Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces

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Abstract. The main aim of this monograph is to survey some recent results obtained by the author related to reverses of the Schwarz, triangle and Bessel inequalities. Some Grüss' type inequalities for orthonormal families of vectors in real or complex inner product spaces are presented as well. Generalizations of the Boas-Bellman, Bombieri, Selberg, Heilbronn and Pečarić inequalities for finite sequences of vectors that are not necessarily orthogonal are also provided. Two extensions of the celebrated Ostrowski's inequalities for sequences or real numbers and the generalization of Wagner's inequality in inner product spaces are pointed out. Finally, some Grüss type inequalities for n-tuples of vectors in inner product spaces and their natural applications for the approximation of the discrete Fourier and Mellin transforms are given as well.

Contents

Preface

The theory of Hilbert spaces plays a central role in contemporary mathematics with numerous applications for Linear Operators, Partial Differential Equations, in Nonlinear Analysis, Approximation Theory, Optimization Theory, Numerical Analysis, Probability Theory, Statistics and other fields.

The Schwarz, triangle, Bessel, Gram and most recently, Grüss type inequalities have been frequently used as powerful tools in obtaining bounds or estimating the errors for various approximation formulae occurring in the domains mentioned above. Therefore, any new advancement related to these fundamental facts will have a flow of important consequences in the mathematical fields where these inequalities have been used before.

The main aim of this monograph is to survey some recent results obtained by the author related to reverses of the Schwarz, triangle and Bessel inequalities. Some Grüss type inequalities for orthonormal families of vectors in real or complex inner product spaces are presented as well. Generalizations of the Boas-Bellman, Bombieri, Selberg, Heilbronn and Pečarić inequalities for finite sequences of vectors that are not necessarily orthogonal are also provided. Two extensions of the celebrated Ostrowski inequalities for sequences of real numbers and the generalization of Wagner's inequality in inner product spaces are pointed out. Finally, some Grüss type inequalities for n-tuples of vectors in inner product spaces and their natural applications for the approximation of the discrete Fourier and Mellin transforms are given as well.

The monograph may be used by researchers in different branches of Mathematical and Functional Analysis where the theory of Hilbert spaces is of relevance. Since it is self-contained and all the results are completely proved, the work may be also used by graduate students interested in Theory of Inequalities and its Applications.

The Author,

February, 2004.

Part 1

Reverse Inequalities

CHAPTER 1

Reverses for the Schwarz Inequality

1. Introduction

Let H be a linear space over the real or complex number field $\mathbb{K}$. The functional $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{K}$ is called an *inner product* on H if it satisfies the conditions

(i) $\langle x, x\rangle \geq 0$ for any $x \in H$ and $\langle x, x\rangle = 0$ iff $x = 0$;
(ii) $\langle \alpha x + \beta y, z\rangle = \alpha \langle x, z\rangle + \beta \langle y, z\rangle$ for any $\alpha, \beta \in \mathbb{K}$ and $x, y, z \in H$;
(iii) $\langle y, x\rangle = \overline{\langle x, y\rangle}$ for any $x, y \in H$.

A first fundamental consequence of the properties (i)-(iii) above, is the *Schwarz inequality:*

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle, \tag{1.1}$$

for any $x, y \in H$. The equality holds in (1.1) if and only if the vectors x and y are *linearly dependent,* i.e., there exists a nonzero constant $\alpha \in \mathbb{K}$ so that $x = \alpha y$.

If we denote $\|x\| := \sqrt{\langle x, x\rangle}, x \in H$, then one may state the following properties

(n) $\|x\| \geq 0$ for any $x \in H$ and $\|x\| = 0$ iff $x = 0$;
(nn) $\|\alpha x\| = |\alpha| \|x\|$ for any $\alpha \in \mathbb{K}$ and $x \in H$;
(nnn) $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in H$ (the triangle inequality);

i.e., $\|\cdot\|$ is a *norm* on H.

In this chapter we present some recent reverse inequalities for the Schwarz and the triangle inequalities. More precisely, we point out upper bounds for the nonnegative quantities

$$\|x\| \|y\| - |\langle x, y\rangle|, \quad \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2$$

and

$$\|x\| + \|y\| - \|x + y\|$$

under various assumptions for the vectors $x, y \in H$.

If the vectors $x, y \in H$ are not *orthogonal*, i.e., $\langle x, y\rangle \neq 0$, then some upper bounds for the supra-unitary quantities

$$\frac{\|x\|\,\|y\|}{|\langle x, y\rangle|}, \quad \frac{\|x\|^2\,\|y\|^2}{|\langle x, y\rangle|^2}$$

are provided as well.

2. An Additive Reverse of the Schwarz Inequality

2.1. Introduction. Let $\overline{\mathbf{a}} = (a_1, \dots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \dots, b_n)$ be two positive $n-$*tuples* with

$$0 < m_1 \leq a_i \leq M_1 < \infty \text{ and } 0 < m_2 \leq b_i \leq M_2 < \infty; \tag{2.1}$$

for each $i \in \{1, \dots, n\}$, and some constants m_1, m_2, M_1, M_2.

The following reverses of the Cauchy-Bunyakovsky-Schwarz inequality for positive sequences of real numbers are well known:

(1) *Pólya-Szegö's inequality* [**20**]

$$\frac{\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2}{\left(\sum_{k=1}^n a_k b_k\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2.$$

(2) *Shisha-Mond's inequality* [**22**]

$$\frac{\sum_{k=1}^n a_k^2}{\sum_{k=1}^n a_k b_k} - \frac{\sum_{k=1}^n a_k b_k}{\sum_{k=1}^n b_k^2} \leq \left[\left(\frac{M_1}{m_2}\right)^{\frac{1}{2}} - \left(\frac{m_1}{M_2}\right)^{\frac{1}{2}}\right]^2.$$

(3) *Ozeki's inequality* [**19**]

$$\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2 - \left(\sum_{k=1}^n a_k b_k\right)^2 \leq \frac{n^2}{4}\left(M_1 M_2 - m_1 m_2\right)^2.$$

(4) *Diaz-Metcalf's inequality* [**2**]

$$\sum_{k=1}^n b_k^2 + \frac{m_2 M_2}{m_1 M_1}\sum_{k=1}^n a_k^2 \leq \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right)\sum_{k=1}^n a_k b_k.$$

If $\overline{\mathbf{w}} = (w_1, \dots, w_n)$ is a positive sequence, then the following weighted inequalities also hold:

(1) *Cassel's inequality* [**23**]. If the positive real sequences $\overline{\mathbf{a}} = (a_1, \dots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \dots, b_n)$ satisfy the condition

$$0 < m \leq \frac{a_k}{b_k} \leq M < \infty \text{ for each } k \in \{1, \dots, n\}, \tag{2.2}$$

then

$$\frac{\left(\sum_{k=1}^{n} w_k a_k^2\right)\left(\sum_{k=1}^{n} w_k b_k^2\right)}{\left(\sum_{k=1}^{n} w_k a_k b_k\right)^2} \leq \frac{(M+m)^2}{4mM}.$$

(2) *Greub-Reinboldt's inequality* [**12**]. We have

$$\left(\sum_{k=1}^{n} w_k a_k^2\right)\left(\sum_{k=1}^{n} w_k b_k^2\right) \leq \frac{(M_1M_2+m_1m_2)^2}{4m_1m_2M_1M_2}\left(\sum_{k=1}^{n} w_k a_k b_k\right)^2,$$

provided $\overline{\mathbf{a}} = (a_1, \dots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \dots, b_n)$ satisfy the condition (2.1) .

(3) *Generalised Diaz-Metcalf's inequality* [**2**], see also [**17**, p. 123]. If $u, v \in [0,1]$ and $v \leq u$, $u + v = 1$ and (2.2) holds, then one has the inequality

$$u\sum_{k=1}^{n} w_k b_k^2 + vMm\sum_{k=1}^{n} w_k a_k^2 \leq (vm+uM)\sum_{k=1}^{n} w_k a_k b_k.$$

(4) *Klamkin-McLenaghan's inequality* [**15**]. If $\overline{\mathbf{a}}, \overline{\mathbf{b}}$ satisfy (2.2), then

$$\left(\sum_{i=1}^{n} w_i a_i^2\right)\left(\sum_{i=1}^{n} w_i b_i^2\right) - \left(\sum_{i=1}^{n} w_i a_i b_i\right)^2 \leq \left(M^{\frac{1}{2}} - m^{\frac{1}{2}}\right)^2 \sum_{i=1}^{n} w_i a_i b_i \sum_{i=1}^{n} w_i a_i^2. \tag{2.3}$$

For other recent results providing discrete reverse inequalities, see the recent monograph online [**5**].

In this section, by following [**3**], we point out a new reverse of Schwarz's inequality in real or complex inner product spaces. Particular cases for isotonic linear functionals, integrals and sequences are also given.

2.2. An Additive Reverse Inequality. The following reverse of Schwarz's inequality in inner product spaces holds [**3**].

THEOREM 1. *Let $A, a \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) and $x, y \in H$. If*

$$\operatorname{Re}\langle Ay - x, x - ay\rangle \geq 0, \tag{2.4}$$

or, equivalently,

$$\left\| x - \frac{a+A}{2}\cdot y \right\| \leq \frac{1}{2}|A-a|\,\|y\|, \tag{2.5}$$

holds, then one has the inequality

$$0 \le \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \le \frac{1}{4} |A - a|^2 \|y\|^4 . \tag{2.6}$$

The constant $\frac{1}{4}$ is sharp in (2.6).

Proof. The equivalence between (2.4) and (2.5) can be easily proved, see for example [**10**].

Let us define

$$I_1 := \operatorname{Re}\left[\left(A \|y\|^2 - \langle x, y\rangle\right)\left(\overline{\langle x, y\rangle} - \overline{a} \|y\|^2\right)\right]$$

and

$$I_2 := \|y\|^2 \operatorname{Re} \langle Ay - x, x - ay\rangle .$$

Then

$$I_1 = \|y\|^2 \operatorname{Re}\left[A\overline{\langle x, y\rangle} + \overline{a} \langle x, y\rangle\right] - |\langle x, y\rangle|^2 - \|y\|^4 \operatorname{Re}(A\overline{a})$$

and

$$I_2 = \|y\|^2 \operatorname{Re}\left[A\overline{\langle x, y\rangle} + \overline{a} \langle x, y\rangle\right] - \|x\|^2 \|y\|^2 - \|y\|^4 \operatorname{Re}(A\overline{a}),$$

which gives

$$I_1 - I_2 = \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 ,$$

for any $x, y \in H$ and $a, A \in \mathbb{K}$. This is an interesting identity in itself as well.

If (2.4) holds, then $I_2 \ge 0$ and thus

$$\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \le \operatorname{Re}\left[\left(A \|y\|^2 - \langle x, y\rangle\right)\left(\overline{\langle x, y\rangle} - \overline{a} \|y\|^2\right)\right]. \tag{2.7}$$

Further, if we use the elementary inequality for $u, v \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$)

$$\operatorname{Re}(u\overline{v}) \le \frac{1}{4} |u + v|^2 ,$$

then we have, for

$$u := A \|y\|^2 - \langle x, y\rangle , \quad v := \langle x, y\rangle - a \|y\|^2 ,$$

that

$$\operatorname{Re}\left[\left(A \|y\|^2 - \langle x, y\rangle\right)\left(\overline{\langle x, y\rangle} - \overline{a} \|y\|^2\right)\right] \le \frac{1}{4} |A - a|^2 \|y\|^4 . \tag{2.8}$$

Making use of the inequalities (2.7) and (2.8), we deduce (2.6).

Now, assume that (2.6) holds with a constant $C > 0$, i.e.,

$$\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \le C |A - a|^2 \|y\|^4 , \tag{2.9}$$

where x, y, a, A satisfy (2.4).

Consider $y \in H$, $\|y\| = 1$, $a \neq A$ and $m \in H$, $\|m\| = 1$ with $m \perp y$. Define

$$x := \frac{A+a}{2} y + \frac{A-a}{2} m.$$

Then

$$\langle Ay - x, x - ay \rangle = \left| \frac{A-a}{2} \right|^2 \langle y - m, y + m \rangle = 0,$$

and thus the condition (2.4) is fulfilled. From (2.9) we deduce

$$\left\| \frac{A+a}{2} y + \frac{A-a}{2} m \right\|^2 - \left| \left\langle \frac{A+a}{2} y + \frac{A-a}{2} m, y \right\rangle \right|^2 \leq C \left| A - a \right|^2, \tag{2.10}$$

and since

$$\left\| \frac{A+a}{2} y + \frac{A-a}{2} m \right\|^2 = \left| \frac{A+a}{2} \right|^2 + \left| \frac{A-a}{2} \right|^2$$

and

$$\left| \left\langle \frac{A+a}{2} y + \frac{A-a}{2} m, y \right\rangle \right|^2 = \left| \frac{A+a}{2} \right|^2$$

then, by (2.10), we obtain

$$\frac{\left| A - a \right|^2}{4} \leq C \left| A - a \right|^2,$$

which gives $C \geq \frac{1}{4}$, and the theorem is completely proved. ∎

2.3. Applications for Isotonic Linear Functionals. Let $F(T)$ be an algebra of real functions defined on T and L a subclass of $F(T)$ satisfying the conditions:

(i) $f, g \in L$ implies $f + g \in L$;
(ii) $f \in L$, $\alpha \in \mathbb{R}$ implies $\alpha f \in L$.

A functional A defined on L is an *isotonic linear functional* on L provided that

(a) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in L$;
(aa) $f \geq g$, that is, $f(t) \geq g(t)$ for all $t \in T$, implies $A(f) \geq A(g)$.

The functional A is *normalised* on L, provided that $\mathbf{1} \in L$, i.e., $\mathbf{1}(t) = 1$ for all $t \in T$, implies $A(\mathbf{1}) = 1$.

Usual examples of isotonic linear functionals are integrals, sums, etc.

Now, suppose that $h \in F(T)$, $h \geq 0$ is given and satisfies the properties that $fgh \in L$, $fh \in L$, $gh \in L$ for all $f, g \in L$. For a given isotonic linear functional $A : L \to \mathbb{R}$ with $A(h) > 0$, define the mapping $(\cdot,\cdot)_{A,h} : L \times L \to \mathbb{R}$ by

$$(f,g)_{A,h} := \frac{A(fgh)}{A(h)}.$$

This functional satisfies the following properties:

(s) $(f,f)_{A,h} \geq 0$ for all $f \in L$;

(ss) $(\alpha f + \beta g, k)_{A,h} = \alpha (f,k)_{A,h} + \beta (g,k)_{A,h}$ for all $f, g, k \in L$ and $\alpha, \beta \in \mathbb{R}$;

(sss) $(f,g)_{A,h} = (g,f)_{A,h}$ for all $f, g \in L$.

The following reverse of Schwarz's inequality for positive linear functionals holds [**3**].

PROPOSITION 1. *Let $f, g, h \in F(T)$ be such that $fgh \in L$, $f^2 h \in L$, $g^2 h \in L$. If m, M are real numbers such that*

$$mg \leq f \leq Mg \ \text{ on } F(T), \tag{2.11}$$

then for any isotonic linear functional $A : L \to \mathbb{R}$, with $A(h) > 0$, we have the inequality

$$0 \leq A\left(hf^2\right) A\left(hg^2\right) - [A(hfg)]^2 \leq \frac{1}{4}(M-m)^2 A^2\left(hg^2\right). \tag{2.12}$$

The constant $\frac{1}{4}$ in (2.12) is sharp.

PROOF. We observe that

$$(Mg - f, f - mg)_{A,h} = A[h(Mg-f)(f-mg)] \geq 0.$$

Applying Theorem 1 for $(\cdot,\cdot)_{A,h}$, we get

$$0 \leq (f,f)_{A,h}(g,g)_{A,h} - (f,g)^2_{A,h} \leq \frac{1}{4}(M-m)^2 (g,g)^2_{A,h},$$

which is clearly equivalent to (2.12). ∎

The following corollary holds.

COROLLARY 1. *Let $f, g \in F(T)$ such that fg, $f^2, g^2 \in F(T)$. If m, M are real numbers such that (2.11) holds, then*

$$0 \leq A\left(f^2\right) A\left(g^2\right) - A^2(fg) \leq \frac{1}{4}(M-m)^2 A^2\left(g^2\right). \tag{2.13}$$

The constant $\frac{1}{4}$ is sharp in (2.13).

REMARK 1. *The condition (2.11) may be replaced with the weaker assumption*

$$(Mg - f, f - mg)_{A,h} \geq 0.$$

2.4. Applications for Integrals. Let (Ω, Σ, μ) be a measure space consisting of a set Ω, Σ a $\sigma-$algebra of subsets of Ω and μ a countably additive and positive measure on Σ with values in $\mathbb{R} \cup \{\infty\}$.

Denote by $L^2_\rho(\Omega, \mathbb{K})$ the Hilbert space of all $\mathbb{K}$-valued functions f defined on Ω that are $2-\rho-$integrable on Ω, i.e., $\int_\Omega \rho(t) |f(s)|^2 d\mu(s) < \infty$, where $\rho : \Omega \to [0, \infty)$ is a measurable function on Ω.

The following proposition contains a reverse of the weighted Cauchy-Bunyakovsky-Schwarz's integral inequality [**3**].

Proposition 2. *Let $A, a \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ and $f, g \in L^2_\rho(\Omega, \mathbb{K})$. If*

$$\int_\Omega \operatorname{Re}\left[(Ag(s) - f(s))\left(\overline{f(s)} - \overline{a}\ \overline{g}(s)\right)\right] \rho(s)\, d\mu(s) \geq 0 \tag{2.14}$$

or, equivalently,

$$\int_\Omega \rho(s) \left| f(s) - \frac{a+A}{2} g(s)\right|^2 d\mu(s) \leq \frac{1}{4}|A-a|^2 \int_\Omega \rho(s) |g(s)|^2 d\mu(s),$$

holds, then one has the inequality

$$\begin{aligned} 0 &\leq \int_\Omega \rho(s) |f(s)|^2 d\mu(s) \int_\Omega \rho(s) |g(s)|^2 d\mu(s) \\ &\qquad - \left| \int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s)\right|^2 \\ &\leq \frac{1}{4}|A-a|^2 \left(\int_\Omega \rho(s) |g(s)|^2 d\mu(s)\right)^2. \end{aligned}$$

The constant $\frac{1}{4}$ is best possible.

Proof. Follows by Theorem 1 applied for the inner product $\langle \cdot, \cdot \rangle_\rho : L^2_\rho(\Omega, \mathbb{K}) \times L^2_\rho(\Omega, \mathbb{K}) \to \mathbb{K}$,

$$\langle f, g \rangle_\rho := \int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s).$$

∎

Remark 2. *A sufficient condition for (2.14) to hold is*

$$\operatorname{Re}\left[(Ag(s) - f(s))\left(\overline{f(s)} - \overline{a}\ \overline{g}(s)\right)\right] \geq 0, \quad \textit{for} \ \ \mu - a.e.\ s \in \Omega.$$

In the particular case $\rho = 1$, we have the following reverse of the Cauchy-Bunyakovsky-Schwarz inequality.

Corollary 2. *Let $a, A \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ and $f, g \in L^2(\Omega, \mathbb{K})$. If*

$$\int_\Omega \operatorname{Re}\left[(Ag(s) - f(s))\left(\overline{f(s)} - \overline{a}\ \overline{g}(s)\right)\right] d\mu(s) \geq 0, \tag{2.15}$$

or, equivalently,

$$\int_{\Omega}\left|f(s)-\frac{a+A}{2}g(s)\right|^{2}d\mu(s)\leq\frac{1}{4}|A-a|^{2}\int_{\Omega}|g(s)|^{2}d\mu(s),$$

holds, then one has the inequality

$$0\leq\int_{\Omega}|f(s)|^{2}d\mu(s)\int_{\Omega}|g(s)|^{2}d\mu(s)-\left|\int_{\Omega}f(s)\overline{g(s)}d\mu(s)\right|^{2}$$
$$\leq\frac{1}{4}|A-a|^{2}\left(\int_{\Omega}|g(s)|^{2}d\mu(s)\right)^{2}.$$

The constant $\frac{1}{4}$ *is best possible*

REMARK 3. *If* $\mathbb{K}=\mathbb{R}$, *then a sufficient condition for either (2.14) or (2.15) to hold is*

$$ag(s)\leq f(s)\leq Ag(s),\quad \textit{for}\ \ \mu-\textit{a.e.}\ s\in\Omega,$$

where, in this case, $a,A\in\mathbb{R}$ *with* $A>a>0$.

2.5. Applications for Sequences. For a given sequence $(w_i)_{i\in\mathbb{N}}$ of nonnegative real numbers, consider the Hilbert space $\ell_w^2(\mathbb{K}),(\mathbb{K}=\mathbb{C},\mathbb{R}),$ where

$$\ell_w^2(\mathbb{K}):=\left\{\overline{\mathbf{x}}=(x_i)_{i\in\mathbb{N}}\subset\mathbb{K}\,\middle|\,\sum_{i=0}^{\infty}w_i|x_i|^2<\infty\right\}.$$

The following proposition that provides a reverse of the weighted Cauchy-Bunyakovsky-Schwarz inequality for complex numbers holds.

PROPOSITION 3. *Let* $a,A\in\mathbb{K}$ *and* $\overline{\mathbf{x}},\overline{\mathbf{y}}\in\ell_w^2(\mathbb{K})$. *If*

$$\sum_{i=0}^{\infty}w_i\operatorname{Re}\left[(Ay_i-x_i)(\overline{x_i}-\overline{a}\,\overline{y_i})\right]\geq 0,\tag{2.16}$$

then one has the inequality

$$0\leq\sum_{i=0}^{\infty}w_i|x_i|^2\sum_{i=0}^{\infty}w_i|y_i|^2-\left|\sum_{i=0}^{\infty}w_ix_i\overline{y_i}\right|^2\leq\frac{1}{4}|A-a|^2\left(\sum_{i=0}^{\infty}w_i|y_i|^2\right)^2.$$

The constant $\frac{1}{4}$ *is sharp.*

PROOF. Follows by Theorem 1 applied for the inner product $\langle\cdot,\cdot\rangle_w:\ell_w^2(\mathbb{K})\times\ell_w^2(\mathbb{K})\to\mathbb{K}$,

$$\langle\overline{\mathbf{x}},\overline{\mathbf{y}}\rangle_w:=\sum_{i=0}^{\infty}w_ix_i\overline{y_i}.$$

∎

REMARK 4. *A sufficient condition for (2.16) to hold is*

$$\operatorname{Re}\left[\left(Ay_i - x_i\right)\left(\overline{x_i} - \overline{ay_i}\right)\right] \geq 0, \quad \text{for all } i \in \mathbb{N}.$$

In the particular case $w_i = 1$, $i \in \mathbb{N}$, we have the following reverse of the Cauchy-Bunyakovsky-Schwarz inequality.

COROLLARY 3. *Let $a, A \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ and $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in \ell^2(\mathbb{K})$. If*

$$\sum_{i=0}^{\infty} \operatorname{Re}\left[\left(Ay_i - x_i\right)\left(\overline{x_i} - \overline{ay_i}\right)\right] \geq 0, \tag{2.17}$$

then one has the inequality

$$0 \leq \sum_{i=0}^{\infty} |x_i|^2 \sum_{i=0}^{\infty} |y_i|^2 - \left|\sum_{i=0}^{\infty} x_i \overline{y_i}\right|^2 \leq \frac{1}{4}|A-a|^2 \left(\sum_{i=0}^{\infty} |y_i|^2\right)^2.$$

REMARK 5. *If $\mathbb{K} = \mathbb{R}$, then a sufficient condition for either (2.16) or (2.17) to hold is*

$$ay_i \leq x_i \leq Ay_i \quad \text{for each} \quad i \in \mathbb{N},$$

with $A > a > 0$.

3. A Generalisation of the Cassels Inequality

3.1. Introduction. The following result was proved by J.W.S. Cassels in 1951 (see Appendix 1 of [**23**]).

THEOREM 2. *Let $\overline{\mathbf{a}} = (a_1, \dots, a_n)$, $\overline{\mathbf{b}} = (b_1, \dots, b_n)$ be sequences of positive real numbers and $\overline{\mathbf{w}} = (w_1, \dots, w_n)$ a sequence of nonnegative real numbers. Suppose that*

$$m = \min_{i=\overline{1,n}} \left\{\frac{a_i}{b_i}\right\} \quad \text{and} \quad M = \max_{i=\overline{1,n}} \left\{\frac{a_i}{b_i}\right\}. \tag{3.1}$$

Then one has the inequality

$$\frac{\sum_{i=1}^n w_i a_i^2 \sum_{i=1}^n w_i b_i^2}{\left(\sum_{i=1}^n w_i a_i b_i\right)^2} \leq \frac{(m+M)^2}{4mM}. \tag{3.2}$$

The equality holds in (3.2) when $w_1 = \frac{1}{a_1 b_1}$, $w_n = \frac{1}{a_n b_n}$, $w_2 = \dots = w_{n-1} = 0$, $m = \frac{a_n}{b_1}$ and $M = \frac{a_1}{b_n}$.

If one assumes that $0 < a \leq a_i \leq A < \infty$ and $0 < b \leq b_i \leq B < \infty$ for each $i \in \{1, \dots, n\}$, then by (3.2) we may obtain *Greub-Reinboldt's inequality* [**12**]

$$\frac{\sum_{i=1}^n w_i a_i^2 \sum_{i=1}^n w_i b_i^2}{\left(\sum_{i=1}^n w_i a_i b_i\right)^2} \leq \frac{(ab+AB)^2}{4abAB}.$$

The following "unweighted" Cassels' inequality also holds

$$\frac{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}{\left(\sum_{i=1}^{n} a_i b_i\right)^2} \le \frac{(m+M)^2}{4mM},$$

provided $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ satisfy (3.1). This inequality will produce the well known *Pólya-Szegö inequality* [**20**, pp. 57, 213-114], [**17**, pp. 71-72, 253-255]:

$$\frac{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}{\left(\sum_{i=1}^{n} a_i b_i\right)^2} \le \frac{(ab+AB)^2}{4abAB},$$

provided $0 < a \le a_i \le A < \infty$ and $0 < b \le b_i \le B < \infty$ for each $i \in \{1, \dots, n\}$.

In [**18**], C.P. Niculescu proved, amongst others, the following generalisation of Cassels' inequality:

THEOREM 3. *Let E be a vector space endowed with a Hermitian product $\langle \cdot, \cdot \rangle$. Then*

$$\frac{\operatorname{Re}\langle x, y\rangle}{\langle x, x\rangle^{\frac{1}{2}} \langle y, y\rangle^{\frac{1}{2}}} \ge \frac{2}{\sqrt{\frac{\omega}{\Omega}} + \sqrt{\frac{\Omega}{\omega}}}, \tag{3.3}$$

for every $x, y \in E$ and every $\omega, \Omega > 0$ for which $\operatorname{Re}\langle x - \omega y, x - \Omega y\rangle \le 0$.

In this section, by following [**4**], we establish a generalisation of (3.3) for the complex numbers ω and Ω for which $\operatorname{Re}(\overline{\omega}\Omega) > 0$. Applications for isotonic linear functionals, integrals and sequences are also given.

3.2. An Inequality in Real or Complex Inner Product Spaces. The following reverse of Schwarz's inequality in inner product spaces holds [**4**].

THEOREM 4. *Let $a, A \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ so that $\operatorname{Re}(\overline{a}A) > 0$. If $x, y \in H$ are such that*

$$\operatorname{Re}\langle Ay - x, x - ay\rangle \ge 0, \tag{3.4}$$

then one has the inequality

$$\|x\| \|y\| \le \frac{1}{2} \cdot \frac{\operatorname{Re}\left[A\overline{\langle x, y\rangle} + \overline{a}\langle x, y\rangle\right]}{\left[\operatorname{Re}(\overline{a}A)\right]^{\frac{1}{2}}} \le \frac{1}{2} \cdot \frac{|A| + |a|}{\left[\operatorname{Re}(\overline{a}A)\right]^{\frac{1}{2}}} |\langle x, y\rangle|. \tag{3.5}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.

PROOF. We have, obviously, that

$$\begin{aligned} I &:= \operatorname{Re}\langle Ay - x, x - ay\rangle \\ &= \operatorname{Re}\left[A\overline{\langle x, y\rangle} + \overline{a}\langle x, y\rangle\right] - \|x\|^2 - [\operatorname{Re}(\overline{a}A)]\|y\|^2 \end{aligned}$$

and, thus, by (3.4), one has

$$\|x\|^2 + [\operatorname{Re}(\overline{a}A)] \cdot \|y\|^2 \leq \operatorname{Re}\left[A\overline{\langle x, y\rangle} + \overline{a}\langle x, y\rangle\right],$$

which gives

$$\frac{1}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}}\|x\|^2 + [\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}\|y\|^2 \leq \frac{\operatorname{Re}\left[A\overline{\langle x, y\rangle} + \overline{a}\langle x, y\rangle\right]}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}}. \tag{3.6}$$

On the other hand, by the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha}q^2 \geq 2pq,$$

valid for $p, q \geq 0$ and $\alpha > 0$, we deduce

$$2\|x\|\|y\| \leq \frac{1}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}}\|x\|^2 + [\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}\|y\|^2. \tag{3.7}$$

Utilizing (3.6) and (3.7) we deduce the first part of (3.5).

The second part is obvious by the fact that for $z \in \mathbb{C}$, $|\operatorname{Re}(z)| \leq |z|$.

Now, assume that the first inequality in (3.5) holds with a constant $c > 0$, i.e.,

$$\|x\|\|y\| \leq c \cdot \frac{\operatorname{Re}\left[A\overline{\langle x, y\rangle} + \overline{a}\langle x, y\rangle\right]}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}}, \tag{3.8}$$

where a, A, x and y satisfy (3.5).

If we choose $a = A = 1$, $y = x \neq 0$, then obviously (3.4) holds and from (3.8) we obtain

$$\|x\|^2 \leq 2c\|x\|^2,$$

giving $c \geq \frac{1}{2}$.

The theorem is completely proved. ∎

The following corollary is a natural consequence of the above theorem [**4**].

COROLLARY 4. *Let $m, M > 0$. If $x, y \in H$ are such that*

$$\operatorname{Re}\langle My - x, x - my\rangle \geq 0,$$

then one has the inequality

$$\|x\|\,\|y\| \leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} \operatorname{Re}\langle x,y\rangle \leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} \left|\langle x,y\rangle\right|. \tag{3.9}$$

The constant $\frac{1}{2}$ *is sharp in (3.9).*

Remark 6. *The inequality (3.9) is equivalent to Niculescu's inequality (3.3).*

The following corollary providing an additive reverse for the Schwarz inequality is also obvious [**4**].

Corollary 5. *With the assumptions of Corollary 4, we have*

$$\begin{aligned} 0 &\leq \|x\|\,\|y\| - |\langle x,y\rangle| \leq \|x\|\,\|y\| - \operatorname{Re}\langle x,y\rangle \\ &\leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\langle x,y\rangle \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} |\langle x,y\rangle| \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} 0 &\leq \|x\|^2\,\|y\|^2 - |\langle x,y\rangle|^2 \leq \|x\|^2\,\|y\|^2 - [\operatorname{Re}\langle x,y\rangle]^2 \\ &\leq \frac{(M-m)^2}{4mM} [\operatorname{Re}\langle x,y\rangle]^2 \leq \frac{(M-m)^2}{4mM} |\langle x,y\rangle|^2. \end{aligned} \tag{3.11}$$

The constants $\frac{1}{2}$ *and* $\frac{1}{4}$ *are sharp.*

Proof. If we subtract $\operatorname{Re}\langle x,y\rangle \geq 0$ from the first inequality in (3.9), we get

$$\begin{aligned} \|x\|\,\|y\| - \operatorname{Re}\langle x,y\rangle &\leq \left(\frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} - 1\right) \operatorname{Re}\langle x,y\rangle \\ &= \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\langle x,y\rangle, \end{aligned}$$

which proves the third inequality in (3.10). The other ones are obvious.

Now, if we square the first inequality in (3.9) and then subtract $[\operatorname{Re}\langle x,y\rangle]^2$, we get

$$\begin{aligned} \|x\|^2\,\|y\|^2 - [\operatorname{Re}\langle x,y\rangle]^2 &\leq \left[\frac{(M+m)^2}{4mM} - 1\right] [\operatorname{Re}\langle x,y\rangle]^2 \\ &= \frac{(M-m)^2}{4mM} [\operatorname{Re}\langle x,y\rangle]^2, \end{aligned}$$

which proves the third inequality in (3.11). The other ones are obvious. ∎

3.3. Applications for Isotonic Linear Functionals. The following proposition holds [**4**].

PROPOSITION 4. *Let $f,g,h \in F(T)$ be such that $fgh \in L$, $f^2h \in L$, $g^2h \in L$. If $m,M>0$ are such that*

$$mg \leq f \leq Mg \text{ on } F(T), \tag{3.12}$$

then for any isotonic linear functional $A: L \to \mathbb{R}$ with $A(h)>0$, we have the inequality

$$1 \leq \frac{A(f^2h)\,A(g^2h)}{A^2(fgh)} \leq \frac{(M+m)^2}{4mM}. \tag{3.13}$$

The constant $\frac{1}{4}$ in (3.13) is sharp.

PROOF. We observe that

$$(Mg-f, f-mg)_{A,h} = A\left[h(Mg-f)(f-mg)\right] \geq 0.$$

Applying Corollary 4 for $(\cdot,\cdot)_{A,h}$ we get

$$1 \leq \frac{(f,f)_{A,h}\,(g,g)_{A,h}}{(f,g)^2_{A,h}} \leq \frac{(M+m)^2}{4mM},$$

which is clearly equivalent to (3.13). ∎

The following additive versions of (3.13) also hold [**4**].

COROLLARY 6. *With the assumption in Proposition 4, one has*

$$\begin{aligned} 0 &\leq \left[A\left(f^2h\right)A\left(g^2h\right)\right]^{\frac{1}{2}} - A(hfg) \\ &\leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} A(hfg) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq A\left(f^2h\right)A\left(g^2h\right) - A^2(fgh) \\ &\leq \frac{(M-m)^2}{4mM} A^2(fgh). \end{aligned}$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ are sharp.

REMARK 7. *The condition (3.12) may be replaced with the weaker assumption*

$$(Mg-f, f-mg)_{A,h} \geq 0. \tag{3.14}$$

Remark 8. *With the assumption (3.12) or (3.14) and if* $f,g\in F(T)$ *with* $fg,f^2,g^2\in L$, *then one has the inequalities*

$$1\le\frac{A(f^2)A(g^2)}{A^2(fg)}\le\frac{(M+m)^2}{4mM},$$

$$\begin{aligned}0&\le\left[A\left(f^2\right)A\left(g^2\right)\right]^{\frac12}-A(fg)\\&\le\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}}A(fg)\end{aligned}$$

and

$$0\le A\left(f^2\right)A\left(g^2\right)-A^2(fg)\le\frac{(M-m)^2}{4mM}A^2(fg).$$

3.4. Applications for Integrals. The following proposition contains a reverse of the weighted Cauchy-Bunyakovsky-Schwarz integral inequality.

