q-2}$. We therefore have $\frac{df(x,p,q)}{dx} > 0$ on $(0, x_0)$ and $\frac{df(x,p,q)}{dx} < 0$ on $(x_0, 1)$. Consequently, we see that

$$\|f(\cdot, p, q)\|_\infty = f(x_0; p, q) = \frac{(p-1)^{p-1}(q-1)^{q-1}}{B(p,q)(p+q-2)^{p+q-2}}.$$

On the other hand we have

$$E(X) = \frac{1}{B(p,q)}\int_0^1 x \cdot x^{p-1}(1-x)^{q-1}\,dx = \frac{B(p+1,q)}{B(p,q)}.$$

Upon employing the familiar relationships $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ and $\Gamma(z+1) = z\Gamma(z)$, $(z \in \mathbb{C}\diagdown\{-1,-2,-3,...\})$, where Γ denotes the well-known Gamma Function, it is easy to see that

$$E(X) = \frac{p}{p+q}.$$

Finally, using Theorem 2, we can state the following.

Let X be a Beta random variable with the parameters (p,q), $(p,q)\in[1,\infty)\times[1,\infty)$, then we have the inequality

$$\left|\Pr(X\le x)-\frac{q}{p+q}\right|\le\left[\frac{1}{4}+\left(x-\frac{1}{2}\right)^{2}\right]\frac{(p-1)^{p-1}(q-1)^{q-1}}{B(p,q)(p+q-2)^{p+q-2}}$$

and

$$\left|\Pr(X\ge x)-\frac{p}{p+q}\right| \le\left[\frac{1}{4}+\left(x-\frac{1}{2}\right)^{2}\right]\frac{(p-1)^{p-1}(q-1)^{q-1}}{B(p,q)(p+q-2)^{p+q-2}}$$

where $x\in[0,1]$. In particular,

$$\left|\Pr\left(X\le\frac{1}{2}\right)-\frac{q}{p+q}\right|\le\frac{1}{4}\cdot\frac{(p-1)^{p-1}(q-1)^{q-1}}{B(p,q)(p+q-2)^{p+q-2}}$$

and

$$\left|\Pr\left(X\ge\frac{1}{2}\right)-\frac{p}{p+q}\right|\le\frac{1}{4}\cdot\frac{(p-1)^{p-1}(q-1)^{q-1}}{B(p,q)(p+q-2)^{p+q-2}}.$$

3. Random Variables whose PDFs Belong to $L_p[a,b], p>1$

3.1. Inequalities.

The following theorem holds [**76**].

THEOREM 3. *Let X be a random variable with the probability density function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}_+$ and with cumulative distribution function $F(x)=\Pr(X\le x)$. If $f\in L_p[a,b], p>1$, then we have the inequalities*

$$\left|\Pr(X\le x)-\frac{b-E(X)}{b-a}\right| \le\frac{q}{q+1}\|f\|_p(b-a)^{\frac{1}{q}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{1+q}{q}}+\left(\frac{b-x}{b-a}\right)^{\frac{1+q}{q}}\right] \le\frac{q}{q+1}\|f\|_p(b-a)^{\frac{1}{q}} \tag{1.32}$$

for all $x\in[a,b]$, where $\frac{1}{p}+\frac{1}{q}=1$.

PROOF. By Hölder's integral inequality we have

$$|F(x) - F(y)| = \left| \int_x^y f(t)\, dt \right| \leq \left| \int_x^y dt \right|^{\frac{1}{q}} \left| \int_x^y |f(t)|^p\, dt \right|^{\frac{1}{p}} \leq |x - y|^{\frac{1}{q}} \|f\|_p, \tag{1.33}$$

for all $x, y \in [a, b]$, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\|f\|_p := \left(\int_a^b |f(t)|^p\, dt \right)^{\frac{1}{p}}$$

is the usual $p-$norm on $L_p [a, b]$.

The inequality (1.33) shows, in fact, that the mapping $F(\cdot)$ is of the $r - H-$Hölder type, i.e.,

$$|F(x) - F(y)| \leq H |x - y|^r, \qquad \text{for all } x, y \in [a, b] \tag{1.34}$$

with $0 < H = \|f\|_p$ and $r = \frac{1}{q} \in (0, 1)$.

Integrating the inequality (1.33) over $y \in [a, b]$ we get successively

$$\begin{aligned} &\left| F(x) - \frac{1}{b-a} \int_a^b F(y)\, dy \right| \\ &\leq \frac{1}{b-a} \int_a^b |F(x) - F(y)|\, dy \leq \frac{1}{b-a} \|f\|_p \int_a^b |x - y|^{\frac{1}{q}}\, dy \\ &= \frac{1}{b-a} \|f\|_p \left[\int_a^x (x-y)^{\frac{1}{q}}\, dy + \int_x^b (y-x)^{\frac{1}{q}}\, dy \right] \\ &= \frac{1}{b-a} \|f\|_p \left[\frac{(x-a)^{\frac{1}{q}+1}}{\frac{1}{q}+1} + \frac{(b-x)^{\frac{1}{q}+1}}{\frac{1}{q}+1} \right] \\ &= \frac{q}{q+1} \cdot \frac{1}{b-a} \|f\|_p \left[(x-a)^{\frac{1}{q}+1} + (b-x)^{\frac{1}{q}+1} \right] \\ &= \frac{q}{q+1} \|f\|_p (b-a)^{\frac{1}{q}} \left[\left(\frac{x-a}{b-a} \right)^{\frac{1}{q}+1} + \left(\frac{b-x}{b-a} \right)^{\frac{1}{q}+1} \right] \end{aligned} \tag{1.35}$$

for all $x \in [a, b]$.

Since

$$E(X) = b - \int_a^b F(t)\, dt$$

then, by (1.35), we get the first inequality in (1.32).

For the second inequality, we observe that

$$\left(\frac{x-a}{b-a}\right)^{\frac{1}{q}+1}+\left(\frac{b-x}{b-a}\right)^{\frac{1}{q}+1}\leq 1, \qquad \text{for all } x\in[a,b]$$

and the theorem is completely proved. ∎

REMARK 9. *The inequalities* (1.32) *are equivalent to*

$$\begin{aligned}(1.36)\qquad &\left|\Pr\left(X\geq x\right)-\frac{E\left(X\right)-a}{b-a}\right| \\ &\leq\frac{q}{q+1}\left\|f\right\|_{p}\left(b-a\right)^{\frac{1}{q}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{1+q}{q}}+\left(\frac{b-x}{b-a}\right)^{\frac{1+q}{q}}\right] \\ &\leq\frac{q}{q+1}\left\|f\right\|_{p}\left(b-a\right)^{\frac{1}{q}}, \qquad \textit{for all } x\in\left[a,b\right].\end{aligned}$$

COROLLARY 8. *([**76**]) Under the above assumptions, we have the double inequality*

$$(1.37)\quad b-\frac{q}{q+1}\left\|f\right\|_{p}\left(b-a\right)^{1+\frac{1}{q}}\leq E\left(X\right)\leq a+\frac{q}{q+1}\left\|f\right\|_{p}\left(b-a\right)^{\frac{1}{q}+1}.$$

PROOF. We know that $a\leq E\left(X\right)\leq b$.

Now, choose in (1.32) $x=a$ to get

$$\left|\frac{b-E\left(X\right)}{b-a}\right|\leq\frac{q}{q+1}\left\|f\right\|_{p}\left(b-a\right)^{\frac{1}{q}}$$

i.e.,

$$b-E\left(X\right)\leq\frac{q}{q+1}\left\|f\right\|_{p}\left(b-a\right)^{1+\frac{1}{q}},$$

which is equivalent to the first inequality in (1.37).

Alternatively, let $x=b$ in (1.32) to give:

$$\left|1-\frac{b-E\left(X\right)}{b-a}\right|\leq\frac{q}{q+1}\left\|f\right\|_{p}\left(b-a\right)^{\frac{1}{q}}$$

so

$$E\left(X\right)-a\leq\frac{q}{q+1}\left\|f\right\|_{p}\left(b-a\right)^{\frac{1}{q}+1},$$

which is equivalent to the second inequality in (1.37). ∎

REMARK 10. *By Hölder's integral inequality,*

$$1=\int_{a}^{b}f\left(t\right)dt\leq\left(b-a\right)^{\frac{1}{q}}\left\|f\right\|_{p},$$

which gives

$$\left\|f\right\|_{p}\geq\frac{1}{\left(b-a\right)^{\frac{1}{q}}}.$$

Now, if we assume that $\|f\|_p$ *is not too large, such that*

$$\|f\|_p \leq \frac{q+1}{q}\cdot\frac{1}{(b-a)^{\frac{1}{q}}} \tag{1.38}$$

then we get

$$a+\frac{q}{q+1}\|f\|_p(b-a)^{\frac{1}{q}+1}\leq b$$

and

$$b-\frac{q}{q+1}\|f\|_p(b-a)^{1+\frac{1}{q}}\geq a,$$

which shows that the inequality (1.37) *is a tighter inequality than* $a \leq E(X) \leq b$ *when* (1.38) *holds.*

Another equivalent inequality to (1.37), which can be more useful in practice, is the following (see also [**76**]):

COROLLARY 9. *With the above assumptions, we have:*

$$\left|E(X)-\frac{a+b}{2}\right| \leq (b-a)\left[\frac{q}{q+1}\|f\|_p(b-a)^{\frac{1}{q}}-\frac{1}{2}\right]. \tag{1.39}$$

PROOF. From (1.37) we have:

$$\begin{aligned} &b-\frac{a+b}{2}-\frac{q}{q+1}\|f\|_p(b-a)^{1+\frac{1}{q}}\\ &\leq E(X)-\frac{a+b}{2}\\ &\leq a-\frac{a+b}{2}+\frac{q}{q+1}\|f\|_p(b-a)^{1+\frac{1}{q}}.\end{aligned}$$

That is,

$$\begin{aligned} &\frac{b-a}{2}-\frac{q}{q+1}\|f\|_p(b-a)^{1+\frac{1}{q}}\\ &\leq E(X)-\frac{a+b}{2}\\ &\leq -\frac{b-a}{2}+\frac{q}{q+1}\|f\|_p(b-a)^{1+\frac{1}{q}},\end{aligned}$$

which is equivalent to

$$\begin{aligned}\left|E(X)-\frac{a+b}{2}\right| &\leq \frac{q}{q+1}\|f\|_p(b-a)^{1+\frac{1}{q}}-\frac{b-a}{2}\\ &=(b-a)\left[\frac{q}{q+1}\|f\|_p(b-a)^{\frac{1}{q}}-\frac{1}{2}\right]\end{aligned}$$

and the inequality (1.39) is proved. ∎

This corollary provides the possibility of finding a sufficient condition, in terms of $\|f\|_p$ $(p > 1)$, for the expectation $E(X)$ to be close to the interval midpoint, $\frac{a+b}{2}$ (see also [**76**]).

COROLLARY 10. *Let X and f be as above and $\varepsilon > 0$. If*

$$\|f\|_p \leq \frac{q+1}{2q} \cdot \frac{1}{(b-a)^{\frac{1}{q}}} + \frac{\varepsilon (q+1)}{q (b-a)^{1+\frac{1}{q}}}$$

then

$$\left| E(X) - \frac{a+b}{2} \right| \leq \varepsilon.$$

The details are omitted.

The following corollary of Theorem 3 also holds (see also [**76**]):

COROLLARY 11. *Let X and f be as above, then:*

$$\left| \Pr\left(X \leq \frac{a+b}{2} \right) - \frac{1}{2} \right| \leq \frac{q}{2^{\frac{1}{q}} (q+1)} \|f\|_p (b-a)^{\frac{1}{q}} + \frac{1}{b-a} \left| E(X) - \frac{a+b}{2} \right|.$$

PROOF. If, in (1.32), $x = \frac{a+b}{2}$, we get

$$\left| \Pr\left(X \leq \frac{a+b}{2} \right) - \frac{b - E(X)}{b-a} \right| \leq \frac{q}{2^{\frac{1}{q}} (q+1)} \|f\|_p (b-a)^{\frac{1}{q}},$$

which is clearly equivalent to:

$$\left| \Pr\left(X \leq \frac{a+b}{2} \right) - \frac{1}{2} + \frac{1}{b-a} \left(E(X) - \frac{a+b}{2} \right) \right| \leq \frac{q}{2^{\frac{1}{q}} (q+1)} \|f\|_p (b-a)^{\frac{1}{q}}.$$

Using the triangle inequality, this becomes:

$$\begin{aligned} &\left| \Pr\left(X \leq \frac{a+b}{2} \right) - \frac{1}{2} \right| \\ &= \left| \Pr\left(X \leq \frac{a+b}{2} \right) - \frac{1}{2} + \frac{1}{b-a} \left(E(X) - \frac{a+b}{2} \right) \right. \\ &\quad \left. - \frac{1}{b-a} \left(E(X) - \frac{a+b}{2} \right) \right| \end{aligned}$$

$$\leq \left|\Pr\left(X \leq \frac{a+b}{2}\right) - \frac{1}{2} + \frac{1}{b-a}\left(E(X) - \frac{a+b}{2}\right)\right| + \frac{1}{b-a}\left|E(X) - \frac{a+b}{2}\right|$$
$$\leq \frac{q}{2^{\frac{1}{q}}(q+1)} \|f\|_p (b-a)^{\frac{1}{q}} + \frac{1}{b-a}\left|E(X) - \frac{a+b}{2}\right|$$

and the corollary is proved. ∎

Finally, the following result also holds (see also [**76**]):

Corollary 12. *With the above assumptions, we have:*

$$\left|E(X) - \frac{a+b}{2}\right| \leq \frac{q}{2^{\frac{1}{q}}(q+1)} \|f\|_p (b-a)^{1+\frac{1}{q}} + (b-a)\left|\Pr\left(X \leq \frac{a+b}{2}\right) - \frac{1}{2}\right|.$$

The proof is similar and we omit the details.
For some related results see [**25**].

3.2. Applications for A Beta Random Variable. Let X be a Beta Random Variable with parameters (s,t) as defined in (1.2). Observe that, for $p>1$,

$$\|f(\cdot;s,t)\|_p = \frac{1}{B(s,t)}\left(\int_0^1 \tau^{p(s-1)}(1-\tau)^{p(t-1)}\,d\tau\right)^{\frac{1}{p}}$$
$$= \frac{1}{B(s,t)}\left(\int_0^1 \tau^{p(s-1)+1-1}(1-\tau)^{p(t-1)+1-1}\,d\tau\right)^{\frac{1}{p}}$$
$$= \frac{1}{B(s,t)}\left[B(p(s-1)+1, p(t-1)+1)\right]^{\frac{1}{p}},$$

provided

$$p(s-1)+1, \quad p(t-1)+1 > 0,$$

namely,

$$s > 1 - \frac{1}{p} \quad \text{and} \quad t > 1 - \frac{1}{p}.$$

Now, using Theorem 3, we can state the following:

Let $p > 1$ and X be a Beta random variable with the parameters (s,t), $s > 1 - \frac{1}{p}$, $t > 1 - \frac{1}{p}$, then we have:

$$\left|\Pr(X \le x) - \frac{t}{s+t}\right| \le \frac{q}{q+1} \frac{\left[x^{\frac{1+q}{q}} + (1-x)^{\frac{1+q}{q}}\right] \left[B(p(s-1)+1, p(t-1)+1)\right]^{\frac{1}{p}}}{B(s,t)} \tag{1.40}$$

for all $x \in [0,1]$.
In particular, we have

$$\left|\Pr\left(X \le \frac{1}{2}\right) - \frac{t}{s+t}\right| \le \frac{q}{2^{\frac{1}{q}}(q+1)} \frac{\left[B(p(s-1)+1, p(t-1)+1)\right]^{\frac{1}{p}}}{B(s,t)}.$$

The proof follows by Theorem 3 choosing $f(x) = f(x;s,t)$, $x \in [0,1]$ and taking $E(X) = \frac{s}{s+t}$.

4. Better Bounds for an Inequality of the Ostrowski Type

4.1. Introduction. In 1938, A. Ostrowski [**112**] (see also [**110**, p. 468]) proved the following inequality

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)M \tag{1.41}$$

for all $x \in [a,b]$, provided that f is differentiable on (a,b) and $|f'(t)| \le M$ for all $t \in (a,b)$.

Using the following representation, which has been obtained by Montgomery in an equivalent form [**110**, p. 565],

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b p(x,t) f'(t)\,dt \tag{1.42}$$

for all $x \in [a,b]$, provided that f is absolutely continuous on $[a,b]$ and

$$p(x,t) := \begin{cases} t-a & \text{if } t \in [a,x] \\ t-b & \text{if } t \in (x,b\} \end{cases}, (x,t) \in [a,b]^2,$$

we can put, in place of M in (1.41), the sup norm of f', namely, $\|f'\|_\infty$ where

$$\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)|,$$

provided that $f' \in L_\infty [a, b]$.

In [**88**], Dragomir and Wang, using the Grüss inequality, proved the following perturbed version of Ostrowski's inequality:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \leq \frac{1}{4} (b-a)(\Gamma - \gamma) \tag{1.43}$$

for all $x \in [a, b]$, provided the derivative f' satisfies the condition

$$\gamma \leq f'(t) \leq \Gamma \text{ on } (a, b). \tag{1.44}$$

Using a pre-Grüss inequality, Matić, Pečaric and Ujević [**108**] improved the constant $\frac{1}{4}$, in the right hand member of (1.43), with the constant $\frac{1}{4\sqrt{3}}$.

An upper bound in terms of the second derivative has been pointed out by Barnett and Dragomir in [**12**].

For two mappings $g, h : [a, b] \to \mathbb{R}$, define the *Čebyšev functional* as

$$T(g, h) := \frac{1}{b-a} \int_a^b g(t)\, h(t)\, dt - \frac{1}{b-a} \int_a^b g(t)\, dt \cdot \frac{1}{b-a} \int_a^b h(t)\, dt, \tag{1.45}$$

provided the involved integrals exist.

By use of (1.45), we improve the Matić-Pečaric-Ujević result by providing a better bound for the first membership of (1.43) in terms of Euclidean norms. Since the bound in (1.43) will apply for absolutely continuous mappings whose derivatives are bounded, the new inequality will also apply for the larger class of absolutely continuous mappings whose derivative $f' \in L_2 [a, b]$.

4.2. The Results. The following theorem holds [**23**].

THEOREM 4. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous mapping whose derivative $f' \in L_2 [a, b]$, then we have the inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \leq \frac{(b-a)}{2\sqrt{3}} \left[\frac{1}{b-a} \|f'\|_2^2 - \left(\frac{f(b) - f(a)}{b-a} \right)^2 \right]^{\frac{1}{2}} \tag{1.46}$$

$$\left(\leq \frac{(b-a)(\Gamma-\gamma)}{4\sqrt{3}} \quad \textit{if} \ \ \gamma \leq f'(t) \leq \Gamma \ \ \textit{for a.e. } t \textit{ on } [a,b] \right)$$

for all $x \in [a,b]$*.*

PROOF. Using Korkine's identity,

$$T(g,h) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (g(t) - g(s))(h(t) - h(s))\, dtds, \tag{1.47}$$

we obtain from (1.45) and (1.47),

$$\begin{aligned} &\frac{1}{b-a} \int_a^b p(x,t) f'(t)\, dt \\ &\qquad - \frac{1}{b-a} \int_a^b p(x,t)\, dt \cdot \frac{1}{b-a} \int_a^b f'(t)\, dt \\ &= \frac{1}{2(b-a)^2} \int_a^b \int_a^b (p(x,t) - p(x,s))(f'(t) - f'(s))\, dtds. \end{aligned} \tag{1.48}$$

As

$$\frac{1}{b-a} \int_a^b p(x,t) f'(t)\, dt = f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt,$$
$$\frac{1}{b-a} \int_a^b p(x,t)\, dt = x - \frac{a+b}{2}$$

and

$$\frac{1}{b-a} \int_a^b f'(t)\, dt = \frac{f(b) - f(a)}{b-a},$$

then, by (1.48), we get the identity,

$$\begin{aligned} &f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \\ &= \frac{1}{2(b-a)^2} \int_a^b \int_a^b (p(x,t) - p(x,s))(f'(t) - f'(s))\, dtds \end{aligned} \tag{1.49}$$

for all $x \in [a,b]$.

Using the Cauchy-Buniakowski-Schwartz inequality for double integrals, we can write,

$$\begin{aligned}(1.50)\quad & \frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))(f'(t)-f'(s))\,dtds \\ & \leq \left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))^2\,dtds\right)^{\frac{1}{2}} \\ & \quad \times \left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b (f'(t)-f'(s))^2\,dtds\right)^{\frac{1}{2}}.\end{aligned}$$

However,

$$\begin{aligned}& \frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))^2\,dtds \\ & = \frac{1}{b-a}\int_a^b p^2(x,t)\,dt - \left(\frac{1}{b-a}\int_a^b p(x,t)\,dt\right)^2 \\ & = \frac{1}{b-a}\left[\int_a^x (t-a)^2\,dt + \int_x^b (t-a)^2\,dt\right] - \left(x-\frac{a+b}{2}\right)^2 \\ & = \frac{1}{b-a}\left[\frac{(x-a)^3+(b-x)^3}{3}\right] - \left(x-\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}\end{aligned}$$

and

$$\begin{aligned}& \frac{1}{2(b-a)^2}\int_a^b\int_a^b (f'(t)-f'(s))^2\,dtds \\ & \qquad = \frac{1}{b-a}\|f'\|_2^2 - \left(\frac{f(b)-f(a)}{b-a}\right)^2.\end{aligned}$$

Consequently, by (1.49) and (1.50), we deduce the first inequality in (1.46).

If $\gamma \leq f'(t) \leq \Gamma$ for a.e. $t \in (a,b)$, then, by the Grüss inequality, we have:

$$0 \leq \frac{1}{b-a}\int_a^b (f'(t))^2\,dt - \left(\frac{1}{b-a}\int_a^b f'(t)\,dt\right)^2 \leq \frac{1}{4}(\Gamma-\gamma)^2,$$

and the last inequality in (1.46) is obtained. ∎

Corollary 13. *([**23**]) With the above assumptions, from (1.46) with $x = \frac{a+b}{2}$, we have the mid-point inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{1.51}$$
$$\leq \frac{(b-a)}{2\sqrt{3}}\left[\frac{1}{b-a}\|f'\|_2^2 - \left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}$$
$$\left(\leq \frac{(b-a)(\Gamma-\gamma)}{4\sqrt{3}} \quad \text{if } \gamma \leq f'(t) \leq \Gamma \text{ a.e. } t \text{ on } [a,b]\right).$$

Remark 11. *Since $L_\infty[a,b] \subset L_2[a,b]$ (and the inclusion is strict), then we remark that the inequality (1.46) can be applied also for the mappings f whose derivatives are unbounded on (a,b), but $f' \in L_2[a,b]$.*

4.3. Applications for CDFs. Let X be a random variable having the PDF $f : [a,b] \to \mathbb{R}_+$ and *cumulative density function* $F : [a,b] \to [0,1]$, i.e.,

$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a,b].$$

We have the following inequality [**23**].

Theorem 5. *With the above assumptions and if the PDF $f \in L_2[a,b]$, then we have the inequality*

$$\left| F(x) - \frac{b-E(X)}{b-a} - \frac{1}{b-a}\left(x - \frac{a+b}{2}\right) \right| \tag{1.52}$$
$$\leq \frac{1}{2\sqrt{3}}\left[(b-a)\|f\|_2^2 - 1\right]^{\frac{1}{2}}$$
$$\left(\leq \frac{(b-a)(M-m)}{4\sqrt{3}} \quad \text{if } m \leq f \leq M \text{ a.e. on } [a,b]\right)$$

for all $x \in [a,b]$, where $E(X)$ is the expectation of X.

Proof. Put F instead of f in (1.46), to get

$$\left| F(x) - \frac{1}{b-a}\int_a^b F(t)\,dt - \frac{F(b)-F(a)}{b-a}\left(x - \frac{a+b}{2}\right) \right| \tag{1.53}$$
$$\leq \frac{(b-a)}{2\sqrt{3}}\left[\frac{1}{b-a}\|f\|_2^2 - \left(\frac{F(b)-F(a)}{b-a}\right)^2\right]^{\frac{1}{2}}$$
$$\left(\leq \frac{(b-a)(M-m)}{4\sqrt{3}} \quad \text{if } m \leq f(t) \leq M \text{ a.e. } t \text{ on } [a,b]\right).$$

As $F(a) = 0$, $F(b) = 1$, and

$$\int_a^b F(t)\,dt = b - E(X),$$

then, by (1.53), we easily deduce (1.52). ∎

COROLLARY 14. *([**23**]) With the above assumptions, we have:*

$$\left|\Pr\left(X \leq \frac{a+b}{2}\right) - \frac{b-E(X)}{b-a}\right| \leq \frac{1}{2\sqrt{3}}\left[(b-a)\|f\|_2^2 - 1\right]^{\frac{1}{2}} \left(\leq \frac{(b-a)(M-m)}{4\sqrt{3}} \quad \text{where } m \leq f \leq M \text{ are as above}\right). \tag{1.54}$$

Let X be a Beta random variable with parameters (p, q), then we know that

$$E(X) = \frac{p}{p+q}$$

and

$$\|f(\cdot; p, q)\|_2^2 = \int_0^1 \frac{x^{2(p-1)}(1-x)^{2(q-1)}}{B^2(p,q)}dx = \frac{B(2p-1, 2q-1)}{B^2(p,q)}.$$

By Theorem 5, *we then have the inequality,*

$$\left|\Pr(X \leq x) - \frac{p}{p+q} - x + \frac{1}{2}\right| \leq \frac{1}{2\sqrt{3}} \cdot \frac{\left[B(2p-1, 2q-1) - B^2(p,q)\right]^{\frac{1}{2}}}{B(p,q)} \tag{1.55}$$

for all $x \in [0, 1]$.

CHAPTER 2

Other Ostrowski Type Results and Applications for PDFs

1. Ostrowski's Inequality for Functions of Bounded Variation

1.1. Introduction. In [**59**], the author proved the following inequality of the Ostrowski type for functions of bounded variation.

THEOREM 6. *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and denote by $\bigvee_a^b (f)$ its total variation on $[a, b]$. For any $x \in [a, b]$, one has the inequality:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f). \tag{2.1}$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller one.

The above inequality (2.1) has, as a particular case, the mid-point inequality.

The corresponding version for the generalised trapezoid inequality was obtained in [**36**].

THEOREM 7. *With the assumptions in Theorem 6,*

$$\left| \frac{(x-a) f(a) + (b-x) f(b)}{b-a} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f) \tag{2.2}$$

for any $x \in [a, b]$.

Here the constant $\frac{1}{2}$ is also the best possible in the above sense.

The above inequality (2.2) incorporates the trapezoid inequality.

In [**99**], Guessab and Schmeisser developed a generalised model incorporating both the mid-point and trapezoid inequality as special cases. They have proved amongst others, the following companion of Ostrowski's inequality.

THEOREM 8. *Assume that the function $f : [a,b] \to \mathbb{R}$ is of the $M - r-$Hölder type with $r \in (0,1]$, namely,*

$$|f(t) - f(s)| \leq M |t-s|^r \quad \text{for any } t,s \in [a,b]. \tag{2.3}$$

For each $x \in \left[a, \frac{a+b}{2}\right]$, one has the inequality

$$\left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{2^{r+1}(x-a)^{r+1} + (a+b-2x)^{r+1}}{2^r (r+1)(b-a)} \right] M. \tag{2.4}$$

This inequality is sharp for each admissible x. Equality is obtained if and only if $f = \pm M f_* + c$, with $c \in \mathbb{R}$ and

$$f_*(t) = \begin{cases} (x-t)^r, & \text{for } a \leq t \leq x \\ (t-x)^r, & \text{for } x \leq t \leq \frac{1}{2}(a+b) \\ f_*(a+b-t), & \text{for } \frac{1}{2}(a+b) \leq t \leq b. \end{cases} \tag{2.5}$$

REMARK 12. *For $r = 1$, f is Lipschitzian with constant $L > 0$, and since*

$$\frac{4(x-a)^2 + (a+b-2x)^2}{4(b-a)} = \left[\frac{1}{8} + 2\left(\frac{x - \frac{3a+b}{4}}{b-a} \right)^2 \right](b-a)$$

then, by (2.4), we get the following companion of Ostrowski's inequality for Lipschitzian functions,

$$\left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{8} + 2\left(\frac{x - \frac{3a+b}{4}}{b-a} \right)^2 \right](b-a) L, \tag{2.6}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

The constant $\frac{1}{8}$ is the best possible in (2.6).

By substituting $x = \frac{3a+b}{4}$ into the above inequality, we obtain the following trapezoid type inequality, which is the best in the class,

$$\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \frac{1}{8}(b-a) L. \tag{2.7}$$

The constant $\frac{1}{8}$ here is also best possible in the above sense.

For a recent monograph devoted to Ostrowski type inequalities, see [**86**].

The main aim of the next section is to provide a sharp bound for the difference

$$\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt,$$

where f is assumed to be of bounded variation.

1.2. Some Integral Inequalities. The following identity holds [**47**].

LEMMA 1. *Assume that the function $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$, then we have the equality*

$$\begin{aligned}(2.8)\quad & \frac{1}{2}[f(x)+f(a+b-x)]-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\left[\int_a^x (t-a)\,df(t)+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)df(t)\right. \\ &\qquad\left.+\int_{a+b-x}^b (t-b)\,df(t)\right]\end{aligned}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

PROOF. Obviously, all the Riemann-Stieltjes integrals from the right hand side of (2.8) exist because the functions $(\cdot-a)$, $\left(\cdot-\frac{a+b}{2}\right)$ and $(\cdot-b)$ are continuous on these intervals and f is of bounded variation.

Using the integration by parts formula for Riemann-Stieltjes integrals, we have, for any $x\in\left[a,\frac{a+b}{2}\right]$, that

$$\int_a^x (t-a)\,df(t)=f(x)(x-a)-\int_a^x f(t)\,dt,$$

$$\begin{aligned}&\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)df(t)\\ &= f(a+b-x)\left(\frac{a+b}{2}-x\right)-f(x)\left(x-\frac{a+b}{2}\right)-\int_x^{a+b-x} f(t)\,dt\end{aligned}$$

and

$$\int_{a+b-x}^b (t-b)\,df(t)=(x-a)f(a+b-x)-\int_{a+b-x}^b f(t)\,dt.$$

Summing the above equalities we deduce (2.8). ∎

REMARK 13. *A version of this identity for piecewise continuously differentiable functions has been obtained in* [**99**, Lemma 3.2].

The following companion of Ostrowski's inequality holds [**47**].

THEOREM 9. *Assume that the function* $f:[a,b]\to\mathbb{R}$ *is of bounded variation on* $[a,b]$, *then we have the inequalities:*

$$\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.9}$$
$$\leq \frac{1}{b-a}\left[(x-a)\bigvee_a^x(f)+\left(\frac{a+b}{2}-x\right)\bigvee_x^{a+b-x}(f)+(x-a)\bigvee_{a+b-x}^{b}(f)\right]$$
$$\leq\begin{cases}\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\bigvee_a^b(f)\\[2mm] \left[2\left(\frac{x-a}{b-a}\right)^{\alpha}+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{\alpha}\right]^{\frac{1}{\alpha}}\\ \quad\times\left[\left[\bigvee_a^x(f)\right]^{\beta}+\left[\bigvee_x^{a+b-x}(f)\right]^{\beta}+\left[\bigvee_{a+b-x}^{b}(f)\right]^{\beta}\right]^{\frac{1}{\beta}},\\ \qquad\text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1\\[2mm] \left[\frac{x-a+\frac{b-a}{2}}{b-a}\right]\max\left\{\bigvee_a^x(f),\bigvee_x^{a+b-x}(f),\bigvee_{a+b-x}^{b}(f)\right\}\end{cases}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$, *where* $\bigvee_c^d(f)$ *denotes the total variation of* f *on* $[c,d]$. *The constant* $\frac{1}{4}$ *is the best possible in the first branch of the second inequality in (2.9).*

PROOF. We use the fact that for a continuous function $p:[c,d]\to\mathbb{R}$ and a function $v:[a,b]\to\mathbb{R}$ of bounded variation, one has the inequality

$$\left|\int_c^d p(t)\,dv(t)\right|\leq\sup_{t\in[c,d]}|p(t)|\bigvee_c^d(v). \tag{2.10}$$

Taking the modulus in (2.8) we have

$$\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$
\begin{aligned}
&\leq \frac{1}{b-a}\left[\left|\int_a^x (t-a)\,df(t)\right| + \left|\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)df(t)\right|\right.\\
&\qquad\left.+\left|\int_{a+b-x}^{b}(t-b)\,df(t)\right|\right]\\
&\leq \frac{1}{b-a}\left[(x-a)\bigvee_a^x(f)+\left(\frac{a+b}{2}-x\right)\bigvee_x^{a+b-x}(f)+(x-a)\bigvee_{a+b-x}^{b}(f)\right]\\
&=: M(x)
\end{aligned}
$$

and the first inequality in (2.9) is obtained.

Now, observe that

$$
\begin{aligned}
&M(x)\\
&\quad\leq \frac{1}{b-a}\max\left\{x-a,\frac{a+b}{2}-x\right\}\left[\bigvee_a^x(f)+\bigvee_x^{a+b-x}(f)+\bigvee_{a+b-x}^{b}(f)\right]\\
&\quad= \frac{1}{b-a}\left[\frac{1}{4}(b-a)+\left|x-\frac{3a+b}{4}\right|\right]\bigvee_a^b(f)
\end{aligned}
$$

and the first branch in the second inequality in (2.9) is proved.

Using Hölder's discrete inequality we have (for $\alpha>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$) that

$$
\begin{aligned}
M(x) &\leq \frac{1}{b-a}\left[(x-a)^\alpha+\left(\frac{a+b}{2}-x\right)^\alpha+(x-a)^\alpha\right]^{\frac{1}{\alpha}}\\
&\qquad\times\left[\left[\bigvee_a^x(f)\right]^\beta+\left[\bigvee_x^{a+b-x}(f)\right]^\beta+\left[\bigvee_{a+b-x}^{b}(f)\right]^\beta\right]^{\frac{1}{\beta}}
\end{aligned}
$$

giving the second branch in the second inequality.

Finally, we have

$$
\begin{aligned}
M(x) &\leq \frac{1}{b-a}\max\left\{\bigvee_a^x(f),\bigvee_x^{a+b-x}(f),\bigvee_{a+b-x}^{b}(f)\right\}\\
&\qquad\times\left[(x-a)+\left(\frac{a+b}{2}-x\right)+(x-a)\right],
\end{aligned}
$$

which is equivalent to the last inequality in (2.9).

The sharpness of the constant $\frac{1}{4}$ in the first branch of the second inequality in (2.9) will be proved in a particular case later. ∎

COROLLARY 15. *With the assumptions in Theorem 9, we have the trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{2}\bigvee_a^b(f). \tag{2.11}$$

The constant $\frac{1}{2}$ is the best possible in (2.11).

PROOF. The proof follows from the first inequality in (2.9) on choosing $x=a$. For the sharpness of the constant, assume that (2.11) holds with a constant $A>0$, i.e.,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq A\bigvee_a^b(f). \tag{2.12}$$

If we choose $f:[a,b]\to\mathbb{R}$ with

$$f(x)=\begin{cases}1 & \text{if} \quad x=a,\\ 0 & \text{if} \quad x\in(a,b),\\ 1 & \text{if} \quad x=b,\end{cases}$$

then f is of bounded variation on $[a,b]$ and

$$\frac{f(a)+f(b)}{2}=1,\quad \int_a^b f(t)\,dt=0,\quad\text{and}\quad \bigvee_a^b(f)=2,$$

giving in (2.12) $1\leq 2A$, thus $A\geq\frac{1}{2}$ and the corollary is proved. ∎

REMARK 14. *The inequality (2.11) was first proved in a different manner in* [**70**].

COROLLARY 16. *With the assumptions in Theorem 9, one has the midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{2}\bigvee_a^b(f). \tag{2.13}$$

The constant $\frac{1}{2}$ is the best possible in (2.13).

PROOF. The proof follows from the first inequality in (2.9) on choosing $x=\frac{a+b}{2}$. For the sharpness of the constant, assume that (2.13) holds with a constant $B>0$, so that

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq B\cdot\bigvee_a^b(f). \tag{2.14}$$

If we choose $f:[a,b]\to\mathbb{R}$ with

$$f(x)=\begin{cases}0 & \text{if} \quad x\in\left[a,\frac{a+b}{2}\right),\\ 1 & \text{if} \quad x=\frac{a+b}{2},\\ 0 & \text{if} \quad x\in\left(\frac{a+b}{2},b\right],\end{cases}$$

then f is of bounded variation on $[a,b]$, and

$$f\left(\frac{a+b}{2}\right)=1,\quad \int_a^b f(t)\,dt=0,\quad \text{and} \quad \bigvee_a^b(f)=2,$$

giving in (2.14), $1\le 2B$, thus $B\ge\frac{1}{2}$. ∎

REMARK 15. *The inequality (2.13) was first proved in a different manner in* [**58**].

The best inequality we can get from Theorem 9, on using the bound provided by the first branch in the second inequality in (2.9), is incorporated in the following corollary [**47**].

COROLLARY 17. *With the assumptions in Theorem 9, one has the inequality:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{4}\bigvee_a^b(f). \tag{2.15}$$

The constant $\frac{1}{4}$ is best possible.

PROOF. Follows by Theorem 9 on choosing $x=\frac{3a+b}{4}$.

To prove the sharpness of the constant $\frac{1}{4}$, assume that (2.15) holds with a constant $C>0$, so that

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le C\cdot\bigvee_a^b(f). \tag{2.16}$$

Consider the function $f:[a,b]\to\mathbb{R}$, given by

$$f(x)=\begin{cases}1 & \text{if} \quad x\in\left\{\frac{3a+b}{4},\frac{a+3b}{4}\right\},\\ 0 & \text{if} \quad x\in[a,b]\setminus\left\{\frac{3a+b}{4},\frac{a+3b}{4}\right\}.\end{cases}$$

Then f is of bounded variation on $[a,b]$,

$$\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}=1,\quad \int_a^b f(t)\,dt=0$$

and

$$\bigvee_a^b (f) = 4,$$

giving in (2.16) $4C \geq 1$, thus $C \geq \frac{1}{4}$.

This example can be used to prove the sharpness of the constant $\frac{1}{4}$ in (2.9) as well. ∎

1.3. Applications for CDFs. Let X be a random variable taking values in the finite interval $[a,b]$, with probability density function $f : [a,b] \to [0,\infty)$ and with cumulative distribution function $F(x) = \Pr(X \leq x) = \int_a^x f(t)\,dt$.

We give the following theorem [**47**].

THEOREM 10. *With the above assumptions,*

$$\begin{aligned} &\left| \frac{1}{2}[F(x) + F(a+b-x)] - \frac{b - E(X)}{b-a} \right| \\ &\leq \frac{1}{b-a} \left\{ \left(2x - \frac{3a+b}{4} \right) [F(x) - F(a+b-x)] + (x-a) \right\} \\ &\leq \frac{1}{4} + \left| \frac{x - \frac{3a+b}{4}}{b-a} \right|, \end{aligned} \tag{2.17}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$, where $E(X)$ denotes the expectation of X.

PROOF. If we apply Theorem 9 for F, which is monotonic nondecreasing, we get

$$\begin{aligned} &\left| \frac{1}{2}[F(x) + F(a+b-x)] - \frac{1}{b-a}\int_a^b F(t)\,dt \right| \\ &\leq \frac{1}{b-a} \left[(x-a)F(x) + \left(\frac{a+b}{2} - x \right) \right. \\ &\qquad \left. \times (F(a+b-x) - F(x)) + (x-a)(1 - F(a+b-x)) \right] \\ &\leq \frac{1}{4} + \left| \frac{x - \frac{3a+b}{4}}{b-a} \right|. \end{aligned} \tag{2.18}$$

Since

$$E(X) = \int_a^b t\,dF(t) = b - \int_a^b F(t)\,dt,$$

then by (2.18) we get (2.17) and the theorem is proved. ∎

In particular, we have [**47**]:

COROLLARY 18. *With the above assumptions,*

$$\left|\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]-\frac{b-E(X)}{b-a}\right|\leq\frac{1}{4}.$$

2. Inequalities for Absolutely Continuous Functions

2.1. Some Integral Inequalities. The following identity holds [**64**].

LEMMA 2. *Assume that* $f:[a,b]\to\mathbb{R}$ *is an absolutely continuous function on* $[a,b]$*, then we have the equality*

$$\begin{aligned}(2.19)\quad &\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &=\frac{1}{b-a}\int_a^x (t-a)\,f'(t)\,dt+\frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)f'(t)\,dt\\ &\qquad+\frac{1}{b-a}\int_{a+b-x}^b (t-b)\,f'(t)\,dt,\end{aligned}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

PROOF. Using the integration by parts formula for Lebesgue integrals, we have

$$\int_a^x (t-a)\,f'(t)\,dt=f(x)(x-a)-\int_a^x f(t)\,dt,$$

$$\begin{aligned}&\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)f'(t)\,dt\\ &\qquad=f(a+b-x)\left(\frac{a+b}{2}-x\right)\\ &\qquad\quad-f(x)\left(x-\frac{a+b}{2}\right)-\int_x^{a+b-x} f(t)\,dt\end{aligned}$$

and

$$\int_{a+b-x}^b (t-b)\,f'(t)\,dt=(x-a)\,f(a+b-x)-\int_{a+b-x}^b f(t)\,dt.$$

Summing the above equalities, we deduce the desired identity (2.19). ∎

REMARK 16. *The identity (2.19) was obtained in* [**99**, Lemma 3.2] *for the case of piecewise continuously differentiable functions on* $[a,b]$.

The following result holds [**64**].

THEOREM 11. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$, then,*

$$\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{b-a}\left[\int_a^x (t-a)\left|f'(t)\right|dt + \int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'(t)\right|dt+\int_{a+b-x}^b (b-t)\left|f'(t)\right|dt\right] := M(x) \tag{2.20}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

If $f'\in L_\infty[a,b]$, then we have the inequalities

$$\begin{aligned} M(x) &\le \frac{1}{b-a}\left[\frac{(x-a)^2}{2}\left\|f'\right\|_{[a,x],\infty}+\left(\frac{a+b}{2}-x\right)^2\left\|f'\right\|_{[x,a+b-x],\infty} + \frac{(x-a)^2}{2}\left\|f'\right\|_{[a+b-x,b],\infty}\right] \\ &\le \begin{cases} \left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\left\|f'\right\|_{[a,b],\infty} \\ \left[\frac{1}{2^{\alpha-1}}\left(\frac{x-a}{b-a}\right)^{2\alpha}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^{2\alpha}\right]^{\frac{1}{\alpha}} \\ \quad\times\left[\left\|f'\right\|_{[a,x],\infty}^\beta+\left\|f'\right\|_{[x,a+b-x],\infty}^\beta+\left\|f'\right\|_{[a+b-x,b],\infty}^\beta\right]^{\frac{1}{\beta}}(b-a) \\ \qquad\qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \max\left\{\frac{1}{2}\left(\frac{x-a}{b-a}\right)^2,\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right\} \\ \quad\times\left[\left\|f'\right\|_{[a,x],\infty}+\left\|f'\right\|_{[x,a+b-x],\infty}+\left\|f'\right\|_{[a+b-x,b],\infty}\right](b-a) \end{cases} \end{aligned} \tag{2.21}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

The inequality (2.20), the first inequality in (2.21) and the constant $\frac{1}{8}$ are sharp.

PROOF. The inequality (2.20) follows by Lemma 2 on taking the modulus and using its properties.

If $f'\in L_\infty[a,b]$, then

$$\int_a^x (t-a)\left|f'(t)\right|dt \le \frac{(x-a)^2}{2}\left\|f'\right\|_{[a,x],\infty},$$

$$\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'(t)\right|dt \le \left(\frac{a+b}{2}-x\right)^2\left\|f'\right\|_{[x,a+b-x],\infty},$$

$$\int_{a+b-x}^{b}(b-t)\left|f'(t)\right|dt \le \frac{(x-a)^2}{2}\left\|f'\right\|_{[a+b-x,b],\infty}$$

and the first inequality in (2.21) is proved.

Denote

$$\tilde{M}(x) := \frac{(x-a)^2}{2}\left\|f'\right\|_{[a,x],\infty}+\left(\frac{a+b}{2}-x\right)^2\left\|f'\right\|_{[x,a+b-x],\infty} + \frac{(x-a)^2}{2}\left\|f'\right\|_{[a+b-x,b],\infty}$$

for $x \in \left[a, \frac{a+b}{2}\right]$.

First, observe that

$$\begin{aligned}\tilde{M}(x) &\le \max\left\{\left\|f'\right\|_{[a,x],\infty}, \left\|f'\right\|_{[x,a+b-x],\infty}, \left\|f'\right\|_{[a+b-x,b],\infty}\right\} \\ &\quad\times\left[\frac{(x-a)^2}{2}+\left(\frac{a+b}{2}-x\right)^2+\frac{(x-a)^2}{2}\right] \\ &= \left\|f'\right\|_{[a,b],\infty}\left[\frac{1}{8}(b-a)^2+2\left(x-\frac{3a+b}{4}\right)^2\right]\end{aligned}$$

and the first inequality in (2.21) is proved.

Using Hölder's inequality for $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, we also have

$$\begin{aligned}\tilde{M}(x) &\le \left\{\left[\frac{(x-a)^2}{2}\right]^{\alpha}+\left(x-\frac{a+b}{2}\right)^{2\alpha}+\left[\frac{(x-a)^2}{2}\right]^{\alpha}\right\}^{\frac{1}{\alpha}} \\ &\quad\times\left[\left\|f'\right\|_{[a,x],\infty}^{\beta}+\left\|f'\right\|_{[x,a+b-x],\infty}^{\beta}+\left\|f'\right\|_{[a+b-x,b],\infty}^{\beta}\right]^{\frac{1}{\beta}},\end{aligned}$$

giving the second inequality in (2.21).

Finally, we also observe that

$$\begin{aligned}\tilde{M}(x) &\le \max\left\{\frac{(x-a)^2}{2}, \left(x-\frac{a+b}{2}\right)^2\right\} \\ &\quad\times\left[\left\|f'\right\|_{[a,x],\infty}+\left\|f'\right\|_{[x,a+b-x],\infty}+\left\|f'\right\|_{[a+b-x,b],\infty}\right].\end{aligned}$$

The sharpness of the inequalities mentioned follows from Theorem 11 for $k = 1$. We omit the details. ∎

REMARK 17. *If in Theorem 11 we choose $x = a$, then we get*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{1}{4}(b-a) \|f'\|_{[a,b],\infty} \tag{2.22}$$

with $\frac{1}{4}$ as a sharp constant (see for example [**86**, p. 25]*).*

If in the same theorem we now choose $x = \frac{a+b}{2}$, then we get

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| & \\ \leq \frac{1}{8}(b-a) \left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty} + \|f'\|_{\left[\frac{a+b}{2},b\right],\infty} \right] & \\ \leq \frac{1}{4}(b-a) \|f'\|_{[a,b],\infty} & \end{aligned} \tag{2.23}$$

with the constants $\frac{1}{8}$ and $\frac{1}{4}$ being sharp. This result was obtained in [**51**].

It is natural to consider the following corollary.

COROLLARY 19. *With the assumptions in Theorem 11, we have the inequality:*

$$\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{1}{8}(b-a) \|f'\|_{[a,b],\infty}. \tag{2.24}$$

The constant $\frac{1}{8}$ is the best possible.

The case when $f' \in L_p[a,b]$, $p > 1$ is embodied in the following theorem.

THEOREM 12. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ so that $f' \in L_p[a,b]$, $p > 1$. If $M(x)$ is as defined in (2.20), then we have the bounds:*

$$\begin{aligned} M(x) \leq \frac{1}{(q+1)^{\frac{1}{q}}} \Bigg[& \left(\frac{x-a}{b-a}\right)^{1+\frac{1}{q}} \|f'\|_{[a,x],p} \\ & + 2^{\frac{1}{q}} \left(\frac{\frac{a+b}{2} - x}{b-a}\right)^{1+\frac{1}{q}} \|f'\|_{[x,a+b-x],p} \\ & + \left(\frac{x-a}{b-a}\right)^{1+\frac{1}{q}} \|f'\|_{[a+b-x,b],p} \Bigg] (b-a)^{\frac{1}{q}} \end{aligned} \tag{2.25}$$

$$\leq \frac{1}{(q+1)^{\frac{1}{q}}} \begin{cases} \left[2\left(\frac{x-a}{b-a}\right)^{1+\frac{1}{q}} + 2^{\frac{1}{q}}\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{1+\frac{1}{q}}\right] \\ \times \max\left\{\|f'\|_{[a,x],p}, \|f'\|_{[x,a+b-x],p}, \|f'\|_{[a+b-x,b],p}\right\}(b-a)^{\frac{1}{q}} \\ \\ \left[2\left(\frac{x-a}{b-a}\right)^{\alpha+\frac{\alpha}{q}} + 2^{\frac{\alpha}{q}}\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{\alpha+\frac{\alpha}{q}}\right]^{\frac{1}{\alpha}} \\ \times\left[\|f'\|^{\beta}_{[a,x],p} + \|f'\|^{\beta}_{[x,a+b-x],p} + \|f'\|^{\beta}_{[a+b-x,b],p}\right]^{\frac{1}{\beta}}(b-a)^{\frac{1}{q}} \\ \qquad\qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \\ \max\left\{\left(\frac{x-a}{b-a}\right)^{1+\frac{1}{q}}, 2^{\frac{1}{q}}\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{1+\frac{1}{q}}\right\} \\ \quad\times\left[\|f'\|_{[a,x],p} + \|f'\|_{[x,a+b-x],p} + \|f'\|_{[a+b-x,b],p}\right](b-a)^{\frac{1}{q}} \end{cases}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

PROOF. Using Hölder's integral inequality for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\int_a^x (t-a)\left|f'(t)\right| dt \leq \left(\int_a^x (t-a)^q\, dt\right)^{\frac{1}{q}} \|f'\|_{[a,x],p} = \frac{(x-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|f'\|_{[a,x],p},$$

$$\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'(t)\right| dt \leq \left(\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|^q dt\right)^{\frac{1}{q}} \|f'\|_{[x,a+b-x],p} = \frac{2^{\frac{1}{q}}\left(\frac{a+b}{2}-x\right)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|f'\|_{[x,a+b-x],p}$$

and

$$\int_{a+b-x}^b (b-t)\left|f'(t)\right| dt \leq \left(\int_{a+b-x}^b (b-t)^q\, dt\right)^{\frac{1}{q}} \|f'\|_{[a+b-x,b],p} = \frac{(x-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|f'\|_{[a+b-x,b],p}.$$

Summing the above inequalities, we deduce the first bound in (2.25).

The last part may be proved in a similar fashion to Theorem 11, and we omit the details. ∎

REMARK 18. *If in (2.25) we choose* $\alpha = q$, $\beta = p$, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, *then we get the inequality*

$$M(x) \le \frac{2^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \left[\left(\frac{x-a}{b-a} \right)^{q+1} + \left(\frac{\frac{a+b}{2} - x}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} \times (b-a)^{\frac{1}{q}} \|f'\|_{[a,b],p} \tag{2.26}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

REMARK 19. *If in Theorem 12 we choose* $x = a$, *then we get the trapezoid inequality*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le \frac{1}{2} \cdot \frac{(b-a)^{\frac{1}{q}} \|f'\|_{[a,b],p}}{(q+1)^{\frac{1}{q}}}. \tag{2.27}$$

The constant $\frac{1}{2}$ *is the best possible (see for example* [**86**, p. 42]*).*

Indeed, if we assume that (2.27) holds with a constant $C > 0$, instead of $\frac{1}{2}$, i.e.,

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le C \cdot \frac{(b-a)^{\frac{1}{q}} \|f'\|_{[a,b],p}}{(q+1)^{\frac{1}{q}}}, \tag{2.28}$$

then for the function $f : [a,b] \to \mathbb{R}$, $f(x) = k \left| x - \frac{a+b}{2} \right|$, $k > 0$, we have

$$\frac{f(a) + f(b)}{2} = k \cdot \frac{b-a}{2},$$

$$\frac{1}{b-a} \int_a^b f(t)\,dt = k \cdot \frac{b-a}{4},$$

$$\|f'\|_{[a,b],p} = k(b-a)^{\frac{1}{p}};$$

and by (2.28) we deduce

$$\left| \frac{k(b-a)}{2} - \frac{k(b-a)}{4} \right| \le \frac{C \cdot k(b-a)}{(q+1)^{\frac{1}{q}}},$$

giving $C \ge \frac{(q+1)^{\frac{1}{q}}}{4}$. Letting $q \to 1+$, we deduce that $C \ge \frac{1}{2}$, and the sharpness of the constant is proved.

REMARK 20. *If in Theorem 12 we choose $x = \frac{a+b}{2}$, then we get the midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{2}\cdot\frac{(b-a)^{\frac{1}{q}}}{2^{\frac{1}{q}}(q+1)^{\frac{1}{q}}}\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],p} + \|f'\|_{\left[\frac{a+b}{2},b\right],p}\right] \leq \frac{1}{2}\cdot\frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[a,b],p}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1. \tag{2.29}$$

In both inequalities the constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

To show this fact, assume that (2.29) holds with $C, D > 0$, i.e.,

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq C\cdot\frac{(b-a)^{\frac{1}{q}}}{2^{\frac{1}{q}}(q+1)^{\frac{1}{q}}}\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],p} + \|f'\|_{\left[\frac{a+b}{2},b\right],p}\right] \leq D\cdot\frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[a,b],p}. \tag{2.30}$$

For the function $f : [a,b] \to \mathbb{R}$, $f(x) = k\left|x - \frac{a+b}{2}\right|$, $k > 0$, we have

$$f\left(\frac{a+b}{2}\right) = 0, \quad \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{k(b-a)}{4},$$

$$\|f'\|_{\left[a,\frac{a+b}{2}\right],p} + \|f'\|_{\left[\frac{a+b}{2},b\right],p} = 2\left(\frac{b-a}{2}\right)^{\frac{1}{p}} k = 2^{\frac{1}{q}}(b-a)^{\frac{1}{p}} k,$$

$$\|f'\|_{[a,b],p} = (b-a)^{\frac{1}{p}} k;$$

and then by (2.30) we deduce

$$\frac{k(b-a)}{4} \leq C\cdot\frac{k(b-a)}{(q+1)^{\frac{1}{q}}} \leq D\cdot\frac{k(b-a)}{(q+1)^{\frac{1}{q}}},$$

giving $C, D \geq \frac{(q+1)^{\frac{1}{q}}}{4}$ for any $q > 1$. Letting $q \to 1+$, we deduce $C, D \geq \frac{1}{2}$ and the sharpness of the constants in (2.29) is proved.

The following result is useful in providing the best quadrature rule in the class for approximating the integral of an absolutely continuous function whose derivative is in $L_p[a,b]$ [**64**].

COROLLARY 20. *Assume that $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function so that $f'\in L_p[a,b]$, $p>1$, then*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{4}\frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left\|f'\right\|_{[a,b],p}, \tag{2.31}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

The constant $\frac{1}{4}$ is the best possible.

PROOF. The inequality follows by Theorem 12 and Remark 18 on choosing $x=\frac{3a+b}{4}$.

To prove the sharpness of the constant, assume that (2.31) holds with a constant $E>0$, and so,

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq E\cdot\frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left\|f'\right\|_{[a,b],p}. \tag{2.32}$$

Consider the function $f:[a,b]\to\mathbb{R}$,

$$f(x)=\begin{cases}\left|x-\dfrac{3a+b}{4}\right| & \text{if } x\in\left[a,\frac{a+b}{2}\right]\\[2ex] \left|x-\dfrac{a+3b}{4}\right| & \text{if } x\in\left(\frac{a+b}{2},b\right].\end{cases}$$

It follows that f is absolutely continuous and $f'\in L_p[a,b]$, $p>1$. We also have

$$\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]=0,\quad \frac{1}{b-a}\int_a^b f(t)\,dt=\frac{b-a}{8},$$

$$\left\|f'\right\|_{[a,b],p}=(b-a)^{\frac{1}{p}},$$

and then, by (2.32), we obtain:

$$\frac{b-a}{8}\leq E\frac{(b-a)}{(q+1)^{\frac{1}{q}}}$$

giving $E\geq\frac{(q+1)^{\frac{1}{q}}}{8}$ for any $q>1$, i.e., $E\geq\frac{1}{4}$, and the corollary is proved. ∎

If one is interested in obtaining bounds in terms of the $1-$norm for the derivative, then the following result may be useful [**64**].

THEOREM 13. *Assume that the function $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. If $M(x)$ is as in equation (2.20), then we have the bounds*

$$M(x)\leq\left(\frac{x-a}{b-a}\right)\|f'\|_{[a,x],1}+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)\|f'\|_{[x,a+b-x],1}+\left(\frac{x-a}{b-a}\right)\|f'\|_{[a+b-x,b],1} \tag{2.33}$$

$$\leq\begin{cases}\left[\dfrac{1}{4}+\left|\dfrac{x-\frac{3a+b}{4}}{b-a}\right|\right]\|f'\|_{[a,b],1}\\[2ex] \left[2\left(\dfrac{x-a}{b-a}\right)^{\alpha}+\left(\dfrac{\frac{a+b}{2}-x}{b-a}\right)^{\alpha}\right]^{\frac{1}{\alpha}}\\ \times\left[\|f'\|_{[a,x],1}^{\beta}+\|f'\|_{[x,a+b-x],1}^{\beta}+\|f'\|_{[a+b-x,b],1}^{\beta}\right]^{\frac{1}{\beta}}\\ \hfill \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\\[2ex] \left[\dfrac{x+\frac{b-3a}{2}}{b-a}\right]\max\left[\|f'\|_{[a,x],1},\|f'\|_{[x,a+b-x],1},\|f'\|_{[a+b-x,b],1}\right].\end{cases}$$

The proof is as in Theorem 11 so we omit the details.

REMARK 21. *By the use of Theorem 12, for $x=a$, we get the trapezoid inequality (see for example* [**86**, p. 55]*)*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{2}\|f'\|_{[a,b],1}. \tag{2.34}$$

If in (2.33) we also choose $x=\frac{a+b}{2}$, then we get the mid point inequality (see for example [**86**, p. 56]*)*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{2}\|f'\|_{[a,b],1}. \tag{2.35}$$

The following corollary also holds [**64**].

COROLLARY 21. *With the assumption in Theorem 12, one has the inequality:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{4}\|f'\|_{[a,b],1}. \tag{2.36}$$

2.2. Applications for PDFs. Summarising some of the results in Section 2, we may state that for $f : [a,b] \to \mathbb{R}$ being an absolutely continuous function, we have the inequality

$$\left| \frac{1}{2} \left[f(x) + f(a+b-x) \right] - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{2.37}$$
$$\leq \begin{cases} \left[\frac{1}{8} + 2 \left(\frac{x - \frac{3a+b}{4}}{b-a} \right)^2 \right] (b-a) \left\| f' \right\|_{[a,b],\infty} \text{ if } f' \in L_\infty [a,b] \\ \frac{2^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \left[\left(\frac{x-a}{b-a} \right)^{q+1} + \left(\frac{\frac{a+b}{2} - x}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} (b-a)^{\frac{1}{q}} \left\| f' \right\|_{[a,b],p}, \\ \qquad \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \text{ and } f' \in L_p [a,b]; \\ \left[\frac{1}{4} + \left| \frac{x - \frac{3a+b}{4}}{b-a} \right| \right] \left\| f' \right\|_{[a,b],1}, \end{cases}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Now, let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f : [a,b] \to [0,\infty)$ and with the cumulative distribution function $F(x) = \Pr(X \leq x) = \int_a^x f(t)\,dt$.

The following result holds [**64**].

Theorem 14. *With the above assumptions, we have the inequality*

$$\left| \frac{1}{2} \left[F(x) + F(a+b-x) \right] - \frac{b - E(X)}{b-a} \right| \tag{2.38}$$
$$\leq \begin{cases} \left[\frac{1}{8} + 2 \left(\frac{x - \frac{3a+b}{4}}{b-a} \right)^2 \right] (b-a) \left\| f \right\|_{[a,b],\infty} \textit{ if } f \in L_\infty [a,b] \\ \frac{2^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \left[\left(\frac{x-a}{b-a} \right)^{q+1} + \left(\frac{\frac{a+b}{2} - x}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} (b-a)^{\frac{1}{q}} \left\| f \right\|_{[a,b],p}, \\ \qquad \textit{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \textit{ and } f \in L_p [a,b]; \\ \left[\frac{1}{4} + \left| \frac{x - \frac{3a+b}{4}}{b-a} \right| \right], \end{cases}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

Proof. The proof follows by (2.37) on choosing $f = F$ and taking into account that

$$E(X) = \int_a^b t dF(t) = b - \int_a^b F(t)\,dt.$$

∎

In particular, we have [**64**]:

COROLLARY 22. *With the above assumptions, we have*

$$\left|\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]-\frac{b-E(X)}{b-a}\right| \tag{2.39}$$
$$\leq \begin{cases} \frac{1}{8}(b-a)\|f\|_{[a,b],\infty} & \text{if } f\in L_\infty[a,b] \\ \frac{1}{4}\cdot\frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f\|_{[a,b],p}\,, \ \text{if } \ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \text{and } f\in L_p[a,b]\,; \\ \frac{1}{4}. \end{cases}$$

3. Ostrowski's Inequality for Convex Functions

3.1. Introduction. The result known in the literature as Ostrowski's inequality [**112**], see (1.41) will again be utilised in this section.

The following Ostrowski type result holds (see [**89**], [**90**] and [**92**]).

THEOREM 15. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$, then, for all $x\in[a,b]$, we have:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.40}$$
$$\leq \begin{cases} \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty \ \text{ if } f'\in L_\infty[a,b]\,; \\ \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{x-a}{b-a}\right)^{p+1}+\left(\frac{b-x}{b-a}\right)^{p+1}\right]^{\frac{1}{p}}(b-a)^{\frac{1}{p}}\|f'\|_q \\ \qquad\qquad \text{if } f'\in L_q[a,b]\,,\ \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1\,; \end{cases}$$

where $\|\cdot\|_r$ $(r\in[1,\infty])$ are the usual Lebesgue norms on $L_r[a,b]$, i.e.,

$$\|g\|_\infty := ess\ \sup_{t\in[a,b]}|g(t)|$$

and

$$\|g\|_r := \left(\int_a^b |g(t)|^r\,dt\right)^{\frac{1}{r}},\ r\in[1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ are all sharp in the sense that they cannot be replaced by smaller quantities.

The above inequalities can also be obtained from Fink's result in [**96**] on choosing $n=1$ and performing some appropriate computations.

If we drop the condition of absolute continuity and assume that f is Hölder continuous, then we can state the following result (see [**78**]):

THEOREM 16. *Let $f : [a,b] \to \mathbb{R}$ be of $r-H-$Hölder type, i.e.,*

$$|f(x) - f(y)| \leq H|x-y|^r, \text{ for all } x, y \in [a,b], \tag{2.41}$$

where $r \in (0,1]$ and $H > 0$ are fixed, then, for all $x \in [a,b]$, we have the inequality:

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1} + \left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r. \tag{2.42}$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r = 1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see [**71**])

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)L. \tag{2.43}$$

Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the continuity condition of the function, and assumes that it is of bounded variation, then the following result may be stated (see [**59**]).

THEOREM 17. *Assume that $f : [a,b] \to \mathbb{R}$ is of bounded variation and denote by $\bigvee_a^b(f)$ its total variation, then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f) \tag{2.44}$$

for all $x \in [a,b]$.
The constant $\frac{1}{2}$ is the best possible.

If we assume that f is monotonically increasing, then the inequality (2.44) may be improved in the following manner [**63**] (see also [**31**]).

THEOREM 18. *Let $f:[a,b]\to\mathbb{R}$ be monotonic nondecreasing, then for all $x\in[a,b]$, we have the inequality:*

$$\begin{aligned}(2.45)\qquad & \left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ & \leq \frac{1}{b-a}\left\{[2x-(a+b)]\,f(x)+\int_a^b sgn\,(t-x)\,f(t)\,dt\right\} \\ & \leq \frac{1}{b-a}\left\{(x-a)[f(x)-f(a)]+(b-x)[f(b)-f(x)]\right\} \\ & \leq \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right][f(b)-f(a)].\end{aligned}$$

All the inequalities in (2.45) are sharp and the constant $\frac{1}{2}$ is the best possible.

In the next section we establish an Ostrowski type inequality for convex functions. Applications for PDFs are also provided.

3.2. The Results. The following theorem providing a lower bound for the Ostrowski difference $\int_a^b f(t)\,dt-(b-a)\,f(x)$ for convex function $f(\cdot)$ holds [**53**].

THEOREM 19. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$, then for any $x\in(a,b)$, we have the inequality:*

$$\begin{aligned}(2.46)\qquad & \frac{1}{2}\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right] \\ & \qquad\leq \int_a^b f(t)\,dt-(b-a)\,f(x).\end{aligned}$$

The constant $\frac{1}{2}$ in the left hand side of (2.46) is sharp.

PROOF. It is easy to see that for any locally absolutely continuous function $f:(a,b)\to\mathbb{R}$, we have the identity

$$(2.47)\qquad \int_a^x (t-a)\,f'(t)\,dt+\int_x^b (t-b)\,f'(t)\,dt=f(x)-\int_a^b f(t)\,dt,$$

for any $x\in(a,b)$ where f' is the derivative of f which exists a.e. on (a,b).

Since f is convex, then it is locally Lipschitzian and thus (2.47) holds. Moreover, for any $x\in(a,b)$, we have the inequalities

$$(2.48)\qquad f'(t)\leq f'_-(x)\ \text{ for a.e. } t\in[a,x]$$

and

$$f'(t) \geq f'_{+}(x) \text{ for a.e. } t \in [x, b]. \tag{2.49}$$

If we multiply (2.48) by $t - a \geq 0$, $t \in [a, x]$, and integrate over $[a, x]$, we get

$$\int_a^x (t-a) f'(t)\, dt \leq \frac{1}{2}(x-a)^2 f'_{-}(x) \tag{2.50}$$

and if we multiply (2.49) by $b - t \geq 0$, $t \in [x, b]$, and integrate over $[x, b]$, we also have

$$\int_x^b (b-t) f'(t)\, dt \geq \frac{1}{2}(b-x)^2 f'_{+}(x). \tag{2.51}$$

Finally, if we subtract (2.51) from (2.50) and use the representation (2.47) we deduce the desired inequality (2.46).

Now, assume that (2.46) holds with a constant $C > 0$ instead of $\frac{1}{2}$, so that,

$$C\left[(b-x)^2 f'_{+}(x) - (x-a)^2 f'_{-}(x)\right] \leq \int_a^b f(t)\, dt - (b-a) f(x). \tag{2.52}$$

Consider the convex function $f_0(t) := k\left|t - \frac{a+b}{2}\right|$, $k > 0$, $t \in [a, b]$, then

$$f'_{0+}\left(\frac{a+b}{2}\right) = k, \quad f'_{0-}\left(\frac{a+b}{2}\right) = -k, \quad f_0\left(\frac{a+b}{2}\right) = 0$$

and

$$\int_a^b f_0(t)\, dt = \frac{1}{4}k(b-a)^2.$$

If in (2.52) we choose f_0 as above and $x = \frac{a+b}{2}$, then we get

$$C\left[\frac{1}{4}(b-a)^2 k + \frac{1}{4}(b-a)^2 k\right] \leq \frac{1}{4}k(b-a)^2,$$

which gives $C \leq \frac{1}{2}$, and the sharpness of the constant is proved. ∎

Now, recall the following inequality, which is well known in the literature as the *Hermite-Hadamard inequality* for convex functions, holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\, dt \leq \frac{f(a) + f(b)}{2}. \tag{HH}$$

The following corollary which improves the first Hermite-Hadamard inequality (HH) holds [**53**].

Corollary 23. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$, then*

$$0 \leq \frac{1}{8}\left[f'_{+}\left(\frac{a+b}{2}\right) - f'_{-}\left(\frac{a+b}{2}\right)\right](b-a) \leq \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right). \tag{2.53}$$

The constant $\frac{1}{8}$ is sharp.

The proof is obvious by the above theorem. The sharpness of the constant is obtained for $f_0(t) := k\left|t - \frac{a+b}{2}\right|$, $t \in [a,b]$, $k > 0$.

When x is a point of differentiability, we can state the following corollary as well.

Corollary 24. *Let f be as in Theorem 19. If $x \in (a,b)$ is a point of differentiability for f, then*

$$\left(\frac{a+b}{2} - x\right) f'(x) \leq \frac{1}{b-a}\int_a^b f(t)\,dt - f(x). \tag{2.54}$$

Remark 22. *If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex on I and if we choose $x \in \mathring{I}$ ($\mathring{I}$ is the interior of I), $b = x + \frac{h}{2}$, $a = x - \frac{h}{2}$, $h > 0$, for $a, b \in I$, from (2.46), we have,*

$$0 \leq \frac{1}{8}h^2\left[f'_{+}(x) - f'_{-}(x)\right] \leq \int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(t)\,dt - hf(x), \tag{2.55}$$

and the constant $\frac{1}{8}$ is sharp in (2.55).

The following result, providing an upper bound for the Ostrowski difference $\int_a^b f(t)\,dt - (b-a) f(x)$, also holds [**53**].

Theorem 20. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$, then for any $x \in [a,b]$, we have the inequality:*

$$\int_a^b f(t)\,dt - (b-a) f(x) \leq \frac{1}{2}\left[(b-x)^2 f'_{-}(b) - (x-a)^2 f'_{+}(a)\right]. \tag{2.56}$$

The constant $\frac{1}{2}$ is sharp.

Proof. If either $f'_{+}(a) = -\infty$ or $f'_{-}(b) = +\infty$, then the inequality (2.56) evidently holds true.

Assume that $f'_{+}(a)$ and $f'_{-}(b)$ are finite.