Proposition 5. *Let* $A,a\in\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ *with* $\operatorname{Re}(\overline{a}A)>0$ *and* $f,g\in L^2_\rho(\Omega,\mathbb{K})$. *If*

$$\int_\Omega\operatorname{Re}\left[(Ag(s)-f(s))\left(\overline{f(s)}-\overline{a}\,\overline{g}(s)\right)\right]\rho(s)\,d\mu(s)\ge0, \tag{3.15}$$

then one has the inequality

$$\begin{aligned}&\left[\int_\Omega\rho(s)|f(s)|^2d\mu(s)\int_\Omega\rho(s)|g(s)|^2d\mu(s)\right]^{\frac12}\\&\le\frac12\cdot\frac{\int_\Omega\rho(s)\operatorname{Re}\left[A\overline{f(s)}g(s)+\overline{a}f(s)\overline{g(s)}\right]d\mu(s)}{[\operatorname{Re}(\overline{a}A)]^{\frac12}}\\&\le\frac12\cdot\frac{|A|+|a|}{[\operatorname{Re}(\overline{a}A)]^{\frac12}}\left|\int_\Omega\rho(s)f(s)\overline{g(s)}d\mu(s)\right|.\end{aligned}\tag{3.16}$$

The constant $\frac12$ *is sharp in (3.16).*

Proof. Follows by Theorem 4 applied for the inner product $\langle\cdot,\cdot\rangle_\rho:L^2_\rho(\Omega,\mathbb{K})\times L^2_\rho(\Omega,\mathbb{K})\to\mathbb{K}$,

$$\langle f,g\rangle:=\int_\Omega\rho(s)f(s)\overline{g(s)}d\mu(s).$$

∎

Remark 9. *A sufficient condition for (3.15) to hold is*

$$\operatorname{Re}\left[(Ag(s)-f(s))\left(\overline{f(s)}-\overline{a}\overline{g(s)}\right)\right]\ge0,\quad \textit{for } \mu\textit{-a.e.}\quad s\in\Omega.$$

In the particular case $\rho = 1$, we have the following result.

COROLLARY 7. *Let $a, A \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ with $\operatorname{Re}(\overline{a}A) > 0$ and $f, g \in L^2(\Omega, \mathbb{K})$. If*

$$\int_\Omega \operatorname{Re}\left[(Ag(s) - f(s))\left(\overline{f(s)} - \overline{a}\overline{g(s)}\right)\right] d\mu(s) \geq 0, \tag{3.17}$$

then one has the inequality

$$\begin{aligned}
&\left[\int_\Omega |f(s)|^2 d\mu(s) \int_\Omega |g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} \\
&\leq \frac{1}{2} \cdot \frac{\int_\Omega \operatorname{Re}\left[A\overline{f(s)}g(s) + \overline{a}f(s)\overline{g(s)}\right] d\mu(s)}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} \\
&\leq \frac{1}{2} \cdot \frac{|A| + |a|}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} \left|\int_\Omega f(s)\overline{g(s)} d\mu(s)\right|.
\end{aligned}$$

REMARK 10. *If $\mathbb{K} = \mathbb{R}$, then a sufficient condition for either (3.15) or (3.17) to hold is*

$$ag(s) \leq f(s) \leq Ag(s), \quad \textit{for } \mu\textit{-a.e. } s \in \Omega,$$

where, in this case, $a, A \in \mathbb{R}$ with $A > a > 0$.

If a, A are real positive constants, then the following proposition holds.

PROPOSITION 6. *Let $m, M > 0$. If $f, g \in L^2_\rho(\Omega, \mathbb{K})$ such that*

$$\int_\Omega \rho(s) \operatorname{Re}\left[(Mg(s) - f(s))\left(\overline{f(s)} - m\overline{g}(s)\right)\right] d\mu(s) \geq 0,$$

then one has the inequality

$$\begin{aligned}
&\left[\int_\Omega \rho(s) |f(s)|^2 d\mu(s) \int_\Omega \rho(s) |g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} \\
&\qquad\qquad \leq \frac{1}{2} \cdot \frac{M + m}{\sqrt{mM}} \int_\Omega \rho(s) \operatorname{Re}\left[f(s)\overline{g(s)}\right] d\mu(s).
\end{aligned}$$

The proof follows by Corollary 4 applied for the inner product

$$\langle f, g\rangle_\rho := \int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s).$$

The following additive versions also hold [**4**].

COROLLARY 8. *With the assumptions in Proposition 6, one has the inequalities*

$$\begin{aligned}0 &\leq \left[\int_{\Omega}\rho(s)\left|f(s)\right|^{2}d\mu(s)\int_{\Omega}\rho(s)\left|g(s)\right|^{2}d\mu(s)\right]^{\frac{1}{2}} \\ &\qquad\qquad - \int_{\Omega}\rho(s)\operatorname{Re}\left[f(s)\overline{g(s)}\right]d\mu(s) \\ &\leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^{2}}{2\sqrt{mM}}\int_{\Omega}\rho(s)\operatorname{Re}\left[f(s)\overline{g(s)}\right]d\mu(s)\end{aligned}$$

and

$$\begin{aligned}0 &\leq \int_{\Omega}\rho(s)\left|f(s)\right|^{2}d\mu(s)\int_{\Omega}\rho(s)\left|g(s)\right|^{2}d\mu(s) \\ &\qquad\qquad - \left(\int_{\Omega}\rho(s)\operatorname{Re}\left[f(s)\overline{g(s)}\right]d\mu(s)\right)^{2} \\ &\leq \frac{(M-m)^{2}}{4mM}\left(\int_{\Omega}\rho(s)\operatorname{Re}\left[f(s)\overline{g(s)}\right]d\mu(s)\right)^{2}.\end{aligned}$$

REMARK 11. *If $\mathbb{K}=\mathbb{R}$, a sufficient condition for (3.15) to hold is*

$$mg(s) \leq f(s) \leq Mg(s), \quad \text{for } \mu\text{-a.e. } s\in\Omega,$$

where $M>m>0$.

3.5. Applications for Sequences. For a given sequence $(w_i)_{i\in\mathbb{N}}$ of nonnegative real numbers, consider the Hilbert space $\ell_w^2(\mathbb{K}),\ (\mathbb{K}=\mathbb{C},\mathbb{R})$, where

$$\ell_w^2(\mathbb{K}) := \left\{\overline{\mathbf{x}}=(x_i)_{i\in\mathbb{N}}\subset\mathbb{K}\,\left|\,\sum_{i=0}^{\infty}w_i\left|x_i\right|^2<\infty\right.\right\}.$$

The following proposition that provides a reverse of the weighted Cauchy-Bunyakovsky-Schwarz inequality for complex numbers holds [**4**].

PROPOSITION 7. *Let $a,A\in\mathbb{K}$ with $\operatorname{Re}(\overline{a}A)>0$ and $\overline{\mathbf{x}},\overline{\mathbf{y}}\in\ell_w^2(\mathbb{K})$. If*

$$\sum_{i=0}^{\infty}w_i\operatorname{Re}\left[(Ay_i-x_i)\left(\overline{x_i}-\overline{ay_i}\right)\right]\geq 0, \tag{3.18}$$

then one has the inequality

$$\left[\sum_{i=0}^{\infty} w_i |x_i|^2 \sum_{i=0}^{\infty} w_i |y_i|^2\right]^{\frac{1}{2}} \leq \frac{1}{2} \cdot \frac{\sum_{i=0}^{\infty} w_i \operatorname{Re}\left[A\overline{x_i} y_i + \overline{a} x_i \overline{y_i}\right]}{\left[\operatorname{Re}\left(\overline{a}A\right)\right]^{\frac{1}{2}}} \leq \frac{1}{2} \cdot \frac{|A| + |a|}{\left[\operatorname{Re}\left(\overline{a}A\right)\right]^{\frac{1}{2}}} \left|\sum_{i=0}^{\infty} w_i x_i \overline{y_i}\right|. \tag{3.19}$$

The constant $\frac{1}{2}$ *is sharp in (3.19).*

Proof. Follows by Theorem 4 applied for the inner product $\langle \cdot, \cdot \rangle_w : \ell_w^2(\mathbb{K}) \times \ell_w^2(\mathbb{K}) \to \mathbb{K}$,

$$\langle \overline{\mathbf{x}}, \overline{\mathbf{y}} \rangle_w := \sum_{i=0}^{\infty} w_i x_i \overline{y_i}.$$

∎

Remark 12. *A sufficient condition for (3.18) to hold is*

$$\operatorname{Re}\left[\left(Ay_i - x_i\right)\left(\overline{x_i} - \overline{ay_i}\right)\right] \geq 0 \quad \textit{for all } i \in \mathbb{N}. \tag{3.20}$$

In the particular case $\rho = 1$, we have the following result.

Corollary 9. *Let* $a, A \in \mathbb{K}$ *with* $\operatorname{Re}(\overline{a}A) > 0$ *and* $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in \ell^2(\mathbb{K})$. *If*

$$\sum_{i=0}^{\infty} \operatorname{Re}\left[\left(Ay_i - x_i\right)\left(\overline{x_i} - \overline{ay_i}\right)\right] \geq 0,$$

then one has the inequality

$$\left[\sum_{i=0}^{\infty} |x_i|^2 \sum_{i=0}^{\infty} |y_i|^2\right]^{\frac{1}{2}} \leq \frac{1}{2} \cdot \frac{\sum_{i=0}^{\infty} \operatorname{Re}\left[A\overline{x_i} y_i + \overline{a} x_i \overline{y_i}\right]}{\left[\operatorname{Re}\left(\overline{a}A\right)\right]^{\frac{1}{2}}} \leq \frac{1}{2} \cdot \frac{|A| + |a|}{\left[\operatorname{Re}\left(\overline{a}A\right)\right]^{\frac{1}{2}}} \left|\sum_{i=0}^{\infty} x_i \overline{y_i}\right|.$$

Remark 13. *If* $\mathbb{K} = \mathbb{R}$, *then a sufficient condition for either (3.18) or (3.20) to hold is*

$$ay_i \leq x_i \leq Ay_i \quad \textit{for each} \quad i \in \{1, \dots, n\},$$

where, in this case, $a, A \in \mathbb{R}$ *with* $A > a > 0$.

For $a = m$, $A = M$, then the following proposition also holds.

Proposition 8. *Let* $m, M > 0$. *If* $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in \ell_w^2(\mathbb{K})$ *such that*

$$\sum_{i=0}^{\infty} w_i \operatorname{Re}\left[\left(My_i - x_i\right)\left(\overline{x_i} - m\overline{y_i}\right)\right] \geq 0, \tag{3.21}$$

then one has the inequality

$$\left[\sum_{i=0}^{\infty} w_i |x_i|^2 \sum_{i=0}^{\infty} w_i |y_i|^2\right]^{\frac{1}{2}} \leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} \sum_{i=0}^{\infty} w_i \operatorname{Re}(x_i \overline{y_i}).$$

The proof follows by Corollary 4 applied for the inner product

$$\langle \overline{\mathbf{x}}, \overline{\mathbf{y}} \rangle_w := \sum_{i=0}^{\infty} w_i x_i, \overline{y_i}.$$

The following additive version also holds [**4**].

Corollary 10. *With the assumptions in Proposition 8, one has the inequalities*

$$\begin{aligned} 0 &\leq \left[\sum_{i=0}^{\infty} w_i |x_i|^2 \sum_{i=0}^{\infty} w_i |y_i|^2\right]^{\frac{1}{2}} - \sum_{i=0}^{\infty} w_i \operatorname{Re}(x_i \overline{y_i}) \\ &\leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \sum_{i=0}^{\infty} w_i \operatorname{Re}(x_i \overline{y_i}) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \sum_{i=0}^{\infty} w_i |x_i|^2 \sum_{i=0}^{\infty} w_i |y_i|^2 - \left[\sum_{i=0}^{\infty} w_i \operatorname{Re}(x_i \overline{y_i})\right]^2 \\ &\leq \frac{(M-m)^2}{4mM} \left[\sum_{i=0}^{\infty} w_i \operatorname{Re}(x_i \overline{y_i})\right]^2. \end{aligned}$$

Remark 14. *If $\mathbb{K} = \mathbb{R}$, a sufficient condition for (3.21) to hold is*

$$my_i \leq x_i \leq My_i \quad \textit{for each} \quad i \in \mathbb{N},$$

where $M > m > 0$.

4. Quadratic Reverses of Schwarz's Inequality

4.1. Two Better Reverse Inequalities. It has been proven in [**7**], that

$$0 \leq \|x\|^2 - |\langle x, e \rangle|^2 \leq \frac{1}{4} |\phi - \varphi|^2 - \left| \frac{\phi + \varphi}{2} - \langle x, e \rangle \right|^2, \tag{4.1}$$

provided, either

$$\operatorname{Re} \langle \phi e - x, x - \varphi e \rangle \geq 0, \tag{4.2}$$

or, equivalently,

$$\left\| x - \frac{\phi + \varphi}{2} e \right\| \leq \frac{1}{2} \left| \phi - \varphi \right|, \tag{4.3}$$

holds, where $e \in H$, $\|e\| = 1$. The constant $\frac{1}{4}$ in (4.1) is best possible.

If we choose $e = \frac{y}{\|y\|}$, $\phi = \Gamma \|y\|$, $\varphi = \gamma \|y\|$ $(y \neq 0)$, $\Gamma, \gamma \in \mathbb{K}$, then by (4.2) and (4.3) we have,

$$\operatorname{Re} \langle \Gamma y - x, x - \gamma y \rangle \geq 0, \tag{4.4}$$

or, equivalently,

$$\left\| x - \frac{\Gamma + \gamma}{2} y \right\| \leq \frac{1}{2} \left| \Gamma - \gamma \right| \|y\|, \tag{4.5}$$

implying the following reverse of Schwarz's inequality:

$$\begin{aligned} 0 &\leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \\ &\leq \frac{1}{4} |\Gamma - \gamma|^2 \|y\|^4 - \left| \frac{\Gamma + \gamma}{2} \|y\|^2 - \langle x, y \rangle \right|^2. \end{aligned} \tag{4.6}$$

The constant $\frac{1}{4}$ in (4.6) is sharp.

Note that, this inequality is an improvement of (2.6), but it may not be very convenient for applications.

In [**10**], it has also been proven that

$$0 \leq \|x\|^2 - |\langle x, e \rangle|^2 \leq \frac{1}{4} |\phi - \varphi|^2 - \operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \tag{4.7}$$

provided either (4.2) or (4.3) holds true.

If we make the same choice for e, Φ and φ as above, then we deduce the inequality

$$\begin{aligned} 0 &\leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \\ &\leq \frac{1}{4} |\Gamma - \gamma|^2 \|y\|^4 - \|y\|^2 \operatorname{Re} \langle \Gamma y - x, x - \gamma y \rangle \end{aligned} \tag{4.8}$$

provided either (4.4) or (4.5) holds true.

The constant $\frac{1}{4}$ is best possible in (4.8).

One may easily realise that the bounds provided by (4.6) and (4.8) cannot be compared in general, meaning that for different choices of variables one may be better than the other.

4.2. A Reverse of Schwarz's Inequality Under More General Assumptions.

The following result holds [**8**].

THEOREM 5. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$, $\mathbb{K}=\mathbb{C}$) and $x,a\in H$, $r>0$ are such that*

$$x\in\overline{B}(a,r):=\{z\in H|\,\|z-a\|\leq r\}.$$

(i) *If $\|a\|>r$, then we have the inequalities*

$$0\leq\|x\|^2\|a\|^2-|\langle x,a\rangle|^2\leq\|x\|^2\|a\|^2-[\operatorname{Re}\langle x,a\rangle]^2\leq r^2\|x\|^2. \tag{4.9}$$

The constant $C=1$ in front of r^2 is best possible in the sense that it cannot be replaced by a smaller one.

(ii) *If $\|a\|=r$, then*

$$\|x\|^2\leq 2\operatorname{Re}\langle x,a\rangle\leq 2|\langle x,a\rangle|. \tag{4.10}$$

The constant 2 is best possible in both inequalities.

(iii) *If $\|a\|<r$, then*

$$\|x\|^2\leq r^2-\|a\|^2+2\operatorname{Re}\langle x,a\rangle\leq r^2-\|a\|^2+2|\langle x,a\rangle|. \tag{4.11}$$

Here the constant 2 is also best possible.

PROOF. Since $x\in\overline{B}(a,r)$, then obviously $\|x-a\|^2\leq r^2$, which is equivalent to

$$\|x\|^2+\|a\|^2-r^2\leq 2\operatorname{Re}\langle x,a\rangle. \tag{4.12}$$

(i) If $\|a\|>r$, then we may divide (4.12) by $\sqrt{\|a\|^2-r^2}>0$ getting

$$\frac{\|x\|^2}{\sqrt{\|a\|^2-r^2}}+\sqrt{\|a\|^2-r^2}\leq\frac{2\operatorname{Re}\langle x,a\rangle}{\sqrt{\|a\|^2-r^2}}. \tag{4.13}$$

Using the elementary inequality

$$\alpha p+\frac{1}{\alpha}q\geq 2\sqrt{pq},\quad \alpha>0,\quad p,q\geq 0,$$

we may state that

$$2\|x\|\leq\frac{\|x\|^2}{\sqrt{\|a\|^2-r^2}}+\sqrt{\|a\|^2-r^2}. \tag{4.14}$$

Making use of (4.13) and (4.14), we deduce

$$\|x\|\sqrt{\|a\|^2-r^2}\leq\operatorname{Re}\langle x,a\rangle, \tag{4.15}$$

which is an interesting inequality in itself as well.

Taking the square in (4.15) and re-arranging the terms, we deduce the third inequality in (4.9). The others are obvious.

To prove the sharpness of the constant, assume, under the hypothesis of the theorem, that, there exists a constant $c > 0$ such that

$$\|x\|^2 \|a\|^2 - [\operatorname{Re}\langle x, a\rangle]^2 \leq cr^2 \|x\|^2, \tag{4.16}$$

provided $x \in \overline{B}(a, r)$ and $\|a\| > r$.

Let $r = \sqrt{\varepsilon} > 0$, $\varepsilon \in (0, 1)$, $a, e \in H$ with $\|a\| = \|e\| = 1$ and $a \perp e$. Put $x = a + \sqrt{\varepsilon} e$. Then obviously $x \in \overline{B}(a, r)$, $\|a\| > r$ and $\|x\|^2 = \|a\|^2 + \varepsilon \|e\|^2 = 1 + \varepsilon$, $\operatorname{Re}\langle x, a\rangle = \|a\|^2 = 1$, and thus $\|x\|^2 \|a\|^2 - [\operatorname{Re}\langle x, a\rangle]^2 = \varepsilon$. Using (4.16), we may write that

$$\varepsilon \leq c\varepsilon (1 + \varepsilon), \quad \varepsilon > 0$$

giving

$$c + c\varepsilon \geq 1 \text{ for any } \varepsilon > 0. \tag{4.17}$$

Letting $\varepsilon \to 0+$, we get from (4.17) that $c \geq 1$, and the sharpness of the constant is proved.

(ii) The inequality (4.10) is obvious by (4.12) since $\|a\| = r$. The best constant follows in a similar way to the above.

(iii) The inequality (4.11) is obvious. The best constant may be proved in a similar way to the above. We omit the details.

∎

The following reverse of Schwarz's inequality holds [**8**].

Theorem 6. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ and $x, y \in H$, $\gamma, \Gamma \in \mathbb{K}$ such that either*

$$\operatorname{Re}\langle \Gamma y - x, x - \gamma y\rangle \geq 0, \tag{4.18}$$

or, equivalently,

$$\left\| x - \frac{\Gamma + \gamma}{2} y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|y\|, \tag{4.19}$$

holds.

(i) *If $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$, then we have the inequalities*

$$\|x\|^2 \|y\|^2 \leq \frac{1}{4} \cdot \frac{\left\{\operatorname{Re}\left[\left(\overline{\Gamma} + \overline{\gamma}\right)\langle x, y\rangle\right]\right\}^2}{\operatorname{Re}(\Gamma\overline{\gamma})} \leq \frac{1}{4} \cdot \frac{|\Gamma + \gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})} |\langle x, y\rangle|^2. \tag{4.20}$$

The constant $\frac{1}{4}$ is best possible in both inequalities.

(ii) *If* $\operatorname{Re}(\Gamma\overline{\gamma}) = 0$, *then*

$$\|x\|^2 \leq \operatorname{Re}\left[\left(\overline{\Gamma} + \overline{\gamma}\right)\langle x, y\rangle\right] \leq |\Gamma + \gamma| \, |\langle x, y\rangle| \, .$$

(iii) *If* $\operatorname{Re}(\Gamma\overline{\gamma}) < 0$, *then*

$$\begin{aligned}\|x\|^2 &\leq -\operatorname{Re}(\Gamma\overline{\gamma})\|y\|^2 + \operatorname{Re}\left[\left(\overline{\Gamma} + \overline{\gamma}\right)\langle x, y\rangle\right] \\ &\leq -\operatorname{Re}(\Gamma\overline{\gamma})\|y\|^2 + |\Gamma + \gamma| \, |\langle x, y\rangle| \, .\end{aligned}$$

Proof. The proof of the equivalence between the inequalities (4.18) and (4.19) follows by the fact that in an inner product space

$$\operatorname{Re}\langle Z - x, x - z\rangle \geq 0$$

for $x, z, Z \in H$ is equivalent with

$$\left\| x - \frac{z + Z}{2} \right\| \leq \frac{1}{2}\|Z - z\|$$

(see for example [**9**]).

Consider, for $y \neq 0$, $a = \frac{\gamma + \Gamma}{2}y$ and $r = \frac{1}{2}|\Gamma - \gamma| \, \|y\|$. Then

$$\|a\|^2 - r^2 = \frac{|\Gamma + \gamma|^2 - |\Gamma - \gamma|^2}{4}\|y\|^2 = \operatorname{Re}(\Gamma\overline{\gamma})\|y\|^2 .$$

(i) If $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$, then the hypothesis of (i) in Theorem 5 is satisfied, and by the second inequality in (4.9) we have

$$\|x\|^2 \frac{|\Gamma + \gamma|^2}{4}\|y\|^2 - \frac{1}{4}\left\{\operatorname{Re}\left[\left(\overline{\Gamma} + \overline{\gamma}\right)\langle x, y\rangle\right]\right\}^2 \leq \frac{1}{4}|\Gamma - \gamma|^2\|x\|^2\|y\|^2$$

from where we derive

$$\frac{|\Gamma + \gamma|^2 - |\Gamma - \gamma|^2}{4}\|x\|^2\|y\|^2 \leq \frac{1}{4}\left\{\operatorname{Re}\left[\left(\overline{\Gamma} + \overline{\gamma}\right)\langle x, y\rangle\right]\right\}^2 ,$$

giving the first inequality in (4.20).

The second inequality is obvious.

To prove the sharpness of the constant $\frac{1}{4}$, assume that the first inequality in (4.20) holds with a constant $c > 0$, i.e.,

$$\|x\|^2\|y\|^2 \leq c \cdot \frac{\left\{\operatorname{Re}\left[\left(\overline{\Gamma} + \overline{\gamma}\right)\langle x, y\rangle\right]\right\}^2}{\operatorname{Re}(\Gamma\overline{\gamma})}, \tag{4.21}$$

provided $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$ and either (4.18) or (4.19) holds.

Assume that $\Gamma, \gamma > 0$, and let $x = \gamma y$. Then (4.18) holds and by (4.21) we deduce

$$\gamma^2 \|y\|^4 \leq c \cdot \frac{(\Gamma+\gamma)^2 \gamma^2 \|y\|^4}{\Gamma\gamma}$$

giving

$$\Gamma\gamma \leq c\,(\Gamma+\gamma)^2 \quad \text{for any} \quad \Gamma, \gamma > 0. \tag{4.22}$$

Let $\varepsilon \in (0,1)$ and choose in (4.22), $\Gamma = 1+\varepsilon$, $\gamma = 1-\varepsilon > 0$ to get $1-\varepsilon^2 \leq 4c$ for any $\varepsilon \in (0,1)$. Letting $\varepsilon \to 0+$, we deduce $c \geq \frac{1}{4}$, and the sharpness of the constant is proved.

(ii) and (iii) are obvious and we omit the details.

∎

Remark 15. *We observe that the second bound in (4.20) for $\|x\|^2 \|y\|^2$ is better than the second bound provided by (3.5).*

The following corollary provides a reverse inequality for the additive version of Schwarz's inequality [**8**].

Corollary 11. *With the assumptions of Theorem 6 and if* $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$, *then we have the inequality:*

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4} \cdot \frac{|\Gamma-\gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})} |\langle x, y\rangle|^2 . \tag{4.23}$$

The constant $\frac{1}{4}$ is best possible in (4.23).

The proof is obvious from (4.20) on subtracting in both sides the same quantity $|\langle x, y\rangle|^2$. The sharpness of the constant may be proven in a similar manner to the one incorporated in the proof of (i), Theorem 6. We omit the details.

For other recent results in connection to Schwarz's inequality, see [**1**], [**11**] and [**13**].

4.3. Reverses of the Triangle Inequality.

The following reverse of the triangle inequality holds [**8**].

Proposition 9. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) and $x, a \in H$, $r > 0$ are such that*

$$\|x - a\| \leq r < \|a\| .$$

Then we have the inequality

$$\begin{aligned} 0 &\leq \|x\| + \|a\| - \|x+a\| \\ &\leq \sqrt{2} r \cdot \sqrt{\frac{\operatorname{Re}\langle x, a\rangle}{\sqrt{\|a\|^2 - r^2}\left(\sqrt{\|a\|^2 - r^2} + \|a\|\right)}}. \end{aligned} \tag{4.24}$$

Proof. Using the inequality (4.15), we may write that

$$\|x\| \, \|a\| \leq \frac{\|a\| \operatorname{Re}\langle x, a\rangle}{\sqrt{\|a\|^2 - r^2}},$$

which gives

$$\begin{aligned} 0 &\leq \|x\| \, \|a\| - \operatorname{Re}\langle x, a\rangle \\ &\leq \frac{\|a\| - \sqrt{\|a\|^2 - r^2}}{\sqrt{\|a\|^2 - r^2}} \operatorname{Re}\langle x, a\rangle \\ &= \frac{r^2 \operatorname{Re}\langle x, a\rangle}{\sqrt{\|a\|^2 - r^2}\left(\sqrt{\|a\|^2 - r^2} + \|a\|\right)}. \end{aligned} \tag{4.25}$$

Since

$$(\|x\| + \|a\|)^2 - \|x+a\|^2 = 2\left(\|x\| \, \|a\| - \operatorname{Re}\langle x, a\rangle\right),$$

then by (4.25), we have

$$\begin{aligned} \|x\| + \|a\| &\leq \sqrt{\|x+a\|^2 + \frac{2r^2 \operatorname{Re}\langle x, a\rangle}{\sqrt{\|a\|^2 - r^2}\left(\sqrt{\|a\|^2 - r^2} + \|a\|\right)}} \\ &\leq \|x+a\| + \sqrt{2} r \cdot \sqrt{\frac{\operatorname{Re}\langle x, a\rangle}{\sqrt{\|a\|^2 - r^2}\left(\sqrt{\|a\|^2 - r^2} + \|a\|\right)}}, \end{aligned}$$

giving the desired inequality (4.24). ∎

The following proposition providing a simpler reverse for the triangle inequality also holds [**8**].

Proposition 10. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be an inner product space over* $\mathbb{K}$ *and* $x, y \in H$, $M > m > 0$ *such that either*

$$\operatorname{Re}\langle My - x, x - my\rangle \geq 0,$$

or, equivalently,

$$\left\|x-\frac{M+m}{2}\cdot y\right\|\leq\frac{1}{2}\left(M-m\right)\left\|y\right\|,$$

holds. Then we have the inequality

$$0\leq\left\|x\right\|+\left\|y\right\|-\left\|x+y\right\|\leq\frac{\sqrt{M}-\sqrt{m}}{\sqrt[4]{mM}}\sqrt{\operatorname{Re}\left\langle x,y\right\rangle}. \tag{4.26}$$

Proof. Choosing in (4.15), $a=\frac{M+m}{2}y$, $r=\frac{1}{2}\left(M-m\right)\left\|y\right\|$ we get

$$\left\|x\right\|\left\|y\right\|\sqrt{Mm}\leq\frac{M+m}{2}\operatorname{Re}\left\langle x,y\right\rangle,$$

giving

$$0\leq\left\|x\right\|\left\|y\right\|-\operatorname{Re}\left\langle x,y\right\rangle\leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^{2}}{2\sqrt{mM}}\operatorname{Re}\left\langle x,y\right\rangle.$$

Following the same argument as in the proof of Proposition 9, we deduce the desired inequality (4.26). ∎

For some results related to triangle inequality in inner product spaces, see [**2**], [**14**], [**16**] and [**21**].

4.4. Integral Inequalities. Denote by $L_{\rho}^{2}\left(\Omega,\mathbb{K}\right)$ the Hilbert space of all real or complex valued functions defined on Ω and $2-\rho-$integrable on Ω, i.e.,

$$\int_{\Omega}\rho\left(s\right)\left|f\left(s\right)\right|^{2}d\mu\left(s\right)<\infty.$$

It is obvious that the following inner product

$$\left\langle f,g\right\rangle_{\rho}:=\int_{\Omega}\rho\left(s\right)f\left(s\right)\overline{g\left(s\right)}d\mu\left(s\right),$$

generates the norm $\left\|f\right\|_{\rho}:=\left(\int_{\Omega}\rho\left(s\right)\left|f\left(s\right)\right|^{2}d\mu\left(s\right)\right)^{\frac{1}{2}}$ of $L_{\rho}^{2}\left(\Omega,\mathbb{K}\right)$, and all the above results may be stated for integrals.

It is important to observe that, if

$$\operatorname{Re}\left[f\left(s\right)\overline{g\left(s\right)}\right]\geq0,\quad\text{for }\mu-\text{a.e. }s\in\Omega,$$

then, obviously,

$$\begin{aligned}\operatorname{Re}\left\langle f,g\right\rangle_{\rho}&=\operatorname{Re}\left[\int_{\Omega}\rho\left(s\right)f\left(s\right)\overline{g\left(s\right)}d\mu\left(s\right)\right]\\&=\int_{\Omega}\rho\left(s\right)\operatorname{Re}\left[f\left(s\right)\overline{g\left(s\right)}\right]d\mu\left(s\right)\geq0.\end{aligned} \tag{4.27}$$

The reverse is evidently not true in general.

Moreover, if the space is real, i.e., $\mathbb{K}=\mathbb{R}$, then a sufficient condition for (4.27) to hold is:

$$f(s)\geq 0,\quad g(s)\geq 0,\quad \text{for } \mu-\text{a.e. } s\in\Omega.$$

We now provide, by the use of certain results obtained above, some integral inequalities that may be used in practical applications.

PROPOSITION 11. *Let $f,g\in L^2_\rho(\Omega,\mathbb{K})$ and $r>0$ with the properties that*

$$|f(s)-g(s)|\leq r\leq |g(s)|,\quad \text{for } \mu-a.e.\ s\in\Omega. \tag{4.28}$$

Then we have the inequalities

$$\begin{aligned}0&\leq \int_\Omega \rho(s)|f(s)|^2 d\mu(s)\int_\Omega \rho(s)|g(s)|^2 d\mu(s)\\ &\qquad\qquad -\left|\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s)\right|^2\\ &\leq \int_\Omega \rho(s)|f(s)|^2 d\mu(s)\int_\Omega \rho(s)|g(s)|^2 d\mu(s)\\ &\qquad\qquad -\left[\int_\Omega \rho(s)\operatorname{Re}\left(f(s)\overline{g(s)}\right)d\mu(s)\right]^2\\ &\leq r^2\int_\Omega \rho(s)|g(s)|^2 d\mu(s).\end{aligned} \tag{4.29}$$

The constant $c=1$ in front of r^2 is best possible.

The proof follows by Theorem 5 and we omit the details [**8**].

PROPOSITION 12. *Let $f,g\in L^2_\rho(\Omega,\mathbb{K})$ and $\gamma,\Gamma\in\mathbb{K}$ such that* $\operatorname{Re}(\Gamma\overline{\gamma})>0$ *and*

$$\operatorname{Re}\left[(\Gamma g(s)-f(s))\left(\overline{f(s)}-\overline{\gamma}\overline{g(s)}\right)\right]\geq 0,\quad \text{for } \mu-a.e.\ s\in\Omega.$$

Then we have the inequalities

$$\begin{aligned}&\int_\Omega \rho(s)|f(s)|^2 d\mu(s)\int_\Omega \rho(s)|g(s)|^2 d\mu(s)\\ &\leq \frac{1}{4}\cdot\frac{\left\{\operatorname{Re}\left[\left(\overline{\Gamma}+\overline{\gamma}\right)\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s)\right]\right\}^2}{\operatorname{Re}(\Gamma\overline{\gamma})}\\ &\leq \frac{1}{4}\cdot\frac{|\Gamma+\gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}\left|\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s)\right|^2.\end{aligned} \tag{4.30}$$

The constant $\frac{1}{4}$ is best possible in both inequalities.

The proof follows by Theorem 6 and we omit the details.

COROLLARY 12. *With the assumptions of Proposition 12, we have the inequality*

$$0 \leq \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) - \left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right|^2 \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{\operatorname{Re}(\Gamma \overline{\gamma})} \left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right|^2 . \tag{4.31}$$

The constant $\frac{1}{4}$ is best possible.

REMARK 16. *If the space is real and we assume, for $M > m > 0$, that*

$$mg(s) \leq f(s) \leq Mg(s), \quad \textit{for } \mu - a.e. \ s \in \Omega,$$

then, by (4.30) and (4.31), we deduce the inequalities

$$\int_{\Omega} \rho(s) [f(s)]^2 d\mu(s) \int_{\Omega} \rho(s) [g(s)]^2 d\mu(s) \leq \frac{1}{4} \cdot \frac{(M+m)^2}{mM} \left[\int_{\Omega} \rho(s) f(s) g(s) d\mu(s) \right]^2 \tag{4.32}$$

and

$$0 \leq \int_{\Omega} \rho(s) [f(s)]^2 d\mu(s) \int_{\Omega} \rho(s) [g(s)]^2 d\mu(s) - \left[\int_{\Omega} \rho(s) f(s) g(s) d\mu(s) \right]^2 \leq \frac{1}{4} \cdot \frac{(M-m)^2}{mM} \left[\int_{\Omega} \rho(s) f(s) g(s) d\mu(s) \right]^2 . \tag{4.33}$$

The inequality (4.32) is known in the literature as Cassel's inequality.

5. More Reverses of Schwarz's Inequality

5.1. General Results. The following result holds [**6**].

THEOREM 7. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x, a \in H$ and $r > 0$. If*

$$x \in \bar{B}(a, r) := \{ z \in H | \, \|z - a\| \leq r \} , \tag{5.1}$$

then we have the inequalities:

$$0 \leq \|x\| \|a\| - |\langle x, a \rangle| \leq \|x\| \|a\| - |\operatorname{Re} \langle x, a \rangle| \leq \|x\| \|a\| - \operatorname{Re} \langle x, a \rangle \leq \frac{1}{2} r^2 . \tag{5.2}$$

The constant $\frac{1}{2}$ *is best possible in (5.2) in the sense that it cannot be replaced by a smaller constant.*

PROOF. The condition (5.1) is clearly equivalent to

$$\|x\|^2 + \|a\|^2 \leq 2\operatorname{Re}\langle x, a\rangle + r^2. \tag{5.3}$$

Using the elementary inequality

$$2\|x\|\|a\| \leq \|x\|^2 + \|a\|^2, \qquad a, x \in H$$

and (5.3), we deduce

$$2\|x\|\|a\| \leq 2\operatorname{Re}\langle x, a\rangle + r^2,$$

giving the last inequality in (5.2). The other inequalities are obvious.

To prove the sharpness of the constant $\frac{1}{2}$, assume that

$$0 \leq \|x\|\|a\| - \operatorname{Re}\langle x, a\rangle \leq cr^2 \tag{5.4}$$

for any $x, a \in H$ and $r > 0$ satisfying (5.1).

Assume that $a, e \in H$, $\|a\| = \|e\| = 1$ and $e \perp a$. If $r = \sqrt{\varepsilon}$, $\varepsilon > 0$ and if we define $x = a + \sqrt{\varepsilon}e$, then $\|x - a\| = \sqrt{\varepsilon} = r$ showing that the condition (5.1) is fulfilled.

On the other hand,

$$\begin{aligned}\|x\|\|a\| - \operatorname{Re}\langle x, a\rangle &= \sqrt{\left\|a + \sqrt{\varepsilon}e\right\|^2} - \operatorname{Re}\left\langle a + \sqrt{\varepsilon}e, a\right\rangle \\ &= \sqrt{\|a\|^2 + \varepsilon\|e\|^2} - \|a\|^2 \\ &= \sqrt{1+\varepsilon} - 1.\end{aligned}$$

Utilising (5.4), we conclude that

$$\sqrt{1+\varepsilon} - 1 \leq c\varepsilon \quad \text{for any } \varepsilon > 0. \tag{5.5}$$

Multiplying (5.5) by $\sqrt{1+\varepsilon} + 1 > 0$ and then dividing by $\varepsilon > 0$, we get

$$\left(\sqrt{1+\varepsilon} + 1\right) c \geq 1 \quad \text{for any} \quad \varepsilon > 0. \tag{5.6}$$

Letting $\varepsilon \to 0+$ in (5.6), we deduce $c \geq \frac{1}{2}$, and the theorem is proved. ∎

The following result also holds [**6**].

THEOREM 8. *Let* $(H; \langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$ *and* $x, y \in H$, $\gamma, \Gamma \in \mathbb{K}$ $(\Gamma \neq -\gamma)$ *so that either*

$$\operatorname{Re}\langle \Gamma y - x, x - \gamma y\rangle \geq 0, \tag{5.7}$$

or, equivalently,

$$\left\|x - \frac{\gamma + \Gamma}{2}y\right\| \leq \frac{1}{2}|\Gamma - \gamma|\,\|y\|, \tag{5.8}$$

holds. Then we have the inequalities

$$
\begin{aligned}
0 &\leq \|x\| \|y\| - |\langle x, y\rangle| \\
&\leq \|x\| \|y\| - \left|\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\langle x, y\rangle\right]\right| \\
&\leq \|x\| \|y\| - \operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\langle x, y\rangle\right] \\
&\leq \frac{1}{4} \cdot \frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\|y\|^2 .
\end{aligned} \tag{5.9}
$$

The constant $\frac{1}{4}$ in the last inequality is best possible.

Proof. Consider for $a, y \neq 0$, $a = \frac{\Gamma+\gamma}{2} \cdot y$ and $r = \frac{1}{2}|\Gamma-\gamma|\|y\|$. Thus from (5.2), we get

$$
\begin{aligned}
0 &\leq \|x\|\left|\frac{\Gamma+\gamma}{2}\right|\|y\| - \left|\frac{\Gamma+\gamma}{2}\right||\langle x, y\rangle| \\
&\leq \|x\|\left|\frac{\Gamma+\gamma}{2}\right|\|y\| - \left|\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{2}\langle x, y\rangle\right]\right| \\
&\leq \|x\|\left|\frac{\Gamma+\gamma}{2}\right|\|y\| - \operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{2}\langle x, y\rangle\right] \\
&\leq \frac{1}{8} \cdot |\Gamma-\gamma|^2\|y\|^2 .
\end{aligned}
$$

Dividing by $\frac{1}{2}|\Gamma+\gamma| > 0$, we deduce the desired inequality (5.9).

To prove the sharpness of the constant $\frac{1}{4}$, assume that there exists a $c > 0$ such that:

$$
\|x\| \|y\| - \operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\langle x, y\rangle\right] \leq c \cdot \frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\|y\|^2 , \tag{5.10}
$$

provided either (5.7) or (5.8) holds.

Consider the real inner product space $(\mathbb{R}^2, \langle\cdot,\cdot\rangle)$ with $\langle \bar{\mathbf{x}}, \bar{\mathbf{y}}\rangle = x_1 y_1 + x_2 y_2$, $\bar{\mathbf{x}} = (x_1, x_2)$, $\bar{\mathbf{y}} = (y_1, y_2) \in \mathbb{R}^2$. Let $\bar{\mathbf{y}} = (1,1)$ and $\Gamma, \gamma > 0$ with $\Gamma > \gamma$. Then, by (5.10), we deduce

$$
\sqrt{2}\sqrt{x_1^2 + x_2^2} - (x_1 + x_2) \leq 2c \cdot \frac{(\Gamma-\gamma)^2}{\Gamma+\gamma}. \tag{5.11}
$$

If $x_1 = \Gamma$, $x_2 = \gamma$, then

$$
\langle \Gamma\bar{\mathbf{y}} - \bar{\mathbf{x}}, \bar{\mathbf{x}} - \gamma\bar{\mathbf{y}}\rangle = (\Gamma - x_1)(x_1 - \gamma) + (\Gamma - x_2)(x_2 - \gamma) = 0,
$$

showing that the condition (5.7) is valid. Replacing x_1 and x_2 in (5.11), we deduce

$$\sqrt{2}\sqrt{\Gamma^2+\gamma^2}-(\Gamma+\gamma)\leq 2c\frac{(\Gamma-\gamma)^2}{\Gamma+\gamma}. \tag{5.12}$$

If in (5.12) we choose $\Gamma=1+\varepsilon$, $\gamma=1-\varepsilon$ with $\varepsilon\in(0,1)$, then we have

$$2\sqrt{1+\varepsilon^2}-2\leq 2c\frac{4\varepsilon^2}{2},$$

giving

$$\sqrt{1+\varepsilon^2}-1\leq 2c\varepsilon^2. \tag{5.13}$$

Finally, multiplying (5.13) with $\sqrt{1+\varepsilon^2}+1>0$ and thus dividing by ε^2, we deduce

$$1\leq 2c\left(\sqrt{1+\varepsilon^2}+1\right)\quad\text{for any}\quad\varepsilon\in(0,1). \tag{5.14}$$

Letting $\varepsilon\to 0+$ in (5.14) we get $c\geq\frac{1}{4}$, and the sharpness of the constant is proved. ∎

5.2. Reverses of the Triangle Inequality. The following reverse of the triangle inequality in inner product spaces holds [**6**].

PROPOSITION 13. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x,a\in H$ and $r>0$. If $\|x-a\|\leq r$, then we have the inequality*

$$0\leq\|x\|+\|a\|-\|x+a\|\leq r. \tag{5.15}$$

PROOF. Since

$$(\|x\|+\|a\|)^2-\|x+a\|^2\leq 2\left(\|x\|\,\|a\|-\operatorname{Re}\langle x,a\rangle\right),$$

then by Theorem 7 we deduce

$$(\|x\|+\|a\|)^2-\|x+a\|^2\leq r^2,$$

from where we obtain

$$\|x\|+\|a\|\leq\sqrt{r^2+\|x+a\|^2}\leq r+\|x+a\|,$$

giving the desired result (5.15). ∎

We may state the following result [**6**].

PROPOSITION 14. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $x,y\in H$, $M>m>0$ such that either*

$$\operatorname{Re}\langle My-x,x-my\rangle\geq 0,$$

or, equivalently,

$$\left\| x - \frac{M+m}{2} y \right\| \leq \frac{1}{2} (M-m) \|y\| ,$$

holds. Then we have the inequality

$$0 \leq \|x\| + \|y\| - \|x+y\| \leq \frac{\sqrt{2}}{2} \cdot \frac{M-m}{\sqrt{M+m}} \|y\| . \tag{5.16}$$

PROOF. By Theorem 8 for $\Gamma = M$, $\gamma = m$, we have the inequality

$$\|x\| \|y\| - \operatorname{Re} \langle x, y \rangle \leq \frac{1}{4} \cdot \frac{(M-m)^2}{M+m} \|y\|^2 .$$

Then we may state that

$$\begin{aligned} (\|x\| + \|y\|)^2 - \|x+y\|^2 &= 2 (\|x\| \|y\| - \operatorname{Re} \langle x, y \rangle) \\ &\leq \frac{1}{2} \cdot \frac{(M-m)^2}{M+m} \|y\|^2 , \end{aligned}$$

from where we get

$$\begin{aligned} \|x\| + \|y\| &\leq \sqrt{\frac{1}{2} \cdot \frac{(M-m)^2}{M+m} \|y\|^2 + \|x+y\|^2} \\ &\leq \|x+y\| + \frac{M-m}{\sqrt{2(M+m)}} \|y\| , \end{aligned}$$

giving the desired inequality (5.16). ∎

5.3. Integral Inequalities. We provide now, by the use of certain results obtained above, some integral inequalities that may be used in practical applications.

PROPOSITION 15. *Let $f, g \in L^2_\rho (\Omega, \mathbb{K})$ and $r > 0$ with the property that*

$$|f(s) - g(s)| \leq r \quad \textit{for} \;\; \mu - \textit{a.e.} \;\; s \in \Omega.$$

Then we have the inequalities

$$0 \leq \left[\int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} - \left|\int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s)\right| \tag{5.17}$$
$$\leq \left[\int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} - \left|\int_{\Omega} \rho(s) \operatorname{Re}\left[f(s) \overline{g(s)}\right] d\mu(s)\right|$$
$$\leq \left[\int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} - \int_{\Omega} \rho(s) \operatorname{Re}\left[f(s) \overline{g(s)}\right] d\mu(s)$$
$$\leq \frac{1}{2} r^2.$$

The constant $\frac{1}{2}$ is best possible in (5.17).

The proof follows by Theorem 7, and we omit the details.

Proposition 16. *Let $f, g \in L^2_\rho(\Omega, \mathbb{K})$ and $\gamma, \Gamma \in \mathbb{K}$ so that $\Gamma \neq -\gamma$, and*

$$\operatorname{Re}\left[(\Gamma g(s) - f(s)) \left(\overline{f(s)} - \overline{\gamma} \overline{g(s)}\right)\right] \geq 0, \quad \text{for } \ \mu - a.e. \ \ s \in \Omega.$$

Then we have the inequalities

$$0 \leq \left[\int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} - \left|\int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s)\right| \tag{5.18}$$
$$\leq \left[\int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} - \left|\operatorname{Re}\left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s)\right]\right|$$

$$
\begin{aligned}
&\leq \left[\int_{\Omega}\rho(s)\left|f(s)\right|^{2}d\mu(s)\int_{\Omega}\rho(s)\left|g(s)\right|^{2}d\mu(s)\right]^{\frac{1}{2}} \\
&\qquad\qquad -\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{\left|\Gamma+\gamma\right|}\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right] \\
&\leq \frac{1}{4}\cdot\frac{\left|\Gamma-\gamma\right|^{2}}{\left|\Gamma+\gamma\right|}\int_{\Omega}\rho(s)\left|g(s)\right|^{2}d\mu(s).
\end{aligned}
$$

The constant $\frac{1}{4}$ *is best possible.*

Remark 17. *If the space is real and we assume, for* $M>m>0$, *that*

$$mg(s)\leq f(s)\leq Mg(s), \quad \textit{for}\ \ \mu-a.e.\ \ s\in\Omega, \tag{5.19}$$

then, by (5.18), we deduce the inequality:

$$
\begin{aligned}
0 &\leq \left[\int_{\Omega}\rho(s)\left|f(s)\right|^{2}d\mu(s)\int_{\Omega}\rho(s)\left|g(s)\right|^{2}d\mu(s)\right]^{\frac{1}{2}} \\
&\qquad -\left|\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right| \\
&\leq \frac{1}{4}\cdot\frac{(M-m)^{2}}{M+m}\int_{\Omega}\rho(s)\left|g(s)\right|^{2}d\mu(s).
\end{aligned}
$$

The constant $\frac{1}{4}$ *is best possible.*

The following reverse of the triangle inequality for integrals holds.

Proposition 17. *Assume that the functions* $f,g\in L_{\rho}^{2}(\Omega,\mathbb{K})$ *satisfy (5.19). Then we have the inequality*

$$
\begin{aligned}
0 &\leq \left(\int_{\Omega}\rho(s)\left|f(s)\right|^{2}d\mu(s)\right)^{\frac{1}{2}}+\left(\int_{\Omega}\rho(s)\left|g(s)\right|^{2}d\mu(s)\right)^{\frac{1}{2}} \\
&\qquad\qquad -\left(\int_{\Omega}\rho(s)\left|f(s)+g(s)\right|^{2}d\mu(s)\right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{2}}{2}\cdot\frac{M-m}{\sqrt{M+m}}\left(\int_{\Omega}\rho(s)\left|g(s)\right|^{2}d\mu(s)\right)^{\frac{1}{2}}.
\end{aligned}
$$

The proof follows by Proposition 14.

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CHAPTER 2

Inequalities of the Grüss Type

1. Introduction

Over the last five years, the development of Grüss type inequalities has experienced a surge, having been stimulated by their applications in different branches of Applied Mathematics including: in perturbed quadrature rules (see for example [**5**], [**2**]) and in the approximation of integral transforms (see [**18**], [**21**]) and the references therein.

For two Lebesgue integrable functions $f, g : [a, b] \to \mathbb{R}$, consider the Čebyšev functional:

$$T(f, g) := \frac{1}{b-a}\int_a^b f(t)\, g(t)\, dt - \frac{1}{b-a}\int_a^b f(t)\, dt \cdot \frac{1}{b-a}\int_a^b g(t)\, dt.$$

In 1934, G. Grüss [**20**] showed that

$$|T(f, g)| \leq \frac{1}{4}(M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property

$$\begin{aligned} &-\infty < m \leq f \leq M < \infty, \\ &-\infty < n \leq g \leq N < \infty \quad \text{a.e. on } [a, b]. \end{aligned} \tag{1.2}$$

The quantity $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

Another less well known inequality for $T(f, g)$ was derived in 1882 by Čebyšev [**4**] under the assumption that f', g' exist and are continuous in $[a, b]$ and is given by

$$|T(f, g)| \leq \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{1.3}$$

where $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)|$.

The constant $\frac{1}{12}$ cannot be improved in the general case.

Čebyšev's inequality (1.3) also holds if $f, g : [a, b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_\infty[a, b]$.

In 1970, A.M. Ostrowski [**23**] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results

$$|T(f,g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_{\infty},$$

provided f is Lebesgue integrable on $[a,b]$ and satisfying (1.2) with $g:[a,b] \to \mathbb{R}$ being absolutely continuous and $g' \in L_{\infty}[a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

Finally, let us recall a result by Lupaş (see for example [**24**, p. 210]), which states that:

$$|T(f,g)| \leq \frac{1}{\pi^2}\|f'\|_2 \|g'\|_2 (b-a),$$

provided f,g are absolutely continuous and $f', g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible here also.

For other Grüss type integral inequalities, see the books [**22**], [**24**], and the papers [**6**] – [**15**] and [**19**], where further references are given.

In [**1**], P. Cerone has obtained the following identity that involves a Stieltjes integral (Lemma 2.1, p. 3):

Lemma 1. *Let $f,g:[a,b] \to \mathbb{R}$, where f is of bounded variation and g is continuous on $[a,b]$, then the $T(f,g)$ satisfies the identity,*

$$T(f,g) = \frac{1}{(b-a)^2}\int_a^b \Psi(t)\,df(t), \tag{1.4}$$

where

$$\Psi(t) := (t-a)A(t,b) - (b-t)A(a,t),$$

with

$$A(c,d) := \int_c^d g(x)\,dx.$$

Using this representation and the properties of Stieltjes integrals he obtained the following result in bounding the functional $T(\cdot,\cdot)$ (Theorem 2.5, p. 4):

Theorem 9. *With the assumptions in Lemma 1, we have:*

$$|T(f,g)| \leq \frac{1}{(b-a)^2}\cdot\begin{cases} \sup\limits_{t\in[a,b]}|\Psi(t)|\bigvee_a^b(f), & \\ L\int_a^b|\Psi(t)|\,dt, & \text{for } L-\text{Lipschitzian;} \\ \int_a^b|\Psi(t)|\,df(t), & \text{for } f \text{ monotonic nondecreasing,}\end{cases}$$

where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$.

Now, if we use the function $\varphi : (a,b) \to \mathbb{R}$,

$$\varphi(t) := D(g;a,t,b) = \frac{\int_t^b g(x)\,dx}{b-t} - \frac{\int_a^t g(x)\,dx}{t-a}, \tag{1.5}$$

then by (1.4) we may obtain the identity:

$$T(f,g) = \frac{1}{(b-a)^2} \int_a^b (t-a)(b-t)\,\varphi(t)\,df(t).$$

In [**3**] various upper bounds for $|T(f,g)|$ have been given, from which we would like to mention only the following ones:

THEOREM 10. *Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation and $g : [a,b] \to \mathbb{R}$ an absolutely continuous function so that φ is bounded on (a,b). Then one has the inequality:*

$$|T(f,g)| \le \frac{1}{4} \|\varphi\|_\infty \bigvee_a^b (f),$$

where φ is as given by (1.5) and

$$\|\varphi\|_\infty := \sup_{t\in(a,b)} |\varphi(t)|.$$

The case of Lipschitzian functions $f : [a,b] \to \mathbb{R}$ is embodied in the following theorem [**3**].

THEOREM 11. *Let $f : [a,b] \to \mathbb{R}$ be an $L-$Lipschitzian function on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ an absolutely continuous function on $[a,b]$. Then*

$$|T(f,g)| \le \begin{cases} L\frac{(b-a)^3}{6} \|\varphi\|_\infty & \text{if } \varphi \in L_\infty[a,b]; \\ L(b-a)^{\frac{1}{q}} \left[B(q+1,q+1)\right]^{\frac{1}{q}} \|\varphi\|_p, & p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{if } \varphi \in L_p[a,b]; \\ \frac{L}{4}\|\varphi\|_1, & \text{if } \varphi \in L_1[a,b], \end{cases}$$

where $\|\cdot\|_p$ are the usual Lebesgue $p-$norms on $[a,b]$ and $B(\cdot,\cdot)$ is Euler's Beta function.

Finally, the following result containing Stieltjes integral holds [**3**]:

THEOREM 12. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing function on $[a,b]$. If g is continuous, then one has the inequality:*

$$|T(f,g)| \leq \begin{cases} \frac{1}{4}\int_a^b |\varphi(t)|\, df(t) \\ \frac{1}{(b-a)^2}\left(\int_a^b [(b-t)(t-a)]^q\, df(t)\right)^{\frac{1}{q}} \left(\int_a^b |\varphi(t)|^p\, df(t)\right)^{\frac{1}{p}}, \\ \qquad\qquad\qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{(b-a)^2}\sup\limits_{t\in[a,b]} |\varphi(t)| \int_a^b (t-a)(b-t)\, df(t). \end{cases}$$

In [**16**], the authors have considered the following functional

$$D(f;u) := \int_a^b f(x)\, du(x) - [u(b)-u(a)] \cdot \frac{1}{b-a}\int_a^b f(t)\, dt,$$

provided that the involved integrals exist.

In the same paper, the following result in estimating the above functional has been obtained.

THEOREM 13. *Let $f,u : [a,b] \to \mathbb{R}$ be such that u is Lipschitzian on $[a,b]$, i.e.,*

$$|u(x)-u(y)| \leq L|x-y| \quad \text{for any } x,y \in [a,b] \quad (L>0)$$

and f is Riemann integrable on $[a,b]$. If $m,M \in \mathbb{R}$ are such that

$$m \leq f(x) \leq M \quad \text{for any } x,y \in [a,b],$$

then we have the inequality

$$|D(f;u)| \leq \frac{1}{2}L(M-m)(b-a).$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In [**15**], the following result complementing the above one was obtained.

THEOREM 14. *Let $f,u : [a,b] \to \mathbb{R}$ be such that $u : [a,b] \to \mathbb{R}$ is of bounded variation in $[a,b]$ and $f : [a,b] \to \mathbb{R}$ is $K-$Lipschitzian $(K>0)$. Then we have the inequality*

$$|D(f;u)| \leq \frac{1}{2}K(b-a)\bigvee_a^b(u).$$

The constant $\frac{1}{2}$ is sharp in the above sense.

The main aim of this chapter is to survey some recent inequalities of the Grüss type holding in the general setting of inner product spaces. Natural applications for Lebesgue integrals in measure spaces are presented as well.

2. Grüss' Inequality in Inner Product Spaces

2.1. Introduction. In [**6**], the author has proved the following Grüss' type inequality in real or complex inner product spaces.

THEOREM 15. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $e \in H, \|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \geq 0 \text{ and } \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0 \tag{2.1}$$

hold, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| . \tag{2.2}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

Some particular cases of interest for integrable functions with real or complex values and the corresponding discrete versions are listed below.

COROLLARY 13. *Let $f, g : [a, b] \to \mathbb{K}\,(\mathbb{K} = \mathbb{R}, \mathbb{C})$ be Lebesgue integrable and such that*

$$\operatorname{Re}\left[(\Phi - f(x))\left(\overline{f(x)} - \overline{\varphi}\right)\right] \geq 0, \quad \operatorname{Re}\left[(\Gamma - g(x))\left(\overline{g(x)} - \overline{\gamma}\right)\right] \geq 0$$

for a.e. $x \in [a, b]$, where $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and $\bar{z}$ denotes the complex conjugate of z. Then we have the inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\overline{g(x)}dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b \overline{g(x)}dx\right| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| .$$

The constant $\frac{1}{4}$ is best possible.

The discrete case is embodied in

COROLLARY 14. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{K}^n$ and $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers such that*

$$\operatorname{Re}\left[(\Phi - x_i)\left(\overline{x_i} - \overline{\varphi}\right)\right] \geq 0, \quad \operatorname{Re}\left[(\Gamma - y_i)\left(\overline{y_i} - \overline{\gamma}\right)\right] \geq 0$$

for each $i \in \{1, \ldots, n\}$. *Then we have the inequality*

$$\left| \frac{1}{n} \sum_{i=1}^{n} x_i \overline{y_i} - \frac{1}{n} \sum_{i=1}^{n} x_i \cdot \frac{1}{n} \sum_{i=1}^{n} \overline{y_i} \right| \leq \frac{1}{4} \left| \Phi - \varphi \right| \cdot \left| \Gamma - \gamma \right|.$$

The constant $\frac{1}{4}$ *is best possible.*

For other applications of Theorem 15, see the recent paper [**17**].

In the present section, by following [**13**], we show that the condition (2.1) may be replaced by an equivalent but simpler assumption and a new proof of Theorem 15 is produced. A refinement of the Grüss type inequality (2.2), some companions and applications for integrals are pointed out as well.

2.2. An Equivalent Assumption. The following lemma holds [**13**].

Lemma 2. *Let* a, x, A *be vectors in the inner product space* $(H, \langle \cdot, \cdot \rangle)$ *over* $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ *with* $a \neq A$. *Then*

$$\operatorname{Re} \langle A - x, x - a \rangle \geq 0$$

if and only if

$$\left\| x - \frac{a + A}{2} \right\| \leq \frac{1}{2} \left\| A - a \right\|.$$

Proof. Define

$$I_1 := \operatorname{Re} \langle A - x, x - a \rangle, \qquad I_2 := \frac{1}{4} \left\| A - a \right\|^2 - \left\| x - \frac{a + A}{2} \right\|^2.$$

A simple calculation shows that

$$I_1 = I_2 = \operatorname{Re} \left[\langle x, a \rangle + \langle A, x \rangle \right] - \operatorname{Re} \langle A, a \rangle - \left\| x \right\|^2$$

and thus, obviously, $I_1 \geq 0$ iff $I_2 \geq 0$ showing the required equivalence. ∎

The following corollary is obvious

Corollary 15. *Let* $x, e \in H$ *with* $\|e\| = 1$ *and* $\delta, \Delta \in \mathbb{K}$ *with* $\delta \neq \Delta$. *Then*

$$\operatorname{Re} \langle \Delta e - x, x - \delta e \rangle \geq 0$$

iff

$$\left\| x - \frac{\delta + \Delta}{2} \cdot e \right\| \leq \frac{1}{2} \left| \Delta - \delta \right|.$$

REMARK 18. *If $H=\mathbb{C}$, then*

$$\operatorname{Re}\left[(A-x)(\bar{x}-\bar{a})\right] \geq 0$$

if and only if

$$\left|x-\frac{a+A}{2}\right| \leq \frac{1}{2}|A-a|,$$

where $a,x,A\in\mathbb{C}$. If $H=\mathbb{R}$, and $A>a$ then $a\leq x\leq A$ if and only if $\left|x-\frac{a+A}{2}\right| \leq \frac{1}{2}|A-a|$.

The following lemma is of interest [**13**].

LEMMA 3. *Let $x,e\in H$ with $\|e\|=1$. Then one has the following representation*

$$\|x\|^2-|\langle x,e\rangle|^2=\inf_{\lambda\in\mathbb{K}}\|x-\lambda e\|^2 \geq 0. \tag{2.3}$$

PROOF. Observe, for any $\lambda\in\mathbb{K}$, that

$$\begin{aligned}\langle x-\lambda e, x-\langle x,e\rangle e\rangle &= \|x\|^2-|\langle x,e\rangle|^2-\lambda\left[\langle e,x\rangle-\langle e,x\rangle\|e\|^2\right]\\ &= \|x\|^2-|\langle x,e\rangle|^2.\end{aligned}$$

Using Schwarz's inequality, we have

$$\begin{aligned}\left[\|x\|^2-|\langle x,e\rangle|^2\right]^2 &= |\langle x-\lambda e, x-\langle x,e\rangle e\rangle|^2\\ &\leq \|x-\lambda e\|^2\|x-\langle x,e\rangle e\|^2\\ &= \|x-\lambda e\|^2\left[\|x\|^2-|\langle x,e\rangle|^2\right],\end{aligned}$$

giving the bound

$$\|x\|^2-|\langle x,e\rangle|^2 \leq \|x-\lambda e\|^2, \qquad \lambda\in\mathbb{K}. \tag{2.4}$$

Taking the infimum in (2.4) over $\lambda\in\mathbb{K}$, we deduce

$$\|x\|^2-|\langle x,e\rangle|^2 \leq \inf_{\lambda\in\mathbb{K}}\|x-\lambda e\|^2.$$

Since, for $\lambda_0=\langle x,e\rangle$, we get $\|x-\lambda_0 e\|^2=\|x\|^2-|\langle x,e\rangle|^2$, then the representation (2.3) is proved. ∎

We are now able to provide a different proof for the Grüss type inequality in inner product spaces (mentioned in the Introduction), than the one from the paper [**6**].

THEOREM 16. *Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $e\in H, \|e\|=1$. If $\varphi,\gamma,\Phi,\Gamma$ are real or complex numbers and x,y are vectors in H such that the conditions* (2.1) *hold, or, equivalently, the following assumptions*

$$\left\|x-\frac{\varphi+\Phi}{2}\cdot e\right\| \leq \frac{1}{2}|\Phi-\varphi|, \quad \left\|y-\frac{\gamma+\Gamma}{2}\cdot e\right\| \leq \frac{1}{2}|\Gamma-\gamma| \tag{2.5}$$

are valid, then one has the inequality

$$|\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle| \leq \frac{1}{4}|\Phi-\varphi|\cdot|\Gamma-\gamma|. \tag{2.6}$$

The constant $\frac{1}{4}$ *is best possible.*

Proof. It can be easily shown (see for example the proof of Theorem 1 from [**6**]) that

$$|\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle| \leq \left[\|x\|^2 - |\langle x,e\rangle|^2\right]^{\frac{1}{2}} \left[\|y\|^2 - |\langle y,e\rangle|^2\right]^{\frac{1}{2}}, \tag{2.7}$$

for any $x,y \in H$ and $e \in H, \|e\| = 1$. Using Lemma 3 and the conditions (2.5) we obviously have that

$$\left[\|x\|^2 - |\langle x,e\rangle|^2\right]^{\frac{1}{2}} = \inf_{\lambda\in\mathbb{K}} \|x-\lambda e\| \leq \left\|x - \frac{\varphi+\Phi}{2}\cdot e\right\| \leq \frac{1}{2}|\Phi-\varphi|$$

and

$$\left[\|y\|^2 - |\langle y,e\rangle|^2\right]^{\frac{1}{2}} = \inf_{\lambda\in\mathbb{K}} \|y-\lambda e\| \leq \left\|y - \frac{\gamma+\Gamma}{2}\cdot e\right\| \leq \frac{1}{2}|\Gamma-\gamma|$$

and by (2.7) the desired inequality (2.6) is obtained.

The fact that $\frac{1}{4}$ is the best possible constant, has been shown in [**6**] and we omit the details. ∎

2.3. A Refinement of the Grüss Inequality. The following result improving (2.1) holds [**13**].

Theorem 17. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$ *and* $e \in H, \|e\| = 1$. *If* $\varphi,\gamma,\Phi,\Gamma$ *are real or complex numbers and* x,y *are vectors in* H *such that the conditions* (2.1)*, or, equivalently,* (2.5) *hold, then we have the inequality*

$$\begin{aligned} &|\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle| \\ &\leq \frac{1}{4}|\Phi-\varphi|\cdot|\Gamma-\gamma| \\ &\qquad - \left[\operatorname{Re}\langle \Phi e - x, x-\varphi e\rangle\right]^{\frac{1}{2}} \left[\operatorname{Re}\langle \Gamma e - y, y-\gamma e\rangle\right]^{\frac{1}{2}} \\ &\leq \left(\frac{1}{4}|\Phi-\varphi|\cdot|\Gamma-\gamma|\right). \end{aligned} \tag{2.8}$$

The constant $\frac{1}{4}$ *is best possible.*

Proof. As in [**6**], we have

$$|\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle|^2 \leq \left[\|x\|^2 - |\langle x,e\rangle|^2\right] \left[\|y\|^2 - |\langle y,e\rangle|^2\right], \tag{2.9}$$

$$\|x\|^2 - |\langle x, e\rangle|^2 = \operatorname{Re}\left[(\Phi - \langle x, e\rangle)\left(\overline{\langle x, e\rangle} - \overline{\varphi}\right)\right] - \operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \tag{2.10}$$

and

$$\|y\|^2 - |\langle y, e\rangle|^2 = \operatorname{Re}\left[(\Gamma - \langle y, e\rangle)\left(\overline{\langle y, e\rangle} - \overline{\gamma}\right)\right] - \operatorname{Re}\langle \Gamma e - x, x - \gamma e\rangle . \tag{2.11}$$

Using the elementary inequality

$$4\operatorname{Re}\left(a\bar{b}\right) \leq |a+b|^2 ; \quad a, b \in \mathbb{K} \ \ (\mathbb{K} = \mathbb{R}, \mathbb{C}),$$

we may state that

$$\operatorname{Re}\left[(\Phi - \langle x, e\rangle)\left(\overline{\langle x, e\rangle} - \overline{\varphi}\right)\right] \leq \frac{1}{4}|\Phi - \varphi|^2 \tag{2.12}$$

and

$$\operatorname{Re}\left[(\Gamma - \langle y, e\rangle)\left(\overline{\langle y, e\rangle} - \overline{\gamma}\right)\right] \leq \frac{1}{4}|\Gamma - \gamma|^2 . \tag{2.13}$$

Consequently, by (2.9) – (2.13) we may state that

$$\begin{aligned} &|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle|^2 \\ &\leq \left[\frac{1}{4}|\Phi - \varphi|^2 - \left([\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle]^{\frac{1}{2}}\right)^2\right] \\ &\qquad \times \left[\frac{1}{4}|\Gamma - \gamma|^2 - \left([\operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle]^{\frac{1}{2}}\right)^2\right]. \end{aligned} \tag{2.14}$$

Finally, using the elementary inequality for positive real numbers

$$\left(m^2 - n^2\right)\left(p^2 - q^2\right) \leq (mp - nq)^2 ,$$

we have

$$\begin{aligned} &\left[\frac{1}{4}|\Phi - \varphi|^2 - \left([\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle]^{\frac{1}{2}}\right)^2\right] \\ &\qquad \times \left[\frac{1}{4}|\Gamma - \gamma|^2 - \left([\operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle]^{\frac{1}{2}}\right)^2\right] \\ &\leq \left(\frac{1}{4}|\Phi - \varphi| \cdot |\Gamma - \gamma| - [\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle]^{\frac{1}{2}} [\operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle]^{\frac{1}{2}}\right)^2 , \end{aligned}$$

giving the desired inequality (2.8).

The fact that $\frac{1}{4}$ is the best constant can be proven in a similar manner to the one in the Grüss inequality (2.2) (see for instance [**6**]) and we omit the details. ∎

2.4. Some Companion Inequalities. The following companion of the Grüss inequality in inner product spaces holds [**13**].