Since f is convex on $[a,b]$, we have

$$f'(t) \geq f'_{+}(a) \text{ for a.e. } t \in [a,x] \tag{2.57}$$

and

$$f'(t) \le f'_-(b) \text{ for a.e. } t \in [x,b]. \tag{2.58}$$

If we multiply (2.57) by $t-a \ge 0$, $t \in [a,x]$, and integrate over $[a,x]$, we deduce

$$\int_a^x (t-a) f'(t)\, dt \ge \frac{1}{2}(x-a)^2 f'_+(a) \tag{2.59}$$

and if we multiply (2.58) by $b-t \ge 0$, $t \in [x,b]$, and integrate over $[x,b]$, we also have

$$\int_x^b (b-t) f'(t)\, dt \le \frac{1}{2}(b-x)^2 f'_-(b). \tag{2.60}$$

Finally, if we subtract (2.59) from (2.60) and use the representation (2.47), we deduce the desired inequality (2.56).

Now, assume that (2.56) holds with a constant $D>0$ instead of $\frac{1}{2}$, i.e.,

$$\begin{aligned} \int_a^b f(t)\, dt - (b-a) f(x) & \\ &\le D\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\right]. \end{aligned} \tag{2.61}$$

If we consider the convex function $f_0 : [a,b] \to \mathbb{R}$, $f_0(t) = k\left|t - \frac{a+b}{2}\right|$, then we have $f'_-(b) = k$, $f'_+(a) = -k$ and by (2.61) we deduce for $x = \frac{a+b}{2}$ that

$$\frac{1}{4}k(b-a)^2 \le D\left[\frac{1}{4}k(b-a)^2 + \frac{1}{4}k(b-a)^2\right],$$

giving $D \ge \frac{1}{2}$, and the sharpness of the constant is proved. ∎

The following corollary, related to the Hermite-Hadamard inequality, is interesting as well [**53**].

Corollary 25. *Let $f:[a,b]\to\mathbb{R}$ be convex on $[a,b]$, then*

$$\begin{aligned} 0 &\le \frac{1}{b-a}\int_a^b f(t)\, dt - f\left(\frac{a+b}{2}\right) \\ &\le \frac{1}{8}\left[f'_-(b) - f'_+(a)\right](b-a) \end{aligned} \tag{2.62}$$

and the constant $\frac{1}{8}$ is sharp.

REMARK 23. *Denote* $B := f'_-(b)$, $A := f'_+(a)$ *and assume that* $B \neq A$, *i.e.,* f *is not constant on* (a,b), *then*

$$(b-x)^2 B - (x-a)^2 A = (B-A)\left[x - \left(\frac{bB-aA}{B-A}\right)\right]^2 - \frac{AB}{B-A}(b-a)^2$$

and by (2.56) we get

$$\int_a^b f(t)\,dt - (b-a) f(x) \leq \frac{1}{2}(B-A)\left\{\left[x - \left(\frac{bB-aA}{B-A}\right)\right]^2 - \frac{AB}{(B-A)^2}(b-a)^2\right\} \tag{2.63}$$

for any $x \in [a,b]$.

If $A \geq 0$ *then* $x_0 = \frac{bB-aA}{B-A} \in [a,b]$ *and by (2.63) we get, choosing* $x = \frac{bB-aA}{B-A}$, *that*

$$0 \leq \frac{1}{2}\frac{AB}{B-A}(b-a) \leq f\left(\frac{bB-aA}{B-A}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt. \tag{2.64}$$

REMARK 24. *If* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *is convex on* I *and if we choose* $x \in \mathring{I}$, $b = x + \frac{h}{2}$, $a = x - \frac{h}{2}$, $h > 0$ *such that* $a, b \in I$, *then from (2.56) we deduce:*

$$0 \leq \int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(t)\,dt - hf(x) \leq \frac{1}{8}h^2\left[f'_-\left(x+\frac{h}{2}\right) - f'_+\left(x-\frac{h}{2}\right)\right], \tag{2.65}$$

and the constant $\frac{1}{8}$ *is sharp.*

3.3. Inequalities for Integral Means. The following result in comparing two integral means has been obtained in [**53**].

THEOREM 21. *Let $f : [a, b] \to \mathbb{R}$ be a convex function and $c, d \in [a, b]$ with $c < d$, then we have the inequalities*

$$\begin{aligned}(2.66)\quad & \frac{a+b}{2} \cdot \frac{f(d) - f(c)}{d-c} - \frac{df(d) - cf(c)}{d-c} + \frac{1}{d-c} \int_c^d f(x)\,dx \\ & \leq \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{1}{d-c} \int_c^d f(x)\,dx \\ & \leq \frac{f'_-(b)\left[(b-d)^2 + (b-d)(b-c) + (b-c)^2\right]}{6(b-a)} \\ & \qquad - \frac{f'_+(a)\left[(d-a)^2 + (d-a)(c-a) + (c-a)^2\right]}{6(b-a)}.\end{aligned}$$

PROOF. Since f is convex, then for *a.e.* $x \in [a, b]$, we have (by (2.54)) that

$$(2.67)\qquad \left(\frac{a+b}{2} - x\right) f'(x) \leq \frac{1}{b-a} \int_a^b f(t)\,dt - f(x).$$

Integrating (2.67) on $[c, d]$ we deduce

$$\begin{aligned}(2.68)\quad & \frac{1}{d-c} \int_c^d \left(\frac{a+b}{2} - x\right) f'(x)\,dx \\ & \qquad \leq \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{1}{d-c} \int_c^d f(x)\,dx.\end{aligned}$$

Since

$$\begin{aligned}& \frac{1}{d-c} \int_c^d \left(\frac{a+b}{2} - x\right) f'(x)\,dx \\ & \quad = \frac{1}{d-c} \left[\left(\frac{a+b}{2} - d\right) f(d) - \left(\frac{a+b}{2} - c\right) f(c) + \int_c^d f(x)\,dx\right]\end{aligned}$$

then by (2.68) we deduce the first part of (2.66).

Using (2.56), we may write for any $x \in [a, b]$ that

$$\begin{aligned}(2.69)\quad & \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \\ & \qquad \leq \frac{1}{2(b-a)} \left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\right].\end{aligned}$$

Integrating (2.69) over $[c,d]$, we deduce

$$\begin{aligned}(2.70)\quad & \frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{d-c}\int_c^d f(x)\,dx \\ & \leq \frac{1}{2(b-a)}\left[f'_-(b)\frac{1}{d-c}\int_c^d (b-x)^2\,dx \right. \\ & \qquad \left. -f'_+(a)\frac{1}{d-c}\int_c^d (x-a)^2\,dx\right].\end{aligned}$$

Since

$$\frac{1}{d-c}\int_c^d (b-x)^2\,dx=\frac{(b-d)^2+(b-d)(b-c)+(b-c)^2}{3}$$

and

$$\frac{1}{d-c}\int_c^d (x-a)^2\,dx=\frac{(d-a)^2+(d-a)(c-a)+(c-a)^2}{3},$$

then, by (2.70), we deduce the second part of (2.66). ∎

Remark 25. *If we choose* $f(x)=x^p$, $p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$ *or* $f(x)=\frac{1}{x}$ *or even* $f(x)=-\ln x$, $x\in[a,b]\subset(0,\infty)$, *in the above inequalities, then a great number of interesting results for* $p-$*logarithmic, logarithmic and identric means may be obtained. We leave this as an exercise for the interested reader.*

3.4. Applications for PDFs. Let X be a random variable with the *probability density function* $f:[a,b]\subset\mathbb{R}\to\mathbb{R}_+$ and with *cumulative distribution function* $F(x)=\Pr(X\leq x)$.

The following theorem holds [**53**].

Theorem 22. *If* $f:[a,b]\subset\mathbb{R}\to\mathbb{R}_+$ *is monotonically increasing on* $[a,b]$, *then we have the inequality:*

$$\begin{aligned}(2.71)\quad & \frac{1}{2}\left[(b-x)^2 f_+(x)-(x-a)^2 f_-(x)\right] \\ & \leq b-E(X)-(b-a)F(x) \\ & \leq \frac{1}{2}\left[(b-x)^2 f_-(b)-(x-a)^2 f_+(a)\right]\end{aligned}$$

for any $x\in(a,b)$, *where* $f_-(\alpha)$ *means the left limit in* α *while* $f_+(\alpha)$ *means the right limit in* α.
The constant $\frac{1}{2}$ *is sharp in both inequalities.*
The second inequality also holds for $x=a$ *of* $x=b$.

PROOF. The proof follows by Theorem 19 and 20 applied to the convex cdf function $F(x) = \int_a^x f(t)\,dt,\ \ x \in [a,b]$ and taking into account that

$$\int_a^b F(x)\,dx = b - E(X).$$

∎

Finally, we state the following corollary for estimating the probability $\Pr\left(X \le \frac{a+b}{2}\right)$ [**53**].

COROLLARY 26. *With the above assumptions, we have*

$$\begin{aligned} & b - E(X) - \frac{1}{8}(b-a)^2\left[f_-(b) - f_+(a)\right] \\ & \le \Pr\left(X \le \frac{a+b}{2}\right) \\ & \le b - E(X) - \frac{1}{8}(b-a)^2\left[f_+\left(\frac{a+b}{2}\right) - f_-\left(\frac{a+b}{2}\right)\right]. \end{aligned} \tag{2.72}$$

4. A New Ostrowski Type Inequality and Applications

4.1. Introduction. Let the functional $S(f;a,b)$ be defined by

$$S(f;a,b) = f(x) - \mathcal{M}(f;a,b), \tag{2.73}$$

where

$$\mathcal{M}(f;a,b) = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{2.74}$$

The functional $S(f;a,b)$ represents the deviation of $f(x)$ from its integral mean over $[a,b]$.

In 1938, A. Ostrowski proved the following integral inequality [**112**] as mentioned earlier.

THEOREM 23. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) and assume $|f'(x)| \le M$ for all $x\in(a,b)$, then the inequality*

$$|S(f;a,b)| \le \left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right]\frac{M}{b-a} \tag{2.75}$$

holds for all $x \in [a,b]$. The constant $\frac{1}{4}$ is best possible.

In a series of papers, Dragomir and Wang [**89**] – [**92**] proved (2.75) and other variants for $f' \in L_p[a,b]$ for $p \ge 1$, the Lebesgue norms making use of a Peano kernel approach and Montgomery's identity [**110**,

p. 585]. Montgomery's identity states that for absolutely continuous mappings $f:[a,b]\to\mathbb{R}$

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{b-a}\int_a^b p(x,t)\,f'(t)\,dt, \tag{2.76}$$

where the kernel $p:[a,b]^2\to\mathbb{R}$ is given by

$$p(x,t)=\begin{cases} t-a, & a\le t\le x\le b,\\ t-b, & a\le x<t\le b.\end{cases}$$

If we assume that $f'\in L_\infty[a,b]$ and $\|f'\|_\infty := ess\sup\limits_{t\in[a,b]}|f'(t)|$ then M in (2.75) may be replaced by $\|f'\|_\infty$.

Dragomir and Wang [**89**] – [**92**], utilising an integration by parts argument, ostensibly Montgomery's identity (2.76), obtained

$$|S(f;a,b)| \le \begin{cases} \left[\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2\right]\frac{\|f'\|_\infty}{b-a}, & f'\in L_\infty[a,b];\\ \left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}\frac{\|f'\|_p}{b-a}, & f'\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\frac{\|f'\|_1}{b-a}, \end{cases} \tag{2.77}$$

where $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ and the constants $\frac{1}{4}$, $\frac{1}{(q+1)^{\frac{1}{q}}}$ and $\frac{1}{2}$ are all sharp.

In this section we obtain bounds for the deviation of a function from integral means that not necessarily cover the whole interval. The Ostrowski type results are recaptured as special cases. Following an identity obtained in Subsection 4.2 and the resulting bounds, perturbed results arising from the Chebychev functional are investigated in Subsection 4.3. The final Subsection 4.4 applies the results to the cumulative distribution function.

4.2. Results. We commence with the following identity which although of interest in itself, will be used to obtain bounds [**28**].

LEMMA 3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping. Denote by $P(x,\cdot):[a,b]\to\mathbb{R}$ the kernel given by*

$$P(x,t)=\begin{cases} \frac{\alpha}{\alpha+\beta}\left(\frac{t-a}{x-a}\right), & t\in[a,x]\\ \frac{-\beta}{\alpha+\beta}\left(\frac{b-t}{b-x}\right), & t\in(x,b]\end{cases} \tag{2.78}$$

where $\alpha, \beta \in \mathbb{R}$ are nonnegative and not both zero, then the identity

$$\int_a^b P(x,t) f'(t)\,dt = f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(t)\,dt + \frac{\beta}{b-x}\int_x^b f(t)\,dt\right] \tag{2.79}$$

holds.

PROOF. From (2.78), we have

$$\begin{aligned}
&\int_a^b P(x,t) f'(t)\,dt \\
&= \frac{\alpha}{\alpha+\beta}\int_a^x \left(\frac{t-a}{x-a}\right) f'(t)\,dt - \frac{\beta}{\alpha+\beta}\int_x^b \left(\frac{b-t}{b-x}\right) f'(t)\,dt \\
&= \frac{\alpha}{\alpha+\beta}\left\{\left(\frac{t-a}{x-a}\right) f(t)\right]_{t=a}^{x} - \frac{1}{x-a}\int_a^x f(t)\,dt\Bigg\} \\
&\quad - \frac{\beta}{\alpha+\beta}\left\{\left(\frac{b-t}{b-x}\right) f(t)\right]_{t=x}^{b} - \frac{1}{b-x}\int_x^b f(t)\,dt\Bigg\},
\end{aligned}$$

where the integration by parts formula has been utilised on the separate intervals $[a,x]$ and $(x,b]$. Simplification of the expressions readily produces the identity as stated. ∎

THEOREM 24. *([**28**]) Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping and define*

$$\mathcal{T}(x;\alpha,\beta) := f(x) - \frac{1}{\alpha+\beta}\left[\alpha\mathcal{M}(f;a,x) + \beta\mathcal{M}(f;x,b)\right], \tag{2.80}$$

where $\mathcal{M}(f;a,b)$ is the integral mean as defined by (2.74), then

$$|\mathcal{T}(x;\alpha,\beta)| \leq \begin{cases} [\alpha(x-a)+\beta(b-x)]\frac{\|f'\|_\infty}{2(\alpha+\beta)}, & f' \in L_\infty[a,b]; \\ [\alpha^q(x-a)+\beta^q(b-x)]^{\frac{1}{q}}\frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}(\alpha+\beta)}, & f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[1+\frac{|\alpha-\beta|}{\alpha+\beta}\right]\frac{\|f'\|_1}{2}, & \end{cases} \tag{2.81}$$

where $\|h\|_.$ are the usual Lebesgue norms for $h \in L_.[a,b]$.

PROOF. Taking the modulus of (2.79) we have, from (2.80) and (2.74),

$$|\mathcal{T}(x;\alpha,\beta)| = \left|\int_a^b P(x,t) f'(t)\,dt\right| \leq \int_a^b |P(x,t)|\,|f'(t)|\,dt, \tag{2.82}$$

where we have used the well known properties of the integral and modulus.

Thus, for $f' \in L_\infty[a,b]$ from (2.82) we have,

$$|\mathcal{T}(x;\alpha,\beta)| \leq \|f'\|_\infty \int_a^b |P(x,t)|\,dt$$

from which a simple calculation using (2.78) gives

$$\begin{aligned}\int_a^b |P(x,t)|\,dt &= \frac{\alpha}{\alpha+\beta}\int_a^x \frac{t-a}{x-a}dt + \frac{\beta}{\alpha+\beta}\int_x^b \frac{b-t}{b-x}dt \\ &= \left[\frac{\alpha}{\alpha+\beta}(x-a) + \frac{\beta}{\alpha+\beta}(b-x)\right]\int_0^1 udu\end{aligned}$$

and hence the first inequality follows.

Further, using Hölder's integral inequality, we have for $f' \in L_p[a,b]$ from (2.82)

$$|\mathcal{T}(x;\alpha,\beta)| \leq \|f'\|_p \left(\int_a^b |P(x,t)|^q\,dt\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$. Now

$$\begin{aligned}&(\alpha+\beta)\left(\int_a^b |P(x,t)|^q\,dt\right)^{\frac{1}{q}} \\ &\quad = \left[\alpha^q \int_a^x \left(\frac{t-a}{x-a}\right)^q dt + \beta^q \int_x^b \left(\frac{b-t}{b-x}\right)^q dt\right]^{\frac{1}{q}} \\ &\quad = \left[\alpha^q (x-a) + \beta^q (b-x)\right]^{\frac{1}{q}} \left(\int_0^1 u^q du\right)^{\frac{1}{q}}\end{aligned}$$

and so the second inequality is obtained.

Finally, for $f' \in L_1[a,b]$ we have from (2.82) and using (2.78)

$$|\mathcal{T}(x;\alpha,\beta)| \leq \sup_{t\in[a,b]} |P(x,t)|\,\|f'\|_1,$$

where

$$(\alpha+\beta)\sup_{t\in[a,b]}|P(x,t)| = \max\{\alpha,\beta\} = \frac{\alpha+\beta}{2} + \left|\frac{\alpha-\beta}{2}\right|$$

and so the proof is completed. ∎

REMARK 26. *It should be noted, that from (2.80) and (2.73),*

$$(\alpha+\beta)\mathcal{T}(x;\alpha,\beta)=\alpha\mathcal{S}(f;a,x)+\beta\mathcal{S}(f;x,b) \tag{2.83}$$

and so from (2.77), using the triangle inequality,

$$\left|(\alpha+\beta)\mathcal{T}(x;\alpha,\beta)\right| \le \begin{cases} \frac{\alpha}{2}(x-a)\|f'\|_{\infty,[a,x]}+\frac{\beta}{2}(b-x)\|f'\|_{\infty,[x,b]}, \\ \alpha\left(\frac{x-a}{q+1}\right)^{\frac{1}{q}}\|f'\|_{p,[a,x]}+\beta\left(\frac{b-x}{q+1}\right)^{\frac{1}{q}}\|f'\|_{p,[x,b]}, \\ \alpha\|f'\|_{1,[a,x]}+\beta\|f'\|_{1,[x,b]}. \end{cases} \tag{2.84}$$

Further,

$$\left|(\alpha+\beta)\mathcal{T}(x;\alpha,\beta)\right| \le \begin{cases} \left[\alpha(x-a)+\beta(b-x)\right]\frac{\|f'\|_\infty}{2}, & f'\in L_\infty[a,b]; \\ \left[\alpha\left(\frac{x-a}{q+1}\right)^{\frac{1}{q}}+\beta\left(\frac{b-x}{q+1}\right)^{\frac{1}{q}}\right]\|f'\|_p, & f'\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ (\alpha+\beta)\|f'\|_1, \end{cases} \tag{2.85}$$

where the expression (2.85) involving the $\|\cdot\|_p$ *norm is coarser.*

The results of (2.84), in which the norms are evaluated over the two subintervals, although finer, do require more work.

REMARK 27. *It is possible to reduce the amount of work alluded to in Remark 26 by observing that,*

$$\begin{aligned} &\alpha\mathcal{M}(f;a,x)+\beta\mathcal{M}(f;x,b) \\ &\quad=\alpha\mathcal{M}(f;a,x)+\frac{\beta}{b-x}\left[\int_a^b f(u)\,du-\int_a^x f(u)\,du\right] \\ &\quad=\left[\alpha-\beta\left(\frac{x-a}{b-x}\right)\right]\mathcal{M}(f;a,x)+\beta\left(\frac{b-a}{b-x}\right)\mathcal{M}(f;a,b) \\ &\quad=\left[\alpha+\beta-\beta\rho(x)\right]\mathcal{M}(f;a,x)+\beta\rho(x)\mathcal{M}(f;a,b), \end{aligned}$$

where

$$\rho(x)=\frac{b-a}{b-x}. \tag{2.86}$$

Thus, from (2.80), $\mathcal{T}(x;\alpha,\beta)$ may be written in the following equivalent form

$$\begin{aligned}(2.87)\quad & \mathcal{T}(x;\alpha,\beta) \\ &= f(x) - \left[\left(1-\frac{\beta}{\alpha+\beta}\rho(x)\right)\mathcal{M}(f;a,x) + \frac{\beta}{\alpha+\beta}\rho(x)\mathcal{M}(f;a,b)\right]\end{aligned}$$

so that for fixed $[a,b]$, $\mathcal{M}(f;a,b)$ is also fixed.

The following uniform bounds are valid [**28**].

Corollary 27. *Let the conditions of Theorem 24 hold, then*

$$(2.88)\quad \left|f(x) - \frac{1}{2}\left[\mathcal{M}(f;a,x)+\mathcal{M}(f;x,b)\right]\right| \le \begin{cases} \frac{(b-a)}{4}\left\|f'\right\|_{\infty}, & f' \in L_{\infty}[a,b]; \\ \left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \cdot \frac{\left\|f'\right\|_p}{2}, & f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\left\|f'\right\|_1}{2}. \end{cases}$$

Proof. The result is readily obtained by allowing $\beta=\alpha$ in (2.81) so that the left hand side is $\mathcal{T}(x;\alpha,\alpha)$ from (2.80). ∎

Corollary 28. *([**28**]) Let the conditions of Theorem 24 hold, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{2}{(b-a)(\alpha+\beta)}\left[\alpha\int_a^{\frac{a+b}{2}} f(u)\,du + \beta\int_{\frac{a+b}{2}}^b f(u)\,du\right]\right|$$

$$(2.89)\quad \le \begin{cases} \frac{(b-a)}{4}\left\|f'\right\|_{\infty}, & f' \in L_{\infty}[a,b]; \\ \left[\alpha^q+\beta^q\right]^{\frac{1}{q}}\left(\frac{b-a}{2(q+1)}\right)^{\frac{1}{q}} \cdot \frac{\left\|f'\right\|_p}{\alpha+\beta}, & f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[1+\frac{|\alpha-\beta|}{\alpha+\beta}\right]\frac{\left\|f'\right\|_1}{2}. \end{cases}$$

Proof. Placing $x=\frac{a+b}{2}$ in (2.80) and (2.81) produces the results as stated in (2.89). ∎

Corollary 29. *([**28**]) If (2.88) is evaluated at the midpoint then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \begin{cases} \frac{(b-a)}{4}\left\|f'\right\|_\infty, & f' \in L_\infty[a,b]; \\ \left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \cdot \frac{\|f'\|_p}{2}, & f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f'\|_1}{2}. \end{cases}$$

which is in agreement with (2.77) when $x = \frac{a+b}{2}$. The above result could also be obtained by taking $\alpha = \beta$ in (2.89) or equivalently $\alpha = \beta$ and $x = \frac{a+b}{2}$ in (2.81).

4.3. Perturbed Results. Perturbed versions of the results of the previous section may be obtained by using Grüss type results involving the Chebychev functional

$$T(f,g) = \mathfrak{M}(fg) - \mathfrak{M}(f)\,\mathfrak{M}(g) \tag{2.90}$$

with $\mathfrak{M}(f)$ being the integral mean of f over $[a,b]$.

For $f,g : [a,b] \to \mathbb{R}$ and integrable on $[a,b]$, as is their product, then

$$\begin{aligned} |T(f,g)| &\leq T^{\frac{1}{2}}(f,f)\,T^{\frac{1}{2}}(g,g), && \text{Dragomir [54]} \\ & && \text{for } f,g \in L_2[a,b]; \\ &\leq \tfrac{\Gamma-\gamma}{2}T^{\frac{1}{2}}(f,f), && \text{Matić et al. [108]} \\ & && \text{for } \gamma \leq g(t) \leq \Gamma, t \in [a,b], \\ &\leq \tfrac{(\Gamma-\gamma)(\Phi-\phi)}{4}, && \text{Grüss (see [109, 295-310]),} \\ & && \phi \leq f \leq \Phi, t \in [a,b]. \end{aligned} \tag{2.91}$$

Dragomir [**54**] obtained numerous results when either f, g or both are known, although the first inequality in (2.91) has a long history (see for example [**109**, pp. 295-310]. The inequalities in (2.91), when proceeding from top to bottom, are in order of decreasing coarseness.

The following theorem is valid [**28**].

THEOREM 25. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping and $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta \neq 0$ then*

$$\left|\mathcal{T}(x,\alpha,\beta) - (x-\gamma)\tfrac{S}{2}\right| \leq (b-a)\,\kappa(x)\left[\tfrac{1}{b-a}\left\|f'\right\|_2^2 - S^2\right]^{\frac{1}{2}}, \quad f' \in L_2[a,b];$$

$$\leq (b-a)\,\kappa(x)\,\tfrac{\Gamma-\gamma}{2}, \quad \text{if } \gamma < f'(t) < \Gamma,\ t \in [a,b]; \tag{2.92}$$

$$\leq (b-a)\,\tfrac{\Gamma-\gamma}{4},$$

where, $\mathcal{T}(x,\alpha,\beta)$ is as given by (2.80) or equivalently (2.87),

$$\gamma = \frac{\alpha a + \beta b}{\alpha+\beta}, \quad S = \frac{f(b)-f(a)}{b-a}, \tag{2.93}$$

$$\kappa^2(x) = \frac{1}{3}\left[\left(\frac{\alpha}{\alpha+\beta}\right)^2 (x-a) + \left(\frac{\beta}{\alpha+\beta}\right)^2 (b-x)\right] - \left(\frac{x-\gamma}{2(b-a)}\right)^2. \tag{2.94}$$

PROOF. Associating $f(t)$ with $P(x,t)$ and $g(t)$ with $f'(t)$, we obtain, from (2.78) and (2.90),

$$T(P(x,\cdot), f'(\cdot)) = \mathfrak{M}(P(x,\cdot), f'(\cdot)) - \mathfrak{M}(P(x,\cdot))\,\mathfrak{M}(f'(\cdot))$$

and so, on using identity (2.79),

$$(b-a)\,T(P(x,\cdot), f'(\cdot)) = \mathcal{T}(x,\alpha,\beta) - (b-a)\,\mathfrak{M}(P(x,\cdot))\,S \tag{2.95}$$

where S is the secant slope of f over $[a,b]$ as given in (2.93). Now, from (2.79),

$$\begin{aligned} (b-a)\,\mathfrak{M}(P(x,\cdot)) &= \int_a^b P(x,t)\,dt \\ &= \frac{\alpha}{\alpha+\beta}\int_a^x \frac{t-a}{x-a}dt - \frac{\beta}{\alpha+\beta}\int_x^b \frac{b-t}{b-x}dt \\ &= (x-\gamma)\int_0^1 u\,du \end{aligned} \tag{2.96}$$

and combining this with (2.94) gives the left hand side of (2.92).

Now, for the bounds on (2.95) from (2.91) we have to determine $T^{\frac{1}{2}}(P(x,\cdot),P(x,\cdot))$ and $\phi \leq P(x,\cdot) \leq \Phi$. First, we note that

$$
\begin{aligned}
(2.97)\qquad 0 \leq T^{\frac{1}{2}}(f'(\cdot),f'(\cdot)) &= \left[\mathfrak{M}\left((f'(\cdot))^2\right) - \mathfrak{M}^2(f'(\cdot))\right]^{\frac{1}{2}} \\
&= \left[\frac{1}{b-a}\int_a^b [f'(t)]^2\,dt - \left(\frac{\int_a^b f'(t)\,dt}{b-a}\right)^2\right]^{\frac{1}{2}} \\
&= \left[\frac{1}{b-a}\left\|f'\right\|_2^2 - S^2\right]^{\frac{1}{2}} \\
&\leq \left(\frac{\Gamma-\gamma}{2}\right), \text{ where } \ \gamma \leq f'(t) \leq \Gamma,\ t\in[a,b].
\end{aligned}
$$

Now from (2.78), the definition of $P(x,t)$, we have

$$
(2.98)\qquad T(P(x,\cdot),P(x,\cdot)) = \mathfrak{M}\left(P^2(x,\cdot)\right) - \mathfrak{M}^2(P(x,\cdot))
$$

where from (2.96),

$$
\mathfrak{M}(P(x,\cdot)) = \frac{x-\gamma}{2(b-a)},
$$

and

$$
\begin{aligned}
&\mathfrak{M}\left(P^2(x,\cdot)\right) \\
&= \left(\frac{\alpha}{\alpha+\beta}\right)^2 \int_a^x \left(\frac{t-a}{x-a}\right)^2 dt + \left(\frac{\beta}{\alpha+\beta}\right)^2 \int_x^b \left(\frac{b-t}{b-x}\right)^2 dt \\
&= \left[\left(\frac{\alpha}{\alpha+\beta}\right)^2 (x-a) + \left(\frac{\beta}{\alpha+\beta}\right)^2 (b-x)\right] \int_0^1 u^2 du.
\end{aligned}
$$

Thus, substituting the above results into (2.98) gives

$$
(2.99)\qquad 0 \leq \kappa(x) = T^{\frac{1}{2}}(P(x,\cdot),P(x,\cdot))
$$

which is given explicitly by (2.94). Combining (2.95), (2.99) and (2.97) give, from the first inequality in (2.91), the first inequality in (2.92). Also, utilising the inequality in (2.97) produces the second result in (2.92).

Further, it may be noticed from the definition of $P(x,t)$ in (2.78) that for $\alpha,\beta \geq 0$ and α and β not zero at the same time,

$$
\Phi = \sup_{t\in[a,b]} P(x,t) \quad \text{and} \quad \phi = \inf_{t\in[a,b]} P(x,t),
$$

giving $\Phi = \frac{\alpha}{\alpha+\beta}$ and $\phi = \frac{-\beta}{\alpha+\beta}$.

Hence, from (2.95) and the last inequality in (2.91) we have the final result in (2.92) and the theorem is now proved. ∎

4.4. An Application to the Cumulative Distribution Function. Let X be a random variable taking values in the finite interval $[a,b]$ with cumulative distribution function $F(x) = \Pr(X \leq x) = \int_a^x f(u)\,du$, where f is a probability density function. The following theorem holds [**28**].

THEOREM 26. *Let X and F be as above, then*

$$\begin{aligned} &|(\alpha(b-x) - \beta(x-a))F(x) \\ &\qquad - (x-a)[(\alpha+\beta)(b-x)f(x) - \beta]| \\ &\leq \begin{cases} (b-x)(x-a)[\alpha(x-a)+\beta(b-x)] \cdot \frac{\|f'\|_\infty}{2}, \ f' \in L_\infty[a,b]; \\ (b-x)(x-a)[\alpha^q(x-a)+\beta^q(b-x)]^{\frac{1}{q}} \cdot \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}}, \\ \qquad\qquad f' \in L_p[a,b], \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ (b-x)(x-a)[\alpha+\beta+|\alpha-\beta|] \cdot \frac{\|f'\|_1}{2}, \qquad f' \in L_1[a,b]. \end{cases} \end{aligned} \tag{2.100}$$

PROOF. The proof follows in a straightforward manner from (2.81) of Theorem 24.

Using (2.87) for $\mathcal{T}(x;\alpha,\beta)$ and (2.88) we obtain, on using the fact that $\int_a^b f(u)\,du = 1$,

$$\begin{aligned} &(\alpha+\beta)(x-a)(b-x)\mathcal{T}(x;\alpha,\beta) \\ &\qquad = (\alpha+\beta)(x-a)(b-x)f(x) \\ &\qquad\qquad - [\alpha(b-x)-\beta(x-a)]F(x) - \beta(x-a). \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{-(\alpha+\beta)(x-a)(b-x)}{\alpha(b-x)-\beta(x-a)}\mathcal{T}(x;\alpha,\beta) \\ &\qquad = F(x) - (x-a)\left[\frac{(\alpha+\beta)(b-x)f(x)-\beta}{\alpha(b-x)-\beta(x-a)}\right] \end{aligned}$$

and so taking the modulus and using (2.81) gives the stated result. ∎

Corollary 30. *([**28**]) Let X be a random variable , $F(x)$ the associated cumulative distribution function and $f(x)$ the associated probability density function. We have,*

$$\left|\left(\frac{a+b}{2}-x\right)F(x)-(x-a)\left[(b-x)f(x)-\frac{1}{2}\right]\right| \tag{2.101}$$
$$\leq \begin{cases} (b-x)(x-a)(b-a)\cdot\frac{\|f'\|_\infty}{2}, & f'\in L_\infty[a,b]; \\ (b-x)(x-a)(b-a)^{\frac{1}{q}}\cdot\frac{\|f'\|_p}{2(q+1)^{\frac{1}{q}}}, & f'\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ (b-x)(x-a)\cdot\frac{\|f'\|_1}{2}, & f'\in L_1[a,b]. \end{cases}$$

Remark 28. *The above results allow the approximation of $F(x)$ in terms of $f(x)$. The approximation of $R(x)=1-F(x)$ could also be obtained by a simple substitution. $R(x)$ is of importance in reliability theory where $f(x)$ is the PDF of failure.*

Remark 29. *We may take $\beta=0$ in (2.80) and (2.81), whilst assuming that $\alpha\neq 0$, to give*

$$|F(x)-(x-a)f(x)| \tag{2.102}$$
$$\leq \begin{cases} \frac{(x-a)^2}{2}\|f'\|_\infty, & f'\in L_\infty[a,b]; \\ (x-a)^{1+\frac{1}{q}}\cdot\frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}}, & f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ (x-a)\|f'\|_1, & f'\in L_1[a,b]. \end{cases}$$

which agrees with (2.77) for $|S(f;a,x)|$.

Remark 30. *The perturbed results of Section 4.3 could also be applied here, however, this will not be pursued further.*

Remark 31. *We may replace f by F (see [**9**] for related results) in any of the equations (2.100) – (2.102) so that the bounds are in terms of $\|f\|_p$, $p\geq 1$.*

5. Some Inequalities Arising from Montgomery's Identity

5.1. Introduction. As mentioned earlier, the following identity, attributed to Montgomery, is well known in the literature (see [**110**, Chapter XVII, p. 565])

$$f(x)=\mathcal{M}(f;a,b)+\kappa(x), \tag{2.103}$$

where, as earlier,

$$\mathcal{M}(f;a,b)=\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.104}$$

is the integral mean,

$$\kappa(x)=\int_a^b P(x,t)\,f'(t)\,dt, \tag{2.105}$$

and the Peano kernel $P(x,t)$ is given by

$$(b-a)\,P(x,t):=\begin{cases} t-a, & a\le t\le x,\\ t-b, & x<t\le b.\end{cases} \tag{2.106}$$

Recently, Dragomir and Wang [**92**] utilized (2.103)-(2.106) to prove Ostrowski's inequality [**96**, p. 469]

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{M}{b-a}\left[\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2\right],$$

where $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is a differential mapping on $\mathring{I}$, the interior of I, and $|f'(x)|\le M$ for all $x\in[a,b]$, $a<b\in\mathring{I}$. Many Ostrowski type results applied to numerical integration and probability have appeared in the literature (see for example [**18**] – [**86**] and the references therein).

It is the intention of the current section to develop, through the framework of Montgomery's identity, a systematic study which produces some novel results and recaptures existing results as special cases. Bounds are obtained in terms of the Lebesgue norms of the first derivative.

In Subsection 5.2, results are obtained for a generalised Chebychev functional involving the integral mean of functions over different intervals. In particular, bounds are obtained for the difference of means over two different intervals, producing a generalisation of Mahajani type inequalities. In Subsection 5.3, we study bounds involving moments about any general parameter producing results for central moments and for moments about the origin as special cases. Bounds for the expectation and the variance are investigated, in particular, recapturing some earlier results and obtaining some previously unknown results.

5.2. Main Results and Some Ramifications. We start with the following theorem [**32**].

THEOREM 27. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping as is $u:[\alpha,\beta]\to\mathbb{R}$ with $[\alpha,\beta]\subseteq[a,b]$. The following inequalities* *hold,*

$$\left|\int_{\alpha}^{\beta} u(x) f(x) dx - \mathcal{M}(f;a,b) \int_{\alpha}^{\beta} u(x) dx\right| \tag{2.107}$$
$$\leq \begin{cases} \frac{\|f'\|_{\infty}}{2(b-a)} \int_{\alpha}^{\beta} |u(x)| \left[(x-a)^2 + (b-x)^2\right] dx, f' \in L_{\infty}[a,b]; \\ \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}(b-a)} \int_{\alpha}^{\beta} |u(x)| \left[(x-a)^{q+1} + (b-x)^{q+1}\right]^{\frac{1}{q}} dx, \\ \qquad f' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f'\|_1}{b-a} \int_{\alpha}^{\beta} |u(x)| \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right] dx, \qquad f' \in L_1[a,b], \end{cases}$$

PROOF. Using identity (2.103), we obtain, for $a \leq \alpha < \beta \leq b$,

$$\int_{\alpha}^{\beta} u(x) f(x) dx = \mathcal{M}(f;a,b) \int_{\alpha}^{\beta} u(x) dx + \int_{\alpha}^{\beta} u(x) \kappa(x) dx, \tag{2.108}$$

and therefore

$$\left|\int_{\alpha}^{\beta} u(x) f(x) dx - \mathcal{M}(f;a,b) \int_{\alpha}^{\beta} u(x) dx\right| = \left|\int_{\alpha}^{\beta} u(x) \kappa(x) dx\right|. \tag{2.109}$$

Now,

$$\left|\int_{\alpha}^{\beta} u(x) \kappa(x) dx\right| \leq \int_{\alpha}^{\beta} |u(x)| |\kappa(x)| dx \tag{2.110}$$

and using the properties of modulus and Hölder's integral inequality

$$|\kappa(x)| \leq \begin{cases} \|f'\|_{\infty} \int_a^b |P(x,t)| dt, & f' \in L_{\infty}[a,b]; \\ \|f'\|_p \left(\int_a^b |P(x,t)|^q dt\right)^{\frac{1}{q}}, & f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \|f'\|_1 \sup_{t\in[a,b]} |P(x,t)|, & f' \in L_1[a,b], \end{cases}$$

and these reduce to,

$$|\kappa(x)| \leq \begin{cases} \frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2+(b-x)^2\right], & f' \in L_\infty[a,b]; \\ \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}(b-a)}\left[(x-a)^{q+1}+(b-x)^{q+1}\right]^{\frac{1}{q}}, & f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f'\|_1}{b-a}\max\{x-a,b-x\}, & f' \in L_1[a,b]. \end{cases} \tag{2.111}$$

Substituting (2.111) into (2.110) and using identity (2.109) gives (2.107) on noting that for $X,Y \in \mathbb{R}$,

$$\max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{X-Y}{2}\right|.$$

Hence, the theorem is proved. ∎

LEMMA 4. *([**32**]) Let f and u satisfy the conditions of Theorem 27, then the following identity is valid,*

$$\int_\alpha^\beta u(x) f(x)\,dx = A(\alpha,\beta)\Bigg\{\mathcal{M}(f;a,b) + \int_a^\alpha \left(\frac{t-a}{b-a}\right) f'(t)\,dt - \int_\beta^b \left(\frac{b-t}{b-a}\right) f'(t)\,dt\Bigg\} + \frac{1}{b-a}\int_\alpha^\beta \left[(t-a)A(t,\beta)-(b-t)A(\alpha,t)\right] f'(t)\,dt, \tag{2.112}$$

where $A(\alpha,\beta)=\int_\alpha^\beta u(x)\,dx$ and $\mathcal{M}(f;a,b)$ is defined in (2.104).

PROOF. The proof is straight forward from identity (2.108) by an interchange of the order of integration of

$$\int_\alpha^\beta u(x)\,\kappa(x)\,dx,$$

where $\kappa(x)$ is defined by (2.105).

$$\int_\alpha^\beta u(x)\,\kappa(x)\,dx \tag{2.113}$$

$$= \int_\alpha^\beta u(x)\int_a^x \left(\frac{t-a}{b-a}\right) f'(t)\,dt dx + \int_\alpha^\beta u(x)\int_x^b \left(\frac{t-b}{b-a}\right) f'(t)\,dt dx$$

$$= A(\alpha,\beta)\int_a^\alpha \left(\frac{t-a}{b-a}\right) f'(t)\,dt + \int_\alpha^\beta \left(\frac{t-a}{b-a}\right) A(t,\beta)\,f'(t)\,dt$$

$$+ A(\alpha,\beta)\int_{\beta}^{b}\left(\frac{t-b}{b-a}\right)f'(t)\,dt + \int_{\alpha}^{\beta}\left(\frac{t-b}{b-a}\right)A(\alpha,t)\,f'(t)\,dt.$$

Substitution into (2.108) produces (2.112). ∎

THEOREM 28. *([**32**]) Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping as is also $u:[\alpha,\beta]\to\mathbb{R}$ with $[\alpha,\beta]\subseteq[a,b]$. The following inequalities are then valid.*

$$\left|\int_{\alpha}^{\beta}u(x)\,f(x)\,dx - \mathcal{M}(f;a,b)\,A(\alpha,\beta)\right| \tag{2.114}$$

$$\leq \begin{cases} \frac{\|f'\|_\infty}{b-a}\left\{\frac{|A(\alpha,\beta)|}{2}\left[(\alpha-a)^2+(b-\beta)^2\right]+\int_\alpha^\beta|\phi(t)|\,dt\right\}, \\ \qquad\qquad f'\in L_\infty[a,b]; \\ \frac{\|f'\|_p}{b-a}\cdot\left\{\frac{|A(\alpha,\beta)|^q}{q+1}\left[(\alpha-a)^{q+1}+(b-\beta)^{q+1}\right]\right. \\ \qquad \left.+\int_\alpha^\beta|\phi(t)|^q\,dt\right\}^{\frac{1}{q}},\ f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f'\|_1}{b-a}\max\left\{|A(\alpha,\beta)|\,\Theta,\ \sup_{t\in[\alpha,\beta]}|\phi(t)|\right\}, \quad f'\in L_1[a,b], \end{cases}$$

where

$$\phi(t) = (t-a)\,A(t,\beta)-(b-t)\,A(\alpha,t) = \begin{vmatrix} t-a & b-t \\ A(\alpha,t) & A(t,\beta) \end{vmatrix} \tag{2.115}$$

and

$$\Theta = \frac{b-a}{2}-\frac{\beta-\alpha}{2}+\left|\frac{b+a}{2}-\frac{\beta+\alpha}{2}\right|. \tag{2.116}$$

PROOF. From identity (2.112)

$$\int_{\alpha}^{\beta}u(x)\,f(x)\,dx-\mathcal{M}(f;a,b)\,A(\alpha,\beta)=R, \tag{2.117}$$

where

$$R = A(\alpha,\beta)\left\{\int_a^\alpha\left(\frac{t-a}{b-a}\right)f'(t)\,dt+\int_\beta^b\left(\frac{t-b}{b-a}\right)f'(t)\,dt\right\} + \frac{1}{b-a}\int_\alpha^\beta\phi(t)\,f'(t)\,dt, \tag{2.118}$$

with $\phi(t)$ being as given by (2.115).

Now, taking the modulus of (2.117) and using the triangle inequality gives, from (2.118),

$$
\begin{aligned}
(2.119)\qquad |R| &\leq |A(\alpha,\beta)| \left\{ \sup_{t\in[a,\alpha]} |f'(t)| \cdot \frac{1}{b-a} \cdot \frac{(\alpha-a)^2}{2} \right. \\
&\quad + \sup_{t\in[\beta,b]} |f'(t)| \cdot \frac{1}{b-a} \cdot \frac{(b-\beta)^2}{2} \\
&\quad \left. + \sup_{t\in(\alpha,\beta)} |f'(t)| \frac{1}{b-a} \int_\alpha^\beta \phi(t)\,dt \right\} \\
&\leq \frac{\|f'\|_\infty}{b-a} \left\{ \left[\frac{(\alpha-a)^2+(b-\beta)^2}{2} \right] + \int_\alpha^\beta |\phi(t)|\,dt \right\}.
\end{aligned}
$$

Substitution of (2.119) into (2.117) produces the first inequality in (2.114).

Further, from (2.117), using Hölder's integral inequality

$$
\begin{aligned}
|R| \leq \frac{\|f'\|_p}{b-a} &\left\{ |A(\alpha,\beta)|^q \left[\int_a^\alpha (t-a)^q\,dt \right.\right. \\
&\qquad \left.\left. + \int_\beta^b (b-t)^q\,dt \right] + \int_\alpha^\beta |\phi(t)|^q\,dt \right\}^{\frac{1}{q}},
\end{aligned}
$$

which, upon some simple calculations and substitution into (2.116), gives the second inequality in (2.114).

For the last inequality, from (2.118)

$$
\begin{aligned}
|R| &\leq \frac{|A(\alpha,\beta)|}{b-a} \left[(\alpha-a) \int_a^\alpha |f'(t)|\,dt + (b-\beta) \int_\beta^b |f'(t)|\,dt \right] \\
&\quad + \frac{1}{b-a} \sup_{t\in[\alpha,\beta]} |\phi(t)| \int_\alpha^\beta |f'(t)|\,dt \\
&\leq \frac{\|f'\|_1}{b-a} \max\left\{ |A(\alpha,\beta)|\,\Theta, \sup_{t\in[\alpha,\beta]} |\phi(t)| \right\},
\end{aligned}
$$

where $\Theta = \max\{\alpha-a, b-\beta\}$.

On substitution of the last inequality into (2.117) and, using the fact that $\max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{X-Y}{2}\right|$, $X, Y \in \mathbb{R}$, we obtain the final inequality in (2.114). ∎

Remark 32. *The bound for the left hand side of (2.117) in terms of R as given by (2.118) was used so that a comparison could be made with the bounds obtained from Theorem 27. In Corollary 31, the first two* *terms in (2.118) disappear on using particular choices of the parameters, and in Theorem 29, the terms constitute part of the expression to be approximated.*

The following corollary gives an estimate of the error for the difference between weighted and unweighted integral means.

COROLLARY 31. *([**32**]) Let $f, u : [a, b] \to \mathbb{R}$ be absolutely continuous mappings on $[a, b]$. Then*

$$\left| \frac{\int_a^b u(x) f(x) \, dx}{\int_a^b u(x) \, dx} - \frac{\int_a^b f(x) \, dx}{b-a} \right| \leq \begin{cases} \frac{\|f'\|_\infty}{b-a} \int_a^b |\Phi(t)| \, dt, & f' \in L_\infty[a, b]; \\ \frac{\|f'\|_p}{b-a} \left[\int_a^b |\Phi(t)|^q \, dt \right]^{\frac{1}{q}}, & f' \in L_p[a, b], \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{\|f'\|_1}{b-a} \sup_{t \in [a,b]} |\Phi(t)|, & f' \in L_1[a, b], \end{cases} \tag{2.120}$$

where

$$\Phi(t) = (t-a) H(t, b) - (b-t) H(a, t) \tag{2.121}$$

with

$$H(a, t) = \frac{\int_a^t u(x) \, dx}{\int_a^b u(x) \, dx} \quad \text{and} \quad H(t, b) = 1 - H(a, t). \tag{2.122}$$

PROOF. Setting $\alpha = a$ and $\beta = b$ in Theorem 28 produces the result (2.120) after some minor rearrangements. ∎

THEOREM 29. *([**32**]) Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous mapping as is $u : [\alpha, \beta] \to \mathbb{R}$ with $[\alpha, \beta] \subseteq [a, b]$, then the following inequalities are valid.*

$$\begin{aligned} |\mathcal{T}| := \Bigg| \int_\alpha^\beta u(t) f(t) \, dt - A(\alpha, \beta) \\ \times \{[1 - (\lambda_1 + \lambda_2)] \mathcal{M}(f; \alpha, \beta) + \lambda_1 f(\alpha) + \lambda_2 f(\beta)\} \Bigg| \end{aligned}$$
$$\leq \begin{cases} \frac{\|f'\|_\infty}{b-a} \int_\alpha^\beta |\phi(t)| \, dt, & f' \in L_\infty[\alpha, \beta]; \\ \frac{\|f'\|_p}{b-a} \left(\int_\alpha^\beta |\phi(t)|^q \, dt \right)^{\frac{1}{q}}, & f' \in L_p[\alpha, \beta], \ p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{\|f'\|_1}{b-a} \sup_{t \in [\alpha,\beta]} |\phi(t)|, & f' \in L_1[\alpha, \beta], \end{cases} \tag{2.123}$$

where

$$A(\alpha,\beta)=\int_{\alpha}^{\beta}u(t)\,dt,\ \lambda_1=\frac{\alpha-a}{b-a},\ \ \lambda_2=\frac{b-\beta}{b-a}, \tag{2.124}$$

and $\phi(t)$ *is as defined by (2.115).*

Proof. From identity (2.112), we have

$$\begin{aligned}\int_{\alpha}^{\beta}u(t)\,f(t)\,dt \\ -A(\alpha,\beta)\Big\{\mathcal{M}(f;a,b)+\int_{a}^{\alpha}\left(\frac{t-a}{b-a}\right)f'(t)\,dt \\ -\int_{\beta}^{b}\left(\frac{b-t}{b-a}\right)f'(t)\,dt\Big\}=\frac{1}{b-a}\int_{\alpha}^{\beta}\phi(t)\,f'(t)\,dt,\end{aligned} \tag{2.125}$$

where $A(\alpha,\beta)$ is as given by (2.124), $\mathcal{M}(f;a,b)$ is as defined in (2.74) and $\phi(t)$ is as given by (2.115).

Simple integration by parts gives

$$\int_{a}^{\alpha}\left(\frac{t-a}{b-a}\right)f'(t)\,dt=\left(\frac{\alpha-a}{b-a}\right)f(\alpha)-\frac{1}{b-a}\int_{a}^{\alpha}f(t)\,dt$$

and

$$-\int_{\beta}^{b}\left(\frac{b-t}{b-a}\right)f'(t)\,dt=\left(\frac{b-\beta}{b-a}\right)f(\beta)-\frac{1}{b-a}\int_{\beta}^{b}f(t)\,dt,$$

which, upon substitution into (2.125) produces the identity

$$\begin{aligned}\int_{\alpha}^{\beta}u(t)\,f(t)\,dt-\frac{A(\alpha,\beta)}{b-a} \\ \times\left[\int_{\alpha}^{\beta}f(t)\,dt+(\alpha-a)\,f(\alpha)+(b-\beta)\,f(\beta)\right] \\ =\frac{1}{b-a}\int_{\alpha}^{\beta}\phi(t)\,f'(t)\,dt.\end{aligned} \tag{2.126}$$

Now, allowing λ_1, λ_2 to be as given in (2.124), then, from (2.126),

$$\begin{aligned}\int_{\alpha}^{\beta}u(t)\,f(t)\,dt-A(\alpha,\beta)\Big\{[1-(\lambda_1+\lambda_2)]\frac{1}{\beta-\alpha} \\ \times\int_{\alpha}^{\beta}f(t)\,dt+\lambda_1 f(\alpha)+\lambda_2 f(\beta)\Big\} \\ =\frac{1}{b-a}\int_{\alpha}^{\beta}\phi(t)\,f'(t)\,dt.\end{aligned} \tag{2.127}$$

Taking the modulus of (2.127) and using the results from the proof of Theorem 28 involving the modulus and integral and, Hölder's inequality, produces the results as stated in (2.123) involving the Lebesgue norms, for $f' \in L_p[\alpha,\beta]$, $p \geq 1$. ∎

REMARK 33. *The left hand side of (2.123) may be written in the form*

$$\mathcal{T} = (\beta-\alpha)\,T(u,f) + A(\alpha,\beta)\left\{\lambda_1\left[\mathcal{M}(f;\alpha,\beta) - f(\alpha)\right] + \lambda_2\left[\mathcal{M}(f;\alpha,\beta) - f(\beta)\right]\right\}, \tag{2.128}$$

where $T(g,h)$ *is the Chebychev functional (see for example* [**109**]*) given by*

$$T(g,h) = \mathcal{M}(gh) - \mathcal{M}(g)\,\mathcal{M}(h), \tag{2.129}$$

where $\mathcal{M}(\cdot)$ *is the mean over some interval. Hence, the bounds of Theorem 29 may be viewed as bounds for a perturbed Chebychev functional.*
If $\lambda_1 = \lambda_2 = 0$*, then there is no perturbation.*
If $\lambda_1 = \lambda_2 = \lambda$*, say, then from (2.128)*

$$(\beta-\alpha)\,T(u,f) + 2\lambda A(\alpha,\beta)\left[\mathcal{M}(f;\alpha,\beta) - \frac{f(\alpha)+f(\beta)}{2}\right],$$

where the perturbation to the Chebychev functional involves the difference between the mean and the trapezoidal approximation of a function $f(\cdot)$.
If $\lambda_1 = \lambda_2 = \frac{1}{2}$*, then, from the left hand side of (2.123), on division by* $\beta-\alpha$ *we obtain the difference between the average of the product of two functions and the average of one by the difference between the average and the trapezoidal approximation of another. If* $\lambda_1 = 0$ $(\alpha = a)$ *and* $\lambda_2 = \frac{b-x}{b-a}$ $(\beta = x)$*, then from (2.123)*

$$\frac{\mathcal{T}}{\beta-\alpha} = \mathcal{M}(uf;a,x) - \mathcal{M}(u;a,x)\left[\left(\frac{x-a}{b-a}\right)\mathcal{M}(f;a,x) + \left(\frac{b-x}{b-a}\right)f(x)\right], \tag{2.130}$$

giving a convex combination between the mean of $f(\cdot)$ *and evaluation at only one end point.*
In fact, from (2.123)

$$\frac{\mathcal{T}}{\beta-\alpha} = \mathcal{M}(uf;\alpha,\beta) - \mathcal{M}(u;\alpha,\beta)\left\{\left[1-(\lambda_1+\lambda_2)\right]\mathcal{M}(f;\alpha,\beta) + \lambda_1 f(\alpha) + \lambda_2 f(\beta)\right\},$$

giving a comparison between the mean of a product of two functions and the product of the mean of a function and a convex combination of the mean and the end point function evaluations of the other function.

The following corollary gives bounds for the difference between the mean of a function and the mean over a subinterval.

COROLLARY 32. *([**32**]) Let the conditions of Theorem 27 hold. Then, for* $[\alpha,\beta]\subseteq[a,b]$

$$|\mathcal{M}(f;\alpha,\beta)-\mathcal{M}(f;a,b)| \tag{2.131}$$
$$\leq\begin{cases}\frac{\|f'\|_\infty}{2(b-a)(\beta-\alpha)}\left[M_2(\alpha-a,\beta-a)+M_2(b-\beta,b-\alpha)\right], & f'\in L_\infty[a,b];\\ \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}(b-a)(\beta-\alpha)}\left[M_{q+1}(\alpha-a,\beta-a)+M_{q+1}(b-\beta,b-\alpha)\right]^{\frac{1}{q}}, & f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{\|f'\|_1}{b-a}\cdot\left[\frac{(b-a)(\beta-\alpha)}{2}+\int_{\alpha-\frac{a+b}{2}}^{\beta-\frac{a+b}{2}}|u|\,du\right], & f'\in L_1[a,b],\end{cases}$$

where $\mathcal{M}(f;\cdot,\cdot)$ *is as defined by (2.74) and*

$$M_r(x_1,x_2)=\int_{x_1}^{x_2}u^r du=\frac{x_2^{r+1}-x_1^{r+1}}{r+1}. \tag{2.132}$$

PROOF. The proof follows from Theorem 27. Placing $u\equiv 1$ gives the above results after some straight forward algebra and noting that from (2.132)

$$\int_\alpha^\beta\left[(x-a)^r+(b-x)^r\right]dx=M_r(\alpha-a,\beta-a)+M_r(b-\alpha,b-\beta).$$

∎

COROLLARY 33. *([**32**]) Let the conditions of Theorem 27 hold, then*

$$\left|\int_a^x f(u)\,du-\left(\frac{x-a}{b-a}\right)\int_a^b f(u)\,du\right| \tag{2.133}$$
$$\leq\begin{cases}\frac{\|f'\|_\infty}{6(b-a)}\left[(x-a)^3+(b-a)^3-(b-x)^3\right], & f'\in L_\infty[a,b];\\ \frac{\|f'\|_p}{[(q+2)(q+1)]^{\frac{1}{q}}(b-a)}\left[(x-a)^{q+2}+(b-a)^{q+2}-(b-x)^{q+2}\right]^{\frac{1}{q}}, & f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{\|f'\|_1}{b-a}(x-a)\left[\frac{(b-a)(x-a)}{2}+\int_{-\frac{b-a}{2}}^{x-\frac{a+b}{2}}|u|\,du\right], & f'\in L_1[a,b].\end{cases}$$

PROOF. Taking $u \equiv 1$ in Theorem 27 and placing $\beta = x$ and $\alpha = a$ or taking $\beta = x$ and $\alpha = a$ in Corollary 32 produces the results stated after some simplification.

$$M_r(0, x-a) + M_r(b-x, b-a) = \frac{(x-a)^{r+1} + (b-a)^{r+1} - (b-x)^{r+1}}{r+1}$$

gives the results as stated in (2.133). ∎

REMARK 34. *An upper bound may be obtained from Corollary 32 when $x_1 \equiv 0$, i.e., if $\alpha = a$ and $\beta = b$. Taking $x = \frac{a+b}{2}$ on the right hand side of Corollary 33 produces the result*

$$\left| \int_a^{\frac{a+b}{2}} f(u)du - \frac{1}{2}\int_a^b f(u)du \right| \le \begin{cases} \frac{(b-a)^2}{6} \|f'\|_\infty, & f' \in L_\infty[a,b]; \\ \frac{(b-a)^{\frac{2}{q}}}{[(q+2)(q+1)]^{\frac{1}{q}}} \|f'\|_p, & f' \in L_p[a,b], \\ & p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \frac{3}{4}\left(\frac{b-a}{2}\right)^2 \|f'\|_1, & f' \in L_1[a,b]. \end{cases}$$

Furthermore, if f is a probability density function such that $f : [a,b] \to \mathbb{R}_+$ and $\int_a^b f(t)\,dt = 1$, then $\int_a^x f(u)\,du = F(x)$, the cumulative density function and so Corollary 33 may be viewed as a first order approximation for F.

COROLLARY 34. *([**32**]) Let the conditions of Theorem 28 hold, then for $[\alpha, \beta] \subset [a,b]$, the following inequalities are valid.*

$$(2.134)\quad |\mathcal{M}(f;\alpha,\beta) - \mathcal{M}(f;a,b)| \le \begin{cases} \frac{\|f'\|_\infty}{2[b-a-(\beta-\alpha)]}\left[(\alpha-a)^2 + (b-\beta)^2\right], & f' \in L_\infty[a,b]; \\ \frac{\|f'\|_p}{b-a}\left[\frac{(\alpha-a)^{q+1}+(b-\beta)^{q+1}}{(q+1)(1-\rho)}\right]^{\frac{1}{q}}, & f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f'\|_1}{b-a}\Theta, & f' \in L_1[a,b], \end{cases}$$

where $\rho = \frac{\beta-\alpha}{b-a}$ and $\Theta = \max\{\alpha - a, b - \beta\}$ as given by (2.116).

PROOF. Taking $u \equiv 1$ in Theorem 28 gives $A(\alpha,\beta) = \beta - \alpha$ and for $k \ge 1$,

$$(2.135)\qquad \int_\alpha^\beta |\Phi(t)|^k\,dt = \int_\alpha^\beta |(t-a)(\beta-t) - (b-t)(t-\alpha)|^k\,dt$$

$$= \int_{\alpha}^{\beta} |\gamma t - c|^k \, dt,$$

where

$$\left\{ \begin{array}{l} \gamma = b - a - (\beta - a) \\ \text{and} \\ c = \alpha b - a\beta. \end{array} \right. \tag{2.136}$$

Thus, from (2.135),

$$\int_{\alpha}^{\beta} |\Phi(t)|^k \, dt = \gamma^k \int_{\alpha}^{\frac{c}{\gamma}} \left(\frac{c}{\gamma} - t \right)^k dt + \int_{\frac{c}{\gamma}}^{\beta} \left(t - \frac{c}{\gamma} \right)^k dt$$
$$= \frac{(c - \alpha\gamma)^{k+1} + (\beta\gamma - c)^{k+1}}{(q+1)\gamma}.$$

Substituting for γ and c from (2.136) gives

$$\int_{\alpha}^{\beta} |\Phi(t)|^k \, dt = \frac{(\beta - a)^{k+1}}{[b - a - (\beta - \alpha)]} \left[\frac{(\alpha - a)^{k+1} + (b - \beta)^{k+1}}{k+1} \right]$$

and thus, from (2.114), after a little algebra, we obtain the first two inequalities for $k = 1$ and q respectively.

Now, for the third inequality, from (2.114) and (2.135),

$$\sup_{t \in [\alpha, \beta]} |\Phi(t)| = \max \{ |\gamma\alpha - c|, |\gamma\beta - c| \}$$
$$= (\beta - \alpha) \max \{ \alpha - a, b - \beta \} = (\beta - \alpha) \Theta,$$

where Θ is as given by (2.116) and hence the corollary is proved. ∎

COROLLARY 35. *([**32**]) Let the conditions of Theorem 28 hold, then*

$$\left| \int_a^x f(u) \, du - \left(\frac{x-a}{b-a} \right) \int_a^b f(u) \, du \right| \tag{2.137}$$
$$\leq \left\{ \begin{array}{l} \|f'\|_{\infty} \frac{(x-a)(b-x)}{2}, \;\; f' \in L_{\infty}[a,b]; \\ \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}} \cdot \frac{(x-a)(b-x)}{(b-a)^{1-\frac{1}{q}}}, \; f' \in L_p[a,b], \\ \qquad\qquad p > 1, \; \frac{1}{p} + \frac{1}{q} = 1; \\ \|f'\|_1 \frac{(x-a)(b-x)}{b-a}, \quad f' \in L_1[a,b]. \end{array} \right.$$

PROOF. Take $u \equiv 1$ in Theorem 28 with $\beta = x$ and $\alpha = a$, or, alternatively, and perhaps the easier route, take $\beta = x$ and $\alpha = a$ in Corollary 34. This produces the result (2.137) on multiplication by $x - a$. ∎

REMARK 35. *The tightest bound from (2.137) is obtained by taking $x = \frac{a+b}{2}$ to give*

$$\left| \int_a^{\frac{a+b}{2}} f(u)\,du - \frac{1}{2}\int_a^b f(u)\,du \right| \leq \begin{cases} \frac{(b-a)^2}{8} \|f'\|_\infty, & f' \in L_\infty[a,b]; \\ \frac{(b-a)^{1+\frac{1}{q}}}{4(q+1)^{\frac{1}{q}}} \|f'\|_p, & f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{b-a}{4}\|f'\|_1, & f' \in L_1[a,b]. \end{cases}$$

It may be noticed that these bounds are sharper than those of Remark 34. As a matter of fact, it may be shown that the bounds given by Corollary 35 are better than those of (2.137) except for the case $f' \in L_1[a,b]$ for $b-a<\frac{4}{3}$.

REMARK 36. *If we allow $\int_a^b f(u)\,du = 0$, then Corollary 35 reproduces the results of a comprehensive article by Fink* [**96**] *dealing with Ostrowski, Mahajani and Iyengar type inequalities. In fact, the first inequality in (2.137) with $\int_a^b f(u)\,du = 0$ is superior to the Mahajani inequality*

$$\left| \int_a^x f(x)\,dx \right| \leq \frac{(b-a)^2}{8}\|f'\|_\infty \tag{2.138}$$

except at $x = \frac{a+b}{2}$.

It is important to note that the Mahajani inequality (2.138) and the Mahajani type generalisations of Fink [**96**], which are recaptured as a special case of (2.137) $\left(\int_a^b f(u)\,du = 0\right)$ effectively involve obtaining bounds on the area over a specific subinterval $[a,x]$ of $[a,b]$. The following corollary may be viewed as Mahajani type inequalities over **any** subinterval.

COROLLARY 36. *(*[**32**]*) Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping with $[\alpha,\beta]\subseteq[a,b]$ and $\int_a^b f(u)\,du=0$, then*

$$\left|\int_\alpha^\beta f(u)\,du\right| \tag{2.139}$$
$$\leq \begin{cases} \frac{\rho}{2[1-\rho]}\left[(\alpha-a)^2+(b-\beta)^2\right]\|f'\|_\infty, & f'\in L_\infty[a,b]; \\ \rho\left[\frac{(\alpha-a)^{q+1}+(b-\beta)^{q+1}}{(q+1)(1-\rho)}\right]^{\frac{1}{q}}\|f'\|_p, & f'\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \rho\left[\frac{b-a}{2}-\frac{\beta-\alpha}{2}+\left|\frac{b+a}{2}-\frac{\beta+\alpha}{2}\right|\right]\|f'\|_1, & f'\in L_1[a,b], \end{cases}$$

where $\rho = \frac{\beta-\alpha}{b-a}$.

PROOF. Corollary 34, putting $\int_a^b f(u)\,du = 0$ and multiplying both sides by $\beta - \alpha$, readily produces the result (2.139). ∎

REMARK 37. *If $\mathcal{M}(f;a,b)$ is taken to be zero in any of the earlier results, then they may be looked upon as weighted Mahajani type inequalities over arbitrary subintervals $[\alpha,\beta]$. Further, the condition of $\int_a^b f(u)\,du = 0$ may be done away with if we consider a function shifted by its mean, that is, taking $f(x) = g(x) - \frac{1}{b-a}\int_a^b g(u)\,du$.*

The following theorem provides bounds in terms of Lebesgue norms over a subinterval [**32**].

THEOREM 30. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping as is also $u : [\alpha,\beta] \subseteq [a,b]$, then*

$$\left|\int_\alpha^\beta u(x) f(x)\,dx - \mathcal{M}(f;a,b)\int_\alpha^\beta u(x)\,dx\right| \leq \begin{cases} \|S(f)\|_{\infty,s}\int_\alpha^\beta |u(x)|\,dx, & f \in L_\infty[\alpha,\beta]; \\ \|S(f)\|_{p,s}\left(\int_\alpha^\beta |u(x)|^q\,dx\right)^{\frac{1}{q}}, & f \in L_p[\alpha,\beta],\ p>1, \\ & \frac{1}{p}+\frac{1}{q}=1; \\ \|S(f)\|_{1,s}\sup\limits_{x\in[\alpha,\beta]}|u(x)|, & f \in L_1[\alpha,\beta], \end{cases} \tag{2.140}$$

where $\mathcal{M}(f;a,b)$ is as given by (2.74),

$$S(f(x)) = f(x) - \mathcal{M}(f;a,b) \tag{2.141}$$

and $\|\cdot\|_{p,s}$, $1 \leq p \leq \infty$ are the Lebesgue norms on the subinterval $[\alpha,\beta]$.

PROOF. From (2.103) and (2.108) we obtain the identity

$$\int_\alpha^\beta u(x) f(x)\,dx - \mathcal{M}(f;a,b)\int_\alpha^\beta u(x)\,dx = \int_\alpha^\beta u(x)\,S(f(x))\,dx, \tag{2.142}$$

where $S(f(\cdot))$ is a shift operator as defined by (2.141). From (2.142), using the properties of modulus and integral together with Hölder's integral inequality, gives

$$\left|\int_{\alpha}^{\beta} u(x) S(f(x)) \, dx\right| \leq \begin{cases} \|S(f)\|_{\infty,s} \int_{\alpha}^{\beta} |u(x)| \, dx, & f \in L_{\infty}[\alpha,\beta]; \\ \|S(f)\|_{p,s} \left(\int_{\alpha}^{\beta} |u(x)|^q \, dx\right)^{\frac{1}{q}}, & f \in L_p[\alpha,\beta], \\ & p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \|S(f)\|_{1,s} \sup_{x \in [\alpha,\beta]} |u(x)|, & f \in L_1[\alpha,\beta]. \end{cases}$$

That is, substitution into the right hand side of the modulus of (2.142) gives (2.140) and the theorem is proved. ∎

Remark 38. *The equivalent of the shifted norms has appeared in the work of Dragomir and McAndrew* [**83**] *in which they obtained bounds for perturbed trapezoidal rules in terms of the norms of functions shifted by their average i.e., the Lebesgue norms of (2.141).*

5.3. Results Involving Moments. In this section we investigate inequalities involving moments. Let

$$\left\{ \begin{array}{ll} & m_n(\gamma) = \int_{\alpha}^{\beta} (x-\gamma)^n f(x) \, dx \\ \text{and} & \\ & M_n(\gamma) = \int_{a}^{b} (x-\gamma)^n f(x) \, dx \end{array} \right. \tag{2.143}$$

with $[\alpha,\beta] \subseteq [a,b]$.

That is, m represents moments about γ over the subinterval $[\alpha,\beta]$ while M represents moments about γ over the interval $[a,b]$. It should be noted that if $\gamma = 0$, then (2.143) produces the moments about the origin, while taking $\gamma = m_1(0)$ (or $\gamma = M_1(0)$) gives the central moments.

The following theorem holds [**32**].

Theorem 31. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping with $\gamma \in \mathbb{R}$ and $[\alpha,\beta] \subseteq [a,b]$, then*

(2.144) $\quad \left|m_n(\gamma) - \mathcal{M}(f;a,b) A(\alpha,\beta;\gamma)\right|$

$$\leq \begin{cases} \frac{\|f'\|_\infty}{2(b-a)} \Psi_1(a,\alpha,\beta,b;\gamma), & f' \in L_\infty[a,b]; \\ \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}(b-a)} \Psi_q(a,\alpha,\beta,b;\gamma), f' \in L_p[a,b], & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \|f'\|_1 \left[\frac{\Theta(\alpha,\beta;\gamma)}{2} + \frac{1}{b-a} \int_{\alpha-\gamma}^{\beta-\gamma} |v|^n \left| v - \left(\frac{a+b}{2} - \gamma\right) \right| dv \right], & \\ & f' \in L_1[a,b], \end{cases}$$

where $\mathcal{M}(f;a,b)$ *is as defined by (2.74),*

$$(n+1) A(\alpha,\beta;\gamma) = (\beta-\gamma)^{n+1} - (\alpha-\gamma)^{n+1}, \tag{2.145}$$

$$\Psi_r(a,\alpha,\beta,b;\gamma) = \int_{\alpha-\gamma}^{\beta-\gamma} |v|^n \left[(v+\gamma-a)^{r+1} + (b-\gamma-v)^{r+1} \right]^{\frac{1}{r}} dv \tag{2.146}$$

and

$$(n+1)\Theta(\alpha,\beta;\gamma) = \begin{cases} (n+1) A(\alpha,\beta;\gamma), & \gamma \leq \alpha; \\ (\beta-\gamma)^{n+1} + (\alpha-\gamma)^{n+1}, & \alpha < \gamma \leq \beta; \\ (\gamma-\alpha)^{n+1} - (\gamma-\beta)^{n+1}, & \gamma > \beta. \end{cases} \tag{2.147}$$

Proof. Taking $u(x) = (x-\gamma)^n$ in (2.107) readily gives the left hand side of (2.144) after some simple algebra. For $1 \leq r < \infty$ then the substitution $v = x - \gamma$ into

$$\int_\alpha^\beta |x-\gamma|^n \left[(x-a)^{r+1} + (b-x)^{r+1} \right]^{\frac{1}{r}} dx$$

produces $\Psi_r(a,\alpha,\beta,b;\gamma)$ as given by (2.146).