THEOREM 18. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$ *and* $e \in H, \|e\| = 1$. *If* $\gamma, \Gamma \in \mathbb{K}$ *and* $x, y \in H$ *are such that*

$$\operatorname{Re}\left\langle \Gamma e - \frac{x+y}{2}, \frac{x+y}{2} - \gamma e \right\rangle \geq 0, \tag{2.15}$$

or, equivalently,

$$\left\| \frac{x+y}{2} - \frac{\gamma+\Gamma}{2} \cdot e \right\| \leq \frac{1}{2} |\Gamma - \gamma|,$$

then we have the inequality

$$\operatorname{Re}\left[\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle\right] \leq \frac{1}{4} |\Gamma - \gamma|^2. \tag{2.16}$$

The constant $\frac{1}{4}$ *is best possible in the sense that it cannot be replaced by a smaller constant.*

PROOF. Start with the obvious inequality

$$\operatorname{Re}\langle z, u\rangle \leq \frac{1}{4} \|z + u\|^2; \quad z, u \in H. \tag{2.17}$$

Since

$$\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle = \langle x - \langle x, e\rangle e, y - \langle y, e\rangle e\rangle,$$

then using (2.17) we may write

$$\begin{aligned} \operatorname{Re}\left[\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle\right] &= \operatorname{Re}\left[\langle x - \langle x, e\rangle e, y - \langle y, e\rangle e\rangle\right] \\ &\leq \frac{1}{4} \|x - \langle x, e\rangle e + y - \langle y, e\rangle e\|^2 \\ &= \left\| \frac{x+y}{2} - \left\langle \frac{x+y}{2}, e \right\rangle \cdot e \right\|^2 \\ &= \left\| \frac{x+y}{2} \right\|^2 - \left| \left\langle \frac{x+y}{2}, e \right\rangle \right|^2. \end{aligned} \tag{2.18}$$

If we apply Grüss' inequality in inner product spaces for, say, $a = b = \frac{x+y}{2}$, we get

$$\left\| \frac{x+y}{2} \right\|^2 - \left| \left\langle \frac{x+y}{2}, e \right\rangle \right|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2. \tag{2.19}$$

Making use of (2.18) and (2.19) we deduce (2.16).

The fact that $\frac{1}{4}$ is the best possible constant in (2.16) follows by the fact that if in (2.15) we choose $x = y$, then it becomes

$$\operatorname{Re}\langle \Gamma e - x, x - \gamma e\rangle \geq 0,$$

implying that $0 \leq \|x\|^2 - |\langle x, e\rangle|^2 \leq \frac{1}{4}|\Gamma - \gamma|^2$, for which, by Grüss' inequality in inner product spaces, we know that the constant $\frac{1}{4}$ is best possible. ∎

The following corollary might be of interest if one wanted to evaluate the absolute value of

$$\operatorname{Re}\left[\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle\right].$$

COROLLARY 16. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $e \in H, \|e\| = 1$. If $\gamma, \Gamma \in \mathbb{K}$ and $x, y \in H$ are such that*

$$\operatorname{Re}\left\langle \Gamma e - \frac{x \pm y}{2}, \frac{x \pm y}{2} - \gamma e \right\rangle \geq 0,$$

or, equivalently,

$$\left\| \frac{x \pm y}{2} - \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2}|\Gamma - \gamma|,$$

holds, then we have the inequality

$$|\operatorname{Re}\left[\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle\right]| \leq \frac{1}{4}|\Gamma - \gamma|^2. \tag{2.20}$$

If the inner product space H is real, then (for $m, M \in \mathbb{R}$, $M > m$)

$$\left\langle Me - \frac{x \pm y}{2}, \frac{x \pm y}{2} - me \right\rangle \geq 0,$$

or, equivalently,

$$\left\| \frac{x \pm y}{2} - \frac{m + M}{2} \cdot e \right\| \leq \frac{1}{2}(M - m),$$

implies

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4}(M - m)^2. \tag{2.21}$$

In both inequalities (2.20) *and* (2.21), *the constant $\frac{1}{4}$ is best possible.*

PROOF. We only remark that, if

$$\operatorname{Re}\left\langle \Gamma e - \frac{x - y}{2}, \frac{x - y}{2} - \gamma e \right\rangle \geq 0$$

holds, then by Theorem 18, we get

$$\operatorname{Re}\left[-\langle x, y\rangle + \langle x, e\rangle \langle e, y\rangle\right] \leq \frac{1}{4}|\Gamma - \gamma|^2,$$

showing that

$$\operatorname{Re}\left[\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle\right] \geq -\frac{1}{4}|\Gamma - \gamma|^2. \tag{2.22}$$

Making use of (2.16) and (2.22) we deduce the desired result (2.20). ∎

Finally, we may state and prove the following dual result as well [**13**].

Proposition 18. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ and $e \in H, \|e\| = 1$. If $\varphi, \Phi \in \mathbb{K}$ and $x, y \in H$ are such that*

$$\operatorname{Re}\left[(\Phi - \langle x, e\rangle)\left(\overline{\langle x, e\rangle} - \overline{\varphi}\right)\right] \leq 0, \tag{2.23}$$

then we have the inequalities

$$\begin{aligned} \|x - \langle x, e\rangle e\| &\leq [\operatorname{Re}\langle x - \Phi e, x - \varphi e\rangle]^{\frac{1}{2}} \\ &\leq \frac{\sqrt{2}}{2}\left[\|x - \Phi e\|^2 + \|x - \varphi e\|^2\right]^{\frac{1}{2}}. \end{aligned} \tag{2.24}$$

Proof. We know that the following identity holds true (see (2.10))

$$\begin{aligned} &\|x\|^2 - |\langle x, e\rangle|^2 \\ &= \operatorname{Re}\left[(\Phi - \langle x, e\rangle)\left(\overline{\langle x, e\rangle} - \overline{\varphi}\right)\right] + \operatorname{Re}\langle x - \Phi e, x - \varphi e\rangle. \end{aligned} \tag{2.25}$$

Using the assumption (2.23) and the fact that

$$\|x\|^2 - |\langle x, e\rangle|^2 = \|x - \langle x, e\rangle e\|^2,$$

by (2.25), we deduce the first inequality in (2.24).

The second inequality in (2.24) follows by the fact that for any $v, w \in H$ one has

$$\operatorname{Re}\langle w, v\rangle \leq \frac{1}{2}\left(\|w\|^2 + \|v\|^2\right).$$

The proposition is thus proved. ∎

2.5. Integral Inequalities. The following proposition holds [**13**].

Proposition 19. *If $f, g, h \in L^2(\Omega, \mathbb{K})$ and $\varphi, \Phi, \gamma, \Gamma \in \mathbb{K}$, are such that $\int_\Omega |h(s)|^2 d\mu(s) = 1$ and*

$$\begin{aligned} &\int_\Omega \operatorname{Re}\left[(\Phi h(s) - f(s))\left(\overline{f(s)} - \overline{\varphi}\overline{h(s)}\right)\right] d\mu(s) \geq 0, \\ &\int_\Omega \operatorname{Re}\left[(\Gamma h(s) - g(s))\left(\overline{g(s)} - \overline{\gamma}\overline{h(s)}\right)\right] d\mu(s) \geq 0 \end{aligned} \tag{2.26}$$

or, equivalently,

$$\left(\int_{\Omega}\left|f(s)-\frac{\Phi+\varphi}{2}h(s)\right|^{2}d\mu(s)\right)^{\frac{1}{2}}\leq\frac{1}{2}\left|\Phi-\varphi\right|,$$
$$\left(\int_{\Omega}\left|g(s)-\frac{\Gamma+\gamma}{2}h(s)\right|^{2}d\mu(s)\right)^{\frac{1}{2}}\leq\frac{1}{2}\left|\Gamma-\gamma\right|,$$

hold, then we have the following refinement of the Grüss integral inequality

$$\begin{aligned}&\left|\int_{\Omega}f(s)\overline{g(s)}d\mu(s)-\int_{\Omega}f(s)\overline{h(s)}d\mu(s)\int_{\Omega}h(s)\overline{g(s)}d\mu(s)\right|\\&\leq\frac{1}{4}\left|\Phi-\varphi\right|\cdot\left|\Gamma-\gamma\right|-\left[\int_{\Omega}\operatorname{Re}\left[\left(\Phi h(s)-f(s)\right)\left(\overline{f(s)}-\overline{\varphi}\overline{h(s)}\right)\right]d\mu(s)\right.\\&\qquad\left.\times\int_{\Omega}\operatorname{Re}\left[\left(\Gamma h(s)-g(s)\right)\left(\overline{g(s)}-\overline{\gamma}\overline{h(s)}\right)\right]d\mu(s)\right]^{\frac{1}{2}}.\end{aligned}$$

The constant $\frac{1}{4}$ is best possible.

The proof follows by Theorem 17 on choosing $H=L^{2}(\Omega,\mathbb{K})$ with the inner product

$$\langle f,g\rangle:=\int_{\Omega}f(s)\overline{g(s)}d\mu(s).$$

We omit the details.

Remark 19. *It is obvious that a sufficient condition for* (2.26) *to hold is*

$$\operatorname{Re}\left[\left(\Phi h(s)-f(s)\right)\left(\overline{f(s)}-\overline{\varphi}\overline{h(s)}\right)\right]\geq 0,$$

and

$$\operatorname{Re}\left[\left(\Gamma h(s)-g(s)\right)\left(\overline{g(s)}-\overline{\gamma}\overline{h(s)}\right)\right]\geq 0,$$

for $\mu-$a.e. $s\in\Omega$, or, equivalently,

$$\left|f(s)-\frac{\Phi+\varphi}{2}h(s)\right|\leq\frac{1}{2}\left|\Phi-\varphi\right|\left|h(s)\right|\quad\text{and}$$
$$\left|g(s)-\frac{\Gamma+\gamma}{2}h(s)\right|\leq\frac{1}{2}\left|\Gamma-\gamma\right|\left|h(s)\right|,$$

for $\mu-$a.e. $s\in\Omega$.

The following result may be stated as well [**13**].

Corollary 17. *If $z, Z, t, T \in \mathbb{K}$, $\mu(\Omega) < \infty$ and $f, g \in L^2(\Omega, \mathbb{K})$ are such that:*

$$\operatorname{Re}\left[(Z - f(s))\left(\overline{f(s)} - \bar{z}\right)\right] \geq 0,$$
$$\operatorname{Re}\left[(T - g(s))\left(\overline{g(s)} - \bar{t}\right)\right] \geq 0,$$

for a.e. $s \in \Omega$, or, equivalently,

$$\left|f(s) - \frac{z+Z}{2}\right| \leq \frac{1}{2}|Z - z|,$$
$$\left|g(s) - \frac{t+T}{2}\right| \leq \frac{1}{2}|T - t|,$$

for a.e. $s \in \Omega$, then we have the inequality

$$\begin{aligned} &\left|\frac{1}{\mu(\Omega)}\int_\Omega f(s)\overline{g(s)}d\mu(s)\right. \\ &\qquad\left. - \frac{1}{\mu(\Omega)}\int_\Omega f(s)\,d\mu(s) \cdot \frac{1}{\mu(\Omega)}\int_\Omega \overline{g(s)}d\mu(s)\right| \\ &\leq \frac{1}{4}|Z - z||T - t| - \frac{1}{\mu(\Omega)}\left[\int_\Omega \operatorname{Re}\left[(Z - f(s))\left(\overline{f(s)} - \bar{z}\right)\right]d\mu(s)\right. \\ &\qquad\left. \times \int_\Omega \operatorname{Re}\left[(T - g(s))\left(\overline{g(s)} - \bar{t}\right)\right]d\mu(s)\right]^{\frac{1}{2}}. \end{aligned} \tag{2.27}$$

Using Theorem 18 we may state the following result as well [**13**].

Proposition 20. *If $f, g, h \in L^2(\Omega, \mathbb{K})$ and $\gamma, \Gamma \in \mathbb{K}$ are such that $\int_\Omega |h(s)|^2 d\mu(s) = 1$ and*

$$\begin{aligned} \int_\Omega \operatorname{Re}\left\{\left[\Gamma h(s) - \frac{f(s) + g(s)}{2}\right]\right. \\ \left. \times \left[\frac{\overline{f(s)} + \overline{g(s)}}{2} - \bar{\gamma}\bar{h}(s)\right]\right\} d\mu(s) \geq 0 \end{aligned} \tag{2.28}$$

or, equivalently,

$$\left(\int_\Omega \left|\frac{f(s) + g(s)}{2} - \frac{\gamma + \Gamma}{2}h(s)\right|^2 d\mu(s)\right)^{\frac{1}{2}} \leq \frac{1}{2}|\Gamma - \gamma|, \tag{2.29}$$

holds, then we have the inequality

$$\begin{aligned} I &:= \int_\Omega \operatorname{Re}\left[f(s)\overline{g(s)}\right] d\mu(s) \\ &\qquad - \operatorname{Re}\left[\int_\Omega f(s)\overline{h(s)}d\mu(s)\cdot\int_\Omega h(s)\overline{g(s)}d\mu(s)\right] \\ &\leq \frac{1}{4}\left|\Gamma-\gamma\right|^2. \end{aligned}$$

If (2.28) and (2.29) hold with " $\pm$ " instead of " $+$ ", then

$$|I| \leq \frac{1}{4}\left|\Gamma-\gamma\right|^2.$$

Remark 20. *It is obvious that a sufficient condition for (2.28) to hold is*

$$\operatorname{Re}\left\{\left[\Gamma h(s)-\frac{f(s)+g(s)}{2}\right]\cdot\left[\frac{\overline{f(s)}+\overline{g(s)}}{2}-\bar{\gamma}\bar{h}(s)\right]\right\}\geq 0,$$

for a.e. $s\in\Omega$, or, equivalently,

$$\left|\frac{f(s)+g(s)}{2}-\frac{\gamma+\Gamma}{2}h(s)\right|\leq\frac{1}{2}\left|\Gamma-\gamma\right|\left|h(s)\right|,$$

for a.e. $s\in\Omega$.

Finally, the following corollary holds.

Corollary 18. *If $Z,z\in\mathbb{K}$, $\mu(\Omega)<\infty$ and $f,g\in L^2(\Omega,\mathbb{K})$ are such that*

$$\operatorname{Re}\left[\left(Z-\frac{f(s)+g(s)}{2}\right)\left(\frac{\overline{f(s)}+\overline{g(s)}}{2}-\overline{z}\right)\right]\geq 0, \tag{2.30}$$

for a.e. $s\in\Omega$, or, equivalently,

$$\left|\frac{f(s)+g(s)}{2}-\frac{z+Z}{2}\right|\leq\frac{1}{2}\left|Z-z\right|, \tag{2.31}$$

for a.e. $s\in\Omega$, then we have the inequality

$$\begin{aligned} J &:= \frac{1}{\mu(\Omega)}\int_\Omega \operatorname{Re}\left[f(s)\overline{g(s)}\right] d\mu(s) \\ &\qquad - \operatorname{Re}\left[\frac{1}{\mu(\Omega)}\int_\Omega f(s)\,d\mu(s)\cdot\frac{1}{\mu(\Omega)}\int_\Omega \overline{g(s)}d\mu(s)\right] \\ &\leq \frac{1}{4}\left|Z-z\right|^2. \end{aligned}$$

If (2.30) and (2.31) hold with " $\pm$ " instead of " $+$ ", then

$$|J| \leq \frac{1}{4} |Z - z|^2 .$$

REMARK 21. *It is obvious that if one chooses the discrete measure above, then all the inequalities in this section may be written for sequences of real or complex numbers. We omit the details.*

3. Companions of Grüss' Inequality

3.1. A General Result. The following Grüss type inequality in inner product spaces holds [**12**].

THEOREM 19. *Let $x, y, e \in H$ with $\|e\| = 1$, and the scalars $a, A, b, B \in \mathbb{K}$ such that $\operatorname{Re}(\bar{a}A) > 0$ and $\operatorname{Re}\left(\bar{b}B\right) > 0$. If*

$$\operatorname{Re} \langle Ae - x, x - ae \rangle \geq 0 \quad \text{and} \quad \operatorname{Re} \langle Be - y, y - be \rangle \geq 0$$

or, equivalently,

$$\left\| x - \frac{a + A}{2} e \right\| \leq \frac{1}{2} |A - a| \quad \text{and} \quad \left\| y - \frac{b + B}{2} e \right\| \leq \frac{1}{2} |B - b| ,$$

holds, then we have the inequality

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} M(a, A) M(b, B) |\langle x, e \rangle \langle e, y \rangle| , \tag{3.1}$$

where $M(\cdot, \cdot)$ is defined by

$$M(a, A) := \left[\frac{(|A| - |a|)^2 + 4 [|A\bar{a}| - \operatorname{Re}(A\bar{a})]}{\operatorname{Re}(\bar{a}A)} \right]^{\frac{1}{2}} .$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

PROOF. Start with the inequality

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle|^2 \leq \left(\|x\|^2 - |\langle x, e \rangle|^2 \right) \left(\|y\|^2 - |\langle y, e \rangle|^2 \right) . \tag{3.2}$$

Now, assume that $u, v \in H$, and $c, C \in \mathbb{K}$ with $\operatorname{Re}(\bar{c}C) > 0$ and $\operatorname{Re} \langle Cv - u, u - cv \rangle \geq 0$. This last inequality is equivalent to

$$\|u\|^2 + \operatorname{Re}(\bar{c}C) \|v\|^2 \leq \operatorname{Re} \left[C\overline{\langle u, v \rangle} + \bar{c} \langle u, v \rangle \right] .$$

Dividing this inequality by $[\operatorname{Re}(C\bar{c})]^{\frac{1}{2}} > 0$, we deduce

$$\frac{1}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}} \|u\|^2 + [\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}} \|v\|^2 \leq \frac{\operatorname{Re} \left[C\overline{\langle u, v \rangle} + \bar{c} \langle u, v \rangle \right]}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}} . \tag{3.3}$$

On the other hand, by the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha} q^2 \geq 2pq, \quad \alpha > 0,\ p, q \geq 0,$$

we deduce

$$2\|u\|\|v\| \leq \frac{1}{[\mathrm{Re}(\bar{c}C)]^{\frac{1}{2}}}\|u\|^2 + [\mathrm{Re}(\bar{c}C)]^{\frac{1}{2}}\|v\|^2. \tag{3.4}$$

Making use of (3.3) and (3.4) and the fact that for any $z \in \mathbb{C}$, $\mathrm{Re}(z) \leq |z|$, we get

$$\|u\|\|v\| \leq \frac{\mathrm{Re}\left[C\overline{\langle u, v\rangle} + \bar{c}\langle u, v\rangle\right]}{2[\mathrm{Re}(\bar{c}C)]^{\frac{1}{2}}} \leq \frac{|c| + |C|}{2[\mathrm{Re}(\bar{c}C)]^{\frac{1}{2}}}|\langle u, v\rangle|.$$

Consequently

$$\begin{aligned}
\|u\|^2\|v\|^2 - |\langle u, v\rangle|^2 & \\
&\leq \left[\frac{(|c| + |C|)^2}{4[\mathrm{Re}(\bar{c}C)]} - 1\right]|\langle u, v\rangle|^2 \\
&= \frac{1}{4}\frac{(|c| - |C|)^2 + 4[|\bar{c}C| - \mathrm{Re}(\bar{c}C)]}{\mathrm{Re}(\bar{c}C)}|\langle u, v\rangle|^2 \\
&= \frac{1}{4}M^2(c, C)|\langle u, v\rangle|^2.
\end{aligned} \tag{3.5}$$

Now, if we write (3.5) for the choices $u = x$, $v = e$ and $u = y$, $v = e$ respectively and use (3.2), we deduce the desired result (3.1). The sharpness of the constant will be proved in the case where H is a real inner product space. ∎

The following corollary which provides a simpler Grüss type inequality for real constants (and in particular, for real inner product spaces) holds [**12**].

COROLLARY 19. *With the assumptions of Theorem 19 and if* $a, b, A, B \in \mathbb{R}$ *are such that* $A > a > 0$, $B > b > 0$ *and*

$$\left\|x - \frac{a + A}{2}e\right\| \leq \frac{1}{2}(A - a) \quad \text{and} \quad \left\|y - \frac{b + B}{2}e\right\| \leq \frac{1}{2}(B - b), \tag{3.6}$$

then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{4} \cdot \frac{(A - a)(B - b)}{\sqrt{abAB}}|\langle x, e\rangle\langle e, y\rangle|. \tag{3.7}$$

The constant $\frac{1}{4}$ *is best possible.*

Proof. To prove the sharpness of the constant $\frac{1}{4}$ assume that the inequality (3.7) holds in real inner product spaces with $x = y$ and for a constant $k > 0$, i.e.,

$$\|x\|^2 - |\langle x, e\rangle|^2 \leq k \cdot \frac{(A-a)^2}{aA} |\langle x, e\rangle|^2 \quad (A > a > 0), \tag{3.8}$$

provided that $\left\|x - \frac{a+A}{2}e\right\| \leq \frac{1}{2}(A-a)$, or, equivalently,

$$\langle Ae - x, x - ae\rangle \geq 0.$$

We choose $H = \mathbb{R}^2$, $x = (x_1, x_2) \in \mathbb{R}^2$, $e = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Then we have

$$\|x\|^2 - |\langle x, e\rangle|^2 = x_1^2 + x_2^2 - \frac{(x_1 + x_2)^2}{2} = \frac{(x_1 - x_2)^2}{2},$$
$$|\langle x, e\rangle|^2 = \frac{(x_1 + x_2)^2}{2},$$

and by (3.8) we get

$$\frac{(x_1 - x_2)^2}{2} \leq k \cdot \frac{(A-a)^2}{aA} \cdot \frac{(x_1 + x_2)^2}{2}. \tag{3.9}$$

Now, if we let $x_1 = \frac{a}{\sqrt{2}}$, $x_2 = \frac{A}{\sqrt{2}}$ $(A > a > 0)$, then obviously

$$\langle Ae - x, x - ae\rangle = \sum_{i=1}^{2} \left(\frac{A}{\sqrt{2}} - x_i\right)\left(x_i - \frac{a}{\sqrt{2}}\right) = 0,$$

which shows that the condition (3.6) is fulfilled, and by (3.9) we get

$$\frac{(A-a)^2}{4} \leq k \cdot \frac{(A-a)^2}{aA} \cdot \frac{(a+A)^2}{4}$$

for any $A > a > 0$. This implies

$$(A+a)^2 k \geq aA \tag{3.10}$$

for any $A > a > 0$.

Finally, let $a = 1 - q$, $A = 1 + q$, $q \in (0, 1)$. Then from (3.10) we get $4k \geq 1 - q^2$ for any $q \in (0,1)$ which produces $k \geq \frac{1}{4}$. ∎

Remark 22. *If $\langle x, e\rangle, \langle y, e\rangle$ are assumed not to be zero, then the inequality (3.1) is equivalent to*

$$\left|\frac{\langle x, y\rangle}{\langle x, e\rangle\langle e, y\rangle} - 1\right| \leq \frac{1}{4} M(a, A) M(b, B),$$

while the inequality (3.7) is equivalent to

$$\left|\frac{\langle x, y\rangle}{\langle x, e\rangle\langle e, y\rangle} - 1\right| \leq \frac{1}{4} \cdot \frac{(A-a)(B-b)}{\sqrt{abAB}}.$$

The constant $\frac{1}{4}$ *is best possible in both inequalities.*

3.2. Some Related Results. The following result holds [**12**].

THEOREM 20. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$. *If* $\gamma,\Gamma\in\mathbb{K}$, $e,x,y\in H$ *with* $\|e\|=1$ *and* $\lambda\in(0,1)$ *are such that*

$$\operatorname{Re}\left\langle \Gamma e-(\lambda x+(1-\lambda)\,y),(\lambda x+(1-\lambda)\,y)-\gamma e\right\rangle \geq 0, \tag{3.11}$$

or, equivalently,

$$\left\|\lambda x+(1-\lambda)\,y-\frac{\gamma+\Gamma}{2}e\right\|\leq\frac{1}{2}\left|\Gamma-\gamma\right|,$$

then we have the inequality

$$\operatorname{Re}\left[\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right]\leq\frac{1}{16}\cdot\frac{1}{\lambda\,(1-\lambda)}\left|\Gamma-\gamma\right|^{2}. \tag{3.12}$$

The constant $\frac{1}{16}$ *is the best possible constant in (3.12) in the sense that it cannot be replaced by a smaller one.*

PROOF. We know that for any $z,u\in H$ one has

$$\operatorname{Re}\langle z,u\rangle\leq\frac{1}{4}\left\|z+u\right\|^{2}.$$

Then for any $a,b\in H$ and $\lambda\in(0,1)$ one has

$$\operatorname{Re}\langle a,b\rangle\leq\frac{1}{4\lambda\,(1-\lambda)}\left\|\lambda a+(1-\lambda)\,b\right\|^{2}. \tag{3.13}$$

Since

$$\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle=\langle x-\langle x,e\rangle e,y-\langle y,e\rangle e\rangle\qquad(\text{as }\ \|e\|=1),$$

using (3.13), we have

$$\begin{aligned}
&\operatorname{Re}\left[\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right]\\
&=\operatorname{Re}\left[\langle x-\langle x,e\rangle e,y-\langle y,e\rangle e\rangle\right]\\
&\leq\frac{1}{4\lambda\,(1-\lambda)}\left\|\lambda\,(x-\langle x,e\rangle e)+(1-\lambda)\,(y-\langle y,e\rangle e)\right\|^{2}\\
&=\frac{1}{4\lambda\,(1-\lambda)}\left\|\lambda x+(1-\lambda)\,y-\langle\lambda x+(1-\lambda)\,y,e\rangle e\right\|^{2}.
\end{aligned} \tag{3.14}$$

Since, for $m,e\in H$ with $\|e\|=1$, one has the equality

$$\left\|m-\langle m,e\rangle e\right\|^{2}=\left\|m\right\|^{2}-\left|\langle m,e\rangle\right|^{2},$$

then, by (3.14), we deduce the inequality

$$\begin{aligned}(3.15)\quad & \operatorname{Re}\left[\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right] \\ & \leq \frac{1}{4\lambda(1-\lambda)}\left[\|\lambda x+(1-\lambda)y\|^2-|\langle\lambda x+(1-\lambda)y,e\rangle|^2\right].\end{aligned}$$

Now, if we apply Grüss' inequality

$$0\leq\|a\|^2-|\langle a,e\rangle|^2\leq\frac{1}{4}|\Gamma-\gamma|^2,$$

provided

$$\operatorname{Re}\langle\Gamma e-a,a-\gamma e\rangle\geq 0,$$

for $a=\lambda x+(1-\lambda)y$, we have

$$(3.16)\qquad \|\lambda x+(1-\lambda)y\|^2-|\langle\lambda x+(1-\lambda)y,e\rangle|^2\leq\frac{1}{4}|\Gamma-\gamma|^2.$$

Utilising (3.15) and (3.16) we deduce the desired inequality (3.12).

To prove the sharpness of the constant $\frac{1}{16}$, assume that (3.12) holds with a constant $C>0$, provided that (3.11) is valid, i.e.,

$$(3.17)\qquad \operatorname{Re}\left[\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right]\leq C\cdot\frac{1}{\lambda(1-\lambda)}|\Gamma-\gamma|^2.$$

If in (3.17) we choose $x=y$, given that (3.11) holds with $x=y$ and $\lambda\in(0,1)$, then

$$(3.18)\qquad \|x\|^2-|\langle x,e\rangle|^2\leq C\cdot\frac{1}{\lambda(1-\lambda)}|\Gamma-\gamma|^2,$$

provided

$$\operatorname{Re}\langle\Gamma e-x,x-\gamma e\rangle\geq 0.$$

Since we know, in Grüss' inequality, that the constant $\frac{1}{4}$ is best possible, then by (3.18), one has

$$\frac{1}{4}\leq\frac{C}{\lambda(1-\lambda)}\quad\text{for}\quad\lambda\in(0,1),$$

giving, for $\lambda=\frac{1}{2}$, $C\geq\frac{1}{16}$.

The theorem is completely proved. ∎

The following corollary is a natural consequence of the above result [**12**].

COROLLARY 20. *Assume that* γ,Γ,e,x,y *and* λ *are as in Theorem 20. If*

$$\operatorname{Re}\langle\Gamma e-(\lambda x\pm(1-\lambda)y),(\lambda x\pm(1-\lambda)y)-\gamma e\rangle\geq 0,$$

or, equivalently,

$$\left\|\lambda x \pm (1-\lambda)\, y - \frac{\gamma+\Gamma}{2} e\right\| \leq \frac{1}{2} \left|\Gamma - \gamma\right|^2,$$

then we have the inequality

$$\left|\operatorname{Re}\left[\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle\right]\right| \leq \frac{1}{16} \cdot \frac{1}{\lambda\,(1-\lambda)} \left|\Gamma - \gamma\right|^2. \tag{3.19}$$

The constant $\frac{1}{16}$ is best possible in (3.19).

Proof. Using Theorem 20 for $(-y)$ instead of y, we have that

$$\operatorname{Re}\left\langle \Gamma e - (\lambda x - (1-\lambda)\, y), (\lambda x - (1-\lambda)\, y) - \gamma e\right\rangle \geq 0,$$

which implies that

$$\operatorname{Re}\left[-\langle x, y\rangle + \langle x, e\rangle \langle e, y\rangle\right] \leq \frac{1}{16} \cdot \frac{1}{\lambda\,(1-\lambda)} \left|\Gamma - \gamma\right|^2$$

giving

$$\operatorname{Re}\left[\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle\right] \geq -\frac{1}{16} \cdot \frac{1}{\lambda\,(1-\lambda)} \left|\Gamma - \gamma\right|^2. \tag{3.20}$$

Consequently, by (3.12) and (3.20) we deduce the desired inequality (3.19). ∎

Remark 23. *If $M, m \in \mathbb{R}$ with $M > m$ and, for $\lambda \in (0,1)$,*

$$\left\|\lambda x + (1-\lambda)\, y - \frac{M+m}{2} e\right\| \leq \frac{1}{2} (M - m), \tag{3.21}$$

then

$$\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle \leq \frac{1}{16} \cdot \frac{1}{\lambda\,(1-\lambda)} (M - m)^2.$$

If (3.21) holds with "$\pm$" instead of "$+$", then

$$\left|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle\right| \leq \frac{1}{16} \cdot \frac{1}{\lambda\,(1-\lambda)} (M - m)^2.$$

Remark 24. *If $\lambda = \frac{1}{2}$ in (3.11), then we obtain the result from* [**13**], *i.e.,*

$$\operatorname{Re}\left\langle \Gamma e - \frac{x+y}{2}, \frac{x+y}{2} - \gamma e\right\rangle \geq 0,$$

or, equivalently,

$$\left\|\frac{x+y}{2} - \frac{\gamma+\Gamma}{2} e\right\| \leq \frac{1}{2} \left|\Gamma - \gamma\right|,$$

implies

$$\operatorname{Re}\left[\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right]\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}. \tag{3.22}$$

The constant $\frac{1}{4}$ is best possible in (3.22).

For $\lambda=\frac{1}{2}$, Corollary 20 and Remark 23 will produce the corresponding results obtained in [**13**]. We omit the details.

3.3. Integral Inequalities. The following proposition holds [**12**].

Proposition 21. *If $f,g,h\in L^{2}(\Omega,\mathbb{K})$ and $\varphi,\Phi,\gamma,\Gamma\in\mathbb{K}$, are such that* $\operatorname{Re}(\Phi\overline{\varphi})>0,\operatorname{Re}(\Gamma\overline{\gamma})>0$, $\int_{\Omega}|h(s)|^{2}d\mu(s)=1$ *and*

$$\begin{aligned}\int_{\Omega}\operatorname{Re}\left[\left(\Phi h(s)-f(s)\right)\left(\overline{f(s)}-\overline{\varphi}\overline{h(s)}\right)\right]d\mu(s)&\geq 0,\\ \int_{\Omega}\operatorname{Re}\left[\left(\Gamma h(s)-g(s)\right)\left(\overline{g(s)}-\overline{\gamma}\overline{h(s)}\right)\right]d\mu(s)&\geq 0,\end{aligned} \tag{3.23}$$

or, equivalently,

$$\begin{aligned}\left(\int_{\Omega}\left|f(s)-\frac{\Phi+\varphi}{2}h(s)\right|^{2}d\mu(s)\right)^{\frac{1}{2}}&\leq\frac{1}{2}\left|\Phi-\varphi\right|,\\ \left(\int_{\Omega}\left|g(s)-\frac{\Gamma+\gamma}{2}h(s)\right|^{2}d\mu(s)\right)^{\frac{1}{2}}&\leq\frac{1}{2}\left|\Gamma-\gamma\right|,\end{aligned}$$

then we have the following Grüss type integral inequality

$$\begin{aligned}&\left|\int_{\Omega}f(s)\overline{g(s)}d\mu(s)-\int_{\Omega}f(s)\overline{h(s)}d\mu(s)\int_{\Omega}h(s)\overline{g(s)}d\mu(s)\right|\\ &\quad\leq\frac{1}{4}M(\varphi,\Phi)M(\gamma,\Gamma)\left|\int_{\Omega}f(s)\overline{h(s)}d\mu(s)\int_{\Omega}h(s)\overline{g(s)}d\mu(s)\right|,\end{aligned} \tag{3.24}$$

where

$$M(\varphi,\Phi):=\left[\frac{\left(|\Phi|-|\varphi|\right)^{2}+4\left[|\Phi\overline{\varphi}|-\operatorname{Re}(\Phi\overline{\varphi})\right]}{\operatorname{Re}(\Phi\overline{\varphi})}\right]^{\frac{1}{2}}.$$

The constant $\frac{1}{4}$ is best possible.

The proof follows by Theorem 20 on choosing $H=L^{2}(\Omega,\mathbb{K})$ with the inner product

$$\langle f,g\rangle:=\int_{\Omega}f(s)\overline{g(s)}d\mu(s).$$

We omit the details.

Remark 25. *It is obvious that a sufficient condition for* (3.23) *to hold is*

$$\operatorname{Re}\left[\left(\Phi h\left(s\right)-f\left(s\right)\right)\left(\overline{f\left(s\right)}-\overline{\varphi}\overline{h\left(s\right)}\right)\right]\geq 0,$$

and

$$\operatorname{Re}\left[\left(\Gamma h\left(s\right)-g\left(s\right)\right)\left(\overline{g\left(s\right)}-\overline{\gamma}\overline{h\left(s\right)}\right)\right]\geq 0,$$

for $\mu-$a.e. $s\in\Omega$, or, equivalently,

$$\left|f\left(s\right)-\frac{\Phi+\varphi}{2}h\left(s\right)\right|\leq\frac{1}{2}\left|\Phi-\varphi\right|\left|h\left(s\right)\right|$$

and

$$\left|g\left(s\right)-\frac{\Gamma+\gamma}{2}h\left(s\right)\right|\leq\frac{1}{2}\left|\Gamma-\gamma\right|\left|h\left(s\right)\right|,$$

for $\mu-$a.e. $s\in\Omega$.

The following result may be stated as well.