The last inequality is obtained on noting that

$$\int_\alpha^\beta |x-\gamma|^n dx = \int_{\alpha-\gamma}^{\beta-\gamma} |v|^n dv = \begin{cases} \int_{\alpha-\gamma}^{\beta-\gamma} v^n dv, & \gamma \leq \alpha; \\ \int_0^{\gamma-\alpha} v^n dv + \int_0^{\beta-\gamma} v^n dv, & \alpha < \gamma \leq \beta; \\ \int_{\gamma-\beta}^{\gamma-\alpha} v^n dv, & \gamma > \beta, \end{cases}$$

which, on evaluation, produces $\Theta(\alpha,\beta;\gamma)$ as given by (2.147).

Further, $\int_\alpha^\beta |x-\gamma|^n \left| x - \frac{a+b}{2} \right| dx$ produces the integral term in the third inequality of (2.144) on making the substitution $v = x - \gamma$. ∎

THEOREM 32. *([**32**]) Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous mapping $\gamma \in \mathbb{R}$ and $[\alpha, \beta] \subseteq [a, b]$, then*

$$|m_n(\gamma) - \mathcal{M}(f; a, b) A(\alpha, \beta; \gamma)| \tag{2.148}$$
$$\leq \begin{cases} \frac{\|f'\|_\infty}{b-a} \left\{ \frac{|A(\alpha,\beta;\gamma)|}{2} \left[(\alpha - a)^2 + (b - \beta)^2\right] \right. \\ \qquad \left. + \chi_1(a, \alpha, \beta, b; \gamma) \right\}, \qquad f' \in L_\infty[a, b]; \\ \frac{\|f'\|_p}{b-a} \left\{ \frac{|A(\alpha,\beta;\gamma)|^q}{q+1} \left[(\alpha - a)^{q+1} + (b - \beta)^{q+1}\right] \right. \\ \left. + \chi_q(a, \alpha, \beta, b; \gamma) \right\}^{\frac{1}{q}}, \ f' \in L_p[a, b], \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{\|f'\|_1}{b-a} \max\left\{ |A(\alpha, \beta; \gamma)| \Theta, \sup\limits_{t \in [\alpha,\beta]} |\phi(t)| \right\}, \ f' \in L_1[a, b], \end{cases}$$

where $\mathcal{M}(f; a, b)$ is as given by (2.74), $(n + 1) A(\alpha, \beta; \gamma)$ is as given by (2.145)

$$\chi_r(a, \alpha, \beta, b; \gamma) \tag{2.149}$$
$$= \frac{1}{(n + 1)^r} \int_{\alpha-\gamma}^{\beta-\gamma} \left| (b - a) v^{n+1} + v \left[(\alpha - \gamma)^{n+1} - (\beta - \gamma)^{n+1}\right] \right.$$
$$\left. + (b - \gamma)(\alpha - \gamma)^{n+1} - (\gamma - a)(\beta - \gamma)^{n+1} \right|^r dv$$

and

$$(n + 1) |\phi(t)| \tag{2.150}$$
$$= \left| (b - a)(t - \gamma)^{n+1} + (t - \gamma) \left[(\alpha - \gamma)^{n+1} - (\beta - \gamma)^{n+1}\right] \right.$$
$$\left. + (b - \gamma)(\alpha - \gamma)^{n+1} - (\gamma - a)(\beta - \gamma)^{n+1} \right|.$$

PROOF. Taking $u(x) = (x - \gamma)^n$ in (2.114) gives the left hand side of (2.148).

From (2.115), with $u(x) = (x - \gamma)^n$, we obtain

$$\phi(t) = (t - a) A(t, \beta; \gamma) - (b - t) A(\alpha, t; \gamma),$$

where $A(\alpha, \beta; \gamma)$ is as given by (2.145) and thus

$$(n + 1) |\phi(t)| = \left|(t - a) \left[(\beta - \gamma)^{n+1} - (t - \gamma)^{n+1}\right] \right.$$
$$\left. - (b - t) \left[(t - \gamma)^{n+1} - (\alpha - \gamma)^{n+1}\right]\right|$$

and so

$$(n+1)\left|\phi(t)\right| = \left|(b-a)(t-\gamma)^{n+1} - \left[(t-a)(\beta-\gamma)^{n+1} + (b-t)(\alpha-\gamma)^{n+1}\right]\right|, \tag{2.151}$$

which produces (2.150) on expressing it as a polynomial in terms of $t-\gamma$.

Hence,

$$\int_\alpha^\beta \left|\phi(t)\right|^r dt = \frac{1}{(n+1)^r}\int_\alpha^\beta \Big|(b-a)(t-\gamma)^{n+1} + (t-\gamma)\left[(\alpha-\gamma)^{n+1} - (\beta-\gamma)^{n+1}\right] + (b-\gamma)(\alpha-\gamma)^{n+1} - (\gamma-a)(\beta-\gamma)^{n+1}\Big|^r dt,$$

which, on substitution of $v = t-\gamma$ produces $\chi_r(a,\alpha,\beta,b;\gamma)$ as defined in (2.149).

The last inequality in (2.148) is obtained from the third inequality in (2.114) and from (2.115) on taking $u(x) = (x-\gamma)^n$. ∎

REMARK 39. *The above two theorems provide quite general results, producing bounds for the moments over a subinterval in terms of the* $L_p[a,b]$ *norms of the derivative of the function. Taking* $\alpha = a$ *and* $\beta = b$ *gives results involving* $M_n(\gamma)$ *rather than* $m_n(\gamma)$ *as defined by (2.143). As stated previously at the start of this section, taking* $\gamma = 0$ *and* $\gamma = m_1(0)$ *(or* $M_1(0)$*) produces the moments about the origin and the central moments respectively. Taking* $n = 0$ *reproduces the corollaries of the previous section.*

The following corollaries investigate in some detail a few specialisations. We restrict the examples to taking $\alpha = a$ and $\beta = b$.

COROLLARY 37. *Let* $f : [a,b] \to \mathbb{R}$ *be absolutely continuous and* $\gamma \in \mathbb{R}$*, then*

$$\left|M_n(\gamma) - \mathcal{M}(f;a,b)\frac{(b-\gamma)^{n+1} - (a-\gamma)^{n+1}}{n+1}\right| \tag{2.152}$$
$$\leq \begin{cases} \frac{\|f'\|_\infty \tilde{\chi}_1(\gamma)}{n+1}, & f' \in L_\infty[a,b]; \\ \frac{\|f'\|_p \tilde{\chi}_q^{\frac{1}{q}}(\gamma)}{n+1}, & f' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f'\|_1 \sup_{t\in[a,b]}\left|\tilde{\phi}(t)\right|}{n+1}, & f' \in L_1[a,b], \end{cases}$$

where

$$\tilde{\phi}(t) = (t-\gamma)^{n+1} - \left[\left(\frac{t-a}{b-a}\right)(b-\gamma)^{n+1} + \left(\frac{b-t}{b-a}\right)(a-\gamma)^{n+1}\right] \tag{2.153}$$

and

$$\tilde{\chi}_r(\gamma) = \int_a^b \left|\tilde{\phi}(t)\right|^r dt. \tag{2.154}$$

Proof. Taking $\alpha = a$ and $\beta = b$ in Theorem 32, we obtain $\tilde{\phi}(t) = \frac{(n+1)\phi(t)}{b-a}$ from (2.151) as given in (2.153) and $\tilde{\chi}_r(\gamma) = \frac{(n+1)^r}{b-a}\chi_r(a,a,b,b;\gamma)$ as shown by (2.154). ∎

The results of Corollary 37 may be simplified if the nature of $\tilde{\phi}(t)$ as given by (2.153) is known. The following lemma examines the behaviour of $\tilde{\phi}(t)$.

Lemma 5. *([**32**]) For $\tilde{\phi}(t)$ given by (2.153) we have*

$$\tilde{\phi}(t)\begin{cases} <0 & \begin{cases} n \text{ odd}, & \text{any } \gamma \text{ and } t\in(a,b) \\ n \text{ even} & \begin{cases} \gamma<a, & t\in(a,b) \\ a<\gamma<b, & t\in[c,b) \end{cases} \end{cases} \\ >0, & n \text{ even} \begin{cases} \gamma>b, & t\in(a,b) \\ a<\gamma<b, & t\in(a,c) \end{cases} \end{cases}. \tag{2.155}$$

where $\tilde{\phi}(c) = 0$, $a<c<b$ and

$$c\begin{cases} >\gamma, & \gamma<\frac{a+b}{2} \\ =\gamma, & \gamma=\frac{a+b}{2} \\ <\gamma, & \gamma>\frac{a+b}{2}. \end{cases}$$

Proof. From (2.153), $\tilde{\phi}(a) = \tilde{\phi}(b) = 0$.
Further,

$$\tilde{\phi}'(t) = (n+1)(t-\gamma)^n - \frac{(b-\gamma)^{n+1} - (a-\gamma)^{n+1}}{b-a} \tag{2.156}$$

and

$$\tilde{\phi}''(t) = (n+1)\,n\,(t-\gamma)^{n-1}. \tag{2.157}$$

Thus, for n odd, $\tilde{\phi}''(t) > 0$, $t\in[a,b]$ and so $\tilde{\phi}(t) < 0$ for $t\in(a,b)$.

For n even, the behaviour depends also on γ and t. $\tilde{\phi}''(t) > 0$ for any $t\in[a,b]$ if $\gamma<a$ and for $t\in(c,b)$ if $a<\gamma<b$, where $\tilde{\phi}(c)=0$. Thus, $\tilde{\phi}(t) < 0$ over these regions.

Now $\tilde{\phi}''(t) < 0$ for $t - \gamma < 0$ i.e., for $t \in [a, b]$ if $\gamma > b$ and for $t \in (a, c)$ if $a < \gamma < b$, where $\tilde{\phi}(c) = 0$. Hence $\tilde{\phi}(t) > 0$ for these cases and the lemma is proved. It is straightforward to see that as $\tilde{\phi}(a) = \tilde{\phi}(b) = 0$ and ϕ is concave, then c relative to γ is as stated in the lemma. ∎

LEMMA 6. *([**32**]) For $\tilde{\chi}_1(t)$ as given by (2.154) and (2.153), we have from*

$$\tilde{\chi}_1(t) = \begin{cases} \frac{B-A}{2}\left[B^{n+1} - A^{n+1}\right] - \frac{B^{n+2}-A^{n+2}}{n+2}, & \left\{\begin{array}{l} n \text{ odd and any } \gamma \\ n \text{ even and } \gamma < a \end{array}\right. ; \\ \frac{2C^{n+2}-B^{n+2}-A^{n+2}}{n+2} & \\ \quad +\frac{1}{2(b-a)}\left\{\left[(b-a)^2 - 2(c-a)^2\right]B^{n+1}\right. & \\ \quad \left. +\left[2(b-c)^2 - (b-a)^2\right]\right\}A^{n+1}, & n \text{ even and } a < \gamma < b; \\ \frac{B^{n+2}-A^{n+2}}{n+2} - \frac{B-A}{2}\left[B^{n+1} - A^{n+1}\right], & n \text{ even and } \gamma > b, \end{cases} \tag{2.158}$$

where

$$\begin{cases} B = b - \gamma,\ A = a - \gamma,\ C = c - \gamma, \\ C_1 = \int_a^c C(t)\,dt, \quad C_2 = \int_c^b C(t)\,dt, \\ \text{with} \quad C(t) = \left(\frac{t-a}{b-a}\right)B^{n+1} + \left(\frac{b-t}{b-a}\right)A^{n+1} \end{cases} \tag{2.159}$$

and $\tilde{\phi}(c) = 0$ with $a < c < b$.

PROOF. From (2.153)

$$\tilde{\phi}(t) = (t-\gamma)^{n+1} - C(t), \tag{2.160}$$

where $C(t)$ is as given in (2.159).

Now,

$$\int_a^c (t-\gamma)^{n+1}\,dt = \frac{C^{n+2} - A^{n+2}}{n+2},$$
$$\int_c^b (t-\gamma)^{n+1}\,dt = \frac{B^{n+2} - C^{n+2}}{n+2}$$

and so

$$\int_a^b (t-\gamma)^{n+1}\,dt = \frac{B^{n+2} - A^{n+2}}{n+2}.$$

In addition,

$$\begin{aligned} C_1 &= \int_a^c C(t)\,dt \\ &= \frac{1}{2(b-a)}\left\{(c-a)^2 B^{n+1} + \left[(b-a)^2 - (b-c)^2\right] A^{n+1}\right\}, \\ C_2 &= \int_\gamma^b C(t)\,dt \\ &= \frac{1}{2(b-a)}\left\{\left[(b-a)^2 - (c-a)^2\right] B^{n+1} + (b-c)^2 A^{n+1}\right\}, \end{aligned}$$

and

$$C_1 + C_2 = \int_a^b C(t)\,dt = \frac{B-A}{2}\left(B^{n+1} + A^{n+1}\right).$$

Thus, using Lemma 5, (2.153), (2.154) and (2.160), we have the results as stated in the lemma, after some algebraic manipulation. ∎

LEMMA 7. *([**32**]) For $\tilde{\phi}(t)$ as defined by (2.153), then*

$$\sup_{t\in[a,b]}\left|\tilde{\phi}(t)\right| = \begin{cases} C(t^*) - \frac{B^{n+1}-A^{n+1}}{(n+1)(B-A)}, & n \text{ odd, } n \text{ even and } \gamma < a; \\ \frac{B^{n+1}-A^{n+1}}{(n+1)(B-A)} - C(t^*) & n \text{ even and } \gamma > b; \\ \frac{m_1+m_2}{2} + \left|\frac{m_1-m_2}{2}\right| & n \text{ even and } a < \gamma < b, \end{cases} \tag{2.161}$$

where $(t^* - \gamma)^n = \frac{B^{n+1}-A^{n+1}}{(n+1)(B-A)}$, $C(t)$ is as defined in (2.159), $m_1 = \tilde{\phi}(t_1^*)$, $m_2 = -\tilde{\phi}(t_2^*)$ and t^*, t_1^*, t_2^* are given by (2.162) and (2.163).

PROOF. From Lemma 5, we know that $\tilde{\phi}(a) = \tilde{\phi}(b) = 0$ and so the maximum occurs at t^* where $\tilde{\phi}'(t^*) = 0$, that is, from (2.156)

$$(n+1)(t^* - \gamma)^n - \frac{B^{n+1} - A^{n+1}}{B-A} = 0. \tag{2.162}$$

For n even and $a < \gamma < b$, then there are two solutions to (2.162). Let these be t_1^* and t_2^* with $t_1^* < t_2^*$, i.e.

$$\begin{aligned} t_1^* &= \gamma - \left(\frac{B^{n+1} - A^{n+1}}{(n+1)(B-A)}\right)^{\frac{1}{n}}, \\ t_2^* &= \gamma + \left(\frac{B^{n+1} - A^{n+1}}{(n+1)(B-A)}\right)^{\frac{1}{n}}. \end{aligned} \tag{2.163}$$

Now, using the fact that

$$\max\{m_1, m_2\} = \frac{m_1+m_2}{2} + \left|\frac{m_1-m_2}{2}\right|,$$

the proof of the lemma is completed. ∎

COROLLARY 38. *([**32**]) Let $f : [a, b] \to \mathbb{R}_+$ be an absolutely continuous PDF associated with a random variable X, then, the expectation $E[X]$ satisfies the inequalities*

$$\left|E(X) - \frac{a+b}{2}\right| \leq \begin{cases} \frac{(b-a)^3}{6} \|f'\|_\infty, & f' \in L_\infty[a,b]; \\ \left(\frac{b-a}{2}\right)^{2+\frac{1}{q}} \left[\int_0^{\frac{\pi}{4}} \sec^{2(q+1)}(\theta)\, d\theta\right]^{\frac{1}{q}} \|f'\|_p, & \\ \qquad f' \in L_p[a,b],\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; & \\ \frac{(b-a)^2}{8} \|f'\|_1, & f' \in L_1[a,b]. \end{cases}$$

PROOF. Taking $n = 1$ in Corollary 37 and Using Lemmas 5 - 7, gives the above results after some elementary algebra. In particular,

$$\tilde{\phi}(t) = t^2 - (a+b)t + ab = \left(t - \frac{a+b}{2}\right)^2 + \left(\frac{b-a}{2}\right)^2.$$

and t^*, the one solution to $\tilde{\phi}'(t) = 0$, is $t^* = \frac{a+b}{2}$. ∎

COROLLARY 39. *([**32**]) Let $f : [a, b] \to \mathbb{R}_+$ be an absolutely continuous PDF associated with a random variable X, then the variance, $\sigma^2(X)$ is such that*

$$\left|\sigma^2(X) - S\right| \tag{2.164}$$
$$\leq \begin{cases} \left\{\frac{C^4}{2} - \frac{1}{b-a}\left[(c-a)^3 B^3 - (b-c)^2 A^3\right] \right. \\ \quad \left. + \left(B^2 + A^2\right)\left(\frac{b-a}{2}\right)^2 - \frac{(AB)^2}{2}\right\} \frac{\|f'\|_\infty}{3}, & f' \in L_\infty[a,b]; \\ \left[\int_a^b \left|\hat{\phi}(t)\right|^q dt\right]^{\frac{1}{q}} \frac{\|f'\|_p}{3}, & f' \in L_p[a,b], \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \left[m_1 + m_2 + |m_2 - m_1|\right] \frac{\|f'\|_1}{6}, & f' \in L_1[a,b], \end{cases}$$

where

$$S = \frac{B^3 - A^3}{3(b-a)},$$

$$\begin{cases} & A = a - \gamma,\ B = b - \gamma,\ C = c - \gamma,\ \gamma = E[X], \\ & c \text{ satisfies } \hat{\phi}(c) = 0,\ a < c < b, \\ \text{with} & m_1 = \hat{\phi}\left[E[X] - S^{\frac{1}{2}}\right],\ m_2 = \hat{\phi}\left[E[X] + S^{\frac{1}{2}}\right] \\ \text{and} & \hat{\phi}(\cdot) \text{ as given by (2.128).} \end{cases} \tag{2.165}$$

$a < \gamma = E(X) < b.$

Proof. Taking $n = 2$ in Corollary 37 gives, from (2.153),

$$\hat{\phi}(t) = (t-\gamma)^3 + \left(\frac{b-t}{b-a}\right)(\gamma - a)^3 - \left(\frac{t-a}{b-a}\right)(b-\gamma)^3, \tag{2.166}$$

where $a < \gamma = E(X) < b$.

Now, from Lemmas 5 and 6 with $n = 2$ and $a < \gamma < b$ gives

$$\begin{aligned}\tilde{\chi}_1(t) &= \frac{2C^4 - B^4 - A^4}{4} \\ &\quad + \frac{1}{2(b-a)}\left\{\left[(b-a)^2 - 2(c-a)^2\right]B^3 \right. \\ &\quad \left. + \left[2(b-c)^2 - (b-a)^2\right]A^3\right\} \\ &= \frac{C^4}{2} - \frac{1}{b-a}\left[(c-a)^2 B^3 - (b-c)^2 A^3\right] \\ &\quad + \frac{B-A}{2}\left[B^3 - A^3\right] - \frac{B^4 + A^4}{4}.\end{aligned} \tag{2.167}$$

Now,

$$\begin{aligned}&\frac{B-A}{2}\left[B^3 - A^3\right] - \frac{B^4+A^4}{4} \\ &= \frac{1}{4}\left\{2(B-A)\left(B^3 - A^3\right) - \left(B^4 + A^4\right)\right\} \\ &= \frac{1}{4}\left[B^4 + A^4 - 2AB\left(B^2 + A^2\right)\right] \\ &= \frac{1}{4}\left[\left(B^2 + A^2\right)^2 - 2(AB)^2 - 2AB\left(B^2 + A^2\right)\right] \\ &= \frac{1}{4}\left[\left(B^2 + A^2\right)(B-A)^2 - 2(AB)^2\right]\end{aligned}$$

and so substitution into (2.167) gives the first inequality in (2.164) for $f' \in L_\infty[a,b]$ on using (2.158) and the fact that $B - A = b - a$.

For $f' \in L_p[a,b]$, the bound is not given explicitly but is as presented in (2.152) with $\tilde{\phi}(t)$ replaced by $\hat{\phi}(t)$ from (2.166).

Now, for $f' \in L_1[a,b]$, using Lemma 6 with $n = 2$ and in particular (2.161) and (2.163) gives the stated result. The corollary is now completely established. ∎

CHAPTER 3

Trapezoidal Type Results and Applications for PDFs

1. The Perturbed Trapezoid Formula and Applications

1.1. Introduction. In [**80**], the authors have pointed out the following trapezoid inequality in terms of the $p-$norms of the second derivative.

THEOREM 33. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b). Then we have the estimate*

$$\left|\int_a^b f(x)\,dx-\frac{b-a}{2}\left[f(a)+f(b)\right]\right| \tag{3.1}$$
$$\leq\begin{cases}\frac{\|f''\|_\infty}{12}(b-a)^3 & \text{if } f''\in L_\infty[a,b]\,;\\ \frac{1}{2}\|f''\|_p\left[B(q+1,q+1)\right]^{\frac{1}{q}}(b-a)^{2+\frac{1}{q}}\,, & \\ \quad \text{if } f''\in L_p[a,b]\,,\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; & \\ \frac{\|f''\|_1}{8}(b-a)^2 & \text{if } f''\in L_1[a,b]\,;\end{cases}$$

and B is the Beta function.

Using Grüss' integral inequality, the following perturbed trapezoid inequality in terms of the upper and lower bounds of the second derivative, may be stated (see [**80**]):

THEOREM 34. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b) and assume that*

$$\gamma:=\inf_{x\in(a,b)}f''(x)>-\infty\quad\text{and}\quad\Gamma:=\sup_{x\in(a,b)}f''(x)<\infty, \tag{3.2}$$

then we have the estimation,

$$\left|\int_a^b f(x)\,dx-\frac{b-a}{2}\left[f(a)+f(b)\right]+\frac{(b-a)^2}{12}\left[f'(b)-f'(a)\right]\right| \tag{3.3}$$
$$\leq\frac{1}{32}(\Gamma-\gamma)(b-a)^3\,.$$

In [**35**], by the use of a finer argument based on a pre-Grüss inequality, the authors have improved (3.3) as follows.

THEOREM 35. *If f is as in Theorem 34, then*

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]\right| \leq \frac{1}{24\sqrt{5}}(b-a)^3(\Gamma-\gamma), \tag{3.4}$$

where γ and Γ are given in (3.2).

REMARK 40. *Atkinson* [**5**] *defines the quadrature rule*

$$PT(f;a,b) := \frac{b-a}{2}[f(a)+f(b)] - \frac{(b-a)^2}{12}[f'(b)-f'(a)]$$

as a corrected trapezoidal rule and obtains it using an asymptotic error estimate approach which does not provide an expression for the error bound.

In this section we point out different bounds for the corrected trapezoidal rule. A natural application for the expectation of a random variable is also given.

1.2. The Results. We have the following representation [**17**].

LEMMA 8. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function so that f' is absolutely continuous on $[a,b]$, then we have the representation:*

$$\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^3}{12}[f';a,b] = \frac{1}{2}\int_a^b (x-a)(b-x)\{[f';a,b]-f''(x)\}\,dx, \tag{3.5}$$

where

$$[f';a,b] := \frac{f'(b)-f'(a)}{b-a}$$

is the divided difference.

PROOF. By twice applying the integration by parts formula, we have (see for example [**5**]) that

$$\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] = -\frac{1}{2}\int_a^b (x-a)(b-x)f''(x)\,dx. \tag{3.6}$$

On the other hand, by the simple identity:

$$\begin{aligned} (3.7)\quad & \frac{1}{b-a}\int_a^b h(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b h(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx \\ &= \frac{1}{b-a}\int_a^b h(x)\left[g(x) - \frac{1}{b-a}\int_a^b g(y)\,dy\right]dx, \end{aligned}$$

we can write

$$\begin{aligned} & \int_a^b (x-a)(b-x)\,f''(x)\,dx \\ & \quad - \int_a^b (x-a)(b-x)\,dx \cdot \frac{1}{b-a}\int_a^b f''(x)\,dx \\ & \quad\quad = \int_a^b (x-a)(b-x)\left[f''(x) - [f';a,b]\right]dx, \end{aligned}$$

which is clearly equivalent to:

$$\begin{aligned} (3.8)\quad \int_a^b (x-a)(b-x)\,f''(x)\,dx &= \frac{(b-a)^2}{6}\left[f'(b) - f'(a)\right] \\ &+ \int_a^b (x-a)(b-x)\left[f''(x) - [f';a,b]\right]dx. \end{aligned}$$

Combining (3.6) with (3.8), we deduce (3.5). ∎

Using the above representation, we can state the following result on the error of the perturbed trapezoid formula [**17**]:

Theorem 36. *With the assumptions of Lemma 8, we have*

$$\begin{aligned} (3.9)\quad & \left|\int_a^b f(x)\,dx - \frac{b-a}{2}\left[f(a)+f(b)\right] + \frac{(b-a)^3}{12}[f';a,b]\right| \\ & \leq \begin{cases} \frac{(b-a)^3}{12}\left\|f'' - [f';a,b]\right\|_\infty & \text{if } f'' \in L_\infty[a,b]; \\ \frac{1}{2}\left[B(q+1,q+1)\right]^{\frac{1}{q}}(b-a)^{2+\frac{1}{q}}\left\|f'' - [f';a,b]\right\|_p, & \\ \qquad \text{if } f'' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; & \\ \frac{1}{8}(b-a)^2\left\|f'' - [f';a,b]\right\|_1 & \text{if } f'' \in L_1[a,b], \end{cases} \end{aligned}$$

PROOF. Using Lemma 8, we have

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}\left[f(a)+f(b)\right] + \frac{(b-a)^3}{12}\left[f';a,b\right]\right| \leq \frac{1}{2}\int_a^b (x-a)(b-x)\left|f''(x)-\left[f';a,b\right]\right|dx =: M. \tag{3.10}$$

It is obvious that

$$\begin{aligned} M &\leq \frac{1}{2}\left\|f''-\left[f';a,b\right]\right\|_\infty \cdot \int_a^b (x-a)(b-x)\,dx \\ &= \frac{(b-a)^3}{12}\left\|f''-\left[f';a,b\right]\right\|_\infty \end{aligned}$$

and the first part of (3.9) is proved.

Using Hölder's integral inequality, we have for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, that

$$M \leq \frac{1}{2}\left(\int_a^b (x-a)^q (b-x)^q\,dx\right)^{\frac{1}{q}} \times \left(\int_a^b \left|f''(x)-\left[f';a,b\right]\right|^p dx\right)^{\frac{1}{p}}. \tag{3.11}$$

Now, using the transformation $x=(1-t)a+tb$, $t\in[0,1]$, we get

$$\begin{aligned} (x-a)^q (b-x)^q &= (b-a)^{2q} t^q (1-t)^q, \\ dx &= (b-a)\,dt \end{aligned}$$

and thus

$$\begin{aligned} \int_a^b (x-a)^q (b-x)^q\,dx &= (b-a)^{2q+1}\int_0^1 t^q(1-t)^q\,dt \\ &= (b-a)^{2q+1} B(q+1,q+1). \end{aligned}$$

Using (3.11) we deduce the second part of (3.9).

Finally, as

$$\begin{aligned} M &\leq \frac{1}{2}\sup_{x\in[a,b]}\left\{(x-a)(b-x)\right\}\int_a^b \left|f''(x)-\left[f';a,b\right]\right|dx \\ &= \frac{(b-a)^2}{8}\left\|f''-\left[f';a,b\right]\right\|_1 \end{aligned}$$

the theorem is completely proved. ∎

The following corollary concerning the Euclidean norm is useful in practice.

COROLLARY 40. *If $f : [a,b] \to \mathbb{R}$ is such that $f'' \in L_2[a,b]$, then we have the inequality:*

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^3}{12}[f';a,b] \right| \le \frac{(b-a)^3}{2\sqrt{30}} \left[\frac{1}{b-a}\|f''\|_2^2 - [f';a,b]^2 \right]^{\frac{1}{2}}. \tag{3.12}$$

PROOF. Choosing in (3.9) $p=q=2$, we get

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^3}{12}[f';a,b] \right| \le \frac{1}{2}[B(3,3)]^{\frac{1}{2}}(b-a)^{2+\frac{1}{2}} \|f'' - [f';a,b]\|_2 . \tag{3.13}$$

However,

$$B(3,3) = \frac{1}{30},$$

and

$$\begin{aligned}
&\|f'' - [f';a,b]\|_2 \\
&= \left[\int_a^b (f''(x) - [f';a,b])^2\,dx \right]^{\frac{1}{2}} \\
&= \left[\int_a^b (f''(x))^2\,dx - 2\int_a^b f''(x)[f';a,b]\,dx + (b-a)[f';a,b]^2 \right]^{\frac{1}{2}} \\
&= \left(\|f''\|_2^2 - 2(b-a)[f';a,b]^2 + (b-a)[f';a,b]^2 \right)^{\frac{1}{2}} \\
&= \sqrt{b-a}\left(\frac{1}{b-a}\|f''\|_2^2 - [f';a,b]^2 \right)^{\frac{1}{2}},
\end{aligned}$$

then, by (3.13) we get (3.12). ∎

REMARK 41. (1) *The Grüss integral inequality for a function $g : [a,b] \to \mathbb{R}$ with $-\infty < m \le g(x) \le M < \infty$ for almost every $x \in [a,b]$ states that (see for example* [**109**, p. 296]*)*

$$0 \le \frac{1}{b-a}\|g\|_2^2 - \left(\frac{1}{b-a}\int_a^b g(x)\,dx \right)^2 \le \frac{1}{4}(M-m)^2 . \tag{3.14}$$

Applying (3.14) for the mapping f'' under the assumption that $\gamma \leq f''(x) \leq \Gamma$ for a.e. $x \in [a,b]$, we deduce

$$\left(\frac{1}{b-a}\left\|f''\right\|_2^2 - [f';a,b]^2\right)^{\frac{1}{2}} \leq \frac{1}{2}(\Gamma-\gamma)$$

and then, by (3.12), we further deduce

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^3}{12}[f';a,b]\right| \leq \frac{(b-a)^3}{4\sqrt{30}}(\Gamma-\gamma). \tag{3.15}$$

which is not as good as the result of (3.3).

(2) *Chebychev's inequality for a differentiable function $g:[a,b]\to\mathbb{R}$, with $g' \in L_\infty[a,b]$ states that (see* [**109**, p. 297]*)*

$$0 \leq \frac{1}{b-a}\left\|g\right\|_2^2 - \left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)^2 \leq \frac{1}{12}(b-a)^2\left\|g'\right\|_\infty^2. \tag{3.16}$$

Applying (3.16) for the mapping f'' under the assumption that $f''' \in L_\infty[a,b]$, we deduce, by (3.12), that

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^3}{12}[f';a,b]\right| \leq \frac{(b-a)^4\left\|f'''\right\|_\infty}{12\sqrt{10}}. \tag{3.17}$$

(3) *Lupaş's inequality for a differentiable function f with $f''' \in L_2[a,b]$ states that (see* [**109**, p. 301]*)*

$$0 \leq \frac{1}{b-a}\left\|g\right\|_2^2 - \left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)^2 \leq \frac{b-a}{\pi^2}\left\|g'\right\|_2^2. \tag{3.18}$$

Applying (3.18) for the mapping f'' under the assumption that $f''' \in L_2[a,b]$, we deduce, by (3.12), that

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^3}{12}[f';a,b]\right| \leq \frac{(b-a)^{\frac{7}{2}}\left\|f'''\right\|_2}{2\pi\sqrt{30}}. \tag{3.19}$$

The following lemma of representation also holds [**17**].

LEMMA 9. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' is absolutely continuous on $[a, b]$. We have the representation:*

$$\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^3}{12}[f';a,b] = \frac{1}{2}\int_a^b \left(x-\frac{a+b}{2}\right)^2 [f''(x) - [f';a,b]]\,dx. \tag{3.20}$$

PROOF. The identity (3.20) may be proven directly.

A simpler proof uses Lemma 8 as follows.

Since

$$(x-a)(b-x) = \left(\frac{a-b}{2}\right)^2 - \left(x-\frac{a+b}{2}\right)^2$$

and

$$\begin{aligned}
&\frac{1}{2}\int_a^b (x-a)(b-x)\{[f';a,b] - f''(x)\}\,dx \\
&= \frac{1}{2}\int_a^b \left[\left(\frac{a-b}{2}\right)^2 - \left(x-\frac{a+b}{2}\right)^2\right]\{[f';a,b] - f''(x)\}\,dx \\
&= \frac{1}{2}\int_a^b \left(\frac{a-b}{2}\right)^2 \{[f';a,b] - f''(x)\}\,dx \\
&\quad - \frac{1}{2}\int_a^b \left(x-\frac{a+b}{2}\right)^2 \{[f';a,b] - f''(x)\}\,dx \\
&= \frac{1}{2}\int_a^b \left(x-\frac{a+b}{2}\right)^2 \{f''(x) - [f';a,b]\}\,dx,
\end{aligned}$$

then by (3.5) we deduce (3.20). ∎

The following result also holds [**17**].

THEOREM 37. *With the assumptions of Lemma 9, we have the inequality*

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^3}{12}[f';a,b]\right| \tag{3.21}$$
$$\le \begin{cases} \frac{(b-a)^3}{24}\left\|f'' - [f';a,b]\right\|_\infty & \text{if } f'' \in L_\infty[a,b]; \\ \frac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}\left\|f'' - [f';a,b]\right\|_p, & \text{if } f'' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^2}{8}\left\|f'' - [f';a,b]\right\|_1. & \end{cases}$$

PROOF. Using Lemma 9, we have:

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}\left[f(a)+f(b)\right] + \frac{(b-a)^3}{12}\left[f';a,b\right]\right| \leq \frac{1}{2}\int_a^b \left(x-\frac{a+b}{2}\right)^2 \left|f''(x)-\left[f';a,b\right]\right| dx =: N. \tag{3.22}$$

It is obvious that

$$N \leq \frac{1}{2}\left\|f''-\left[f';a,b\right]\right\|_\infty \cdot \int_a^b \left(x-\frac{a+b}{2}\right)^2 dx = \frac{(b-a)^3}{24}\left\|f''-\left[f';a,b\right]\right\|_\infty .$$

Using Hölder's integral inequality, we have for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, that

$$N \leq \frac{1}{2}\left(\int_a^b \left|x-\frac{a+b}{2}\right|^{2q} dx\right)^{\frac{1}{q}} \times \left(\int_a^b \left|f''(x)-\left[f';a,b\right]\right|^p dx\right)^{\frac{1}{p}}. \tag{3.23}$$

However,

$$\int_a^b \left|x-\frac{a+b}{2}\right|^{2q} dx = 2\int_{\frac{a+b}{2}}^b \left(x-\frac{a+b}{2}\right)^{2q} dx = \frac{(b-a)^{2q+1}}{4^q(2q+1)}$$

and then, by (3.23), we deduce the second part of (3.21).

Finally, as

$$\sup_{x\in[a,b]}\left|x-\frac{a+b}{2}\right|^2 = \frac{(b-a)^2}{4},$$

then

$$N \leq \frac{(b-a)^2}{8}\left\|f''-\left[f';a,b\right]\right\|_1 ,$$

proving the last part of (3.21). ∎

REMARK 42. *It is obvious that the first inequality in (3.21) is better than the similar one in (3.9), while the last ones are identical.*

REMARK 43. *A computer simulation for the functions* $\frac{1}{2}\left[B(q+1,q+1)\right]^{\frac{1}{q}}$ *and* $\frac{1}{8}\cdot\frac{1}{(2q+1)^{\frac{1}{q}}}$ *shows that the latter is smaller for* *any $q > 1$, but we do not have an analytic proof of this. We conjecture that the second inequality in (3.11) is better than the second inequality in (3.9) for every $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.*

For $p = q = 2$, we get the following particular case for the Euclidean norm [**17**]:

COROLLARY 41. *If $f : [a, b] \to \mathbb{R}$ is such that $f'' \in L_2[a, b]$, then*

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^3}{12}[f'; a, b] \right| \leq \frac{(b-a)^3}{8\sqrt{5}} \left[\frac{1}{b-a} \|f''\|_2^2 - [f'; a, b]^2 \right]^{\frac{1}{2}}. \tag{3.24}$$

REMARK 44. *We note that (3.24) is a better result than the corresponding one in (3.12), and thus, we note the following better results via Grüss type inequalities.*

If f'' is such that $\gamma \leq f'' \leq \Gamma$ for a.e. $x \in [a, b]$, then by Grüss' (3.14), we have

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^3}{12}[f'; a, b] \right| \leq \frac{(b-a)^3 (\Gamma - \gamma)}{16\sqrt{5}}. \tag{3.25}$$

If $f''' \in L_\infty[a, b]$, then by the Chebychev inequality (3.16), we have

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^3}{12}[f'; a, b] \right| \leq \frac{(b-a)^4 \|f'''\|_\infty}{16\sqrt{15}}. \tag{3.26}$$

Finally, if $f''' \in L_2[a, b]$, then by the Lupaş inequality (3.18), we have:

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^3}{12}[f'; a, b] \right| \leq \frac{(b-a)^{\frac{7}{2}} \|f'''\|_2}{8\pi\sqrt{5}}. \tag{3.27}$$

1.3. Applications for Expectation. Let X be a random variable having the PDF, $f : [a,b] \to \mathbb{R}$ and the *cumulative distribution function* $F : [a,b] \to [0,1]$.

The following result holds [**17**].

THEOREM 38. *With the above assumptions and assuming, additionally, that the PDF is absolutely continuous on* $[a,b]$, *then we have the inequality:*

$$\left| E(X) - \frac{a+b}{2} - \frac{(b-a)^2}{12}[f(b) - f(a)] \right| \tag{3.28}$$

$$\leq \begin{cases} \frac{(b-a)^3}{24} \|f' - [f;a,b]\|_\infty & \text{if } f' \in L_\infty[a,b] \\ \frac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}} \|f' - [f;a,b]\|_p & \text{if } f' \in L_p[a,b], \\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^2}{8} \|f' - [f;a,b]\|_1 . \end{cases}$$

PROOF. Applying Theorem 37 for the CDF, F, we may write

$$\left| \int_a^b F(t)\,dt - \frac{F(a)+F(b)}{2}(b-a) + \frac{(b-a)^2}{12}[f(b)-f(a)] \right| \tag{3.29}$$

$$\leq \begin{cases} \frac{(b-a)^3}{24} \|f' - [f;a,b]\|_\infty \\ \frac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}} \|f' - [f;a,b]\|_p \\ \frac{(b-a)^2}{8} \|f' - [f;a,b]\|_1 . \end{cases}$$

However, $F(a) = 0$, $F(b) = 1$ and

$$\int_a^b F(t)\,dt = b - E(X),$$

and then, by (3.29) we deduce the desired inequality (3.28). ∎

2. A Perturbed Inequality Using the Third Derivative

2.1. Introduction. In [**80**], by the use of Grüss' integral inequality, the authors obtained the following perturbed trapezoid inequality.

THEOREM 39. *Let* $f : [a,b] \to \mathbb{R}$ *be a twice differentiable function on* (a,b) *and asume that*

$$\gamma := \inf_{x\in(a,b)} f''(x) > -\infty \text{ and } \Gamma := \sup_{x\in(a,b)} f''(x) < \infty,$$

then we have the inequality

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]\right| \leq \frac{(b-a)^3}{12}(\Gamma-\gamma). \tag{3.30}$$

Using a finer argument based on a pre-Grüss inequality, Cerone and Dragomir [**35**, p. 121] improved the above result as follows.

Theorem 40. *Let f have the properties of Theorem 39, then*

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]\right| \leq \frac{1}{24\sqrt{5}}(b-a)^3(\Gamma-\gamma). \tag{3.31}$$

The main aim of the next section is to point out some bounds for the left part of (3.31) in terms of the $p-$norms of f''' assuming that the function f is twice differentiable on (a,b) and that the second derivative is absolutely continuous on (a,b).

A number of applications are also pointed out.

2.2. A Perturbed Trapezoid Formula. The following representation lemma holds [**14**].

Lemma 10. *Let $f:[a,b]\to\mathbb{R}$ be such that the second derivative is absolutely continuous on $[a,b]$, then we have the equality:*

$$\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a) + \frac{(b-a)^2}{12}[f'(b)-f'(a)] = \frac{1}{4(b-a)}\int_a^b\int_a^b\left(\int_s^t\left(u-\frac{t+s}{2}\right)f'''(u)\,du\right)(t-s)\,dtds. \tag{3.32}$$

Proof. Integrating by parts, we have

$$\begin{aligned} I &:= \int_a^b\int_a^b\left(\int_s^t\left(u-\frac{t+s}{2}\right)f'''(u)\,du\right)(t-s)\,dtds \\ &= \int_a^b\int_a^b\left[\frac{f''(t)+f''(s)}{2}(t-s) - \int_s^t f''(u)\,du\right](t-s)\,dtds \\ &= \int_a^b\int_a^b\left[\frac{f''(t)(t-s)^2+f''(s)(t-s)^2}{2}\right. \end{aligned}$$

$$
\begin{aligned}
&\quad - \left(f'(t) - f'(s)\right)(t-s)\Big] dtds \\
&= \frac{1}{2}\left[\int_a^b \int_a^b f''(t)(t-s)^2 \, dtds + \int_a^b \int_a^b f''(s)(t-s)^2 \, dtds\right] \\
&\quad - \int_a^b \int_a^b \left(f'(t) - f'(s)\right)(t-s) \, dtds.
\end{aligned}
$$

By symmetry,

$$
J := \int_a^b \int_a^b f''(t)(t-s)^2 \, dtds = \int_a^b \int_a^b f''(s)(t-s)^2 \, dtds,
$$

and using Korkine's identity or direct computation, we have

$$
\begin{aligned}
K &:= \int_a^b \int_a^b \left(f'(t) - f'(s)\right)(t-s) \, dtds \\
&= 2\left[(b-a)\int_a^b f'(t) \, tdt - \int_a^b f'(t) \, dt \cdot \int_a^b tdt\right].
\end{aligned}
$$

Then, $I = J - K$.

Since

$$
\begin{aligned}
J &= \int_a^b f''(t)\left(\int_a^b (t-s)^2 \, ds\right) dt \\
&= \frac{1}{3}\left[\int_a^b f''(t)(b-t)^3 \, dt + \int_a^b (t-a)^3 f''(t) \, dt\right] \\
&= \frac{1}{3}\Bigg[f'(t)(b-t)^3\Big|_a^b + 3\int_a^b (b-t)^2 f'(t) \, dt \\
&\qquad + f'(t)(t-a)^3\Big|_a^b - 3\int_a^b (t-a)^2 f'(t) \, dt\Bigg] \\
&= \frac{1}{3}\Bigg[-f'(a)(b-a)^3 \\
&\qquad + 3\left[f(t)(b-t)^2\Big|_a^b - 2\int_a^b (b-t) f(t) \, dt\right] \\
&\qquad + f'(b)(b-a)^3 - 3\left[f(t)(t-a)^2\Big|_a^b - 2\int_a^b (t-a) f(t) \, dt\right]\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{3}\Big[\left[f'(b) - f'(a)\right](b-a)^3 \\
&\qquad + 3\left[-f(a)(b-a)^2 + 2\int_a^b (b-t) f(t)\,dt\right] \\
&\qquad - 3\left[f(b)(b-a)^2 - 2\int_a^b (t-a) f(t)\,dt\right]\Big] \\
&= \frac{1}{3}\left[f'(b) - f'(a)\right](b-a)^3 \\
&\qquad - \left[\frac{f(a)+f(b)}{2}\right](b-a)^2 + 2(b-a)\int_a^b f(t)\,dt
\end{aligned}
$$

and

$$
\begin{aligned}
K &= 2\left[(b-a)\left[f(t)\,t\Big|_a^b - \int_a^b f(t)\,dt\right] - \left[f(b)-f(a)\right]\frac{b^2-a^2}{2}\right] \\
&= 2\Big[(b-a)\left[f(b)\,b - f(a)\,a - \int_a^b f(t)\,dt\right] \\
&\qquad - (b-a)\left[f(b)-f(a)\right]\frac{a+b}{2}\Big] \\
&= 2(b-a)\Big[f(b)\,b - f(a)\,a - \int_a^b f(t)\,dt \\
&\qquad - f(b)\frac{a+b}{2} + f(a)\frac{a+b}{2}\Big] \\
&= (b-a)^2\left[f(a)+f(b)\right] - 2(b-a)\int_a^b f(t)\,dt,
\end{aligned}
$$

then

$$
\begin{aligned}
I &= \frac{1}{3}\left[f'(b) - f'(a)\right](b-a)^3 \\
&\qquad - 2\left[f(a)+f(b)\right](b-a)^2 + 4(b-a)\int_a^b f(t)\,dt.
\end{aligned}
$$

Dividing by $4(b-a)$ we deduce the desired equality (3.32). ∎

The following perturbed version of the trapezoid inequality holds [**14**].

Theorem 41. *Let $f:[a,b]\to\mathbb{R}$ be such that the second derivative is absolutely continuous on $[a,b]$, then we have*

$$\left|\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)+\frac{(b-a)^2}{12}\left[f'(b)-f'(a)\right]\right| \tag{3.33}$$

$$\leq\begin{cases}\frac{1}{16(b-a)}\int_a^b\int_a^b|t-s|^3\left\|f'''\right\|_{[t,s],\infty}dtds & \text{if } f'''\in L_\infty[a,b];\\ \frac{1}{8(q+1)^{\frac{1}{q}}(b-a)}\int_a^b\int_a^b|t-s|^{2+\frac{1}{q}}\left\|f'''\right\|_{[t,s],p}dtds & \\ \qquad\text{if } f'''\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; & \\ \frac{1}{8(b-a)}\int_a^b\int_a^b(t-s)^2\left\|f'''\right\|_{[t,s],1}dtds & \end{cases}$$

$$\leq\begin{cases}\frac{(b-a)^4}{160}\left\|f'''\right\|_{[a,b],\infty} & \text{if } f'''\in L_\infty[a,b];\\ \frac{q^2(b-a)^{3+\frac{1}{q}}}{4(3q+1)(4q+1)(q+1)^{\frac{1}{q}}}\left\|f'''\right\|_{[a,b],p} & \text{if } f'''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1.\\ \frac{(b-a)^3}{48}\left\|f'''\right\|_{[a,b],1}, & \end{cases}$$

Proof. Denote

$$R(f;a,b) := \frac{1}{4(b-a)}\int_a^b\int_a^b\left(\int_s^t\left(u-\frac{t+s}{2}\right)f'''(u)\,du\right)(t-s)\,dtds.$$

As

$$\left|\int_s^t\left(u-\frac{t+s}{2}\right)f'''(u)\,du\right|\leq\left\|f'''\right\|_{[t,s],\infty}\left|\int_s^t\left|u-\frac{t+s}{2}\right|du\right| = \frac{(t-s)^2}{4}\left\|f'''\right\|_{[t,s],\infty},$$

$$\left|\int_s^t\left(u-\frac{t+s}{2}\right)f'''(u)\,du\right| \leq\left\|f'''\right\|_{[t,s],p}\left|\int_s^t\left|u-\frac{t+s}{2}\right|^q du\right|^{\frac{1}{q}} = \frac{|t-s|^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\left\|f'''\right\|_{[t,s],q},\ p>1,\ \frac{1}{p}+\frac{1}{q}=1$$

and

$$\left|\int_s^t \left(u-\frac{t+s}{2}\right) f'''(u)\,du\right| \leq \sup_{\substack{u\in[t,s]\\(u\in[t,s])}} \left|u-\frac{t+s}{2}\right| \|f'''\|_{[t,s],1}$$

then we can state that

$$\left|\int_s^t \left(u-\frac{t+s}{2}\right) f'''(u)\,du\right| \leq \begin{cases} \frac{(t-s)^2}{4}\|f'''\|_{[t,s],\infty} & \text{if } f'''\in L_\infty[a,b]; \\ \frac{|t-s|^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\|f'''\|_{[t,s],q} & \text{if } f'''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{|t-s|}{2}\|f'''\|_{[t,s],1}. \end{cases} \tag{3.34}$$

Taking the modulus of $R(f;a,b)$ we get, by (3.34),

$$\begin{aligned} &|R(f;a,b)| \\ &\leq \frac{1}{4(b-a)}\int_a^b\int_a^b |t-s|\left|\int_s^t\left(u-\frac{t+s}{2}\right)f'''(u)\,du\right|dtds \\ &\leq \frac{1}{4(b-a)}\begin{cases} \frac{1}{4}\int_a^b\int_a^b|t-s|^3\|f'''\|_{[t,s],\infty}\,dtds \text{ if } f'''\in L_\infty[a,b]; \\ \frac{1}{2(q+1)^{\frac{1}{q}}}\int_a^b\int_a^b|t-s|^{2+\frac{1}{q}}\|f'''\|_{[t,s],p}\,dtds \\ \qquad \text{if } f'''\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\int_a^b\int_a^b|t-s|^2\|f'''\|_{[t,s],1}\,dtds \end{cases} \end{aligned}$$

which proves the first inequality in (3.33).

Now, consider the double integral

$$\begin{aligned} I_m &:= \int_a^b\int_a^b |t-s|^m\,dtds \\ &= \int_a^b\left[\int_a^t (t-s)^m\,ds + \int_t^b (s-t)^m\,ds\right]dt \\ &= \int_a^b\left[\frac{(t-a)^{m+1}+(b-t)^{m+1}}{m+1}\right]dt = \frac{2(b-a)^{m+2}}{(m+1)(m+2)} \end{aligned}$$

for all $m>0$.

Using the above calculation for I_m, we have:

$$\int_a^b \int_a^b |t-s|^3 \|f'''\|_{[t,s],\infty} \, dtds \leq \|f'''\|_{[a,b],\infty} \int_a^b \int_a^b |t-s|^3 \, dtds = \frac{(b-a)^5}{10} \cdot \|f'''\|_{[a,b],\infty},$$

$$\int_a^b \int_a^b |t-s|^{2+\frac{1}{q}} \|f'''\|_{[t,s],p} \, dtds \leq \|f'''\|_{[a,b],p} \int_a^b \int_a^b |t-s|^{2+\frac{1}{q}} \, dtds = \frac{2q^2 (b-a)^{4+\frac{1}{q}}}{(3q+1)(4q+1)} \cdot \|f'''\|_{[a,b],p}$$

and

$$\int_a^b \int_a^b (t-s)^2 \|f'''\|_{[t,s],1} \, dtds \leq \|f'''\|_{[a,b],1} \int_a^b \int_a^b (t-s)^2 \, dtds = \frac{(b-a)^4}{6} \cdot \|f'''\|_{[a,b],1},$$

which give the last part of (3.33). ∎

2.3. Applications for Expectation. Let X be a random variable having the PDF, $f : [a,b] \to \mathbb{R}$ and the *cumulative distribution function* $F : [a,b] \to [0,1]$.

THEOREM 42. ([**14**]) *With the above assumptions and, additionally, that the PDF, f is differentiable on $[a,b]$ and f' is absolutely continuous, then*

$$\left| E(X) - \frac{a+b}{2} - \frac{(b-a)^2}{12} [f(b) - f(a)] \right| \leq \begin{cases} \frac{(b-a)^4}{160} \|f''\|_{[a,b],\infty} & \text{if } f'' \in L_\infty [a,b]; \\ \frac{q^2 (b-a)^{3+\frac{1}{q}}}{4(3q+1)(4q+1)(q+1)^{\frac{1}{q}}} \|f''\|_{[a,b],p} & \text{if } f'' \in L_p [a,b], \\ & \quad p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^3}{48} \|f''\|_{[a,b],1}. \end{cases} \tag{3.35}$$

PROOF. Applying Theorem 41 for F, we may write that

$$\left|\int_a^b F(t)\,dt - \frac{F(a)+F(b)}{2}(b-a) + \frac{(b-a)^2}{12}[f(b)-f(a)]\right| \tag{3.36}$$

$$\leq \begin{cases} \frac{(b-a)^4}{160}\|f''\|_{[a,b],\infty} & \text{if } f'' \in L_\infty[a,b]; \\ \frac{q^2(b-a)^{3+\frac{1}{q}}}{4(3q+1)(4q+1)(q+1)^{\frac{1}{q}}}\|f''\|_{[a,b],p} & \text{if } f'' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^3}{48}\|f''\|_{[a,b],1}. \end{cases}$$

However, $F(a)=0$, $F(b)=1$ and

$$\int_a^b F(t)\,dt = b - E(X),$$

and then, by (3.36), we obtain the desired inequality (3.35). ∎

3. Bounds in Terms of the Fourth Derivative

3.1. Introduction. Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b) and assume that

$$\gamma := \inf_{x\in(a,b)} f''(x) > -\infty \text{ and } \Gamma := \sup_{x\in(a,b)} f''(x) < \infty.$$

Denote the 'Corrected Trapezoid' rule [**5**] by

$$CT(f,a,b,f') \tag{3.37}$$
$$:= \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)].$$

In [**80**], by the use of Grüss' integral inequality, the authors have proved

$$|CT(f,a,b,f')| \leq K(b-a)^3(\Gamma-\gamma) \tag{3.38}$$

with $K=\frac{1}{12}$.

Using a more careful analysis based on a pre-Grüss inequality, Cerone and Dragomir [**35**, p. 121], have shown that (3.38) holds with the better constant $K=\frac{1}{24\sqrt{5}}$.

A completely different upper bound for $|CT(f,a,b,f')|$ has been obtained by the present authors by assuming the existence of absolutely continuous third derivatives [**14**],

$$|CT(f,a,b,f')| \tag{3.39}$$

$$\leq \begin{cases} \frac{1}{16(b-a)} \int_a^b \int_a^b |t-s|^3 \|f'''\|_{[t,s],\infty} \, dtds \text{ if } f''' \in L_\infty [a,b]; \\ \frac{1}{8(q+1)^{\frac{1}{q}}(b-a)} \int_a^b \int_a^b |t-s|^{2+\frac{1}{q}} \|f'''\|_{[t,s],p} \, dtds \\ \qquad \text{if } f''' \in L_p [a,b], \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{8(b-a)} \int_a^b \int_a^b (t-s)^2 \|f'''\|_{[t,s],1} \, dtds \end{cases}$$

$$\leq \begin{cases} \frac{(b-a)^4}{160} \|f'''\|_{[a,b],\infty} \qquad \text{if } f''' \in L_\infty [a,b]; \\ \frac{q^2 (b-a)^{3+\frac{1}{q}}}{4(3q+1)(4q+1)(q+1)^{\frac{1}{q}}} \|f'''\|_{[a,b],p} \\ \qquad \text{if } f''' \in L_p [a,b], \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-a)^3}{48} \|f'''\|_{[a,b],1}. \end{cases}$$

In [**15**], the authors have established the following bounds in terms of the fourth derivatives as well,

$$|CT(f,a,b,f')| \tag{3.40}$$

$$\leq \begin{cases} \frac{1}{4(b-a)} \int_a^b \int_a^b \left| \int_s^t \left| u - \frac{t+s}{2} \right|^2 \left\| f^{(4)} \right\|_{\left[\frac{t+s}{2},u\right],\infty} du \right| |t-s| \, dtds \\ \qquad \text{if } f^{(4)} \in L_\infty [a,b]; \\ \frac{1}{4(b-a)} \int_a^b \int_a^b \left| \int_s^t \left| u - \frac{t+s}{2} \right|^{1+\frac{1}{q}} \left\| f^{(4)} \right\|_{\left[\frac{t+s}{2},u\right],p} du \right| |t-s| \, dtds \\ \qquad \text{if } f^{(4)} \in L_p [a,b], \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{4(b-a)} \int_a^b \int_a^b \left| \int_s^t \left| u - \frac{t+s}{2} \right| \left\| f^{(4)} \right\|_{\left[\frac{t+s}{2},u\right],1} du \right| |t-s| \, dtds \end{cases}$$

$$\leq \begin{cases} \frac{1}{48(b-a)} \int_a^b \int_a^b |t-s|^4 \left\| f^{(4)} \right\|_{[t,s],\infty} dtds \ \text{ if } f^{(4)} \in L_\infty [a,b]; \\ \frac{q}{2^{3+\frac{1}{q}}(2q+1)(b-a)} \int_a^b \int_a^b |t-s|^{3+\frac{1}{q}} \left\| f^{(4)} \right\|_{[t,s],p} dtds \\ \qquad \text{if } f^{(4)} \in L_p [a,b], \text{ and } \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{16(b-a)} \int_a^b \int_a^b |t-s|^3 \left\| f^{(4)} \right\|_{[t,s],1} dtds \end{cases}$$

$$\leq \begin{cases} \frac{\left\| f^{(4)} \right\|_{[a,b],\infty} (b-a)^5}{720} & \text{if } \ f^{(4)} \in L_\infty [a,b]; \\ \frac{q^3 \left\| f^{(4)} \right\|_{[a,b],p}}{2^{2+\frac{1}{q}}(2q+1)(4q+1)(5q+1)} (b-a)^{4+\frac{1}{q}} & \text{if } \ f^{(4)} \in L_p [a,b], \text{ and} \\ & \quad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-a)^4 \left\| f^{(4)} \right\|_{[a,b],1}}{160} \end{cases}$$

The main aim of the next section is to point out some new and better bounds for the functional $CT(f,a,b,f')$ defined by (3.37) in terms of the $p-$norms of $f^{(4)}$ assuming that the function f is three-times differentiable on (a,b) and the third derivative is absolutely continuous on (a,b).

Applications for estimating the expectation of a random variable are also pointed out.

3.2. The Results. The following lemma of representation holds [**16**].

LEMMA 11. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(3)}:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$, then we have the equality:*

$$CT(f,a,b,f') = \frac{1}{8(b-a)}\int_a^b\int_a^b\left(\int_s^t (u-s)(t-u)f^{(4)}(u)\,du\right)(t-s)\,dtds \tag{3.41}$$

PROOF. The following identity has been proven by Dragomir and Mabizela in [**82**]

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2(b-a)^2}\int_a^b\int_a^b [f'(t)-f'(s)](t-s)\,dsdt, \tag{3.42}$$

provided $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$.

This identity can be easily verified by direct computation.

Indeed, we have, successively,

$$\begin{aligned}
&\int_a^b\int_a^b [f'(x)-f'(y)](x-y)\,dxdy \\
&= \int_a^b\int_a^b [xf'(x)+yf'(y)-xf'(y)-yf'(x)]\,dxdy \\
&= 2\int_a^b\int_a^b [xf'(x)-xf'(y)]\,dxdy \\
&= 2\int_a^b\int_a^b xf'(x)\,dxdy - 2\int_a^b\int_a^b xf'(y)\,dxdy
\end{aligned}$$

$$= 2\left[bf(b) - af(a) - (b-a)\int_a^b f(x)\,dx\right] - \left(b^2 - a^2\right)[f(b) - f(a)]$$
$$= (b-a)^2 [f(a) + f(b)] - 2(b-a)\int_a^b f(x)\,dx.$$

Dividing both sides by $2(b-a)^2$ yields the required result.

Also, using the integration by parts formula twice, we obtain the identity

$$\frac{1}{2}\int_a^b (x-a)(b-x)\,g''(x)\,dx = \frac{g(a)+g(b)}{2}(b-a) - \int_a^b g(x)\,dx. \tag{3.43}$$

We have, by (3.43), that

$$f'(t) - f'(s) = \int_s^t f''(u)\,du$$
$$= \frac{f''(t)+f''(s)}{2}(t-s) - \frac{1}{2}\int_s^t (u-s)(t-u)\,f^{(4)}(u)\,du$$

and then, by (3.42)

$$\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)\,dx \tag{3.44}$$
$$= \frac{1}{2(b-a)}\int_a^b\int_a^b \left[\frac{f''(t)+f''(s)}{2}(t-s) - \frac{1}{2}\int_s^t (u-s)(t-u)\,f^{(4)}(u)\,du\right](t-s)\,dsdt$$
$$= \frac{1}{2(b-a)}\int_a^b\int_a^b \left[\frac{f''(t)+f''(s)}{2}(t-s)^2\right] dsdt$$
$$- \frac{1}{4(b-a)} \times \int_a^b\int_a^b \left(\int_s^t (u-s)(t-u)\,f^{(4)}(u)\,du\right)(t-s)\,dsdt$$
$$= I - J,$$

where

$$I := \frac{1}{2(b-a)}\int_a^b\int_a^b \left[\frac{f''(t)+f''(s)}{2}(t-s)^2\right] dsdt$$

and

$$J := -\frac{1}{4(b-a)} \int_a^b \int_a^b \left(\int_s^t (u-s)(t-u) f^{(4)}(u)\, du \right) (t-s)\, ds dt.$$

Taking into account the symmetry of the integrand, we have

$$\begin{aligned} I &:= \frac{1}{2(b-a)} \int_a^b \int_a^b \left[f''(t)(t-s)^2 \right] ds dt \\ &= \frac{1}{2(b-a)} \int_a^b f''(t) \left(\int_a^b (t-s)^2\, ds \right) dt \\ &= \frac{1}{2(b-a)} \int_a^b f''(t) \left[\frac{(b-t)^3 + (t-a)^3}{3} \right] dt \\ &= \frac{1}{2(b-a)} \left[f'(t) \frac{(b-t)^3 + (t-a)^3}{3} \bigg|_a^b \right. \\ &\qquad \left. - \int_a^b f'(t) \left[(t-a)^2 - (b-t)^2 \right] dt \right] \\ &= \frac{1}{2(b-a)} \left\{ \frac{(b-a)^3}{3} \left[f'(b) - f'(a) \right] \right. \\ &\qquad - f(t) \left[(t-a)^2 - (b-t)^2 \right] \bigg|_a^b \\ &\qquad \left. - \int_a^b f(t) \left[2(t-a) - 2(t-b) \right] dt \right\} \\ &= \frac{1}{2(b-a)} \left\{ \frac{(b-a)^3}{3} \left[f'(b) - f'(a) \right] \right. \\ &\qquad \left. - \left[f(b)(b-a)^2 + f(a)(b-a)^2 + 2(b-a) \int_a^b f(t)\, dt \right] \right\} \\ &= \frac{(b-a)^2}{6} \left[f'(b) - f'(a) \right] - \frac{f(a) + f(b)}{2}(b-a) + \int_a^b f(t)\, dt. \end{aligned}$$

If we insert I in (3.44), we deduce:

$$\begin{aligned} &\frac{f(a) + f(b)}{2}(b-a) - \int_a^b f(x)\, dx \\ &= \frac{(b-a)^2}{6} \left[f'(b) - f'(a) \right] - \frac{f(a) + f(b)}{2}(b-a) + \int_a^b f(t)\, dt - J, \end{aligned}$$

which is equivalent to (3.41). ∎

The following inequality holds [**16**].

THEOREM 43. *Let $f : [a,b] \to \mathbb{R}$ be such that the third derivative is absolutely continuous on $[a,b]$, then*

$$|CT(f,a,b,f')| \tag{3.45}$$

$$\leq \begin{cases} \frac{1}{48(b-a)} \int_a^b \int_a^b (t-s)^4 \left\|f^{(4)}\right\|_{[t,s],\infty} dtds \text{ if } f^{(4)} \in L_\infty[a,b]; \\ \frac{[B(q+1,q+1)]^{\frac{1}{q}}}{8(b-a)} \int_a^b \int_a^b |t-s|^{3+\frac{1}{q}} \left\|f^{(4)}\right\|_{[t,s],p} dtds \\ \qquad \text{if } f^{(4)} \in L_p[a,b], \quad \text{and } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{32(b-a)} \int_a^b \int_a^b |t-s|^3 \left\|f^{(4)}\right\|_{[t,s],1} dtds \end{cases}$$

$$\leq \begin{cases} \frac{(b-a)^5}{720} \left\|f^{(4)}\right\|_{[a,b],\infty} & \text{if} \quad f^{(4)} \in L_\infty[a,b]; \\ \frac{q^2(b-a)^{4+\frac{1}{q}}[B(q+1,q+1)]^{\frac{1}{q}}}{4(4q+1)(5q+1)} \left\|f^{(4)}\right\|_{[a,b],p} & \text{if} \quad f^{(4)} \in L_p[a,b], \text{ and} \\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^4}{320} \left\|f^{(4)}\right\|_{[a,b],1}. \end{cases}$$

PROOF. Define

$$U(t,s) := \int_s^t (u-s)(t-u) f^{(4)}(u)\, du,$$

then

$$|U(t,s)| \leq \left| \int_s^t |(u-s)(t-u)| \left|f^{(4)}(u)\right| du \right| \tag{3.46}$$

$$\leq \begin{cases} \frac{|t-s|^3}{6} \left\|f^{(4)}\right\|_{[t,s],\infty} & \text{if } f^{(4)} \in L_\infty[a,b]; \\ |t-s|^{2+\frac{1}{q}} [B(q+1,q+1)]^{\frac{1}{q}} \left\|f^{(4)}\right\|_{[t,s],p} \\ \qquad \text{if } f^{(4)} \in L_p[a,b], \text{ and } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(t-s)^2}{4} \left\|f^{(4)}\right\|_{[t,s],1}. \end{cases}$$

Using (3.41), we have

$$\left| \int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a) + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \right| \tag{3.47}$$

$$\leq \frac{1}{8(b-a)} \int_a^b \int_a^b |U(t,s)|\, |t-s|\, dtds.$$

Now combining (3.46) with (3.47), we deduce the first inequality in (3.45).

Also, for a general $r>0$, we have

$$\int_a^b\int_a^b |t-s|^r\,dtds = \int_a^b\left(\int_a^t (t-s)^r\,ds + \int_t^b (s-t)^r\,ds\right)dt$$
$$= \frac{2\,(b-a)^{r+2}}{(r+1)\,(r+2)}.$$

Applying this equality, we may state:

$$\int_a^b\int_a^b (t-s)^4\left\|f^{(4)}\right\|_{[t,s],\infty} dtds \le \left\|f^{(4)}\right\|_{[a,b],\infty}\int_a^b\int_a^b (t-s)^4\,dsdt$$
$$= \frac{(b-a)^6}{15}\left\|f^{(4)}\right\|_{[a,b],\infty},$$

$$\int_a^b\int_a^b |t-s|^{3+\frac{1}{q}}\left\|f^{(4)}\right\|_{[t,s],p} dtds \le \left\|f^{(4)}\right\|_{[a,b],p}\int_a^b\int_a^b |t-s|^{3+\frac{1}{q}}\,dtds$$
$$= \frac{2q^2\,(b-a)^{5+\frac{1}{q}}}{(4q+1)\,(5q+1)}\left\|f^{(4)}\right\|_{[a,b],p}$$

and

$$\int_a^b\int_a^b |t-s|^3\left\|f^{(4)}\right\|_{[t,s],1} dtds \le \left\|f^{(4)}\right\|_{[a,b],1}\int_a^b\int_a^b |t-s|^3\,dtds$$
$$= \frac{(b-a)^5}{10}\left\|f^{(4)}\right\|_{[a,b],1},$$

and then the last part of (3.45) is obtained. ∎

REMARK 45. *We observe that inequality (3.40) provides for the error estimate of the corrected trapezoidal formula, the bounds*

$$B_1 := \begin{cases} \frac{1}{720}\,(b-a)^5\left\|f^{(4)}\right\|_{[a,b],\infty} & \textit{if}\quad f^{(4)}\in L_\infty[a,b]\,; \\ \frac{q^3(b-a)^{4+\frac{1}{q}}}{2^{2+\frac{1}{q}}(2q+1)(4q+1)(5q+1)}\left\|f^{(4)}\right\|_{[a,b],p} & \textit{if}\quad f^{(4)}\in L_p[a,b]\,,\ \textit{and} \\ & \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{160}\,(b-a)^4\left\|f^{(4)}\right\|_{[a,b],1}, & \end{cases}$$

while Theorem 43 provides the bounds

$$B_2 := \begin{cases} \frac{1}{720}(b-a)^5 \left\|f^{(4)}\right\|_{[a,b],\infty} & \text{if } f^{(4)} \in L_\infty [a,b]; \\ \frac{q^2[B(q+1,q+1)]^{\frac{1}{q}}}{4(4q+1)(5q+1)}(b-a)^{4+\frac{1}{q}} \left\|f^{(4)}\right\|_{[a,b],p} & \text{if } f^{(4)} \in L_p [a,b], \text{ and} \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{320}(b-a)^4 \left\|f^{(4)}\right\|_{[a,b],1}. \end{cases}$$

Comparing the first lines in B_1 and B_2 we see that they are equal, while comparing the last lines, we observe that the bound predicted by Theorem 43 is better.

If we define the following functions:

$$f(q) := \frac{q^3}{2^{2+\frac{1}{q}}(2q+1)(4q+1)(5q+1)},$$

$$g(q) := \frac{q^2[B(q+1,q+1)]^{\frac{1}{q}}}{4(4q+1)(5q+1)},$$

then

$$h(q) = \frac{g(q)}{f(q)} = [B(q+1,q+1)]^{\frac{1}{q}}\, 2^{\frac{1}{q}}\left(2+\frac{1}{q}\right),\ q>1.$$

The graph of $h(q)$, $q \in (1,\infty)$ produced using Maple 6 shows that for all $q>1$ we have $h(q)<1$, suggesting that the bound B_2 of the new theorem is better than B_1 for $q \in (1,\infty)$. At this stage we do not have any analytic proof of this fact.

For $p=q=2$, we obtain the following bounds in terms of the Euclidean norm:

$$\begin{aligned} B_1 &= \frac{\sqrt{2}}{495}(b-a)^{\frac{9}{2}} \left\|f^{(4)}\right\|_{[a,b],2} \\ &\approx 0.002851237020\,(b-a)^{\frac{9}{2}} \left\|f^{(4)}\right\|_{[a,b],2}, \\ B_2 &= \frac{\sqrt{30}}{2970}(b-a)^{\frac{9}{2}} \left\|f^{(4)}\right\|_{[a,b],2} \\ &\approx 0.001844183695\,(b-a)^{\frac{9}{2}} \left\|f^{(4)}\right\|_{[a,b],2}, \end{aligned}$$

which shows that the bound B_2 is around 1.546 times better than B_1.

3.3. Applications for Expectation. Let X be a random variable having the PDF, $f:[a,b]\to\mathbb{R}$ and the *cumulative distribution function* $F:[a,b]\to[0,1]$.

We may state the following result [**16**].

THEOREM 44. *With the above assumptions and with the PDF, f being twice differentiable on* $[a,b]$ *and with* f'' *absolutely continuous on* $[a,b]$, *then*

$$\left|E(X)-\frac{a+b}{2}-\frac{(b-a)^2}{12}\left[f(b)-f(a)\right]\right| \tag{3.48}$$
$$\leq\begin{cases}\frac{(b-a)^5}{720}\left\|f'''\right\|_{[a,b],\infty} & \text{if } f'''\in L_\infty[a,b];\\ \frac{q^2[B(q+1,q+1)]^{\frac{1}{q}}(b-a)^{4+\frac{1}{q}}}{4(4q+1)(5q+1)}\left\|f'''\right\|_{[a,b],p} & \text{if } f'''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{(b-a)^4}{320}\left\|f'''\right\|_{[a,b],1}. \end{cases}$$

PROOF. Applying Theorem 43 for F, we may write:

$$\left|\int_a^b F(t)\,dt-\frac{F(a)+F(b)}{2}(b-a)+\frac{(b-a)^2}{12}\left[f(b)-f(a)\right]\right| \tag{3.49}$$
$$\leq\begin{cases}\frac{\left\|f'''\right\|_{[a,b],\infty}\cdot(b-a)^5}{720} & \text{if } f'''\in L_\infty[a,b];\\ \frac{q^2[B(q+1,q+1)]^{\frac{1}{q}}\left\|f'''\right\|_{[a,b],p}}{4(4q+1)(5q+1)}(b-a)^{4+\frac{1}{q}} & \text{if } f'''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{(b-a)^4\left\|f'''\right\|_{[a,b],1}}{320}. \end{cases}$$

However, $F(a)=0$, $F(b)=1$ and

$$\int_a^b F(t)\,dt=b-E(X),$$

and then by (3.49), we obtain (3.48). ∎

4. More Bounds in Terms of the Fourth Derivative

4.1. Introduction. In [**14**], the authors pointed out the following estimate of the error in the perturbed trapezoid formula.

THEOREM 45. *Let $f : [a,b] \to \mathbb{R}$ be such that the second derivative is absolutely continuous on $[a,b]$, then we have:*

$$\left| \int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a) + \frac{(b-a)^2}{12}\left[f'(b)-f'(a)\right] \right| \tag{3.50}$$

$$\leq \begin{cases} \frac{1}{16(b-a)} \int_a^b \int_a^b |t-s|^3 \|f'''\|_{[t,s],\infty}\, dtds \text{ if } f''' \in L_\infty[a,b]; \\ \frac{1}{8(q+1)^{\frac{1}{q}}(b-a)} \int_a^b \int_a^b |t-s|^{2+\frac{1}{q}} \|f'''\|_{[t,s],p}\, dtds \\ \qquad \text{if } f''' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{8(b-a)} \int_a^b \int_a^b (t-s)^2 \|f'''\|_{[t,s],1}\, dtds \end{cases}$$

$$\leq \begin{cases} \frac{(b-a)^4}{160} \|f'''\|_{[a,b],\infty} \qquad \text{if } f''' \in L_\infty[a,b]; \\ \frac{q^2(b-a)^{3+\frac{1}{q}}}{4(3q+1)(4q+1)(q+1)^{\frac{1}{q}}} \|f'''\|_{[a,b],p} \\ \qquad \text{if } f''' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^3}{48} \|f'''\|_{[a,b],1}. \end{cases}$$

The main aim of this section is to point out some bounds for the left part of (3.50) in terms of the $p-$norms of $f^{(4)}$ assuming that the function f is three-times differentiable on (a,b) and the third derivative is absolutely continuous on (a,b).

Applications for estimating the expectation of a random variable are also pointed out.

4.2. The Results.

The following representation lemma holds [**15**].

LEMMA 12. *Let $f : [a,b] \to \mathbb{R}$ be such that the second derivative is absolutely continuous on $[a,b]$, then we have the equality:*

$$\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a) + \frac{(b-a)^2}{12}\left[f'(b)-f'(a)\right] = \frac{1}{4(b-a)} \int_a^b \int_a^b K(t,s)(t-s)\,dtds, \tag{3.51}$$

where $K : [a,b]^2 \to \mathbb{R}$, and

$$K(t,s) = \int_s^t \left(u - \frac{t+s}{2}\right)\left(f'''(u) - f'''\left(\frac{t+s}{2}\right)\right) du.$$

PROOF. We have:

$$\int_a^b \int_a^b K(t,s)(t-s)\,dtds = \int_a^b \int_a^b \left(\int_s^t \left(u - \frac{t+s}{2} \right) f'''(u)\,du \right) (t-s)\,dtds =: I$$

as

$$\int_s^t \left(u - \frac{t+s}{2} \right) f''' \left(\frac{t+s}{2} \right) du = 0.$$

Integrating by parts, we have

$$\begin{aligned} I &= \int_a^b \int_a^b \left(\int_s^t \left(u - \frac{t+s}{2} \right) f'''(u)\,du \right) (t-s)\,dtds \\ &= \int_a^b \int_a^b \left[\frac{f''(t)+f''(s)}{2}(t-s) - \int_s^t f''(u)\,du \right] (t-s)\,dtds \\ &= \int_a^b \int_a^b \left[\frac{f''(t)(t-s)^2 + f''(s)(t-s)^2}{2} \right. \\ &\qquad \left. - (f'(t)-f'(s))(t-s) \right] dtds \\ &= \frac{1}{2} \left[\int_a^b \int_a^b f''(t)(t-s)^2\,dtds + \int_a^b \int_a^b f''(s)(t-s)^2\,dtds \right] \\ &\qquad - \int_a^b \int_a^b (f'(t)-f'(s))(t-s)\,dtds. \end{aligned}$$

By symmetry, we have

$$J := \int_a^b \int_a^b f''(t)(t-s)^2\,dtds = \int_a^b \int_a^b f''(s)(t-s)^2\,dtds,$$

and using *Korkine's identity* or direct computation, we have

$$\begin{aligned} K &:= \int_a^b \int_a^b (f'(t)-f'(s))(t-s)\,dtds \\ &= 2\left[(b-a) \int_a^b f'(t)\,tdt - \int_a^b f'(t)\,dt \cdot \int_a^b tdt \right]. \end{aligned}$$

$I = J - K$, since

$$\begin{aligned}
J &= \int_a^b f''(t)\left(\int_a^b (t-s)^2\,ds\right)dt \\
&= \frac{1}{3}\left[\int_a^b f''(t)(b-t)^3\,dt + \int_a^b (t-a)^3 f''(t)\,dt\right] \\
&= \frac{1}{3}\Bigg[f'(t)(b-t)^3\Big|_a^b + 3\int_a^b (b-t)^2 f'(t)\,dt \\
&\qquad + f'(t)(t-a)^3\Big|_a^b - 3\int_a^b (t-a)^2 f'(t)\,dt\Bigg] \\
&= \frac{1}{3}\Bigg[-f'(a)(b-a)^3 + 3\left[f(t)\,(b-t)^2\Big|_a^b - 2\int_a^b (b-t)\,f(t)\,dt\right] \\
&\qquad + f'(b)(b-a)^3 - 3\left[f(t)\,(t-a)^2\Big|_a^b - 2\int_a^b (t-a)\,f(t)\,dt\right]\Bigg] \\
&= \frac{1}{3}\Bigg[\left[f'(b)-f'(a)\right](b-a)^3 \\
&\qquad + 3\left[-f(a)(b-a)^2 + 2\int_a^b (b-t)\,f(t)\,dt\right] \\
&\qquad - 3\left[f(b)(b-a)^2 - 2\int_a^b (t-a)\,f(t)\,dt\right]\Bigg] \\
&= \frac{1}{3}\left[f'(b)-f'(a)\right](b-a)^3 \\
&\qquad - \left[\frac{f(a)+f(b)}{2}\right](b-a)^2 + 2(b-a)\int_a^b f(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
K &= 2\left[(b-a)\left[f(t)\,t\Big|_a^b - \int_a^b f(t)\,dt\right] - \left[f(b)-f(a)\right]\frac{b^2-a^2}{2}\right] \\
&= 2\Bigg[(b-a)\left[f(b)\,b - f(a)\,a - \int_a^b f(t)\,dt\right] \\
&\qquad - (b-a)\left[f(b)-f(a)\right]\frac{a+b}{2}\Bigg] \\
&= (b-a)^2\left[f(a)+f(b)\right] - 2(b-a)\int_a^b f(t)\,dt.
\end{aligned}$$

Consequently,

$$I = \frac{1}{3}\left[f'(b) - f'(a)\right](b-a)^3 - 2\left[f(a) + f(b)\right](b-a)^2 + 4(b-a)\int_a^b f(t)\,dt.$$

Dividing by $4(b-a)$ we deduce the desired equality (3.51). ∎

The following result can be stated [**15**]:

THEOREM 46. *Let $f:[a,b]\to\mathbb{R}$ be such that the third derivative is monotonically increasing (decreasing) on $[a,b]$, then:*

$$\int_a^b f(t)\,dt \geq (\leq) \frac{f(a)+f(b)}{2}(b-a) - \frac{(b-a)^2}{12}\left[f'(b) - f'(a)\right]. \tag{3.52}$$

PROOF. Since f''' is increasing (decreasing) on $[a,b]$, then

$$\left(u - \frac{t+s}{2}\right)\left(f'''(u) - f'''\left(\frac{t+s}{2}\right)\right) \geq (\leq)\, 0 \tag{3.53}$$

for all $u,t,s\in[a,b]$.

Now, using (3.53), we may state that

$$K(t,s)(t-s) = (t-s)\int_s^t \left(u - \frac{t+s}{2}\right)\left(f'''(u) - f'''\left(\frac{t+s}{2}\right)\right)du \geq (\leq)\, 0$$

and using the representation (3.51) we deduce the desired inequality (3.52). ∎

If we assume Hölder continuity for the third derivative, we may state the following result as well [**15**].

THEOREM 47. *Assume that $f:[a,b]\to\mathbb{R}$ is such that*

$$\left|f'''(t) - f'''(s)\right| \leq H\left|t-s\right|^r \quad \text{for all } t,s\in[a,b], \tag{3.54}$$

where $H>0$ and $r\in(0,1]$ are given, then we have the inequality

$$\left|\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a) + \frac{(b-a)^2}{12}\left[f'(b)-f'(a)\right]\right| \leq \frac{H(b-a)^{r+4}}{2^{r+2}(r+2)(r+4)(r+5)}. \tag{3.55}$$

PROOF. Using the representation (3.52) we have

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a) + \frac{(b-a)^2}{12}\left[f'(b)-f'(a)\right]\right| \\
&\le \frac{1}{4(b-a)}\int_a^b\int_a^b \left|\int_s^t \left|u-\frac{t+s}{2}\right|\right. \\
&\qquad \times \left.\left|f'''(u)-f'''\left(\frac{t+s}{2}\right)\right| du\right| |t-s|\,dsdt \\
&\le \frac{H}{4(b-a)}\int_a^b\int_a^b \left|\int_s^t \left|u-\frac{t+s}{2}\right|^{r+1} du\right| |t-s|\,dsdt \\
&= \frac{H}{4(b-a)}\int_a^b\int_a^b \frac{|t-s|^{r+2}}{2^{r+1}(r+2)}|t-s|\,dsdt \\
&= \frac{H}{4(b-a)}\int_a^b\int_a^b \frac{|t-s|^{r+3}}{2^{r+1}(r+2)}dsdt =: B.
\end{aligned}$$

Now, consider the double integral

$$\begin{aligned}
\int_a^b\int_a^b |t-s|^m\,dtds &= \int_a^b\left[\int_a^t (t-s)^m\,ds + \int_t^b (s-t)^m\,ds\right]dt \\
&= \int_a^b\left[\frac{(t-a)^{m+1}+(b-t)^{m+1}}{m+1}\right]dt \\
&= \frac{2(b-a)^{m+2}}{(m+1)(m+2)}.
\end{aligned}$$

Using this, we have

$$\int_a^b\int_a^b |t-s|^{r+3}\,dsdt = \frac{2(b-a)^{r+5}}{(r+4)(r+5)}$$

and then

$$B = \frac{H}{4(b-a)2^{r+1}(r+2)}\cdot\frac{2(b-a)^{r+5}}{(r+4)(r+5)} = \frac{H(b-a)^{r+4}}{2^{r+2}(r+2)(r+4)(r+5)},$$

proving the desired inequality. ∎

COROLLARY 42. *If $f:[a,b]\to\mathbb{R}$ is such that f''' is Lipschitzian with the constant $L>0$, then*

$$\left|\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)+\frac{(b-a)^2}{12}\left[f'(b)-f'(a)\right]\right| \leq \frac{L(b-a)^5}{720}. \tag{3.56}$$

Now, if we assume absolute continuity for the third derivative, we may point out the following estimate for the remainder in terms of the fourth derivative [**15**].

THEOREM 48. *Assume that $f:[a,b]\to\mathbb{R}$ is such that the third derivative f''' is absolutely continuous on $[a,b]$, then*

$$\left|\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)+\frac{(b-a)^2}{12}\left[f'(b)-f'(a)\right]\right| \tag{3.57}$$

$$\leq\begin{cases}\frac{1}{4(b-a)}\int_a^b\int_a^b\left|\int_s^t\left|u-\frac{t+s}{2}\right|^2\left\|f^{(4)}\right\|_{\left[\frac{t+s}{2},u\right],\infty}du\right||t-s|\,dtds \\ \qquad\text{if } f^{(4)}\in L_\infty[a,b]; \\ \frac{1}{4(b-a)}\int_a^b\int_a^b\left|\int_s^t\left|u-\frac{t+s}{2}\right|^{1+\frac{1}{q}}\left\|f^{(4)}\right\|_{\left[\frac{t+s}{2},u\right],p}du\right||t-s|\,dtds \\ \qquad\text{if } f^{(4)}\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{4(b-a)}\int_a^b\int_a^b\left|\int_s^t\left|u-\frac{t+s}{2}\right|\left\|f^{(4)}\right\|_{\left[\frac{t+s}{2},u\right],1}du\right||t-s|\,dtds\end{cases}$$

$$\leq\begin{cases}\frac{1}{48(b-a)}\int_a^b\int_a^b|t-s|^4\left\|f^{(4)}\right\|_{[t,s],\infty}dtds \text{ if } f^{(4)}\in L_\infty[a,b]; \\ \frac{q}{2^{3+\frac{1}{q}}(2q+1)(b-a)}\int_a^b\int_a^b|t-s|^{3+\frac{1}{q}}\left\|f^{(4)}\right\|_{[t,s],p}dtds \\ \qquad\text{if } f^{(4)}\in L_p[a,b],\text{ and } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{16(b-a)}\int_a^b\int_a^b|t-s|^3\left\|f^{(4)}\right\|_{[t,s],1}dtds\end{cases}$$

$$\leq\begin{cases}\frac{\left\|f^{(4)}\right\|_{[a,b],\infty}(b-a)^5}{720} \qquad \text{if } f^{(4)}\in L_\infty[a,b]; \\ \frac{q^3\left\|f^{(4)}\right\|_{[a,b],p}}{2^{2+\frac{1}{q}}(2q+1)(4q+1)(5q+1)}(b-a)^{4+\frac{1}{q}} \\ \qquad\text{if } f^{(4)}\in L_p[a,b],\text{ and } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^4\left\|f^{(4)}\right\|_{[a,b],1}}{160}\end{cases}$$

PROOF. Using the representation (3.51), we have

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a) + \frac{(b-a)^2}{12}\left[f'(b)-f'(a)\right]\right| \\
&\leq \frac{1}{4(b-a)}\int_a^b\int_a^b\left|\int_s^t\left|u-\frac{t+s}{2}\right|\left|\int_{\frac{t+s}{2}}^u f^{(4)}(\tau)\,d\tau\right|du\right||t-s|\,dtds \\
&=: C.
\end{aligned}$$

However,

$$\left|\int_{\frac{t+s}{2}}^u f^{(4)}(\tau)\,d\tau\right| \leq \left|u-\frac{t+s}{2}\right|\left\|f^{(4)}\right\|_{\left[\frac{t+s}{2},u\right],\infty},$$

$$\left|\int_{\frac{t+s}{2}}^u f^{(4)}(\tau)\,d\tau\right| \leq \left|u-\frac{t+s}{2}\right|^{\frac{1}{q}}\left\|f^{(4)}\right\|_{\left[\frac{t+s}{2},u\right],p}$$

and

$$\left|\int_{\frac{t+s}{2}}^u f^{(4)}(\tau)\,d\tau\right| \leq \left\|f^{(4)}\right\|_{\left[\frac{t+s}{2},u\right],1}$$

and thus, we may state that

$$C \leq \frac{1}{4(b-a)} \times \begin{cases} \int_a^b\int_a^b\left|\int_s^t\left|u-\frac{t+s}{2}\right|^2\left\|f^{(4)}\right\|_{\left[\frac{t+s}{2},u\right],\infty}du\right||t-s|\,dtds; \\[2ex] \int_a^b\int_a^b\left|\int_s^t\left|u-\frac{t+s}{2}\right|^{1+\frac{1}{q}}\left\|f^{(4)}\right\|_{\left[\frac{t+s}{2},u\right],p}du\right||t-s|\,dtds; \\[2ex] \int_a^b\int_a^b\left|\int_s^t\left|u-\frac{t+s}{2}\right|\left\|f^{(4)}\right\|_{\left[\frac{t+s}{2},u\right],1}du\right||t-s|\,dtds, \end{cases}$$

proving the first inequality in (3.57).

Now, observe that

$$\begin{aligned}
&\left|\int_s^t\left|u-\frac{t+s}{2}\right|^2\left\|f^{(4)}\right\|_{\left[\frac{t+s}{2},u\right],\infty}du\right| \\
&\qquad\leq \left\|f^{(4)}\right\|_{[t,s],\infty}\left|\int_s^t\left|u-\frac{t+s}{2}\right|^2du\right| \\
&\qquad= \left\|f^{(4)}\right\|_{[t,s],\infty}\frac{|t-s|^3}{12},
\end{aligned}$$

$$\begin{aligned}
&\left|\int_s^t \left|u-\frac{t+s}{2}\right|^{1+\frac{1}{q}} \left\|f^{(4)}\right\|_{\left[\frac{t+s}{2},u\right],p} du\right| \\
&\qquad \le \left\|f^{(4)}\right\|_{[t,s],p} \left|\int_s^t \left|u-\frac{t+s}{2}\right|^{1+\frac{1}{q}} du\right| \\
&\qquad = \left\|f^{(4)}\right\|_{[t,s],p} \frac{|t-s|^{2+\frac{1}{q}}}{2^{1+\frac{1}{q}}\left(2+\frac{1}{q}\right)} \\
&\qquad = \left\|f^{(4)}\right\|_{[t,s],p} \frac{q\,|t-s|^{2+\frac{1}{q}}}{2^{1+\frac{1}{q}}\,(2q+1)}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_s^t \left|u-\frac{t+s}{2}\right| \left\|f^{(4)}\right\|_{\left[\frac{t+s}{2},u\right],1} du\right| \\
&\qquad \le \left\|f^{(4)}\right\|_{[t,s],1} \left|\int_s^t \left|u-\frac{t+s}{2}\right| du\right| \\
&\qquad = \left\|f^{(4)}\right\|_{[t,s],1} \frac{|t-s|^2}{4}
\end{aligned}$$

and the second part of (3.57) is proved.

For the last part, we observe that

$$\begin{aligned}
&\int_a^b \int_a^b |t-s|^4 \left\|f^{(4)}\right\|_{[t,s],\infty} dtds \\
&\qquad \le \left\|f^{(4)}\right\|_{[a,b],\infty} \int_a^b \int_a^b (t-s)^4\, dtds \\
&\qquad = \left\|f^{(4)}\right\|_{[a,b],\infty} \cdot \frac{(b-a)^6}{15},
\end{aligned}$$

$$\begin{aligned}
&\int_a^b \int_a^b |t-s|^{3+\frac{1}{q}} \left\|f^{(4)}\right\|_{[t,s],p} dtds \\
&\qquad \le \left\|f^{(4)}\right\|_{[a,b],p} \int_a^b \int_a^b |t-s|^{3+\frac{1}{q}}\, dtds \\
&\qquad = \frac{2q^2\,(b-a)^{5+\frac{1}{q}}}{(4q+1)\,(5q+1)} \left\|f^{(4)}\right\|_{[a,b],p}
\end{aligned}$$

and

$$\int_a^b \int_a^b |t-s|^3 \left\| f^{(4)} \right\|_{[t,s],1} dtds$$
$$\leq \left\| f^{(4)} \right\|_{[a,b],1} \int_a^b \int_a^b |t-s|^3 \, dtds$$
$$= \left\| f^{(4)} \right\|_{[a,b],1} \frac{(b-a)^5}{10}.$$

The proof is completed. ∎

4.3. Applications for Expectation. Let X be a random variable having the PDF, $f : [a,b] \to \mathbb{R}$ and the *cumulative distribution function* $F : [a,b] \to [0,1]$.

We may state the following result [**15**].

THEOREM 49. *With the above assumptions and of the PDF, f is twice differentiable on $[a,b]$ and f'' is absolutely continuous on $[a,b]$, then*

$$\left| E(X) - \frac{a+b}{2} - \frac{(b-a)^2}{12} [f(b) - f(a)] \right| \tag{3.58}$$
$$\leq \begin{cases} \frac{(b-a)^5}{720} \left\| f''' \right\|_{[a,b],\infty} & \text{if } \; f''' \in L_\infty [a,b]; \\ \frac{q^3 (b-a)^{4+\frac{1}{q}}}{2^{2+\frac{1}{q}} (2q+1)(4q+1)(5q+1)^{\frac{1}{q}}} \left\| f''' \right\|_{[a,b],p} & \text{if } \; f''' \in L_p [a,b], \\ & p>1, \; \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-a)^4}{160} \left\| f''' \right\|_{[a,b],1}. \end{cases}$$

PROOF. Applying Theorem 48 for F, we may write:

$$\left| \int_a^b F(t)\, dt - \frac{F(a) + F(b)}{2} (b-a) + \frac{(b-a)^2}{12} [f(b) - f(a)] \right| \tag{3.59}$$
$$\leq \begin{cases} \frac{\left\| f''' \right\|_{[a,b],\infty} \cdot (b-a)^5}{720} & \text{if } \; f''' \in L_\infty [a,b]; \\ \frac{q^3 \left\| f''' \right\|_{[a,b],p}}{2^{2+\frac{1}{q}} (2q+1)(4q+1)(5q+1)^{\frac{1}{q}}} (b-a)^{4+\frac{1}{q}} & \text{if } \; f''' \in L_p [a,b], \\ & p>1, \; \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-a)^4 \left\| f''' \right\|_{[a,b],1}}{160}. \end{cases}$$

However, $F(a) = 0$, $F(b) = 1$ and $\int_a^b F(t)\, dt = b - E(X)$, and so by (3.59), we obtain (3.58). ∎

5. A Trapezoid Inequality for Convex Functions

5.1. Introduction. The following integral inequality for the generalised trapezoid formula was obtained in [**36**] (see also [**35**, p. 68]):

THEOREM 50. *Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation. We have the inequality*

$$\left|\int_a^b f(t)\,dt-[(x-a)f(a)+(b-x)f(b)]\right| \leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f), \tag{3.60}$$

holding for all $x\in[a,b]$, where $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a,b]$.
The constant $\frac{1}{2}$ is the best possible.

This result may be improved if one assumes the monotonicity of f as follows (see [**35**, p. 76])

THEOREM 51. *Let $f:[a,b]\to\mathbb{R}$ be a monotonic nondecreasing function on $[a,b]$, then we have the inequality:*

$$\begin{aligned}\left|\int_a^b f(t)\,dt-[(x-a)f(a)+(b-x)f(b)]\right| &\leq (b-x)f(b)-(x-a)f(a)+\int_a^b sgn(x-t)f(t)\,dt\\ &\leq (x-a)[f(x)-f(a)]+(b-x)[f(b)-f(x)]\\ &\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right][f(b)-f(a)]\end{aligned} \tag{3.61}$$

for all $x\in[a,b]$.
The above inequalities are sharp.

If the mapping is Lipschitzian, then the following result holds as well [**62**] (see also [**35**, p. 82]).

THEOREM 52. *Let $f:[a,b]\to\mathbb{R}$ be an $L-$Lipschitzian function on $[a,b]$, i.e.., f satisfies the condition:*

$$\text{(L)}\quad |f(s)-f(t)|\leq L|s-t| \quad \textit{for any } s,t\in[a,b] \quad (L>0 \textit{ is given}),$$

then we have the inequality:

$$\left|\int_a^b f(t)\,dt - [(x-a) f(a) + (b-x) f(b)]\right| \leq \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] L \tag{3.62}$$

for any $x \in [a, b]$.
The constant $\frac{1}{4}$ *is best in (3.62).*

If we assume absolute continuity for the function f, then the following estimates in terms of the Lebesgue norms of the derivative f' hold [**35**, p. 93].

THEOREM 53. *Let* $f : [a, b] \to \mathbb{R}$ *be an absolutely continuous function on* $[a, b]$, *then for any* $x \in [a, b]$, *we have*

$$\left|\int_a^b f(t)\,dt - [(x-a) f(a) + (b-x) f(b)]\right| \tag{3.63}$$
$$\leq \begin{cases} \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \|f'\|_\infty & \text{if } f' \in L_\infty [a, b]; \\ \frac{1}{(q+1)^{\frac{1}{q}}} \left[(x-a)^{q+1} + (b-x)^{q+1}\right]^{\frac{1}{q}} \|f'\|_p & \text{if } f' \in L_p [a, b], \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \|f'\|_1, \end{cases}$$

The next section points out some similar results for convex functions. Applications for probability density functions are also considered.

5.2. The Results. The following theorem providing a lower bound for the difference

$$(x-a) f(a) + (b-x) f(b) - \int_a^b f(t)\,dt$$

holds [**49**].

THEOREM 54. *Let* $f : [a, b] \to \mathbb{R}$ *be a convex function on* $[a, b]$, *then for any* $x \in (a, b)$ *we have the inequality*

$$\frac{1}{2}\left[(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x)\right] \leq (x-a) f(a) + (b-x) f(b) - \int_a^b f(t)\,dt. \tag{3.64}$$

The constant $\frac{1}{2}$ *in the left hand side of (3.64) is sharp.*

The proof follows along similar lines to that of Theorem 19.

Proof. It is easy to see that for any locally absolutely continuous function $f:(a,b)\to\mathbb{R}$, we have the identity

$$(x-a)f(a)+(b-x)f(b)-\int_a^b f(t)\,dt=\int_a^b (t-x)f'(t)\,dt \tag{3.65}$$

for any $x\in(a,b)$, where f' is the derivative of f which exists a.e. on $[a,b]$.

Since f is convex, then it is locally Lipschitzian and thus (3.65) holds. Moreover, for any $x\in(a,b)$, we have the inequalities:

$$f'(t)\leq f'_-(x) \quad \text{for a.e. } t\in[a,x] \tag{3.66}$$

and

$$f'(t)\geq f'_+(x) \quad \text{for a.e. } t\in[x,b]. \tag{3.67}$$

If we multiply (3.66) by $x-t\geq 0$, $t\in[a,x]$ and integrate over $[a,x]$, we get

$$\int_a^x (x-t)f'(t)\,dt\leq \frac{1}{2}(x-a)^2 f'_-(x) \tag{3.68}$$

and if we multiply (3.67) by $t-x\geq 0$, $t\in[x,b]$ and integrate over $[x,b]$, we also have

$$\int_x^b (t-x)f'(t)\,dt\geq \frac{1}{2}(b-x)^2 f'_+(x). \tag{3.69}$$

Finally, if we subtract (3.68) from (3.69) and use the representation (3.65), we deduce the desired inequality (3.64).

Now, assume that (3.64) holds with a constant $C>0$ instead of $\frac{1}{2}$, i.e.,

$$C\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right] \leq (x-a)f(a)+(b-x)f(b)-\int_a^b f(t)\,dt. \tag{3.70}$$

Consider the convex function $f_0(t):=k\left|t-\frac{a+b}{2}\right|$, $k>0$, $t\in[a,b]$, then

$$f'_{0+}\left(\frac{a+b}{2}\right)=k,\quad f'_{0-}\left(\frac{a+b}{2}\right)=-k,$$

$$f_0(a)=\frac{k(b-a)}{2}=f_0(b),\quad \int_a^b f_0(t)\,dt=\frac{1}{4}k(b-a)^2.$$

If in (3.70) we choose f_0 as above and $x = \frac{a+b}{2}$, then we get

$$C\left[\frac{1}{4}(b-a)^2 k + \frac{1}{4}(b-a)^2 k\right] \le \frac{1}{4}k(b-a)^2,$$

giving $C \le \frac{1}{2}$, and the sharpness of the constant is proved. ∎

Now, recall that the following inequality holds, which is well known in the literature as the *Hermite-Hadamard inequality* for convex functions,

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}. \tag{H-H}$$

The following corollary gives a sharp lower bound for the difference

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt.$$

COROLLARY 43. *([**49**]) Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$, then*

$$\begin{aligned} 0 &\le \frac{1}{8}\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right](b-a) \\ &\le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt. \end{aligned} \tag{3.71}$$

The constant $\frac{1}{8}$ is sharp.

The proof is obvious by the above theorem. The sharpness of the constant is obtained for $f_0(t) = k\left|t - \frac{a+b}{2}\right|$, $t \in [a,b]$, $k > 0$.

When x is a point of differentiability, we may state the following corollary as well [**49**].

COROLLARY 44. *Let f be as in Theorem 54. If $x \in (a,b)$ is a point of differentiability for f, then*

$$\begin{aligned} &(b-a)\left(\frac{a+b}{2} - x\right) f'(x) \\ &\qquad \le (x-a) f(a) + (b-x) f(b) - \int_a^b f(t)\,dt. \end{aligned} \tag{3.72}$$

REMARK 46. *If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex on I and if we choose $x \in \mathring{I}$ ($\mathring{I}$ is the interior of I), $b = x + \frac{h}{2}$, $a = x - \frac{h}{2}$, $h > 0$ is such that $a, b \in I$, then from (3.64) we may write*

$$0 \le \frac{1}{8}h^2\left[f'_+(x) - f'_-(x)\right] \le \frac{f(a)+f(b)}{2} \cdot h - \int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(t)\,dt \tag{3.73}$$

and the constant $\frac{1}{8}$ *is sharp in (3.73).*

The following result providing an upper bound for the difference

$$(x-a)\, f(a) + (b-x)\, f(b) - \int_a^b f(t)\, dt$$

also holds [**49**].

THEOREM 55. *Let* $f : [a,b] \to \mathbb{R}$ *be a convex function on* $[a,b]$, *then for any* $x \in [a,b]$, *we have the inequality:*

$$(x-a)\, f(a) + (b-x)\, f(b) - \int_a^b f(t)\, dt \leq \frac{1}{2}\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\right]. \tag{3.74}$$

The constant $\frac{1}{2}$ *is sharp.*

PROOF. If either $f'_+(a) = -\infty$ or $f'_-(b) = +\infty$, then the inequality (3.74) evidently holds true.

Assume that $f'_+(a)$ and $f'_-(b)$ are finite.

Since f is convex on $[a,b]$, we have

$$f'(t) \geq f'_+(a) \quad \text{for a.e.} \quad t \in [a,x] \tag{3.75}$$

and

$$f'(t) \leq f'_-(b) \quad \text{for a.e.} \quad t \in [x,b]. \tag{3.76}$$

If we multiply (3.75) by $(x-t) \geq 0$, $t \in [a,x]$ and integrate over $[a,x]$, then we deduce

$$\int_a^x (x-t)\, f'(t)\, dt \geq \frac{1}{2}(x-a)^2 f'_+(a) \tag{3.77}$$

and if we multiply (3.76) by $t-x \geq 0$, $t \in [x,b]$ and integrate over $[x,b]$, then we also have

$$\int_x^b (t-x)\, f'(t)\, dt \leq \frac{1}{2}(b-x)^2 f'_-(b). \tag{3.78}$$

Finally, if we subtract (3.77) from (3.78) and use the representation (3.65), we deduce the desired inequality (3.74).

Now, assume that (3.74) holds with a constant $D > 0$ instead of $\frac{1}{2}$, i.e.,

$$(x-a)\, f(a) + (b-x)\, f(b) - \int_a^b f(t)\, dt \leq D\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\right]. \tag{3.79}$$

If we consider the convex function $f_0 : [a, b] \to \mathbb{R}$, $f_0(t) = k\left|t - \frac{a+b}{2}\right|$, then we have $f'_-(b) = k$, $f'_+(a) = -k$ and by (3.79) we deduce, for $x = \frac{a+b}{2}$, that

$$\frac{1}{4}k(b-a)^2 \le D\left[\frac{1}{4}k(b-a)^2 + \frac{1}{4}k(b-a)^2\right]$$

giving $D \ge \frac{1}{2}$, and the sharpness of the constant is proved. ∎

The following corollary related to the Hermite-Hadamard inequality is interesting as well [**49**].

COROLLARY 45. *Let $f : [a, b] \to \mathbb{R}$ be convex on $[a, b]$, then*

$$\begin{aligned} 0 &\le \frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le \frac{1}{8}\left[f'_-(b) - f'_+(a)\right](b-a) \end{aligned} \tag{3.80}$$

and the constant $\frac{1}{8}$ is sharp.

REMARK 47. *Denote $B := f'_-(b)$, $A := f'_+(a)$ and assume that $B \ne A$, i.e., f is not constant on (a, b), then*

$$\begin{aligned} &(b-x)^2 B - (x-a)^2 A \\ &= (B-A)\left[x - \left(\frac{bB - aA}{B-A}\right)\right]^2 - \frac{AB}{B-A}(b-a)^2 \end{aligned}$$

and by (3.74) we get

$$\begin{aligned} &(x-a) f(a) + (b-x) f(b) - \int_a^b f(t)\,dt \\ &\le (B-A)\left[x - \left(\frac{bB - aA}{B-A}\right)\right]^2 - \frac{AB}{(B-A)^2}(b-a)^2 \end{aligned} \tag{3.81}$$

for any $x \in [a, b]$.
If $A \ge 0$, then $x_0 = \frac{bB-aA}{B-A} \in [a, b]$, and by (3.81) for $x = \frac{bB-aA}{B-A}$ we get

$$0 \le \frac{1}{2} \cdot \frac{AB}{B-A}(b-a) \le \frac{Bf(a) - Af(b)}{B-A} - \frac{1}{b-a}\int_a^b f(t)\,dt \tag{3.82}$$

which is of intrinsic interest itself.

5.3. Applications for PDFs. Let X be a random variable with *probability density function* $f:[a,b]\subset\mathbb{R}\rightarrow[0,\infty)$ and with *cumulative distribution function* $F(x)=\Pr(X\leq x)$.

The following theorem holds [**49**].

THEOREM 56. *If* $f:[a,b]\subset\mathbb{R}\rightarrow\mathbb{R}_+$ *is monotonically increasing on* $[a,b]$*, then we have the inequality:*

$$\begin{aligned}(3.83)\qquad &\frac{1}{2}\left[(b-x)^2 f_+(x)-(x-a)^2 f_-(x)\right]+x\\ &\leq E(X)\\ &\leq\frac{1}{2}\left[(b-x)^2 f_+(b)-(x-a)^2 f_-(a)\right]+x\end{aligned}$$

for any $x\in(a,b)$, *where* $f_{\pm}(\alpha)$ *represent respectively the right and left limits of* f *in* α.
The constant $\frac{1}{2}$ *is sharp in both inequalities.*
The second inequality also holds for $x=a$ *or* $x=b$.

PROOF. Follows by Theorem 54 and 55 applied for the convex cdf function $F(x)=\int_a^x f(t)\,dt,\ x\in[a,b]$ and taking into account that $\int_a^b F(x)\,dx=b-E(X)$. ∎

Finally, we may state the following corollary in estimating the expectation of X [**49**].

COROLLARY 46. *With the above assumptions, we have*

$$\begin{aligned}(3.84)\qquad &\frac{1}{8}\left[f_+\left(\frac{a+b}{2}\right)-f_-\left(\frac{a+b}{2}\right)\right](b-a)^2+\frac{a+b}{2}\\ &\leq E(X)\\ &\leq\frac{1}{8}\left[f_+(b)-f_-(a)\right](b-a)^2+\frac{a+b}{2}.\end{aligned}$$

6. Generalizations of the Weighted Trapezoidal Inequality

6.1. Introduction. The classical trapezoid inequality states that if f'' exists and is bounded on (a,b), then

$$(3.85)\qquad \left|\int_a^b f(x)dx-\frac{b-a}{2}[f(a)+f(b)]\right|\leq\frac{(b-a)^3}{12}\left\|f''\right\|_{\infty}.$$

Cerone-Dragomir-Pearce [**36**] proved the following trapezoid type inequality:

THEOREM 57. *Let $f : [a, b] \to \mathbb{R}$ be a mapping of bounded variation, then*

$$\left|\int_a^b f(t)dt - [(x-a)f(a) + (b-x)f(b)]\right| \leq \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f), \tag{3.86}$$

for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

In this section, by following [**119**], we establish weighted generalizations of Theorem 57, and give several applications for $r-$moments, the expectation of a continuous random variable and for the Beta mapping.

6.2. Some Integral Inequalities. We may state the following result [**119**].

THEOREM 58. *Let $g : [a, b] \to \mathbb{R}$ be non-negative and continuous and let $h : [a, b] \to \mathbb{R}$ be differentiable such that $h'(t) = g(t)$ on $[a, b]$. Suppose f is defined as in Theorem 57, then*

$$\left|\int_a^b f(t)g(t)\,dt - [(x-h(a))f(a) + (h(b)-x)f(b)]\right| \leq \left[\frac{1}{2}\int_a^b g(t)\,dt + \left|x - \frac{h(a)+h(b)}{2}\right|\right] \bigvee_a^b (f), \tag{3.87}$$

for all $x \in [h(a), h(b)]$. The constant $\frac{1}{2}$ is the best possible.

PROOF. Let $x \in [h(a), h(b)]$. Using integration by parts, we have the following identity

$$\begin{aligned} \int_a^b (x-h(t))\,df(t) &= (x-h(t))f(t)\Big|_a^b + \int_a^b f(t)g(t)\,dt \\ &= \int_a^b f(t)g(t)\,dt - [(x-h(a))f(a) + (h(b)-x)f(b)]. \end{aligned} \tag{3.88}$$

It is well known [**4**, p.159] that if $\mu, \nu : [a, b] \to \mathbb{R}$ are such that μ is continuous on $[a, b]$ and ν is of bounded variation on $[a, b]$, then $\int_a^b \mu(t)\,d\nu(t)$ exists and [**4**, p.177]

$$\left|\int_a^b \mu(t)\,d\nu(t)\right| \leq \sup_{x\in[a,b]} |\mu(t)| \bigvee_a^b (\nu). \tag{3.89}$$

Now, using identity (3.88) and inequality (3.89), we have

$$\left|\int_a^b f(t)g(t)\,dt - [(x-h(a))\,f(a) + (h(b)-x)\,f(b)]\right| \leq \sup_{t\in[a,b]} |x-h(t)| \bigvee_a^b (f). \tag{3.90}$$

Since $x-h(t)$ is decreasing on $[a,b]$, $h(a) \leq x \leq h(b)$ and $h'(t) = g(t)$ on $[a,b]$, we have

$$\begin{aligned} \sup_{t\in[a,b]} |x-h(t)| &= \max\{x-h(a), h(b)-x\} \\ &= \frac{h(b)-h(a)}{2} + \left|x - \frac{h(a)+h(b)}{2}\right| \\ &= \frac{1}{2}\int_a^b g(t)\,dt + \left|x - \frac{h(a)+h(b)}{2}\right|. \end{aligned} \tag{3.91}$$

Thus, by (3.90) and (3.91), we obtain (3.87).

Let

$$g(t) \equiv 1,\ t \in [a,b],$$
$$h(t) = t,\ t \in [a,b],$$
$$f(t) = \begin{cases} 0 & \text{as } t = a \\ 1 & \text{as } t \in (a,b) \\ 0 & \text{as } t = b, \end{cases}$$

and $x = \frac{a+b}{2}$, then we can see that the constant $\frac{1}{2}$ is the best possible. This completes the proof. ∎

REMARK 48. *(1) If we choose* $g(t) \equiv 1, h(t) = t$ *on* $[a,b]$*, then the inequality* (3.87) *reduces to* (3.86).

(2) If we choose $x = \frac{h(a)+h(b)}{2}$*, then we get*

$$\left|\int_a^b f(t)g(t)\,dt - \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt\right| \leq \frac{1}{2}\int_a^b g(t)\,dt \cdot \bigvee_a^b (f), \tag{3.92}$$

which is the "weighted trapezoid" inequality.

Under the conditions of Theorem 58, we have the following corollaries.

COROLLARY 47. *Let $f \in C^{(1)}[a,b]$, then we have the inequality*

$$\left|\int_a^b f(t)g(t)\,dt - [(x-h(a))f(a) + (h(b)-x)f(b)]\right| \leq \left[\frac{1}{2}\int_a^b g(t)\,dt + \left|x - \frac{h(a)+h(b)}{2}\right|\right]\|f'\|_1, \tag{3.93}$$

for all $x \in [h(a), h(b)]$.

COROLLARY 48. *Let $f : [a,b] \to \mathbb{R}$ be a Lipschitzian mapping with the constant $L > 0$, then we have the inequality*

$$\left|\int_a^b f(t)g(t)\,dt - [(x-h(a))f(a) + (h(b)-x)f(b)]\right| \leq \left[\frac{1}{2}\int_a^b g(t)\,dt + \left|x - \frac{h(a)+h(b)}{2}\right|\right](b-a)L, \tag{3.94}$$

for all $x \in [h(a), h(b)]$.

COROLLARY 49. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic mapping, then we have the inequality*

$$\left|\int_a^b f(t)g(t)\,dt - [(x-h(a))f(a) + (h(b)-x)f(b)]\right| \leq \left[\frac{1}{2}\int_a^b g(t)\,dt + \left|x - \frac{h(a)+h(b)}{2}\right|\right] \cdot |f(b) - f(a)|, \tag{3.95}$$

for all $x \in [h(a), h(b)]$.

REMARK 49. *The following inequality is well-known in the literature as the Fejér inequality (see for example* [**94**]*):*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(t)\,dt \leq \int_a^b f(t)g(t)\,dt \leq \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt, \tag{3.96}$$

where $f : [a,b] \to \mathbb{R}$ is convex and $g : [a,b] \to \mathbb{R}$ is non-negative integrable and symmetric to $\frac{a+b}{2}$. Using the above results and (3.92)*, we obtain the following error bound of the second inequality in* (3.96)*,*

$$0 \leq \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt - \int_a^b f(t)g(t)\,dt \leq \frac{1}{2}\int_a^b g(t)\,dt \cdot \bigvee_a^b (f), \tag{3.97}$$

provided that f is of bounded variation on $[a,b]$.

REMARK 50. *If f is convex and Lipschitzian with the constant L on $[a,b]$, g is defined as in Remark 49 and $x = \frac{h(a)+h(b)}{2}$, then we get from (3.94) and (3.96),*

$$0 \leq \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt - \int_a^b f(t)g(t)\,dt \leq \frac{(b-a)L}{2}\int_a^b g(t)\,dt. \tag{3.98}$$

REMARK 51. *If f is convex and monotonic on $[a,b]$, g is defined as in Remark 49 and $x = \frac{h(a)+h(b)}{2}$, then we get, from (3.95) and (3.96),*

$$0 \leq \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt - \int_a^b f(t)g(t)\,dt \leq \frac{|f(b)-f(a)|}{2}\int_a^b g(t)\,dt. \tag{3.99}$$

REMARK 52. *If f is continuous, differentiable and convex on $[a,b]$ and $f' \in L_1(a,b)$, g is defined as in Remark 49 and $x = \frac{h(a)+h(b)}{2}$, then we get, from (3.93) and (3.96),*

$$0 \leq \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt - \int_a^b f(t)g(t)\,dt \leq \frac{\|f'\|_1}{2}\int_a^b g(t)\,dt. \tag{3.100}$$

6.3. Some Inequalities for Random Variables. Throughout this section, let $0 < a < b$, $r \in \mathbb{R}$, and let X be a continuous random variable having the continuous probability density function $g : [a,b] \to \mathbb{R}$ and the $r-$moment

$$E_r(X) := \int_a^b t^r g(t)\,dt,$$

which is assumed to be finite. The following result holds [**119**].

THEOREM 59. *The inequality*

$$\left| E_r(X) - \frac{a^r+b^r}{2} \right| \leq \frac{1}{2}|b^r - a^r|, \tag{3.101}$$

holds.

Proof. If we put $f(t) = t^r$, $h(t) = \int_a^t g(x)\,dx$ ($t \in [a,b]$) and $x = \frac{h(a)+h(b)}{2}$ in Corollary 49, we obtain the inequality

$$\left|\int_a^b f(t)g(t)\,dt - \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt\right| \leq \frac{1}{2}\int_a^b g(t)\,dt \cdot |f(b)-f(a)|. \tag{3.102}$$

Since

$$\int_a^b f(t)g(t)\,dt = E_r(X), \qquad \int_a^b g(t)\,dt = 1,$$

$$\frac{f(a)+f(b)}{2} = \frac{a^r+b^r}{2}, \text{ and } |f(b)-f(a)| = |b^r - a^r|,$$

(3.101) follows from (3.102), immediately. This completes the proof. ∎

If we choose $r = 1$ in Theorem 59, then we have the following familiar inequality

$$\left|E(X) - \frac{a+b}{2}\right| \leq \frac{b-a}{2}. \tag{3.103}$$

Theorem 60. ([**119**]) *Let p, $q \geq 1$, then the inequality*

$$\left|\beta(p,q) - \frac{1}{2np}\sum_{i=0}^{n-1}\left\{\left[1-\left(\frac{i}{n}\right)^{\frac{1}{p}}\right]^{q-1} + \left[1-\left(\frac{i+1}{n}\right)^{\frac{1}{p}}\right]^{q-1}\right\}\right| \leq \frac{1}{2np}, \tag{3.104}$$

holds for any positive integer n.

Proof. Let p, $q \geq 1$. If we put $a = 0$, $b = 1$, $f(t) = (1-t)^{q-1}$, $g(t) = t^{p-1}$ and $h(t) = \frac{t^p}{p}$ ($t \in [0,1]$) in Corollary 49, we obtain the inequality (3.104). This completes the proof. ∎

7. More Generalizations for Monotone Mappings

7.1. Introduction. The *trapezoid inequality*, states that if f'' exists and is bounded on (a,b), then

$$\left|\int_a^b f(x)dx - \frac{b-a}{2}[f(a)+f(b)]\right| \leq \frac{(b-a)^3}{12}\left\|f''\right\|_\infty. \tag{3.105}$$

Cerone and Dragomir [**35**] proved the following trapezoid type inequality:

Theorem A. *Let $f:[a,b]\to\mathbb{R}$ be a monotonic non-decreasing mapping, then*

$$\left|\int_a^b f(t)dt-[(x-a)f(a)+(b-x)f(b)]\right| \tag{3.106}$$
$$\leq (b-x)f(b)-(x-a)f(a)+\int_a^b \operatorname{sgn}(x-t)f(t)\,dt$$
$$\leq (x-a)[f(x)-f(a)]+(b-x)[f(b)-f(x)]$$
$$\leq \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right][f(b)-f(a)],$$

for all $x\in[a,b]$. The above inequalities are sharp.

In the next section, by following [**120**], we establish weighted generalizations of Theorem A, and give several applications for $r-$moments and the expectation of a continuous random variable , the Beta mapping and the Gamma mapping.

7.2. Some Integral Inequalities. The following result holds [**120**].

THEOREM 61. *Let $g:[a,b]\to\mathbb{R}$ be non-negative and continuous with $g(t)>0$ on (a,b) and let $h:[a,b]\to\mathbb{R}$ be differentiable such that $h'(t)=g(t)$ on $[a,b]$.*

(a) *Suppose $f:[a,b]\to\mathbb{R}$ is a monotonic non-decreasing mapping, then*

$$\left|\int_a^b f(t)g(t)\,dt-[(x-h(a))f(a)+(h(b)-x)f(b)]\right| \tag{3.107}$$
$$\leq (h(b)-x)f(b)-(x-h(a))f(a)$$
$$+\int_a^b \operatorname{sgn}\left(h^{-1}(x)-t\right)f(t)\,g(t)\,dt$$
$$\leq (x-h(a))\cdot\left[f\left(h^{-1}(x)\right)-f(a)\right]$$
$$+(h(b)-x)\cdot\left[f(b)-f\left(h^{-1}(x)\right)\right]$$
$$\leq \left[\frac{1}{2}\int_a^b g(t)\,dt+\left|x-\frac{h(a)+h(b)}{2}\right|\right]\cdot[f(b)-f(a)]$$

for all $x\in[h(a),h(b)]$.

(b) *Suppose $f:[a,b]\to\mathbb{R}$ is a monotonic non-increasing mapping, then*

$$\left|\int_a^b f(t)g(t)\,dt-[(x-h(a))f(a)+(h(b)-x)f(b)]\right| \tag{3.108}$$

$$
\begin{aligned}
&\leq (x-h(a))\, f(a) - (h(b)-x)\, f(b) \\
&\qquad + \int_a^b \operatorname{sgn}\left(t-h^{-1}(x)\right) f(t)\, g(t)\, dt \\
&\leq (x-h(a))\cdot\left[f(a) - f\left(h^{-1}(x)\right)\right] \\
&\qquad + (h(b)-x)\cdot\left[f\left(h^{-1}(x)\right) - f(b)\right] \\
&\leq \left[\frac{1}{2}\int_a^b g(t)\, dt + \left|x - \frac{h(a)+h(b)}{2}\right|\right]\cdot\left[f(a) - f(b)\right]
\end{aligned}
$$

for all $x \in [h(a), h(b)]$.

The above inequalities are sharp.

PROOF. (1)

(a) Let $x \in [h(a), h(b)]$. Using integration by parts, we have the following identity

$$
\begin{aligned}
&\int_a^b (x-h(t))\, df(t) \\
&= (x-h(t))\, f(t)\Big|_a^b + \int_a^b f(t)g(t)\, dt \\
&= \int_a^b f(t)g(t)\, dt - \left[(x-h(a))\, f(a) + (h(b)-x)\, f(b)\right].
\end{aligned} \tag{3.109}
$$

It is well known [**4**, p. 813] that if $\mu, \nu : [a,b] \to \mathbb{R}$ are such that μ is continuous on $[a,b]$ and ν is monotonic non-decreasing on $[a,b]$, then

$$
\left|\int_a^b \mu(t)\, d\nu(t)\right| \leq \int_a^b |\mu(t)|\, d\nu(t). \tag{3.110}
$$

Now, using identity (3.109) and inequality (3.110), we have

$$
\begin{aligned}
&\left|\int_a^b f(t)g(t)\, dt - \left[(x-h(a))\, f(a) + (h(b)-x)\, f(b)\right]\right| \\
&\leq \int_a^b |x-h(t)|\, df(t) \\
&= \int_a^{h^{-1}(x)} (x-h(t))\, df(t) + \int_{h^{-1}(x)}^b (h(t)-x)\, df(t)
\end{aligned} \tag{3.111}
$$

$$
\begin{aligned}
&= (x - h(t)) f(t) \Big|_a^{h^{-1}(x)} + \int_a^{h^{-1}(x)} f(t) g(t) \, dt \\
&\qquad + (h(t) - x) \Big|_{h^{-1}(x)}^{b} - \int_{h^{-1}(x)}^{b} f(t) g(t) \, dt \\
&= (h(b) - x) f(b) - (x - h(a)) f(a) \\
&\qquad + \int_a^b \operatorname{sgn}\left(h^{-1}(x) - t\right) f(t) g(t) \, dt
\end{aligned}
$$

and the first inequalities in (3.107) are proved.

As f is monotonic non-decreasing on $[a, b]$, we obtain

$$
\begin{aligned}
\int_a^{h^{-1}(x)} f(t) g(t) \, dt &\le f\left(h^{-1}(x)\right) \int_a^{h^{-1}(x)} g(t) \, dt \\
&= (x - h(a)) f\left(h^{-1}(x)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{h^{-1}(x)}^{b} f(t) g(t) \, dt &\ge f\left(h^{-1}(x)\right) \int_{h^{-1}(x)}^{b} g(t) \, dt \\
&= (h(b) - x) f\left(h^{-1}(x)\right),
\end{aligned}
$$

then

$$
\begin{aligned}
&\int_a^b \operatorname{sgn}\left(h^{-1}(x) - t\right) f(t) g(t) \, dt \\
&\qquad \le (x - h(a)) f\left(h^{-1}(x)\right) + (x - h(b)) f\left(h^{-1}(x)\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(h(b) - x) f(b) - (x - h(a)) f(a) \\
&\qquad + \int_a^b \operatorname{sgn}\left(h^{-1}(x) - t\right) f(t) g(t) \, dt \\
&\le (h(b) - x) f(b) - (x - h(a)) f(a) \\
&\qquad + (x - h(a)) f\left(h^{-1}(x)\right) + (x - h(b)) f\left(h^{-1}(x)\right) \\
&= (x - h(a)) \cdot \left[f\left(h^{-1}(x)\right) - f(a)\right] \\
&\qquad + (h(b) - x) \cdot \left[f(b) - f\left(h^{-1}(x)\right)\right]
\end{aligned} \tag{3.112}
$$

which proves that the second inequality in (3.107).

As f is monotonic non-decreasing on $[a, b]$, we have

$$
f(a) \le f\left(h^{-1}(x)\right) \le f(b)
$$

and

$$
\begin{aligned}
(3.113)\qquad & (x-h(a))\cdot\left[f\left(h^{-1}(x)\right)-f(a)\right] \\
&\qquad +(h(b)-x)\cdot\left[f(b)-f\left(h^{-1}(x)\right)\right] \\
&\le \max\{x-h(a),h(b)-x\} \\
&\qquad \times\left[f\left(h^{-1}(x)\right)-f(a)+f(b)-f\left(h^{-1}(x)\right)\right] \\
&=\left[\frac{h(b)-h(a)}{2}+\left|x-\frac{h(a)+h(b)}{2}\right|\right]\cdot[f(b)-f(a)] \\
&=\left[\frac{1}{2}\int_a^b g(t)\,dt+\left|x-\frac{h(a)+h(b)}{2}\right|\right]\cdot[f(b)-f(a)].
\end{aligned}
$$

Thus, by (3.111), (3.112) and (3.113), we obtain (3.107).

Let

$$
\begin{aligned}
g(t) &\equiv 1,\ \ t\in[a,b] \\
h(t) &= t,\ \ t\in[a,b] \\
f(t) &= \begin{cases} 0, & t\in[a,b) \\ 1, & t=b \end{cases}
\end{aligned}
$$

and $x=\frac{a+b}{2}$, then

$$
\begin{aligned}
&\left|\int_a^b f(t)g(t)\,dt-[(x-h(a))f(a)+(h(b)-x)f(b)]\right| \\
&=(h(b)-x)f(b)-(x-h(a))f(a) \\
&\qquad +\int_a^b \operatorname{sgn}\left(h^{-1}(x)-t\right)f(t)g(t)\,dt \\
&=(x-h(a))\cdot\left[f\left(h^{-1}(x)\right)-f(a)\right] \\
&\qquad +(h(b)-x)\cdot\left[f(b)-f\left(h^{-1}(x)\right)\right] \\
&=\left[\frac{1}{2}\int_a^b g(t)\,dt+\left|x-\frac{h(a)+h(b)}{2}\right|\right]\cdot[f(b)-f(a)] \\
&=\frac{b-a}{2}
\end{aligned}
$$

which proves that the inequalities (3.107) are sharp.

(b) If f is replaced by $-f$ in (a), then (3.108) is obtained from (3.107).

This completes the proof. ∎

Remark 53. *If we choose $g(t)\equiv 1, h(t)=t$ on $[a,b]$, then the inequalities (3.107) reduce to (3.106).*

COROLLARY 50. *([**120**]) If we choose* $x = \frac{h(a)+h(b)}{2}$*, then we get*

$$\left|\int_a^b f(t)g(t)\,dt - \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt\right| \leq \frac{1}{2}\int_a^b g(t)\,dt \cdot [f(b)-f(a)] + \int_a^b \operatorname{sgn}\left(h^{-1}\left(\frac{h(a)+h(b)}{2}\right)-t\right) f(t)\,g(t)\,dt \leq \frac{1}{2}\int_a^b g(t)\,dt \cdot [f(b)-f(a)] \tag{3.114}$$

where f *and* g *are defined as in (a) of Theorem 61, and*

$$\left|\int_a^b f(t)g(t)\,dt - \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt\right| \leq \frac{1}{2}\int_a^b g(t)\,dt \cdot [f(a)-f(b)] + \int_a^b \operatorname{sgn}\left(t-h^{-1}\left(\frac{h(a)+h(b)}{2}\right)\right) f(t)\,g(t)\,dt \leq \frac{1}{2}\int_a^b g(t)\,dt \cdot [f(a)-f(b)] \tag{3.115}$$

where f *and* g *are defined as in (b) of Theorem 61.*

The inequalities (3.114) and (3.115) are the "*weighted trapezoid*" inequalities.

Note that the trapezoid inequality (3.114) and (3.115) are, in a sense, the best possible inequalities we can obtain from (3.107) and (3.108). Moreover, the constant $\frac{1}{2}$ is the best possible for both inequalities in (3.114) and (3.115), respectively.

REMARK 54. *The following inequality is well-known in the literature as the Fejér inequality (see for example [**94**]):*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(t)\,dt \leq \int_a^b f(t)g(t)\,dt \leq \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt, \tag{3.116}$$

where $f:[a,b]\to\mathbb{R}$ *is convex and* $g:[a,b]\to\mathbb{R}$ *is positive integrable and symmetric with respect to* $\frac{a+b}{2}$*.*

Using the above results and (3.114) – (3.115), we obtain the following error bound of the second inequality in (3.116):

$$
\begin{aligned}
(3.117)\qquad 0 &\le \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt \\
&\le \frac{1}{2}\int_a^b g(t)\,dt\cdot[f(b)-f(a)] \\
&\qquad +\int_a^b \operatorname{sgn}\left(h^{-1}(\frac{h(a)+h(b)}{2})-t\right)f(t)\,g(t)\,dt \\
&\le \frac{1}{2}\int_a^b g(t)\,dt\cdot[f(b)-f(a)]
\end{aligned}
$$

provided that f is monotonic non-decreasing on $[a,b]$.

$$
\begin{aligned}
(3.118)\qquad 0 &\le \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt \\
&\le \frac{1}{2}\int_a^b g(t)\,dt\cdot[f(a)-f(b)] \\
&\qquad +\int_a^b \operatorname{sgn}\left(t-h^{-1}(\frac{h(a)+h(b)}{2})\right)f(t)\,g(t)\,dt \\
&\le \frac{1}{2}\int_a^b g(t)\,dt\cdot[f(a)-f(b)]
\end{aligned}
$$

provided that f is monotonic non-increasing on $[a,b]$.

7.3. Some Inequalities for Random Variables. Throughout this section, let $0<a<b$, $r\in\mathbb{R}$, and let X be a continuous random variable having the continuous probability density mapping $g:[a,b]\to\mathbb{R}$ with $g(t)>0$ on (a,b) , $h:[a,b]\to\mathbb{R}$ with $h'(t)=g(t)$ for $t\in(a,b)$ and the $r-$moment

$$E_r(X):=\int_a^b t^r g(t)\,dt,$$

which is assumed to be finite [**120**].

Theorem 62. *The inequalities*

$$
\begin{aligned}
(3.119)\qquad &\left|E_r(X)-\frac{a^r+b^r}{2}\right| \\
&\le \frac{1}{2}(b^r-a^r)+\int_a^b \operatorname{sgn}\left(h^{-1}\left(\frac{1}{2}\right)-t\right)t^r g(t)\,dt \\
&\le \frac{1}{2}(b^r-a^r) \qquad\qquad \textit{as } r\ge 0
\end{aligned}
$$

and

$$\left|E_r(X)-\frac{a^r+b^r}{2}\right| \le \frac{1}{2}(a^r-b^r)+\int_a^b \operatorname{sgn}\left(t-h^{-1}\left(\frac{1}{2}\right)\right)t^r g(t)\,dt \le \frac{1}{2}(a^r-b^r) \qquad \text{as } r<0, \tag{3.120}$$

hold.

Proof. If we put $f(t)=t^r\ (t\in[a,b])$, $h(t)=\int_a^t g(x)\,dx\ (t\in[a,b])$ and $x=\frac{h(a)+h(b)}{2}=\frac{1}{2}$ in Corollary 50, then we obtain (3.119) and (3.120). This completes the proof. ∎

The following corollary which is a special case of Theorem 62.

Corollary 51. *The inequalities*

$$\left|E(X)-\frac{a+b}{2}\right| \le \frac{b-a}{2}+\int_a^b \operatorname{sgn}\left(h^{-1}\left(\frac{1}{2}\right)-t\right)tg(t)\,dt \le \frac{b-a}{2} \tag{3.121}$$

hold.

The following inequality, which is an application of Theorem 61 for the Beta mapping, holds:

Theorem 63. *Let p, $q>0$, then we have the inequality*

$$\begin{aligned}|\beta(p+1,q+1)-x| &\le x+\int_a^b \operatorname{sgn}\left[t-((p+1)x)^{\frac{1}{p+1}}\right]t^p(1-t)^q\,dt \\ &\le x+\left(\frac{1}{p+1}-2x\right)\left[1-((p+1)x)^{\frac{1}{p+1}}\right]^q \\ &\le \frac{1}{2(p+1)}+\left|x-\frac{1}{2(p+1)}\right|\end{aligned} \tag{3.122}$$

for all $x\in\left[0,\frac{1}{p+1}\right]$.

Proof. If we put $a=0$, $b=1$, $f(t)=(1-t)^q$, $g(t)=t^p$ and $h(t)=\frac{t^{p+1}}{p+1}\ (t\in[0,1])$ in Theorem 61, we obtain the inequality (3.122) for all $x\in\left[0,\frac{1}{p+1}\right]$. This completes the proof. ∎

The following remark, which is an application of Theorem 63 for the Gamma mapping, applies:

REMARK 55. *Taking into account that* $\beta(p+1,q+1)=\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}$, *the inequality (3.122) is equivalent to*

$$\begin{aligned}
&\left|\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}-x\right| \\
&\qquad\le x+\int_a^b \operatorname{sgn}\left[t-((p+1)\,x)^{\frac{1}{p+1}}\right]t^p\,(1-t)^q\,dt \\
&\qquad\le x+\left(\frac{1}{p+1}-2x\right)\left[1-((p+1)\,x)^{\frac{1}{p+1}}\right]^q \\
&\qquad\le \frac{1}{2\,(p+1)}+\left|x-\frac{1}{2(p+1)}\right|
\end{aligned}$$

i.e.,

$$\begin{aligned}
&|(p+1)\Gamma(p+1)\Gamma(q+1)-x(p+1)\Gamma(p+q+2)| \\
&\le\left[x+\int_a^b \operatorname{sgn}\left[t-((p+1)\,x)^{\frac{1}{p+1}}\right]t^p\,(1-t)^q\,dt\right](p+1)\Gamma(p+q+2) \\
&\le\left[x+\left(\frac{1}{p+1}-2x\right)\left[1-((p+1)\,x)^{\frac{1}{p+1}}\right]^q\right](p+1)\Gamma(p+q+2) \\
&\le\left[\frac{1}{2}+\left|x(p+1)-\frac{1}{2}\right|\right]\cdot\Gamma(p+q+2)
\end{aligned}$$

and as $(p+1)\Gamma(p+1)=\Gamma(p+2)$, *we get*

$$\begin{aligned}
&|\Gamma(p+2)\Gamma(q+1)-x(p+1)\Gamma(p+q+2)| \\
&\qquad\le\left[x+\int_a^b \operatorname{sgn}\left[t-((p+1)\,x)^{\frac{1}{p+1}}\right]t^p\,(1-t)^q\,dt\right] \\
&\qquad\qquad\times(p+1)\Gamma(p+q+2) \\
&\qquad\le\left[x+\left(\frac{1}{p+1}-2x\right)\left[1-((p+1)\,x)^{\frac{1}{p+1}}\right]^q\right] \\
&\qquad\qquad\times(p+1)\Gamma(p+q+2) \\
&\qquad\le\left[\frac{1}{2}+\left|x(p+1)-\frac{1}{2}\right|\right]\cdot\Gamma(p+q+2).
\end{aligned}\tag{3.123}$$

CHAPTER 4

Inequalities for CDFs Via Grüss Type Results

1. Random Variables whose PDFs are Bounded

1.1. Introduction. In papers [**107, 108**], Matić, Pečarić and Ujević proved the following inequality, which has been called the *pre-Grüss inequality* in [**34**]

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt\right| \leq \frac{1}{2}(\phi-\gamma)\left[\frac{1}{b-a}\int_a^b g^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)^2\right]^{\frac{1}{2}}, \tag{4.1}$$

provided that $\gamma \leq f(t) \leq \phi$ a.e. on $[a,b]$ and the integrals exist and are finite.

In [**108**], the authors used (4.1) to obtain some bounds for the remainder in certain Taylor like formulae whilst in [**34**], the authors applied (4.1) to estimation of the remainder in three point quadrature formulae.

Basically, (4.1) is a pre-Grüss inequality since, if we assume that $\alpha \leq g(t) \leq \beta$ a.e. in $[a,b]$, then, (see for example [**67**])

$$\frac{1}{b-a}\int_a^b g^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)^2 \leq \frac{1}{4}(\beta-\alpha)^2, \tag{4.2}$$

which, together with (4.1), gives the original *Grüss inequality,*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt\right| \leq \frac{1}{4}(\phi-\gamma)(\beta-\alpha). \tag{4.3}$$

In [**108**], Matić, Pečarić and Ujević observed that if a factor is known, for example $g(t)$, $t\in[a,b]$, then instead of using (4.3) in estimating the difference

$$\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt,$$

it is better to use (4.1) [**107**].

In this section, by adopting this same approach, we obtain some inequalities for the expectation $E(X)$ and cumulative distribution function $F(\cdot)$ of a random variable having the probability distribution function $f:[a,b]\to\mathbb{R}$. It is assumed that we know the lower and the upper bound for f, i.e., the real numbers γ,ϕ such that $0\leq\gamma\leq f(t)\leq\phi\leq 1$ a.e. t on $[a,b]$. Some related results are also established.

1.2. Some Inequalities for Expectation and Dispersion. We start with the following result for expectation [**20**].

THEOREM 64. *Let X be a random variable having the probability density function $f:[a,b]\to\mathbb{R}$. Assume that there exist constants γ,ϕ such that $0\leq\gamma\leq f(t)\leq\phi\leq 1$ a.e. t on $[a,b]$, then,*

$$\left|E(X)-\frac{a+b}{2}\right|\leq\frac{1}{4\sqrt{3}}(\phi-\gamma)(b-a)^2. \tag{4.4}$$

PROOF. If we put $g(t)=t$ in (4.1), we obtain

$$\left|\frac{1}{b-a}\int_a^b tf(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b tdt\right| \tag{4.5}$$
$$\leq\frac{1}{2}(\phi-\gamma)\left[\frac{1}{b-a}\int_a^b t^2dt-\left(\frac{1}{b-a}\int_a^b tdt\right)^2\right]^{\frac{1}{2}}$$

and as

$$\int_a^b tf(t)\,dt=E(X),$$
$$\int_a^b f(t)\,dt=1,\quad \frac{1}{b-a}\int_a^b tdt=\frac{a+b}{2}$$

and

$$\frac{1}{b-a}\int_a^b t^2dt-\left(\frac{1}{b-a}\int_a^b tdt\right)^2=\frac{(b-a)^2}{12},$$

then by (4.5) we deduce (4.4). ∎

To point out a result for the $p-$moments of the random variable X, $p\in\mathbb{R}\backslash\{-1,0\}$, we need the following $p-$*Logarithmic mean,*

$$M_p(a,b):=\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}},$$

where $0<a<b$.

THEOREM 65. *([**20**]) Let X and f be as in Theorem 64 and $E_p(X)$ be the p-moment of X, i.e.,*

$$E_p(X) := \int_a^b t^p f(t)\,dt,$$

which is assumed to be finite, then:

$$\left|E_p(X) - M_p^p(a,b)\right| \leq \frac{1}{2}(\phi - \gamma)\left[M_{2p}^{2p}(a,b) - M_p^{2p}(a,b)\right]^{\frac{1}{2}}. \tag{4.6}$$

The proof is obvious by (4.1) in which we choose $g(t) = t^p,\ p \in \mathbb{R}\setminus\{-1,0\}$.

If we consider the Logarithmic mean

$$M_{-1}(a,b) := L(a,b) = \frac{b-a}{\ln b - \ln a},\ 0 < a < b$$

and define the (-1)-moment of the random variable X by

$$E_{-1}(X) := \int_a^b \frac{f(t)}{t}dt,$$

then we can also state the following theorem [**20**].

THEOREM 66. *Let X and f be as in Theorem 64, then:*

$$\left|E_{-1}(X) - M_{-1}^{-1}(a,b)\right| \leq \frac{1}{2}(\phi - \gamma)\left[M_{-2}^{-2}(a,b) - M_{-1}^{-2}(a,b)\right]^{\frac{1}{2}}, \tag{4.7}$$

provided the (-1)-moment of X is finite.

The proof is obvious by (4.1) and so we omit the details.

The following theorem also holds [**20**].