Corollary 21. *If $z,Z,t,T\in\mathbb{K}$, $\mu\left(\Omega\right)<\infty$ and $f,g\in L^{2}\left(\Omega,\mathbb{K}\right)$ are such that:*

$$\operatorname{Re}\left[\left(Z-f\left(s\right)\right)\left(\overline{f\left(s\right)}-\bar{z}\right)\right]\geq 0,$$
$$\operatorname{Re}\left[\left(T-g\left(s\right)\right)\left(\overline{g\left(s\right)}-\bar{t}\right)\right]\geq 0,$$

for a.e. $s\in\Omega$, or, equivalently,

$$\left|f\left(s\right)-\frac{z+Z}{2}\right|\leq\frac{1}{2}\left|Z-z\right|,$$
$$\left|g\left(s\right)-\frac{t+T}{2}\right|\leq\frac{1}{2}\left|T-t\right|,$$

for a.e. $s\in\Omega$, then we have the inequality

$$\begin{aligned}(3.25)\quad &\left|\frac{1}{\mu\left(\Omega\right)}\int_{\Omega}f\left(s\right)\overline{g\left(s\right)}d\mu\left(s\right)\right.\\ &\qquad\left.-\frac{1}{\mu\left(\Omega\right)}\int_{\Omega}f\left(s\right)d\mu\left(s\right)\cdot\frac{1}{\mu\left(\Omega\right)}\int_{\Omega}\overline{g\left(s\right)}d\mu\left(s\right)\right|\\ &\leq\frac{1}{4}M\left(z,Z\right)M\left(t,T\right)\left|\frac{1}{\mu\left(\Omega\right)}\int_{\Omega}f\left(s\right)d\mu\left(s\right)\cdot\frac{1}{\mu\left(\Omega\right)}\int_{\Omega}\overline{g\left(s\right)}d\mu\left(s\right)\right|.\end{aligned}$$

Remark 26. *The case of real functions incorporates the following interesting inequality*

$$\left|\frac{\mu\left(\Omega\right)\int_{\Omega}f\left(s\right)g\left(s\right)d\mu\left(s\right)}{\int_{\Omega}f\left(s\right)d\mu\left(s\right)\int_{\Omega}g\left(s\right)d\mu\left(s\right)}-1\right|\leq\frac{1}{4}\cdot\frac{\left(Z-z\right)\left(T-t\right)}{\sqrt{ztZT}},$$

provided $\mu(\Omega) < \infty$,

$$z \leq f(s) \leq Z, \quad t \leq g(s) \leq T$$

for $\mu-$*a.e.* $s \in \Omega$, *where* z, t, Z, T *are real numbers and the integrals at the denominator are not zero. Here the constant* $\frac{1}{4}$ *is best possible in the sense mentioned above.*

Using Theorem 20 we may state the following result as well [**12**].

PROPOSITION 22. *If* $f, g, h \in L^2(\Omega, \mathbb{K})$ *and* $\gamma, \Gamma \in \mathbb{K}$ *are such that* $\int_\Omega |h(s)|^2 d\mu(s) = 1$ *and*

$$\int_\Omega \{\operatorname{Re}[\Gamma h(s) - (\lambda f(s) + (1-\lambda) g(s))] \times \left[\lambda\overline{f(s)} + (1-\lambda)\overline{g(s)} - \bar{\gamma}\bar{h}(s)\right]\} d\mu(s) \geq 0, \tag{3.26}$$

or, equivalently,

$$\left(\int_\Omega \left|\lambda f(s) + (1-\lambda) g(s) - \frac{\gamma+\Gamma}{2} h(s)\right|^2 d\mu(s)\right)^{\frac{1}{2}} \leq \frac{1}{2}|\Gamma - \gamma|, \tag{3.27}$$

then we have the inequality

$$\begin{aligned} I &:= \int_\Omega \operatorname{Re}\left[f(s)\overline{g(s)}\right] d\mu(s) \\ &\quad - \operatorname{Re}\left[\int_\Omega f(s)\overline{h(s)} d\mu(s) \cdot \int_\Omega h(s)\overline{g(s)} d\mu(s)\right] \\ &\leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |\Gamma - \gamma|^2. \end{aligned}$$

The constant $\frac{1}{16}$ *is best possible.*

If (3.26) and (3.27) hold with " $\pm$ *" instead of "* $+$ *" (see Corollary 20), then*

$$|I| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |\Gamma - \gamma|^2.$$

REMARK 27. *It is obvious that a sufficient condition for (3.26) to hold is*

$$\operatorname{Re}\left\{ [\Gamma h(s) - (\lambda f(s) + (1-\lambda) g(s))] \times \left[\lambda\overline{f(s)} + (1-\lambda)\overline{g(s)} - \bar{\gamma}\bar{h}(s)\right]\right\} \geq 0$$

for a.e. $s \in \Omega$, *or, equivalently,*

$$\left| \lambda f(s) + (1-\lambda) g(s) - \frac{\gamma + \Gamma}{2} h(s) \right| \leq \frac{1}{2} |\Gamma - \gamma| \, |h(s)|,$$

for a.e. $s \in \Omega$.

Finally, the following corollary holds.

COROLLARY 22. *If* $Z, z \in \mathbb{K}$, $\mu(\Omega) < \infty$ *and* $f, g \in L^2(\Omega, \mathbb{K})$ *are such that*

$$\operatorname{Re}\Big[(Z - (\lambda f(s) + (1-\lambda) g(s))) \times \left(\lambda \overline{f(s)} + (1-\lambda) \overline{g(s)} - \overline{z} \right) \Big] \geq 0 \tag{3.28}$$

for a.e. $s \in \Omega$, *or, equivalently*

$$\left| \lambda f(s) + (1-\lambda) g(s) - \frac{z + Z}{2} \right| \leq \frac{1}{2} |Z - z|, \tag{3.29}$$

for a.e. $s \in \Omega$, *then we have the inequality*

$$\begin{aligned} J &:= \frac{1}{\mu(\Omega)} \int_\Omega \operatorname{Re}\left[f(s) \overline{g(s)} \right] d\mu(s) \\ &\qquad - \operatorname{Re}\left[\frac{1}{\mu(\Omega)} \int_\Omega f(s)\, d\mu(s) \cdot \frac{1}{\mu(\Omega)} \int_\Omega \overline{g(s)} d\mu(s) \right] \\ &\leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |Z - z|^2. \end{aligned}$$

If (3.28) and (3.29) hold with " $\pm$ " instead of " $+$ ", then

$$|J| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |Z - z|^2.$$

REMARK 28. *It is obvious that if one chooses the discrete measure above, then all the inequalities in this section may be written for sequences of real or complex numbers. We omit the details.*

4. Other Grüss Type Inequalities

4.1. General Results. We may state the following result [**10**].

THEOREM 21. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$ *and* $x, y, e \in H$ *with* $\|e\| = 1$. *If* $r_1, r_2 \in (0, 1)$ *and*

$$\|x - e\| \leq r_1, \qquad \|y - e\| \leq r_2,$$

then we have the inequality

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq r_1 r_2 \|x\| \|y\|. \tag{4.1}$$

The inequality (4.1) is sharp in the sense that the constant $c = 1$ in front of $r_1 r_2$ cannot be replaced by a smaller quantity.

PROOF. Start with the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle|^2 \leq \left(\|x\|^2 - |\langle x, e\rangle|^2\right)\left(\|y\|^2 - |\langle y, e\rangle|^2\right). \tag{4.2}$$

Using Theorem 5 for $a = e$, we may state that

$$\|x\|^2 - |\langle x, e\rangle|^2 \leq r_1^2 \|x\|^2, \qquad \|y\|^2 - |\langle y, e\rangle|^2 \leq r_2^2 \|y\|^2. \tag{4.3}$$

Utilizing (4.2) and (4.3), we deduce

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle|^2 \leq r_1^2 r_2^2 \|x\|^2 \|y\|^2,$$

which is clearly equivalent to the desired inequality (4.1).

The sharpness of the constant follows by the fact that for $x = y$, $r_1 = r_2 = r$, we get from (4.1) that

$$\|x\|^2 - |\langle x, e\rangle|^2 \leq r^2 \|x\|^2, \tag{4.4}$$

provided $\|e\| = 1$ and $\|x - e\| \leq r < 1$. The inequality (4.4) is sharp, as shown in Theorem 5, and the proof is completed. ∎

Another companion of the Grüss inequality may be stated as well [**10**].

THEOREM 22. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $x, y, e \in H$ with $\|e\| = 1$. Suppose also that $a, A, b, B \in \mathbb{K}$ such that $\operatorname{Re}(A\overline{a}), \operatorname{Re}(B\overline{b}) > 0$. If either*

$$\operatorname{Re}\langle Ae - x, x - ae\rangle \geq 0, \quad \operatorname{Re}\langle Be - y, y - be\rangle \geq 0,$$

or, equivalently,

$$\left\|x - \frac{a + A}{2} e\right\| \leq \frac{1}{2}|A - a|, \quad \left\|y - \frac{b + B}{2} e\right\| \leq \frac{1}{2}|B - b|,$$

holds, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4} \cdot \frac{|A - a|\,|B - b|}{\sqrt{\operatorname{Re}(A\overline{a})\operatorname{Re}(B\overline{b})}} |\langle x, e\rangle \langle e, y\rangle|. \tag{4.5}$$

The constant $\frac{1}{4}$ is best possible.

PROOF. We know, that

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle|^2 \leq \left(\|x\|^2 - |\langle x, e\rangle|^2\right)\left(\|y\|^2 - |\langle y, e\rangle|^2\right). \tag{4.6}$$

If we use Corollary 11, then we may state that

$$\|x\|^2 - |\langle x, e\rangle|^2 \leq \frac{1}{4} \cdot \frac{|A - a|^2}{\operatorname{Re}(A\overline{a})} |\langle x, e\rangle|^2 \tag{4.7}$$

and

$$\|y\|^2 - |\langle y, e\rangle|^2 \le \frac{1}{4} \cdot \frac{|B-b|^2}{\operatorname{Re}\left(B\bar{b}\right)} |\langle y, e\rangle|^2 . \tag{4.8}$$

Utilizing (4.6) – (4.8), we deduce

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle|^2 \le \frac{1}{16} \cdot \frac{|A-a|^2 |B-b|^2}{\operatorname{Re}(A\bar{a}) \operatorname{Re}\left(B\bar{b}\right)} |\langle x, e\rangle \langle e, y\rangle|^2 ,$$

which is clearly equivalent to the desired inequality (4.5).

The sharpness of the constant follows from Corollary 11, and we omit the details. ∎

Remark 29. *With the assumptions of Theorem 22 and if $\langle x, e\rangle, \langle y, e\rangle \ne 0$ (that is actually the interesting case), then one has the inequality*

$$\left|\frac{\langle x, y\rangle}{\langle x, e\rangle \langle e, y\rangle} - 1\right| \le \frac{1}{4} \cdot \frac{|A-a| |B-b|}{\sqrt{\operatorname{Re}(A\bar{a}) \operatorname{Re}\left(B\bar{b}\right)}}.$$

The constant $\frac{1}{4}$ is best possible.

We may state the following result [**9**].

Theorem 23. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $x, y, e \in H$ with $\|e\| = 1$. If $r_1, r_2 > 0$ and*

$$\|x - e\| \le r_1, \qquad \|y - e\| \le r_2,$$

then we have the inequalities

$$\begin{aligned} |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| &\le \frac{1}{2} r_1 r_2 \sqrt{\|x\| + |\langle x, e\rangle|} \cdot \sqrt{\|y\| + |\langle y, e\rangle|} \\ &\le r_1 r_2 \|x\| \|y\| . \end{aligned} \tag{4.9}$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Start with the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle|^2 \le \left(\|x\|^2 - |\langle x, e\rangle|^2\right)\left(\|y\|^2 - |\langle y, e\rangle|^2\right) . \tag{4.10}$$

Using Theorem 7 for $a = e$, we have

$$\begin{aligned} 0 &\le \|x\|^2 - |\langle x, e\rangle|^2 \\ &= (\|x\| - |\langle x, e\rangle|)(\|x\| + |\langle x, e\rangle|) \\ &\le \frac{1}{2} r_1^2 (\|x\| + |\langle x, e\rangle|) \le r_1^2 \|x\| , \end{aligned} \tag{4.11}$$

and, in a similar way

$$0 \le \|y\|^2 - |\langle y, e\rangle|^2 \le \frac{1}{2} r_2^2 (\|y\| + |\langle y, e\rangle|) \le r_2^2 \|y\| . \tag{4.12}$$

Utilising (4.10) – (4.12), we may state that

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle|^2 \le \frac{1}{4} r_1^2 r_2^2 (\|x\| + |\langle x, e\rangle|) (\|y\| + |\langle y, e\rangle|) \le r_1^2 r_2^2 \|x\| \|y\| , \tag{4.13}$$

giving the desired inequality (4.9).

To prove the sharpness of the constant $\frac{1}{2}$, let us assume that $x = y$ in (4.9), to get

$$\|x\|^2 - |\langle x, e\rangle|^2 \le \frac{1}{2} r_1^2 (\|x\| + |\langle x, e\rangle|) , \tag{4.14}$$

provided $\|x - e\| \le r_1$. If $x \ne 0$, then dividing (4.14) with $\|x\| + |\langle x, e\rangle| > 0$ we get

$$\|x\| - |\langle x, e\rangle| \le \frac{1}{2} r_1^2 \tag{4.15}$$

provided $\|x - e\| \le r_1$, $\|e\| = 1$. However, (4.15) is in fact (5.2) for $a = e$, for which we have shown that $\frac{1}{2}$ is the best possible constant. ∎

The following result also holds [**9**].

Theorem 24. *With the assumptions of Theorem 23, we have the inequality*

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \le r_1 r_2 \sqrt{\frac{1}{4} r_1^2 + |\langle x, e\rangle|} \cdot \sqrt{\frac{1}{4} r_2^2 + |\langle y, e\rangle|}. \tag{4.16}$$

Proof. Note that, from Theorem 8, we have

$$\|x\| \|a\| \le |\langle x, a\rangle| + \frac{1}{2} r^2, \tag{4.17}$$

provided $\|x - a\| \le r$.

Taking the square of (4.17) and re-arranging the terms, we obtain:

$$0 \le \|x\|^2 \|a\|^2 - |\langle x, a\rangle|^2 \le r^2 \left(\frac{1}{4} r^2 + |\langle x, a\rangle| \right) ,$$

provided $\|x - a\| \le r$.

Using the assumption of the theorem, we then have

$$0 \le \|x\|^2 - |\langle x, e\rangle|^2 \le r_1^2 \left(\frac{1}{4} r_1^2 + |\langle x, e\rangle| \right) , \tag{4.18}$$

and

$$0 \leq \|y\|^2 - |\langle y, e\rangle|^2 \leq r_2^2 \left(\frac{1}{4} r_2^2 + |\langle y, e\rangle| \right). \tag{4.19}$$

Utilising (4.10), (4.18) and (4.19), we deduce the desired inequality (4.16). ∎

The following result may be stated as well [**9**].

THEOREM 25. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ and $x, y, e \in H$ with $\|e\| = 1$. Suppose also that $a, A, b, B \in \mathbb{K}$ such that $A \neq -a, B \neq -b$. If either*

$$\operatorname{Re} \langle Ae - x, x - ae\rangle \geq 0, \quad \operatorname{Re} \langle Be - y, y - be\rangle \geq 0,$$

or, equivalently,

$$\left\| x - \frac{a + A}{2} e \right\| \leq \frac{1}{2} |A - a|, \quad \left\| y - \frac{b + B}{2} e \right\| \leq \frac{1}{2} |B - b|,$$

holds, then we have the inequality

$$\begin{aligned} & |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \\ & \leq \frac{1}{4} \cdot \frac{|A - a| \, |B - b|}{\sqrt{|A + a| \, |B + b|}} \sqrt{\|x\| + |\langle x, e\rangle|} \cdot \sqrt{\|y\| + |\langle y, e\rangle|} \\ & \leq \frac{1}{2} \cdot \frac{|A - a| \, |B - b|}{\sqrt{|A + a| \, |B + b|}} \sqrt{\|x\| \, \|y\|}. \end{aligned} \tag{4.20}$$

The constant $\frac{1}{4}$ is best possible in (4.20).

PROOF. From Theorem 8, we may state that

$$\begin{aligned} 0 & \leq \|x\|^2 - |\langle x, e\rangle|^2 \\ & = (\|x\| - |\langle x, e\rangle|)(\|x\| + |\langle x, e\rangle|) \\ & \leq \frac{1}{4} \cdot \frac{|A - a|^2}{|A + a|} (\|x\| + |\langle x, e\rangle|), \end{aligned} \tag{4.21}$$

and

$$0 \leq \|y\|^2 - |\langle y, e\rangle|^2 \leq \frac{1}{4} \cdot \frac{|B - b|^2}{|B + b|} (\|y\| + |\langle y, e\rangle|). \tag{4.22}$$

Making use of (4.10) and (4.21), (4.22), we deduce the first inequality in (4.20).

The best constant follows by the use of Theorem 8, and we omit the details. ∎

Finally, we may state the following theorem as well [**9**].

Theorem 26. *With the assumptions of Theorem 25, we have the inequality*

$$(4.23)\quad \begin{aligned}&\left|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right|\\&\leq\frac{1}{2}\cdot\frac{|A-a|\,|B-b|}{\sqrt{|A+a|\,|B+b|}}\sqrt{\frac{1}{8}\cdot\frac{|A-a|^2}{|A+a|}+|\langle x,e\rangle|}\\&\qquad\times\sqrt{\frac{1}{8}\cdot\frac{|B-b|^2}{|B+b|}+|\langle y,e\rangle|}.\end{aligned}$$

Proof. Using Theorem 8, we may state that

$$0\leq\|x\|-|\langle x,e\rangle|\leq\frac{1}{4}\cdot\frac{|A-a|^2}{|A+a|}.$$

This inequality implies that

$$\|x\|^2\leq|\langle x,e\rangle|^2+\frac{1}{2}|\langle x,e\rangle|\cdot\frac{|A-a|^2}{|A+a|}+\frac{1}{16}\cdot\frac{|A-a|^4}{|A+a|^2},$$

giving

$$(4.24)\quad 0\leq\|x\|^2-|\langle x,e\rangle|^2\leq\frac{1}{2}\cdot\frac{|A-a|^2}{|A+a|}\left[|\langle x,e\rangle|+\frac{1}{8}\cdot\frac{|A-a|^2}{|A+a|}\right].$$

Similarly, we have

$$(4.25)\quad 0\leq\|y\|^2-|\langle y,e\rangle|^2\leq\frac{1}{2}\cdot\frac{|B-b|^2}{|B+b|}\left[|\langle y,e\rangle|+\frac{1}{8}\cdot\frac{|B-b|^2}{|B+b|}\right].$$

By making use of (4.10) and (4.24), (4.25), we deduce the desired inequality (4.23). ∎

4.2. Integral Inequalities. The following Grüss type integral inequality for real or complex-valued functions also holds [**10**].

Proposition 23. *Let* $f,g,h\in L^2_\rho(\Omega,\mathbb{K})$ *with* $\int_\Omega\rho(s)|h(s)|^2d\mu(s)=1$ *and* $a,A,b,B\in\mathbb{K}$ *such that* $\operatorname{Re}(A\overline{a}),\operatorname{Re}(B\overline{b})>0$ *and*

$$\begin{aligned}\operatorname{Re}\left[(Ah(s)-f(s))\left(\overline{f(s)}-\overline{a}\overline{h(s)}\right)\right]&\geq 0,\\ \operatorname{Re}\left[(Bh(s)-g(s))\left(\overline{g(s)}-\overline{b}\overline{h(s)}\right)\right]&\geq 0,\end{aligned}$$

for $\mu-$a.e. $s \in \Omega$. Then we have the inequalities

$$\left|\int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) - \int_{\Omega} \rho(s) f(s) \overline{h(s)} d\mu(s) \int_{\Omega} \rho(s) h(s) \overline{g(s)} d\mu(s)\right| \leq \frac{1}{4} \cdot \frac{|A-a||B-b|}{\sqrt{\operatorname{Re}(A\overline{a}) \operatorname{Re}\left(B\overline{b}\right)}} \times \left|\int_{\Omega} \rho(s) f(s) \overline{h(s)} d\mu(s) \int_{\Omega} \rho(s) h(s) \overline{g(s)} d\mu(s)\right|.$$

The constant $\frac{1}{4}$ is best possible.

The proof follows by Theorem 22.

By making use of Theorem 25, we may also state:

Proposition 24. *Let $f, g, h \in L_{\rho}^{2}(\Omega, \mathbb{K})$ be such that*

$$\int_{\Omega} \rho(s) |h(s)|^2 d\mu(s) = 1.$$

Suppose also that $a, A, b, B \in \mathbb{K}$ with $A \neq -a, B \neq -b$ and

$$\begin{aligned} \operatorname{Re}\left[(Ah(s) - f(s))\left(\overline{f(s)} - \overline{a}\overline{h(s)}\right)\right] &\geq 0, \\ \operatorname{Re}\left[(Bh(s) - g(s))\left(\overline{g(s)} - \overline{b}\overline{h(s)}\right)\right] &\geq 0, \end{aligned}$$

for $\mu-$a.e. $s \in \Omega$. Then we have the inequality

$$\left|\int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) - \int_{\Omega} \rho(s) f(s) \overline{h(s)} d\mu(s) \int_{\Omega} \rho(s) h(s) \overline{g(s)} d\mu(s)\right| \leq \frac{1}{4} \cdot \frac{|A-a||B-b|}{\sqrt{|A+a||B+b|}} \times \sqrt{\left(\int_{\Omega} \rho(s) |f(s)|^2 d\mu(s)\right)^{\frac{1}{2}} + \left|\int_{\Omega} \rho(s) f(s) \overline{h(s)} d\mu(s)\right|} \times \sqrt{\left(\int_{\Omega} \rho(s) |g(s)|^2 d\mu(s)\right)^{\frac{1}{2}} + \left|\int_{\Omega} \rho(s) g(s) \overline{h(s)} d\mu(s)\right|}.$$

The constant $\frac{1}{4}$ is best possible.

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CHAPTER 3

Reverses of Bessel's Inequality

1. Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) and $\{e_i\}_{i \in I}$ a finite or infinite family of *orthonormal vectors* in H, i.e.,

$$\langle e_i, e_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ \\ 1 & \text{if } i = j \end{cases} \quad ; \quad i, j \in I.$$

It is well known that, the following inequality due to Bessel, holds

$$\sum_{i \in I} |\langle x, e_i \rangle|^2 \leq \|x\|^2 ,$$

for any $x \in H$, where the meaning of the sum is:

$$\sum_{i \in I} |\langle x, e_i \rangle|^2 := \sup_{F \subset I} \left\{ \sum_{i \in F} |\langle x, e_i \rangle|^2 , \quad F \text{ is a finite part of } I \right\}.$$

If $(H, \langle \cdot, \cdot \rangle)$ is an infinite dimensional Hilbert space and $\{e_i\}_{i \in \mathbb{N}}$ an orthonormal family in H, then we also have

$$\sum_{i=0}^{\infty} |\langle x, e_i \rangle|^2 \leq \|x\|^2$$

for any $x \in H$. Here the meaning of the series is the usual one.

In this chapter we establish reverses of the Bessel inequality and some Grüss type inequalities for orthonormal families, namely, upper bounds for the expressions

$$\|x\|^2 - \sum_{i \in I} |\langle x, e_i \rangle|^2 , \quad \|x\| - \left(\sum_{i \in I} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} , \quad x \in X$$

and

$$\left| \langle x, y \rangle - \sum_{i \in I} \langle x, e_i \rangle \langle e_i, y \rangle \right| , \quad x, y \in H,$$

under various assumptions for the vectors x, y and the orthonormal family $\{e_i\}_{i \in I}$.

2. Reverses of Bessel's Inequality

2.1. Introduction. In [**3**], the author has proved the following Grüss type inequality in real or complex inner product spaces.

Theorem 27. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $e \in H$, $\|e\| = 1$. If $\phi, \Phi, \gamma, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re}\langle \Phi e - x, x - \phi e\rangle \geq 0 \quad \textit{and} \quad \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0 \tag{2.1}$$

hold, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4}|\Phi - \phi|\,|\Gamma - \gamma|\,. \tag{2.2}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

In [**8**], the following refinement of (2.2) has been pointed out.

Theorem 28. *Let $H, \mathbb{K}$ and e be as in Theorem 27. If $\phi, \Phi, \gamma, \Gamma, x, y$ satisfy (2.1) or, equivalently,*

$$\left\| x - \frac{\phi + \Phi}{2} e \right\| \leq \frac{1}{2}|\Phi - \phi|\,, \quad \left\| y - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2}|\Gamma - \gamma|\,,$$

then

$$\begin{aligned} |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| &\leq \frac{1}{4}|\Phi - \phi|\,|\Gamma - \gamma| \\ &\quad - [\operatorname{Re}\langle \Phi e - x, x - \phi e\rangle]^{\frac{1}{2}} [\operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle]^{\frac{1}{2}}\,. \end{aligned} \tag{2.3}$$

In [**12**], N. Ujević has generalised Theorem 27 for the case of real inner product spaces as follows.

Theorem 29. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real number field $\mathbb{R}$, and $\{e_i\}_{i\in\{1,\dots,n\}}$ an orthornormal family in H. If $\phi_i, \gamma_i, \Phi_i, \Gamma_i \in \mathbb{R}$, $i \in \{1, \dots, n\}$ satisfy the condition*

$$\left\langle \sum_{i=1}^{n} \Phi_i e_i - x, x - \sum_{i=1}^{n} \phi_i e_i \right\rangle \geq 0, \qquad \left\langle \sum_{i=1}^{n} \Gamma_i e_i - y, y - \sum_{i=1}^{n} \gamma_i e_i \right\rangle \geq 0,$$

then one has the inequality:

$$\begin{aligned} &\left| \langle x, y\rangle - \sum_{i=1}^{n} \langle x, e_i\rangle \langle e_i, y\rangle \right| \\ &\qquad\qquad \leq \frac{1}{4}\left[\sum_{i=1}^{n} (\Phi_i - \phi_i)^2 \cdot \sum_{i=1}^{n} (\Gamma_i - \gamma_i)^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{2.4}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

We note that the key point in his proof is the following identity:

$$\sum_{i=1}^{n}\left(\langle x,e_i\rangle-\phi_i\right)\left(\Phi_i-\langle x,e_i\rangle\right)-\left\langle x-\sum_{i=1}^{n}\phi_i e_i,\sum_{i=1}^{n}\Phi_i e_i-x\right\rangle=\|x\|^2-\sum_{i=1}^{n}\langle x,e_i\rangle^2,$$

holding for $x\in H$, $\phi_i,\Phi_i\in\mathbb{R}$, $i\in\{1,\dots,n\}$ and $\{e_i\}_{i\in\{1,\dots,n\}}$ an orthornormal family of vectors in the real inner product space H.

In this section, by following [**2**], we point out a reverse of Bessel's inequality in both real and complex inner product spaces. This result will then be employed to provide a refinement of the Grüss type inequality (2.4) for real or complex inner products. Related results as well as integral inequalities for general measure spaces are also given.

2.2. A General Result. The following lemma holds [**2**].

LEMMA 4. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, F a finite part of I and ϕ_i,Φ_i $(i\in F)$, real or complex numbers. The following statements are equivalent for $x\in H$:*

(i) $\operatorname{Re}\left\langle\sum_{i\in F}\Phi_i e_i-x,x-\sum_{i\in F}\phi_i e_i\right\rangle\geq 0$,

(ii) $\left\|x-\sum_{i\in F}\frac{\phi_i+\Phi_i}{2}e_i\right\|\leq\frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}$.

PROOF. It is easy to see that for $y,a,A\in H$, the following are equivalent (see [**8**, Lemma 1])

(b) $\operatorname{Re}\langle A-y,y-a\rangle\geq 0$ and

(bb) $\left\|y-\frac{a+A}{2}\right\|\leq\frac{1}{2}\|A-a\|$.

Now, for $a=\sum_{i\in F}\phi_i e_i$, $A=\sum_{i\in F}\Phi_i e_i$, we have

$$\|A-a\|=\left\|\sum_{i\in F}\left(\Phi_i-\phi_i\right)e_i\right\|=\left(\left\|\sum_{i\in F}\left(\Phi_i-\phi_i\right)e_i\right\|^2\right)^{\frac{1}{2}}=\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\|e_i\|^2\right)^{\frac{1}{2}}=\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}},$$

giving, for $y=x$, the desired equivalence. ∎

The following reverse of Bessel's inequality holds [**2**].

THEOREM 30. *Let $\{e_i\}_{i\in I}$, F, ϕ_i, Φ_i, $i \in F$ and $x \in H$ such that either (i) or (ii) of Lemma 4 holds. Then we have the inequality:*

$$
\begin{aligned}
(2.5)\qquad 0 &\leq \|x\|^2 - \sum_{i\in F} |\langle x, e_i\rangle|^2 \\
&\leq \frac{1}{4}\sum_{i\in F} |\Phi_i - \phi_i|^2 - \operatorname{Re}\left\langle \sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F}\phi_i e_i \right\rangle \\
&\leq \frac{1}{4}\sum_{i\in F} |\Phi_i - \phi_i|^2 .
\end{aligned}
$$

The constant $\frac{1}{4}$ is best in both inequalities.

PROOF. Define

$$I_1 := \sum_{i\in H} \operatorname{Re}\left[(\Phi_i - \langle x, e_i\rangle)\left(\overline{\langle x, e_i\rangle} - \overline{\phi_i}\right)\right]$$

and

$$I_2 := \operatorname{Re}\left[\left\langle \sum_{i\in H}\Phi_i e_i - x, x - \sum_{i\in H}\phi_i e_i\right\rangle\right].$$

Observe that

$$
\begin{aligned}
I_1 \quad = \quad & \sum_{i\in H}\operatorname{Re}\left[\Phi_i\overline{\langle x, e_i\rangle}\right] + \sum_{i\in H}\operatorname{Re}\left[\overline{\phi_i}\langle x, e_i\rangle\right] \\
& - \sum_{i\in H}\operatorname{Re}\left[\Phi_i\overline{\phi_i}\right] - \sum_{i\in H}|\langle x, e_i\rangle|^2
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \operatorname{Re}\left[\sum_{i\in H}\Phi_i\overline{\langle x, e_i\rangle} + \sum_{i\in H}\overline{\phi_i}\langle x, e_i\rangle - \|x\|^2 - \sum_{i\in H}\sum_{j\in H}\Phi_i\overline{\phi_i}\langle e_i, e_j\rangle\right] \\
&= \sum_{i\in H}\operatorname{Re}\left[\Phi_i\overline{\langle x, e_i\rangle}\right] + \sum_{i\in H}\operatorname{Re}\left[\overline{\phi_i}\langle x, e_i\rangle\right] - \|x\|^2 - \sum_{i\in H}\operatorname{Re}\left[\Phi_i\overline{\phi_i}\right].
\end{aligned}
$$

Consequently, subtracting I_2 from I_1, we deduce the following equality that is interesting in its turn

$$
\begin{aligned}
(2.6)\qquad \|x\|^2 - \sum_{i\in F}|\langle x, e_i\rangle|^2 = & \sum_{i\in H}\operatorname{Re}\left[(\Phi_i - \langle x, e_i\rangle)\left(\overline{\langle x, e_i\rangle} - \overline{\phi_i}\right)\right] \\
& - \operatorname{Re}\left[\left\langle\sum_{i\in H}\Phi_i e_i - x, x - \sum_{i\in H}\phi_i e_i\right\rangle\right].
\end{aligned}
$$

Using the following elementary inequality for complex numbers

$$\operatorname{Re}\left[a\bar{b}\right] \leq \frac{1}{4}|a+b|^2, \qquad a, b \in \mathbb{K},$$

for the choices $a = \Phi_i - \langle x, e_i \rangle$, $b = \langle x, e_i \rangle - \phi_i$ $(i \in F)$, we deduce

$$\sum_{i \in H} \operatorname{Re}\left[\left(\Phi_i - \langle x, e_i \rangle\right)\left(\overline{\langle x, e_i \rangle} - \overline{\phi_i}\right)\right] \leq \frac{1}{4}\sum_{i \in H}|\Phi_i - \phi_i|^2. \tag{2.7}$$

Making use of (2.6), (2.7) and the assumption (i), we deduce (2.5).

The sharpness of the constant $\frac{1}{4}$ was proved for a single element e, $\|e\| = 1$ in [**3**], or for the real case in [**12**].

We can give here a simple proof as follows.

Assume that there is a $c > 0$ such that

$$\begin{aligned} 0 &\leq \|x\|^2 - \sum_{i \in F}|\langle x, e_i \rangle|^2 \\ &\leq c\sum_{i \in F}|\Phi_i - \phi_i|^2 - \operatorname{Re}\left\langle \sum_{i \in F}\Phi_i e_i - x, x - \sum_{i \in F}\phi_i e_i \right\rangle, \end{aligned} \tag{2.8}$$

provided ϕ_i, Φ_i, x and F satisfy (i) or (ii).

We choose $F = \{1\}$, $e_1 = e_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \in \mathbb{R}^2$, $x = (x_1, x_2) \in \mathbb{R}^2$, $\Phi_1 = \Phi = m > 0$, $\phi_1 = \phi = -m$, $H = \mathbb{R}^2$ to get from (2.8) that

$$\begin{aligned} 0 &\leq x_1^2 + x_2^2 - \frac{(x_1 + x_2)^2}{2} \\ &\leq 4cm^2 - \left(\frac{m}{\sqrt{2}} - x_1\right)\left(x_1 + \frac{m}{\sqrt{2}}\right) \\ &\qquad - \left(\frac{m}{\sqrt{2}} - x_2\right)\left(x_2 + \frac{m}{\sqrt{2}}\right), \end{aligned} \tag{2.9}$$

provided

$$\begin{aligned} 0 &\leq \langle me - x, x + me \rangle \\ &= \left(\frac{m}{\sqrt{2}} - x_1\right)\left(x_1 + \frac{m}{\sqrt{2}}\right) + \left(\frac{m}{\sqrt{2}} - x_2\right)\left(x_2 + \frac{m}{\sqrt{2}}\right). \end{aligned} \tag{2.10}$$

If we choose $x_1 = \frac{m}{\sqrt{2}}$, $x_2 = -\frac{m}{\sqrt{2}}$, then (2.10) is fulfilled and by (2.9) we get $m^2 \leq 4cm^2$, giving $c \geq \frac{1}{4}$. ∎

2.3. A Refinement of the Grüss Inequality for Orthonormal Families. The following result holds [**2**].

THEOREM 31. *Let* $\{e_i\}_{i\in I}$ *be a family of orthornormal vectors in* H, F *a finite part of* I *and* $\phi_i, \Phi_i, \gamma_i, \Gamma_i \in \mathbb{K}$, $i \in F$ *and* $x, y \in H$. *If either*

$$\operatorname{Re}\left\langle \sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \phi_i e_i \right\rangle \geq 0,$$
$$\operatorname{Re}\left\langle \sum_{i\in F} \Gamma_i e_i - y, y - \sum_{i\in F} \gamma_i e_i \right\rangle \geq 0,$$

or, equivalently,

$$\left\| x - \sum_{i\in F} \frac{\Phi_i + \phi_i}{2} e_i \right\| \leq \frac{1}{2} \left(\sum_{i\in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}},$$
$$\left\| y - \sum_{i\in F} \frac{\Gamma_i + \gamma_i}{2} e_i \right\| \leq \frac{1}{2} \left(\sum_{i\in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}},$$

hold, then we have the inequalities

$$\begin{aligned}
(2.11)\qquad & \left| \langle x, y\rangle - \sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle \right| \\
&\leq \frac{1}{4} \left(\sum_{i\in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i\in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \\
&\quad - \left[\operatorname{Re}\left\langle \sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \phi_i e_i \right\rangle \right]^{\frac{1}{2}} \\
&\qquad \times \left[\operatorname{Re}\left\langle \sum_{i\in F} \Gamma_i e_i - y, y - \sum_{i\in F} \gamma_i e_i \right\rangle \right]^{\frac{1}{2}} \\
&\leq \frac{1}{4} \left(\sum_{i\in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i\in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

The constant $\frac{1}{4}$ *is best possible.*

Proof. Using Schwarz's inequality in the inner product space $(H,\langle\cdot,\cdot\rangle)$ one has

$$\left|\left\langle x-\sum_{i\in F}\langle x,e_i\rangle e_i, y-\sum_{i\in F}\langle y,e_i\rangle e_i\right\rangle\right|^2 \leq \left\|x-\sum_{i\in F}\langle x,e_i\rangle e_i\right\|^2\left\|y-\sum_{i\in F}\langle y,e_i\rangle e_i\right\|^2 \tag{2.12}$$

and since a simple calculation shows that

$$\left\langle x-\sum_{i\in F}\langle x,e_i\rangle e_i, y-\sum_{i\in F}\langle y,e_i\rangle e_i\right\rangle = \langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle$$

and

$$\left\|x-\sum_{i\in F}\langle x,e_i\rangle e_i\right\|^2=\|x\|^2-\sum_{i\in F}|\langle x,e_i\rangle|^2,$$

for any $x,y\in H$, then by (2.12) and by the reverse of Bessel's inequality in Theorem 30, we have

$$\begin{aligned}
&\left|\langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right|^2 \\
&\leq\left(\|x\|^2-\sum_{i\in F}|\langle x,e_i\rangle|^2\right)\left(\|y\|^2-\sum_{i\in F}|\langle y,e_i\rangle|^2\right)\\
&\leq\left[\frac14\sum_{i\in F}|\Phi_i-\phi_i|^2-\operatorname{Re}\left\langle\sum_{i\in F}\Phi_ie_i-x,x-\sum_{i\in F}\phi_ie_i\right\rangle\right]\\
&\quad\times\left[\frac14\sum_{i\in F}|\Gamma_i-\gamma_i|^2-\operatorname{Re}\left\langle\sum_{i\in F}\Gamma_ie_i-y,y-\sum_{i\in F}\gamma_ie_i\right\rangle\right]\\
&\leq\left[\frac14\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac12}\cdot\left(\sum_{i\in F}|\Gamma_i-\gamma_i|^2\right)^{\frac12}\right.\\
&\quad-\left[\operatorname{Re}\left\langle\sum_{i\in F}\Phi_ie_i-x,x-\sum_{i\in F}\phi_ie_i\right\rangle\right]^{\frac12}\\
&\quad\left.\times\left[\operatorname{Re}\left\langle\sum_{i\in F}\Gamma_ie_i-y,y-\sum_{i\in F}\gamma_ie_i\right\rangle\right]^{\frac12}\right]
\end{aligned}\tag{2.13}$$

where, for the last inequality, we have made use of the inequality

$$\left(m^2-n^2\right)\left(p^2-q^2\right) \le (mp-nq)^2,$$

holding for any $m,n,p,q>0$.

Taking the square root in (2.13) and observing that the quantity in the last square bracket is nonnegative (see for example (2.5)), we deduce the desired result (2.11).

The best constant has been proved in [**3**] for one element and we omit the details. ∎

2.4. Some Companion Inequalities. The following companion of the Grüss inequality also holds [**2**].

Theorem 32. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, F a finite part of I and $\phi_i, \Phi_i \in \mathbb{K}$, $i\in F$ and $x,y\in H$ such that*

$$\operatorname{Re}\left\langle \sum_{i\in F}\Phi_i e_i - \frac{x+y}{2}, \frac{x+y}{2} - \sum_{i\in F}\phi_i e_i\right\rangle \ge 0, \tag{2.14}$$

or, equivalently,

$$\left\|\frac{x+y}{2} - \sum_{i\in F}\frac{\Phi_i+\phi_i}{2}\cdot e_i\right\| \le \frac{1}{2}\left(\sum_{i\in F}\left|\Phi_i-\phi_i\right|^2\right)^{\frac{1}{2}},$$

holds, then we have the inequality

$$\operatorname{Re}\left[\langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right] \le \frac{1}{4}\sum_{i\in F}\left|\Phi_i-\phi_i\right|^2. \tag{2.15}$$

The constant $\frac{1}{4}$ is best possible.

Proof. Start with the well known inequality

$$\operatorname{Re}\langle z,u\rangle \le \frac{1}{4}\|z+u\|^2, \quad z,u\in H. \tag{2.16}$$

Since

$$\langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle = \left\langle x-\sum_{i\in F}\langle x,e_i\rangle e_i, y-\sum_{i\in F}\langle y,e_i\rangle e_i\right\rangle,$$

for any $x, y \in H$, then, by (2.16), we get

$$\begin{aligned}
&\operatorname{Re}\left[\langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right] \\
&= \operatorname{Re}\left[\left\langle x - \sum_{i\in F}\langle x,e_i\rangle e_i, y - \sum_{i\in F}\langle y,e_i\rangle e_i\right\rangle\right] \\
&\le \frac{1}{4}\left\| x - \sum_{i\in F}\langle x,e_i\rangle e_i + y - \sum_{i\in F}\langle y,e_i\rangle e_i\right\|^2 \\
&= \left\| \frac{x+y}{2} - \sum_{i\in F}\left\langle \frac{x+y}{2}, e_i\right\rangle e_i\right\|^2 \\
&= \left\|\frac{x+y}{2}\right\|^2 - \sum_{i\in F}\left|\left\langle\frac{x+y}{2}, e_i\right\rangle\right|^2 .
\end{aligned} \tag{2.17}$$

If we apply the reverse of Bessel's inequality in Theorem 30 for $\frac{x+y}{2}$, we may state that

$$\left\|\frac{x+y}{2}\right\|^2 - \sum_{i\in F}\left|\left\langle\frac{x+y}{2}, e_i\right\rangle\right|^2 \le \frac{1}{4}\left(\sum_{i\in F}|\Phi_i - \phi_i|^2\right)^{\frac{1}{2}} . \tag{2.18}$$

Now, by making use of (2.17) and (2.18), we deduce (2.15).

The fact that $\frac{1}{4}$ is the best constant in (2.15) follows by the fact that if in (2.14) we choose $x = y$, then it becomes (i) of Lemma 4, implying (2.5), for which, we have shown that $\frac{1}{4}$ was the best constant. ∎

The following corollary may be of interest if we wish to evaluate the absolute value of

$$\operatorname{Re}\left[\langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right].$$

Corollary 23. *With the assumptions of Theorem 32 and if*

$$\operatorname{Re}\left\langle \sum_{i\in F}\Phi_i e_i - \frac{x\pm y}{2}, \frac{x\pm y}{2} - \sum_{i\in F}\phi_i e_i\right\rangle \ge 0,$$

or, equivalently,

$$\left\|\frac{x\pm y}{2} - \sum_{i\in F}\frac{\Phi_i+\phi_i}{2}\cdot e_i\right\| \le \frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}},$$

holds, then we have the inequality

$$\left|\operatorname{Re}\left[\langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right]\right|\leq\frac{1}{4}\sum_{i\in F}\left|\Phi_i-\phi_i\right|^2. \tag{2.19}$$

Proof. We only remark that, if

$$\operatorname{Re}\left\langle\sum_{i\in F}\Phi_ie_i-\frac{x-y}{2},\frac{x-y}{2}-\sum_{i\in F}\phi_ie_i\right\rangle\geq 0$$

holds, then by Theorem 32 for $(-y)$ instead of y, we have

$$\operatorname{Re}\left[-\langle x,y\rangle+\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right]\leq\frac{1}{4}\sum_{i\in F}\left|\Phi_i-\phi_i\right|^2,$$

showing that

$$\operatorname{Re}\left[\langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right]\geq-\frac{1}{4}\sum_{i\in F}\left|\Phi_i-\phi_i\right|^2. \tag{2.20}$$

Making use of (2.15) and (2.20), we deduce the desired inequality (2.19). ∎

Remark 30. *If H is a real inner product space and $m_i,M_i\in\mathbb{R}$ with the property that*

$$\left\langle\sum_{i\in F}M_ie_i-\frac{x\pm y}{2},\frac{x\pm y}{2}-\sum_{i\in F}m_ie_i\right\rangle\geq 0,$$

or, equivalently,

$$\left\|\frac{x\pm y}{2}-\sum_{i\in F}\frac{M_i+m_i}{2}\cdot e_i\right\|\leq\frac{1}{2}\left(\sum_{i\in F}\left(M_i-m_i\right)^2\right)^{\frac{1}{2}},$$

then we have the Grüss type inequality

$$\left|\langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right|\leq\frac{1}{4}\sum_{i\in F}\left(M_i-m_i\right)^2.$$

2.5. Integral Inequalities. Let (Ω,Σ,μ) be a measure space consisting of a set Ω, Σ a $\sigma-$algebra of parts and μ a countably additive and positive measure on Σ with values in $\mathbb{R}\cup\{\infty\}$. Let $\rho\geq 0$ be a $\mu-$measurable function on Ω. Denote by $L_\rho^2(\Omega,\mathbb{K})$ the Hilbert space of all real or complex valued functions defined on Ω and $2-\rho-$integrable on Ω, i.e.,

$$\int_\Omega\rho(s)\left|f(s)\right|^2d\mu(s)<\infty.$$

Consider the family $\{f_i\}_{i\in I}$ of functions in $L^2_\rho(\Omega,\mathbb{K})$ with the properties that

$$\int_\Omega \rho(s) f_i(s)\overline{f_j}(s)\,d\mu(s) = \delta_{ij}, \quad i,j\in I,$$

where δ_{ij} is 0 if $i\neq j$ and $\delta_{ij}=1$ if $i=j$. $\{f_i\}_{i\in I}$ is an orthornormal family in $L^2_\rho(\Omega,\mathbb{K})$.

The following proposition holds [**2**].

Proposition 25. *Let $\{f_i\}_{i\in I}$ be an orthornormal family of functions in $L^2_\rho(\Omega,\mathbb{K})$, F a finite subset of I, $\phi_i,\Phi_i\in\mathbb{K}$ $(i\in F)$ and $f\in L^2_\rho(\Omega,\mathbb{K})$, such that either*

$$\begin{aligned}(2.21)\quad \int_\Omega \rho(s)\operatorname{Re}\Bigg[\Bigg(\sum_{i\in F}\Phi_i f_i(s)-f(s)\Bigg)\\ \times\Bigg(\overline{f}(s)-\sum_{i\in F}\overline{\phi_i}\;\overline{f_i}(s)\Bigg)\Bigg]d\mu(s)\geq 0,\end{aligned}$$

or, equivalently,

$$\int_\Omega \rho(s)\left|f(s)-\sum_{i\in F}\frac{\Phi_i+\phi_i}{2}f_i(s)\right|^2 d\mu(s)\leq \frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2.$$

Then we have the inequality

$$\begin{aligned}(2.22)\quad 0 &\leq \int_\Omega \rho(s)|f(s)|^2\,d\mu(s)-\sum_{i\in F}\left|\int_\Omega \rho(s) f(s)\overline{f_i}(s)\,d\mu(s)\right|^2\\ &\leq \frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2\\ &\quad -\int_\Omega \rho(s)\operatorname{Re}\Bigg[\Bigg(\sum_{i\in F}\Phi_i f_i(s)-f(s)\Bigg)\\ &\qquad\times\Bigg(\overline{f}(s)-\sum_{i\in F}\overline{\phi_i}\;\overline{f_i}(s)\Bigg)\Bigg]d\mu(s)\\ &\leq \frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2.\end{aligned}$$

The constant $\frac{1}{4}$ is best possible in both inequalities.

The proof follows by Theorem 30 applied for the Hilbert space $L^2_\rho(\Omega,\mathbb{K})$ and the orthornormal family $\{f_i\}_{i\in I}$.

The following Grüss type inequality also holds [**2**].

Proposition 26. *Let $\{f_i\}_{i\in I}$ and F be as in Proposition 25. If $\phi_i, \Phi_i, \gamma_i, \Gamma_i \in \mathbb{K}$ $(i \in F)$ and $f, g \in L^2_\rho(\Omega, \mathbb{K})$ so that either*

$$\int_\Omega \rho(s) \operatorname{Re}\left[\left(\sum_{i\in F} \Phi_i f_i(s) - f(s)\right)\left(\overline{f}(s) - \sum_{i\in F} \overline{\phi_i}\ \overline{f_i}(s)\right)\right] d\mu(s) \geq 0,$$

$$\int_\Omega \rho(s) \operatorname{Re}\left[\left(\sum_{i\in F} \Gamma_i f_i(s) - g(s)\right)\left(\overline{g}(s) - \sum_{i\in F} \overline{\gamma_i}\ \overline{f_i}(s)\right)\right] d\mu(s) \geq 0,$$

or, equivalently,

$$\int_\Omega \rho(s) \left| f(s) - \sum_{i\in F} \frac{\Phi_i + \phi_i}{2} f_i(s)\right|^2 d\mu(s) \leq \frac{1}{4}\sum_{i\in F} |\Phi_i - \phi_i|^2,$$

$$\int_\Omega \rho(s) \left| g(s) - \sum_{i\in F} \frac{\Gamma_i + \gamma_i}{2} f_i(s)\right|^2 d\mu(s) \leq \frac{1}{4}\sum_{i\in F} |\Gamma_i - \gamma_i|^2,$$

hold, then we have the inequalities

$$\begin{aligned} (2.23)\quad & \left|\int_\Omega \rho(s) f(s) g(s) d\mu(s) \right. \\ & \qquad \left. - \sum_{i\in F} \int_\Omega \rho(s) f(s) \overline{f_i}(s) d\mu(s) \int_\Omega \rho(s) f_i(s) \overline{g(s)} d\mu(s)\right| \\ & \leq \frac{1}{4}\left(\sum_{i\in F} |\Phi_i - \phi_i|^2\right)^{\frac{1}{2}} \left(\sum_{i\in F} |\Gamma_i - \gamma_i|^2\right)^{\frac{1}{2}} \\ & \quad - \left[\int_\Omega \rho(s) \operatorname{Re}\left[\left(\sum_{i\in F} \Phi_i f_i(s) - f(s)\right)\left(\overline{f}(s) - \sum_{i\in F} \overline{\phi_i} \overline{f_i}(s)\right)\right] d\mu(s)\right]^{\frac{1}{2}} \\ & \quad \times \left[\int_\Omega \rho(s) \operatorname{Re}\left[\left(\sum_{i\in F} \Gamma_i f_i(s) - g(s)\right)\left(\overline{g}(s) - \sum_{i\in F} \overline{\gamma_i} \overline{f_i}(s)\right)\right] d\mu(s)\right]^{\frac{1}{2}} \\ & \leq \frac{1}{4}\left(\sum_{i\in F} |\Phi_i - \phi_i|^2\right)^{\frac{1}{2}} \left(\sum_{i\in F} |\Gamma_i - \gamma_i|^2\right)^{\frac{1}{2}}. \end{aligned}$$

The constant $\frac{1}{4}$ is the best possible.

The proof follows by Theorem 31 and we omit the details.

Remark 31. *Similar results may be stated if we apply the other inequalities obtained above. We omit the details.*

In the case of real spaces, the following corollaries provide much simpler sufficient conditions for the reverse of Bessel's inequality (2.22) or for the Grüss type inequality (2.23) to hold.

Corollary 24. *Let $\{f_i\}_{i\in I}$ be an orthornormal family of functions in the real Hilbert space $L_\rho^2(\Omega)$, F a finite part of I, $M_i, m_i \in \mathbb{R}$ $(i \in F)$ and $f \in L_\rho^2(\Omega)$ such that*

$$\sum_{i\in F} m_i f_i(s) \leq f(s) \leq \sum_{i\in F} M_i f_i(s) \quad \text{for} \quad \mu - a.e. \ s \in \Omega.$$

Then we have the inequalities

$$\begin{aligned}
0 &\leq \int_\Omega \rho(s) f^2(s)\, d\mu(s) - \sum_{i\in F}\left[\int_\Omega \rho(s) f(s) f_i(s)\, d\mu(s)\right]^2 \\
&\leq \frac{1}{4}\sum_{i\in F}(M_i - m_i)^2 \\
&\qquad - \int_\Omega \rho(s)\left(\sum_{i\in F} M_i f_i(s) - f(s)\right)\left(f(s) - \sum_{i\in F} m_i f_i(s)\right) d\mu(s) \\
&\leq \frac{1}{4}\sum_{i\in F}(M_i - m_i)^2.
\end{aligned}$$

The constant $\frac{1}{4}$ is best possible.

Corollary 25. *Let $\{f_i\}_{i\in I}$ and F be as in Corollary 24. If M_i, m_i, $N_i, n_i \in \mathbb{R}$ $(i \in F)$ and $f, g \in L_\rho^2(\Omega)$ are such that*

$$\sum_{i\in F} m_i f_i(s) \leq f(s) \leq \sum_{i\in F} M_i f_i(s)$$

and

$$\sum_{i\in F} n_i f_i(s) \leq g(s) \leq \sum_{i\in F} N_i f_i(s), \quad \text{for} \quad \mu - a.e. \ s \in \Omega,$$

then we have the inequalities

$$\begin{aligned}
&\left|\int_\Omega \rho(s) f(s) g(s)\, d\mu(s)\right. \\
&\qquad\left. - \sum_{i\in F}\int_\Omega \rho(s) f(s) f_i(s)\, d\mu(s) \int_\Omega \rho(s) g(s) f_i(s)\, d\mu(s)\right|
\end{aligned}$$

$$\leq \frac{1}{4}\left(\sum_{i\in F}(M_i-m_i)^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}(N_i-n_i)^2\right)^{\frac{1}{2}}$$
$$-\left[\int_\Omega \rho(s)\left(\sum_{i\in F}M_if_i(s)-f(s)\right)\left(f(s)-\sum_{i\in F}m_if_i(s)\right)d\mu(s)\right]^{\frac{1}{2}}$$
$$\times\left[\int_\Omega \rho(s)\left(\sum_{i\in F}N_if_i(s)-g(s)\right)\left(g(s)-\sum_{i\in F}n_if_i(s)\right)d\mu(s)\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\left(\sum_{i\in F}(M_i-m_i)^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}(N_i-n_i)^2\right)^{\frac{1}{2}}.$$

3. Another Reverse for Bessel's Inequality

3.1. A General Result. The following lemma holds [**6**].

LEMMA 5. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, F a finite part of I, $\lambda_i\in\mathbb{K}$, $i\in F$, $r>0$ and $x\in H$. If*

$$\left\|x-\sum_{i\in F}\lambda_ie_i\right\|\leq r,$$

then we have the inequality

$$0\leq\|x\|^2-\sum_{i\in F}|\langle x,e_i\rangle|^2\leq r^2-\sum_{i\in F}|\lambda_i-\langle x,e_i\rangle|^2. \tag{3.1}$$

PROOF. Consider

$$\begin{aligned}I_1:=\left\|x-\sum_{i\in F}\lambda_ie_i\right\|^2&=\left\langle x-\sum_{i\in F}\lambda_ie_i,x-\sum_{j\in F}\lambda_je_j\right\rangle\\&=\|x\|^2-\sum_{i\in F}\lambda_i\overline{\langle x,e_i\rangle}-\sum_{i\in F}\overline{\lambda_i}\langle x,e_i\rangle+\sum_{i\in F}\sum_{j\in F}\lambda_i\overline{\lambda_j}\langle e_i,e_j\rangle\\&=\|x\|^2-\sum_{i\in F}\lambda_i\overline{\langle x,e_i\rangle}-\sum_{i\in F}\overline{\lambda_i}\langle x,e_i\rangle+\sum_{i\in F}|\lambda_i|^2\end{aligned}$$

and

$$\begin{aligned} I_2 &:= \sum_{i\in F}|\lambda_i - \langle x, e_i\rangle|^2 = \sum_{i\in F}(\lambda_i - \langle x, e_i\rangle)\left(\overline{\lambda_i} - \overline{\langle x, e_i\rangle}\right) \\ &= \sum_{i\in F}\left[|\lambda_i|^2 + |\langle x, e_i\rangle|^2 - \overline{\lambda_i}\langle x, e_i\rangle - \lambda_i\overline{\langle x, e_i\rangle}\right] \\ &= \sum_{i\in F}|\lambda_i|^2 + \sum_{i\in F}|\langle x, e_i\rangle|^2 - \sum_{i\in F}\overline{\lambda_i}\langle x, e_i\rangle - \sum_{i\in F}\lambda_i\overline{\langle x, e_i\rangle}. \end{aligned}$$

If we subtract I_2 from I_1 we deduce the following identity that is interesting in its own right

$$\left\|x - \sum_{i\in F}\lambda_i e_i\right\|^2 - \sum_{i\in F}|\lambda_i - \langle x, e_i\rangle|^2 = \|x\|^2 - \sum_{i\in F}|\langle x, e_i\rangle|^2,$$

from which we easily deduce (3.1). ∎

The following reverse of Bessel's inequality holds [**6**].

THEOREM 33. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, F a finite part of I, ϕ_i, Φ_i, $i \in I$ real or complex numbers. For $x \in H$, if either*

(i) $\operatorname{Re}\left\langle \sum_{i\in F}\Phi_i e_i - x, x - \sum_{i\in F}\phi_i e_i\right\rangle \geq 0$;

or, equivalently,

(ii) $\left\|x - \sum_{i\in F}\frac{\phi_i+\Phi_i}{2}e_i\right\| \leq \frac{1}{2}\left(\sum_{i\in F}|\Phi_i - \phi_i|^2\right)^{\frac{1}{2}}$;

holds, then the following reverse of Bessel's inequality

$$\begin{aligned} 0 &\leq \|x\|^2 - \sum_{i\in F}|\langle x, e_i\rangle|^2 \\ &\leq \frac{1}{4}\sum_{i\in F}|\Phi_i - \phi_i|^2 - \sum_{i\in F}\left|\frac{\phi_i+\Phi_i}{2} - \langle x, e_i\rangle\right|^2 \\ &\leq \frac{1}{4}\sum_{i\in F}|\Phi_i - \phi_i|^2, \end{aligned} \tag{3.2}$$

is valid.

The constant $\frac{1}{4}$ is best possible in both inequalities.

PROOF. If we apply Lemma 5 for $\lambda_i = \frac{\phi_i+\Phi_i}{2}$ and

$$r := \frac{1}{2}\left(\sum_{i\in F}|\Phi_i - \phi_i|^2\right)^{\frac{1}{2}},$$

we deduce the first inequality in (3.2).

Let us prove that $\frac{1}{4}$ is best possible in the second inequality in (3.2).

Assume that there is a $c > 0$ such that

$$0 \le \|x\|^2 - \sum_{i\in F} |\langle x, e_i\rangle|^2 \le c \sum_{i\in F} |\Phi_i - \phi_i|^2 - \sum_{i\in F} \left|\frac{\phi_i + \Phi_i}{2} - \langle x, e_i\rangle\right|^2, \tag{3.3}$$

provided that ϕ_i, Φ_i, x and F satisfy (i) and (ii).

We choose $F = \{1\}$, $e_1 = e = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \in \mathbb{R}^2$, $x = (x_1, x_2) \in \mathbb{R}^2$, $\Phi_1 = \Phi = m > 0$, $\phi_1 = \phi = -m$, $H = \mathbb{R}^2$ to get from (3.3) that

$$0 \le x_1^2 + x_2^2 - \frac{(x_1 + x_2)^2}{2} \le 4cm^2 - \frac{(x_1 + x_2)^2}{2}, \tag{3.4}$$

provided

$$0 \le \langle me - x, x + me\rangle = \left(\frac{m}{\sqrt{2}} - x_1\right)\left(x_1 + \frac{m}{\sqrt{2}}\right) + \left(\frac{m}{\sqrt{2}} - x_2\right)\left(x_2 + \frac{m}{\sqrt{2}}\right). \tag{3.5}$$

From (3.4) we get

$$x_1^2 + x_2^2 \le 4cm^2 \tag{3.6}$$

provided (3.5) holds.

If we choose $x_1 = \frac{m}{\sqrt{2}}$, $x_2 = -\frac{m}{\sqrt{2}}$, then (3.5) is fulfilled and by (3.6) we get $m^2 \le 4cm^2$, giving $c \ge \frac{1}{4}$. ∎

REMARK 32. *If $F = \{1\}$, $e_1 = 1$, $\|e\| = 1$ and for $\phi, \Phi \in \mathbb{K}$ and $x \in H$ one has either*

$$\operatorname{Re}\langle \Phi e - x, x - \phi e\rangle \ge 0,$$

or, equivalently,

$$\left\|x - \frac{\phi + \Phi}{2} e\right\| \le \frac{1}{2}|\Phi - \phi|,$$

then

$$0 \le \|x\|^2 - |\langle x, e\rangle|^2 \le \frac{1}{4}|\Phi - \phi|^2 - \left|\frac{\phi + \Phi}{2} - \langle x, e\rangle\right|^2 \le \frac{1}{4}|\Phi - \phi|^2.$$

The constant $\frac{1}{4}$ is best possible in both inequalities.

REMARK 33. *It is important to compare the bounds provided by Theorem 30 and Theorem 33.*

For this purpose, consider

$$B_1(x,e,\phi,\Phi) := \frac{1}{4}(\Phi-\phi)^2 - \langle \Phi e - x, x - \phi e\rangle$$

and

$$B_2(x,e,\phi,\Phi) := \frac{1}{4}(\Phi-\phi)^2 - \left(\frac{\phi+\Phi}{2} - \langle x, e\rangle\right)^2,$$

where H is a real inner product, $e \in H$, $\|e\| = 1$, $x \in H$, $\phi, \Phi \in \mathbb{R}$ with

$$\langle \Phi e - x, x - \phi e\rangle \geq 0,$$

or, equivalently,

$$\left\| x - \frac{\phi+\Phi}{2} e \right\| \leq \frac{1}{2}|\Phi - \phi|.$$

If we choose $\phi = -1$, $\Phi = 1$, then we have

$$B_1(x,e) = 1 - \langle e - x, x + e\rangle = 1 - \left(\|e\|^2 - \|x\|^2\right) = \|x\|^2,$$
$$B_2(x,e) = 1 - \langle x, e\rangle^2,$$

provided $\|x\| \leq 1$.

Consider $H = \mathbb{R}^2$, $\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 + x_2y_2, \mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$ and $\mathbf{e} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Then $\|\mathbf{e}\| = 1$ and we must compare

$$B_1(\mathbf{x}) = x_1^2 + x_2^2$$

with

$$B_2(\mathbf{x}) = 1 - \frac{(x_1+x_2)^2}{2},$$

provided $x_1^2 + x_2^2 \leq 1$.

If we choose $\mathbf{x}_0 = (1,0)$, then $\|\mathbf{x}_0\| = 1$ and $B_1(\mathbf{x}_0) = 1, B_2(\mathbf{x}_0) = \frac{1}{2}$ showing that $B_1 > B_2$. If we choose $\mathbf{x}_{00} = \left(-\frac{1}{2}, \frac{1}{2}\right)$, then $B_1(\mathbf{x}_{00}) = \frac{1}{2}$, $B_2(\mathbf{x}_{00}) = 1$, showing that $B_1 < B_2$.

We may state the following proposition.

PROPOSITION 27. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, F a finite part of I, ϕ_i, $\Phi_i \in \mathbb{K}$ $(i \in F)$. If $x \in H$ either satisfies (i), or, equivalently, (ii) of Theorem 33, then the upper bounds*

$$B_1(x,e,\boldsymbol{\phi},\boldsymbol{\Phi},F) := \frac{1}{4}\sum_{i\in F}|\Phi_i - \phi_i|^2 - \operatorname{Re}\left\langle \sum_{i\in F}\Phi_i e_i - x, x - \sum_{i\in F}\phi_i e_i\right\rangle,$$

$$B_2(x,e,\boldsymbol{\phi},\boldsymbol{\Phi},F) := \frac{1}{4}\sum_{i\in F}|\Phi_i - \phi_i|^2 - \sum_{i\in F}\left|\frac{\phi_i+\Phi_i}{2} - \langle x, e_i\rangle\right|^2,$$

for the Bessel's difference $B_s(x, \mathbf{e}, F) := \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2$, *cannot be compared in general.*

3.2. A Refinement of the Grüss Inequality for Orthonormal Families. The following result holds [**6**].

THEOREM 34. *Let* $\{e_i\}_{i \in I}$ *be a family of orthornormal vectors in* H, F *a finite part of* I, $\phi_i, \Phi_i, \gamma_i, \Gamma_i \in \mathbb{K}$, $i \in F$ *and* $x, y \in H$. *If either*

$$\operatorname{Re}\left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle \geq 0,$$
$$\operatorname{Re}\left\langle \sum_{i \in F} \Gamma_i e_i - y, y - \sum_{i \in F} \gamma_i e_i \right\rangle \geq 0,$$

or, equivalently,

$$\left\| x - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} e_i \right\| \leq \frac{1}{2} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}},$$
$$\left\| y - \sum_{i \in F} \frac{\Gamma_i + \gamma_i}{2} e_i \right\| \leq \frac{1}{2} \left(\sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}},$$

hold, then we have the inequalities

$$\begin{aligned} 0 &\leq \left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right| \\ &\leq \frac{1}{4} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \\ &\qquad - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle x, e_i \rangle \right| \left| \frac{\Gamma_i + \gamma_i}{2} - \langle y, e_i \rangle \right| \\ &\leq \frac{1}{4} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{3.7}$$

The constant $\frac{1}{4}$ *is best possible.*

Proof. Using Schwarz's inequality in the inner product space $(H, \langle\cdot,\cdot\rangle)$ one has

$$\left|\left\langle x-\sum_{i\in F}\langle x,e_i\rangle e_i, y-\sum_{i\in F}\langle y,e_i\rangle e_i\right\rangle\right|^2 \leq \left\|x-\sum_{i\in F}\langle x,e_i\rangle e_i\right\|^2\left\|y-\sum_{i\in F}\langle y,e_i\rangle e_i\right\|^2 \tag{3.8}$$

and since a simple calculation shows that

$$\left\langle x-\sum_{i\in F}\langle x,e_i\rangle e_i, y-\sum_{i\in F}\langle y,e_i\rangle e_i\right\rangle = \langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle$$

and

$$\left\|x-\sum_{i\in F}\langle x,e_i\rangle e_i\right\|^2 = \|x\|^2-\sum_{i\in F}|\langle x,e_i\rangle|^2$$

for any $x, y \in H$, then by (3.8) and by the reverse of Bessel's inequality in Theorem 33, we have

$$\begin{aligned}
&\left|\langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right|^2 \\
&\leq \left(\|x\|^2-\sum_{i\in F}|\langle x,e_i\rangle|^2\right)\left(\|y\|^2-\sum_{i\in F}|\langle y,e_i\rangle|^2\right) \\
&\leq \left[\frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2-\sum_{i\in F}\left|\frac{\Phi_i+\phi_i}{2}-\langle x,e_i\rangle\right|^2\right] \\
&\qquad\times\left[\frac{1}{4}\sum_{i\in F}|\Gamma_i-\gamma_i|^2-\sum_{i\in F}\left|\frac{\Gamma_i+\gamma_i}{2}-\langle y,e_i\rangle\right|^2\right] \\
&:= K.
\end{aligned} \tag{3.9}$$

Using Aczél's inequality for real numbers, i.e., we recall that

$$\left(a^2-\sum_{i\in F}a_i^2\right)\left(b^2-\sum_{i\in F}b_i^2\right)\leq\left(ab-\sum_{i\in F}a_ib_i\right)^2, \tag{3.10}$$

provided that $a, b, a_i, b_i > 0$, $i \in F$, we may state that

$$K \leq \left[\frac{1}{4} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle x, e_i \rangle \right| \left| \frac{\Gamma_i + \gamma_i}{2} - \langle y, e_i \rangle \right| \right]^2 . \tag{3.11}$$

Using (3.9) and (3.11) we conclude that

$$\begin{aligned} &\left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right|^2 \\ &\leq \left[\frac{1}{4} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle x, e_i \rangle \right| \left| \frac{\Gamma_i + \gamma_i}{2} - \langle y, e_i \rangle \right| \right]^2 . \end{aligned} \tag{3.12}$$

Taking the square root in (3.12) and taking into account that the quantity in the last square bracket is nonnegative (this follows by (3.2) and by the Cauchy-Bunyakovsky-Schwarz inequality), we deduce the second inequality in (3.7).

The fact that $\frac{1}{4}$ is the best possible constant follows by Theorem 33 and we omit the details. ∎

The following corollary may be stated [**6**].

COROLLARY 26. *Let $e \in H$, $\|e\| = 1$, $\phi, \Phi, \gamma, \Gamma \in \mathbb{K}$ and $x, y \in H$ are such that either*

$$\operatorname{Re} \langle \Phi e - x, x - \phi e \rangle \geq 0 \quad \text{and} \quad \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0,$$

or, equivalently,

$$\left\| x - \frac{\phi + \Phi}{2} e \right\| \leq \frac{1}{2} |\Phi - \phi| , \quad \left\| y - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma| ,$$

hold. Then we have the following refinement of Grüss' inequality

$$\begin{aligned} 0 &\leq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \\ &\leq \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma| - \left| \frac{\phi + \Phi}{2} - \langle x, e \rangle \right| \left| \frac{\gamma + \Gamma}{2} - \langle y, e \rangle \right| \\ &\leq \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma| . \end{aligned}$$

The constant $\frac{1}{4}$ is best possible in both inequalities.

3.3. Some Companion Inequalities. The following companion of the Grüss inequality also holds [**6**].

THEOREM 35. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, F a finite part of I and $\phi_i, \Phi_i \in \mathbb{K}$, $i \in F$, $x, y \in H$ and $\lambda \in (0,1)$, such that either*

$$\operatorname{Re}\left\langle \sum_{i\in F} \Phi_i e_i - (\lambda x + (1-\lambda) y), \right. \qquad \left. \lambda x + (1-\lambda) y - \sum_{i\in F} \phi_i e_i \right\rangle \geq 0 \tag{3.13}$$

or, equivalently,

$$\left\| \lambda x + (1-\lambda) y - \sum_{i\in F} \frac{\Phi_i + \phi_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left(\sum_{i\in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}},$$

holds. Then we have the inequality

$$\begin{aligned} &\operatorname{Re}\left[\langle x, y\rangle - \sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle \right] \\ &\leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} \sum_{i\in F} |\Phi_i - \phi_i|^2 \\ &\qquad - \frac{1}{4} \frac{1}{\lambda(1-\lambda)} \sum_{i\in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle \lambda x + (1-\lambda) y, e_i\rangle \right|^2 \\ &\leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} \sum_{i\in F} |\Phi_i - \phi_i|^2 . \end{aligned} \tag{3.14}$$

The constant $\frac{1}{16}$ is the best possible constant in (3.14) in the sense that it cannot be replaced by a smaller constant.