THEOREM 67. *Let X and f be as above. If*

$$\sigma_\mu(X) := \left[\int_a^b (t-\mu)^2 f(t)\,dt\right]^{\frac{1}{2}},\ \mu \in [a,b],$$

then we have the inequality,

$$\begin{aligned} &\left|\sigma_\mu^2(X) - \left(\mu - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12}\right| \\ &\leq \frac{1}{2}(\phi - \gamma)(b-a)^2\left[\frac{1}{3}\left(\mu - \frac{a+b}{2}\right)^2 + \frac{1}{180}(b-a)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{3\sqrt{5}}(\phi - \gamma)(b-a)^3. \end{aligned} \tag{4.8}$$

PROOF. If we put $g(t) = (t-\mu)^2$ in (4.1) we get

$$\begin{aligned}(4.9)\quad & \left|\frac{1}{b-a}\int_a^b f(t)(t-\mu)^2\,dt \right.\\ & \left. - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b (t-\mu)^2\,dt\right| \\ & \leq \frac{1}{2}(\phi-\gamma)\left[\frac{1}{b-a}\int_a^b (t-\mu)^4\,dt - \left(\frac{1}{b-a}\int_a^b (t-\mu)^2\,dt\right)^2\right]^{\frac{1}{2}},\end{aligned}$$

and as

$$\begin{aligned}\int_a^b f(t)\,dt &= 1,\\ \frac{1}{b-a}\int_a^b (t-\mu)^2\,dt &= \frac{(b-\mu)^3+(\mu-a)^3}{3(b-a)}\\ &= \frac{(b-\mu)^2-(b-\mu)(\mu-a)+(\mu-a)^2}{3}\\ &= \left(\mu-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{12},\end{aligned}$$

$$\begin{aligned}&\frac{1}{b-a}\int_a^b (t-\mu)^4\,dt - \left(\frac{1}{b-a}\int_a^b (t-\mu)^2\,dt\right)^2\\ &= \frac{1}{45}\Big[4\left[(b-\mu)^2-(\mu-a)^2\right]^2\\ &\qquad + 2(b-\mu)^2(\mu-a)^2 + (\mu-a)(b-\mu)\left[(b-\mu)^2+(\mu-a)^2\right]\Big]\\ &:= A.\end{aligned}$$

(after considerable algebraic manipulation). However,

$$\begin{aligned}(b-\mu)^2-(\mu-a)^2 &= (b-a)(b+a-2\mu)\\ &= 2(b-a)\left(\frac{b+a}{2}-\mu\right),\\ (b-\mu)(\mu-a) &= \frac{1}{4}(b-a)^2-\left(\mu-\frac{a+b}{2}\right)^2,\\ (b-\mu)^2+(\mu-a)^2 &= \frac{1}{2}(b-a)^2+2\left(\mu-\frac{a+b}{2}\right)^2,\end{aligned}$$

giving,

$$A = \frac{(b-a)^2}{45}\left[15\left(\mu - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]$$
$$= (b-a)^2\left[\frac{1}{3}\left(\mu - \frac{a+b}{2}\right)^2 + \frac{1}{180}(b-a)^2\right].$$

Using the inequality (4.9), we deduce the desired inequality (4.8). ∎

The best inequality we can obtain from (4.8) is that for which $\mu = \frac{a+b}{2}$ and, therefore, we can state the following corollary (see also [**20**]).

Corollary 52. *With the above assumptions and denoting* $\sigma_0(X) := \sigma_{\frac{a+b}{2}}(X)$*, we have the inequality:*

$$\left|\sigma_0^2(X) - \frac{(b-a)^2}{12}\right| \leq \frac{1}{12\sqrt{5}}(\phi - \gamma)(b-a)^3. \tag{4.10}$$

The following theorem also holds [**20**].

Theorem 68. *Let X and f be as above. If*

$$A_\mu(X) := \int_a^b |t-\mu| f(t)\,dt,\ \mu \in [a,b], \tag{4.11}$$

then we have the inequality

$$\left|A_\mu(X) - \frac{1}{b-a}\left[\frac{(b-a)^2}{4} + \left(\mu - \frac{a+b}{2}\right)^2\right]\right| \tag{4.12}$$
$$\leq \frac{1}{2}(\phi-\gamma)(b-a)\left[\frac{(b-a)^2}{48}\right.$$
$$\left.+\left(\frac{\mu - \frac{a+b}{2}}{b-a}\right)^2\left[\frac{1}{2}(b-a)^2 + \left(\mu - \frac{a+b}{2}\right)^2\right]\right]^{\frac{1}{2}}.$$

for all $\mu \in [a,b]$.

PROOF. If we put $g(t) = |t-\mu|$ in (4.1), we have

$$\begin{aligned}(4.13)\quad & \left|\frac{1}{b-a}\int_a^b |t-\mu| f(t)\,dt \right. \\ & \left. \qquad - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b |t-\mu|\,dt\right| \\ & \leq \frac{1}{2}(\phi-\gamma)\left[\frac{1}{b-a}\int_a^b |t-\mu|^2\,dt - \left(\frac{1}{b-a}\int_a^b |t-\mu|\,dt\right)^2\right]^{\frac{1}{2}}\end{aligned}$$

and as $\int_a^b f(t)\,dt = 1$,

$$\begin{aligned}\frac{1}{b-a}\int_a^b |t-\mu|\,dt &= \frac{1}{b-a}\left[\int_a^{\mu}(\mu-t)\,dt + \int_{\mu}^b (t-\mu)\,dt\right] \\ &= \frac{1}{b-a}\left[\frac{(b-\mu)^2+(\mu-a)^2}{2}\right] \\ &= \frac{1}{b-a}\left[\frac{(b-a)^2}{4} + \left(\mu-\frac{a+b}{2}\right)^2\right],\end{aligned}$$

$$\begin{aligned}\frac{1}{b-a}\int_a^b (t-\mu)^2\,dt &= \frac{(b-\mu)^3+(\mu-a)^3}{3(b-a)} \\ &= \frac{(b-a)^2}{12} + \left(\mu-\frac{a+b}{2}\right)^2,\end{aligned}$$

$$\begin{aligned}&\frac{1}{b-a}\int_a^b (t-\mu)^2\,dt - \left(\frac{1}{b-a}\int_a^b |t-\mu|\,dt\right)^2 \\ &= \frac{(b-a)^2}{12} + \left(\mu-\frac{a+b}{2}\right)^2 - \left[\frac{(b-a)}{4} + \frac{1}{b-a}\left(\mu-\frac{a+b}{2}\right)^2\right]^2 \\ &= \frac{(b-a)^2}{48} + \frac{1}{2}\left(\mu-\frac{a+b}{2}\right)^2 - \frac{1}{(b-a)^2}\left(\mu-\frac{a+b}{2}\right)^4 \\ &= \frac{(b-a)^2}{48} + \left(\frac{\mu-\frac{a+b}{2}}{b-a}\right)^2\left[\frac{1}{2}(b-a)^2 - \left(\mu-\frac{a+b}{2}\right)^2\right].\end{aligned}$$

Finally, using (4.13) we deduce the desired inequality. ∎

COROLLARY 53. *([**20**]) The best inequality we can get from (4.12) is for $\mu = \mu_0 := \frac{a+b}{2}$, giving:*

$$\left| A_{\mu_0}(X) - \frac{b-a}{4} \right| \leq \frac{1}{8\sqrt{3}} (\phi - \gamma)(b-a)^2, \tag{4.14}$$

where $A_{\mu_0}(X)$ is as defined in (4.11).

PROOF. Consider the mapping

$$g(\mu) := \frac{(b-a)^2}{48} + \frac{1}{2}\left(\mu - \frac{a+b}{2}\right)^2 - \frac{1}{(b-a)^2}\left(\mu - \frac{a+b}{2}\right)^4.$$

We have

$$\begin{aligned} \frac{dg(\mu)}{d\mu} &= \left(\mu - \frac{a+b}{2}\right) - \frac{4}{(b-a)^2}\left(\mu - \frac{a+b}{2}\right)^3 \\ &= \left(\mu - \frac{a+b}{2}\right)\left[1 - \frac{4}{(b-a)^2}\left(\mu - \frac{a+b}{2}\right)^2\right]. \end{aligned}$$

Note that $\frac{dg(\mu)}{d\mu} = 0$ if $\mu = a$ or $\mu = \frac{a+b}{2}$ or $\mu = b$ and as

$$\frac{dg(\mu)}{d\mu} < 0 \text{ for } \mu \in \left(a, \frac{a+b}{2}\right) \text{ and } \frac{dg(\mu)}{d\mu} > 0 \text{ for } \mu \in \left(\frac{a+b}{2}, \mu\right),$$

we deduce that $\mu = \frac{a+b}{2}$ is the point realizing the global minimum on (a,b) and as $g(\mu_0) = \frac{(b-a)^2}{48}$, the inequality (4.14) is indeed the best inequality we can get from (4.12). ∎

Another inequality that can be useful for obtaining different inequalities for dispersion is the following weighted Grüss type result (see for example [**50**]).

LEMMA 13. *Let $g, p : [a,b] \to \mathbb{R}$ be measurable functions such that $\alpha \leq g \leq \beta$ a.e., $p \geq 0$ a.e. on $[a,b]$ and $\int_a^b p(x)\,dx > 0$. Then*

$$0 \leq \frac{\int_a^b p(x) g^2(x)\,dx}{\int_a^b p(x)\,dx} - \left(\frac{\int_a^b p(x) g(x)\,dx}{\int_a^b p(x)\,dx}\right)^2 \leq \frac{1}{4}(\beta - \alpha)^2, \tag{4.15}$$

provided that all the integrals in (4.15) exist and are finite.

Using the above lemma we prove the following result for dispersion [**20**].

THEOREM 69. *Let X be a random variable whose probability density function f is defined on the finite interval $[a,b]$ and $\sigma(X) < \infty$. Then we have the inequality*

$$0 \leq \sigma_\mu^2(X) - (E(X) - \mu)^2 \leq \frac{1}{4}(b-a)^2 \tag{4.16}$$

for all $\mu \in [a,b]$, or, equivalently,

$$0 \leq \sigma(X) \leq \frac{1}{2}(b-a). \tag{4.17}$$

PROOF. Choose in (4.15), $g(x) = x - \mu$, $p(x) = f(x)$, then, obviously, $\sup_{x\in[a,b]} g(x) = b - \mu$, $\inf_{x\in[a,b]} g(x) = a - \mu$, $\int_a^b f(x)\,dx = 1$, and by (4.15),

$$0 \leq \int_a^b (x-\mu)^2 f(x)\,dx - \left(\int_a^b (x-\mu) f(x)\,dx\right)^2 \leq \frac{1}{4}(b-a)^2$$

and the inequality (4.16) is proved. ∎

The following inequality connecting $\sigma_\mu(X)$ and $A_\mu(X)$ also holds (see also [**20**]).

THEOREM 70. *Let X be as in Theorem 69 and assume that $\sigma_\mu(X)$, $A_\mu(X) < \infty$ for all $\mu \in [a,b]$. We have the inequality,*

$$0 \leq \sigma_\mu^2(X) - A_\mu^2(X) \leq \frac{1}{2}\left|\mu - \frac{a+b}{2}\right| \tag{4.18}$$

for all $\mu \in [a,b]$ with $A_\mu(X)$ given in (4.11).

PROOF. Choose in Lemma 13, $p(x) = f(x)$, $g(x) = |x-\mu|$, $\mu \in [a,b]$, then

$$\beta = \sup_{x\in[a,b]} g(x) = \max\{\mu - a, b - \mu\} = \frac{b - a + |\mu - a - b + \mu|}{2},$$

$$\alpha = \inf_{x\in[a,b]} g(x) = \min\{\mu - a, b - \mu\} = \frac{b - a - |\mu - a - b + \mu|}{2},$$

which gives us

$$\beta - \alpha = 2\left|\mu - \frac{a+b}{2}\right|.$$

Applying (4.15), we deduce (4.18). ∎

1.3. Some Inequalities for CDFs. The following theorem contains an inequality which connects the expectation $E(X)$, the cumulative distribution function $F(X) := \int_a^x f(t)\,dt$ and the bounds γ and ϕ of the probability density function $f : [a,b] \to \mathbb{R}$ (see also [**20**]).

THEOREM 71. *Let $X,\ f,\ E(X),\ F(\cdot)$ and γ, ϕ be as above, then:*

$$\left| E(X) + (b-a) F(x) - x - \frac{b-a}{2} \right| \leq \frac{1}{4\sqrt{3}} (\phi - \gamma)(b-a)^2, \tag{4.19}$$

for all $x \in [a,b]$.

PROOF. The following identity was established by Barnett and Dragomir in [**9**]

$$\begin{aligned} (b-a) F(x) + E(X) - b &= \int_a^b p(x,t)\, dF(t) \\ &= \int_a^b p(x,t) f(t)\, dt, \end{aligned} \tag{4.20}$$

where

$$p(x,t) := \begin{cases} t-a & \text{if} \quad a \leq t \leq x \leq b \\ \\ t-b & \text{if} \quad a \leq x < t \leq b \end{cases}.$$

Applying the inequality (4.1) for $g(t) = p(x,t)$, we get

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b p(x,t) f(t)\, dt \right. \\ &\qquad \left. - \frac{1}{b-a} \int_a^b p(x,t)\, dt \cdot \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ &\leq \frac{1}{2}(\phi - \gamma) \left[\frac{1}{b-a} \int_a^b p^2(x,t)\, dt - \left(\frac{1}{b-a} \int_a^b p(x,t)\, dt \right)^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{4.21}$$

Observe that

$$\frac{1}{b-a} \int_a^b p(x,t)\, dt = x - \frac{a+b}{2},$$
$$\int_a^b f(t)\, dt = 1,$$

and

$$D := \frac{1}{b-a}\int_a^b p^2(x,t)\,dt - \left(\frac{1}{b-a}\int_a^b p(x,t)\,dt\right)^2$$
$$= \frac{1}{b-a}\left[\frac{(b-x)^3+(x-a)^3}{3}\right] - \left(x-\frac{a+b}{2}\right)^2$$
$$= \frac{(b-x)^2-(b-x)(x-a)+(x-a)^2}{3} - \left(x-\frac{a+b}{2}\right)^2.$$

As a simple calculation shows that

$$(b-x)^2-(b-x)(x-a)+(x-a)^2 = 3\left(x-\frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2,$$

we get,

$$D = \frac{1}{12}(b-a)^2.$$

Using (4.21), we deduce (4.19). ∎

Remark 56. *If in (4.19) we choose either $x=a$ or $x=b$, we get*

$$\left|E(X)-\frac{a+b}{2}\right| \le \frac{1}{4\sqrt{3}}(\phi-\gamma)(b-a)^2,$$

which is the inequality (4.4).

Remark 57. *If in (4.19) we choose $x=\frac{a+b}{2}$, then we get the inequality*

$$\left|E(X)+(b-a)\Pr\left(X\le\frac{a+b}{2}\right)-b\right| \le \frac{1}{4\sqrt{3}}(\phi-\gamma)(b-a)^2. \tag{4.22}$$

The following theorem also holds (see also [**20**]).

Theorem 72. *Let X, f, γ,ϕ and $F(\cdot)$ be as above, then we have:*

$$\left|E(X)+\frac{b-a}{2}F(x)-\frac{b+x}{2}\right| \le \frac{1}{2\sqrt{3}}(\phi-\gamma)\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right] \le \frac{1}{4\sqrt{3}}(\phi-\gamma)(b-a)^2, \tag{4.23}$$

for all $x \in [a,b]$.

PROOF. We use the identity (4.20).

Applying the pre-Grüss inequality (4.1), we get, for $x \in [a,b]$,

$$\begin{aligned}(4.24)\qquad &\left|\frac{1}{x-a}\int_a^x (t-a) f(t)\,dt - \frac{1}{x-a}\int_a^x (t-a)\,dt \cdot \frac{1}{x-a}\int_a^x f(t)\,dt\right| \\ &\leq \frac{1}{2}(\phi-\gamma)\left[\frac{1}{x-a}\int_a^x (t-a)^2\,dt - \left[\frac{1}{x-a}\int_a^x (t-a)\,dt\right]^2\right]^{\frac{1}{2}} \\ &= \frac{1}{4\sqrt{3}}(\phi-\gamma)(x-a)\end{aligned}$$

and, similarly,

$$\begin{aligned}(4.25)\qquad &\left|\frac{1}{b-x}\int_x^b (t-b) f(t)\,dt - \frac{1}{b-x}\int_x^b (t-b)\,dt \cdot \frac{1}{b-x}\int_x^b f(t)\,dt\right| \\ &\leq \frac{1}{4\sqrt{3}}(\phi-\gamma)(b-x), \quad x \in (a,b).\end{aligned}$$

From (4.24) and (4.25) we can write

$$(4.26)\qquad \left|\int_a^x (t-a) f(t)\,dt - \frac{x-a}{2}F(x)\right| \leq \frac{1}{4\sqrt{3}}(\phi-\gamma)(x-a)^2$$

and

$$(4.27)\qquad \left|\int_x^b (t-b) f(t)\,dt + \frac{b-x}{2}(1-F(x))\right| \leq \frac{1}{4\sqrt{3}}(\phi-\gamma)(b-x)^2,$$

for all $x \in [a,b]$.

Summing (4.26) and (4.27) and using the triangle inequality, we deduce that

$$\left|\int_a^x (t-a) f(t)\,dt + \int_x^b (t-b) f(t)\,dt - \frac{b-a}{2}F(x) + \frac{b-x}{2}\right| \tag{4.28}$$
$$\leq \frac{1}{4\sqrt{3}}(\phi-\gamma)\left[(x-a)^2+(b-x)^2\right]$$
$$= \frac{1}{2\sqrt{3}}(\phi-\gamma)\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right].$$

Evaluation at $x = a$ or b produces the final inequality in (4.23). ∎

REMARK 58. *If we choose in (4.23), either $x = a$ or $x = b$, we get the inequality*

$$\left|E(X) - \frac{a+b}{2}\right| \leq \frac{1}{4\sqrt{3}}(\phi-\gamma)(b-a)^2 \tag{4.29}$$

and thus recapture (4.4).

REMARK 59. *If we choose in (4.23), $x = \frac{a+b}{2}$, then we get*

$$\left|E(X) + \left(\frac{b-a}{2}\right)\Pr\left(X \leq \frac{a+b}{2}\right) - \frac{a+3b}{4}\right| \tag{4.30}$$
$$\leq \frac{1}{8\sqrt{3}}(\phi-\gamma)(b-a)^2,$$

which is the best inequality of this type that can be obtained.

2. The Case of Absolutely Continuous PDFs

2.1. Introduction. In [**108**], Matić, Pečarić and Ujević proved the following refinement of Čebyšev's inequality which we call the "*pre-Čebyšev*" inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \right. \tag{4.31}$$
$$\left. - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right|$$
$$\leq \frac{1}{2\sqrt{3}}(b-a)\left\|f'\right\|_\infty$$
$$\times\left[\frac{1}{b-a}\int_a^b g^2(x)\,dx - \left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)^2\right]^{\frac{1}{2}},$$

provided that f is absolutely continuous on $[a,b]$ and all the integrals in (4.31) exist and are finite.

Matić, Pečarić and Ujević observed that: if a factor is known, say $g(t)$, $t \in [a,b]$, then instead of using Čebyšev's inequality to estimate the difference

$$\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt,$$

it is better to use (4.31). They demonstrated this by improving some results of the second author in [**108**] related to Taylor's formula with integral remainder.

Using the same approach here, we obtain some inequalities for the expectation, $E(X)$, and cumulative distribution function $F(x)$ of a random variable having the probability density function $f:[a,b]\to\mathbb{R}$ which is assumed to be absolutely continuous and whose derivative $f' \in L_\infty[a,b]$.

2.2. Some Inequalities. We start with the following result for expectation [**21**].

THEOREM 73. *Let X be a random variable having the probability density function $f:[a,b]\to\mathbb{R}$. Assume that f is absolutely continuous on $[a,b]$ and $f' \in L_\infty[a,b]$, then,*

$$\left|E(X) - \frac{a+b}{2}\right| \le \frac{1}{12}(b-a)^2 \|f'\|_\infty . \tag{4.32}$$

PROOF. If we put $g(t) = t$ in (4.31),

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b tf(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b tdt\right| \\ &\le \frac{1}{2\sqrt{3}}(b-a)\|f'\|_\infty \left[\frac{1}{b-a}\int_a^b t^2dt - \left(\frac{1}{b-a}\int_a^b tdt\right)^2\right]^{\frac{1}{2}} . \end{aligned} \tag{4.33}$$

However,

$$\frac{1}{b-a}\int_a^b t^2dt - \left(\frac{1}{b-a}\int_a^b tdt\right)^2 = \frac{(b-a)^2}{12}$$

and so (4.32) is true. ∎

REMARK 60. *We could obtain the same inequality by applying Čebyšev's inequality. Note, however, that for further results, the pre-Čebyšev inequality provides better estimates than would be obtained using the classical result.*

THEOREM 74. *([**21**]) Let X and f be as above. If*

$$\sigma_\mu(X) := \left[\int_a^b (t-\mu)^2 f(t)\,dt\right]^{\frac{1}{2}}, \quad \mu \in [a,b],$$

then,

$$\left|\sigma_\mu^2(X) - \left(\mu - \frac{a+b}{2}\right)^2 - \frac{1}{12}(b-a)^2\right| \tag{4.34}$$
$$\le \frac{1}{2\sqrt{3}}(b-a)^2\left[\frac{1}{3}\left(\mu-\frac{a+b}{2}\right)^2 + \frac{1}{180}(b-a)^2\right]\|f'\|_\infty$$
$$\le \frac{1}{3\sqrt{15}}(b-a)^3\|f'\|_\infty,$$

for all $\mu \in [a,b]$.

PROOF. If $g(t) = (t-\mu)^2$ in (4.31), then,

$$\left|\frac{1}{b-a}\int_a^b (t-\mu)^2 f(t)\,dt \right. \tag{4.35}$$
$$\left. - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b (t-\mu)^2\,dt\right|$$
$$\le \frac{1}{2\sqrt{3}}\|f'\|_\infty\left[\frac{1}{b-a}\int_a^b (t-\mu)^4\,dt - \left(\frac{1}{b-a}\int_a^b (t-\mu)^2\,dt\right)^2\right]^{\frac{1}{2}}.$$

However,

$$\frac{1}{b-a}\int_a^b (t-\mu)^2\,dt = \left(\mu-\frac{a+b}{2}\right)^2 + \frac{1}{12}(b-a)^2$$

and

$$\frac{1}{b-a}\int_a^b (t-\mu)^4\,dt - \left(\frac{1}{b-a}\int_a^b (t-\mu)^2\,dt\right)^2$$
$$= \frac{1}{5}\cdot\frac{(b-\mu)^5+(\mu-a)^5}{b-a} - \left[\frac{(b-\mu)^3+(\mu-a)^3}{3(b-a)}\right]^2$$
$$= \frac{1}{45}\Big[4\left[(b-\mu)^2-(\mu-a)^2\right]^2 + 2(b-\mu)^2(\mu-a)^2$$
$$+ (\mu-a)(b-\mu)\left[(b-\mu)^2+(\mu-a)^2\right]\Big]$$
$$:= A,$$

which simplifies further to give:

$$A = \frac{(b-a)^2}{45}\left[15\left(\mu - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]$$
$$= (b-a)^2\left[\frac{1}{3}\left(\mu - \frac{a+b}{2}\right)^2 + \frac{1}{180}(b-a)^2\right].$$

Using (4.35), we deduce the desired inequality (4.34). ∎

The best inequality we can obtain from (4.34) is that for which $\mu = \frac{a+b}{2}$, giving the following corollary (see also [**21**]).

COROLLARY 54. *With the above assumptions and denoting* $\sigma_0(X) := \sigma_{\frac{a+b}{2}}(X)$,

$$\left|\sigma_0^2(X) - \frac{(b-a)^2}{12}\right| \le \frac{1}{12\sqrt{15}}(b-a)^3 \|f'\|_\infty . \tag{4.36}$$

The following theorem provides an inequality that connects the expectation $E(X)$ and the cumulative distribution function $F(x) := \int_a^x f(t)\,dt$ of a random variable X having the PDF $f : [a,b] \to \mathbb{R}$ (see also [**21**]).

THEOREM 75. *Let X be a random variable whose PDF, $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ and $f' \in L_\infty[a,b]$, then,*

$$\left|E(X) + (b-a)F(x) - x - \frac{b-a}{2}\right| \le \frac{1}{12}(b-a)^3 \|f'\|_\infty \tag{4.37}$$

for all $x \in [a,b]$.

PROOF. We use the following equality established by Barnett and Dragomir in [**9**]

$$(b-a)F(x) + E(X) - b = \int_a^b p(x,t)\,dF(t) = \int_a^b p(x,t) f(t)\,dt, \tag{4.38}$$

where

$$p(x,t) := \begin{cases} t-a & \text{if} \quad a \le t \le x \le b \\ t-b & \text{if} \quad a \le x < t \le b \end{cases}.$$

Now, if we apply the inequality (4.31) for $g(t) = p(x,t)$, we obtain

$$\begin{aligned}(4.39)\quad & \left|\frac{1}{b-a}\int_a^b p(x,t) f(t)\,dt - \frac{1}{b-a}\int_a^b p(x,t)\,dt \cdot \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ & \leq \frac{1}{2\sqrt{3}}(b-a)\left\|f'\right\|_\infty \\ & \times \left[\frac{1}{b-a}\int_a^b p^2(x,t)\,dt - \left(\frac{1}{b-a}\int_a^b p(x,t)\,dt\right)^2\right]^{\frac{1}{2}}.\end{aligned}$$

Observe that

$$\frac{1}{b-a}\int_a^b p(x,t)\,dt = x - \frac{a+b}{2},$$

and

$$\begin{aligned} D &:= \frac{1}{b-a}\int_a^b p^2(x,t)\,dt - \left(\frac{1}{b-a}\int_a^b p(x,t)\,dt\right)^2 \\ &= \frac{1}{b-a}\left[\frac{(b-x)^3 + (x-a)^3}{3}\right] - \left(x - \frac{a+b}{2}\right)^2 \\ &= \frac{1}{12}(b-a)^2. \end{aligned}$$

Using (4.39), we deduce (4.37). ∎

Remark 61. *If in (4.37) either $x = a$ or $x = b$,*

$$\left|E(X) - \frac{a+b}{2}\right| \leq \frac{1}{12}(b-a)^3 \left\|f'\right\|_\infty,$$

which is inequality (4.32).

Remark 62. *If in (4.37) $x = \frac{a+b}{2}$, then*

$$(4.40)\quad \left|E(X) + (b-a)\Pr\left(X \leq \frac{a+b}{2}\right) - b\right| \leq \frac{1}{12}(b-a)^3 \left\|f'\right\|_\infty.$$

THEOREM 76. *([**21**]) Let X, F and f be as above, then,*

$$\begin{aligned}(4.41)\qquad &\left|E(X)+\frac{b-a}{2}F(x)-\frac{x+b}{2}\right| \\ &\leq \frac{1}{4}(b-a)\left\|f'\right\|_{\infty}\left[\left(x-\frac{a+b}{2}\right)^{2}+\frac{1}{12}(b-a)^{2}\right] \\ &\leq \frac{1}{12}(b-a)^{3}\left\|f'\right\|_{\infty}\end{aligned}$$

for all $x \in [a,b]$.

PROOF. Using the same identity of Barnett and Dragomir [**9**] as in Theorem 75 and applying the pre-Čebyšev inequality (4.31), for $x \in [a,b]$ we get:

$$\begin{aligned}(4.42)\qquad &\left|\frac{1}{x-a}\int_{a}^{x}(t-a)f(t)\,dt\right. \\ &\qquad\left.-\frac{1}{x-a}\int_{a}^{x}(t-a)\,dt\cdot\frac{1}{x-a}\int_{a}^{x}f(t)\,dt\right| \\ &\leq \frac{1}{2\sqrt{3}}(x-a)\left\|f'\right\|_{\infty} \\ &\qquad\times\left[\frac{1}{x-a}\int_{a}^{x}(t-a)^{2}\,dt-\left(\frac{1}{x-a}\int_{a}^{x}(t-a)\,dt\right)^{2}\right]^{\frac{1}{2}} \\ &= \frac{1}{12}(x-a)^{2}\left\|f'\right\|_{\infty}\end{aligned}$$

and, similarly,

$$\begin{aligned}(4.43)\qquad &\left|\frac{1}{b-x}\int_{x}^{b}(t-b)f(t)\,dt\right. \\ &\qquad\left.-\frac{1}{b-x}\int_{x}^{b}(t-b)\,dt\cdot\frac{1}{b-x}\int_{x}^{b}f(t)\,dt\right| \\ &\qquad\qquad\leq \frac{1}{12}(b-x)^{2}\left\|f'\right\|_{\infty},\end{aligned}$$

for all $x \in [a,b)$.

From (4.42) and (4.43) we can write

$$(4.44)\qquad \left|\int_{a}^{x}(t-a)f(t)\,dt-\frac{x-a}{2}F(x)\right| \leq \frac{1}{12}(x-a)^{3}\left\|f'\right\|_{\infty}$$

and

$$\left|\int_x^b (t-b) f(t)\, dt + \frac{b-x}{2}(1-F(x))\right| \leq \frac{1}{12}(b-x)^3 \|f'\|_\infty , \tag{4.45}$$

for all $x \in [a, b]$.

Summing (4.44) and (4.45) and using the triangle inequality, we deduce

$$\begin{aligned}
&\left|\int_a^x (t-a) f(t)\, dt + \int_x^b (t-b) f(t)\, dt - \frac{b-a}{2} F(x) + \frac{b-x}{2}\right| \\
&\leq \frac{1}{12}\|f'\|_\infty \left[(x-a)^3 + (b-x)^3\right] \\
&= \frac{1}{12}(b-a)\|f'\|_\infty \left[3\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] \\
&= \frac{1}{4}(b-a)\|f'\|_\infty \left[\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{12}(b-a)^2\right].
\end{aligned}$$

Using the identity (4.38), the desired result (4.41) is obtained. ∎

Remark 63. *If in (4.41) either $x = a$ or $x = b$, the inequality (4.32) is recaptured.*

Remark 64. *If in (4.41), $x = \frac{a+b}{2}$, then the best inequality of this type that can be obtained is:*

$$\left|E(X) + \frac{b-a}{2} \Pr\left(X \leq \frac{a+b}{2}\right) - \frac{a+3b}{4}\right| \leq \frac{1}{48}(b-a)^3 \|f'\|_\infty .$$

3. Some Elementary Inequalities

3.1. Introduction. Let X be a continuous random variable having the probability density function f defined on a finite interval $[a, b]$.

Using some tools from the theory of inequalities, namely Hölder's inequality, pre-Grüss inequality, pre-Čebyšev inequality, Taylor's formula with integral remainder, we point out some elementary inequalities linking the expectation and variance.

3.2. The Results. The following inequalities for the dispersion $\sigma(X)$ hold [**18**].

Theorem 77. *Let X be a continuous random variable defined on $[a, b]$ having PDF, f, then:*

(i) *we have the inequality*

$$0 \leq \sigma(X) \leq [b - E(X)]^{\frac{1}{2}} [E(X) - a]^{\frac{1}{2}} \leq \frac{1}{2}(b-a) \tag{4.46}$$

and

$$0 \leq [b - E(X)][E(X) - a] - \sigma^2(X) \leq \begin{cases} \dfrac{(b-a)^3}{6} \|f\|_\infty \\ \\ [B(q+1, q+1)]^{\frac{1}{q}} (b-a)^{2+\frac{1}{q}} \|f\|_p \\ \text{if } f \in L_p[a,b],\ p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \tag{4.47}$$

where $B(\cdot, \cdot)$ is Euler's Beta function.

(ii) *If $m \leq f \leq M$ a.e. on $[a, b]$, then*

$$\frac{m(b-a)^3}{6} \leq [b - E(X)][E(X) - a] - \sigma^2(X) \leq \frac{M(b-a)^3}{6} \tag{4.48}$$

and

$$\left| [b - E(X)][E(X) - a] - \sigma^2(X) - \frac{(b-a)^2}{6} \right| \leq \frac{\sqrt{5}(b-a)^3 (M-m)}{60}. \tag{4.49}$$

Proof. Note that:

$$\begin{aligned} &\int_a^b (b-t)(t-a) f(t)\, dt \\ &= \int_a^b [(b - E(X)) + (E(X) - t)] \\ &\quad \times [(E(X) - a) + (t - E(X))] f(t)\, dt \\ &= (b - E(X))(E(X) - a) \int_a^b f(t)\, dt \\ &\quad + (E(X) - a) \int_a^b (E(X) - t) f(t)\, dt \\ &\quad + (b - E(X)) \int_a^b (t - E(X)) f(t)\, dt \\ &\quad - \int_a^b (t - E(X))^2 f(t)\, dt \\ &= [b - E(X)][E(X) - a] - \sigma^2(X) \end{aligned} \tag{4.50}$$

since

$$\int_a^b f(t)\, dt = 1 \quad \text{and} \quad \int_a^b (t - E(X)) f(t)\, dt = 0.$$

(i) Using the fact that

$$\int_a^b (t-a)(b-t) f(t)\, dt \geq 0,$$

it follows that

$$\sigma^2(X) \leq [b - E(X)][E(X) - a]$$

and so the first inequality in (4.46) is established.

The second inequality in (4.46) follows from the elementary result that

$$\alpha\beta \leq \frac{1}{4}(\alpha+\beta)^2, \ \alpha, \beta \in \mathbb{R}$$

where $\alpha = b - E(X)$, $\beta = E(X) - a$.

The first inequality in (4.47) follows, since

$$\int_a^b (t-a)(b-t) f(t)\, dt \leq \|f\|_\infty \int_a^b (t-a)(b-t)\, dt = \frac{(b-a)^3}{6}\|f\|_\infty .$$

The second inequality is obvious by Hölder's integral inequality,

$$\int_a^b (t-a)(b-t) f(t)\, dt \leq \left(\int_a^b f^p(t)\, dt\right)^{\frac{1}{p}} \left(\int_a^b (t-a)^q (b-t)^q\, dt\right)^{\frac{1}{q}} = \|f\|_p (b-a)^{2+\frac{1}{q}} [B(q+1, q+1)]^{\frac{1}{q}} .$$

(ii) The inequality (4.48) is obvious, taking into account that if $m \leq f \leq M$ a.e. on $[a,b]$, then

$$m(t-a)(b-t) \leq (t-a)(b-t) f(t) \leq M(t-a)(b-t)$$

a.e. on $[a,b]$, and by integrating over $[a,b]$.

To prove (4.49), we use the following *"pre-Grüss"* inequality established in [**108**]

$$\left|\frac{1}{b-a}\int_a^b h(t) g(t)\, dt - \frac{1}{b-a}\int_a^b h(t)\, dt \cdot \frac{1}{b-a}\int_a^b g(t)\, dt\right| \leq \frac{1}{2}(\phi-\gamma)\left[\frac{1}{b-a}\int_a^b g^2(t)\, dt - \left(\frac{1}{b-a}\int_a^b g(t)\, dt\right)^2\right]^{\frac{1}{2}}, \tag{4.51}$$

provided that the mappings $h, g : [a, b] \to \mathbb{R}$ are measurable, all the integrals involved in (4.51) exist and are finite and $\gamma \leq h \leq \phi$ a.e. on $[a, b]$.
Choose in (4.51), $h(t) = f(t)$ and $g(t) = (t-a)(b-t)$, which then gives

$$\left|\frac{1}{b-a}\int_a^b (t-a)(b-t) f(t)\,dt - \frac{1}{b-a}\int_a^b (t-a)(b-t)\,dt \cdot \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{2}(M-m)\left[\frac{1}{b-a}\int_a^b (t-a)^2 (b-t)^2\,dt - \left(\frac{1}{b-a}\int_a^b (t-a)(b-t)\,dt\right)^2\right]^{\frac{1}{2}}. \tag{4.52}$$

However,

$$\int_a^b (t-a)(b-t)\,dt = \frac{(b-a)^3}{6}, \quad \int_a^b f(t)\,dt = 1,$$

$$\int_a^b (t-a)^2 (b-t)^2\,dt = (b-a)^5 \int_0^1 t^2 (1-t)^2\,dt = \frac{(b-a)^5}{30}$$

and

$$\frac{1}{b-a}\int_a^b (t-a)^2 (b-t)^2\,dt - \left(\frac{1}{b-a}\int_a^b (t-a)(b-t)\,dt\right)^2 = \frac{(b-a)^4}{180}.$$

Consequently, by (4.52), we deduce that

$$\left|\int_a^b (t-a)(b-t) f(t)\,dt - \frac{(b-a)^2}{6}\right| \leq \frac{1}{2}(b-a)(M-m)\left[\frac{(b-a)^4}{180}\right]^{\frac{1}{2}} = \frac{(b-a)^3 (M-m)}{12\sqrt{5}}.$$

Using (4.50), we deduce (4.49).

∎

With additional information about the derivative of f, we can state the following result which complements (4.49) (see also [**18**]).

THEOREM 78. *Assume that the PDF of X is absolutely continuous on $[a,b]$.*

(i) *If $f' \in L_\infty[a,b]$, then we have:*

$$\left|[b-E(X)][E(X)-a]-\sigma^2(X)-\frac{(b-a)^2}{6}\right| \leq \frac{\sqrt{30}}{720}\|f'\|_\infty (b-a)^3. \tag{4.53}$$

(ii) *If $f' \in L_2[a,b]$, then we have:*

$$\left|[b-E(X)][E(X)-a]-\sigma^2(X)-\frac{(b-a)^2}{6}\right| \leq \frac{\sqrt{5}}{60\pi}\|f'\|_2 (b-a)^3. \tag{4.54}$$

PROOF.

(i) Use is made of the "pre-Čebyšev" inequality proved in [**108**] and given in (4.31). Now, if we choose $h(t)=f(t)$, $g(t)=(t-a)(b-t)$ in (4.31), we get

$$\begin{aligned}&\left|\int_a^b (t-a)(b-t)f(t)\,dt-\frac{(b-a)^2}{6}\right|\\ &\leq \frac{\|h'\|_\infty (b-a)}{2\sqrt{3}}\cdot\frac{(b-a)^2}{12\sqrt{5}}\\ &=\frac{(b-a)^3\|h'\|_\infty}{24\sqrt{30}}.\end{aligned}$$

Using (4.50), we deduce (4.53).

(ii) For the second part of the theorem, we use the following "pre-Lupaş" inequality as stated in [**108**]

$$\left|\frac{1}{b-a}\int_a^b h(t)g(t)\,dt-\frac{1}{b-a}\int_a^b h(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt\right| \leq \frac{b-a}{\pi}\|h'\|_2\left[\frac{1}{b-a}\int_a^b g^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)^2\right]^{\frac{1}{2}}, \tag{4.55}$$

provided that g, h are as above and $h' \in L_2[a,b]$.
Now if we choose in (4.55) $h(t) = f(t)$, $g(t) = (t-a)(b-t)$, we obtain the desired inequality (4.54). The details are omitted.

∎

THEOREM 79. *([**18**]) Let X be a random variable and $f : [a,b] \to \mathbb{R}$ its PDF. If f is such that $f^{(n)}$ $(n \geq 0)$ is absolutely continuous on $[a,b]$, then we have the inequality*

$$\left| [E(X)-a][b-E(X)] - \sigma^2(X) - \sum_{k=0}^{n} \frac{(k+1)(b-a)^{k+3} f^{(k)}(a)}{(k+3)!} \right| \leq \begin{cases} \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!(n+3)(n+4)} (b-a)^{n+4} & \text{if } f^{(n+1)} \in L_\infty[a,b] \\ \frac{\left\|f^{(n+1)}\right\|_p (b-a)^{n+3+\frac{1}{q}}}{n!(nq+1)^{\frac{1}{q}}\left(n+2+\frac{1}{q}\right)\left(n+3+\frac{1}{q}\right)} & \text{if } f^{(n+1)} \in L_p[a,b],\ p>1 \\ \frac{\left\|f^{(n+1)}\right\|_1 (b-a)^{n+3}}{n!(n+2)(n+3)} & \text{if } f^{(n+1)} \in L_1[a,b]. \end{cases} \tag{4.56}$$

PROOF. The following Taylor's formula with integral remainder is well known in the literature (see for example [**4**]):

$$f(t) = \sum_{k=0}^{n} \frac{(t-a)^k}{k!} f^{(k)}(a) + \frac{1}{n!}\int_a^t (t-s)^n f^{(n+1)}(s)\,ds \tag{4.57}$$

for all $t \in [a,b]$.
Since

$$[E(X)-a][b-E(X)] - \sigma^2(X) = \int_a^b (t-a)(b-t) f(t)\,dt, \tag{4.58}$$

we have:

$$\begin{aligned} &[E(X)-a][b-E(X)] - \sigma^2(X) \\ &= \int_a^b (t-a)(b-t)\left[\sum_{k=0}^{n} \frac{(t-a)^k}{k!} f^{(k)}(a) \right. \\ &\quad \left. + \frac{1}{n!}\int_a^t (t-s)^n f^{(n+1)}(s)\,ds\right] dt \end{aligned} \tag{4.59}$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} \int_a^b (t-a)^{k+1} (b-t)\, dt + \frac{1}{n!} \int_a^b \left[(t-a)(b-t) \int_a^t (t-s)^n f^{(n+1)}(s)\, ds \right] dt.$$

Using the transformation, $t = (1-u)\,a + ub$, we have

$$\int_a^b (t-a)^{k+1} (b-t)\, dt = (b-a)^{k+3} \int_0^1 u^{k+1} (1-u)\, du = \frac{1}{(k+2)(k+3)}$$

and by (4.59), we deduce that

$$\left| [E(X)-a][b-E(X)] - \sigma^2(X) - \sum_{k=0}^{n} \frac{(k+1)(b-a)^{k+3} f^{(k)}(a)}{(k+3)!} \right| \leq \frac{1}{n!} \int_a^b (t-a)(b-t) \left| \int_a^t (t-s)^n f^{(n+1)}(s)\, ds \right| dt =: M(a,b).$$

However, for all $t \in [a,b]$ we have

$$\left| \int_a^t (t-s)^n f^{(n+1)}(s)\, ds \right| \leq \int_a^t |t-s|^n \left| f^{(n+1)}(s) \right| ds \leq \sup_{s\in[a,b]} \left| f^{(n+1)}(s) \right| \int_a^t (t-s)^n\, ds \leq \left\| f^{(n+1)} \right\|_\infty \frac{(t-a)^{n+1}}{n+1}.$$

By Hölder's integral inequality we have,

$$\left| \int_a^t (t-s)^n f^{(n+1)}(s)\, ds \right| \leq \left(\int_a^t \left| f^{(n+1)}(s) \right|^p ds \right)^{\frac{1}{p}} \left(\int_a^t (t-s)^{nq}\, ds \right)^{\frac{1}{q}} \leq \left\| f^{(n+1)} \right\|_p \left[\frac{(t-a)^{nq+1}}{nq+1} \right]^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1,\ p > 1$$

for all $t \in [a,b]$.

Finally, we observe that

$$\left|\int_a^t (t-s)^n f^{(n+1)}(s)\,ds\right| \le \int_a^t (t-s)^n \left|f^{(n+1)}(s)\right| ds$$
$$\le (t-a)^n \int_a^t \left|f^{(n+1)}(s)\right| ds$$
$$\le (t-a)^n \left\|f^{(n+1)}\right\|_1$$

for all $t \in [a,b]$.

Consequently,

$$M(a,b) \le \frac{1}{n!}\begin{cases} \frac{\|f^{(n+1)}\|_\infty}{n+1}\int_a^b (t-a)^{n+2}(b-t)\,dt \\ \frac{\|f^{(n+1)}\|_p}{(nq+1)^{\frac{1}{q}}}\int_a^b (t-a)^{n+1+\frac{1}{q}}(b-t)\,dt \\ \left\|f^{(n+1)}\right\|_1 \int_a^b (t-a)^{n+1}(b-t)\,dt \end{cases}$$

$$= \begin{cases} \frac{\|f^{(n+1)}\|_\infty}{n+1}(b-a)^{n+4}\int_0^1 u^{n+2}(1-u)\,du \\ \frac{\|f^{(n+1)}\|_p}{(nq+1)^{\frac{1}{q}}}(b-a)^{n+3+\frac{1}{q}}\int_0^1 u^{n+1+\frac{1}{q}}(1-u)\,du \\ \left\|f^{(n+1)}\right\|_1 (b-a)^{n+3}\int_0^1 u^{n+1}(1-u)\,du \end{cases}$$

and as

$$\int_0^1 u^{n+2}(1-u)\,du = \frac{1}{(n+3)(n+4)},$$
$$\int_0^1 u^{n+1+\frac{1}{q}}(1-u)\,du = \frac{1}{\left(n+2+\frac{1}{q}\right)\left(n+3+\frac{1}{q}\right)}$$

and

$$\int_0^1 u^{n+1}(1-u)\,du = \frac{1}{(n+2)(n+3)},$$

the inequality (4.56) is proved. ∎

Remark 65. *A similar result can be obtained if use is made of a Taylor expansion around the point b.*

4. On an Identity for the Čebyšev Functional

4.1. Introduction. For two measurable functions $f, g : [a,b] \to \mathbb{R}$, recall the Čebyšev functional,

$$T(f,g) := \mathcal{M}(fg) - \mathcal{M}(f)\,\mathcal{M}(g), \tag{4.60}$$

where the integral mean is given by

$$\mathcal{M}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{4.61}$$

The integrals in (4.60) are assumed to exist.

Further, the weighted Čebyšev functional is defined by

$$\mathfrak{T}(f,g;p) := \mathfrak{M}(f,g;p) - \mathfrak{M}(f;p)\,\mathfrak{M}(g;p), \tag{4.62}$$

where the weighted integral mean is given by

$$\mathfrak{M}(f;p) = \frac{\int_a^b p(x) f(x)\,dx}{\int_a^b p(x)\,dx}. \tag{4.63}$$

We note that,

$$\mathfrak{T}(f,g;1) \equiv T(f,g)$$

and

$$\mathfrak{M}(f;1) \equiv \mathcal{M}(f).$$

Here we obtain bounds on the functionals (4.60) and (4.62) in terms of one of the functions, say f, being of bounded variation, Lipschitzian or monotonic nondecreasing.

This is accomplished by developing identities involving a Riemann-Stieltjes integral. The main results are obtained in Section 4.2, while in Section 4.3 bounds for moments about a general point γ are obtained for functions of bounded variation, Lipschitzian and monotonic. Cerone and Dragomir [**32**] obtained bounds in terms of the $\|f'\|_p$, $p \geq 1$ where it necessitated the differentiability of the function f. There is no need for such assumptions in the work covered by the current development. A further application is given in Section 4.4 in which the moment generating function is approximated.

4.2. An Identity for the Čebyšev Functional. It is worthwhile noting that a number of identities relating to the Čebyšev functional already exist. The reader is referred to [**109**] Chapters IX and X. Korkine's identity is well known, see [**109**, p. 296] and is given by

$$T(f,g) = \frac{1}{2(b-a)^2}\int_a^b\int_a^b (f(x)-f(y))(g(x)-g(y))\,dxdy. \tag{4.64}$$

It is identity (4.64) that is often used to prove an inequality of Grüss for functions bounded above and below, [**109**].

The Grüss inequality is given by

$$|T(f,g)| \leq \frac{1}{4}\left(\Phi_f - \phi_f\right)\left(\Phi_g - \phi_g\right) \tag{4.65}$$

where $\phi_f \leq f(x) \leq \Phi_f$ for $x \in [a,b]$.

If we let $S(f)$ be an operator defined by

$$S(f)(x) := f(x) - \mathcal{M}(f), \tag{4.66}$$

which shifts a function by its integral mean, then the following identity holds. Namely,

$$T(f,g) = T(S(f),g) = T(f,S(g)) = T(S(f),S(g)), \tag{4.67}$$

and so

$$T(f,g) = \mathcal{M}(S(f)g) = \mathcal{M}(fS(g)) = \mathcal{M}(S(f)S(g)) \tag{4.68}$$

since $\mathcal{M}(S(f)) = \mathcal{M}(S(g)) = 0$.

For the last term in (4.67) or (4.68) only one of the functions needs to be shifted by its integral mean. If the other is shifted by any other quantity, the identities still hold. A weighted version of (4.68) related to $\mathfrak{T}(f,g) = \mathcal{M}((f(x)-\kappa)S(g))$ for κ arbitrary was given by Sonin [**118**] (see [**109**, p. 246]).

The interested reader is also referred to Dragomir [**67**] and Fink [**95**] for extensive treatments of the Grüss and related inequalities.

The following lemma presents an identity for the Čebyšev functional that involves a Riemann-Stieltjes integral which was first introduced by Cerone in [**29**].

Lemma 14. *Let $f,g:[a,b]\to\mathbb{R}$, where f is of bounded variation and g is continuous on $[a,b]$, then*

$$T(f,g) = \frac{1}{(b-a)^2}\int_a^b \psi(t)\,df(t), \tag{4.69}$$

where

$$\psi(t) = (t-a)A(t,b) - (b-t)A(a,t) \tag{4.70}$$

with

$$A(a,b) = \int_a^b g(x)\,dx. \tag{4.71}$$

Proof. From (4.69) integrating the Riemann-Stieltjes integral by parts produces

$$\begin{aligned}
&\frac{1}{(b-a)^2}\int_a^b \psi(t)\,df(t)\\
&\qquad = \frac{1}{(b-a)^2}\left\{\psi(t)f(t)\Big]_a^b - \int_a^b f(t)\,d\psi(t)\right\}\\
&\qquad = \frac{1}{(b-a)^2}\left\{\psi(b)f(b) - \psi(a)f(a) - \int_a^b f(t)\psi'(t)\,dt\right\}
\end{aligned}$$

since $\psi(t)$ is differentiable. Thus, from (4.70), $\psi(a) = \psi(b) = 0$ and so

$$\begin{aligned}
\frac{1}{(b-a)^2}\int_a^b \psi(t)\,df(t) &= \frac{1}{(b-a)^2}\int_a^b [(b-a)\,g(t) - A(a,b)]\,f(t)\,dt \\
&= \frac{1}{b-a}\int_a^b [g(t) - \mathcal{M}(g)]\,f(t)\,dt \\
&= \mathcal{M}(fS(g))
\end{aligned}$$

from which the result (4.69) is obtained on noting identity (4.68). ∎

The following well known lemmas will prove useful and are stated here for lucidity.

Lemma 15. *Let $g, v : [a,b] \to \mathbb{R}$ be such that g is continuous and v is of bounded variation on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b g(t)\,dv(t)$ exists and is such that*

$$\left|\int_a^b g(t)\,dv(t)\right| \le \sup_{t\in[a,b]} |g(t)| \bigvee_a^b (v)\,. \tag{4.72}$$

Lemma 16. *Let $g, v : [a,b] \to \mathbb{R}$ be such that g is Riemann-integrable on $[a,b]$ and v is $L-$Lipschitzian on $[a,b]$, then*

$$\left|\int_a^b g(t)\,dv(t)\right| \le L\int_a^b |g(t)|\,dt \tag{4.73}$$

with v $L-$Lipschitzian if it satisfies

$$|v(x) - v(y)| \le L\,|x - y|$$

for all $x, y \in [a,b]$.

Lemma 17. *Let $g, v : [a,b] \to \mathbb{R}$ be such that g is continuous on $[a,b]$ and v is monotonic nondecreasing on $[a,b]$, then*

$$\left|\int_a^b g(t)\,dv(t)\right| \le \int_a^b |g(t)|\,dv(t)\,. \tag{4.74}$$

It should be noted that if v is nonincreasing then $-v$ is nondecreasing.

THEOREM 80. *Let $f, g : [a, b] \to \mathbb{R}$, where f is of bounded variation and g is continuous on $[a, b]$, then*

$$(b-a)^2 |T(f,g)| \leq \begin{cases} \sup_{t\in[a,b]} |\psi(t)| \bigvee_a^b (f), \\ L \int_a^b |\psi(t)|\, dt, \quad \text{for } f \ L-\text{Lipschitzian}, \\ \int_a^b |\psi(t)|\, df(t), \ \text{for } f \text{ monotonic nondecreasing.} \end{cases} \tag{4.75}$$

PROOF. Follows directly from Lemmas 14 – 17, that is, from the identity (4.69) and (4.72) – (4.74). ∎

The following lemma gives an identity for the weighted Chebychev functional that involves a Riemann-Stieltjes integral [**29**].

LEMMA 18. *Let $f, g, p : [a, b] \to \mathbb{R}$, where f is of bounded variation and g, p are continuous on $[a, b]$. Further, let $P(b) = \int_a^b p(x)\, dx > 0$, then*

$$\mathfrak{T}(f, g; p) = \frac{1}{P^2(b)} \int_a^b \Psi(t)\, df(t), \tag{4.76}$$

where $\mathfrak{T}(f, g; p)$ is as given in (4.62),

$$\Psi(t) = P(t)\bar{G}(t) - \bar{P}(t) G(t) \tag{4.77}$$

with

$$\begin{cases} P(t) = \int_a^t p(x)\, dx, & \bar{P}(t) = P(b) - P(t) \\ \text{and} & \\ G(t) = \int_a^t p(x) g(x)\, dx, & \bar{G}(t) = G(b) - G(t). \end{cases} \tag{4.78}$$

PROOF. The proof follows closely that of Lemma 14.

We first note that $\Psi(t)$ may be represented in terms of only $P(\cdot)$ and $G(\cdot)$, namely,

$$\Psi(t) = P(t) G(b) - P(b) G(t). \tag{4.79}$$

It may further be noticed that $\Psi(a) = \Psi(b) = 0$. Thus, integrating from (4.76) and using either (4.77) or (4.79) gives

$$\begin{aligned}
&\frac{1}{P^2(b)}\int_a^b \Psi(t)\,df(t) \\
&= \frac{-1}{P^2(b)}\int_a^b f(t)\,d\Psi(t) \\
&= \frac{1}{P^2(b)}\int_a^b \left[P(b)\,G'(t) - P'(t)\,G(b)\right] f(t)\,dt \\
&= \frac{1}{P(b)}\int_a^b \left[p(t)\,g(t) - \frac{G(b)}{P(b)}p(t)\right] f(t)\,dt \\
&= \frac{1}{P(b)}\int_a^b p(t)\,g(t)\,f(t)\,dt - \frac{G(b)}{P(b)}\cdot\frac{1}{P(b)}\int_a^b p(t)\,f(t)\,dt \\
&= \mathfrak{M}(f,g;p) - \mathfrak{M}(g;p)\,\mathfrak{M}(f;p) = \mathfrak{T}(f,g;p),
\end{aligned}$$

where we have used the fact that $\frac{G(b)}{P(b)} = \mathfrak{M}(g;p)$. ∎

THEOREM 81. *([**29**]). Let the conditions of Lemma 18 on f, g and p continue to hold, then*

$$(4.80) \qquad P^2(b)\,|\mathfrak{T}(f,g;p)| \le \begin{cases} \sup_{t\in[a,b]} |\Psi(t)| \bigvee_a^b (f), & \\ L\int_a^b |\Psi(t)|\,dt, & \text{for } f\ L-\text{Lipschitzian}, \\ \int_a^b |\Psi(t)|\,df(t), & \text{for } f \text{ monotonic nondecreasing.} \end{cases}$$

where $\mathfrak{T}(f,g;p)$ is as given by (4.62) and $\Psi(t) = P(t)\,G(b) - P(b)\,G(t)$, with $P(t) = \int_a^t p(x)\,dx$, $G(t) = \int_a^t p(x)\,g(x)\,dx$.

PROOF. The proof uses Lemmas 14 – 17 and follows closely that of Theorem 80. ∎

REMARK 66. *If we take $p(x) \equiv 1$ in the above results involving the weighted Chebychev functional, then the results obtained earlier for the unweighted Chebychev functional are recaptured.*

Grüss type inequalities obtained from bounds on the Chebychev functional have been applied in a variety of areas including obtaining perturbed rules in numerical integration, see for example [**34**]. In the following section the above work is applied to the approximation of moments. For other related results see also [**18**] and [**33**].

Remark 67. *If f is differentiable then the identity (4.69) becomes*

$$T(f,g)=\frac{1}{(b-a)^2}\int_a^b \psi(t)\,f'(t)\,dt \tag{4.81}$$

and so

$$(b-a)^2\,|T(f,g)| \le \begin{cases} \|\psi\|_1\,\|f'\|_\infty\,, & f'\in L_\infty[a,b]\,;\\ \|\psi\|_q\,\|f'\|_p\,, & f'\in L_p[a,b]\,,\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \|\psi\|_\infty\,\|f'\|_1\,, & f'\in L_1[a,b]\,; \end{cases}$$

where the Lebesgue norms $\|\cdot\|$ are defined in the usual way.

The identity for the weighted integral means (4.76) and the corresponding bounds (4.80) will not be examined further here [**29**].

Theorem 82. *Let $g:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$ then for*

$$D(g;a,t,b):=\mathcal{M}(g;t,b)-\mathcal{M}(g;a,t)\,, \tag{4.82}$$

$$|D(g;a,t,b)| \le \begin{cases} \left(\frac{b-a}{2}\right)\|g'\|_\infty\,, & g'\in L_\infty[a,b]\,;\\ \left[\frac{(t-a)^q+(b-t)^q}{q+1}\right]^{\frac{1}{q}}\|g'\|_p\,, & g'\in L_p[a,b]\,,\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \|g'\|_1\,, & g'\in L_1[a,b]\,;\\ \bigvee_a^b(g)\,, & g \text{ of bounded variation;}\\ \left(\frac{b-a}{2}\right)L, & g \text{ is } L-\text{Lipschitzian.} \end{cases} \tag{4.83}$$

Proof. Let the kernel $r(t,u)$ be defined by

$$r(t,u):=\begin{cases} \dfrac{u-a}{t-a}, & u\in[a,t]\,,\\[2ex] \dfrac{b-u}{b-t}, & u\in(t,b] \end{cases} \tag{4.84}$$

then a straight forward integration by parts argument of the Riemann-Stieltjes integral over each of the intervals $[a,t]$ and $(t,b]$ gives the identity

$$\int_a^b r(t,u)\,dg(u)=D(g;a,t,b)\,. \tag{4.85}$$

Now for g absolutely continuous then

$$D(g;a,t,b)=\int_a^b r(t,u)\,g'(u)\,du \tag{4.86}$$

and so

$$|D(g;a,t,b)|\le ess\sup_{u\in[a,b]}|r(t,u)|\int_a^b |g'(u)|\,du,\quad \text{for}\ \ g'\in L_1[a,b],$$

where from (4.84)

$$ess\sup_{u\in[a,b]}|r(t,u)|=1 \tag{4.87}$$

and so the third inequality in (4.83) results. Further, using the Hölder inequality gives

$$|D(g;a,t,b)|\le\left(\int_a^b |r(t,u)|^q\,du\right)^{\frac{1}{q}}\left(\int_a^b |g'(t)|^p\,dt\right)^{\frac{1}{p}} \tag{4.88}$$
$$\text{for}\ \ p>1,\ \frac{1}{p}+\frac{1}{q}=1,$$

where explicitly from (4.84)

$$\begin{aligned}
&\left(\int_a^b |r(t,u)|^q\,du\right)^{\frac{1}{q}} \\
&=\left[\int_a^t\left(\frac{u-a}{t-a}\right)^q du+\int_t^b\left(\frac{b-u}{b-t}\right)^q du\right]^{\frac{1}{q}} \\
&=\left[(t-a)^q+(b-t)^q\right]^{\frac{1}{q}}\left(\int_0^1 u^q du\right)^{\frac{1}{q}} \\
&=\left[\frac{(t-a)^q+(b-t)^q}{q+1}\right]^{\frac{1}{q}}.
\end{aligned} \tag{4.89}$$

Also,

$$|D(g;a,t,b)|\le ess\sup_{u\in[a,b]}|g'(u)|\int_a^b |r(t,u)|\,du, \tag{4.90}$$

and so from (4.89) with $q=1$ we get the first inequality in (4.83).

Now, for $g(u)$ of bounded variation on $[a,b]$ then from Lemma 15, equation (4.72) and identity (4.85) we have

$$|D(g;a,t,b)|\le ess\sup_{u\in[a,b]}|r(t,u)|\bigvee_a^b(g)$$

producing the fourth inequality in (4.83) on using (4.87). From (4.73) and (4.85) we have, by associating g with v and $r(t,\cdot)$ with $g(\cdot)$,

$$|D(g;a,t,b)| \le L\int_a^b |r(t,u)|\,du$$

and so this, from (4.89) with $q=1$, gives the final inequality in (4.83). ∎

REMARK 68. *The results of Theorem 82 may be used to obtain bounds on $\psi(t)$ since from (4.70) and (4.82)*

$$\psi(t) = (t-a)(b-t)\,D(g;a,t,b).$$

Hence, upper bounds on the Čebyšev functional may be obtained from (4.75) and (4.81) for general functions g. The following two sections investigate the exact evaluation (4.75) for specific functions of $g(\cdot)$.

4.3. Results Involving Moments. In this section bounds on n^{th} moments about a point γ are investigated. Define for n a nonnegative integer,

$$M_n(\gamma) := \int_a^b (x-\gamma)^n h(x)\,dx,\quad \gamma\in\mathbb{R}. \tag{4.91}$$

If $\gamma=0$ then $M_n(0)$ are the moments about the origin and $\gamma=M_1(0)$ are the central moments. Further, the expectation of a continuous random variable can be written as $E(X)=M_1(0)$. Also, the variance of the random variable X, $\sigma^2(X)=M_2(M_1(0))$.

The following corollary is valid [**29**].

COROLLARY 55. *Let $f:[a,b]\to\mathbb{R}$ be integrable on $[a,b]$, then*

$$\left|M_n(\gamma) - \frac{B^{n+1}-A^{n+1}}{n+1}\mathcal{M}(f)\right| \tag{4.92}$$
$$\le \begin{cases} \sup\limits_{t\in[a,b]}|\phi(t)|\cdot\frac{1}{n+1}\bigvee\limits_a^b(f), & \text{for } f \text{ of bounded variation on } [a,b],\\ \dfrac{L}{n+1}\int_a^b|\phi(t)|\,dt, & \text{for } f\ L-\text{Lipschitzian,}\\ \dfrac{1}{n+1}\int_a^b|\phi(t)|\,df(t), & \text{for } f \text{ monotonic nondecreasing.}\end{cases}$$

where $M_n(\gamma)$ is as given by (4.91), $\mathcal{M}(f)$ is the integral mean of f as defined in (4.61),

$$B=b-\gamma,\quad A=a-\gamma$$

and

$$\phi(t) = (t-\gamma)^n - \left[\left(\frac{t-a}{b-a} \right) (b-\gamma)^{n+1} + \left(\frac{b-t}{b-a} \right) (a-\gamma)^{n+1} \right]. \tag{4.93}$$

PROOF. From (4.75) taking $g(t) = (t-\gamma)^n$ and using (4.60) and (4.61) gives

$$(b-a) |T(f, (t-\gamma)^n)| = \left| M_n(\gamma) - \frac{B^{n+1} - A^{n+1}}{n+1} \mathcal{M}(f) \right|.$$

The right hand side is obtained on noting that for $g(t) = (t-\gamma)^n$, $\phi(t) = -\frac{\psi(t)}{b-a}$. ∎

REMARK 69. *It should be noted here that Cerone and Dragomir* [**32**] *obtained bounds on the left hand expression for $f' \in L_p[a,b]$, $p \geq 1$. They obtained the following Lemmas which are useful in procuring expressions for the bounds in (4.92) in more explicit form* [**29**].

LEMMA 19. *Let $\phi(t)$ be as defined by (4.93), then*

$$\phi(t) \begin{cases} < 0 & \begin{cases} n \text{ odd}, & \text{any } \gamma \text{ and } t \in (a,b) \\ n \text{ even} & \begin{cases} \gamma < a, & t \in (a,b) \\ a < \gamma < b, & t \in [c,b) \end{cases} \end{cases} \\ > 0, & n \text{ even} \begin{cases} \gamma > b, & t \in (a,b) \\ a < \gamma < b, & t \in (a,c) \end{cases} \end{cases} \tag{4.94}$$

where $\phi(c) = 0$, $a < c < b$ and

$$c \begin{cases} > \gamma, & \gamma < \frac{a+b}{2} \\ = \gamma, & \gamma = \frac{a+b}{2} \\ < \gamma, & \gamma > \frac{a+b}{2}. \end{cases}$$

Lemma 20. *([**29**]). For $\phi(t)$ as given by (4.93) then*

$$\int_a^b |\phi(t)|\,dt \tag{4.95}$$

$$= \begin{cases} \frac{B-A}{2}\left[B^{n+1}-A^{n+1}\right]-\frac{B^{n+2}-A^{n+2}}{n+2}, & \begin{cases} n \text{ odd and any } \gamma \\ n \text{ even and } \gamma < a \end{cases}; \\ \frac{2C^{n+2}-B^{n+2}-A^{n+2}}{n+2}+\frac{1}{2(b-a)}\left\{\left[(b-a)^2-2(c-a)^2\right]B^{n+1}\right. & \\ \qquad \left.+\left[2(b-c)^2-(b-a)^2\right]\right\}A^{n+1}, & n \text{ even and } a<\gamma<b; \\ \frac{B^{n+2}-A^{n+2}}{n+2}-\frac{B-A}{2}\left[B^{n+1}-A^{n+1}\right], & n \text{ even and } \gamma > b, \end{cases}$$

where

$$\begin{cases} B=b-\gamma,\ A=a-\gamma,\ C=c-\gamma, \\ C_1=\int_a^c C(t)\,dt,\quad C_2=\int_c^b C(t)\,dt, \\ \text{with}\quad C(t)=\left(\frac{t-a}{b-a}\right)B^{n+1}+\left(\frac{b-t}{b-a}\right)A^{n+1} \end{cases} \tag{4.96}$$

and $\phi(c)=0$ with $a<c<b$.

Lemma 21. *([**29**]). For $\phi(t)$ as defined by (4.93), then*

$$\sup_{t\in[a,b]}\left|\tilde{\phi}(t)\right| \tag{4.97}$$

$$= \begin{cases} C(t^*)-\frac{B^{n+1}-A^{n+1}}{(n+1)(B-A)}, & n \text{ odd, } n \text{ even and } \gamma<a; \\ \frac{B^{n+1}-A^{n+1}}{(n+1)(B-A)}-C(t^*) & n \text{ even and } \gamma>b; \\ \frac{m_1+m_2}{2}+\left|\frac{m_1-m_2}{2}\right| & n \text{ even and } a<\gamma<b, \end{cases}$$

where

$$(t^*-\gamma)^n=\frac{B^{n+1}-A^{n+1}}{(n+1)(B-A)}, \tag{4.98}$$

$C(t)$ is as defined in (4.96), $m_1=\tilde{\phi}(t_1^*)$, $m_2=-\tilde{\phi}(t_2^*)$ and t^*, t_1^*, t_2^* satisfy (4.98) with $t_1^*<t_2^*$.

The following lemma is required to determine the bound in (4.92) when f is monotonic nondecreasing (Cerone and Dragomir [**32**] obtained bounds assuming that f was differentiable).

Lemma 22. *The following result holds for $\phi(t)$ as defined by (4.93),*

$$\frac{1}{n+1}\int_a^b |\phi(t)|\,df \tag{4.99}$$
$$= \begin{cases} \chi_n(a,b), & n \text{ odd or } n \text{ even and } \gamma < a, \\ -\chi_n(a,b), & n \text{ even and } \gamma > b, \\ \chi_n(c,b) - \chi_n(a,c), & n \text{ even and } a < \gamma < b \end{cases}$$

and for $f : [a,b] \to \mathbb{R}$, monotonic nondecreasing,

$$\frac{1}{n+1}\int_a^b |\phi(t)|\,df \tag{4.100}$$
$$\leq \begin{cases} \frac{B(B^n-1)-A(A^n-1)}{n+1} f(b), & n \text{ odd or } n \text{ even and } \gamma < a; \\ \frac{A(A^n-1)-B(B^n-1)}{n+1} f(b), & n \text{ even and } \gamma > b; \\ \left[B^{n+1} - C^{n+1} - \frac{(B^n-A^n)}{b-a}(b-c)\right]\frac{f(b)}{n+1} & n \text{ even and} \\ + \left[\frac{(B^n-A^n)}{b-a}(c-a) - \left(C^{n+1} - A^{n+1}\right)\right]\frac{f(a)}{n+1}, & a < \gamma < b, \end{cases}$$

where

$$\chi_n(a,b) = \int_a^b \left[(t-\gamma)^n - \frac{(B^n - A^n)}{(n+1)(b-a)}\right] f(t)\,dt, \tag{4.101}$$
$$A = a - \gamma, \quad B = b - \gamma, \quad C = c - \gamma.$$

Proof. Let $\alpha, \beta \in [a,b]$ and

$$\begin{aligned} \chi_n(\alpha,\beta) &= \frac{1}{n+1}\int_\alpha^\beta |\phi(t)|\,df \\ &= \frac{\phi(\alpha) f(\alpha) - \phi(\beta) f(\beta)}{n+1} \\ &\quad - \int_\alpha^\beta \left[(t-\gamma)^n - \frac{(B^n - A^n)}{(n+1)(b-a)}\right] f(t)\,dt \end{aligned}$$

and $\chi_n(a,b)$ be as given by (4.101) since $\phi(a) = \phi(b) = 0$.

Further, using the results of Lemma 19 as represented in (4.94), and, the fact that

$$\frac{1}{n+1}\int_\alpha^\beta |\phi(t)|\,df = \begin{cases} \chi(\alpha,\beta), & \phi(t) < 0,\ t \in [\alpha,\beta] \\ -\chi(\alpha,\beta), & \phi(t) > 0,\ t \in [\alpha,\beta] \end{cases}$$

gives the results as stated.

We now use the fact that f is monotonic nondecreasing so that from (4.101)

$$\chi_n(a,b) \leq f(b) \int_a^b \left[(t-\gamma)^n - \frac{B^n - A^n}{(n+1)(b-a)} \right] dt.$$

Further,

$$\begin{aligned} \chi_n(c,b) &\leq f(b) \int_c^b \left[(t-\gamma)^n - \frac{B^n - A^n}{(n+1)(b-a)} \right] dt \\ &= f(b) \left[\frac{B^{n+1} - C^{n+1}}{n+1} - \frac{(B^n - A^n)(b-c)}{(n+1)(b-a)} \right] \end{aligned}$$

and

$$\begin{aligned} \chi_n(a,c) &\geq f(a) \int_a^c \left[(t-\gamma)^n - \frac{B^n - A^n}{(n+1)(b-a)} \right] dt \\ &= \left[\frac{C^{n+1} - A^{n+1}}{n+1} - \frac{(B^n - A^n)(c-a)}{(n+1)(b-a)} \right] f(a) \end{aligned}$$

so the proof of the lemma is complete. ∎

The following corollary gives bounds for the expectation [**29**].