PROOF. We know that for any $z, u \in H$, one has

$$\operatorname{Re}\langle z, u\rangle \leq \frac{1}{4} \|z + u\|^2 .$$

Then for any $a, b \in H$ and $\lambda \in (0,1)$ one has

$$\operatorname{Re}\langle a, b\rangle \leq \frac{1}{4\lambda(1-\lambda)} \|\lambda a + (1-\lambda) b\|^2 . \tag{3.15}$$

Since

$$\langle x, y\rangle - \sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle = \left\langle x - \sum_{i\in F} \langle x, e_i\rangle e_i, y - \sum_{i\in F} \langle y, e_i\rangle e_i \right\rangle,$$

for any $x, y \in H$, then, by (3.15), we get

$$\begin{aligned}
(3.16) \qquad \operatorname{Re}\left[\langle x, y\rangle - \sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle\right] \\
= \operatorname{Re}\left[\left\langle x - \sum_{i\in F} \langle x, e_i\rangle e_i, y - \sum_{i\in F} \langle y, e_i\rangle e_i \right\rangle\right]
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{4\lambda(1-\lambda)} \left\| \lambda\left(x - \sum_{i\in F} \langle x, e_i\rangle e_i\right) + (1-\lambda)\left(y - \sum_{i\in F} \langle y, e_i\rangle e_i\right) \right\|^2 \\
&= \frac{1}{4\lambda(1-\lambda)} \left\| \lambda x + (1-\lambda) y - \sum_{i\in F} \langle \lambda x + (1-\lambda) y, e_i\rangle e_i \right\|^2 \\
&= \frac{1}{4\lambda(1-\lambda)} \left[\|\lambda x + (1-\lambda) y\|^2 - \sum_{i\in F} |\langle \lambda x + (1-\lambda) y, e_i\rangle|^2 \right].
\end{aligned}$$

If we apply the reverse of Bessel's inequality in Theorem 33 for $\lambda x + (1-\lambda) y$, we may state that

$$\begin{aligned}
(3.17) \qquad & \|\lambda x + (1-\lambda) y\|^2 - \sum_{i\in F} |\langle \lambda x + (1-\lambda) y, e_i\rangle|^2 \\
&\le \frac{1}{4} \sum_{i\in F} |\Phi_i - \phi_i|^2 - \sum_{i\in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle \lambda x + (1-\lambda) y, e_i\rangle \right|^2 \\
&\le \frac{1}{4} \sum_{i\in F} |\Phi_i - \phi_i|^2 .
\end{aligned}$$

Now, by making use of (3.16) and (3.17), we deduce (3.14).

The fact that $\frac{1}{16}$ is the best possible constant in (3.14) follows by the fact that, if in (3.13) we choose $x = y$, then it becomes (i) of Theorem 33, implying for $\lambda = \frac{1}{2}$ (3.2), for which, we have shown that $\frac{1}{4}$ was the best constant. ∎

Remark 34. *In practical applications we may use only the inequality between the first and the last terms in (3.14).*

REMARK 35. *If in Theorem 35, we choose $\lambda = \frac{1}{2}$, then we get*

$$\begin{aligned}
&\operatorname{Re}\left[\langle x, y\rangle - \sum_{i\in F}\langle x, e_i\rangle\langle e_i, y\rangle\right] \\
&\leq \frac{1}{4}\sum_{i\in F}|\Phi_i - \phi_i|^2 - \sum_{i\in F}\left|\frac{\Phi_i + \phi_i}{2} - \left\langle\frac{x+y}{2}, e_i\right\rangle\right|^2 \\
&\leq \frac{1}{4}\sum_{i\in F}|\Phi_i - \phi_i|^2,
\end{aligned}$$

provided

$$\operatorname{Re}\left\langle\sum_{i\in F}\Phi_i e_i - \frac{x+y}{2}, \frac{x+y}{2} - \sum_{i\in F}\phi_i e_i\right\rangle \geq 0$$

or, equivalently,

$$\left\|\frac{x+y}{2} - \sum_{i\in F}\frac{\Phi_i + \phi_i}{2}\cdot e_i\right\| \leq \frac{1}{2}\left(\sum_{i\in F}|\Phi_i - \phi_i|^2\right)^{\frac{1}{2}}.$$

COROLLARY 27. *With the assumptions of Theorem 35 and if*

$$\operatorname{Re}\left\langle\sum_{i\in F}\Phi_i e_i - (\lambda x \pm (1-\lambda)y), \lambda x \pm (1-\lambda)y - \sum_{i\in F}\phi_i e_i\right\rangle \geq 0$$

or, equivalently,

$$\left\|\lambda x \pm (1-\lambda)y - \sum_{i\in F}\frac{\Phi_i + \phi_i}{2}\cdot e_i\right\| \leq \frac{1}{2}\left(\sum_{i\in F}|\Phi_i - \phi_i|^2\right)^{\frac{1}{2}},$$

then we have the inequality

$$\left|\operatorname{Re}\left[\langle x, y\rangle - \sum_{i\in F}\langle x, e_i\rangle\langle e_i, y\rangle\right]\right| \leq \frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}\sum_{i\in F}|\Phi_i - \phi_i|^2. \tag{3.18}$$

The constant $\frac{1}{16}$ is best possible in (3.18).

REMARK 36. *If H is a real inner product space and $m_i, M_i \in \mathbb{R}$ with the property*

$$\left\langle\sum_{i\in F}M_i e_i - (\lambda x \pm (1-\lambda)y), \lambda x \pm (1-\lambda)y - \sum_{i\in F}m_i e_i\right\rangle \geq 0$$

or, equivalently,

$$\left\|\lambda x \pm (1-\lambda) y - \sum_{i\in F} \frac{M_i + m_i}{2} \cdot e_i\right\| \leq \frac{1}{2}\left[\sum_{i\in F} (M_i - m_i)^2\right]^{\frac{1}{2}},$$

then we have the Grüss type inequality

$$\left|\langle x, y\rangle - \sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle\right| \leq \frac{1}{16} \cdot \frac{1}{\lambda (1-\lambda)} \sum_{i\in F} (M_i - m_i)^2 .$$

3.4. Integral Inequalities. The following proposition holds [**6**].

PROPOSITION 28. *Let $\{f_i\}_{i\in I}$ be an orthornormal family of functions in $L_\rho^2(\Omega, \mathbb{K})$, F a finite subset of I, $\phi_i, \Phi_i \in \mathbb{K}$ $(i \in F)$ and $f \in L_\rho^2(\Omega, \mathbb{K})$, so that either*

$$\int_\Omega \rho(s) \operatorname{Re}\left[\left(\sum_{i\in F} \Phi_i f_i(s) - f(s)\right)\left(\overline{f}(s) - \sum_{i\in F} \overline{\phi_i}\ \overline{f_i}(s)\right)\right] d\mu(s) \geq 0$$

or, equivalently,

$$\int_\Omega \rho(s) \left|f(s) - \sum_{i\in F} \frac{\Phi_i + \phi_i}{2} \cdot f_i(s)\right|^2 d\mu(s) \leq \frac{1}{4} \sum_{i\in F} |\Phi_i - \phi_i|^2 .$$

Then we have the inequality

$$
\begin{aligned}
(3.19)\qquad 0 &\leq \int_\Omega \rho(s) |f(s)|^2 d\mu(s) - \sum_{i\in F} \left|\int_\Omega \rho(s) f(s) \overline{f_i}(s) d\mu(s)\right|^2 \\
&\leq \frac{1}{4} \sum_{i\in F} |\Phi_i - \phi_i|^2 \\
&\qquad - \sum_{i\in F} \left|\frac{\Phi_i + \phi_i}{2} - \int_\Omega \rho(s) f(s) \overline{f_i}(s) d\mu(s)\right|^2 \\
&\leq \frac{1}{4} \sum_{i\in F} |\Phi_i - \phi_i|^2 .
\end{aligned}
$$

The constant $\frac{1}{4}$ is best possible in both inequalities.

The proof follows by Theorem 33 applied for the Hilbert space $L_\rho^2(\Omega, \mathbb{K})$ and the orthornormal family $\{f_i\}_{i\in I}$.

The following Grüss type inequality also holds [**6**].

Proposition 29. *Let* $\{f_i\}_{i\in I}$ *and* F *be as in Proposition 28. If* $\phi_i, \Phi_i, \gamma_i, \Gamma_i \in \mathbb{K}$ $(i \in F)$ *and* $f, g \in L^2_\rho(\Omega, \mathbb{K})$ *so that either*

$$\int_\Omega \rho(s)\operatorname{Re}\left[\left(\sum_{i\in F}\Phi_i f_i(s) - f(s)\right)\left(\overline{f}(s) - \sum_{i\in F}\overline{\phi_i}\,\overline{f_i}(s)\right)\right]d\mu(s) \geq 0,$$

$$\int_\Omega \rho(s)\operatorname{Re}\left[\left(\sum_{i\in F}\Gamma_i f_i(s) - g(s)\right)\left(\overline{g}(s) - \sum_{i\in F}\overline{\gamma_i}\,\overline{f_i}(s)\right)\right]d\mu(s) \geq 0,$$

or, equivalently,

$$\int_\Omega \rho(s)\left|f(s) - \sum_{i\in F}\frac{\Phi_i+\phi_i}{2}f_i(s)\right|^2 d\mu(s) \leq \frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2,$$

$$\int_\Omega \rho(s)\left|g(s) - \sum_{i\in F}\frac{\Gamma_i+\gamma_i}{2}f_i(s)\right|^2 d\mu(s) \leq \frac{1}{4}\sum_{i\in F}|\Gamma_i-\gamma_i|^2,$$

hold, then we have the inequalities

$$\begin{aligned}(3.20)\quad &\left|\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s)\right.\\ &\qquad\left.-\sum_{i\in F}\int_\Omega \rho(s) f(s)\overline{f_i}(s)\,d\mu(s)\int_\Omega \rho(s) f_i(s)\overline{g(s)}d\mu(s)\right|\\ &\leq \frac{1}{4}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}|\Gamma_i-\gamma_i|^2\right)^{\frac{1}{2}}\\ &\qquad -\sum_{i\in F}\left|\frac{\Phi_i+\phi_i}{2} - \int_\Omega \rho(s) f(s)\overline{f_i}(s)\,d\mu(s)\right|\\ &\qquad \times\left|\frac{\Gamma_i+\gamma_i}{2} - \int_\Omega \rho(s) g(s)\overline{f_i}(s)\,d\mu(s)\right|\\ &\leq \frac{1}{4}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}|\Gamma_i-\gamma_i|^2\right)^{\frac{1}{2}}.\end{aligned}$$

The constant $\frac{1}{4}$ *is the best possible.*

The proof follows by Theorem 34 and we omit the details.

Remark 37. *Similar results may be stated if one applies the inequalities in the above subsections. We omit the details.*

In the case of real spaces, the following corollaries provide much simpler sufficient conditions for the reverse of Bessel's inequality (3.19) or for the Grüss type inequality (3.20) to hold.

COROLLARY 28. *Let $\{f_i\}_{i\in I}$ be an orthornormal family of functions in the real Hilbert space $L^2_\rho(\Omega)$, F a finite part of I, $M_i, m_i \in \mathbb{R}$ $(i \in F)$ and $f \in L^2_\rho(\Omega)$ so that*

$$\sum_{i\in F} m_i f_i(s) \leq f(s) \leq \sum_{i\in F} M_i f_i(s) \quad \text{for} \quad \mu-a.e.\ s \in \Omega.$$

Then we have the inequalities

$$\begin{aligned}
0 &\leq \int_\Omega \rho(s) f^2(s)\, d\mu(s) - \sum_{i\in F}\left[\int_\Omega \rho(s) f(s) f_i(s)\, d\mu(s)\right]^2 \\
&\leq \frac{1}{4}\sum_{i\in F}(M_i - m_i)^2 - \sum_{i\in F}\left[\frac{M_i + m_i}{2} - \int_\Omega \rho(s) f(s) f_i(s)\, d\mu(s)\right]^2 \\
&\leq \frac{1}{4}\sum_{i\in F}(M_i - m_i)^2 .
\end{aligned}$$

The constant $\frac{1}{4}$ is best possible.

COROLLARY 29. *Let $\{f_i\}_{i\in I}$ and F be as in Corollary 28. If M_i, m_i, $N_i, n_i \in \mathbb{R}$ $(i \in F)$ and $f, g \in L^2_\rho(\Omega)$ are such that*

$$\sum_{i\in F} m_i f_i(s) \leq f(s) \leq \sum_{i\in F} M_i f_i(s)$$

and

$$\sum_{i\in F} n_i f_i(s) \leq g(s) \leq \sum_{i\in F} N_i f_i(s) \quad \text{for} \quad \mu-a.e.\ s \in \Omega,$$

hold, then we have the inequalities

$$\begin{aligned}
&\left|\int_\Omega \rho(s) f(s) g(s)\, d\mu(s)\right. \\
&\qquad \left. - \sum_{i\in F}\int_\Omega \rho(s) f(s) f_i(s)\, d\mu(s) \int_\Omega \rho(s) g(s) f_i(s)\, d\mu(s)\right|
\end{aligned}$$

$$\leq \frac{1}{4}\left(\sum_{i\in F}(M_i - m_i)^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}(N_i - n_i)^2\right)^{\frac{1}{2}}$$
$$-\sum_{i\in F}\left|\frac{M_i + m_i}{2} - \int_\Omega \rho(s) f(s) f_i(s)\, d\mu(s)\right|$$
$$\times\left|\frac{N_i + n_i}{2} - \int_\Omega \rho(s) g(s) f_i(s)\, d\mu(s)\right|$$
$$\leq \frac{1}{4}\left(\sum_{i\in F}(M_i - m_i)^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}(N_i - n_i)^2\right)^{\frac{1}{2}}.$$

4. More Reverses of Bessel's Inequality

4.1. A General Result. The following reverse of Bessel's inequality holds [**9**].

Theorem 36. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, F a finite part of I, and ϕ_i, Φ_i $(i \in F)$, real or complex numbers such that $\sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right) > 0$. If $x \in H$ is such that either*

(i) $\operatorname{Re}\left\langle \sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \phi_i e_i \right\rangle \geq 0$;
or, equivalently,

(ii) $\left\| x - \sum_{i\in F} \frac{\phi_i + \Phi_i}{2} e_i \right\| \leq \frac{1}{2}\left(\sum_{i\in F} |\Phi_i - \phi_i|^2\right)^{\frac{1}{2}}$;

holds, then one has the inequality

$$\|x\|^2 \leq \frac{1}{4} \cdot \frac{\sum_{i\in F}\left(|\Phi_i| + |\phi_i|\right)^2}{\sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)} \sum_{i\in F} |\langle x, e_i\rangle|^2. \tag{4.1}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Observe that

$$\operatorname{Re}\left\langle \sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \phi_i e_i \right\rangle$$
$$= \sum_{i\in F} \operatorname{Re}\left[\Phi_i \overline{\langle x, e_i\rangle} + \overline{\phi_i}\langle x, e_i\rangle\right] - \|x\|^2 - \sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right),$$

giving, from (i), that

$$\|x\|^2 + \sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right) \leq \sum_{i\in F} \operatorname{Re}\left[\Phi_i \overline{\langle x, e_i\rangle} + \overline{\phi_i}\langle x, e_i\rangle\right]. \tag{4.2}$$

On the other hand, by the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha} q^2 \geq 2pq, \quad \alpha > 0,\ p, q \geq 0;$$

we deduce

$$2\|x\| \leq \frac{\|x\|^2}{\left[\sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)\right]^{\frac{1}{2}}} + \left[\sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)\right]^{\frac{1}{2}}. \tag{4.3}$$

Dividing (4.2) by $\left[\sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)\right]^{\frac{1}{2}} > 0$ and using (4.3), we obtain

$$\|x\| \leq \frac{1}{2} \frac{\sum_{i\in F} \operatorname{Re}\left[\Phi_i \overline{\langle x, e_i\rangle} + \overline{\phi_i} \langle x, e_i\rangle\right]}{\left[\sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)\right]^{\frac{1}{2}}}, \tag{4.4}$$

which is also an interesting inequality in itself.

Using the Cauchy-Bunyakovsky-Schwarz inequality for real numbers, we get

$$\begin{aligned} \sum_{i\in F} \operatorname{Re}\left[\Phi_i \overline{\langle x, e_i\rangle} + \overline{\phi_i} \langle x, e_i\rangle\right] & \\ &\leq \sum_{i\in F} \left|\Phi_i \overline{\langle x, e_i\rangle} + \overline{\phi_i} \langle x, e_i\rangle\right| \\ &\leq \sum_{i\in F} \left(|\Phi_i| + |\phi_i|\right) |\langle x, e_i\rangle| \\ &\leq \left[\sum_{i\in F} \left(|\Phi_i| + |\phi_i|\right)^2\right]^{\frac{1}{2}} \left[\sum_{i\in F} |\langle x, e_i\rangle|^2\right]^{\frac{1}{2}}. \end{aligned} \tag{4.5}$$

Making use of (4.4) and (4.5), we deduce the desired result (4.1).

To prove the sharpness of the constant $\frac{1}{4}$, let us assume that (4.1) holds with a constant $c > 0$, i.e.,

$$\|x\|^2 \leq c \cdot \frac{\sum_{i\in F} \left(|\Phi_i| + |\phi_i|\right)^2}{\sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)} \sum_{i\in F} |\langle x, e_i\rangle|^2, \tag{4.6}$$

provided x, $\phi_i, \Phi_i, i \in F$ satisfies (i).

Choose $F = \{1\}$, $e_1 = e$, $\|e\| = 1$, $\phi_i = m$, $\Phi_i = M$ with $m, M > 0$, then, by (4.6), we get

$$\|x\|^2 \leq c \frac{(M+m)^2}{mM} |\langle x, e\rangle|^2 \tag{4.7}$$

provided

$$\operatorname{Re}\langle Me - x, x - me\rangle \geq 0. \tag{4.8}$$

If $x = me$, then obviously (4.8) holds, and by (4.7) we get

$$m^2 \leq c\frac{(M+m)^2}{mM}m^2$$

giving $mM \leq c\,(M+m)^2$ for $m, M > 0$. Now, if in this inequality we choose $m = 1-\varepsilon$, $M = 1+\varepsilon$ $(\varepsilon \in (0,1))$, then we get $1-\varepsilon^2 \leq 4c$ for $\varepsilon \in (0,1)$, from where we deduce $c \geq \frac{1}{4}$. ∎

REMARK 38. *By the use of (4.4), the second inequality in (4.5) and the Hölder inequality, we may state the following reverses of Bessel's inequality as well:*

$$\|x\|^2 \leq \frac{1}{2} \cdot \frac{1}{\left[\sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)\right]^{\frac{1}{2}}} \times \begin{cases} \max\limits_{i\in F} \left\{|\Phi_i| + |\phi_i|\right\} \sum\limits_{i\in F} |\langle x, e_i\rangle| ; \\ \\ \left[\sum\limits_{i\in F} \left(|\Phi_i| + |\phi_i|\right)^p\right]^{\frac{1}{p}} \left(\sum\limits_{i\in F} |\langle x, e_i\rangle|^q\right)^{\frac{1}{q}}, \\ \qquad\qquad \textit{for } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \max_{i\in F} |\langle x, e_i\rangle| \sum\limits_{i\in F} \left[|\Phi_i| + |\phi_i|\right]. \end{cases}$$

The following corollary holds [**9**].

COROLLARY 30. *With the assumption of Theorem 36 and if either (i) or (ii) holds, then*

$$0 \leq \|x\|^2 - \sum_{i\in F} |\langle x, e_i\rangle|^2 \leq \frac{1}{4} M^2\left(\mathbf{\Phi}, \boldsymbol{\phi}, F\right) \sum_{i\in F} |\langle x, e_i\rangle|^2, \tag{4.9}$$

where

$$M\left(\mathbf{\Phi}, \boldsymbol{\phi}, F\right) := \left[\frac{\sum_{i\in F} \left\{\left(|\Phi_i| - |\phi_i|\right)^2 + 4\left[\left|\Phi_i \overline{\phi_i}\right| - \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)\right]\right\}}{\sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)}\right]^{\frac{1}{2}}.$$

The constant $\frac{1}{4}$ is best possible.

PROOF. The inequality (4.9) follows by (4.1) on subtracting the same quantity $\sum_{i\in F} |\langle x, e_i\rangle|^2$ from both sides.

To prove the sharpness of the constant $\frac{1}{4}$, assume that (4.9) holds with $c > 0$, i.e.,

$$0 \leq \|x\|^2 - \sum_{i\in F} |\langle x, e_i\rangle|^2 \leq c M^2\left(\mathbf{\Phi}, \boldsymbol{\phi}, F\right) \sum_{i\in F} |\langle x, e_i\rangle|^2 \tag{4.10}$$

provided the condition (i) holds.

Choose $F = \{1\}$, $e_1 = e$, $\|e\| = 1$, $\phi_i = \phi$, $\Phi_i = \Phi$, $\phi, \Phi > 0$ in (4.10) to get

$$0 \leq \|x\|^2 - |\langle x, e\rangle|^2 \leq c\frac{(\Phi - \phi)^2}{\phi\Phi} |\langle x, e\rangle|^2, \tag{4.11}$$

provided

$$\langle \Phi e - x, x - \phi e\rangle \geq 0. \tag{4.12}$$

If $H = \mathbb{R}^2$, $x = (x_1, x_2) \in \mathbb{R}^2$, $e = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ then we have

$$\begin{aligned} \|x\|^2 - |\langle x, e\rangle|^2 &= x_1^2 + x_2^2 - \frac{(x_1 + x_2)^2}{2} = \frac{1}{2}(x_1 - x_2)^2, \\ |\langle x, e\rangle|^2 &= \frac{(x_1 + x_2)^2}{2} \end{aligned}$$

and by (4.11) we get

$$\frac{(x_1 - x_2)^2}{2} \leq c\frac{(\Phi - \phi)^2}{\phi\Phi} \cdot \frac{(x_1 + x_2)^2}{2}. \tag{4.13}$$

Now, if we let $x_1 = \frac{\phi}{\sqrt{2}}$, $x_2 = \frac{\Phi}{\sqrt{2}}$ $(\phi, \Phi > 0)$ then obviously

$$\langle \Phi e - x, x - \phi e\rangle = \sum_{i=1}^{2} \left(\frac{\Phi}{\sqrt{2}} - x_i\right)\left(x_i - \frac{\phi}{\sqrt{2}}\right) = 0,$$

which shows that (4.12) is fulfilled, and thus by (4.13) we obtain

$$\frac{(\Phi - \phi)^2}{4} \leq c\frac{(\Phi - \phi)^2}{\phi\Phi} \cdot \frac{(\Phi + \phi)^2}{4}$$

for any $\Phi > \phi > 0$. This implies

$$c(\Phi + \phi)^2 \geq \phi\Phi \tag{4.14}$$

for any $\Phi > \phi > 0$.

Finally, let $\phi = 1 - \varepsilon$, $\Phi = 1 + \varepsilon$, $\varepsilon \in (0, 1)$. Then from (4.14) we get $4c \geq 1 - \varepsilon^2$ for any $\varepsilon \in (0, 1)$ which produces $c \geq \frac{1}{4}$. ∎

Remark 39. *If $\{e_i\}_{i\in I}$ is an orthornormal family in the real inner product space $(H; \langle\cdot,\cdot\rangle)$ and $M_i, m_i \in \mathbb{R}$, $i \in F$ (F is a finite part of I) and $x \in H$ are such that $M_i, m_i \geq 0$ for $i \in F$ with $\sum_{i\in F} M_i m_i \geq 0$ and*

$$\left\langle \sum_{i\in F} M_i e_i - x, x - \sum_{i\in F} m_i e_i \right\rangle \geq 0,$$

then we have the inequality

$$0 \leq \|x\|^2 - \sum_{i \in F} [\langle x, e_i \rangle]^2 \leq \frac{1}{4} \cdot \frac{\sum_{i \in F} (M_i - m_i)^2}{\sum_{i \in F} M_i m_i} \cdot \sum_{i \in F} [\langle x, e_i \rangle]^2 .$$

The constant $\frac{1}{4}$ is best possible.

The following reverse of the Schwarz's inequality in inner product spaces holds.

COROLLARY 31. *Let $x, y \in H$ and $\delta, \Delta \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ with the property that* $\operatorname{Re}\left(\Delta \overline{\delta}\right) > 0$. *If either*

$$\operatorname{Re} \langle \Delta y - x, x - \delta y \rangle \geq 0,$$

or, equivalently,

$$\left\| x - \frac{\delta + \Delta}{2} \cdot y \right\| \leq \frac{1}{2} |\Delta - \delta| \, \|y\| ,$$

holds, then we have the inequalities

$$\begin{aligned} \|x\| \, \|y\| &\leq \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\Delta \overline{\langle x, y \rangle} + \overline{\delta} \langle x, y \rangle\right]}{\sqrt{\Delta \overline{\delta}}} \\ &\leq \frac{1}{2} \cdot \frac{|\Delta| + |\delta|}{\sqrt{\Delta \overline{\delta}}} |\langle x, y \rangle| , \end{aligned} \tag{4.15}$$

$$\begin{aligned} 0 &\leq \|x\| \, \|y\| - |\langle x, y \rangle| \\ &\leq \frac{1}{2} \cdot \frac{\left(\sqrt{|\Delta|} - \sqrt{|\delta|}\right)^2 + 2\left(\sqrt{\Delta \overline{\delta}} - \sqrt{\operatorname{Re}\left(\Delta \overline{\delta}\right)}\right)}{\sqrt{\Delta \overline{\delta}}} |\langle x, y \rangle| , \end{aligned} \tag{4.16}$$

$$\|x\|^2 \, \|y\|^2 \leq \frac{1}{4} \cdot \frac{(|\Delta| + |\delta|)^2}{\operatorname{Re}\left(\Delta \overline{\delta}\right)} |\langle x, y \rangle|^2 , \tag{4.17}$$

and

$$\begin{aligned} 0 &\leq \|x\|^2 \, \|y\|^2 - |\langle x, y \rangle|^2 \\ &\leq \frac{1}{4} \cdot \frac{(|\Delta| + |\delta|)^2 + 4\left(\left|\Delta \overline{\delta}\right| - \operatorname{Re}\left(\Delta \overline{\delta}\right)\right)}{\operatorname{Re}\left(\Delta \overline{\delta}\right)} |\langle x, y \rangle|^2 . \end{aligned} \tag{4.18}$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible.

PROOF. The inequality (4.15) follows from (4.4) on choosing $F = \{1\}$, $e_1 = e = \frac{y}{\|y\|}$, $\Phi_1 = \Phi = \Delta \|y\|$, $\phi_1 = \phi = \delta \|y\|$ $(y \neq 0)$. The inequality (4.16) is equivalent with (4.15). The inequality (4.17) follows from (4.1) for $F = \{1\}$ and the same choices as above. Finally, (4.18) is obviously equivalent with (4.17). ∎

4.2. Some Grüss Type Inequalities for Orthonormal Families. The following result holds [**9**].

THEOREM 37. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, F a finite part of I, $\phi_i, \Phi_i, \gamma_i, \Gamma_i \in \mathbb{K}$, $i \in F$ and $x, y \in H$. If either*

$$\operatorname{Re}\left\langle \sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \phi_i e_i \right\rangle \geq 0,$$
$$\operatorname{Re}\left\langle \sum_{i\in F} \Gamma_i e_i - y, y - \sum_{i\in F} \gamma_i e_i \right\rangle \geq 0,$$

or, equivalently,

$$\left\| x - \sum_{i\in F} \frac{\Phi_i + \phi_i}{2} e_i \right\| \leq \frac{1}{2} \left(\sum_{i\in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}},$$
$$\left\| y - \sum_{i\in F} \frac{\Gamma_i + \gamma_i}{2} e_i \right\| \leq \frac{1}{2} \left(\sum_{i\in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}},$$

hold, then we have the inequality

$$\begin{aligned} (4.19) \qquad 0 &\leq \left| \langle x, y\rangle - \sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle \right| \\ &\leq \frac{1}{4} M(\boldsymbol{\Phi}, \boldsymbol{\phi}, F)\, M(\boldsymbol{\Gamma}, \boldsymbol{\gamma}, F) \\ &\qquad \times \left(\sum_{i\in F} |\langle x, e_i\rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{i\in F} |\langle y, e_i\rangle|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where $M(\boldsymbol{\Phi}, \boldsymbol{\phi}, F)$ is defined in Corollary 30.

The constant $\frac{1}{4}$ is best possible.

PROOF. By the reverse of Bessel's inequality in Corollary 30, we have

$$\left|\langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right|^2 \tag{4.20}$$
$$\leq\left(\|x\|^2-\sum_{i\in F}|\langle x,e_i\rangle|^2\right)\left(\|y\|^2-\sum_{i\in F}|\langle y,e_i\rangle|^2\right)$$
$$\leq\frac{1}{4}M^2(\boldsymbol{\Phi},\boldsymbol{\phi},F)\sum_{i\in F}|\langle x,e_i\rangle|^2\cdot\frac{1}{4}M^2(\boldsymbol{\Gamma},\boldsymbol{\gamma},F)\sum_{i\in F}|\langle y,e_i\rangle|^2.$$

Taking the square root in (4.20), we deduce (4.19).

The fact that $\frac{1}{4}$ is the best possible constant follows by Corollary 30 and we omit the details. ∎

The following corollary for real inner product spaces holds [**9**].

COROLLARY 32. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, F a finite part of I, $M_i,m_i,\ N_i,n_i\geq 0,\ i\in F$ and $x,y\in H$ such that $\sum_{i\in F}M_im_i>0$, $\sum_{i\in F}N_in_i>0$ and*

$$\left\langle\sum_{i\in F}M_ie_i-x,x-\sum_{i\in F}m_ie_i\right\rangle\geq 0,\quad\left\langle\sum_{i\in F}N_ie_i-y,y-\sum_{i\in F}n_ie_i\right\rangle\geq 0.$$

Then we have the inequality

$$0\leq\left|\langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle y,e_i\rangle\right|^2$$
$$\leq\frac{1}{16}\cdot\frac{\sum_{i\in F}(M_i-m_i)^2\sum_{i\in F}(N_i-n_i)^2\sum_{i\in F}|\langle x,e_i\rangle|^2\sum_{i\in F}|\langle y,e_i\rangle|^2}{\sum_{i\in F}M_im_i\sum_{i\in F}N_in_i}.$$

The constant $\frac{1}{16}$ is best possible.

In the case where the family $\{e_i\}_{i\in I}$ reduces to a single vector, we may deduce from Theorem 37 the following particular case.

COROLLARY 33. *Let $e\in H$, $\|e\|=1$, $\phi,\Phi,\gamma,\Gamma\in\mathbb{K}$ with $\mathrm{Re}\left(\Phi\overline{\phi}\right)$, $\mathrm{Re}\left(\Gamma\overline{\gamma}\right)>0$ and $x,y\in H$ such that either*

$$\mathrm{Re}\langle\Phi e-x,x-\phi e\rangle\geq 0,\quad\mathrm{Re}\langle\Gamma e-y,y-\gamma e\rangle\geq 0,$$

or, equivalently,

$$\left\|x-\frac{\phi+\Phi}{2}e\right\|\leq\frac{1}{2}|\Phi-\phi|,\qquad\left\|y-\frac{\gamma+\Gamma}{2}e\right\|\leq\frac{1}{2}|\Gamma-\gamma|$$

hold, then

$$0 \leq |\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle| \leq \frac{1}{4} M(\Phi,\phi) M(\Gamma,\gamma) |\langle x,e\rangle \langle e,y\rangle|,$$

where

$$M(\Phi,\phi) := \left[\frac{(|\Phi|-|\phi|)^2 + 4\left[|\phi\Phi| - \operatorname{Re}\left(\Phi\overline{\phi}\right)\right]}{\operatorname{Re}\left(\Phi\overline{\phi}\right)}\right]^{\frac{1}{2}}.$$

The constant $\frac{1}{4}$ is best possible.

Remark 40. *If H is real, $e \in H$, $\|e\| = 1$ and $a,b,A,B \in \mathbb{R}$ are such that $A > a > 0$, $B > b > 0$ and*

$$\left\|x - \frac{a+A}{2}e\right\| \leq \frac{1}{2}(A-a), \quad \left\|y - \frac{b+B}{2}e\right\| \leq \frac{1}{2}(B-b),$$

then

$$|\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle| \leq \frac{1}{4} \cdot \frac{(A-a)(B-b)}{\sqrt{abAB}} |\langle x,e\rangle \langle e,y\rangle|. \tag{4.21}$$

The constant $\frac{1}{4}$ is best possible.

If $\langle x,e\rangle$, $\langle y,e\rangle \neq 0$, then the following equivalent form of (4.21) also holds

$$\left|\frac{\langle x,y\rangle}{\langle x,e\rangle \langle e,y\rangle} - 1\right| \leq \frac{1}{4} \cdot \frac{(A-a)(B-b)}{\sqrt{abAB}}.$$

4.3. Some Companion Inequalities. The following companion of the Grüss inequality also holds [**9**].

Theorem 38. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, F a finite part of I, $\phi_i, \Phi_i \in \mathbb{K}$, $(i \in F)$, $x,y \in H$ and $\lambda \in (0,1)$, such that either*

$$\operatorname{Re}\left\langle \sum_{i\in F} \Phi_i e_i - (\lambda x + (1-\lambda) y), \lambda x + (1-\lambda) y - \sum_{i\in F} \phi_i e_i \right\rangle \geq 0, \tag{4.22}$$

or, equivalently,

$$\left\|\lambda x + (1-\lambda) y - \sum_{i\in F} \frac{\Phi_i + \phi_i}{2} \cdot e_i\right\| \leq \frac{1}{2}\left(\sum_{i\in F} |\Phi_i - \phi_i|^2\right)^{\frac{1}{2}},$$

holds. Then we have the inequality

$$\operatorname{Re}\left[\langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right]\leq\frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}\sum_{i\in F}M^2(\mathbf{\Phi},\boldsymbol{\phi},F)\sum_{i\in F}|\langle\lambda x+(1-\lambda)y,e_i\rangle|^2. \tag{4.23}$$

The constant $\frac{1}{16}$ is the best possible constant in (4.23) in the sense that it cannot be replaced by a smaller constant.