Corollary 56. *Let $f : [a,b] \to \mathbb{R}_+$ be a probability density function associated with a random variable X, then the expectation $E(X)$ satisfies the inequalities*

$$\left| E(X) - \frac{a+b}{2} \right| \leq \begin{cases} \frac{(b-a)^3}{6} \bigvee_a^b (f), & f \text{ of bounded variation,} \\ \left(\frac{b-a}{2}\right)^2 \cdot \frac{L}{2}, & f \ L - \text{Lipschitzian,} \\ \frac{b-a}{2} [a+b-1] f(b), & f \text{ monotonic nondecreasing.} \end{cases} \tag{4.102}$$

Proof. Taking $n = 1$ in Corollary 55 and using Lemmas 19 – 22 gives the results after some straightforward algebra. In particular,

$$\phi(t) = t^2 - (a+b)t + ab = \left(t - \frac{a+b}{2}\right)^2 + \left(\frac{b-a}{2}\right)^2$$

and t^* the one solution of $\phi'(t) = 0$ is $t^* = \frac{a+b}{2}$. ∎

The following corollary gives bounds for the variance.

We shall assume that $a < \gamma = E[X] < b$.

Corollary 57. *([**29**]). Let $f : [a, b] \to \mathbb{R}_+$ be the PDF associated with a random variable X. The variance $\sigma^2(X)$ is such that*

$$
(4.103)\quad \left|\sigma^2(X) - S\right| \leq \begin{cases} [m_1 + m_2 + |m_2 - m_1|] \frac{\bigvee_a^b(f)}{6}, & f \text{ of bounded variation,} \\ \left\{\frac{C^2}{4} - \frac{1}{b-a}\left[(c-a)^3 B^3 - (b-c)^2 A^3\right] \right. \\ \left. \quad + \left(B^2 + A^2\right)\left(\frac{b-a}{2}\right)^2 - \frac{(AB)^2}{2}\right\} \cdot \frac{L}{3}, & f \text{ is } L - \text{Lipschitzian,} \\ \left[B^3 - C^3 - (a+b)(b-c)\right] \frac{f(b)}{3} \\ \quad + \left[(a+b)(c-a) - \left(C^3 - A^3\right)\right] \frac{f(a)}{3}, & f \text{ monotonic nondecreasing.} \end{cases}
$$

where

$$
\begin{aligned}
S &= \frac{(b - E(X))^3 + (E(X) - a)^3}{3(b-a)}, \\
m_1 &= \phi\left(E(X) - S^{\frac{1}{2}}\right), \quad m_2 = \phi\left(E(X) + S^{\frac{1}{2}}\right), \\
\phi(t) &= (t-\gamma)^3 + \left(\frac{b-t}{b-a}\right)(\gamma - a)^3 - \left(\frac{t-a}{b-a}\right)(b-\gamma)^3, \\
A &= a - \gamma, \quad B = b - \gamma, \quad C = c - \gamma, \quad \phi(c) = 0, \ a < c < b
\end{aligned}
$$

and $\gamma = E(X)$.

Proof. Taking $n = 2$ in Corollary 55 gives, from (4.93),

$$
\phi(t) = (t-\gamma)^3 + \left(\frac{b-t}{b-a}\right) A^3 - \left(\frac{t-a}{b-a}\right) B^3
$$

where $a < \gamma = E(X) < b$.

From Lemma 21 and the third inequality in (4.97) with $n = 2$ we have,

$$
t_1^* = E[X] - S^{\frac{1}{2}}, \ \ t_2^* = E[X] + S^{\frac{1}{2}},
$$

and hence the first inequality is shown from the first inequality of (4.92).

Now, if f is Lipschitzian, then from the second inequality of (4.92) with $n = 2$ and $a < \gamma = E(X) < b$, the second identity in (4.95) produces the reported result given in (4.103) after some simplification.

The last inequality is obtained from (4.100) of Lemma 22 with $n = 2$ and hence the corollary is proved. ∎

4.4. Approximations for the Moment Generating Function. Let X be a random variable on $[a, b]$ with probability density function $h(x)$ then the moment generating function $M_X(p)$ is given by

$$M_X(p) = E\left[e^{pX}\right] = \int_a^b e^{px} h(x)\, dx. \tag{4.104}$$

The following lemma will prove useful in the proof of the subsequent corollary, as it examines the behaviour of the function $\theta(t)$

$$(b-a)\,\theta(t) = tA_p(a,b) - \left[aA_p(t,b) + bA_p(a,t)\right], \tag{4.105}$$

where

$$A_p(a,b) = \frac{e^{bp} - e^{ap}}{p}. \tag{4.106}$$

Lemma 23. *([**29**]). Let $\theta(t)$ be as defined by (4.105) and (4.106) then for any $a, b \in \mathbb{R}$, $\theta(t)$ has the following characteristics:*

(i) $\theta(a) = \theta(b) = 0$,
(ii) *$\theta(t)$ is convex for $p < 0$ and concave for $p > 0$,*
(iii) *there is one turning point at $t^* = \frac{1}{p}\ln\left(\frac{A_p(a,b)}{b-a}\right)$ and $a \le t^* \le b$.*

Proof. The result (i) is trivial from (4.105) using standard properties of the definite integral to give $\theta(a) = \theta(b) = 0$.

Now,

$$\theta'(t) = \frac{A_p(a,b)}{b-a} - e^{pt}, \quad \theta''(t) = -pe^{pt} \tag{4.107}$$

giving $\theta''(t) > 0$ for $p < 0$ and $\theta''(t) < 0$ for $p > 0$ and (ii) holds.

Further, from (4.107), $\theta'(t^*) = 0$ where

$$t^* = \frac{1}{p}\ln\left(\frac{A_p(a,b)}{b-a}\right).$$

To show that $a \le t^* \le b$ it suffices to show that

$$\theta'(a)\,\theta'(b) < 0$$

since the exponential is continuous. Here $\theta'(a)$ is the right derivative at a and $\theta'(b)$ is the left derivative at b.

Now,

$$\theta'(a)\,\theta'(b) = \left(\frac{A_p(a,b)}{b-a} - e^{ap}\right)\left(\frac{A_p(a,b)}{b-a} - e^{bp}\right)$$

but

$$\frac{A_p(a,b)}{b-a} = \frac{1}{b-a}\int_a^b e^{pt}\,dt,$$

the integral mean over $[a,b]$ so that $\theta'(a) > 0$, and $\theta'(b) < 0$ for $p > 0$ and $\theta'(a) < 0$ and $\theta'(b) > 0$ for $p < 0$, giving $t^* \in [a,b]$ where $\theta(t^*) = 0$.

Thus the lemma is now completely proved. ∎

Corollary 58. *([29]). Let $f : [a,b] \to \mathbb{R}$ be of bounded variation on $[a,b]$ then*

$$\left| \int_a^b e^{pt} f(t)\,dt - A_p(a,b)\,\mathcal{M}(f) \right| \tag{4.108}$$
$$\leq \begin{cases} \left(m(\ln(m) - 1) + \dfrac{be^{ap} - ae^{bp}}{b-a} \right) \frac{\bigvee_a^b(f)}{|p|}, \\ (b-a)\,m\left[\left(\frac{b-a}{2}\right)p - 1\right]\frac{L}{|p|} \quad \textit{for } f \ L-\textit{Lipschitzian on } [a,b], \\ \frac{p}{|p|}(b-a)\,m\,[f(b) - f(a)], \quad f \quad \textit{monotonic nondecreasing,} \end{cases}$$

where

$$m = \frac{A_p(a,b)}{b-a} = \frac{e^{bp} - e^{ap}}{p(b-a)}. \tag{4.109}$$

Proof. From (4.75) taking $g(t) = e^{pt}$ and using (4.60) and (4.61) gives

$$(b-a)\left|T\left(f, e^{pt}\right)\right| = \left| \int_a^b e^{pt} f(t)\,dt - A_p(a,b)\,\mathcal{M}(f) \right| \tag{4.110}$$

$$\leq \begin{cases} \sup\limits_{t\in[a,b]} |\theta(t)| \bigvee_a^b(f), & \text{for } f \text{ of bounded variation on } [a,b], \\ L\int_a^b |\theta(t)|\,dt, & \text{for } f \ L-\text{Lipschitzian on } [a,b], \\ \int_a^b |\theta(t)|\,df(t), & f \ \text{ monotonic nondecreasing on } [a,b], \end{cases}$$

where the bounds are obtained from (4.75) on noting that for $g(t) = e^{pt}$, $\theta(t) = \frac{\psi(t)}{b-a}$ is as given by (4.105) – (4.106).

The properties of $\theta(t)$, expounded in Lemma 23, aid in obtaining explicit bounds from (4.110).

First, from (4.105), (4.106) and (4.109)

$$\begin{aligned}\sup_{t\in[a,b]}|\theta(t)| &= |\theta(t^*)| \\ &= \left|t^*m - \left[a\frac{A_p(t^*,b)}{b-a} + b\frac{A_p(a,t^*)}{b-a}\right]\right| \\ &= \left|\frac{m}{p}\ln(m) - \frac{a}{p}\left(\frac{e^{bp}-m}{b-a}\right) - \frac{b}{p}\left(\frac{m-e^{ap}}{b-a}\right)\right| \\ &= \left|\frac{m}{p}(\ln(m)-1) + \frac{be^{ap}-ae^{bp}}{p(b-a)}\right|.\end{aligned}$$

In the above we have used the fact that $m \geq 0$ and that $pt^* = \ln(m)$. Using (from Lemma 23) the result that $\theta(t)$ is positive or negative for $t \in [a,b]$ depending on whether $p > 0$ or $p < 0$ respectively, the first inequality in (4.108) results.

For the second inequality, we have, from (4.105), (4.106) and Lemma 23,

$$\begin{aligned}\int_a^b |\theta(t)|\,dt &= \frac{1}{|p|}\int_a^b \left[pmt - \frac{a\left(e^{bp}-e^{tp}\right)+b\left(e^{tp}-e^{ap}\right)}{b-a}\right]dt \\ &= \frac{1}{|p|}\left[pm\left(\frac{b^2-a^2}{2}\right) - \left(ae^{bp}-be^{ap}\right) - \int_a^b e^{pt}dt\right] \\ &= \frac{1}{|p|}\left[pm\left(\frac{b^2-a^2}{2}\right) - \left(ae^{bp}-be^{ap}\right) - (b-a)\,m\right] \\ &= \frac{1}{|p|}\left[(b-a)\,m\left(\frac{a+b}{2}p-1\right) - \left(ae^{bp}-be^{ap}\right)\right] \\ &= \frac{1}{|p|}\left[\frac{e^{bp}-e^{ap}}{p}\left(\frac{a+b}{2}p-1\right) - \left(ae^{bp}-be^{ap}\right)\right] \\ &= \frac{1}{|p|}\left(e^{bp}-e^{ap}\right)\left(\frac{b-a}{2}-\frac{1}{p}\right).\end{aligned}$$

Using (4.109) gives the second result in (4.108).

For the final inequality in (4.108) we need to determine $\int_a^b |\theta(t)|\,df(t)$ for f monotonic nondecreasing. Now, from (4.105) and (4.106)

$$\begin{aligned}\int_a^b |\theta(t)|\,df(t) &= \int_a^b \left[mt - \frac{be^{ap}-ae^{bp}}{p(b-a)} - \frac{e^{pt}}{p}\right]df(t) \\ &= \frac{1}{|p|}\int_a^b \left[pmt + \frac{be^{ap}-ae^{bp}}{b-a} - e^{pt}\right]df(t),\end{aligned}$$

where we have used the fact that $sgn(\theta(t)) = sgn(p)$.

Integration by parts of the Riemann-Stieltjes integral gives

$$\int_a^b |\theta(t)|\, df(t) \tag{4.111}$$

$$= \frac{1}{|p|}\left\{\left(pmt + \frac{be^{ap} - ae^{bp}}{b-a} - e^{pt}\right) f(t)\right]_a^b$$

$$\left. - p\int_a^b \left[m - e^{pt}\right] f(t)\, dt\right\}$$

$$= \frac{p}{|p|}\int_a^b \left(e^{pt} - m\right) f(t)\, dt.$$

Now,

$$\int_a^b e^{tp} f(t)\, dt \le f(b)\int_a^b e^{tp} dt = \frac{e^{bp} - e^{ap}}{p} f(b) = (b-a)\, m f(b)$$

and

$$-m\int_a^b f(t)\, dt \le -m(b-a)\, f(a)$$

so that combining with (4.111) gives the inequalities for f monotonic nondecreasing. ∎

REMARK 70. *If f is a probability density function then $\mathcal{M}(f) = \frac{1}{b-a}$ and f is non-negative.*

CHAPTER 5

Elementary Inequalities for the Variance

1. Elementary Inequalities

1.1. Introduction. In [**18**], the authors point out a number of inequalities for the expectation, $E(X)$ and the variance, $\sigma^2(X)$ from which we cite only the following:

$$0 \leq \sigma^2(X) \leq [b - E(X)][E(X) - a] \leq \frac{1}{4}(b-a)^2; \tag{5.1}$$

$$\begin{aligned} 0 &\leq [b - E(X)][E(X) - a] - \sigma^2(X), \\ &\leq \begin{cases} \dfrac{(b-a)^3}{6}\|f\|_\infty, \\ \\ [B(q+1,q+1)]^{\frac{1}{q}}(b-a)^{2+\frac{1}{q}}\|f\|_p, \\ \\ \text{provided } f \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \end{cases} \end{aligned} \tag{5.2}$$

where $B(\cdot,\cdot)$ is Euler's Beta function.

Moreover, if $m \leq f \leq M$ a.e. on $[a,b]$, then

$$\frac{m(b-a)^3}{6} \leq [b - E(X)][E(X) - a] - \sigma^2(X) \leq \frac{M(b-a)^3}{6}, \tag{5.3}$$

and

$$\begin{aligned} &\left|[b - E(X)][E(X) - a] - \sigma^2(X) - \frac{(b-a)^3}{6}\right| \\ &\qquad\qquad \leq \frac{\sqrt{5}(b-a)^3(M-m)}{60}. \end{aligned} \tag{5.4}$$

In this section, following [**6**], we point out some additional elementary results.

1.2. The Results. The following lemma holds [**6**].

Lemma 24. *Let X be a continuous random variable having the cumulative distribution function $F:[a,b]\to[0,1]$, then,*

$$\sigma^2(X)=(b-E(X))(E(X)-a)+\frac{1}{b-a}\int_a^b\int_a^b(t-\tau)(F(\tau)-F(t))\,d\tau dt. \tag{5.5}$$

Proof. Using integration by parts, we have

$$\begin{aligned}\sigma^2(X)&=\int_a^b(t-E(X))^2\,dF(t)\\&=(t-E(X))^2F(t)\Big|_a^b-2\int_a^b(t-E(X))F(t)\,dt\\&=(b-E(X))^2-2\int_a^b(t-E(X))F(t)\,dt.\end{aligned} \tag{5.6}$$

Further, using Korkine's identity [**105**]

$$\frac{1}{b-a}\int_a^b h(t)g(t)\,dt=\frac{1}{b-a}\int_a^b h(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt+\frac{1}{2(b-a)^2}\int_a^b\int_a^b(h(t)-h(\tau))(g(t)-g(\tau))\,d\tau dt,$$

we have

$$\begin{aligned}\int_a^b(t-E(X))F(t)\,dt&=\frac{1}{b-a}\int_a^b(t-E(X))\,dt\int_a^b F(t)\,dt\\&\quad+\frac{1}{2(b-a)}\int_a^b\int_a^b(t-\tau)(F(t)-F(\tau))\,d\tau dt.\end{aligned} \tag{5.7}$$

Since,

$$\begin{aligned}\int_a^b(t-E(X))\,dt&=\frac{(b-E(X))^2-(E(X)-a)^2}{2}\\&=(b-a)\left(\frac{b+a}{2}-E(X)\right),\end{aligned}$$

and

$$\int_a^b F(t)\,dt=b-E(X),$$

then, by (5.6) and (5.7),

$$\begin{aligned}\sigma^{2}(X) &= (b-E(X))^{2}-2\left[\frac{b+a-2E(X)}{2}\cdot(b-E(X))\right.\\&\qquad\left.+\frac{1}{2(b-a)}\int_{a}^{b}\int_{a}^{b}(t-\tau)(F(t)-F(\tau))\,d\tau dt\right]\\&= (b-E(X))^{2}-(b+a-2E(X))(b-E(X))\\&\qquad-\frac{1}{b-a}\int_{a}^{b}\int_{a}^{b}(t-\tau)(F(t)-F(\tau))\,d\tau dt\\&= (b-E(X))(E(X)-a)\\&\qquad-\frac{1}{b-a}\int_{a}^{b}\int_{a}^{b}(t-\tau)(F(t)-F(\tau))\,d\tau dt,\end{aligned}$$

and the lemma is proved. ∎

REMARK 71. *Since the mapping F is monotonic nondecreasing on $[a,b]$, then*

$$(t-\tau)(F(\tau)-F(t))\leq 0 \quad \textit{for all} \quad t,\tau\in[a,b]; \tag{5.8}$$

which implies that

$$\sigma^{2}(X)\leq[b-E(X)][E(X)-a], \tag{5.9}$$

an inequality that was proved in [**18**] *and* [**7**] *using two different methods.*

The inequality (5.9) can be improved as follows [**6**].

THEOREM 83. *With the assumptions in Lemma 24,*

$$\begin{aligned}&(b-E(X))(E(X)-a)-\sigma^{2}(X)\\&\qquad\geq 2\left|\int_{a}^{b}|t|\,F(t)\,dt-\frac{1}{b-a}(b-E(X))\int_{a}^{b}|t|\,dt\right|\geq 0.\end{aligned} \tag{5.10}$$

PROOF. In [**66**], Dragomir proved the following refinement of Čebyšev's inequality

$$T(h,g)\geq\max\{|T(h,|g|)|,|T(|h|,g)|,|T(|h|,|g|)|\}\geq 0, \tag{5.11}$$

provided (h,g) are synchronous on $[a,b]$, so that

$$(h(t)-h(\tau))(g(t)-g(\tau))\geq 0 \quad \text{for all} \quad t,\tau\in[a,b]$$

and

$$T(h,g):=\frac{1}{b-a}\int_{a}^{b}h(t)\,g(t)\,dt-\frac{1}{b-a}\int_{a}^{b}h(t)\,dt\cdot\frac{1}{b-a}\int_{a}^{b}g(t)\,dt.$$

If we define $h(t) = t$, $t \in [a,b]$, then from (5.5)

$$\begin{aligned} T(h,F) &= \frac{1}{b-a}\int_a^b\int_a^b (t-\tau)(F(t)-F(\tau))\,d\tau dt \\ &= \frac{1}{b-a}\left[(b-E(X))(E(X)-a)-\sigma^2(X)\right]. \end{aligned}$$

Now, from (5.11),

$$T(|h|,F) = \frac{1}{b-a}\int_a^b\int_a^b |t|\,F(t)\,dt - \frac{1}{(b-a)^2}\int_a^b |t|\,dt\int_a^b F(t)\,dt,$$

$$\begin{aligned} T(h,|F|) &= T(h,F), \\ T(|h|,|F|) &= T(|h|,F). \end{aligned}$$

Using the result (5.11), we get (5.10). ∎

Remark 72. *If $a \le b \le 0$ or $0 \le a \le b$, then the first inequality in (5.10) becomes an identity and is of no special interest.*
If $a < 0 < b$, however, then,

$$\int_a^b |t|\,F(t)\,dt = -\int_a^0 tF(t)\,dt + \int_0^b tF(t)\,dt;$$

$$\frac{1}{b-a}\int_a^b |t|\,dt = \frac{1}{b-a}\left[-\int_a^0 tdt + \int_0^b tdt\right] = \frac{1}{b-a}\left[\frac{a^2+b^2}{2}\right],$$

and by (5.10), we get

$$\begin{aligned} &(b-E(X))(E(X)-a)-\sigma^2(X) \\ &\quad\ge 2\left|\int_0^b tF(t)\,dt - \int_a^0 tF(t)\,dt - \frac{a^2+b^2}{2(b-a)}(b-E(X))\right| \ge 0. \end{aligned} \tag{5.12}$$

Assume that f, $f:[a,b]\to(0,\infty)$ is the PDF of X, then the following theorem holds [**6**].

Theorem 84. *With the assumptions in Lemma 24,*

$$\begin{aligned} &(b-E(X))(E(X)-a)-\sigma^2(X) \\ &\le \begin{cases} \frac{(b-a)^3}{6}\|f\|_\infty & \text{if } f\in L_\infty[a,b]; \\ \frac{2q^2(b-a)^{2+\frac{1}{q}}}{(2q+1)(3q+1)}\|f\|_p & \text{if } f\in L_p[a,b], \\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^2}{3}; & \end{cases} \end{aligned} \tag{5.13}$$

where $\|\cdot\|_p$ $(p \ge 1)$ are the usual Lebesgue norms.

PROOF. Using (5.5), we may state that

$$0 \le (b - E(X))(E(X) - a) - \sigma^2(X) = \frac{1}{b-a}\int_a^b\int_a^b (t-\tau)\left(\int_\tau^t f(u)\,du\right) dt d\tau. \tag{5.14}$$

By the modulus property, we have

$$\begin{aligned} 0 &\le (b - E(X))(E(X) - a) - \sigma^2(X) \\ &= \frac{1}{b-a}\left|\int_a^b\int_a^b (t-\tau)\left(\int_\tau^t f(u)\,du\right) dt d\tau\right| \\ &\le \frac{1}{b-a}\int_a^b\int_a^b |t-\tau|\left|\int_\tau^t f(u)\,du\right| dt d\tau \\ &=: M. \end{aligned} \tag{5.15}$$

If $f \in L_\infty[a,b]$, then we can write,

$$\left|\int_\tau^t f(u)\,du\right| \le |t-\tau|\,\|f\|_\infty,$$

for all $t, \tau \in [a,b]$, and so

$$\begin{aligned} M &\le \frac{1}{b-a}\int_a^b\int_a^b |t-\tau|\,|t-\tau|\,\|f\|_\infty\, dt d\tau \\ &= \frac{\|f\|_\infty}{b-a}\int_a^b\int_a^b (t-\tau)^2\, dt d\tau \\ &= \frac{\|f\|_\infty (b-a)^3}{6}. \end{aligned}$$

For the second part, we apply Hölder's integral inequality to write:

$$\begin{aligned} \left|\int_\tau^t f(u)\,du\right| &\le \left|\int_\tau^t du\right|^{\frac{1}{q}}\left|\int_\tau^t f^p(u)\,du\right|^{\frac{1}{p}} \\ &\le |t-\tau|^{\frac{1}{q}}\left(\int_a^b f^p(u)\,du\right)^{\frac{1}{p}} \\ &= |t-\tau|^{\frac{1}{q}}\,\|f\|_p, \end{aligned}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

In addition,

$$\begin{aligned}
M &\leq \frac{1}{b-a}\int_a^b\int_a^b |t-\tau|\,|t-\tau|^{\frac{1}{q}}\,\|f\|_p\,dtd\tau\\
&= \frac{\|f\|_p}{b-a}\int_a^b\left[\int_a^t (t-\tau)^{1+\frac{1}{q}}\,d\tau+\int_t^b(\tau-t)^{1+\frac{1}{q}}\,d\tau\right]dt\\
&= \frac{\|f\|_p}{b-a}\int_a^b\left[\frac{(t-a)^{2+\frac{1}{q}}+(b-t)^{2+\frac{1}{q}}}{\left(2+\frac{1}{q}\right)}\right]dt\\
&= \frac{2\,\|f\|_p\,(b-a)^{2+\frac{1}{q}}}{\left(2+\frac{1}{q}\right)\left(3+\frac{1}{q}\right)},
\end{aligned}$$

and the second inequality in (5.13) is proved.

Finally,

$$\begin{aligned}
M &\leq \frac{1}{b-a}\int_a^b\int_a^b |t-\tau|\left(\int_a^b f(u)\,du\right)dtd\tau\\
&= \frac{1}{b-a}\int_a^b\left[\frac{(t-a)^2+(b-t)^2}{2}\right]dt\\
&= \frac{1}{2(b-a)}\left[\frac{(b-a)^3}{3}+\frac{(b-a)^3}{3}\right]\\
&= \frac{(b-a)^2}{3},
\end{aligned}$$

and the theorem is completely proved. ∎

Using the Cauchy-Buniakowski-Schwartz inequality, we have the following inequality [**6**].

Theorem 85. *If X and F are as in Lemma 24, then,*

$$\begin{aligned}
0 &\leq (b-E(X))(E(X)-a)-\sigma^2(X)\\
&\leq \frac{(b-a)^2}{\sqrt{3}}\left[(b-a)\|F\|_2^2-(b-E(X))^2\right]^{\frac{1}{2}}\\
&\leq \frac{(b-a)^3}{2\sqrt{3}}.
\end{aligned} \tag{5.16}$$

PROOF. Using the Cauchy-Buniakowski-Schwartz integral inequality for double integrals, we have

$$\begin{aligned}(5.17)\quad & \left|\int_a^b\int_a^b (t-\tau)(F(\tau)-F(t))\,dtd\tau\right| \\ & \leq \left(\int_a^b\int_a^b (t-\tau)^2\,dtd\tau\right)^{\frac{1}{2}}\left(\int_a^b\int_a^b (F(t)-F(\tau))^2\,dtd\tau\right)^{\frac{1}{2}}.\end{aligned}$$

However,

$$\int_a^b\int_a^b (t-\tau)^2\,dtd\tau = \frac{(b-a)^4}{6},$$

$$\begin{aligned}&\int_a^b\int_a^b (F(\tau)-F(t))^2\,dtd\tau \\ &= 2\left[(b-a)\int_a^b F^2(t)\,dt - \left(\int_a^b F(t)\,dt\right)^2\right] \\ &= 2\left[(b-a)\|F\|_2^2 - (b-E(X))^2\right],\end{aligned}$$

and, by (5.17),

$$\begin{aligned}&\left|\int_a^b\int_a^b (t-\tau)(F(\tau)-F(t))\,d\tau dt\right| \\ &\leq \frac{(b-a)^2}{\sqrt{3}}\left[(b-a)\|F\|_2^2-(b-E(X))^2\right]^{\frac{1}{2}},\end{aligned}$$

and the first inequality in (5.16) is proved.

To prove the last part of (5.16), we use the following Grüss type inequality:

$$(5.18)\qquad \frac{1}{b-a}\int_a^b g^2(t)\,dt - \left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)^2 \leq \frac{1}{4}(\phi-\gamma)^2,$$

provided that $g \in L_2(a,b)$ and $\gamma \leq g(t) \leq \phi$ a.e. for $t \in (a,b)$.

From (5.18),

$$(b-a)\int_a^b F^2(t)\,dt - \left(\int_a^b F(t)\,dt\right)^2 \leq \frac{1}{4}(b-a)^2,$$

since

$$\sup_{t\in[a,b]} F(t) = 1 \quad\text{and}\quad \inf_{t\in[a,b]} F(t) = 0.$$

∎

If it is assumed that the mapping f is convex on $[a,b]$, then the following result can be obtained as well [**6**].

THEOREM 86. *Assume that the PDF, $f:[a,b]\to(0,\infty)$ is convex, then, we have the inequality*

$$\begin{aligned}(5.19)\qquad &\frac{1}{b-a}\int_a^b\int_a^b(t-\tau)^2 f\left(\frac{t+\tau}{2}\right)d\tau dt\\ &\leq [b-E(X)][E(X)-a]-\sigma^2(X)\\ &\leq \frac{(b-a)^2}{3}+\sigma^2(X)-(b-E(X))(E(X)-a).\end{aligned}$$

PROOF. Using the Hermite-Hadamard inequality,

$$(5.20)\qquad f\left(\frac{t+\tau}{2}\right)\leq\frac{\int_t^\tau f(u)\,du}{\tau-t}\leq\frac{f(t)+f(\tau)}{2},$$

for all $t,\tau\in[a,b]$, $t\neq\tau$, we have

$$\begin{aligned}(5.21)\qquad (t-\tau)^2 f\left(\frac{t+\tau}{2}\right)&\leq(t-\tau)(F(t)-F(\tau))\\ &\leq\frac{f(t)+f(\tau)}{2}(t-\tau)^2,\end{aligned}$$

for all $t,\tau\in[a,b]$.

Integrating (5.21) on $[a,b]^2$ and using the representation (5.5), gives

$$\begin{aligned}(5.22)\qquad &\frac{1}{b-a}\int_a^b\int_a^b(t-\tau)^2 f\left(\frac{t+\tau}{2}\right)dtd\tau\\ &\leq\frac{1}{b-a}\int_a^b\int_a^b(t-\tau)(F(t)-F(\tau))\,dtd\tau\\ &=[b-E(X)][E(X)-a]-\sigma^2(X)\\ &\leq\frac{1}{b-a}\int_a^b\int_a^b\frac{f(t)+f(\tau)}{2}(t-\tau)^2\,dtd\tau.\end{aligned}$$

Now,

$$\begin{aligned}(5.23)\qquad &\int_a^b\int_a^b(t-\tau)^2\left[\frac{f(t)+f(\tau)}{2}\right]dtd\tau\\ &=\int_a^b\int_a^b(t-\tau)^2 f(t)\,d\tau dt=\int_a^b\left[\int_a^b(t-\tau)^2\,d\tau\right]f(t)\,dt\\ &=\int_a^b\left[\frac{(b-t)^3+(t-a)^3}{3}\right]f(t)\,dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{(b-a)}{3} \int_a^b \left[(b-t)^2 - (b-t)(t-a) + (t-a)^2 \right] f(t)\,dt \\
&= \frac{b-a}{3} \int_a^b \left[(b-a)^2 - 3(b-t)(t-a) \right] f(t)\,dt \\
&= \frac{(b-a)^3}{3} - (b-a) \int_a^b (b-t)(t-a) f(t)\,dt \\
&= \frac{(b-a)^3}{3} + (b-a) \left[\sigma^2(X) - (b-E(X))(E(X)-a) \right],
\end{aligned}
$$

using an identity (see [**18**]).

Hence,

$$
\begin{aligned}
&\frac{1}{b-a} \int_a^b \int_a^b (t-\tau)^2 \left[\frac{f(t)+f(\tau)}{2} \right] dt d\tau \\
&\qquad = \frac{(b-a)^2}{3} + \left[\sigma^2(X) - (b-E(X))(E(X)-a) \right],
\end{aligned}
$$

and the second part of (5.19) is proved. ∎

Remark 73. *The second inequality in (5.19) is equivalent to:*

$$
[b-E(X)][E(X)-a] \le \sigma^2(X) + \frac{1}{6}(b-a)^2. \tag{5.24}
$$

Remark 74. *For $b-a < \frac{1}{\sqrt{3}}$, the result of Theorem 87 is better than that of Theorem 86. For $b-a > \frac{1}{\sqrt{3}}$, the opposite applies. It must be remembered that Theorem 86 relies on f being convex whereas Theorem 85 does not.*

The following representation for the absolutely continuous PDF, $f : [a,b] \to \mathbb{R}$ holds [**6**].

Lemma 25. *Let X be a random variable having the PDF, $f : [a,b] \to \mathbb{R}$ absolutely continuous on $[a,b]$, then, we have*

$$
\begin{aligned}
\sigma^2(X) &= (b-E(X))(E(X)-a) - \frac{(b-a)^2}{6} \\
&\quad + \frac{1}{2(b-a)} \int_a^b \int_a^b (t-\tau) \left(\int_\tau^t \left(u - \frac{t+\tau}{2} \right) f'(u)\,du \right) dt d\tau.
\end{aligned} \tag{5.25}
$$

Proof. We use the following identity which holds for the absolutely continuous mapping $g : [a,b] \to \mathbb{R}$

$$
\int_a^b g(u)\,du = \frac{g(a)+g(b)}{2}(b-a) - \int_a^b \left(u - \frac{a+b}{2} \right) g'(u)\,du, \tag{5.26}
$$

and can be easily proven by using the integration by parts formula.

We know that

$$(5.27)\quad \begin{aligned} &(E(X)-a)(b-E(X))-\sigma^2(X) \\ &= \frac{1}{b-a}\int_a^b\int_a^b (t-\tau)\int_\tau^t f(u)\,du\,dtd\tau \\ &= \frac{1}{b-a}\int_a^b\int_a^b (t-\tau)\left[\frac{f(t)+f(\tau)}{2}(t-\tau)\right. \\ &\qquad \left. -\int_\tau^t\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right]dtd\tau \\ &= \frac{1}{b-a}\int_a^b\int_a^b (t-\tau)^2\left(\frac{f(t)+f(\tau)}{2}\right)dtd\tau \\ &\quad -\frac{1}{b-a}\int_a^b\int_a^b (t-\tau)\left(\int_\tau^t\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right)dtd\tau. \end{aligned}$$

However, observe that (see the proof of Theorem 86)

$$\begin{aligned} L &:= \frac{1}{b-a}\int_a^b\int_a^b (t-\tau)^2\left(\frac{f(t)+f(\tau)}{2}\right)dtd\tau \\ &= \sigma^2(X)+\frac{1}{3}\left[(E(X)-b)^2-(E(X)-a)\right. \\ &\qquad \left.\times(b-E(X))+(E(X)-a)^2\right]. \end{aligned}$$

Using (5.27), we have

$$\begin{aligned} (E(X)-a)(b-E(X))-\sigma^2(X) &= \sigma^2(X)+\frac{1}{3}\left[(E(X)-b)^2\right. \\ &\quad \left.-(E(X)-a)(b-E(X))+(E(X)-a)^2\right] \\ &\quad -\frac{1}{b-a}\int_a^b\int_a^b (t-\tau)\left(\int_\tau^t\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right)dtd\tau, \end{aligned}$$

which is clearly equivalent to (5.25). ∎

Using Lemma 25, we are able to obtain the following bounds [**6**].

THEOREM 87. *Assume that f is as in Lemma 25, then, we have the inequality*

$$(5.28)\quad \left|[b-E(X)][E(X)-a]-\sigma^2(X)-\frac{(b-a)^2}{6}\right|$$

$$\leq \begin{cases} \frac{\|f'\|_\infty}{80}(b-a)^4 & \text{if } f' \in L_\infty[a,b]; \\ \frac{q^2\|f'\|_p}{2(3q+1)(4q+1)(q+1)^{\frac{1}{q}}}(b-a)^{3+\frac{1}{q}} & \text{if } f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{\|f'\|_1}{24}(b-a)^3. \end{cases}$$

Proof. Using the equality (5.25), we may write

$$\left|\sigma^2(X) - (b - E(X))(E(X) - a) + \frac{(b-a)^2}{6}\right| \tag{5.29}$$

$$\leq \frac{1}{2(b-a)} \int_a^b \int_a^b |t-\tau| \left| \int_\tau^t \left(u - \frac{t+\tau}{2}\right) f'(u)\, du \right| dt d\tau := N.$$

Now, it may be easily shown that,

$$\left| \int_\tau^t \left(u - \frac{t+\tau}{2}\right) f'(u)\, du \right| \leq \|f'\|_\infty \left| \int_\tau^t \left|u - \frac{t+\tau}{2}\right| du \right|$$
$$= \|f'\|_\infty \frac{(t-\tau)^2}{4},$$

for all $t, \tau \in [a,b]$.

Also, by Hölder's integral inequality, we may write

$$\left| \int_\tau^t \left(u - \frac{t+\tau}{2}\right) f'(u)\, du \right|$$
$$\leq \left| \int_\tau^t |f'(u)|^p\, du \right|^{\frac{1}{p}} \left| \int_\tau^t \left|u - \frac{t+\tau}{2}\right|^q du \right|^{\frac{1}{q}}$$
$$\leq \|f'\|_p \left[\frac{|t-\tau|^{q+1}}{2q(q+1)} \right]^{\frac{1}{q}}$$
$$= \|f'\|_p \frac{|t-\tau|^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}$$

for all $t, \tau \in [a,b]$, and further,

$$\left| \int_\tau^t \left(u - \frac{t+\tau}{2}\right) f'(u)\, du \right| \leq \sup \left|u - \frac{t+\tau}{2}\right| \int_\tau^t |f'(u)|\, du$$
$$\leq \frac{|t-\tau|}{2} \|f'\|_1.$$

Consequently,

$$\left|\int_{\tau}^{t}\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right| \tag{5.30}$$
$$\leq\begin{cases}\|f'\|_{\infty}\frac{(t-\tau)^2}{4} & \text{if } f'\in L_{\infty}[a,b];\\ \|f'\|_{p}\frac{|t-\tau|^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}} & \text{if } f'\in L_{p}[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \|f'\|_{1}\frac{|t-\tau|}{2}. & \text{if } f'\in L_{1}[a,b].\end{cases}$$

Using (5.30), we may write, for f' belonging to the obvious Lebesgue space $L_p[a,b]$, $p\geq 1$,

$$N\leq\begin{cases}\frac{\|f'\|_{\infty}}{8(b-a)}\int_a^b\int_a^b|t-\tau|^3\,dtd\tau,\\ \frac{\|f'\|_{p}}{4(q+1)^{\frac{1}{q}}(b-a)}\int_a^b\int_a^b|t-\tau|^{2+\frac{1}{q}}\,dtd\tau,\\ \frac{\|f'\|_{1}}{4(b-a)}\int_a^b\int_a^b(t-\tau)^2\,dtd\tau.\end{cases} \tag{5.31}$$

Now, since some straight forward algebra shows that

$$\begin{aligned}\int_a^b\int_a^b|t-\tau|^3\,dtd\tau &= \int_a^b\left[\int_a^t(t-\tau)^3\,d\tau+\int_t^b(\tau-t)^3\,d\tau\right]dt\\ &= \int_a^b\left[\frac{(t-a)^4+(b-t)^4}{4}\right]dt=\frac{(b-a)^5}{10},\end{aligned}$$

$$\begin{aligned}&\int_a^b\int_a^b|t-\tau|^{2+\frac{1}{q}}\,dtd\tau\\ &\quad=\int_a^b\left[\int_a^t(t-\tau)^{2+\frac{1}{q}}\,d\tau+\int_t^b(\tau-t)^{2+\frac{1}{q}}\,d\tau\right]dt\\ &\quad=\int_a^b\left[\frac{(t-a)^{3+\frac{1}{q}}+(b-t)^{3+\frac{1}{q}}}{3+\frac{1}{q}}\right]dt=\frac{2q^2(b-a)^{4+\frac{1}{q}}}{(3q+1)(4q+1)},\end{aligned}$$

and

$$\begin{aligned}\int_a^b\int_a^b(t-\tau)^2\,dtd\tau &= \int_a^b\left[\int_a^t(t-\tau)^2\,d\tau+\int_t^b(\tau-t)^2\,d\tau\right]dt\\ &= \int_a^b\left[\frac{(t-a)^3+(b-t)^3}{3}\right]dt=\frac{(b-a)^4}{6},\end{aligned}$$

we obtain the desired inequality (5.28) from using (5.31) and (5.29). ∎

The following representation for the mappings whose derivatives are absolutely continuous on $[a,b]$ also holds [**6**].

LEMMA 26. *Let X be a random variable having the PDF $f:[a,b]\to\mathbb{R}$ and with the property that $f':[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$, then, we have*

$$\begin{aligned}(5.32)\quad \sigma^2(X) &= (b-E(X))(E(X)-a)-\frac{(b-a)^2}{6}\\ &\quad+\frac{1}{4(b-a)}\int_a^b\int_a^b(t-\tau)\\ &\qquad\times\int_\tau^t(t-u)(u-\tau)f''(u)\,du\,dt\,d\tau.\end{aligned}$$

PROOF. We use the following identity which holds for the mappings g whose derivatives are absolutely continuous:

$$\begin{aligned}(5.33)\quad \int_a^b g(u)\,du &= \frac{g(a)+g(b)}{2}(b-a)\\ &\quad-\frac{1}{2}\int_a^b(b-u)(u-a)g''(u)\,du,\end{aligned}$$

(proved by using the integration by parts formula twice).

We know that,

$$\begin{aligned}&(b-E(X))(E(X)-a)-\sigma^2(X)\\ &\qquad=\frac{1}{b-a}\int_a^b\int_a^b(t-\tau)\int_t^\tau f(u)\,du\,dt\,d\tau,\end{aligned}$$

and then, using the representation (5.33) written for f instead of g, and proceeding as in the proof of Lemma 25, we end up with the identity (5.32). ∎

Using the representation of Lemma 26, we are able to obtain the following bounds [**6**].

THEOREM 88. *Assume that f is as in Lemma 26, then, we have the inequality*

$$(5.34)\quad \left|[b-E(X)][E(X)-a]-\sigma^2(X)-\frac{(b-a)^2}{6}\right|$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{360}(b-a)^5 & \textit{if } f'' \in L_\infty[a,b] \\ \frac{\|f''\|_q}{2(4p+1)(5p+1)}[B(p+1,p+1)]^{\frac{1}{p}}(b-a)^{4+\frac{1}{p}} & \\ \qquad \textit{if } f'' \in L_q[a,b],\ \frac{1}{p}+\frac{1}{q}=1,\ p>1; & \\ \frac{\|f''\|_1}{160}(b-a)^4, & \end{cases}$$

where the $p-$*norms are taken on the interval* $[a,b]$.

Proof. Using the equality (5.32), we may write

$$\begin{aligned} &\left|\sigma^2(X) - [b-E(X)][E(X)-a] - \frac{(b-a)^2}{6}\right| \\ &\leq \frac{1}{4(b-a)}\int_a^b\int_a^b |t-\tau|\left|\int_\tau^t (t-u)(u-\tau)f''(u)\,du\right| dt\,d\tau \\ &:= K. \end{aligned}$$

First of all, let us observe that

$$\begin{aligned} \left|\int_\tau^t (t-u)(u-\tau)f''(u)\,du\right| &\leq \|f''\|_\infty \left|\int_\tau^t (t-u)(u-\tau)\,du\right| \\ &\leq \frac{\|f''\|_\infty}{6}|t-\tau|^3, \end{aligned}$$

for all $t,\tau \in [a,b]$.

Further, by Hölder's integral inequality, we obtain

$$\begin{aligned} \left|\int_\tau^t (t-u)(u-\tau)f''(u)\,du\right| &\leq \|f''\|_q \left|\int_\tau^t |t-u|^p|u-\tau|^p\,du\right|^{\frac{1}{p}} \\ &= \|f''\|_q |t-\tau|^{2+\frac{1}{p}}[B(p+1,p+1)]^{\frac{1}{p}}, \end{aligned}$$

for all $t,\tau \in [a,b]$, where B is the Beta function of Euler and $\frac{1}{p}+\frac{1}{q}=1$; $p>1$.

Also, we have

$$\begin{aligned} \left|\int_\tau^t (t-u)(u-\tau)f''(u)\,du\right| &\leq \|f''\|_1 \max_{u\in[\tau,t]}|(t-u)(u-\tau)| \\ &= \frac{|t-\tau|^2}{4}\|f''\|_1, \end{aligned}$$

for all $t,\tau \in [a,b]$.

Consequently, we may state the inequality

$$\left|\int_{\tau}^{t}(t-u)(u-\tau)f''(u)\,du\right| \tag{5.35}$$
$$\leq \begin{cases} \frac{\|f''\|_{\infty}}{6}|t-\tau|^{3} & \text{if } f''\in L_{\infty}[a,b]; \\ \|f''\|_{q}\left[B(p+1,p+1)\right]^{\frac{1}{p}}|t-\tau|^{2+\frac{1}{p}} & \text{if } f''\in L_{q}[a,b], \\ & \quad \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \frac{|t-\tau|^{2}}{4}\|f''\|_{1}, \end{cases}$$

for all $t,\tau\in[a,b]$.

Using (5.35) and the definition of K above, we may write

$$K\leq\begin{cases} \frac{\|f''\|_{\infty}}{24(b-a)}\int_a^b\int_a^b(t-\tau)^4\,dt\,d\tau \quad \text{if } f''\in L_{\infty}[a,b]; \\ \frac{\|f''\|_{q}}{4(b-a)}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\int_a^b\int_a^b|t-\tau|^{3+\frac{1}{p}}\,dt\,d\tau \\ \qquad \text{if } f''\in L_{q}[a,b],\ \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \frac{\|f''\|_{1}}{16(b-a)}\int_a^b\int_a^b|t-\tau|^{3}\,dt\,d\tau. \end{cases} \tag{5.36}$$

Now, since some straightforward algebra shows that

$$\int_a^b\int_a^b(t-\tau)^4\,dt\,d\tau=\frac{(b-a)^6}{15},$$

$$\begin{aligned}\int_a^b\int_a^b|t-\tau|^{3+\frac{1}{p}}\,dt\,d\tau &= \int_a^b\left[\int_a^t(t-\tau)^{3+\frac{1}{p}}\,d\tau+\int_t^b(\tau-t)^{3+\frac{1}{p}}\,d\tau\right]dt \\ &= \int_a^b\left[\frac{(t-a)^{4+\frac{1}{p}}+(b-t)^{4+\frac{1}{p}}}{4+\frac{1}{p}}\right]dt \\ &= 2\frac{(b-a)^{5+\frac{1}{p}}}{\left(4+\frac{1}{p}\right)\left(5+\frac{1}{p}\right)}=\frac{2p^2(b-a)^{5+\frac{1}{p}}}{(4p+1)(5p+1)},\end{aligned}$$

and

$$\begin{aligned}\int_a^b\int_a^b|t-\tau|^3\,dt\,d\tau &= \int_a^b\left[\int_a^t(t-\tau)^3\,d\tau+\int_t^b(\tau-t)^3\,d\tau\right]dt \\ &= \int_a^b\left[\frac{(t-a)^4+(b-t)^4}{4}\right]dt=\frac{(b-a)^5}{10},\end{aligned}$$

then by (5.36), we deduce the desired inequality (5.34). ∎

2. Perturbed Inequalities

2.1. Introduction. In this section, we obtain some inequalities for the dispersion of a continuous random variable X having the PDF f defined on a finite interval $[a,b]$.

Tools used include: Korkine's identity, which plays a central role in the proof of Čebyšev's integral inequality for synchronous mappings [**109**], Hölder's weighted inequality for double integrals and an integral identity connecting the variance $\sigma^2(X)$ and the expectation $E(X)$. Perturbed results are also obtained by using Grüss, Čebyšev and Lupaş inequalities. In the last part of this section, results from an identity involving a double integral are obtained for a variety of norms.

2.2. Some Inequalities for Dispersion. The following theorem holds [**7**].

THEOREM 89. *With the above assumptions, we have*

$$\sigma(X) \leq \begin{cases} \frac{\sqrt{3}(b-a)^2}{6}\|f\|_\infty & \text{provided } f \in L_\infty[a,b]; \\ \frac{\sqrt{2}(b-a)^{1+\frac{1}{q}}}{2[(q+1)(2q+1)]^{\frac{2}{q}}}\|f\|_p & \text{provided } f \in L_p[a,b] \\ & \text{and } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\sqrt{2}(b-a)}{2}. \end{cases} \tag{5.37}$$

PROOF. Korkine's identity [**105**], is

$$\begin{aligned} &\int_a^b p(t)\,dt \int_a^b p(t)\,g(t)\,h(t)\,dt \\ &\qquad - \int_a^b p(t)\,g(t)\,dt \cdot \int_a^b p(t)\,h(t)\,dt \\ &= \frac{1}{2}\int_a^b\int_a^b p(t)\,p(s)\,(g(t)-g(s))\,(h(t)-h(s))\,dtds\,, \end{aligned} \tag{5.38}$$

which holds for the measurable mappings $p,g,h:[a,b]\to\mathbb{R}$ for which the integrals involved in (5.38) exist and are finite. Choose in (5.38) $p(t)=f(t)$, $g(t)=h(t)=t-E(X)$, $t\in[a,b]$ to get

$$\sigma^2(X) = \frac{1}{2}\int_a^b\int_a^b f(t)\,f(s)\,(t-s)^2\,dtds\,. \tag{5.39}$$

It is obvious that

$$\begin{aligned}(5.40)\qquad &\int_a^b\int_a^b f(t)\,f(s)\,(t-s)^2\,dtds\\ &\le \sup_{(t,s)\in[a,b]^2}|f(t)\,f(s)|\int_a^b\int_a^b (t-s)^2\,dtds\\ &= \frac{(b-a)^4}{6}\,\|f\|_\infty^2\,,\end{aligned}$$

and then, by (5.39), we obtain the first part of (5.37).

For the second part, we apply Hölder's integral inequality for double integrals to obtain

$$\begin{aligned}&\int_a^b\int_a^b f(t)\,f(s)\,(t-s)^2\,dtds\\ &\le \left(\int_a^b\int_a^b f^p(t)\,f^p(s)\,dtds\right)^{\frac{1}{p}}\left(\int_a^b\int_a^b (t-s)^{2q}\,dtds\right)^{\frac{1}{q}}\\ &= \|f\|_p^2\left[\frac{(b-a)^{2q+2}}{(q+1)(2q+1)}\right]^{\frac{1}{q}},\end{aligned}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, and the second inequality in (5.37) is proved.

For the last part, observe that

$$\begin{aligned}\int_a^b\int_a^b f(t)\,f(s)\,(t-s)^2\,dtds &\le \sup_{(t,s)\in[a,b]^2}|(t-s)|^2\int_a^b\int_a^b f(t)\,f(s)\,dtds\\ &= (b-a)^2\,,\end{aligned}$$

as

$$\int_a^b\int_a^b f(t)\,f(s)\,dtds = \int_a^b f(t)\,dt\int_a^b f(s)\,ds = 1.$$

∎

Using a finer argument, the last inequality in (5.37) can be improved as follows [**7**].

Theorem 90. *Under the above assumptions, we have*

$$(5.41)\qquad 0\le\sigma(X)\le\frac{1}{2}(b-a)\,.$$

Proof. We use the following Grüss type inequality:

$$(5.42)\qquad 0\le\frac{\int_a^b p(t)\,g^2(t)\,dt}{\int_a^b p(t)\,dt}-\left(\frac{\int_a^b p(t)\,g(t)\,dt}{\int_a^b p(t)\,dt}\right)^2\le\frac{1}{4}(M-m)^2\,,$$

provided that p and g are measurable on $[a,b]$ and that all the integrals in (5.42) exist and are finite, $\int_a^b p(t)\,dt > 0$ and $m \le g \le M$, a.e., on $[a,b]$.

Choose in (5.42), $p(t) = f(t)$, $g(t) = t - E(X)$, $t \in [a,b]$. Observe that in this case $m = a - E(X)$, $M = b - E(X)$ and then, by (5.42) we deduce (5.41). ∎

REMARK 75. *The same conclusion can be obtained for the choice* $p(t) = f(t)$ *and* $g(t) = t$, $t \in [a,b]$.

The following result holds [**7**].

THEOREM 91. *Let X be a random variable having the PDF given by $f : [a,b] \subset \mathbb{R} \to \mathbb{R}_+$, then, for any $x \in [a,b]$ we have the inequality:*

$$\sigma^2(X) + (x - E(X))^2 \le \begin{cases} (b-a)\left[\frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2\right]\|f\|_\infty, & \textit{provided } f \in L_\infty[a,b]; \\ \left[\frac{(b-x)^{2q+1} + (x-a)^{2q+1}}{2q+1}\right]^{\frac{1}{q}} \|f\|_p & \textit{provided } f \in L_p[a,b],\ p>1, \\ & \textit{and } \frac{1}{p} + \frac{1}{q} = 1; \\ \left(\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right)^2. \end{cases} \tag{5.43}$$

PROOF. We observe that

$$\int_a^b (x-t)^2 f(t)\,dt = \int_a^b \left(x^2 - 2xt + t^2\right) f(t)\,dt = x^2 - 2xE(X) + \int_a^b t^2 f(t)\,dt, \tag{5.44}$$

and as

$$\sigma^2(X) = \int_a^b t^2 f(t)\,dt - [E(X)]^2, \tag{5.45}$$

we get, by (5.44) and (5.45), that

$$[x - E(X)]^2 + \sigma^2(X) = \int_a^b (x-t)^2 f(t)\,dt, \tag{5.46}$$

which is of intrinsic interest itself.

We observe that

$$\begin{aligned}\int_a^b (x-t)^2 f(t)\,dt &\le ess \sup_{t\in[a,b]} |f(t)| \int_a^b (x-t)^2\,dt \\ &= \|f\|_\infty \frac{(b-x)^3+(x-a)^3}{3} \\ &= (b-a)\|f\|_\infty \left[\frac{(b-a)^2}{12}+\left(x-\frac{a+b}{2}\right)^2\right],\end{aligned}$$

and the first inequality in (5.43) is proved.

For the second inequality, observe that by Hölder's integral inequality,

$$\begin{aligned}\int_a^b (x-t)^2 f(t)\,dt &\le \left(\int_a^b f^p(t)\,dt\right)^{\frac{1}{p}} \left(\int_a^b (x-t)^{2q}\,dt\right)^{\frac{1}{q}} \\ &= \|f\|_p \left[\frac{(b-x)^{2q+1}+(x-a)^{2q+1}}{2q+1}\right]^{\frac{1}{q}},\end{aligned}$$

and the second inequality in (5.43) is established.

Finally, observe that,

$$\begin{aligned}\int_a^b (x-t)^2 f(t)\,dt &\le \sup_{t\in[a,b]} (x-t)^2 \int_a^b f(t)\,dt \\ &= \max\left\{(x-a)^2,(b-x)^2\right\} \\ &= (\max\{x-a,b-x\})^2 \\ &= \left(\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right)^2,\end{aligned}$$

and the theorem is proved. ∎

The following corollaries are easily deduced [**7**].

Corollary 59. *With the above assumptions, we have*

$$\sigma(X) \le \begin{cases} (b-a)^{\frac{1}{2}}\left[\frac{(b-a)^2}{12}+\left(E(X)-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}} \|f\|_\infty^{\frac{1}{2}} \\ \qquad\qquad \textit{provided} \;\; f\in L_\infty[a,b]; \\ \left[\frac{(b-E(X))^{2q+1}+(E(X)-a)^{2q+1}}{2q+1}\right]^{\frac{1}{2q}} \|f\|_p^{\frac{1}{2}} \\ \qquad \textit{if} \;\; f\in L_p[a,b],\; p>1 \;\; \textit{and} \;\; \frac{1}{p}+\frac{1}{q}=1; \\ \frac{b-a}{2}+\left|E(X)-\frac{a+b}{2}\right|. \end{cases} \tag{5.47}$$

REMARK 76. *The last inequality in (5.47) is worse than the inequality (5.41), obtained by a technique based on Grüss' inequality.*

The best inequality we can get from (5.43) is the one for which $x = \frac{a+b}{2}$, and this applies for all the bounds since

$$\min_{x\in[a,b]}\left[\frac{(b-a)^2}{12}+\left(x-\frac{a+b}{2}\right)^2\right]=\frac{(b-a)^2}{12},$$

$$\min_{x\in[a,b]}\frac{(b-x)^{2q+1}+(x-a)^{2q+1}}{2q+1}=\frac{(b-a)^{2q+1}}{2^{2q}(2q+1)},$$

and

$$\min_{x\in[a,b]}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]=\frac{b-a}{2}.$$

Consequently, we can state the following corollary as well [**7**].

COROLLARY 60. *With the above assumptions, we have the inequality:*

$$\sigma^2(X)+\left[E(X)-\frac{a+b}{2}\right]^2 \tag{5.48}$$

$$\leq\begin{cases}\frac{(b-a)^3}{12}\|f\|_\infty & \text{provided } f\in L_\infty[a,b];\\ \frac{(b-a)^{2q+1}}{4(2q+1)^{\frac{1}{q}}}\|f\|_p & \text{if } f\in L_p[a,b],\ p>1,\\ & \text{and } \frac{1}{p}+\frac{1}{q}=1;\\ \frac{(b-a)^2}{4}. \end{cases}$$

REMARK 77. *From the last inequality in (5.48), we obtain*

$$\sigma^2(X)\leq(b-E(X))(E(X)-a)\leq\frac{1}{4}(b-a)^2, \tag{5.49}$$

which is an improvement on (5.41).

2.3. Perturbed Results Using Grüss Type inequalities. Grüss (see [**98**] or for example [**114**]) proved the following integral inequality which gives an approximation for the integral of a product in terms of the product of the integrals.

THEOREM 92. *Let $h,g:[a,b]\to\mathbb{R}$ be two integrable mappings such that $\phi\leq h(x)\leq\Phi$ and $\gamma\leq g(x)\leq\Gamma$ for all $x\in[a,b]$, where ϕ,Φ,γ,Γ are real numbers, then,*

$$|T(h,g)|\leq\frac{1}{4}(\Phi-\phi)(\Gamma-\gamma), \tag{5.50}$$

where

$$T(h,g)=\frac{1}{b-a}\int_a^b h(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b h(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx \tag{5.51}$$

and the inequality is sharp.

For a simple proof of this as well as for extensions, generalizations, discrete variants and other associated material, see [**114**], and [**48**], [**85**] where further references are given.

A 'pre-Grüss' inequality is embodied in the following theorem which was proved in [**108**]. It provides a sharper bound than the above Grüss inequality.

THEOREM 93. *Let h,g be integrable functions defined on $[a,b]$ and let $d\le g(t)\le D$. Then,*

$$|T(h,g)|\le\frac{D-d}{2}\left[T(h,h)\right]^{\frac{1}{2}}, \tag{5.52}$$

where $T(h,g)$ is as defined in (5.51).

Theorem 93 will now be used to provide a perturbed rule involving the variance and mean of a PDF (see [**7**]).

THEOREM 94. *Let X be a random variable having the PDF given by $f:[a,b]\subset\mathbb{R}\to\mathbb{R}_+$, then for any $x\in[a,b]$ and $m\le f(x)\le M$ we have the inequality*

$$\begin{aligned}
|P_V(x)| &\\
&:=\left|\sigma^2(X)+(x-E(X))^2-\frac{(b-a)^2}{12}-\left(x-\frac{a+b}{2}\right)^2\right|\\
&\le\frac{M-m}{2}\cdot\frac{(b-a)^2}{\sqrt{45}}\left[\left(\frac{b-a}{2}\right)^2+15\left(x-\frac{a+b}{2}\right)\right]^{\frac{1}{2}}\\
&\le(M-m)\frac{(b-a)^3}{\sqrt{45}}.
\end{aligned} \tag{5.53}$$

PROOF. Applying the 'pre-Grüss' result (5.52) by associating $g(t)$ with $f(t)$ and $h(t) = (x-t)^2$, gives, from (5.50)-(5.52)

$$\left|\int_a^b (x-t)^2 f(t)\,dt - \frac{1}{b-a}\int_a^b (x-t)^2\,dt \cdot \int_a^b f(t)\,dt\right| \leq (b-a)\frac{M-m}{2}[T(h,h)]^{\frac{1}{2}}, \tag{5.54}$$

where from (5.51)

$$T(h,h) = \frac{1}{b-a}\int_a^b (x-t)^4\,dt - \left[\frac{1}{b-a}\int_a^b (x-t)^2\,dt\right]^2. \tag{5.55}$$

Now,

$$\frac{1}{b-a}\int_a^b (x-t)^2\,dt = \frac{(x-a)^3+(b-x)^3}{3(b-a)} = \frac{1}{3}\left(\frac{b-a}{2}\right)^2 + \left(x-\frac{a+b}{2}\right)^2, \tag{5.56}$$

and

$$\frac{1}{b-a}\int_a^b (x-t)^4\,dt = \frac{(x-a)^5+(b-x)^5}{5(b-a)},$$

giving, from (5.55),

$$45T(h,h) = 9\left[\frac{(x-a)^5+(b-x)^5}{b-a}\right] - 5\left[\frac{(x-a)^3+(b-x)^3}{b-a}\right]^2. \tag{5.57}$$

Let $A = x-a$ and $B = b-x$ in (5.57) to give

$$\begin{aligned} 45T(h,h) &= 9\left(\frac{A^5+B^5}{A+B}\right) - 5\left(\frac{A^3+B^3}{A+B}\right)^2 \\ &= 9\left[A^4 - A^3B + A^2B^2 - AB^3 + B^4\right] \\ &\quad - 5\left[A^2 - AB + B^2\right]^2 \\ &= \left(4A^2 - 7AB + 4B^2\right)(A+B)^2 \\ &= \left[\left(\frac{A+B}{2}\right)^2 + 15\left(\frac{A-B}{2}\right)^2\right](A+B)^2. \end{aligned}$$

Using the facts that $A+B=b-a$ and $A-B=2x-(a+b)$ gives

$$T(h,h)=\frac{(b-a)^2}{45}\left[\left(\frac{b-a}{2}\right)^2+15\left(x-\frac{a+b}{2}\right)^2\right], \tag{5.58}$$

and from (5.56)

$$\begin{aligned}\frac{1}{b-a}\int_a^b(x-t)^2\,dt=\frac{A^3+B^3}{3(A+B)}&=\frac{1}{3}\left[A^2-AB+B^2\right]\\&=\frac{1}{3}\left[\left(\frac{A+B}{2}\right)^2+3\left(\frac{A-B}{2}\right)^2\right],\end{aligned}$$

giving

$$\frac{1}{b-a}\int_a^b(x-t)^2\,dt=\frac{(b-a)^2}{12}+\left(x-\frac{a+b}{2}\right)^2. \tag{5.59}$$

Hence, from (5.54), (5.58) (5.59) and (5.46), the first inequality in (5.53) results. The coarsest uniform bound is obtained by taking x at either end point. Thus, the theorem is completely proved. ∎

Remark 78. *The best inequality obtainable from (5.53) is at* $x=\frac{a+b}{2}$ *giving*

$$\left|\sigma^2(X)+\left[E(X)-\frac{a+b}{2}\right]^2-\frac{(b-a)^2}{12}\right|\le\frac{M-m}{12}\frac{(b-a)^3}{\sqrt{5}}. \tag{5.60}$$

The result (5.60) is a tighter bound than that obtained in the first inequality of (5.48) since $0<M-m<2\|f\|_\infty$.

For a symmetric PDF $E(X)=\frac{a+b}{2}$ and so the above results would give bounds on the variance.

The following results hold if the PDF $f(x)$ is absolutely continuous [**7**].

Theorem 95. *Let the conditions on Theorem 92 be satisfied. Further, suppose that* f *is differentiable and is such that*

$$\|f'\|_\infty:=\sup_{t\in[a,b]}|f'(t)|<\infty,$$

then,

$$|P_V(x)|\le\frac{b-a}{\sqrt{12}}\|f'\|_\infty\cdot I(x), \tag{5.61}$$

where $P_V(x)$ *is given by the left hand side of (5.53) and,*

$$I(x)=\frac{(b-a)^2}{\sqrt{45}}\left[\left(\frac{b-a}{2}\right)^2+15\left(x-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}}. \tag{5.62}$$

PROOF. Let $h,g:[a,b]\to\mathbb{R}$ be absolutely continuous and h',g' be bounded, then, Čebyšev's inequality holds (see [**108**])

$$T(h,g)\le\frac{(b-a)^2}{\sqrt{12}}\sup_{t\in[a,b]}|h'(t)|\cdot\sup_{t\in[a,b]}|g'(t)|.$$

Matić, (et al.) [**108**] proved that

$$T(h,g)\le\frac{(b-a)}{\sqrt{12}}\sup_{t\in[a,b]}|g'(t)|\sqrt{T(h,h)}. \tag{5.63}$$

Associating $f(\cdot)$ with $g(\cdot)$ and $(x-\cdot)^2$ with $h(\cdot)$ in (5.62) gives, from (5.54) and (5.58), $I(x)=(b-a)[T(h,h)]^{\frac{1}{2}}$, which simplifies to (5.62) and the theorem is proved. ∎

THEOREM 96. *Let the conditions of Theorem 94 be satisfied. Further, suppose that f is locally absolutely continuous on (a,b) and let $f'\in L_2(a,b)$, then*

$$|P_V(x)|\le\frac{b-a}{\pi}\|f'\|_2\cdot I(x), \tag{5.64}$$

where $P_V(x)$ is the left hand side of (5.53) and $I(x)$ is as given in (5.62).

PROOF. The following result was obtained by Lupaş (see [**108**]). For $h,g:(a,b)\to\mathbb{R}$ locally absolutely continuous on (a,b) and $h',g'\in L_2(a,b)$, then

$$|T(h,g)|\le\frac{(b-a)^2}{\pi^2}\|h'\|_2\|g'\|_2,$$

where

$$\|k\|_2:=\left(\frac{1}{b-a}\int_a^b|k(t)|^2dt\right)^{\frac{1}{2}}\quad\text{for } k\in L_2(a,b).$$

[**108**] further shows that

$$|T(h,g)|\le\frac{b-a}{\pi}\|g'\|_2\sqrt{T(h,h)}. \tag{5.65}$$

Associating $f(\cdot)$ with $g(\cdot)$ and $(x-\cdot)^2$ with h in (5.65) gives (5.64), where $I(x)$ is as found in (5.62), since from (5.54) and (5.58), $I(x)=(b-a)[T(h,h)]^{\frac{1}{2}}$. ∎

Let

$$S(h(x))=h(x)-\mathcal{M}(h), \tag{5.66}$$

where

$$\mathcal{M}(h) = \frac{1}{b-a}\int_a^b h(u)\,du, \tag{5.67}$$

then, from (5.51),

$$\mathcal{T}(h,g) = \mathcal{M}(hg) - \mathcal{M}(h)\,\mathcal{M}(g)\,. \tag{5.68}$$

Dragomir and Mc Andrew [**83**] have shown, that

$$\mathcal{T}(h,g) = \mathcal{T}(S(h), S(g)) \tag{5.69}$$

and have proceeded to obtain bounds for a trapezoidal rule. Identity (5.69) is now applied to obtain bounds for the variance [**7**].

THEOREM 97. *Let X be a random variable having the PDF $f : [a,b] \subset \mathbb{R} \to \mathbb{R}_+$, then, for any $x \in [a,b]$ the following inequality holds, namely,*

$$|P_V(x)| \le \frac{8}{3}\nu^3(x)\left\| f(\cdot) - \frac{1}{b-a}\right\|_\infty, \quad \textit{if } f \in L_\infty[a,b], \tag{5.70}$$

where $P_V(x)$ is as defined by the left hand side of (5.53), and $\nu(x) = \frac{1}{3}\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2$.

PROOF. Using identity (5.69), associated with $h(\cdot)$, $(x-\cdot)^2$ and $f(\cdot)$ with $g(\cdot)$, then,

$$\begin{aligned} &\int_a^b (x-t)^2 f(t)\,dt - \mathcal{M}\left((x-\cdot)^2\right) \\ &\qquad = \int_a^b \left[(x-t)^2 - \mathcal{M}\left((x-\cdot)^2\right)\right]\left[f(t) - \frac{1}{b-a}\right]dt, \end{aligned} \tag{5.71}$$

where, from (5.67),

$$\begin{aligned} \mathcal{M}\left((x-\cdot)^2\right) &= \frac{1}{b-a}\int_a^b (x-t)^2\,dt \\ &= \frac{1}{3(b-a)}\left[(x-a)^3 + (b-x)^3\right], \end{aligned}$$

and so

$$3\mathcal{M}\left((x-\cdot)^2\right) = \left(\frac{b-a}{2}\right)^2 + 3\left(x - \frac{a+b}{2}\right)^2. \tag{5.72}$$

Further, from (5.66),

$$S\left((x-\cdot)^2\right) = (x-t)^2 - \mathcal{M}\left((x-\cdot)^2\right),$$

and so, using (5.72)

$$S\left((x-\cdot)^2\right)=(x-t)^2-\frac{1}{3}\left(\frac{b-a}{2}\right)^2-\left(x-\frac{a+b}{2}\right)^2. \tag{5.73}$$

Now, from (5.71) and using (5.46), (5.72) and (5.73), the following identity is obtained

$$\begin{aligned}\sigma^2(X)+[x-E(X)]^2-\frac{1}{3}\left[\left(\frac{b-a}{2}\right)^2+3\left(x-\frac{a+b}{2}\right)^2\right]\\ =\int_a^b S\left((x-t)^2\right)\left(f(t)-\frac{1}{b-a}\right)dt,\end{aligned} \tag{5.74}$$

where $S(\cdot)$ is as given by (5.73). Taking the modulus of (5.74) gives

$$|P_V(x)|=\left|\int_a^b S\left((x-t)^2\right)\left(f(t)-\frac{1}{b-a}\right)dt\right|. \tag{5.75}$$

Observe that, under different assumptions with regard to the norms of the PDF $f(x)$, we may obtain a variety of bounds.

For $f\in L_\infty[a,b]$ then,

$$|P_V(x)|\le\left\|f(\cdot)-\frac{1}{b-a}\right\|_\infty\int_a^b\left|S\left((x-t)^2\right)\right|dt. \tag{5.76}$$

Now, let

$$S\left((x-t)^2\right)=(t-x)^2-\nu^2=(t-X_-)(t-X_+), \tag{5.77}$$

where

$$\begin{aligned}\nu^2=\mathcal{M}\left((x-\cdot)^2\right)&=\frac{(x-a)^3+(b-x)^3}{3(b-a)}\\ &=\frac{1}{3}\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2,\end{aligned} \tag{5.78}$$

and

$$X_-=x-\nu,\quad X_+=x+\nu, \tag{5.79}$$

then,

$$\begin{aligned}H(t)=\int S\left((x-t)^2\right)dt&=\int\left[(t-x)^2-\nu^2\right]dt\\ &=\frac{(t-x)^3}{3}-\nu^2t+k,\end{aligned} \tag{5.80}$$

and so from (5.80) and using (5.77) - (5.78) gives,

$$
\begin{aligned}
(5.81)\quad & \int_a^b \left|S\left((x-t)^2\right)\right| dt \\
&= H(X_-) - H(a) - [H(X_+) - H(X_-)] + [H(b) - H(X_+)] \\
&= 2[H(X_-) - H(X_+)] + H(b) - H(a) \\
&= 2\left\{-\frac{\nu^3}{3} - \nu^2 X_- - \frac{\nu^3}{3} + \nu^2 X_+\right\} \\
&\qquad + \frac{(b-x)^3}{3} - \nu^2 b + \frac{(x-a)^3}{3} + \nu^2 a \\
&= 2\left[2\nu^3 - \frac{2}{3}\nu^3\right] + \frac{(b-x)^3 + (x-a)^3}{3} - \nu^2(b-a) = \frac{8}{3}\nu^3.
\end{aligned}
$$

Thus, substituting into (5.76), (5.75) and using (5.78) readily produces the result (5.70) and the theorem is proved. ∎

Remark 79. *Other bounds may be obtained for* $f \in L_p[a,b]$, $p \geq 1$, *however, obtaining explicit expressions for these bounds is somewhat intricate and will not be considered further here. They involve the calculation of*

$$
\sup_{t\in[a,b]} \left|(t-x)^2 - \nu^2\right| = \max\left\{\left|(x-a)^2 - \nu^2\right|, \nu^2, \left|(b-x)^2 - \nu^2\right|\right\},
$$

for $f \in L_1[a,b]$ *and*

$$
\left(\int_a^b \left|(t-x)^2 - \nu^2\right|^q dt\right)^{\frac{1}{q}},
$$

for $f \in L_q[a,b]$, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, *where* ν^2 *is given by (5.78).*

2.4. Some Inequalities for Absolutely Continuous PDFs.

Lemma 27. *Let* X *be a random variable whose probability density function* $f : [a,b] \to \mathbb{R}_+$ *is absolutely continuous on* $[a,b]$, *then, we have the identity*

$$
\begin{aligned}
(5.82)\quad & \sigma^2(X) + [E(X) - x]^2 \\
&= \frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2 \\
&\qquad + \frac{1}{b-a}\int_a^b \int_a^b (t-x)^2 p(t,s) f'(s)\, ds dt,
\end{aligned}
$$

where the kernel $p:[a,b]^2 \to \mathbb{R}$ *is given by*

$$p(t,s) := \begin{cases} s-a & \text{if } a \le s \le t \le b, \\ \\ s-b & \text{if } a \le t < s \le b, \end{cases}$$

for all $x \in [a,b]$.

PROOF. We use the identity (see (5.46))

$$\sigma^2(X) + [E(X) - x]^2 = \int_a^b (x-t)^2 f(t)\,dt, \tag{5.83}$$

for all $x \in [a,b]$.

On the other hand, we know that (see for example [**88**] for a simple proof using integration by parts)

$$f(t) = \frac{1}{b-a}\int_a^b f(s)\,ds + \frac{1}{b-a}\int_a^b p(t,s) f'(s)\,ds, \tag{5.84}$$

for all $t \in [a,b]$.

Substituting (5.84) in (5.83) we obtain

$$\begin{aligned} \sigma^2(X) &+ [E(X)-x]^2 \\ &= \int_a^b (t-x)^2 \left[\frac{1}{b-a}\int_a^b f(s)\,ds \right. \\ &\qquad \left. + \frac{1}{b-a}\int_a^b p(t,s) f'(s)\,ds\right] dt \\ &= \frac{1}{b-a}\cdot\frac{1}{3}\left[(x-a)^3 + (b-x)^3\right] \\ &\qquad + \frac{1}{b-a}\int_a^b\int_a^b (t-x)^2 p(t,s) f'(s)\,dsdt. \end{aligned} \tag{5.85}$$

Taking into account the fact that

$$\frac{1}{3}\left[(x-a)^3 + (b-x)^3\right] = \frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2, \quad x \in [a,b],$$

then, by (5.85) we deduce the desired result (5.82). ∎

The following inequality for PDFs which are absolutely continuous and have the derivatives essentially bounded holds [**7**].

THEOREM 98. *If* $f:[a,b] \to \mathbb{R}_+$ *is absolutely continuous on* $[a,b]$ *and* $f' \in L_\infty[a,b]$, *i.e.,* $\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)| < \infty$, *then we have* *the inequality:*

$$\text{(5.86)}\quad \left|\sigma^2(X)+[E(X)-x]^2-\frac{(b-a)^2}{12}-\left(x-\frac{a+b}{2}\right)^2\right| \le \frac{(b-a)^2}{3}\left[\frac{(b-a)^2}{10}+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_\infty$$

for all $x\in[a,b]$.

Proof. Using Lemma 27, we have

$$\begin{aligned}&\left|\sigma^2(X)+[E(X)-x]^2-\frac{(b-a)^2}{12}-\left(x-\frac{a+b}{2}\right)^2\right|\\&=\frac{1}{b-a}\left|\int_a^b\int_a^b(t-x)^2p(t,s)f'(s)\,dsdt\right|\\&\le\frac{1}{b-a}\int_a^b\int_a^b(t-x)^2|p(t,s)||f'(s)|\,dsdt\\&\le\frac{\|f'\|_\infty}{b-a}\int_a^b\int_a^b(t-x)^2|p(t,s)|\,dsdt.\end{aligned}$$

We have

$$\begin{aligned}I&:=\int_a^b\int_a^b(t-x)^2|p(t,s)|\,dsdt\\&=\int_a^b(t-x)^2\left[\int_a^t(s-a)\,ds+\int_t^b(b-s)\,ds\right]dt\\&=\int_a^b(t-x)^2\left[\frac{(t-a)^2+(b-t)^2}{2}\right]dt\\&=\frac{1}{2}\left[\int_a^b(t-x)^2(t-a)^2\,dt+\int_a^b(t-x)^2(b-t)^2\,dt\right]\\&=\frac{(I_a+I_b)}{2}.\end{aligned}$$

Let $A=x-a$, $B=b-x$ then

$$\begin{aligned}I_a&=\int_a^b(t-x)^2(t-a)^2\,dt\\&=\int_0^{b-a}\left(u^2-2Au+A^2\right)u^2du\end{aligned}$$

$$= \frac{(b-a)^3}{3}\left[A^2 - \frac{3}{2}A(b-a) + \frac{3}{5}(b-a)^2\right],$$

and

$$\begin{aligned} I_b &= \int_a^b (t-x)^2 (b-t)^2\, dt \\ &= \int_0^{b-a} \left(u^2 - 2Bu + B^2\right) u^2 du \\ &= \frac{(b-a)^3}{3}\left[B^2 - \frac{3}{2}B(b-a) + \frac{3}{5}(b-a)^2\right]. \end{aligned}$$

Now,

$$\begin{aligned} \frac{I_a + I_b}{2} &= \frac{(b-a)^3}{3}\left[\frac{A^2+B^2}{2} - \frac{3}{4}(A+B)(b-a) + \frac{3}{5}(b-a)^2\right] \\ &= \frac{(b-a)^3}{3}\left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2 - 3\frac{(b-a)^2}{20}\right] \\ &= \frac{(b-a)^3}{3}\left[\frac{(b-a)^2}{10} + \left(x - \frac{a+b}{2}\right)^2\right], \end{aligned}$$

and the theorem is proved. ∎

The best inequality we can get from (5.86) is embodied in the following corollary [**7**].

Corollary 61. *If f is as in Theorem 98, then we have*

$$\left|\sigma^2(X) + \left[E(X) - \frac{a+b}{2}\right]^2 - \frac{(b-a)^2}{12}\right| \leq \frac{(b-a)^4}{30}\|f'\|_\infty . \tag{5.87}$$

We now analyze the case where f' is a Lebesgue $p-$integrable mapping with $p \in (1, \infty)$.

Remark 80. *The results of Theorem 98 may be compared with those of Theorem 95. It may be shown that both bounds are convex and symmetric about $x = \frac{a+b}{2}$. Further, the bound given by the 'pre-Čebyšev' approach, namely from (5.61)-(5.62) is tighter than that obtained by the current approach (5.86) which may be shown from the following. Let these bounds be described by B_p and B_c so that, neglecting the common terms*

$$B_p = \frac{b-a}{2\sqrt{15}}\left[\left(\frac{b-a}{2}\right)^2 + 15Y\right]^{\frac{1}{2}},$$

and

$$B_c = \frac{(b-a)^2}{100} + Y,$$

where

$$Y = \left(x - \frac{a+b}{2}\right)^2.$$

It may be shown through some straightforward algebra that $B_c^2 - B_p^2 > 0$ *for all* $x \in [a,b]$*, so that* $B_c > B_p$.
The current development does, however, have the advantage that the identity (5.82) is satisfied, thus allowing bounds for $L_p[a,b]$, $p \geq 1$ *rather than the infinity norm.*

The following result also holds [**7**].

THEOREM 99. *If* $f : [a,b] \to \mathbb{R}_+$ *is absolutely continuous on* $[a,b]$ *and* $f' \in L_p$*, i.e.,*

$$\|f'\|_p := \left(\int_a^b |f'(t)|^p \, dt\right)^{\frac{1}{p}} < \infty, \quad p \in (1,\infty),$$

then we have the inequality

$$\left|\sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2\right| \tag{5.88}$$
$$\leq \frac{\|f'\|_p}{(b-a)^{\frac{1}{p}}(q+1)^{\frac{1}{q}}}\left[(x-a)^{3q+2}\tilde{B}\left(\frac{b-a}{x-a}, 2q+1, q+2\right)\right.$$
$$\left. + (b-x)^{3q+2}\tilde{B}\left(\frac{b-a}{b-x}, 2q+1, q+2\right)\right],$$

for all $x \in [a,b]$*, when* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\tilde{B}(\cdot,\cdot,\cdot)$ *is the quasi incomplete Euler's Beta mapping:*

$$\tilde{B}(z;\alpha,\beta) := \int_0^z (u-1)^{\alpha-1} u^{\beta-1} du, \quad \alpha,\beta > 0, \; z \geq 1.$$

PROOF. Using Lemma 27, we have, as in Theorem 98, that

$$\left|\sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2\right| \tag{5.89}$$
$$\leq \frac{1}{b-a}\int_a^b \int_a^b (t-x)^2 |p(t,s)| \, |f'(s)| \, ds dt.$$

Using Hölder's integral inequality for double integrals, we have

$$\begin{aligned}(5.90)\quad & \int_a^b \int_a^b (t-x)^2 |p(t,s)| |f'(s)| \, dsdt \\ & \leq \left(\int_a^b \int_a^b |f'(s)|^p \, dsdt \right)^{\frac{1}{p}} \left(\int_a^b \int_a^b (t-x)^{2q} |p(t,s)|^q \, dsdt \right)^{\frac{1}{q}} \\ & = (b-a)^{\frac{1}{p}} \|f'\|_p \left(\int_a^b \int_a^b (t-x)^{2q} |p(t,s)|^q \, dsdt \right)^{\frac{1}{q}},\end{aligned}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

We have to compute the integral

$$\begin{aligned}(5.91)\quad D & := \int_a^b \int_a^b (t-x)^{2q} |p(t,s)|^q \, dsdt \\ & = \int_a^b (t-x)^{2q} \left[\int_a^t (s-a)^q \, ds + \int_t^b (b-s)^q \, ds \right] dt \\ & = \int_a^b (t-x)^{2q} \left[\frac{(t-a)^{q+1} + (b-t)^{q+1}}{q+1} \right] dt \\ & = \frac{1}{q+1} \left[\int_a^b (t-x)^{2q} (t-a)^{q+1} \, dt \right. \\ & \qquad \left. + \int_a^b (t-x)^{2q} (b-t)^{q+1} \, dt \right].\end{aligned}$$

Define

$$(5.92)\qquad E := \int_a^b (t-x)^{2q} (t-a)^{q+1} \, dt.$$

If we consider the change of variable $t = (1-u)a + ux$, we have $t = a$ implies $u = 0$ and $t = b$ implies $u = \frac{b-a}{x-a}$, $dt = (x-a) \, du$, and then

$$\begin{aligned}(5.93)\quad E & = \int_0^{\frac{b-a}{x-a}} [(1-u)a + ux - x]^{2q} \\ & \qquad \times [(1-u)a + ux - a] (x-a) \, du \\ & = (x-a)^{3q+2} \int_0^{\frac{b-a}{x-a}} (u-1)^{2q} u^{q+1} du \\ & = (x-a)^{3q+2} \tilde{B}\left(\frac{b-a}{x-a}, 2q+1, q+2 \right).\end{aligned}$$

Define

$$F := \int_a^b (t-x)^{2q} (b-t)^{q+1} \, dt. \tag{5.94}$$

If we consider the change of variable $t = (1-v)\, b + vx$, we have $t = b$ implies $v = 0$, and $t = a$ implies $v = \frac{b-a}{b-x}$, $dt = (x-b)\, dv$, and then

$$\begin{aligned} F &= \int_{\frac{b-a}{b-x}}^{0} [(1-v)\, b + vx - x]^{2q} \\ &\qquad \times [b - (1-v)\, b - vx]^{q+1} (x-b)\, dv \\ &= (b-x)^{3q+2} \int_0^{\frac{b-a}{b-x}} (v-1)^{2q}\, v^{q+1} dv \\ &= (b-x)^{3q+2}\, \tilde{B}\left(\frac{b-a}{b-x}, 2q+1, q+2\right). \end{aligned} \tag{5.95}$$

Now, using the inequalities (5.89)-(5.90) and the relations (5.91)-(5.95), since $D = \frac{1}{q+1}(E+F)$, we deduce the desired estimate (5.88). ∎

It is natural to consider the following corollary [**7**].

Corollary 62. *Let f be as in Theorem 99, then, we have the inequality:*

$$\begin{aligned} &\left|\sigma^2(X) + \left[E(X) - \frac{a+b}{2}\right]^2 - \frac{(b-a)^2}{12}\right| \\ &\qquad \leq \frac{\|f'\|_p (b-a)^{2+\frac{3}{q}}}{(q+1)^{\frac{1}{q}}\, 2^{3+\frac{2}{q}}} \left[B(2q+1, q+1) + \Psi(2q+1, q+2)\right]^{\frac{1}{q}}, \end{aligned} \tag{5.96}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$ and $B(\cdot,\cdot)$ is Euler's Beta mapping and $\Psi(\alpha, \beta) := \int_0^1 u^{\alpha-1} (u+1)^{\beta-1}\, du$, $\alpha, \beta > 0$.

Proof. In (5.88) put $x = \frac{a+b}{2}$ and the right hand side is,

$$\begin{aligned} \tilde{B}(2, 2q+1, q+2) &= \int_0^2 (u-1)^{2q}\, u^{q+1} du \\ &= \int_0^1 (u-1)^{2q}\, u^{q+1} du + \int_1^2 (u-1)^{2q}\, u^{q+1} du \\ &= B(2q+1, q+2) + \Psi(2q+1, q+2). \end{aligned}$$

The right hand side of (5.88) is thus

$$\frac{\|f'\|_p \left(\frac{b-a}{2}\right)^{\frac{3q+2}{q}}}{(b-a)^{\frac{1}{p}}(q+1)^{\frac{1}{q}}}\left[2B(2q+1,q+2)+2\Psi(2q+1,q+2)\right]^{\frac{1}{q}}$$
$$=\frac{\|f'\|_p (b-a)^{2+\frac{3}{q}}}{(q+1)^{\frac{1}{q}}\,2^{3+\frac{2}{q}}}\left[B(2q+1,q+2)+\Psi(2q+1,q+2)\right]^{\frac{1}{q}},$$

and the corollary is proved. ∎

Finally, as f is absolutely continuous, $f' \in L_1[a,b]$ and $\|f'\|_1 = \int_a^b |f'(t)|\,dt$, and we can state the following theorem [**7**].

THEOREM 100. *If the PDF, $f:[a,b]\to\mathbb{R}_+$ is absolutely continuous on $[a,b]$, then*

$$\left|\sigma^2(X)+[E(X)-x]^2-\frac{(b-a)^2}{12}-\left(x-\frac{a+b}{2}\right)^2\right| \le \|f'\|_1 (b-a)\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^2, \tag{5.97}$$

for all $x\in[a,b]$.

PROOF. As above, we can state that

$$\left|\sigma^2(X)+[E(X)-x]^2-\frac{(b-a)^2}{12}-\left(x-\frac{a+b}{2}\right)^2\right|$$
$$\le \frac{1}{b-a}\int_a^b\int_a^b (t-x)^2|p(t,s)||f'(s)|\,dsdt$$
$$\le \sup_{(t,s)\in[a,b]^2}\left[(t-x)^2|p(t,s)|\right]\frac{1}{b-a}\int_a^b\int_a^b |f'(s)|\,dsdt$$
$$=\|f'\|_1 G,$$

where

$$G:=\sup_{(t,s)\in[a,b]^2}\left[(t-x)^2|p(t,s)|\right]\le (b-a)\sup_{t\in[a,b]}(t-x)^2$$
$$=(b-a)\left[\max(x-a,b-x)\right]^2$$
$$=(b-a)\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^2,$$

and the theorem is proved. ∎

It is clear that the best inequality we can get from (5.97) is the one when $x = \frac{a+b}{2}$, giving the following corollary [**7**].

COROLLARY 63. *With the assumptions of Theorem 100, we have:*

$$\left|\sigma^2(X) + \left[E(X) - \frac{a+b}{2}\right]^2 - \frac{(b-a)^2}{12}\right| \le \frac{(b-a)^3}{4}\|f'\|_1. \tag{5.98}$$

3. Further Inequalities for Univariate Moments

3.1. Introduction. The aim of this section is to provide some additional inequalities on expectation utilising a generalisation of

$$E(X) = b - \int_a^b F(x)dx \tag{5.99}$$

to higher moments, as well as providing some specific results for the extreme order statistics.

In addition, some results are obtained involving the covariance of two random variables using a bivariate generalisation of (5.99) and generalisations of the inequalities of Grüss and Ostrowski.

3.2. Univariate Results. Denote by M_n the n^{th} moment $\int_a^b x^n f(x)\,dx$ which, using integration by parts, can be expressed as

$$b^n - n\int_a^b x^{n-1}F(x)\,dx. \tag{5.100}$$

Consider now

$$\left|\frac{1}{b-a}\int_a^b x^{n-1}F(x)\,dx - \frac{1}{(b-a)^2}\int_a^b x^{n-1}dx\int_a^b F(x)\,dx\right|$$

for which various inequalities can be found. Before exploiting some of these, we express the difference in terms of the distributional moments.

$$\frac{1}{(b-a)^2}\int_a^b x^{n-1}dx\int_a^b F(x)\,dx = \frac{(b^n-a^n)}{n(b-a)^2}\{b - E(X)\}.$$

We therefore have:

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b x^{n-1}F(x)\,dx - \frac{(b^n-a^n)}{n(b-a)^2}(b - E(X))\right| \\ &= \left|\frac{b_n}{n(b-a)} - \frac{M_n}{n(b-a)} - \frac{b(b^n-a^n)}{n(b-a)^2} + \frac{M_1(b^n-a^n)}{n(b-a)^2}\right| \\ &= \left|\frac{ab(a^{n-1}-b^{n-1})}{n(b-a)^2} - \frac{M_n}{n(b-a)} + \frac{M_1(b^n-a^n)}{n(b-a)^2}\right| \end{aligned} \tag{5.101}$$

$$= \frac{1}{n(b-a)^2} \left| a^n (b - M_1) - b^n (a - M_1) - M_n (b-a) \right|.$$

Now utilising various inequalities, we can obtain a number of results.

Pre-Grüss.

Using an inequality of [**108**] applied to (5.101), we have

$$\frac{1}{n(b-a)^2} \left| a^n (b - M_1) - b^n (a - M_1) - M_n (b-a) \right| \tag{5.102}$$
$$\leq \frac{1}{2} \left[\frac{1}{(b-a)} \int_a^b x^{2(n-1)} dx - \left(\frac{1}{(b-a)} \int_a^b x^{n-1} dx \right)^2 \right]^{\frac{1}{2}}$$
$$= \frac{1}{2} \left[\frac{b^{2n-1} - a^{2n-1}}{(b-a)(2n-1)} - \left(\frac{b^n - a^n}{n(b-a)} \right)^2 \right]^{\frac{1}{2}}.$$

The special case when $n = 2$ gives:

$$\left| a^2 (b - E(X)) - b^2 (a - E(X)) - (b-a) \left(\sigma^2 + (E(X))^2 \right) \right|$$
$$\leq (b-a)^2 \left[\frac{b^3 - a^3}{3(b-a)} - \left(\frac{b^2 - a^2}{2(b-a)} \right)^2 \right]^{\frac{1}{2}},$$

that is [**19**],

$$\left| (b - E(X)) (E(X) - a) - \sigma^2 \right|$$
$$\leq (b-a)^2 \left[\frac{b^2 + ab + a^2}{3} - \frac{(b^2 + 2ab + a^2)}{4} \right]^{\frac{1}{2}}$$
$$= \frac{(b-a)^3}{2\sqrt{3}},$$

which is Theorem 3 of [**6**].

Pre-Chebychev.

Using a further result of [**108**] and (5.101) we can obtain

$$\frac{1}{n(b-a)^2} \left| a^n (b - M_1) - b^n (a - M_1) - M_n (b-a) \right|$$
$$\leq \frac{(b-a)}{2\sqrt{3}} \|f\|_\infty \left[\frac{1}{(b-a)} \int_a^b x^{2(n-1)} dx - \left(\frac{1}{(b-a)} \int_a^b x^{n-1} dx \right)^2 \right]^{\frac{1}{2}},$$

giving

$$|a^n (b - M_1) - b^n (a - M_1) - M_n (b - a)| \le \frac{n (b-a)^3}{2\sqrt{3}} \|f\|_\infty \left[\frac{b^{2n-1} - a^{2n-1}}{(b-a)(2n-1)} - \left(\frac{b^n - a^n}{n(b-a)} \right)^2 \right]^{\frac{1}{2}} . \tag{5.103}$$

The special case where $n = 2$ gives [**19**]:

$$\left| (b - E(X)) (E(X) - a) - \sigma^2 \right| \le \frac{(b-a)^3}{6} \|f\|_\infty$$

which was obtained by Barnett and Dragomir in [**18**].

3.3. Lipschitzian Mappings. If $x^{n-1} F(x)$ is of the Lipschitzian type, then

$$\left| x^{n-1} F(x) - y^{n-1} F(y) \right| \le L |x - y| ,$$

where $L \ge 0$ in which case

$$\left| x^{n-1} F(x) - \frac{1}{b-a} \int_a^b x^{n-1} F(x)\, dx \right| \le \frac{L}{2} \left[\left(\frac{x-a}{b-a} \right)^2 + \left(\frac{b-x}{b-a} \right)^2 \right] (b-a) .$$

(Ostrowski's inequality, [**75**].)
Now,

$$\left| x^{n-1} F(x) - \frac{1}{b-a} \int_a^b x^{n-1} F(x)\, dx \right| = \left| x^{n-1} F(x) - \left\{ \frac{b^n - M_n}{n(b-a)} \right\} \right|$$

and thus we have:

$$\left| x^{n-1} F(x) - \left\{ \frac{b^n - M_n}{n(b-a)} \right\} \right| \le \frac{L}{2} \left[\left(\frac{x-a}{b-a} \right)^2 + \left(\frac{b-x}{b-a} \right)^2 \right] (b-a) .$$

If $n = 2$, then

$$\left| x F(x) - \left\{ \frac{b^2 - M_2}{2(b-a)} \right\} \right| \le \frac{L}{2} \left[\left(\frac{x-a}{b-a} \right)^2 + \left(\frac{b-x}{b-a} \right)^2 \right] (b-a) .$$

Consider now the mapping $F(x)$, $x \in [a, b]$, then the mapping is Lipschitzian if there exists $L > 0$ such that

$$|F(x) - F(y)| \le L |x - y| .$$

Now, if $F(\cdot)$ is a cumulative distribution function, it is monotonic increasing between 0 and 1 over $[a,b]$. It is apparent that there exists $z \in [x,y]$ such that

$$\frac{F(x)-F(y)}{x-y} \leq \left[\frac{dF(x)}{dx}\right]_{x=z}.$$

Thus, if we choose $L = \max\left[\frac{dF(x)}{dx}\right]_{x=z}$ for $z \in [a,b]$, then this implies that $F(x)$ is Lipschitzian, since

$$|F(x)-F(y)| \leq \|f\|_{\infty}|x-y|.$$

Consider similarly the mapping $xF(x)$, it is also monotonic increasing and by the same token, there exists $z \in [x,y]$ such that

$$\left|\frac{xF(x)-yF(y)}{x-y}\right| \leq \left|\left[\frac{d(xF(x))}{dx}\right]_{x=z}\right|.$$

In addition, we have that

$$\left|\left[\frac{d(xF(x))}{dx}\right]_{x=z}\right| = \left|\left[F(x)+x\frac{dF(x)}{dx}\right]_{x=z}\right|$$

and hence

$$\begin{aligned}\left|F(z)+z\left[\frac{dF(x)}{dx}\right]_{x=z}\right| &\leq |F(z)| + \left|z\left[\frac{dF(x)}{dx}\right]_{x=z}\right| \\ &\leq 1+\|f\|_{\infty}\max\{|a|,|b|\}.\end{aligned}$$

Thus, L can be taken to be

$$1+\|f\|_{\infty}\max\{|a|,|b|\}$$

and then

$$|xF(x)-yF(y)| < [1+\|f\|_{\infty}\max\{|a|,|b|\}]\,|x-y|,$$

and so $xF(x)$ is Lipschitzian.

Similarly it can be shown that $x^{n-1}F(x)$ is Lipschitzian for $n = 3,4,\ldots$

Thus [**19**],

$$\begin{aligned}&\left|xF(x)-\left\{\frac{b^2-M_2}{2(b-a)}\right\}\right| \\ &\quad\leq \frac{1}{2}[1+\|f\|_{\infty}\max\{|a|,|b|\}]\left[\left(\frac{x-a}{b-a}\right)^2+\left(\frac{b-x}{b-a}\right)^2\right](b-a).\end{aligned}$$

For $x = a$ we get [**19**]

$$\begin{aligned} \left|M_2 - b^2\right| &= \left|\sigma^2 + \left((E(X))^2\right) - b^2\right| \\ &\leq (b-a)^2 \left[1 + \|f\|_\infty \max\{|a|, |b|\}\right] \end{aligned}$$

and for $x = b$ we have [**19**]

$$\begin{aligned} \left|2b(b-a) - b^2 + M_2\right| &= \left|b^2 - 2ab + M_2\right| \\ &= \left|b(b-2a) + \sigma^2 + \left((E(X))^2\right)\right| \\ &\leq (b-a)^2 \left[1 + \|f\|_\infty \max\{|a|, |b|\}\right]. \end{aligned}$$

3.4. Distributions of the Maximum, Minimum and Range of a Sample. Consider a continuous random variable X with a non-zero probability density function over a finite interval $[a, b]$ and let X_1, X_2,..., X_n be a random sample . We investigate the distribution function of the maximum, minimum and the range of the random sample.

Maximum

Let the cumulative distribution function of the maximum be $G(x)$, the probability density function be $g(x)$, and the corresponding functions for X be $F(x)$ and $f(x)$. Then

$$G(x) = \Pr[\max \leq x] = \Pr[\text{all } X_1, \ldots, X_n \leq x].$$

Therefore $G(x) = [F(x)]^n$ and $g(x) = n[F(x)]^{n-1} f(x)$.

Minimum

Let the cumulative distribution function of the minimum and the probability density function be $H(x)$ and $h(x)$ respectively. Therefore

$$\begin{aligned} H(x) &= \Pr[\min \leq x] = 1 - \Pr[\min \geq x] \\ &= 1 - \Pr[\text{all } X_1, \ldots, X_n \geq x] = 1 - [1 - F(x)]^n \end{aligned}$$

and

$$h(x) = n[1 - F(x)]^{n-1} f(x).$$

Range

Let the distribution function of the range be $R(x)$ and the probability density function be $r(x)$.

Now, consider

$$\begin{aligned} K(s,t) &= \Pr[\max \leq s, \min \leq t] \\ &= \Pr[\max \leq s] - \Pr[\max \leq s, \min \geq t] \\ &= [F(s)]^n - \Pr\{t \leq \text{all } X_1, \ldots, X_n \leq s\} \\ &= [F(s)]^n - [F(s) - F(t)]^n. \end{aligned}$$

Therefore, the joint probability density function of the extreme order statistics can be found by differentiating this with respect to s and t, giving

$$k(s,t) = n(n-1) f(s) f(t) [F(s) - F(t)]^{n-2}, \ s > t$$

which for a random variable defined on $[a,b]$ gives

$$r(x) = n(n-1) \int_{s=a}^{b-x} f(s+x) f(s) [F(s+x) - F(s)]^{n-2} ds,$$

where $0 < x < b-a$.

3.5. Application of the Grüss Inequality to Positive Integer Powers of a Function. As a preliminary to the proof of Grüss' inequality we can establish the identity:

$$\frac{1}{b-a} \int_a^b g(x) f(x) dx = p + \left(\frac{1}{b-a}\right)^2 \int_a^b f(x) dx \cdot \int_a^b g(x) dx, \tag{5.104}$$

where it can be subsequently shown that

$$|p| \leq \frac{1}{4} [\Gamma - \gamma] [\Phi - \phi],$$

and $\gamma, \Gamma, \phi, \Phi$ are respectively lower and upper bounds of $f(x)$ and $g(x)$.

Applying this same identity to the square of a function, we have

$$\frac{1}{b-a} \int_a^b f^2(x) dx = p_1 + \left(\frac{1}{b-a}\right)^2 \left(\int_a^b f(x) dx\right)^2 \tag{5.105}$$

Similarly,

$$\frac{1}{b-a} \int_a^b f^3(x) dx = p_2 + \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b f^2(x) dx$$

and using (5.104), we have

$$\frac{1}{b-a} \int_a^b f^3(x) dx = p_2 + \frac{p_1}{b-a} \int_a^b f(x) dx + \left(\frac{1}{b-a}\right)^3 \left(\int_a^b f(x) dx\right)^3. \tag{5.106}$$

Continuing, we can show that for positive integers n, $n \geq 2$

$$(5.107)\quad \frac{1}{b-a}\int_a^b f^n(x)\,dx - \left(\frac{1}{b-a}\right)^n \left(\int_a^b f(x)\,dx\right)^n$$
$$= p_{n-1} + p_{n-2}\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) + p_{n-3}\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2 + \cdots + p_1\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^{n-2},$$

giving:

$$\left|\frac{1}{b-a}\int_a^b f^n(x)\,dx - \left(\frac{1}{b-a}\right)^n \left(\int_a^b f(x)\,dx\right)^n\right|$$
$$\leq |p_{n-1}| + |p_{n-2}|\left|\frac{1}{b-a}\int_a^b f(x)\,dx\right| + \dots + |p_1|\left|\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^{n-2}\right|,$$

where

$$|p_1| \leq \frac{1}{4}(\Gamma-\gamma)^2, \qquad \gamma < f(x) < \Gamma,$$
$$|p_2| \leq \frac{1}{4}(\Gamma-\gamma)(\Phi_1-\phi_1), \quad \phi_1 < f^2(x) < \Phi_1,$$
$$|p_3| \leq \frac{1}{4}(\Gamma-\gamma)(\Phi_2-\phi_2), \quad \phi_2 < f^3(x) < \Phi_2,$$

and so on.

Assuming that $f(x) \geq 0$ and denoting $\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^n$ by λ^n, we have

$$(5.108)\quad \left|\frac{1}{b-a}\int_a^b f^n(x)\,dx - \left(\frac{1}{b-a}\right)^n \left(\int_a^b f(x)\,dx\right)^n\right|$$
$$\leq \frac{1}{4}(\Gamma-\gamma)\left[\Gamma^{n-1}-\gamma^{n-1}\right] + \frac{1}{4}(\Gamma-\gamma)\left[\Gamma^{n-2}-\gamma^{n-2}\right]\lambda + \frac{1}{4}(\Gamma-\gamma)\left[\Gamma^{n-3}-\gamma^{n-3}\right]\lambda^2 + \dots + \frac{1}{4}(\Gamma-\gamma)(\Gamma-\gamma)\lambda^{n-2}$$

$$\le \frac{1}{4}(\Gamma - \gamma) \sum_{i=1}^{n-1} \left(\Gamma^{n-i} - \gamma^{n-i}\right) \lambda^{i-1}.$$

Now, if f is a PDF, the right hand side reduces to

$$\begin{aligned}
&\frac{1}{4}\Gamma \sum_{i=1}^{n-1} \Gamma^{n-i} \left(\frac{1}{b-a}\right)^{i-1} \\
&= \frac{1}{4}\Gamma^{n} \sum_{i=1}^{n-1} \left(\frac{1}{\Gamma(b-a)}\right)^{i-1} = \frac{1}{4}\Gamma^{n} \left(\frac{1-\left(\frac{1}{\Gamma(b-a)}\right)^{n-1}}{1-\frac{1}{\Gamma(b-a)}}\right) \\
&= \frac{\Gamma^{n}}{4\Gamma^{n-2}(b-a)^{n-2}} \left(\frac{\Gamma^{n-1}(b-a)^{n-1}-1}{\Gamma(b-a)-1}\right) \\
&= \frac{\Gamma^{2}}{4(b-a)^{n-2}} \left(\frac{\Gamma^{n-1}(b-a)^{n-1}-1}{\Gamma(b-a)-1}\right).
\end{aligned}$$

If we now consider this inequality for an associated cumulative distribution function $F(\cdot)$, we have that

$$\lambda = \frac{b-E(X)}{b-a}$$

and the right hand side of (5.108) becomes

$$\frac{1}{4}\sum_{i=1}^{n-1} \left(\frac{b-E(X)}{b-a}\right)^{i-1} = \frac{\left[(b-a)^{n-1} - (b-E(X))^{n-1}\right]}{4(E(X)-a)(b-a)^{n-2}}.$$

Thus, we have the two inequalities [**19**]:

$$\left|\frac{1}{b-a}\int_a^b f^n(x)\,dx - \left(\frac{1}{b-a}\right)^n\right| \le \frac{\Gamma^2}{4(b-a)^{n-2}} \left(\frac{\Gamma^{n-1}(b-a)^{n-1}-1}{\Gamma(b-a)-1}\right) \tag{5.109}$$

and

$$\left|\frac{1}{b-a}\int_a^b F^n(x)\,dx - \left(\frac{b-E(X)}{b-a}\right)^n\right| \le \frac{\left[(b-a)^{n-1} - (b-E(X))^{n-1}\right]}{4(E(X)-a)(b-a)^{n-2}}. \tag{5.110}$$

Similarly, we can develop an inequality for $1 - F(x)$ by suitable substitution in (5.108), that is,

$$\lambda = \frac{E(X) - a}{b - a}$$

which gives [**19**]

$$\left| \frac{1}{b-a} \int_a^b (1 - F(x))^n \, dx - \left(\frac{1}{b-a} \int_a^b (1 - F(x)) \, dx \right)^n \right| \tag{5.111}$$

$$= \left| \frac{1}{b-a} \int_a^b (1 - F(x))^n \, dx - \left(\frac{E(X) - a}{b-a} \right)^n \right|$$

$$\leq \frac{\left[(b-a)^{n-1} - (E(X) - a)^{n-1} \right]}{4 (b-a)^{n-2} (b - E(X))}.$$

3.6. Inequalities for the Expectation of the Extreme Order Statistics. As the PDF of the maximum is

$$g(x) = n [F(x)]^{n-1} f(x),$$

then

$$E[X_{\max}] = n \int_a^b x [F(x)]^{n-1} f(x) \, dx.$$

Integrating by parts gives

$$E[X_{\max}] = n \left[\left[\frac{x}{n} [F(x)]^n \right]_a^b - \frac{1}{n} \int_a^b (F(x))^n \, dx \right]$$

$$= b - \int_a^b F^n(x) \, dx$$

giving, from (5.110)

$$\left| \frac{b - E[X_{\max}]}{b-a} - \left(\frac{b - E(X)}{b-a} \right)^n \right| \tag{5.112}$$

$$\leq \frac{\left[(b-a)^{n-1} - (b - E(X))^{n-1} \right]}{4 (E(X) - a) (b-a)^{n-2}}$$

and when $E(X) = \frac{a+b}{2}$, we have [**19**]

$$\left| \frac{b - E[X_{\max}]}{b-a} - \frac{1}{2^n} \right| \leq \left(\frac{2^{n-1} - 1}{2^n} \right). \tag{5.113}$$

Consider now $E(X_{\min}) = n\int_{x=a}^{b} xf(x)[1-F(x)]^{n-1}\,dx$. Integration by parts gives

$$E[X_{\min}] = n\left[\left[-\frac{x}{n}[1-F(x)]^n\right]_a^b + \frac{1}{n}\int_a^b (1-F(x))^n\,dx\right]$$

and so

$$E(X_{\min}) = a + \int_a^b (1-F(x))^n\,dx.$$

Utilising (5.111) we have [**19**]

$$\left|\frac{E(X_{\min})-a}{b-a} - \left(\frac{E(X)-a}{b-a}\right)^n\right| \leq \frac{\left[(b-a)^{n-1} - (E(X)-a)^{n-1}\right]}{4(b-a)^{n-2}(b-E(X))} \tag{5.114}$$

and when $E(X) = \frac{a+b}{2}$, we have, [**19**]

$$\left|\frac{E(X_{\min})-a}{b-a} - \frac{1}{2^n}\right| \leq \left(\frac{2^{n-1}-1}{2^n}\right). \tag{5.115}$$

3.7. Applications to the Beta Distribution. The Beta probability density function is given by

$$\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1},\ 0<x<1,\ \alpha,\beta>0,$$

then clearly,

$$\begin{aligned}
E(X) &= \frac{1}{B(\alpha,\beta)}\int_0^1 x^{\alpha+1-1}(1-x)^{\beta-1}\,dx \\
&= \frac{1}{B(\alpha,\beta)}B(\alpha+1,\beta) \\
&= \frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha)} \\
&= \frac{\alpha\Gamma(\alpha)\Gamma(\alpha+\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)\Gamma(\alpha)} = \frac{\alpha}{(\alpha+\beta)}.
\end{aligned}$$

Substituting $a=0$, $b=1$ and letting 'Γ'$\equiv m$, from (5.109) we obtain

$$\left|\int_0^1 f^n(x)\,dx - 1\right| \leq \frac{m^2(1-m^{n-1})}{4(1-m)}$$

and further,

$$\begin{aligned}&\left|\frac{1}{B^{n}(\alpha,\beta)}\int_{0}^{1}x^{n(\alpha-1)}(1-x)^{n(\beta-1)}\,dx-1\right|\\&\qquad=\left|\frac{B(n(\alpha-1)+1,n(\beta-1)+1)}{B^{n}(\alpha,\beta)}-1\right|\leq\frac{m^{2}(1-m^{n-1})}{4(1-m)}\end{aligned}$$

and m is the value of x for which $f'(x)=0$, that is

$$\begin{aligned}&(1-x)^{\beta-1}(\alpha-1)x^{\alpha-2}+x^{\alpha-1}(\beta-1)(1-x)^{\beta-2}(-1)=0.\\&x^{\alpha-2}(1-x)^{\beta-2}\{(\alpha-1)(1-x)-x(\beta-1)\},\\&\text{i.e. } m=\frac{\alpha-1}{\alpha+\beta-2},\quad \alpha,\beta>1.\end{aligned}$$

We then have the inequality [**19**]:

$$\begin{aligned}&\left|\frac{B(n(\alpha-1)+1,n(\beta-1)+1)}{B^{n}(\alpha,\beta)}-1\right|\\&\qquad\leq\frac{(\alpha-1)^{2}}{4(\alpha+\beta-2)(\beta-1)}\left\{1-\left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{n-1}\right\}.\end{aligned}$$

When $\alpha=\beta$, the right hand side becomes $\frac{1}{8}\left(1-\frac{1}{2^{n-1}}\right)$.

Consider now (5.112) when $f(x)$ is the PDF of a Beta distribution. This gives:

$$\begin{aligned}\left|1-E(X_{\max})-\left(1-\frac{\alpha}{\alpha+\beta}\right)^{n}\right|&=\left|1-\left(\frac{\beta}{\alpha+\beta}\right)^{n}-E(X_{\max})\right|\\&\leq\frac{\left[1-\left(1-\frac{\alpha}{\alpha+\beta}\right)^{n-1}\right]}{4\left(\frac{\alpha}{\alpha+\beta}\right)}\\&=\frac{(\alpha+\beta)^{n-1}-\beta^{n-1}}{4\alpha(\alpha+\beta)^{n-2}}\end{aligned}$$

and when $\alpha=\beta$, this becomes:

$$\left|1-\left(\frac{1}{2}\right)^{n}-E(X_{\max})\right|\leq\frac{1-\left(\frac{1}{2}\right)^{n-1}}{2}$$

and from (5.114)

$$\left|E(X_{\min})-\left(\frac{\alpha}{\alpha+\beta}\right)^{n}\right|\leq\frac{\left[1-\left(\frac{\alpha}{\alpha+\beta}\right)^{n-1}\right]}{4\left(1-\frac{\alpha}{\alpha+\beta}\right)}=\frac{(\alpha+\beta)^{n-1}-\alpha^{n-1}}{4\beta(\alpha+\beta)^{n-2}}$$

and when $\alpha = \beta$

$$\left| E\left(X_{\min}\right) - \frac{1}{2^n} \right| \le \frac{1 - \left(\frac{1}{2}\right)^{n-1}}{2}.$$

From these we can obtain further $E\left(X_{\min}\right) \le \frac{1}{2}$ and $\frac{1}{2} \le E\left(X_{\max}\right) \le \frac{3}{2} - \left(\frac{1}{2}\right)^{n-1}$.

3.8. Some Bounds for Joint Moments and Probabilities Using Ostrowski Type Inequalities for Double Integrals. The following result holds [**19**].

THEOREM 101. *Let X, Y be two continuous random variables $x \in [a,b]$, $y \in [c,d]$ with probability density function s $f_1(\cdot)$ and $f_2(\cdot)$ respectively and with joint probability density function $f(\cdot,\cdot)$ with associated cumulative distribution functions $F_1(\cdot)$, $F_2(\cdot)$ and $F(\cdot,\cdot)$. It follows that*

$$E(XY) = bE(Y) + dE(X) - bd + \int_{s=a}^{b}\int_{t=c}^{d} F(s,t)\,dsdt. \tag{5.116}$$

This is a generalisation of the result

$$E(X) = b - \int_a^b F(s)\,ds$$

and is equivalent to:

$$E(XY) = bd - d\int_a^b F_1(s)\,ds - b\int_c^d F_2(t)\,dt + \int_{s=a}^{b}\int_{t=c}^{d} F(s,t)\,dsdt. \tag{5.117}$$

PROOF. We have

$$E(XY) = \int_{t=c}^{d} t\left\{\int_a^b sf(s,t)\,ds\right\}dt$$

and

$$\begin{aligned}\int_a^b sf(s,t)\,ds &= \left[s\int_{u=a}^{s} f(u,t)\,du\right]_{s=a}^{b} - \int_{s=a}^{b}\left(\int_{u=a}^{s} f(u,t)\,du\right)ds \\ &= bf_2(t) - \int_{s=a}^{b}\left(\int_{u=a}^{s} f(u,t)\,du\right)ds,\end{aligned}$$

so

$$\begin{aligned} E(XY) &= b\int_c^d tf_2(t)\,dt - \int_{t=c}^d t\left[\int_{s=a}^b \left(\int_{u=a}^s f(u,t)\,du\right)ds\right]dt \\ &= bE(Y) - \int_{s=a}^b \left(\int_{u=a}^s \left(\int_{t=c}^d tf(u,t)\,dt\right)du\right)ds \end{aligned}$$

Now

$$\begin{aligned} \int_{t=c}^d tf(u,t)\,dt &= \left[t\int_{v=c}^t f(u,v)\,dv\right]_{t=c}^d - \int_{t=c}^d \left(\int_{v=c}^t f(u,v)\,dv\right)dt \\ &= df_1(u) - \int_{t=c}^d \left(\int_{v=c}^t f(u,v)\,dv\right)dt, \end{aligned}$$

$$\begin{aligned} &E(XY) \\ &= bE(Y) - \int_{s=a}^b \left(\int_{u=a}^s \left\{df_1(u) - \int_{t=c}^d \left(\int_{v=c}^t f(u,v)\,dv\right)dt\right\}du\right)ds, \\ &= bE(Y) - d\int_{s=a}^b F_1(s)\,ds \\ &\quad + \int_{t=c}^d \left(\int_{s=a}^b \left(\int_{v=c}^t \left(\int_{u=a}^s f(u,v)\,du\right)dv\right)ds\right)dt \\ &= bE(Y) + dE(X) - bd + \int_{s=a}^b \int_{t=c}^d F(s,t)\,dsdt \end{aligned}$$

and, equivalently,

$$E(XY) = bd - d\int_a^b F_1(s)\,ds - b\int_c^d F_2(t)\,dt + \int_a^b \int_c^d F(s,t)\,dsdt.$$

∎

In [**11**] Barnett and Dragomir proved the following theorem.

THEOREM 102. *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be continuous on* $[a,b] \times [c,d]$, $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$ *exists on* $(a,b) \times (c,d)$ *and is bounded, i.e.,*

$$\left\|f''_{s,t}\right\|_\infty := \sup_{(x,y)\in(a,b)\times(c,d)} \left|\frac{\partial^2 f(x,y)}{\partial x \partial y}\right| < \infty$$

then we have the inequality:

$$\left|\int_a^b \int_c^d f(s,t)\,dsdt - \left[(b-a)\int_c^d f(x,t)\,dt + (d-c)\int_a^b f(s,y)\,ds - (d-c)(b-a) f(x,y)\right]\right| \leq \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \times \left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2}\right)^2\right] \left\|f''_{s,t}\right\|_\infty \tag{5.118}$$

for all $(x,y) \in [a,b] \times [c,d]$.

If we apply this taking $f(\cdot,\cdot)$ to be a joint cumulative distribution function $F(\cdot,\cdot)$ with $x = b$, $y = d$ we obtain

$$\left|\int_a^b \int_c^d F(s,t)\,dsdt - (b-a)\int_c^d F(b,t)\,dt - (d-c)\int_a^b F(s,d)\,ds + (d-c)(b-a)\right| \leq \frac{1}{4}(b-a)^2(d-c)^2 \left\|F''_{s,t}\right\|_\infty,$$

that is

$$\left|\int_a^b \int_c^d F(s,t)\,dsdt - (b-a)\int_c^d F_2(t)\,dt - (d-c)\int_a^b F_1(s)\,ds + (d-c)(b-a)\right| \leq \frac{1}{4}(b-a)^2(d-c)^2 \left\|F''_{s,t}\right\|_\infty.$$

Using (5.117), this gives

$$\begin{aligned}
&\left|E(XY) + a\int_c^d F_2(t)\,dt + c\int_a^b F_1(s)\,ds - ad - bc + ac\right| \\
&= |E(XY) + aE(Y) - cE(X) + ac| \\
&\leq \frac{1}{4}(b-a)^2(d-c)^2 \left\|F''_{s,t}\right\|_\infty \\
&= \frac{1}{4}(b-a)^2(d-c)^2 \|f\|_\infty,
\end{aligned}$$

providing bounds for $E(XY)$ in terms of $E(X)$ and $E(Y)$.

Since $Cov(X,Y) = E(XY) - E(X)\cdot E(Y)$, we can write the left hand side alternatively as:

$$Cov(X,Y) + [c - E(Y)][a - E(X)].$$

We can similarly extract other bounds from (5.118) in the situations where

(i) $x = b,\ y = c,$
(ii) $x = a,\ y = d,$ and
(iii) $x = a,\ y = c$

giving respectively

$$|E(XY) - dE(X) - aE(Y) + ad| \le \frac{1}{4}(b-a)^2(d-c)^2\|f\|_\infty,$$
$$|E(XY) - cE(X) - bE(Y) + bc| \le \frac{1}{4}(b-a)^2(d-c)^2\|f\|_\infty$$

and

$$|E(XY) - dE(X) - bE(Y) + bd| \le \frac{1}{4}(b-a)^2(d-c)^2\|f\|_\infty.$$

We can use the results of [**77**] by Dragomir, Cerone, Barnett and Roumeliotis to obtain further inequalities relating the first single and joint moments as well as some involving joint probabilities.

In [**77**], bounds were obtained for:

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)\,dsdt - f(x,y)\right|$$

namely $M_1(x) + M_2(y) + M_3(x,y)$ where these are as defined in [**77**]. For one particular case we have

$$M_1(x) = \frac{\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]}{(b-a)(d-c)}\left\|\frac{\partial f(s,t)}{\partial t}\right\|_1,$$
$$M_2(y) = \frac{\left[\frac{1}{2}(d-c) + \left|y - \frac{c+d}{2}\right|\right]}{(b-a)(d-c)}\left\|\frac{\partial f(s,t)}{\partial s}\right\|_1$$

and

$$M_3(x,y) = \frac{\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c) + \left|y - \frac{c+d}{2}\right|\right]}{(b-a)(d-c)} \times \left\|\frac{\partial^2 f(s,t)}{\partial s\partial t}\right\|_1.$$

It follows then that if we choose f to be the joint cumulative distribution function, $F(x,y)$, we can generate the following inequalities

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(s,t)\,dsdt\right| \leq M_1(a)+M_2(c)+M_3(a,c),$$

and

$$\left|\int_a^b\int_c^d F(s,t)\,dsdt - \Pr\{X\leq x, Y\leq y\}\right| \\ \leq M_1(x)+M_2(y)+M_3(x,y).$$

The first of these simplifies to give [**19**]:

$$|E(XY)-bE(Y)-dE(X)+bd| \\ \leq (b-a)+(d-c)+(d-c)(b-a).$$

3.9. Further Inequalities for the Covariance Using the Grüss Inequality. Consider functions $f_1(\cdot)$, $f_2(\cdot)$ and $f(\cdot,\cdot)$ where f_1 is integrable over $[a,b]$, f_2 is integrable over $[c,d]$ and $f(x,y)$ is integrable for $x\in[a,b]$ and $y\in[c,d]$. Consider the integral

$$\int_a^b\int_c^d f_1(s)\,f_2(t)\,f(s,t)\,dsdt = \int_{s=a}^b f_1(s)\left(\int_{t=c}^d f_2(t)\,f(s,t)\,dt\right)ds.$$

We have:

$$\frac{1}{d-c}\int_c^d f_2(t)\,f(s,t)\,dt - \left(\frac{1}{d-c}\right)^2\int_c^d f_2(t)\,dt\int_{t=c}^d f(s,t)\,dt \\ = p_1(s) \quad \text{(say)}.$$

Hence,

$$\int_a^b\int_c^d f_1(s)\,f_2(t)\,f(s,t)\,dsdt \\ = \int_a^b f_1(s)\left\{p_1(s)(d-c)+\frac{1}{d-c}\int_c^d f_2(t)\,dt\int_c^d f(s,t)\,dt\right\}ds \\ = (d-c)\int_a^b p_1(s)\,f_1(s)\,ds \\ + \frac{1}{d-c}\int_c^d f_2(t)\,dt\int_c^d\left(\int_a^b f_1(s)\,f(s,t)\,ds\right)dt$$

and where

$$\frac{1}{b-a}\int_a^b f_1(s)\cdot f(s,t)\,ds - \frac{1}{(b-a)^2}\int_a^b f_1(s)\,ds\int_a^b f(s,t)\,ds = p_2(t).$$

Therefore

$$\begin{aligned}
&\int_a^b \int_c^d f_1(s) f_2(t) f(s,t)\, dsdt \\
&= (d-c) \int_a^b p_1(s) f_1(s)\, ds + \frac{1}{(d-c)} \int_c^d f_2(t)\, dt \\
&\quad \times \int_c^d \left[(b-a) p_2(t) + \frac{1}{b-a} \int_a^b f_1(s)\, ds \int_a^b f(s,t)\, ds \right] dt \\
&= (d-c) \int_a^b p_1(s) f_1(s)\, ds + \frac{b-a}{(d-c)} \int_c^d f_2(t)\, dt \int_c^d p_2(t)\, dt \\
&\quad + \frac{1}{(b-a)(d-c)} \int_c^d f_2(t)\, dt \int_a^b f_1(s)\, ds \int_a^b \int_c^d f(s,t)\, dsdt
\end{aligned}$$

and thus

$$\begin{aligned}
&\left| \int_a^b \int_c^d f_1(s) f_2(t) f(s,t)\, dsdt \right. \\
&\quad \left. - \frac{1}{(b-a)(d-c)} \int_c^d f_2(t)\, dt \int_a^b f_1(s)\, ds \int_a^b \int_c^d f(s,t)\, dsdt \right| \\
&\qquad \leq (d-c) \|p_1\|_\infty \int_a^b |f_1(s)|\, ds + (b-a) \|p_2\|_\infty \int_c^d |f_2(t)|\, dt.
\end{aligned} \tag{5.119}$$

CASE 1. *Now, if $f_1(s) = s$ and $f_2(t) = t$, and $f(\cdot,\cdot)$ is a joint probability density function, the left hand side becomes*

$$\begin{aligned}
&\left| E(XY) - \frac{1}{4(d-c)(b-a)} \left(d^2 - c^2\right)\left(b^2 - a^2\right) \right| \\
&\qquad\qquad = \left| E(XY) - \frac{1}{4}(b+a)(d+c) \right|.
\end{aligned}$$

$$\begin{aligned}
|p_1(s)| &\leq \frac{1}{2} \|f\|_\infty \left[\frac{(d^3 - c^3)}{3(d-c)} - \left(\frac{1}{d-c} \int_c^d t dt \right)^2 \right]^{\frac{1}{2}} \quad \text{(see [}\mathbf{108}\text{])} \\
&= \frac{1}{2} \|f\|_\infty \left[\frac{(d-c)(d^2 + dc + c^2)}{3(d-c)} - \left(\frac{(d^2 - c^2)}{2(d-c)} \right)^2 \right]^{\frac{1}{2}} \\
&= \frac{1}{2} \|f\|_\infty \left[\frac{d^2 + dc + c^2}{3} - \frac{(d+c)^2}{4} \right]^{\frac{1}{2}} \\
&= \frac{1}{2} \|f\|_\infty \times \frac{1}{2\sqrt{3}} \left[d^2 - 2dc + c^2 \right]^{\frac{1}{2}} = \frac{1}{4\sqrt{3}} \|f\|_\infty (d-c).
\end{aligned}$$

Similarly,

$$p_2 \le \frac{a^2 - b^2}{2}, \ a < 0, \ b < 0.$$

Now

$$\int_a^b |f_1(s)|\, ds = \frac{a^2 + b^2}{2}, \ a < 0, \ b > 0$$

and similarly,

$$\int_c^d |f_2(t)|\, dt \le \frac{b^2 - a^2}{2}, \ a > 0, \ b > 0.$$

Thus, we then have for $a < 0$, $b > 0$, $c < 0$, $d > 0$:

$$\begin{aligned}
&\left| E(XY) - \frac{1}{4}(b+a)(d+c) \right| \\
&\le \frac{1}{4\sqrt{3}} \|f\|_\infty \frac{(d-c)^2 (a^2 + b^2)}{2} + \frac{1}{4\sqrt{3}} \|f\|_\infty \frac{(b-a)^2 (c^2 + d^2)}{2} \\
&= \frac{1}{8\sqrt{3}} \left[(d-c)^2 \left(a^2 + b^2\right) + (b-a)^2 \left(c^2 + d^2\right) \right] \|f\|_\infty \\
&= \frac{1}{4\sqrt{3}} \left[\left(a^2 + b^2\right)\left(c^2 + d^2\right) + (ac + db)(ad + bc) \right] \|f\|_\infty .
\end{aligned}$$

CASE 2. *If $f_1(s) = s$ and $f_2(t) = 1$, and $f(\cdot, \cdot) = t\phi(s,t)$ where $\phi(\cdot, \cdot)$ is a joint probability density function, then the left hand side is:*

$$\left| \int_a^b \int_c^d st\phi(s,t)\, dsdt - \frac{(d-c)(b^2 - a^2)}{2(d-c)(b-a)} E(Y) \right| = \left| E(XY) - \frac{1}{2}(a+b) E(Y) \right| .$$

p_2 is as above and

$$p_1 \le \frac{1}{2} \|f\|_\infty \left[\frac{1}{d-c} \int_c^d dt - \left(\frac{1}{d-c} \int_c^d dt \right)^2 \right]^{\frac{1}{2}} = 0$$

and hence [**19**]

$$\left| E(XY) - \frac{1}{2}(a+b) E(Y) \right| \le \frac{(b-a)^2 (c^2 + d^2)}{8\sqrt{3}} \|f\|_\infty$$

when $a < 0, \ b > 0, \ c < 0, \ d > 0$.

CHAPTER 6

Inequalities for n-Time Differentiable PDFs

1. Random Variable whose PDF is n-Times Differentiable

1.1. Introduction. In [**7**], using the identity

$$[x-E(X)]^2+\sigma^2(X)=\int_a^b (x-t)^2 f(t)\,dt \tag{6.1}$$

and applying a variety of inequalities such as: Hölder's inequality, pre-Grüss, pre-Čebyšev, pre-Lupaş, or Ostrowski type inequalities, a number of results concerning the expectation and variance of the random variable X have been obtained.

For example,

$$\sigma^2(X)+[x-E(X)]^2 \tag{6.2}$$
$$\le \begin{cases} (b-a)\left[\frac{(b-a)^2}{12}+\left(x-\frac{a+b}{2}\right)^2\right]\|f\|_\infty, & \text{if } f\in L_\infty[a,b]; \\ \left[\frac{(b-x)^{2q+1}+(x-a)^{2q+1}}{2q+1}\right]^{\frac{1}{q}}\|f\|_p, & \text{if } f\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left(\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right)^2, & \end{cases}$$

for all $x\in[a,b]$, which imply, amongst other things, that

$$\sigma(X)\le \begin{cases} (b-a)^{\frac{1}{2}}\left[\frac{(b-a)^2}{12}+\left[E(X)-\frac{a+b}{2}\right]^2\right]^{\frac{1}{2}}\|f\|_\infty^{\frac{1}{2}}, \\ \qquad \text{if } f\in L_\infty[a,b]; \\ \left\{\frac{[b-E(X)]^{2q+1}+[E(X)-a]^{2q+1}}{2q+1}\right\}^{\frac{1}{2q}}\|f\|_p^{\frac{1}{2}}, \\ \qquad \text{if } f\in L_p[a,b],\ p>1,\ \ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{b-a}{2}+\left|E(X)-\frac{a+b}{2}\right|, \end{cases} \tag{6.3}$$

and

$$\sigma^2(X)\le[b-E(X)][E(X)-a]\le\frac{1}{4}(b-a)^2. \tag{6.4}$$

In this section more accurate inequalities are obtained by assuming that the PDF of X is n-time differentiable and that $f^{(n)}$ is absolutely continuous on $[a, b]$.

1.2. Some Preliminary Integral Identities. The following lemma, which is interesting in itself, holds [**8**].

LEMMA 28. *Let X be a random variable whose PDF $f : [a, b] \to \mathbb{R}_+$ is n-time differentiable and $f^{(n)}$ is absolutely continuous on $[a, b]$, then,*

$$\begin{aligned}(6.5)\quad & \sigma^2(X) + [E(X) - x]^2 \\ &= \sum_{k=0}^{n} \frac{(b-x)^{k+3} + (-1)^k (x-a)^{k+3}}{(k+3)\,k!} f^{(k)}(x) \\ &\quad + \frac{1}{n!} \int_a^b (t-x)^2 \left(\int_x^t (t-s)^n f^{(n+1)}(s)\,ds \right) dt\end{aligned}$$

for all $x \in [a, b]$.

PROOF. By Taylor's formula with integral remainder, we recall that,

$$f(t) = \sum_{k=0}^{n} \frac{(t-x)^k}{k!} f^{(k)}(x) + \frac{1}{n!} \int_x^t (t-s)^n f^{(n+1)}(s)\,ds, \tag{6.6}$$

for all $t, x \in [a, b]$.

Together with (6.1), we obtain

$$\begin{aligned}(6.7)\quad & \sigma^2(X) + [E(X) - x]^2 \\ &= \int_a^b (t-x)^2 \left[\sum_{k=0}^{n} \frac{(t-x)^k}{k!} f^{(k)}(x) \right. \\ &\quad \left. + \frac{1}{n!} \int_x^t (t-s)^n f^{(n+1)}(s)\,ds \right] dt \\ &= \sum_{k=0}^{n} f^{(k)}(x) \int_a^b \frac{(t-x)^{k+2}}{k!} dt \\ &\quad + \frac{1}{n!} \int_a^b (t-x)^2 \left(\int_x^t (t-s)^n f^{(n+1)}(s)\,ds \right) dt,\end{aligned}$$

and since

$$\int_a^b \frac{(t-x)^{k+2}}{k!} dt = \frac{(b-x)^{k+3} + (-1)^k (x-a)^{k+3}}{(k+3)\,k!}, \tag{6.8}$$

the identity (6.7) readily produces (6.5) ∎

We may state the following corollary as well [**8**].

COROLLARY 64. *Under the above assumptions, we have*

$$\sigma^2(X) + \left[E(X) - \frac{a+b}{2}\right]^2 = \sum_{k=0}^{n} \frac{\left[1+(-1)^k\right](b-a)^{k+3}}{2^{k+3}(k+3)k!} f^{(k)}\left(\frac{a+b}{2}\right) + \frac{1}{n!}\int_a^b \left(t - \frac{a+b}{2}\right)^2 \left(\int_{\frac{a+b}{2}}^t (t-s)^n f^{(n+1)}(s)\,ds\right) dt. \tag{6.9}$$

The proof follows by using (6.7) with $x = \frac{a+b}{2}$.
Another result is embodied in the following (see [**8**]).

COROLLARY 65. *Under the above assumptions,*

$$\sigma^2(X) + \frac{1}{2}\left[(E(X)-a)^2 + (E(X)-b)^2\right] = \sum_{k=0}^{n} \frac{(b-a)^{k+3}}{(k+3)k!}\left[\frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2}\right] + \frac{1}{n!}\int_a^b \int_a^b K(t,s)(t-s)^n f^{(n+1)}(s)\,ds dt, \tag{6.10}$$

where

$$K(t,s) := \begin{cases} \frac{(t-a)^2}{2} & \text{if } a \le s \le t \le b, \\ -\frac{(t-b)^2}{2} & \text{if } a \le t < s \le b. \end{cases}$$

PROOF. In (6.5), choose $x = a$ and $x = b$, giving

$$\sigma^2(X) + [E(X) - a]^2 = \sum_{k=0}^{n} \frac{(b-a)^{k+3}}{(k+3)k!} f^{(k)}(a) + \frac{1}{n!}\int_a^b (t-a)^2 \left(\int_a^t (t-s)^n f^{(n+1)}(s)\,ds\right) dt, \tag{6.11}$$

and

$$\sigma^2(X) + [E(X) - b]^2 = \sum_{k=0}^{n} \frac{(-1)^k (b-a)^{k+3}}{(k+3)\,k!} f^{(k)}(b) + \frac{1}{n!} \int_a^b (t-b)^2 \left(\int_b^t (t-s)^n f^{(n+1)}(s)\,ds \right) dt. \tag{6.12}$$

Adding these and dividing by 2 gives (6.10). ∎

Taking into account that $\mu = E(X) \in [a,b]$, then we also obtain the following [**8**].

Corollary 66. *With the above assumptions,*

$$\sigma^2(X) = \sum_{k=0}^{n} \frac{(b-\mu)^{k+3} + (-1)^k (\mu - a)^{k+3}}{(k+3)\,k!} f^{(k)}(\mu) + \frac{1}{n!} \int_a^b (t-\mu)^2 \left(\int_\mu^t (t-s)^n f^{(n+1)}(s)\,ds \right) dt. \tag{6.13}$$

Proof. The proof follows from (6.5) with $x = \mu \in [a,b]$. ∎

We state the following lemma which is interesting in itself as well [**8**].

Lemma 29. *Let the conditions of Lemma 28 relating to f hold, then, the following identity is valid*

$$\sigma^2(X) + [E(X) - x]^2 = \sum_{k=0}^{n} \frac{(b-x)^{k+3} + (-1)^k (x-a)^{k+3}}{k+3} \cdot \frac{f^{(k)}(x)}{k!} + \frac{1}{n!} \int_a^b K_n(x,s) f^{(n+1)}(s)\,ds, \tag{6.14}$$

where

$$K(x,s) = \begin{cases} (-1)^{n+1} \psi_n(s-a, x-s), & a \le s \le x, \\ \psi_n(b-s, s-x), & x < s \le b, \end{cases} \tag{6.15}$$

with

$$\psi_n(u,v) = \frac{u^{n+1}}{(n+3)(n+2)(n+1)} \cdot \left[(n+2)(n+1)u^2 + 2(n+3)(n+1)uv + (n+3)(n+2)v^2 \right]. \tag{6.16}$$

PROOF. From (6.5), an interchange of the order of integration gives

$$\begin{aligned}
&\frac{1}{n!}\int_a^b (t-x)^2\,dt\int_x^t (t-s)^n f^{(n+1)}(s)\,ds \\
&\qquad = \frac{1}{n!}\Big\{-\int_a^x\int_a^s (t-x)^2 (t-s)^n f^{(n+1)}(s)\,dtds \\
&\qquad\qquad + \int_x^b\int_s^b (t-x)^2 (t-s)^n f^{(n+1)}(s)\,dtds\Big\} \\
&\qquad = \frac{1}{n!}\int_a^b \tilde{K}_n(x,s)\, f^{(n+1)}(s)\,ds,
\end{aligned}$$

where

$$\tilde{K}_n(x,s) = \begin{cases} p_n(x,s) = -\int_a^s (t-x)^2 (t-s)^n\,dt, & a \le s \le x, \\ q_n(x,s) = \int_s^b (t-x)^2 (t-s)^n\,dt, & x < s < b. \end{cases}$$

To prove the lemma it is sufficient to show that $K \equiv \tilde{K}$.

Now,

$$\begin{aligned}
\tilde{p}_n(x,s) &= -\int_a^s (t-x)^2 (t-s)^n\,dt \\
&= (-1)^{n+1}\int_0^{s-a} (u+x-s)^2 u^n du \\
&= (-1)^{n+1}\int_0^{s-a} \left[u^2 + 2(x-s)u + (x-s)^2\right] u^n du \\
&= (-1)^{n+1}\,\psi_n(s-a, x-s),
\end{aligned}$$

where $\psi(\cdot,\cdot)$ is as given by (6.16).

Further,

$$\begin{aligned}
\tilde{q}_n(x,s) &= \int_s^b (t-x)^2 (t-s)^n\,dt \\
&= \int_0^{b-s} [u+(s-x)]^2 u^n du \\
&= \psi_n(b-s, s-x),
\end{aligned}$$

where, again, $\psi(\cdot,\cdot)$ is as given by (6.16). Hence, $K \equiv \tilde{K}$ and the lemma is proved. ∎

1.3. Some Estimates. We are now able to obtain the following inequalities [**8**].

THEOREM 103. *Let X be a random variable whose probability density function $f:[a,b]\to\mathbb{R}_+$ is n-time differentiable and $f^{(n)}$ is absolutely continuous on $[a,b]$, then*

$$\left|\sigma^2(X)+[E(X)-x]^2-\sum_{k=0}^{n}\frac{(b-x)^{k+3}+(-1)^k(x-a)^{k+3}}{(k+3)\,k!}f^{(k)}(x)\right| \tag{6.17}$$
$$\le\begin{cases}\frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!(n+4)}\left[(x-a)^{n+4}+(b-x)^{n+4}\right], & \text{if } f^{(n+1)}\in L_\infty[a,b];\\ \frac{\left\|f^{(n+1)}\right\|_p}{n!\left(n+3+\frac{1}{q}\right)}\frac{\left[(x-a)^{n+3+\frac{1}{q}}+(b-x)^{n+3+\frac{1}{q}}\right]}{(nq+1)^{\frac{1}{q}}}, & \text{if } f^{(n+1)}\in L_p[a,b],\\ & p>1,\frac{1}{p}+\frac{1}{q}=1;\\ \frac{\left\|f^{(n+1)}\right\|_1}{n!(n+3)}\left[(x-a)^{n+3}+(b-x)^{n+3}\right], & \text{if } f^{(n+1)}\in L_1[a,b],\end{cases}$$

for all $x\in[a,b]$, where $\|\cdot\|_p$ $(1\le p\le\infty)$ are the usual Lebesgue norms on $[a,b]$.

PROOF. By Lemma 28,

$$\begin{aligned}&\sigma^2(X)+[E(X)-x]^2\\&\qquad-\sum_{k=0}^{n}\frac{(b-x)^{k+3}+(-1)^k(x-a)^{k+3}}{k!\,(k+3)}f^{(k)}(x)\\&=\frac{1}{n!}\int_a^b(t-x)^2\left(\int_x^t(t-s)^n f^{(n+1)}(s)\,ds\right)dt\\&:=M(a,b;x).\end{aligned}\tag{6.18}$$

Clearly,

$$\begin{aligned}&|M(a,b;x)|\\&\le\frac{1}{n!}\int_a^b(t-x)^2\left|\int_x^t(t-s)^n f^{(n+1)}(s)\,ds\right|dt\\&\le\frac{1}{n!}\int_a^b(t-x)^2\left[\sup_{s\in[x,t]}\left|f^{(n+1)}(s)\right|\left|\int_x^t|t-s|^n\,ds\right|\right]dt\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\left\|f^{(n+1)}\right\|_{\infty}}{n!} \int_a^b \frac{(t-x)^2 |t-x|^{n+1}}{n+1} dt \\
&= \frac{\left\|f^{(n+1)}\right\|_{\infty}}{(n+1)!} \int_a^b |t-x|^{n+3} \, dt \\
&= \frac{\left\|f^{(n+1)}\right\|_{\infty}}{(n+1)!} \left[\int_a^x (x-t)^{n+3} \, dt + \int_x^b (t-x)^{n+3} \, dt\right] \\
&= \frac{\left\|f^{(n+1)}\right\|_{\infty} \left[(x-a)^{n+4} + (b-x)^{n+4}\right]}{(n+1)! \, (n+4)},
\end{aligned}
$$

and the first inequality in (6.17) is obtained.

For the second, we use Hölder's integral inequality to obtain

$$
\begin{aligned}
&|M(a,b;x)| \\
&\leq \frac{1}{n!} \int_a^b (t-x)^2 \left|\int_x^t |t-s|^{nq} \, ds\right|^{\frac{1}{q}} \left|\int_x^t \left|f^{(n+1)}(s)\right|^p ds\right|^{\frac{1}{p}} dt \\
&\leq \frac{1}{n!} \left(\int_a^b \left|f^{(n+1)}(s)\right|^p ds\right)^{\frac{1}{p}} \int_a^b (t-x)^2 |t-x|^{\frac{nq+1}{q}} \, dt \\
&= \frac{1}{n!} \frac{\left\|f^{(n+1)}\right\|_p}{(nq+1)^{\frac{1}{q}}} \int_a^b |t-x|^{n+2+\frac{1}{q}} \, dt \\
&= \frac{1}{n!} \frac{\left\|f^{(n+1)}\right\|_p}{(nq+1)^{\frac{1}{q}}} \left[\frac{(b-x)^{n+3+\frac{1}{q}} + (x-a)^{n+3+\frac{1}{q}}}{n+3+\frac{1}{q}}\right].
\end{aligned}
$$

Finally, note that

$$
\begin{aligned}
|M(a,b;x)| &\leq \frac{1}{n!} \int_a^b (t-x)^2 |t-x|^n \left|\int_x^t \left|f^{(n+1)}(s)\right| ds\right| dt \\
&\leq \frac{\left\|f^{(n+1)}\right\|_1}{n!} \int_a^b |t-x|^{n+2} \, dt \\
&= \frac{\left\|f^{(n+1)}\right\|_1}{n!} \left[\frac{(x-a)^{n+3} + (b-x)^{n+3}}{n+3}\right],
\end{aligned}
$$

and the third part of (6.17) is obtained. ∎

It is obvious, that the best inequality in (6.17) is when $x = \frac{a+b}{2}$, giving Corollary 67 (see also [**8**]).

COROLLARY 67. *With the above assumptions on X and f,*

$$\begin{aligned}(6.19)\quad &\left|\sigma^2(X)+\left[E(X)-\frac{a+b}{2}\right]^2\right.\\ &\left.-\sum_{k=0}^{n}\frac{\left[1+(-1)^k\right](b-a)^{k+3}}{2^{k+3}(k+3)\,k!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\ &\leq\begin{cases}\frac{\left\|f^{(n+1)}\right\|_\infty}{2^{n+3}(n+1)!(n+4)}(b-a)^{n+4}, & \text{if } f^{(n+1)}\in L_\infty[a,b];\\ \frac{\left\|f^{(n+1)}\right\|_p}{2^{n+2+\frac{1}{q}}n!\left(n+3+\frac{1}{q}\right)}\cdot\frac{(b-a)^{n+3+\frac{1}{q}}}{(nq+1)^{\frac{1}{q}}}, & \text{if } f^{(n+1)}\in L_p[a,b],\\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1;\\ \frac{\left\|f^{(n+1)}\right\|_1}{2^{n+2}n!(n+3)}(b-a)^{n+3}, & \text{if } f^{(n+1)}\in L_1[a,b].\end{cases}\end{aligned}$$

The following corollary is interesting as it provides the opportunity to approximate the variance when the values of $f^{(k)}(\mu)$ are known, $k=0,...,n$ (cf. [**8**]).

COROLLARY 68. *With the above assumptions and $\mu=\frac{a+b}{2}$, we have*

$$\begin{aligned}(6.20)\quad &\left|\sigma^2(X)-\sum_{k=0}^{n}\frac{(b-\mu)^{k+3}+(-1)^k(\mu-a)^{k+3}}{(k+3)\,k!}f^{(k)}(\mu)\right|\\ &\leq\begin{cases}\frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!(n+4)}\left[(\mu-a)^{n+4}+(b-\mu)^{n+4}\right], & \text{if } f^{(n+1)}\in L_\infty[a,b];\\ \frac{\left\|f^{(n+1)}\right\|_p}{n!\left(n+3+\frac{1}{q}\right)}\cdot\frac{\left[(\mu-a)^{n+3+\frac{1}{q}}+(b-\mu)^{n+3+\frac{1}{q}}\right]}{(nq+1)^{\frac{1}{q}}}, & \\ & \text{if } f^{(n+1)}\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{\left\|f^{(n+1)}\right\|_1}{n!(n+3)}\left[(\mu-a)^{n+3}+(b-\mu)^{n+3}\right], & \text{if } f^{(n+1)}\in L_1[a,b].\end{cases}\end{aligned}$$

The following result also holds [**8**].

THEOREM 104. *Let X be a random variable whose probability density function $f:[a,b]\to\mathbb{R}_+$ is n-time differentiable and $f^{(n)}$ is absolutely continuous on $[a,b]$, then*

$$\begin{aligned}(6.21)\quad &\left|\sigma^2(X)+\frac{1}{2}\left[(E(X)-a)^2+(E(X)-b)^2\right]\right.\\ &\left.-\sum_{k=0}^{n}\frac{(b-a)^{k+3}}{(k+3)\,k!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\right|\end{aligned}$$

$$\leq \begin{cases} \frac{1}{(n+4)(n+1)!} \left\| f^{(n+1)} \right\|_{\infty} (b-a)^{n+4}, \\ \qquad\qquad\qquad\qquad \text{if } \ f^{(n+1)} \in L_{\infty}[a,b]; \\ \frac{1}{n!(qn+1)^{\frac{1}{q}}[(n+2)q+2]^{\frac{1}{q}}} \left\| f^{(n+1)} \right\|_p \frac{(b-a)^{n+3+\frac{1}{q}}}{(nq+1)^{\frac{1}{q}}}, \\ \text{if } \ f^{(n+1)} \in L_p[a,b], \text{ where } \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2n!} \left\| f^{(n+1)} \right\|_1 (b-a)^{n+3}. \end{cases}$$

PROOF. Using Corollary 65,

$$\begin{aligned} &\left| \sigma^2(X) + \frac{1}{2}\left[(E(X)-a)^2 + (E(X)-b)^2 \right] \right. \\ &\qquad \left. - \sum_{k=0}^{n} \frac{(b-a)^{k+3}}{(k+3)\,k!} \left[\frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] \right| \\ &\qquad \leq \frac{1}{n!} \int_a^b \int_a^b |K(t,s)|\, |t-s|^n \left| f^{(n+1)}(s) \right| ds dt =: N(a,b). \end{aligned}$$

It is obvious that,

$$\begin{aligned} &N(a,b) \\ &\leq \left\| f^{(n+1)} \right\|_{\infty} \frac{1}{n!} \int_a^b \int_a^b |K(t,s)|\, |t-s|^n \, ds dt \\ &= \left\| f^{(n+1)} \right\|_{\infty} \frac{1}{n!} \int_a^b \left(\int_a^t |K(t,s)|\, |t-s|^n \, ds + \int_t^b |K(t,s)|\, |t-s|^n \, ds \right) dt \\ &= \frac{1}{n!} \left\| f^{(n+1)} \right\|_{\infty} \int_a^b \left[\frac{(t-a)^2}{2} \cdot \frac{(t-a)^{n+1}}{n+1} + \frac{(t-b)^2}{2} \cdot \frac{(b-t)^{n+1}}{n+1} \right] dt \\ &= \frac{1}{2(n+1)!} \left\| f^{(n+1)} \right\|_{\infty} \int_a^b \left[(t-a)^{n+3} + (b-t)^{n+3} \right] dt \\ &= \frac{1}{2(n+1)!} \left\| f^{(n+1)} \right\|_{\infty} \left[\frac{(b-a)^{n+4}}{n+4} + \frac{(b-a)^{n+4}}{n+4} \right] \\ &= \frac{\left\| f^{(n+1)} \right\|_{\infty}}{(n+4)(n+1)!} (b-a)^{n+4}, \end{aligned}$$

so the first part of (6.21) is proved.

Using Hölder's integral inequality for double integrals,

$$
\begin{aligned}
&N(a,b)\\
&\leq \frac{1}{n!}\left(\int_a^b\int_a^b \left|f^{(n+1)}(s)\right|^p dsdt\right)^{\frac{1}{p}}\\
&\qquad\times\left(\int_a^b\int_a^b |K(t,s)|^q\,|t-s|^{qn}\,dsdt\right)^{\frac{1}{q}}\\
&=\frac{(b-a)^{\frac{1}{p}}\left\|f^{(n+1)}\right\|_p}{n!}\cdot\left[\int_a^b\left(\int_a^t |K(t,s)|^q\,|t-s|^{qn}\,ds\right.\right.\\
&\qquad\left.\left.+\int_t^b |K(t,s)|^q\,|t-s|^{qn}\,ds\right)dt\right]^{\frac{1}{q}}\\
&=\frac{(b-a)^{\frac{1}{p}}\left\|f^{(n+1)}\right\|_p}{n!}\cdot\left[\int_a^b\left[\frac{(t-a)^{2q}}{2^q}\int_a^t |t-s|^{qn}\,ds\right.\right.\\
&\qquad\left.\left.+\frac{(t-b)^{2q}}{2^q}\int_t^b |t-s|^{qn}\,ds\right]dt\right]^{\frac{1}{q}}\\
&=\frac{(b-a)^{\frac{1}{p}}\left\|f^{(n+1)}\right\|_p}{n!}\cdot\left[\int_a^b\left[\frac{(t-a)^{2q}(t-a)^{qn+1}}{2^q(qn+1)}\right.\right.\\
&\qquad\left.\left.+\frac{(t-b)^{2q}(b-t)^{qn+1}}{2^q(qn+1)}\right]dt\right]^{\frac{1}{q}}\\
&=\frac{(b-a)^{\frac{1}{p}}\left\|f^{(n+1)}\right\|_p}{n!}\cdot\left[\frac{1}{2^q(qn+1)}\right]^{\frac{1}{q}}\\
&\qquad\times\left[\int_a^b (t-a)^{(n+2)q+1}\,dt+\int_a^b (b-t)^{(n+2)q+1}\,dt\right]^{\frac{1}{q}}\\
&=\frac{(b-a)^{\frac{1}{p}}\left\|f^{(n+1)}\right\|_p}{n!}\cdot\left[\frac{1}{2^q(qn+1)}\right]^{\frac{1}{q}}\\
&\qquad\times\left[\frac{(b-a)^{(n+2)q+2}}{(n+2)q+2}+\frac{(b-a)^{(n+2)q+2}}{(n+2)q+2}\right]^{\frac{1}{q}}\\
&=\frac{2\left\|f^{(n+1)}\right\|_p(b-a)^{n+2+\frac{1}{p}+\frac{2}{q}}}{n!2(qn+1)^{\frac{1}{q}}((n+2)q+2)^{\frac{1}{q}}}=\frac{\left\|f^{(n+1)}\right\|_p\left[(b-a)^{n+3+\frac{1}{q}}\right]}{n!(qn+1)^{\frac{1}{q}}[(n+2)q+2]^{\frac{1}{q}}},
\end{aligned}
$$

and the second part of (6.21) is proved.

Finally, we observe that

$$\begin{aligned} N(a,b) &\leq \frac{1}{n!}\sup_{(t,s)\in[a,b]^2}|K(t,s)|\,|t-s|^n\int_a^b\int_a^b\left|f^{(n+1)}(s)\right|dsdt \\ &= \frac{1}{n!}\frac{(b-a)^2}{2}\cdot(b-a)^n(b-a)\int_a^b\left|f^{(n+1)}(s)\right|ds \\ &= \frac{1}{2n!}(b-a)^{n+3}\left\|f^{(n+1)}\right\|_1, \end{aligned}$$

which is the final result of (6.21). ∎

The following particular cases can be useful in practical applications (see also [**8**]).

(1) For $n=0$, (6.17) becomes

$$\begin{aligned} &\left|\sigma^2(X)+[E(X)-x]^2\right. \\ &\qquad\left.-(b-a)\left[\left(x-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{12}\right]f(x)\right| \\ &\leq\begin{cases}\frac{\|f'\|_\infty}{4}\left[(x-a)^4+(b-x)^4\right], & \text{if } f'\in L_\infty[a,b]; \\ \frac{q\|f'\|_p}{3q+1}\left[(x-a)^{3+\frac{1}{q}}+(b-x)^{3+\frac{1}{q}}\right], & \text{if } f'\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \|f'\|_1\left[\frac{(b-a)^2}{12}+\left(x-\frac{a+b}{2}\right)^2\right], & \text{if } f'\in L_1[a,b],\end{cases} \end{aligned} \tag{6.22}$$

for all $x\in[a,b]$. In particular, for $x=\frac{a+b}{2}$,

$$\begin{aligned} &\left|\sigma^2(X)+\left[E(X)-\frac{a+b}{2}\right]^2-\frac{(b-a)^3}{12}f\left(\frac{a+b}{2}\right)\right| \\ &\leq\begin{cases}\frac{\|f'\|_\infty}{32}(b-a)^4, & \text{if } f'\in L_\infty[a,b]; \\ \frac{q\|f'\|_p(b-a)^{3+\frac{1}{q}}}{2^{2+\frac{1}{q}}(3q+1)}, & \text{if } f'\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f'\|_1}{12}(b-a)^3, & \end{cases} \end{aligned} \tag{6.23}$$

which is, in a sense, the best inequality that can be obtained from (6.22). If in (6.22) $x=\mu=E(X)$, then,

$$\left|\sigma^2(X)-(b-a)\left[\left(E(X)-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{12}\right]f(E(X))\right| \tag{6.24}$$
$$\leq\begin{cases}\frac{\|f'\|_\infty}{4}\left[(E(X)-a)^4+(b-E(X))^4\right], & \text{if } f'\in L_\infty[a,b];\\ \frac{\|f'\|_p}{\left(3+\frac{1}{q}\right)}\left[(E(X)-a)^4+(b-E(X))^4\right], & \text{if } f'\in L_p[a,b],\ p>1,\\ \|f'\|_1\left[\frac{(b-a)^2}{12}+\left(E(X)-\frac{a+b}{2}\right)^2\right], & \text{if } f'\in L_1[a,b].\end{cases}$$

In addition, from (6.21),

$$\left|\sigma^2(X)+\frac{1}{2}\left[(E(X)-a)^2+(E(X)-b)^2\right]-\frac{(b-a)^3}{3}\left[\frac{f(a)+f(b)}{2}\right]\right| \tag{6.25}$$
$$\leq\begin{cases}\frac{1}{4}\|f'\|_\infty(b-a)^4, & \text{if } f'\in L_\infty[a,b];\\ \frac{1}{n!2^{\frac{1}{q}}(q+1)^{\frac{1}{q}}}\|f'\|_p(b-a)^{3+\frac{1}{q}}, & \text{if } f'\in L_p[a,b],\ p>1,\\ \frac{1}{2}\|f'\|_1(b-a)^3,\end{cases}$$

which provides an approximation for the variance in terms of the expectation and the values of f at the end points a and b.

We may now state and prove the following result as well (cf. [**8**]).

THEOREM 105. *Let X be a random variable whose PDF $f:[a,b]\to\mathbb{R}_+$ is $n-$time differentiable and $f^{(n)}$ is absolutely continuous on $[a,b]$, then,*

$$\left|\sigma^2(X)+(E(X)-x)^2-\sum_{k=0}^{n}\frac{(b-x)^{k+3}+(-1)^k(x-a)^{k+3}}{k+3}\cdot\frac{f^{(k)}(x)}{k!}\right| \tag{6.26}$$
$$\leq\begin{cases}\left[(x-a)^{n+4}+(b-x)^{n+4}\right]\frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!(n+4)},\\ C^{\frac{1}{q}}\left[(x-a)^{(n+3)q+1}+(b-x)^{(n+3)q+1}\right]^{\frac{1}{q}}\frac{\left\|f^{(n+1)}\right\|_p}{n!},\\ \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^{n+3}\cdot\frac{\left\|f^{(n+1)}\right\|_1}{n!(n+3)},\end{cases}$$

where

$$C=\int_0^1\left[\frac{u^{n+3}}{n+3}+2\left(1-u\right)\frac{u^{n+2}}{n+2}+\left(1-u\right)^2\frac{u^{n+1}}{n+1}\right]^q du. \tag{6.27}$$

PROOF. From (6.14),

$$\begin{aligned}\Bigg|\sigma^2\left(X\right)+\left(E\left(X\right)-x\right)^2 \\ -\sum_{k=0}^{n}\frac{\left(b-x\right)^{k+3}+\left(-1\right)^k\left(x-a\right)^{k+3}}{k+3}\cdot\frac{f^{(k)}\left(x\right)}{k!}\Bigg| \\ =\left|\frac{1}{n!}\int_a^b K_n\left(x,s\right)f^{(n+1)}\left(s\right)ds\right|.\end{aligned} \tag{6.28}$$

Now, on using the fact that from (6.15), (6.16), $\psi_n\left(u,v\right)\geq 0$ for $u,v\geq 0$,

$$\begin{aligned}\left|\frac{1}{n!}\int_a^b K_n\left(x,s\right)f^{(n+1)}\left(s\right)ds\right| \\ \leq\frac{\left\|f^{(n+1)}\right\|_\infty}{n!}\left\{\int_a^x\psi_n\left(s-a,x-s\right)ds\right. \\ \left.+\int_x^b\psi_n\left(b-s,s-x\right)ds\right\}.\end{aligned} \tag{6.29}$$

Further,

$$\psi_n\left(u,v\right)=\frac{u^{n+3}}{n+3}+2v\frac{u^{n+2}}{n+2}+v^2\frac{u^{n+1}}{n+1}, \tag{6.30}$$

and so

$$\begin{aligned}&\int_a^x\psi_n\left(s-a,x-s\right)ds \\ &=\int_a^x\left[\frac{\left(s-a\right)^{n+3}}{n+3}+2\left(x-s\right)\frac{\left(s-a\right)^{n+2}}{n+2}\right. \\ &\qquad\left.+\left(x-s\right)^2\frac{\left(s-a\right)^{n+1}}{n+1}\right]ds \\ &=\left(x-a\right)^{n+4}\int_0^1\left[\frac{\lambda^{n+3}}{n+3}+2\left(1-\lambda\right)\frac{\lambda^{n+2}}{n+2}\right. \\ &\qquad\left.+\left(1-\lambda\right)^2\frac{\lambda^{n+1}}{n+1}\right]d\lambda,\end{aligned} \tag{6.31}$$

where we have made the substitution $\lambda=\frac{s-a}{x-a}$.

Collecting powers of λ gives

$$\lambda^{n+3}\left[\frac{1}{n+3}-\frac{2}{n+2}+\frac{1}{n+1}\right]-\frac{2\lambda^{n+2}}{(n+2)(n+1)}+\frac{\lambda^{n+1}}{n+1},$$

and so, from (6.31),

$$\begin{aligned}(6.32)\qquad &\int_a^x \psi_n(s-a,x-s)\,ds\\ &=(x-a)^{n+4}\left\{\frac{1}{n+4}\left[\frac{1}{n+3}-\frac{2}{n+2}+\frac{1}{n+1}\right]\right.\\ &\qquad\left.-\frac{2}{(n+3)(n+2)(n+1)}+\frac{1}{(n+2)(n+1)}\right\}\\ &=\frac{(x-a)^{n+4}}{(n+4)(n+1)}.\end{aligned}$$

Similarly, on using (6.30),

$$\begin{aligned}&\int_x^b \psi_n(b-s,s-x)\,ds\\ &\qquad=\int_x^b\left[\frac{(b-s)^{n+3}}{n+3}+2(s-x)\frac{(b-s)^{n+2}}{n+2}\right.\\ &\qquad\qquad\left.+(s-x)^2\frac{(b-s)^{n+1}}{n+1}\right]ds,\end{aligned}$$

and making the substitution $\nu=\frac{b-s}{b-x}$ gives

$$\begin{aligned}(6.33)\qquad &\int_x^b \psi_n(b-s,s-x)\,ds\\ &=(b-x)^{n+4}\int_0^1\left[\frac{\nu^{n+3}}{n+3}+2(1-\nu)\frac{\nu^{n+2}}{n+2}\right.\\ &\qquad\left.+(1-\nu)^2\frac{\nu^{n+1}}{n+1}\right]d\nu\\ &=\frac{(b-x)^{n+4}}{(n+4)(n+1)},\end{aligned}$$

where we have used (6.31) and (6.32). Combining (6.32) and (6.33) gives the first inequality in (6.26).

For the second inequality in (6.26), we use Hölder's integral inequality to obtain

$$\left|\frac{1}{n!}\int_a^b K_n(x,s)\, f^{(n+1)}(s)\, ds\right| \le \frac{\left\|f^{(n+1)}(s)\right\|_p}{n!}\left(\int_a^b |K_n(x,s)|^q\, ds\right)^{\frac{1}{q}}. \tag{6.34}$$

Now, from (6.15) and (6.30)

$$\begin{aligned}\int_a^b |K_n(x,s)|^q\, ds &= \int_a^x \psi^q(s-a,x-s)\, ds + \int_x^b \psi^q(b-s,s-x)\, ds\\ &= C\left[(x-a)^{(n+3)q+1} + (b-x)^{(n+3)q+1}\right],\end{aligned}$$

where C is as defined in (6.27) and we have used (6.31) and (6.32). Substitution into (6.34) gives the second inequality in (6.26).

Finally, for the third inequality in (6.26), from (6.28),

$$\begin{aligned}&\left|\frac{1}{n!}\int_a^b K_n(x,s)\, f^{(n+1)}(s)\, ds\right|\\ &\quad\le \frac{1}{n!}\left\{\int_a^x \psi_n(s-a,x-s)\left|f^{(n+1)}(s)\right| ds + \int_x^b \psi_n(b-s,s-x)\left|f^{(n+1)}(s)\right| ds\right\}\\ &\quad\le \frac{1}{n!}\left\{\psi_n(x-a,0)\int_a^x \left|f^{(n+1)}(s)\right| ds + \psi_n(b-x,0)\int_x^b \left|f^{(n+1)}(s)\right| ds\right\},\end{aligned} \tag{6.35}$$

where, from (6.30),

$$\psi_n(u,0) = \frac{u^{n+3}}{n+3}. \tag{6.36}$$

Hence, from (6.35) and (6.36)

$$\begin{aligned}&\left|\frac{1}{n!}\int_a^b K_n(x,s)\, f^{(n+1)}(s)\, ds\right|\\ &\quad\le \frac{1}{n!}\max\left\{\frac{(x-a)^{n+3}}{n+3}, \frac{(b-x)^{n+3}}{n+3}\right\}\left\|f^{(n+1)}(\cdot)\right\|_1\\ &\quad= \frac{1}{n!\,(n+3)}\left[\max\{x-a,b-x\}\right]^{n+3}\left\|f^{(n+1)}(\cdot)\right\|_1,\end{aligned}$$

which, on using the fact that for X, $Y \in \mathbb{R}$

$$\max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{X-Y}{2}\right|$$

gives, from (6.28), the third inequality in (6.26). The theorem is now completely proved. ∎

REMARK 81. *The results of Theorem 105 may be compared with those of Theorem 103. Theorem 105 is based on the single integral identity developed in Lemma 29, while Theorem 103 is based on the double integral identity representation for the bound. It may be noticed from (6.17) and (6.26) that the bounds are the same for $f^{(n+1)} \in L_\infty[a,b]$, that for $f^{(n+1)} \in L_1[a,b]$ the bound obtained in (6.17) is better and for $f^{(n+1)} \in L_p[a,b]$, $p > 1$, the result is inconclusive.*

2. Other Inequalities for the Expectation and Variance

2.1. Introduction. Based on the identity (see (5.46))

$$\sigma^2(T) + [x - E(T)]^2 = \int_a^b (x-t)^2 f(t)\,dt, \tag{6.37}$$

Barnett et al. [**7**] obtained a variety of bounds on the left hand side of (6.37). Bounds involving higher order derivatives were obtained in [**7**], by substituting a Taylor series expansion for $f(t)$ in (6.37), in terms of the $L_p[a,b]$ norms of the resulting double integral.

Barnett and Dragomir [**18**] obtained further results for the variance based on the identity

$$\sigma^2(T) + (E(T)-b)(E(T)-a) = \int_a^b (t-a)(t-b) f(t)\,dt. \tag{6.38}$$

The aim of the next section is to obtain a variety of bounds for the variance from an identity which regains (6.37) and (6.38) as special cases. Pre-Grüss, Čebyšev and Lupaş results are also obtained. Further, substitution of a Taylor expansion with integral remainder allows bounds to be obtained for the situation in which the PDF is n-time differentiable. Taking a convex combination of expansions about two separate points allows for further generalisations and a number of novel results.

2.2. Integral Identities. The following lemma is interesting in itself (see [**33**]).

LEMMA 30. *Let $f : [a,b] \to \mathbb{R}$ be a PDF of the random variable T, then, the following integral identity holds, involving the variance and expectation*

$$\sigma^2(T) + (E(T) - \alpha)(E(T) - \beta) = \int_a^b (t-\alpha)(t-\beta) f(t)\,dt, \tag{6.39}$$

where $\alpha,\ \beta \in [a,b]$ and $\alpha < \beta$.

PROOF. A simple expansion gives

$$\int_a^b (t-\alpha)(t-\beta) f(t)\,dt = \int_a^b \left[t^2 - (\alpha+\beta)t + \alpha\beta\right] f(t)\,dt,$$

which, upon using the definitions of the second and first moment together with the fact that $f(\cdot)$ is a PDF over $[a,b]$, gives

$$\begin{aligned} \int_a^b (t-\alpha)(t-\beta) f(t)\,dt &= \sigma^2(T) + M_1^2 - (\alpha+\beta) M_1 + \alpha\beta \\ &= \sigma^2(T) + (M_1 - \alpha)(M_1 - \beta), \end{aligned} \tag{6.40}$$

and hence (6.39) results on noting that $M_1 = E[T]$ and $M_2 = \sigma^2(T) + M_1^2$. ∎

REMARK 82. *If we take $\alpha = \beta = x$, then identity (6.37) is recaptured from (6.39). If further, $x = E(T)$, then σ^2 results. Taking $\alpha = a$ and $\beta = b$ in (6.39) gives the identity (6.38).*

Another interesting identity follows (see [**33**]).

LEMMA 31. *Let T be a random variable whose PDF $f : [a,b] \to \mathbb{R}$ is n-time differentiable and $f^{(n)}$ is absolutely continuous on $[a,b]$, then, the following identity holds for $z \in [a,b]$*

$$\begin{aligned} &\sigma^2(T) + (E(T) - \alpha)(E(T) - \beta) \\ &\quad = \sum_{k=0}^{n} \left[U_{k+3}(b-z) - U_{k+3}(a-z)\right] \frac{f^{(k)}(z)}{k!} + R_{n+1}(z), \end{aligned} \tag{6.41}$$

where

$$\begin{aligned} U_{r+1}(u) = \frac{u^{r-1}}{r(r^2-1)} \Big\{ & r(r-1)u^2 \\ & + 2\left(r^2-1\right)\left[z - \frac{\alpha+\beta}{2}\right] u + r(r+1)\alpha\beta \Big\}, \end{aligned} \tag{6.42}$$

and

$$R_{n+1}(z) = \frac{1}{n!} \int_a^b (t-\alpha)(t-\beta)\rho_n(t,z)\,dt, \tag{6.43}$$

with

$$\rho_n(t,z) = \int_z^t (t-s)^n f^{(n+1)}(s)\,ds. \tag{6.44}$$

PROOF. Using Taylor's formula with integral remainder and expanding about $t = z$ gives

$$f(t) = \sum_{k=0}^{n} \frac{(t-z)^k}{k!} f^{(k)}(z) + \frac{1}{n!}\rho_n(t,z), \tag{6.45}$$

for all $t, z \in [a,b]$ with $\rho_n(t,z)$ being given by (6.44).

Substitution of (6.45) into (6.39) gives

$$\begin{aligned} &\sigma^2(T) + (E(T)-\alpha)(E(T)-\beta) \\ &= \int_a^b (t-\alpha)(t-\beta)\left\{\sum_{k=0}^{n} \frac{(t-z)^k}{k!} f^{(k)}(z) + \frac{1}{n!}\rho_n(t,z)\right\} dt \\ &= \sum_{k=0}^{n}\left[\int_a^b (t-\alpha)(t-\beta)(t-z)^k\,dt\right]\frac{f^{(k)}(z)}{k!} + R_{n+1}(z), \end{aligned} \tag{6.46}$$

where $R_{n+1}(z)$ is as given by (6.43).

Now,

$$\begin{aligned} &\int_a^b (t-\alpha)(t-\beta)(t-z)^k\,dt \\ &\qquad= \int_{a-z}^{b-z} u^k (u+z-\alpha)(u+z-\beta)\,du \\ &\qquad= \int_{a-z}^{b-z} u^k \left[u^2 - 2\left[z - \frac{\alpha+\beta}{2}\right]u + \alpha\beta\right] du \\ &\qquad= \left.\frac{u^{k+3}}{k+3} - 2\left[z-\frac{\alpha+\beta}{2}\right]\frac{u^{k+2}}{k+2} + \alpha\beta\frac{u^{k+1}}{k+1}\right]_{a-z}^{b-z}, \end{aligned}$$

and therefore

$$\int_a^b (t-\alpha)(t-\beta)(t-z)^k\,dt = \left.U_{k+3}(u)\right]_{a-z}^{b-z}, \tag{6.47}$$

where, after some simplification, $U_{r+1}(u)$ is as given in (6.42). Substitution of (6.47) into (6.46) readily produces the result (6.41). ∎

REMARK 83. *Taking $\alpha = \beta = z = x$ reproduces an identity obtained by Barnett et al.* [**7**]. *Placing $\alpha = a$ and $\beta = b$ with $z = x$ gives an n-time differentiable generalisation of identity (6.38) and is thus a generalisation of the result of Barnett and Dragomir* [**18**].

We may also state the identity (see [**33**]) incorporated in the following.

LEMMA 32. *Let T be a random variable with PDF $f : [a,b] \to \mathbb{R}$ being n-time differentiable and $f^{(n)}$ absolutely continuous on $[a,b]$, then, the following identity holds*

$$\begin{aligned}(6.48)\quad & \sigma^2(T) + (E(T)-\alpha)(E(T)-\beta) \\ &= \sum_{k=0}^{n}\Big\{\lambda\left[V_{k+3}(b-\alpha) - V_{k+3}(a-\alpha)\right] f^{(k)}(\alpha) \\ &\quad + (1-\lambda)\left[W_{k+3}(b-\beta) - W_{k+3}(a-\beta)\right] f^{(k)}(\beta)\Big\} \\ &\quad + \lambda R_{n+1}(\alpha) + (1-\lambda) R_{n+1}(\beta),\end{aligned}$$

where

$$(6.49)\quad \begin{cases} V_{k+3}(u) = \dfrac{u^{k+2}}{(k+3)(k+2)}\left[(k+2)u - (\beta-\alpha)(k+3)\right], \\ \\ W_{k+3}(u) = \dfrac{u^{k+2}}{(k+3)(k+2)}\left[(k+2)u + (\beta-\alpha)(k+3)\right], \end{cases}$$

and $R_{n+1}(\cdot)$ is as given by (6.43).

PROOF. From (6.44), on letting $z=\alpha$, we obtain

$$(6.50)\qquad f(t) = \sum_{k=0}^{n} \frac{(t-\alpha)^k}{k!} f^{(k)}(\alpha) + \frac{1}{n!}\rho_n(t,\alpha),$$

where $\rho_n(t,\cdot)$ is as given in (6.44).

Additionally, taking $z=\beta$ in (6.44) produces

$$(6.51)\qquad f(t) = \sum_{k=0}^{n} \frac{(t-\beta)^k}{k!} f^{(k)}(\beta) + \frac{1}{n!}\rho_n(t,\beta).$$

If we let $\lambda \in [0,1]$ and evaluate $\lambda \cdot (6.50) + (1-\lambda)\cdot(6.51)$, we obtain

$$\begin{aligned}(6.52)\quad f(t) &= \sum_{k=0}^{n}\left[\lambda p_k(t-\alpha) f^{(k)}(\alpha) + (1-\lambda) p_k(t-\beta) f^{(k)}(\beta)\right] \\ &\quad + \frac{\lambda}{n!}\rho_n(t,\alpha) + \frac{1-\lambda}{n!}\rho_n(t,\beta),\end{aligned}$$

where

$$(6.53)\qquad p_k(u) = \frac{u^k}{k!},$$

and $\rho_n(t,\cdot)$ is as given by (6.44).

Substitution of (6.52) into (6.39) gives

$$\begin{aligned}
&\sigma^2(T)+(E(T)-\alpha)(E(T)-\beta) \\
&=\int_a^b (t-\alpha)(t-\beta)\left\{\sum_{k=0}^{n}\left[\lambda p_k(t-\alpha)f^{(k)}(\alpha)\right.\right. \\
&\qquad \left.\left.+(1-\lambda)p_k(t-\beta)f^{(k)}(\beta)\right]+\frac{\lambda}{n!}\rho_n(t,\alpha)+\frac{1-\lambda}{n!}\rho_n(t,\beta)\right\} \\
&=\sum_{k=0}^{n}(k+1)\int_a^b\left[\lambda(t-\beta)p_{k+1}(t-\alpha)f^{(k)}(\alpha)\right. \\
&\qquad \left.+(1-\lambda)(t-\alpha)p_{k+1}(t-\beta)f^{(k)}(\beta)\right]dt \\
&\qquad +\lambda R_{n+1}(\alpha)+(1-\lambda)R_{n+1}(\beta),
\end{aligned}$$

with $R_{n+1}(\cdot)$ as given by (6.43).

Now, using (6.53)

$$\begin{aligned}
\int_a^b (t-\beta)p_{k+1}(t-\alpha)\,dt &= \frac{1}{(k+1)!}\int_a^b (t-\beta)(t-\alpha)^{k+1}\,dt \\
&= \frac{1}{(k+1)!}\int_{a-\alpha}^{b-\alpha} u^{k+1}\left[u-(\beta-\alpha)\right]du,
\end{aligned}$$

and so,

$$\int_a^b (t-\beta)p_{k+1}(t-\alpha)\,dt = V_{k+3}(u)\Big]_{a-\alpha}^{b-\alpha},$$

where $V_{k+3}(u)$ is as given by (6.49).

Similarly, interchanging α and β,

$$\int_a^b (t-\alpha)p_{k+1}(t-\beta)\,dt = \frac{1}{(k+1)!}\int_{a-\beta}^{b-\beta} u^{k+1}\left[u+(\beta-\alpha)\right]du,$$

giving

$$\int_a^b (t-\alpha)p_{k+1}(t-\beta)\,dt = W_{k+3}(u)\Big]_{a-\beta}^{b-\beta},$$

where $W_{k+3}(u)$ is as given by (6.49).

The lemma is thus completely proved. ∎

REMARK 84. *It may be noted that, identity (6.48) is a generalization of (6.42) if $\alpha=\beta=z$.*

2.3. Bounds Involving Lebesgue Norms of a Function . A number of bounds can be derived using the identities developed in Section 2.2 in terms of various norms. Here, $\|\cdot\|_p$, $1 \leq p \leq \infty$ are the usual Lebesgue norms on $[a, b]$ (see Theorem 84 and formula (5.49)).

The following result holds [**33**].

THEOREM 106. *Let $f : [a, b] \to \mathbb{R}_+$ be the PDF of the random variable T. Then,*

$$\left|\sigma^2(T) + (E(T) - \alpha)(E(T) - \beta)\right| \tag{6.54}$$

$$\leq \begin{cases} \left\{\frac{1}{3}\left[(\alpha - a)^3 + (b - \beta)^3\right] \right. \\ \left. + \frac{\beta - \alpha}{6}\left[3(\alpha - a)^2 + (b - \beta)^2\right]\right\} \|f\|_\infty, & \text{for } f \in L_\infty[a, b]; \\ \left[\psi_q(\alpha - a) + B_q(\beta - \alpha) + \psi_q(b - \beta)\right]^{\frac{1}{q}} \|f\|_p, & \\ & \text{if } f \in L_p[a, b],\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \theta(a, \alpha, \beta, b)\|f\|_1, & f \in L_1[a, b], \end{cases}$$

where $\alpha, \beta \in [a, b]$ and $\alpha \leq \beta$,

$$\psi_q(X) = \int_0^X u^q (u + \beta - \alpha)^q \, du, \tag{6.55}$$

$$B_q(X) = \int_0^X u^q (\beta - \alpha - u)^q \, du,$$

and

$$\theta(a, \alpha, \beta, b) \tag{6.56}$$

$$= \max\left\{(\alpha - a)(\beta - a), \left(\frac{\beta - \alpha}{2}\right)^2, (b - \alpha)(b - \beta)\right\}.$$

PROOF. From identity (6.39), let

$$R_0(a, \alpha, \beta, b) = \int_a^b (t - \alpha)(t - \beta) f(t) \, dt, \tag{6.57}$$

and thus taking the modulus gives

$$|R_0(a, \alpha, \beta, b)| \leq \|f\|_\infty \int_a^b |(t - \alpha)(t - \beta)| \, dt. \tag{6.58}$$

Now,

$$\begin{aligned}
(6.59)\qquad & \int_a^b |(t-\alpha)(t-\beta)|\,dt \\
&= \int_a^\alpha (\alpha-t)(\beta-t)\,dt + \int_\alpha^\beta (t-\alpha)(\beta-t)\,dt \\
&\quad + \int_\beta^b (t-\alpha)(t-\beta)\,dt \\
&= \int_0^{\alpha-a} u(u+\beta-\alpha)\,du + \int_0^{\beta-\alpha} u(\beta-\alpha-u)\,du \\
&\quad + \int_0^{b-\beta} u(u+\beta-\alpha)\,du \\
&= \frac{1}{3}\left[(\alpha-a)^3+(b-\beta)^3\right] \\
&\quad + \frac{\beta-\alpha}{2}\left[(\alpha-a)^2+(b-\beta)^2\right] + \frac{(\beta-\alpha)^3}{6}.
\end{aligned}$$

A simple rearrangement of (6.59) and using (6.58) and (6.39) readily produces the first inequality in (6.54).

From (6.55), by Hölder's integral inequality, we obtain

$$\begin{aligned}
(6.60)\qquad |R_0(a,\alpha,\beta,b)| &\le \|f\|_p \left(\int_a^b |(t-\alpha)(t-\beta)|^q\,dt\right)^{\frac{1}{q}} \\
&:= \|f\|_p\, E_q^{\frac{1}{q}}(a,\alpha,\beta,b).
\end{aligned}$$

Then,

$$\begin{aligned}
E_q(a,\alpha,\beta,b) &= \int_a^\alpha (\alpha-t)^q(\beta-t)^q\,dt + \int_\alpha^\beta (t-\alpha)^q(\beta-t)^q\,dt \\
&\quad + \int_\beta^b (t-\alpha)^q(t-\beta)^q\,dt \\
&= \int_0^{\alpha-a} [u(u+\beta-\alpha)]^q\,du + \int_0^{\beta-\alpha} [u(\beta-\alpha-u)]^q\,du \\
&\quad + \int_0^{b-\beta} [u(u+\beta-\alpha)]^q\,du.
\end{aligned}$$

Hence, from (6.60), the second inequality in (6.54) results, where $\psi_q(\cdot)$ and $B_q(\cdot)$ are as defined in (6.55).

For the last inequality in (6.54), observe from identity (6.39) and the inequality

$$|R_0(a,\alpha,\beta,b)| \leq \sup_{t\in[a,b]} |(t-\alpha)(t-\beta)| \, \|f\|_1 ,$$

that we have,

$$\begin{aligned}
&\sup_{t\in[a,b]} |(t-\alpha)(t-\beta)| \\
&= \max\Bigg\{ \sup_{t\in[a,\alpha)} (\alpha-t)(\beta-t), \sup_{t\in(\alpha,\beta)} (t-\alpha)(\beta-t), \\
&\quad \sup_{t\in(\beta,b]} (t-\alpha)(t-\beta) \Bigg\} \\
&= \max\left\{ (\alpha-a)(\beta-a), \left(\frac{\beta-\alpha}{2}\right)^2, (b-\alpha)(b-\beta) \right\} \\
&= \theta(a,\alpha,\beta,b),
\end{aligned}$$

as given by (6.57) and hence the theorem is completely proved. ∎

Remark 85. *If $\alpha = \beta = x$ is taken in (6.54), then the results of Barnett et al.* [**7**] *based around the identity (6.37) are recaptured. In addition, if $x = E(T)$, then the bounds are on the variance alone. Taking $\alpha = a$ and $\beta = b$, the results of Barnett and Dragomir* [**18**] *are obtained. Some simplifications occur that have not as yet been developed, such as the result obtained from taking $\alpha = a$ and $\beta = x$.*

Remark 86. *The Euclidean norm is of special interest so that if $p = 2$ and $f \in L_2[a,b]$, then from (6.54),*

$$\begin{aligned}
&\left|\sigma^2(T) + (E(T)-\alpha)(E(T)-\beta)\right| \\
&\qquad \leq \|f\|_2 \left[\psi_2(\alpha-a) + B_2(\beta-\alpha) + \psi_2(b-\beta)\right]^{\frac{1}{2}},
\end{aligned}$$

where, from (6.55),

$$\psi_2(X) = \frac{X^3}{30}\left[6X^2 + 15(\beta-\alpha)X + 10(\beta-\alpha)^2\right],$$

and

$$B_2(X) = \frac{X^3}{30}\left[6X^2 - 15(\beta-\alpha)X + 10(\beta-\alpha)^2\right].$$

In addition, if we take $\alpha = \beta = x$, we obtain

$$\left|\sigma^2(T) + (E(T)-x)^2\right| \leq \frac{1}{\sqrt{5}}\left[(x-a)^5 + (b-x)^5\right]^{\frac{1}{2}} \|f\|_2 ,$$

and, for $x = E(T)$,

$$\sigma^2(T) \le \frac{1}{\sqrt{5}}\left[(E(T)-a)^5 + (b-E(T))^5\right]^{\frac{1}{2}} \|f\|_2 .$$

Taking $\alpha = a$, $\beta = b$ *gives*

$$\left|\sigma^2(T) + (E(T)-a)(b-E(T))\right| \le \frac{(b-a)^{\frac{5}{2}}}{\sqrt{30}} \|f\|_2 .$$

A pre-Grüss inequality is embodied in the following theorem. It provides a sharper bound than the Grüss inequality (see [**109**] for a statement of the Grüss inequality).

The following theorem was proven in [**108**] and is repeated here for convenience.

THEOREM 107. *Let* h, g *be integrable functions defined on* $[a,b]$ *and let* $m \le g(t) \le M$. *Then,*

$$|T(h,g)| \le \frac{M-m}{2}[T(h,h)]^{\frac{1}{2}}, \tag{6.61}$$

where the Čebyšev functional,

$$T(h,g) = \mathcal{M}(hg) - \mathcal{M}(h)\mathcal{M}(g), \tag{6.62}$$

with

$$\mathcal{M}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{6.63}$$

We may now state and prove the following result [**33**].

THEOREM 108. *Let* $f : [a,b] \to \mathbb{R}_+$ *be a PDF of the random variable* T *and such that, for* $m \le f \le M$, *then*

$$\begin{aligned} |\mathcal{T}_p| := \Bigg|&\sigma^2(T) + (E(T)-\alpha)(E(T)-\beta) \\ &- \left[\frac{(b-a)^2}{3} - \left(\frac{\alpha+\beta}{2}-a\right)(b-a) + (\alpha-a)(\beta-a)\right]\Bigg| \\ &\le \frac{M-m}{2} I(a,\alpha,\beta,b), \end{aligned} \tag{6.64}$$

where

$$\begin{aligned} &I(a,\alpha,\beta,b) \\ &= \frac{(b-a)^2}{\sqrt{3}}\left[\frac{4}{15}(b-a)^2 - \left(\frac{\alpha+\beta}{2}-a\right)\left(b-\frac{\alpha+\beta}{2}\right)\right]^{\frac{1}{2}}. \end{aligned} \tag{6.65}$$

PROOF. Applying the pre-Grüss result (6.61) by associating $f(t)$ with $g(t)$ and taking

$$h(t) = (t-\alpha)(t-\beta) \tag{6.66}$$

gives, on noting that $\mathcal{M}(f) = \frac{1}{b-a}$ since f is a PDF,

$$\left| \int_a^b (t-\alpha)(t-\beta) f(t)\, dt - \mathcal{M}(h) \right| \leq (b-a) \frac{M-m}{2} [T(h,h)]^{\frac{1}{2}}, \tag{6.67}$$

where, from (6.62),

$$T(h,h) = \mathcal{M}\left(h^2\right) - [\mathcal{M}(h)]^2. \tag{6.68}$$

Now, from (6.66) and (6.68)

$$\begin{aligned} \mathcal{M}(h) &= \frac{1}{b-a} \int_a^b (t-\alpha)(t-\beta)\, dt \\ &= \frac{1}{D} \int_0^D (u-A)(u-B)\, du, \end{aligned} \tag{6.69}$$

where

$$u = t-a,\ D = b-a,\ A = \alpha - a,\ B = \beta - a,$$

giving,

$$\mathcal{M}(h) = \frac{D^2}{3} - \frac{A+B}{2} D + AB. \tag{6.70}$$

Further, following a similar argument to the above,

$$\begin{aligned} \mathcal{M}\left(h^2\right) &= \frac{1}{D} \int_0^D (u-A)^2 (u-B)^2\, du \\ &= \frac{1}{D} \int_0^D \left[u^2 - (A+B)u + AB\right]^2 du \\ &= \frac{1}{D} \int_0^D \Big\{u^4 + (A+B)^2 u^2 + (AB)^2 \\ &\qquad + 2\left[ABu^2 - AB(A+B)u - (A+B)u^3\right] \Big\} du \end{aligned} \tag{6.71}$$

$$= \frac{1}{D}\int_0^D \left\{u^4 - 2(A+B)u^3 + \left[(A+B)^2 + 2AB\right]u^2 - AB(A+B)u + (AB)^2\right\}du$$
$$= \frac{D^4}{5} - \frac{(A+B)}{2}D^3 + \left[(A+B)^2 + 2AB\right]\frac{D^2}{3} - AB\frac{(A+B)}{2}D + (AB)^2.$$

Thus, from (6.67), (6.69) and (6.70), we have, after some algebra

$$T(h,h) = \frac{D^2}{3}\left[\frac{4}{15}D^2 - \frac{A+B}{2}D + \left(\frac{A+B}{2}\right)^2\right].$$

Using the definitions (6.68), the inequality (6.66) and the identity (6.39), gives the result (6.64) and, after some algebra, the theorem is thus proved. ∎

REMARK 87. *Taking $\alpha = a$, $\beta = b$ in (6.64)-(6.65) recaptures the results obtained by Barnett and Dragomir* [**18**] *while allowing $\alpha = \beta = x$ reproduces the results in Barnett et al.* [**7**]. *Note, from (6.65), that $I(a,\alpha,\beta,b) \le \frac{2(b-a)^3}{3\sqrt{5}}$. In addition, note that if $\frac{\alpha+\beta}{2} = \frac{a+b}{2}$ in (6.65), then*

$$I(a,\alpha,\beta,b) = \frac{(b-a)^3}{6\sqrt{3}},$$

which is 4 times better.

The following theorem holds [**33**].

THEOREM 109. *Let $f : [a,b] \to \mathbb{R}$ and suppose that $f(\cdot)$ is differentiable and is such that*

$$\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty,$$

then,

$$|\mathcal{T}_p| \le \frac{b-a}{\sqrt{12}}\|f'\|_\infty I(a,\alpha,\beta,b), \tag{6.72}$$

where $\mathcal{T}_p$ is the perturbed result given by the left hand side of (6.64) and $I(a,\alpha,\beta,b)$ is as given by (6.65).

PROOF. Let $h, g : [a,b] \to \mathbb{R}$ be absolutely continuous and h', g' be bounded. Then, Čebyšev's inequality holds (see [**109**])

$$|T(h,g)| \le \frac{(b-a)^2}{\sqrt{12}} \sup_{t\in[a,b]}|h'(t)| \cdot \sup_{t\in[a,b]}|g'(t)|.$$

Matić et al. [**108**], using a pre-Grüss type argument proved that

$$|T(h,g)| \leq \frac{b-a}{\sqrt{12}} \sup_{t\in[a,b]} |g'(t)| \sqrt{T(h,h)}.$$

Thus, associating $f(\cdot)$ with $g(\cdot)$ and $h(\cdot)$ with (6.66) produces (6.72) where $I(a,\alpha,\beta,b)$ is as given by (6.65). ∎

Another result is embodied in the following (see also [**33**]).

Theorem 110. *Let $f:[a,b]\to\mathbb{R}$ and suppose $[\alpha,\beta]\subseteq[a,b]$. Further, suppose that f is locally absolutely continuous on (a,b) and let $f'\in L_2(a,b)$, then,*

$$|\mathcal{T}_p| \leq \frac{b-a}{\pi}\|f'\|_2 I(a,\alpha,\beta,b), \tag{6.73}$$

where $\mathcal{T}_p$ is the perturbed result given by the left hand side of (6.64) and $I(a,\alpha,\beta,b)$ is as given by (6.65).

Proof. The following result was obtained by Lupaş (see [**109**]). For $h,g:(a,b)\to\mathbb{R}$ locally absolutely continuous on (a,b) and $h',g'\in L_2(a,b)$, then,

$$|T(h,g)| \leq \frac{(b-a)^2}{\pi^2}\|h'\|_2\|g'\|_2,$$

where

$$\|k\|_2 := \left(\frac{1}{b-a}\int_a^b |k(t)|^2\,dt\right)^{\frac{1}{2}} \quad \text{for} \quad k\in L_2(a,b).$$

Moreover, Matić et al. [**108**] showed that

$$|T(h,g)| \leq \frac{b-a}{\pi}\|g'\|_2\sqrt{T(h,h)}.$$

Now, associating $f(\cdot)$ with $g(\cdot)$ and $h(\cdot)$ as given by (6.66) produces (6.73) where $I(a,\alpha,\beta,b)$ is as found in (6.65). ∎

2.4. Bounds Involving Lebesgue Norms of the n−th Derivative of a Function. In this section, bounds are obtained for $f^{(n)}\in L_p[a,b]$, $p\geq 1$ and n a non-negative integer.

The following result holds [**33**].

Theorem 111. *Let T be a random variable whose PDF $f:[a,b]\to\mathbb{R}$ is n-time differentiable and $f^{(n)}$ is absolutely continuous on $[a,b]$.*

The following inequalities hold for $z \in [a,b]$,

$$(6.74)\quad \mathcal{T}_n := \left| \sigma^2(T) + (E(T)-\alpha)(E(T)-\beta) - \sum_{k=0}^{n} [U_{k+3}(b-z) - U_{k+3}(a-z)] \frac{f^{(k)}(z)}{k!} \right|$$
$$\le |R_{n+1}(z)|$$
$$\le \begin{cases} [\phi_{n+1}(a,\alpha,z) - \phi_{n+1}(\alpha,\beta,z) + \phi_{n+1}(\beta,b,z)] \\ \qquad \times \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}, \qquad f^{(n+1)} \in L_\infty[a,b]; \\ \left[\phi_{n+\frac{1}{q}}(a,\alpha,z) - \phi_{n+\frac{1}{q}}(\alpha,\beta,z) + \phi_{n+\frac{1}{q}}(\beta,b,z)\right] \\ \times \frac{\|f^{(n+1)}\|_p}{n!(nq+1)^{\frac{1}{q}}}, \quad f^{(n+1)} \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ [\phi_n(a,\alpha,z) - \phi_n(\alpha,\beta,z) + \phi_n(\beta,b,z)] \frac{\|f^{(n+1)}\|_1}{n!}, \\ \qquad f^{(n+1)} \in L_1[a,b], \end{cases}$$

where $U_{k+3}(\cdot)$ are as defined by (6.42),

$$(6.75)\quad \phi_{n+\gamma}(x_1,x_2,z) = \int_{x_1-z}^{x_2-z} |u|^{n+\gamma} (u+z-\alpha)(u+z-\beta)\,du, \quad x_1 \le x_2.$$

PROOF. From identity (6.41), on taking the modulus, we have

$$(6.76)\qquad \mathcal{T}_n = |R_{n+1}(z)|,$$

where $R_{n+1}(z)$ is as given by (6.43) and (6.44).
Now,

$$(6.77)\quad |R_{n+1}(z)| \le \frac{1}{n!}\int_a^b |(t-\alpha)(t-\beta)\rho_n(t,z)|\,dt$$
$$\le \frac{1}{n!}\left\{ \int_a^\alpha (\alpha-t)(\beta-t)|\rho_n(t,z)|\,dt + \int_\alpha^\beta (t-\alpha)(\beta-t)|\rho_n(t,z)|\,dt + \int_\beta^b (t-\alpha)(t-\beta)|\rho_n(t,z)|\,dt \right\}.$$

Further, using properties relating to the modulus and integral, and Hölder's integral inequality, gives

$$\left|\rho_n(t,z)\right| \le \begin{cases} \sup\limits_{s\in[z,t]} \left|f^{(n+1)}(s)\right| \left|\int_z^t |t-s|^n\,ds\right|, \\ \left|\int_z^t \left|f^{(n+1)}(s)\right|^p ds\right|^{\frac{1}{p}} \left|\int_z^t |t-s|^{nq}\,ds\right|^{\frac{1}{q}}, \\ |t-z|^n \left|\int_z^t \left|f^{(n+1)}(s)\right| ds\right|, \end{cases}$$

and hence

$$\left|\rho_n(t,z)\right| \le \begin{cases} \sup\limits_{s\in[z,t]} \left|f^{(n+1)}(s)\right| \frac{|t-z|^{n+1}}{n+1} \\ \left|\int_z^t \left|f^{(n+1)}(s)\right|^p ds\right|^{\frac{1}{p}} \left(\frac{|t-z|^{nq+1}}{nq+1}\right)^{\frac{1}{q}}, \\ \left|\int_z^t \left|f^{(n+1)}(s)\right| ds\right| |t-z|^n. \end{cases} \tag{6.78}$$

For $f^{(n+1)} \in L_\infty[a,b]$ using (6.78) and (6.77) gives

$$\left|R_{n+1}(z)\right| \le \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!}\left[\phi_{n+3}(a,\alpha,z) - \phi_{n+3}(\alpha,\beta,z) + \phi_{n+3}(\beta,b,z)\right],$$

where

$$\phi_{n+1}(x_1,x_2,z) = \int_{x_1}^{x_2} (t-\alpha)(t-\beta)\,|t-z|^{n+1}\,dt,$$

which, on substitution of $u = t - z$, produces (6.75) with $\gamma = 1$ and so the first inequality in (6.74) is obtained.

For the second inequality in (6.74), substitution of the second inequality from (6.78) into (6.77) gives, after substitution of $u = t - z$,

$$\left|R_{n+1}(z)\right| \le \frac{\left\|f^{(n+1)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}} \times \left[\phi_{n+\frac{1}{q}}(a,\alpha,z) - \phi_{n+\frac{1}{q}}(\alpha,\beta,z) + \phi_{n+\frac{1}{q}}(\beta,b,z)\right],$$

where ϕ is as defined in (6.75).

Finally, the third inequality in (6.74) is obtained by placing the third inequality in (6.78) into (6.77). In the above, we have used the fact that the respective norm over any subinterval, as represented in (6.78), is less than or equal to the equivalent norm over $[a,b]$. ∎

REMARK 88. *Result (6.74) is very general, containing three parameters α, β and z to be specified besides the degree of differentiability of the PDF f.*

Perturbed results on $\mathcal{T}_n$ as defined by (6.74) will now be obtained [**33**].

THEOREM 112. *Let $f : [a,b] \to \mathbb{R}_+$, a PDF of the random variable T, be such that $d_{n+1} \le f^{(n+1)}(t) \le D_{n+1}$ for $t \in [a,b]$, then,*

$$\begin{aligned} &\Bigg|\sigma^2(T) + (E(T)-\alpha)(E(T)-\beta) \\ &\quad - \sum_{k=0}^{n} \left[U_{k+3}(b-z) - U_{k+3}(a-z)\right] \frac{f^{(k)}(z)}{k!} + (-1)^n \mathcal{M}(h) \\ &\quad \times \left[1 - \sum_{k=0}^{n} \frac{(b-z)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(z) \Bigg| \\ &\qquad \le \frac{\theta_n(z)}{2} \cdot I(a,\alpha,\beta,b), \end{aligned} \tag{6.79}$$

where

$\mathcal{M}(h) = \frac{(b-a)^2}{3} - \left(\frac{\alpha+\beta}{2} - a\right)(b-a) + (\alpha-a)(\beta-a),$

$U_{k+3}(\cdot)$ *are as defined in (6.42),*

$I(a,\alpha,\beta,b)$ *is as given by (6.65),*

(6.80) *and*

$$\theta_n(z) = \begin{cases} \frac{D_{n+1}}{(n+1)!}\left[(z-a)^{n+1} + (b-z)^{n+1}\right], & n \text{ even}, \\ \frac{1}{(n+1)!}\max\left\{(z-a)^{n+1} d_{n+1}, (b-z)^{n+1} D_{n+1}\right\}, & n \text{ odd} \end{cases}.$$

PROOF. Applying the pre-Grüss result (6.61) and associating $\frac{1}{n!}\rho_n(t,z)$ as given by (6.44) with $g(t)$ and taking $h(t)$ as defined in (6.66), gives

$$\begin{aligned} &\left|\int_a^b (t-\alpha)(t-\beta)\frac{\rho_n(t,z)}{n!}dt - \mathcal{M}(h) \cdot \frac{1}{n!}\mathcal{M}(\rho_n(\cdot,z))\right| \\ &\qquad \le \frac{\Gamma(z)-\gamma(z)}{2}(b-a)\left[T(h,h)\right]^{\frac{1}{2}}, \end{aligned} \tag{6.81}$$

where $T(h,h)$ is as defined in (6.68) and

$$\gamma(z) \le \frac{\rho_n(t,z)}{n!} \le \Gamma(z), \text{ for } t \in [a,b]. \tag{6.82}$$

Further, $\mathcal{M}(h)$ is as given by (6.70) with $A = \alpha - a$, $B = \beta - a$ and $D = b - a$.

Now,

$$\begin{aligned}
(6.83)\quad & \frac{(b-a)}{n!}\mathcal{M}(\rho_n(\cdot, z)) \\
&= \frac{1}{n!}\int_a^b \int_z^t (t-s)^n f^{(n+1)}(s)\,dsdt \\
&= \frac{1}{n!}\left[\int_a^z \int_z^t (t-s)^n f^{(n+1)}(s)\,dsdt \right.\\
&\qquad \left. + \int_z^b \int_z^t (t-s)^n f^{(n+1)}(s)\,dsdt\right] \\
&= \frac{1}{n!}\left[-\int_a^z \int_a^s (t-s)^n f^{(n+1)}(s)\,dtds \right.\\
&\qquad \left. + \int_z^b \int_s^b (t-s)^n f^{(n+1)}(s)\,dtds\right] \\
&= \frac{1}{n!}\left[(-1)^{n+1}\int_a^z \frac{(s-a)^{n+1}}{n+1} f^{(n+1)}(s)\,ds \right.\\
&\qquad \left. + \int_z^b \frac{(b-s)^{n+1}}{n+1} f^{(n+1)}(s)\,ds\right] \\
&= (-1)^{n+1}\left[\int_a^b f(t)\,dt \right.\\
&\qquad \left. - \sum_{k=0}^{n} \frac{(b-z)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(z),
\end{aligned}$$

where, to obtain the last result, we have used an identity obtained in Cerone et al. [**41**] (Lemma 2.1, equation (2.1)) involving an Ostrowski result for n-time differentiable functions.

We need to obtain the bounds on $\rho_n(t, z)$ for all $t \in [a, b]$. We are given that

$$(6.84)\qquad d_{n+1} \le f^{(n+1)}(t) \le D_{n+1}.$$

For the case $t \ge z$, from (6.84) we have

$$d_{n+1}\int_z^t \frac{(t-s)^n}{n!}ds \le \frac{\rho_n(t,z)}{n!} \le D_{n+1}\int_z^t \frac{(t-s)^n}{n!}ds,$$

that is,

$$\frac{(t-z)^{n+1}}{(n+1)!}d_{n+1} \leq \frac{\rho_n(t,z)}{n!} \leq D_{n+1}\frac{(t-z)^{n+1}}{(n+1)!}, \quad t \in [z,b]$$

and so for $t \geq z$,

$$0 \leq \frac{\rho_n(t,z)}{n!} \leq D_{n+1}\frac{(b-z)^{n+1}}{(n+1)!}. \tag{6.85}$$

For the situation $t < z$, two separate cases need to be considered, namely, whether n is even or odd.

From (6.84) we have

$$d_{n+1}\int_t^z \frac{(t-s)^n}{n!}ds \leq -\frac{\rho_n(t,z)}{n!} \leq D_{n+1}\int_t^z \frac{(t-s)^n}{n!}ds, \tag{6.86}$$

and so, for n even

$$\frac{(z-t)^{n+1}}{(n+1)!}d_{n+1} \leq -\frac{\rho_n(t,z)}{n!} \leq \frac{(z-t)^{n+1}}{(n+1)!}D_{n+1},$$

$$-\frac{(z-t)^{n+1}}{(n+1)!}D_{n+1} \leq \frac{\rho_n(t,z)}{n!} \leq -\frac{(z-t)^{n+1}}{(n+1)!}d_{n+1}, \quad t \in [a,z],$$

giving for any $t \leq z$ and n even

$$-\frac{(z-a)^{n+1}}{(n+1)!}D_{n+1} \leq \frac{\rho_n(t,z)}{n!} \leq 0. \tag{6.87}$$

If n is odd, then from (6.86)

$$-\frac{(z-t)^{n+1}}{(n+1)!}d_{n+1} \leq -\frac{\rho_n(t,z)}{n!} \leq -\frac{(z-t)^{n+1}}{(n+1)!}D_{n+1},$$

giving

$$\frac{(z-t)^{n+1}}{(n+1)!}D_{n+1} \leq \frac{\rho_n(t,z)}{n!} \leq \frac{(z-t)^{n+1}}{(n+1)!}d_{n+1}, \quad t \in [a,z],$$

and so for $t < z$ and n odd

$$0 \leq \frac{\rho_n(t,z)}{n!} \leq \frac{(z-a)^{n+1}}{(n+1)!}d_{n+1}. \tag{6.88}$$

Thus, for n even, from (6.85) and (6.87), for all $t \in [a,b]$

$$-\frac{(z-a)^{n+1}}{(n+1)!}D_{n+1} \leq \frac{\rho_n(t,z)}{n!} \leq \frac{(b-z)^{n+1}}{(n+1)!}D_{n+1}. \tag{6.89}$$

For n odd, from (6.85) and (6.88), for all $t \in [a,b]$

$$\begin{aligned} 0 &\le \frac{\rho_n(t,z)}{n!} \\ &\le \frac{1}{(n+1)!}\max\left\{(z-a)^{n+1}d_{n+1},(b-z)^{n+1}D_{n+1}\right\}. \end{aligned} \tag{6.90}$$

Using (6.89) and (6.90) gives, from (6.81) and (6.82), $\theta_n(z) = \Gamma(z) - \gamma(z)$ as defined in (6.80). Substitution of identity (6.41) into (6.81) and using the fact that $I(a,\alpha,\beta,b) = (b-a)[T(h,h)]^{\frac{1}{2}}$, where h is as defined by (6.68), produces (6.79). We have further, in (6.83), used the fact that f is a PDF. ∎

Remark 89. *Čebyšev and Lupaş of Theorems 108 and 109 could be obtained here in a straightforward fashion for the expressions on the left of (6.79). The bound would be different and involve behaviour of* $f^{(n+2)}(\cdot)$ *instead of* $f^{(n+1)}(\cdot)$*. This, however, will not be pursued further.*

Finally, we have (see also [**33**])

Theorem 113. *Let T be a random variable with PDF $f:[a,b]\to\mathbb{R}$ being $n-$time differentiable and $f^{(n)}$ is absolutely continuous on $[a,b]$. The following inequality holds*

$$\begin{aligned} \kappa_n := \Bigg| &\sigma^2(T) + (E(T)-\alpha)(E(T)-\beta) \\ &- \sum_{k=0}^{n}\Big\{\lambda\left[V_{k+3}(b-\alpha)-V_{k+3}(a-\alpha)\right]f^{(k)}(\alpha) \\ &+ (1-\lambda)\left[W_{k+3}(b-\beta)-W_{k+3}(a-\beta)\right]f^{(k)}(\beta)\Big\}\Bigg| \end{aligned} \tag{6.91}$$

$$\le \lambda\left|R_{n+1}(\alpha)\right| + (1-\lambda)\left|R_{n+1}(\beta)\right|$$

$$\leq \begin{cases} \left[\lambda Y_{n+1}(\alpha) + (1-\lambda) Y_{n+1}(\beta) + \zeta_{n+1}(\beta-\alpha)\right] \\ \qquad \times \frac{\left\|f^{(n+1)}\right\|_{\infty}}{(n+1)!}, \qquad f^{(n+1)} \in L_{\infty}[a,b]; \\ \left[\lambda Y_{n+\frac{1}{q}}(\alpha) + (1-\lambda) Y_{n+\frac{1}{q}}(\beta) + \zeta_{n+\frac{1}{q}}(\beta-\alpha)\right] \\ \qquad \times \frac{\left\|f^{(n+1)}\right\|_{p}}{n!(nq+1)^{\frac{1}{q}}}, f^{(n+1)} \in L_p[a,b], \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\lambda Y_n(\alpha) + (1-\lambda) Y_n(\beta) + \zeta_n(\beta-\alpha)\right] \frac{\left\|f^{(n+1)}\right\|_{1}}{n!}, \\ \qquad f^{(n+1)} \in L_1[a,b]; \end{cases} \tag{6.92}$$

where $V_{k+3}(\cdot)$, $W_{k+3}(\cdot)$ *are as defined in (6.49) and*

$$Y_{n+\gamma}(\cdot) = A(\cdot - a) + B(b - \cdot) \tag{6.93}$$

with

$$\begin{cases} A(u) \\ = \frac{u^{n+\gamma+2}}{(n+\gamma+3)(n+\gamma+2)} \left[(n+\gamma+2) u + (\beta-\alpha)(n+\gamma+3)\right], \\ B(u) \\ = \frac{u^{n+\gamma+2}}{(n+\gamma+3)(n+\gamma+2)} \left[(n+\gamma+2) u - (\beta-\alpha)(n+\gamma+3)\right], \\ \textit{and} \\ \zeta_{n+\gamma}(\beta-\alpha) = -2B(\beta-\alpha) = \frac{-2(\beta-\alpha)^{n+\gamma+2}}{(n+\gamma+3)(n+\gamma+2)}. \end{cases} \tag{6.94}$$

Proof. Rearranging identity (6.48) and using the triangle inequality produces inequality (6.91).

Now, from the right hand side of (6.74), let

$$\begin{aligned} X_{n+\gamma}(\alpha) &= \chi_{n+\gamma}(a, \alpha, \beta, b, \alpha) \\ &= \phi_{n+\gamma}(a, \alpha, \alpha) - \phi_{n+\gamma}(\alpha, \beta, \alpha) + \phi_{n+\gamma}(\beta, b, \alpha). \end{aligned} \tag{6.95}$$

From (6.75),

$$\begin{aligned} \phi_{n+\gamma}(a, \alpha, \alpha) &= \int_{a-\alpha}^{0} |u|^{n+\gamma} u (u - (\beta-\alpha))\, du \\ &= \int_{0}^{a-\alpha} u^{n+\gamma+1} (u + \beta - \alpha)\, du = A(\alpha - a), \end{aligned}$$

$$\phi_{n+\gamma}(\alpha, \beta, \alpha) = \int_{0}^{\beta-\alpha} u^{n+\gamma+1} (u - (\beta-\alpha))\, du = B(\beta-\alpha),$$

and

$$\phi_{n+\gamma}(\beta, b, \alpha) = \int_{\beta-\alpha}^{b-\alpha} u^{n+\gamma+1}(u - (\beta - \alpha))\,du = B(b-\alpha) - B(\beta-\alpha).$$

Hence, substitution into (6.95) gives

$$X_{n+\gamma}(\alpha) = A(\alpha - a) - 2B(\beta - \alpha) + B(b - \alpha)$$

and so

$$X_{n+\gamma}(\alpha) = Y_{n+\gamma}(\alpha) + \zeta_{n+\gamma}(\beta - \alpha), \tag{6.96}$$

as defined in (6.93) and (6.94).

Again, from the right hand side of (6.74) and (6.93), let

$$X_{n+\gamma}(\beta) = \chi_{n+\gamma}(a, \alpha, \beta, b, \beta) = \phi_{n+\gamma}(a, \alpha, \beta) - \phi_{n+\gamma}(\alpha, \beta, \beta) + \phi_{n+\gamma}(\beta, b, \beta). \tag{6.97}$$

From (6.75)

$$\begin{aligned} \phi_{n+\gamma}(a, \alpha, \beta) &= \int_{a-\beta}^{\alpha-\beta} \left|u^{n+\gamma}\right| u(u + \beta - \alpha)\,du \\ &= \int_{\beta-a}^{\beta-\alpha} u^{n+\gamma+1}(\beta - \alpha - u)\,du \\ &= \int_{\beta-\alpha}^{\beta-a} u^{n+\gamma+1}(u - (\beta - \alpha))\,du \\ &= B(\beta - a) - B(\beta - \alpha), \end{aligned}$$

$$\begin{aligned} \phi_{n+\gamma}(\alpha, \beta, \beta) &= \int_{\alpha-\beta}^{0} \left|u^{n+\gamma}\right| u(u + \beta - \alpha)\,du \\ &= \int_{0}^{\beta-\alpha} u^{n+\gamma+1}(u - (\beta - \alpha))\,du = B(\beta - \alpha), \end{aligned}$$

and

$$\phi_{n+\gamma}(\beta, b, \beta) = \int_{0}^{b-\beta} |u|^{n+\gamma+1}(u + \beta - \alpha)\,du = A(b - \beta).$$

Hence, substitution into (6.97) gives

$$X_{n+\gamma}(\beta) = B(\beta - a) - 2B(\beta - \alpha) + A(b - \beta),$$

and so,

$$X_{n+\gamma}(\beta) = Y_{n+\gamma}(\beta) + \zeta_{n+\gamma}(\beta - \alpha). \tag{6.98}$$

On using (6.74) and (6.91), we have, from (6.96) and (6.98),

$$\lambda X_{n+\gamma}(\alpha) + (1-\lambda) X_{n+\gamma}(\beta) = \lambda Y_{n+\gamma}(\alpha) + (1-\lambda) Y_{n+\gamma}(\beta) + \zeta_{n+\gamma}(\beta-\alpha),$$

and so (6.92) is obtained for $\gamma = 1, \frac{1}{q}$ and 0 respectively. ∎

Remark 90. *Perturbed results on κ_n as defined in (6.92) may be obtained here in a similar fashion to those of Theorem 112. This, however, will not be pursued further.*

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