PROOF. Using the known inequality

$$\operatorname{Re}\langle z,u\rangle\leq\frac{1}{4}\|z+u\|^2$$

we may state that for any $a,b\in H$ and $\lambda\in(0,1)$

$$\operatorname{Re}\langle a,b\rangle\leq\frac{1}{4\lambda(1-\lambda)}\|\lambda a+(1-\lambda)b\|^2. \tag{4.24}$$

Since

$$\langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle=\left\langle x-\sum_{i\in F}\langle x,e_i\rangle e_i,y-\sum_{i\in F}\langle y,e_i\rangle e_i\right\rangle,$$

for any $x,y\in H$, then, by (4.24), we get

$$\begin{aligned}
&\operatorname{Re}\left[\langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right] \\
&=\operatorname{Re}\left[\left\langle x-\sum_{i\in F}\langle x,e_i\rangle e_i,y-\sum_{i\in F}\langle y,e_i\rangle e_i\right\rangle\right] \\
&\leq\frac{1}{4\lambda(1-\lambda)}\left\|\lambda\left(x-\sum_{i\in F}\langle x,e_i\rangle e_i\right)\right. \\
&\qquad\qquad\left.+(1-\lambda)\left(y-\sum_{i\in F}\langle y,e_i\rangle e_i\right)\right\|^2 \\
&=\frac{1}{4\lambda(1-\lambda)}\left\|\lambda x+(1-\lambda)y-\sum_{i\in F}\langle\lambda x+(1-\lambda)y,e_i\rangle e_i\right\|^2 \\
&=\frac{1}{4\lambda(1-\lambda)}\left[\|\lambda x+(1-\lambda)y\|^2-\sum_{i\in F}|\langle\lambda x+(1-\lambda)y,e_i\rangle|^2\right].
\end{aligned} \tag{4.25}$$

If we apply the reverse of Bessel's inequality from Corollary 30 for $\lambda x+(1-\lambda)y$, we may state that

$$\|\lambda x+(1-\lambda)y\|^2-\sum_{i\in F}|\langle\lambda x+(1-\lambda)y,e_i\rangle|^2 \leq \frac{1}{4}M^2(\mathbf{\Phi},\boldsymbol{\phi},F)\sum_{i\in F}|\langle\lambda x+(1-\lambda)y,e_i\rangle|^2. \tag{4.26}$$

Now, by making use of (4.25) and (4.26), we deduce (4.23).

The fact that $\frac{1}{16}$ is the best possible constant in (4.23) follows by the fact that if in (4.22) we choose $x=y$, then it becomes (i) of Theorem 36, implying for $\lambda=\frac{1}{2}$ the inequality (4.9), for which, we have shown that $\frac{1}{4}$ is the best constant. ∎

REMARK 41. *If in Theorem 38, we choose $\lambda=\frac{1}{2}$, then we get*

$$\operatorname{Re}\left[\langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right]\leq\frac{1}{4}M^2(\mathbf{\Phi},\boldsymbol{\phi},F)\sum_{i\in F}\left|\left\langle\frac{x+y}{2},e_i\right\rangle\right|^2,$$

provided

$$\operatorname{Re}\left\langle\sum_{i\in F}\Phi_ie_i-\frac{x+y}{2},\frac{x+y}{2}-\sum_{i\in F}\phi_ie_i\right\rangle\geq 0$$

or, equivalently,

$$\left\|\frac{x+y}{2}-\sum_{i\in F}\frac{\Phi_i+\phi_i}{2}\cdot e_i\right\|\leq\frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}.$$

4.4. Integral Inequalities.

The following proposition holds [**9**].

PROPOSITION 30. *Let $\{f_i\}_{i\in I}$ be an orthornormal family of functions in $L^2_\rho(\Omega,\mathbb{K})$, F a finite subset of I, $\phi_i,\Phi_i\in\mathbb{K}$ $(i\in F)$ such that $\sum_{i\in F}\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)>0$ and $f\in L^2_\rho(\Omega,\mathbb{K})$, so that either*

$$\int_\Omega\rho(s)\operatorname{Re}\left[\left(\sum_{i\in F}\Phi_if_i(s)-f(s)\right)\left(\overline{f}(s)-\sum_{i\in F}\overline{\phi_i}\,\overline{f_i}(s)\right)\right]d\mu(s)\geq 0,$$

or, equivalently,

$$\int_\Omega\rho(s)\left|f(s)-\sum_{i\in F}\frac{\Phi_i+\phi_i}{2}f_i(s)\right|^2d\mu(s)\leq\frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2.$$

Then we have the inequality

$$\left(\int_{\Omega}\rho(s)|f(s)|^{2}d\mu(s)\right)^{\frac{1}{2}}\leq\frac{1}{2}\cdot\frac{1}{\left[\sum_{i\in F}\operatorname{Re}\left(\Phi_{i}\overline{\phi_{i}}\right)\right]^{\frac{1}{2}}}$$

$$\times\begin{cases}\max_{i\in F}\{|\Phi_{i}|+|\phi_{i}|\}\sum_{i\in F}\left|\int_{\Omega}\rho(s)f(s)\overline{f_{i}}(s)d\mu(s)\right| \\ \left[\sum_{i\in F}(|\Phi_{i}|+|\phi_{i}|)^{p}\right]^{\frac{1}{p}}\left(\sum_{i\in F}\left|\int_{\Omega}\rho(s)f(s)\overline{f_{i}}(s)d\mu(s)\right|^{q}\right)^{\frac{1}{q}}, \\ \qquad\qquad\text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \max_{i\in F}\left|\int_{\Omega}\rho(s)f(s)\overline{f_{i}}(s)d\mu(s)\right|\sum_{i\in F}[|\Phi_{i}|+|\phi_{i}|].\end{cases}$$

In particular, we have

$$\int_{\Omega}\rho(s)|f(s)|^{2}d\mu(s)\leq\frac{1}{4}\cdot\frac{\sum_{i\in F}(|\Phi_{i}|+|\phi_{i}|)^{2}}{\sum_{i\in F}\operatorname{Re}\left(\Phi_{i}\overline{\phi_{i}}\right)}\sum_{i\in F}\left|\int_{\Omega}\rho(s)f(s)\overline{f_{i}}(s)d\mu(s)\right|^{2}. \tag{4.27}$$

The constant $\frac{1}{4}$ *is best possible.*

The proof is obvious by Theorem 36 and Remark 38. We omit the details.

The following proposition also holds.

Proposition 31. *Assume that* f_i, f, ϕ_i, Φ_i *and* F *satisfy the assumptions of Proposition 30. Then we have the following reverse of Bessel's inequality:*

$$0\leq\int_{\Omega}\rho(s)f^{2}(s)d\mu(s)-\sum_{i\in F}\left|\int_{\Omega}\rho(s)f(s)\overline{f_{i}}(s)d\mu(s)\right|^{2}\leq\frac{1}{4}M^{2}(\boldsymbol{\Phi},\boldsymbol{\phi},F)\cdot\sum_{i\in F}\left|\int_{\Omega}\rho(s)f(s)\overline{f_{i}}(s)d\mu(s)\right|^{2}, \tag{4.28}$$

where, as above,

$$M(\boldsymbol{\Phi},\boldsymbol{\phi},F):=\left[\frac{\sum_{i\in F}\left\{(|\Phi_{i}|-|\phi_{i}|)^{2}+4\left[|\phi_{i}\Phi_{i}|-\operatorname{Re}\left(\Phi_{i}\overline{\phi_{i}}\right)\right]\right\}}{\operatorname{Re}\left(\Phi_{i}\overline{\phi_{i}}\right)}\right]^{\frac{1}{2}}. \tag{4.29}$$

The constant $\frac{1}{4}$ *is the best possible.*

The following Grüss type inequality also holds.

PROPOSITION 32. *Let* $\{f_i\}_{i\in I}$ *and* F *be as in Proposition 30. If* $\phi_i, \Phi_i, \gamma_i, \Gamma_i \in \mathbb{K}$ $(i \in F)$ *and* $f, g \in L^2_\rho(\Omega, \mathbb{K})$ *such that either*

$$\int_\Omega \rho(s)\operatorname{Re}\left[\left(\sum_{i\in F}\Phi_i f_i(s) - f(s)\right)\left(\overline{f}(s) - \sum_{i\in F}\overline{\phi_i}\,\overline{f_i}(s)\right)\right] d\mu(s) \geq 0,$$

$$\int_\Omega \rho(s)\operatorname{Re}\left[\left(\sum_{i\in F}\Gamma_i f_i(s) - g(s)\right)\left(\overline{g}(s) - \sum_{i\in F}\overline{\gamma_i}\,\overline{f_i}(s)\right)\right] d\mu(s) \geq 0,$$

or, equivalently,

$$\int_\Omega \rho(s)\left|f(s) - \sum_{i\in F}\frac{\Phi_i + \phi_i}{2}\cdot f_i(s)\right|^2 d\mu(s) \leq \frac{1}{4}\sum_{i\in F}|\Phi_i - \phi_i|^2,$$

$$\int_\Omega \rho(s)\left|g(s) - \sum_{i\in F}\frac{\Gamma_i + \gamma_i}{2}\cdot f_i(s)\right|^2 d\mu(s) \leq \frac{1}{4}\sum_{i\in F}|\Gamma_i - \gamma_i|^2,$$

hold, then we have the inequality

$$\begin{aligned}(4.30)\quad &\left|\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s)\right. \\ &\quad\left. - \sum_{i\in F}\int_\Omega \rho(s) f(s)\overline{f_i}(s)\,d\mu(s)\int_\Omega \rho(s) f_i(s)\overline{g(s)}d\mu(s)\right| \\ &\leq \frac{1}{4}M(\boldsymbol{\Phi}, \boldsymbol{\phi}, F)\,M(\boldsymbol{\Gamma}, \boldsymbol{\gamma}, F)\left(\sum_{i\in F}\left|\int_\Omega \rho(s) f(s)\overline{f_i}(s)\,d\mu(s)\right|^2\right)^{\frac{1}{2}} \\ &\qquad\times\left(\sum_{i\in F}\left|\rho(s) f_i(s)\overline{g(s)}d\mu(s)\right|^2\right)^{\frac{1}{2}},\end{aligned}$$

where $M(\boldsymbol{\Phi}, \boldsymbol{\phi}, F)$ *is as defined in (4.29).*

The constant $\frac{1}{4}$ *is the best possible.*

The proof follows by Theorem 37 and we omit the details.

In the case of real spaces, the following corollaries provide much simpler sufficient conditions for the reverse of Bessel's inequality (4.28) or for the Grüss type inequality (4.30) to hold.

COROLLARY 34. *Let* $\{f_i\}_{i\in I}$ *be an orthornormal family of functions in the real Hilbert space* $L^2_\rho(\Omega)$, F *a finite part of* I, $M_i, m_i \geq 0$

$(i \in F)$, with $\sum_{i\in F} M_i m_i > 0$ and $f \in L^2_\rho(\Omega)$ so that

$$\sum_{i\in F} m_i f_i(s) \le f(s) \le \sum_{i\in F} M_i f_i(s) \quad \text{for } \mu-a.e.\ s\in\Omega.$$

Then we have the inequality

$$\begin{aligned} 0 &\le \int_\Omega \rho(s) f^2(s)\,d\mu(s) - \sum_{i\in F}\left[\int_\Omega \rho(s) f(s) f_i(s)\,d\mu(s)\right]^2 \\ &\le \frac{1}{4}\cdot\frac{\sum_{i\in F}(M_i-m_i)^2}{\sum_{i\in F} M_i m_i}\cdot\sum_{i\in F}\left[\int_\Omega \rho(s) f(s) f_i(s)\,d\mu(s)\right]^2. \end{aligned}$$

The constant $\frac{1}{4}$ is best possible.

Corollary 35. *Let $\{f_i\}_{i\in I}$ and F be as above. If $M_i, m_i, N_i, n_i \ge 0$ $(i\in F)$ with $\sum_{i\in F} M_i m_i, \sum_{i\in F} N_i n_i > 0$ and $f, g \in L^2_\rho(\Omega)$ such that*

$$\sum_{i\in F} m_i f_i(s) \le f(s) \le \sum_{i\in F} M_i f_i(s)$$

and

$$\sum_{i\in F} n_i f_i(s) \le g(s) \le \sum_{i\in F} N_i f_i(s) \quad \text{for } \mu-a.e.\ s\in\Omega,$$

then we have the inequality

$$\begin{aligned} &\left|\int_\Omega \rho(s) f(s) g(s)\,d\mu(s)\right. \\ &\left. - \sum_{i\in F}\int_\Omega \rho(s) f(s) f_i(s)\,d\mu(s)\int_\Omega \rho(s) g(s) f_i(s)\,d\mu(s)\right| \\ &\le \frac{1}{4}\left(\frac{\sum_{i\in F}(M_i-m_i)^2}{\sum_{i\in F} M_i m_i}\right)^{\frac{1}{2}}\left(\frac{\sum_{i\in F}(N_i-n_i)^2}{\sum_{i\in F} N_i n_i}\right)^{\frac{1}{2}} \\ &\times\left[\sum_{i\in F}\left(\int_\Omega \rho(s) f(s) f_i(s)\,d\mu(s)\right)^2\sum_{i\in F}\left(\int_\Omega \rho(s) g(s) f_i(s)\,d\mu(s)\right)^2\right]^{\frac{1}{2}}. \end{aligned}$$

5. General Reverses of Bessel's Inequality

5.1. Some Reverses of Bessel's Inequality. Let $(H;\langle\cdot,\cdot\rangle)$ be a real or complex infinite dimensional Hilbert space and $(e_i)_{i\in\mathbb{N}}$ an orthornormal family in H, i.e., we recall that $\langle e_i, e_j\rangle = 0$ if $i,j\in\mathbb{N}$, $i\ne j$ and $\|e_i\| = 1$ for $i\in\mathbb{N}$.

It is well known that, if $x \in H$, then the sum $\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2$ is convergent and the following inequality, called *Bessel's inequality*

$$\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \le \|x\|^2, \tag{5.1}$$

holds.

If $\ell^2(\mathbb{K}) := \left\{ \mathbf{a} = (a_i)_{i\in\mathbb{N}} \subset \mathbb{K} \,\middle|\, \sum_{i=1}^{\infty} |a_i|^2 < \infty \right\}$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, is the Hilbert space of all complex or real sequences that are 2-summable and $\boldsymbol{\lambda} = (\lambda_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{K})$, then the sum $\sum_{i=1}^{\infty} \lambda_i e_i$ is convergent in H and if $y := \sum_{i=1}^{\infty} \lambda_i e_i \in H$, then $\|y\| = \left(\sum_{i=1}^{\infty} |\lambda_i|^2\right)^{\frac{1}{2}}$.

We may state the following result [**7**].

Theorem 39. *Let $(H; \langle\cdot,\cdot\rangle)$ be an infinite dimensional Hilbert space over the real or complex number field $\mathbb{K}$, $(e_i)_{i\in\mathbb{N}}$ an orthornormal family in H, $\boldsymbol{\lambda} = (\lambda_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{K})$ and $r > 0$ with the property that*

$$\sum_{i=1}^{\infty} |\lambda_i|^2 > r^2.$$

If $x \in H$ is such that

$$\left\| x - \sum_{i=1}^{\infty} \lambda_i e_i \right\| \le r,$$

then we have the inequality

$$\begin{aligned} \|x\|^2 &\le \frac{\left(\sum_{i=1}^{\infty} \operatorname{Re}\left[\overline{\lambda_i}\langle x, e_i\rangle\right]\right)^2}{\sum_{i=1}^{\infty} |\lambda_i|^2 - r^2} \\ &\le \frac{\left|\sum_{i=1}^{\infty} \overline{\lambda_i}\langle x, e_i\rangle\right|^2}{\sum_{i=1}^{\infty} |\lambda_i|^2 - r^2} \\ &\le \frac{\sum_{i=1}^{\infty} |\lambda_i|^2}{\sum_{i=1}^{\infty} |\lambda_i|^2 - r^2} \sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} 0 &\le \|x\|^2 - \sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 \\ &\le \frac{r^2}{\sum_{i=1}^{\infty} |\lambda_i|^2 - r^2} \sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2. \end{aligned} \tag{5.3}$$

PROOF. Applying the third inequality in (4.9) for $a = \sum_{i=1}^{\infty} \lambda_i e_i \in H$, we have

$$\|x\|^2 \left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\|^2 - \left[\operatorname{Re} \left\langle x, \sum_{i=1}^{\infty} \lambda_i e_i \right\rangle \right]^2 \leq r^2 \|x\|^2 \tag{5.4}$$

and since

$$\left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\|^2 = \sum_{i=1}^{\infty} |\lambda_i|^2,$$
$$\operatorname{Re} \left\langle x, \sum_{i=1}^{\infty} \lambda_i e_i \right\rangle = \sum_{i=1}^{\infty} \operatorname{Re} \left[\overline{\lambda_i} \langle x, e_i \rangle \right],$$

then, by (5.4), we deduce

$$\|x\|^2 \sum_{i=1}^{\infty} |\lambda_i|^2 - \left[\operatorname{Re} \left\langle x, \sum_{i=1}^{\infty} \lambda_i e_i \right\rangle \right]^2 \leq r^2 \|x\|^2,$$

giving the first inequality in (5.2).

The second inequality is obvious by the modulus property.

The last inequality follows by the Cauchy-Bunyakovsky-Schwarz inequality

$$\left| \sum_{i=1}^{\infty} \overline{\lambda_i} \langle x, e_i \rangle \right|^2 \leq \sum_{i=1}^{\infty} |\lambda_i|^2 \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2.$$

The inequality (5.3) follows by the last inequality in (5.2) on subtracting from both sides the quantity $\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 < \infty$. ∎

The following result provides a generalization for the reverse of Bessel's inequality obtained in [**9**].

THEOREM 40. *Let* $(H; \langle \cdot, \cdot \rangle)$ *and* $(e_i)_{i \in \mathbb{N}}$ *be as in Theorem 39. Suppose that* $\mathbf{\Gamma} = (\Gamma_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{K})$, $\boldsymbol{\gamma} = (\gamma_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{K})$ *are sequences of real or complex numbers such that*

$$\sum_{i=1}^{\infty} \operatorname{Re}\left(\Gamma_i \overline{\gamma_i}\right) > 0.$$

If $x \in H$ *is such that either*

$$\left\| x - \sum_{i=1}^{\infty} \frac{\Gamma_i + \gamma_i}{2} e_i \right\| \leq \frac{1}{2} \left(\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \tag{5.5}$$

or, equivalently,

$$\operatorname{Re}\left\langle \sum_{i=1}^{\infty}\Gamma_i e_i - x, x - \sum_{i=1}^{\infty}\gamma_i e_i \right\rangle \geq 0 \tag{5.6}$$

holds, then we have the inequalities

$$\begin{aligned} \|x\|^2 &\leq \frac{1}{4}\cdot\frac{\left(\sum_{i=1}^{\infty}\operatorname{Re}\left[\left(\overline{\Gamma_i}+\overline{\gamma_i}\right)\langle x, e_i\rangle\right]\right)^2}{\sum_{i=1}^{\infty}\operatorname{Re}\left(\Gamma_i\overline{\gamma_i}\right)} \\ &\leq \frac{1}{4}\cdot\frac{\left|\sum_{i=1}^{\infty}\left(\overline{\Gamma_i}+\overline{\gamma_i}\right)\langle x, e_i\rangle\right|^2}{\sum_{i=1}^{\infty}\operatorname{Re}\left(\Gamma_i\overline{\gamma_i}\right)} \\ &\leq \frac{1}{4}\cdot\frac{\sum_{i=1}^{\infty}\left|\Gamma_i+\gamma_i\right|^2}{\sum_{i=1}^{\infty}\operatorname{Re}\left(\Gamma_i\overline{\gamma_i}\right)}\sum_{i=1}^{\infty}\left|\langle x, e_i\rangle\right|^2. \end{aligned} \tag{5.7}$$

The constant $\frac{1}{4}$ *is best possible in all inequalities in (5.7).*

We also have the inequalities:

$$0 \leq \|x\|^2 - \sum_{i=1}^{\infty}\left|\langle x, e_i\rangle\right|^2 \leq \frac{1}{4}\cdot\frac{\sum_{i=1}^{\infty}\left|\Gamma_i-\gamma_i\right|^2}{\sum_{i=1}^{\infty}\operatorname{Re}\left(\Gamma_i\overline{\gamma_i}\right)}\sum_{i=1}^{\infty}\left|\langle x, e_i\rangle\right|^2. \tag{5.8}$$

Here the constant $\frac{1}{4}$ *is also best possible.*

Proof. Since $\mathbf{\Gamma}, \boldsymbol{\gamma} \in \ell^2(\mathbb{K})$, then also $\frac{1}{2}(\mathbf{\Gamma} \pm \boldsymbol{\gamma}) \in \ell^2(\mathbb{K})$, showing that the series

$$\sum_{i=1}^{\infty}\left|\frac{\Gamma_i+\gamma_i}{2}\right|^2, \ \sum_{i=1}^{\infty}\left|\frac{\Gamma_i-\gamma_i}{2}\right|^2 \text{ and } \sum_{i=1}^{\infty}\operatorname{Re}\left(\Gamma_i\overline{\gamma_i}\right)$$

are convergent. Also, the series

$$\sum_{i=1}^{\infty}\Gamma_i e_i, \ \sum_{i=1}^{\infty}\gamma_i e_i \text{ and } \sum_{i=1}^{\infty}\frac{\gamma_i+\Gamma_i}{2}e_i$$

are convergent in the Hilbert space H.

Now, we observe that the inequalities (5.7) and (5.8) follow from Theorem 39 on choosing $\lambda_i = \frac{\gamma_i+\Gamma_i}{2}$, $i \in \mathbb{N}$ and $r = \frac{1}{2}\left(\sum_{i=1}^{\infty}\left|\Gamma_i-\gamma_i\right|^2\right)^{\frac{1}{2}}$.

The fact that $\frac{1}{4}$ is the best constant in both (5.7) and (5.8) follows from Theorem 6 and Corollary 11, and we omit the details. ∎

For some recent results related to the Bessel inequality, see [**1**], [**4**], [**10**], and [**11**].

5.2. Some Grüss Type Inequalities for Orthonormal Families. The following result related to the Grüss inequality in inner product spaces holds [**7**].

THEOREM 41. *Let $(H; \langle \cdot, \cdot \rangle)$ be an infinite dimensional Hilbert space over the real or complex number field $\mathbb{K}$, and $(e_i)_{i \in \mathbb{N}}$ an orthornormal family in H. Assume that $\boldsymbol{\lambda} = (\lambda_i)_{i \in \mathbb{N}}$, $\boldsymbol{\mu} = (\mu_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{K})$ and $r_1, r_2 > 0$ with the properties that*

$$\sum_{i=1}^{\infty} |\lambda_i|^2 > r_1^2, \quad \sum_{i=1}^{\infty} |\mu_i|^2 > r_2^2.$$

If $x, y \in H$ are such that

$$\left\| x - \sum_{i=1}^{\infty} \lambda_i e_i \right\| \le r_1, \qquad \left\| y - \sum_{i=1}^{\infty} \mu_i e_i \right\| \le r_2,$$

then we have the inequalities

$$(5.9) \qquad \left| \langle x, y \rangle - \sum_{i=1}^{\infty} \langle x, e_i \rangle \langle e_i, y \rangle \right| \le \frac{r_1 r_2}{\sqrt{\sum_{i=1}^{\infty} |\lambda_i|^2 - r_1^2} \sqrt{\sum_{i=1}^{\infty} |\mu_i|^2 - r_2^2}}$$

$$\times \sqrt{\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \sum_{i=1}^{\infty} |\langle y, e_i \rangle|^2}$$
$$\leq \frac{r_1 r_2 \|x\| \|y\|}{\sqrt{\sum_{i=1}^{\infty} |\lambda_i|^2 - r_1^2}\sqrt{\sum_{i=1}^{\infty} |\mu_i|^2 - r_2^2}}.$$

Proof. Applying Schwarz's inequality for the vectors

$$x - \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i, \qquad y - \sum_{i=1}^{\infty} \langle y, e_i \rangle e_i,$$

we have

$$\left| \left\langle x - \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i, y - \sum_{i=1}^{\infty} \langle y, e_i \rangle e_i \right\rangle \right|^2 \leq \left\| x - \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \right\|^2 \left\| y - \sum_{i=1}^{\infty} \langle y, e_i \rangle e_i \right\|^2. \tag{5.10}$$

Since

$$\left\langle x - \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i, y - \sum_{i=1}^{\infty} \langle y, e_i \rangle e_i \right\rangle = \langle x, y \rangle - \sum_{i=1}^{\infty} \langle x, e_i \rangle \langle e_i, y \rangle$$

and

$$\left\| x - \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \right\|^2 = \|x\|^2 - \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2,$$

then by (5.3) applied for x and y, and from (5.10), we deduce the first part of (5.9).

The second part follows by Bessel's inequality. ∎

The following Grüss type inequality may be stated as well.

Theorem 42. *Let $(H; \langle \cdot, \cdot \rangle)$ be an infinite dimensional Hilbert space and $(e_i)_{i \in \mathbb{N}}$ an orthornormal family in H. Suppose that $(\Gamma_i)_{i \in \mathbb{N}}$, $(\gamma_i)_{i \in \mathbb{N}}$, $(\phi_i)_{i \in \mathbb{N}}$, $(\Phi_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{K})$ are sequences of real and complex numbers such that*

$$\sum_{i=1}^{\infty} \operatorname{Re}(\Gamma_i \overline{\gamma_i}) > 0, \quad \sum_{i=1}^{\infty} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right) > 0.$$

If $x, y \in H$ are such that either

$$\left\| x - \sum_{i=1}^{\infty} \frac{\Gamma_i + \gamma_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left(\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}},$$

$$\left\| y - \sum_{i=1}^{\infty} \frac{\Phi_i + \phi_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left(\sum_{i=1}^{\infty} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}},$$

or, equivalently,

$$\operatorname{Re} \left\langle \sum_{i=1}^{\infty} \Gamma_i e_i - x, x - \sum_{i=1}^{\infty} \gamma_i e_i \right\rangle \geq 0,$$

$$\operatorname{Re} \left\langle \sum_{i=1}^{\infty} \Phi_i e_i - y, y - \sum_{i=1}^{\infty} \phi_i e_i \right\rangle \geq 0,$$

holds, then we have the inequality

$$\begin{aligned}
&\left| \langle x, y \rangle - \sum_{i=1}^{\infty} \langle x, e_i \rangle \langle e_i, y \rangle \right| \\
&\leq \frac{1}{4} \cdot \frac{\left(\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}}{\left(\sum_{i=1}^{\infty} \operatorname{Re} \left(\Gamma_i \overline{\gamma_i} \right) \right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} \operatorname{Re} \left(\Phi_i \overline{\phi_i} \right) \right)^{\frac{1}{2}}} \\
&\qquad \times \left(\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |\langle y, e_i \rangle|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{1}{4} \cdot \frac{\left(\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}}{\left[\sum_{i=1}^{\infty} \operatorname{Re} \left(\Gamma_i \overline{\gamma_i} \right) \right]^{\frac{1}{2}} \left[\sum_{i=1}^{\infty} \operatorname{Re} \left(\Phi_i \overline{\phi_i} \right) \right]^{\frac{1}{2}}} \|x\| \|y\|.
\end{aligned}$$

The constant $\frac{1}{4}$ is best possible in the first inequality.

PROOF. Follows by (5.8) and (5.10). The best constant follows from Theorem 22, and we omit the details. ∎

5.3. Other Reverses of Bessel's Inequality. We may state the following result [**5**].

THEOREM 43. *Let $(H; \langle \cdot, \cdot \rangle)$ be an infinite dimensional Hilbert space over the real or complex number field $\mathbb{K}$, $(e_i)_{i \in \mathbb{N}}$ an orthornormal family in H, $\boldsymbol{\lambda} = (\lambda_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{K})$, $\boldsymbol{\lambda} \neq 0$ and $r > 0$. If $x \in H$ is such that*

$$\left\| x - \sum_{i=1}^{\infty} \lambda_i e_i \right\| \leq r,$$

then we have the inequality

$$0 \le \|x\| - \left(\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} \le \frac{1}{2} \cdot \frac{r^2}{\left(\sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}}}. \tag{5.11}$$

The constant $\frac{1}{2}$ is best possible in (5.11) in the sense that it cannot be replaced by a smaller constant.

Proof. Let $a := \sum_{i=1}^{\infty} \lambda_i e_i \in H$. Then by Theorem 7, we have

$$\|x\| \left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\| - \left| \sum_{i=1}^{\infty} \bar{\lambda}_i \langle x, e_i \rangle \right| \le \frac{1}{2} r^2,$$

giving

$$\|x\| \left(\sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}} \le \frac{1}{2} r^2 + \left| \sum_{i=1}^{\infty} \bar{\lambda}_i \langle x, e_i \rangle \right|, \tag{5.12}$$

since

$$\left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\| = \left(\sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}}.$$

Using the Cauchy-Bunyakovsky-Schwarz inequality, we may state that

$$\left| \sum_{i=1}^{\infty} \bar{\lambda}_i \langle x, e_i \rangle \right| \le \left(\sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}}, \tag{5.13}$$

and thus, by (5.12) and (5.13), we may state that

$$\|x\| \left(\sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}} \le \frac{1}{2} r^2 + \left(\sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}},$$

from where we get the desired inequality in (5.11).

The best constant, follows by Theorem 7 on choosing $(e_i)_{i \in \mathbb{N}} = \{e\}$, with $\|e\| = 1$ and we omit the details. ∎

Remark 42. *Under the assumptions of Theorem 43, and if we multiply by $\|x\| + \left(\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} > 0$, then we deduce, from (5.11),* *that*

$$0 \leq \|x\|^2 - \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \tag{5.14}$$
$$\leq \frac{1}{2} \cdot \frac{r^2 \left(\|x\| + \left(\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} \right)}{\left(\sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}}}$$
$$\leq \frac{r^2 \|x\|}{\left(\sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}}},$$

where, for the last inequality, we have used Bessel's inequality

$$\left(\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} \leq \|x\|, \qquad x \in H.$$

The following result also holds [**5**].

Theorem 44. *Assume that $(H; \langle \cdot, \cdot \rangle)$ and $(e_i)_{i\in\mathbb{N}}$ are as in Theorem 43. If $\mathbf{\Gamma} = (\Gamma_i)_{i\in\mathbb{N}}$, $\boldsymbol{\gamma} = (\gamma_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{K})$, with $\mathbf{\Gamma} \neq -\boldsymbol{\gamma}$, and $x \in H$ with the property that, either*

$$\left\| x - \sum_{i=1}^{\infty} \frac{\Gamma_i + \gamma_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left(\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}},$$

or, equivalently,

$$\operatorname{Re} \left\langle \sum_{i=1}^{\infty} \Gamma_i e_i - x, x - \sum_{i=1}^{\infty} \gamma_i e_i \right\rangle \geq 0,$$

holds, then we have the inequality

$$0 \leq \|x\| - \left(\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} \leq \frac{1}{4} \cdot \frac{\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2}{\left(\sum_{i=1}^{\infty} |\Gamma_i + \gamma_i|^2 \right)^{\frac{1}{2}}}. \tag{5.15}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Since $\mathbf{\Gamma}, \boldsymbol{\gamma} \in \ell^2(\mathbb{K})$, then we have that $\frac{1}{2}(\mathbf{\Gamma} \pm \boldsymbol{\gamma}) \in \ell^2(\mathbb{K})$, showing that the series

$$\sum_{i=1}^{\infty} \left| \frac{\Gamma_i + \gamma_i}{2} \right|^2, \quad \sum_{i=1}^{\infty} \left| \frac{\Gamma_i - \gamma_i}{2} \right|^2$$

are convergent. In addition, the series $\sum_{i=1}^{\infty} \Gamma_i e_i$, $\sum_{i=1}^{\infty} \gamma_i e_i$ and $\sum_{i=1}^{\infty} \frac{\Gamma_i + \gamma_i}{2} e_i$ are also convergent in the Hilbert space H.

Now, we observe that the inequality (5.15) follows from Theorem 43 on choosing $\lambda_i = \frac{\Gamma_i+\gamma_i}{2}$, $i \in \mathbb{N}$ and $r = \frac{1}{2}\left(\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2\right)^{\frac{1}{2}}$.

The fact that $\frac{1}{4}$ is the best possible constant in (5.15) follows from Theorem 8, and we omit the details. ∎

Remark 43. *With the assumptions of Theorem 44, we have*

$$0 \le \|x\|^2 - \sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 \le \frac{1}{4} \cdot \frac{\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2}{\left(\sum_{i=1}^{\infty} |\Gamma_i + \gamma_i|^2\right)^{\frac{1}{2}}} \left[\|x\| + \left(\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2\right)^{\frac{1}{2}}\right] \le \frac{1}{2} \cdot \frac{\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2}{\left(\sum_{i=1}^{\infty} |\Gamma_i + \gamma_i|^2\right)^{\frac{1}{2}}} \|x\|. \tag{5.16}$$

5.4. More Grüss Type Inequalities for Orthonormal Families. The following result holds [**5**].

Theorem 45. *Let $(H; \langle\cdot,\cdot\rangle)$ be an infinite dimensional Hilbert space over the real or complex number field $\mathbb{K}$ and $(e_i)_{i\in\mathbb{N}}$ an orthornormal family in H. If $\boldsymbol{\lambda} = (\lambda_i)_{i\in\mathbb{N}}$, $\boldsymbol{\mu} = (\mu_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{K})$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu} \neq 0$, $r_1, r_2 > 0$ and $x, y \in H$ are such that*

$$\left\|x - \sum_{i=1}^{\infty} \lambda_i e_i\right\| \le r_1, \quad \left\|y - \sum_{i=1}^{\infty} \mu_i e_i\right\| \le r_2,$$

then we have the inequality

$$\left|\langle x, y\rangle - \sum_{i=1}^{\infty} \langle x, e_i\rangle \langle e_i, y\rangle\right| \le \frac{1}{2} r_1 r_2 \frac{\left[\|x\| + \left(\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \left[\|y\| + \left(\sum_{i=1}^{\infty} |\langle y, e_i\rangle|^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}}{\left(\sum_{i=1}^{\infty} |\lambda_i|^2\right)^{\frac{1}{4}} \left(\sum_{i=1}^{\infty} |\mu_i|^2\right)^{\frac{1}{4}}} \le r_1 r_2 \frac{\|x\|^{\frac{1}{2}} \|y\|^{\frac{1}{2}}}{\left(\sum_{i=1}^{\infty} |\lambda_i|^2\right)^{\frac{1}{4}} \left(\sum_{i=1}^{\infty} |\mu_i|^2\right)^{\frac{1}{4}}}.$$

Proof. It follows by (5.14) applied for x and y. We omit the details. ∎

Finally we may state the following theorem [**5**].

THEOREM 46. *Assume that* $(H;\langle\cdot,\cdot\rangle)$ *and* $(e_i)_{i\in\mathbb{N}}$ *are as in Theorem 45. If* $\mathbf{\Gamma}=(\Gamma_i)_{i\in\mathbb{N}}$, $\mathbf{\Gamma}=(\Gamma_i)_{i\in\mathbb{N}}$, $\phi=(\boldsymbol{\phi}_i)_{i\in\mathbb{N}}$, $\mathbf{\Phi}=(\Phi_i)_{i\in\mathbb{N}}\in\ell^2(\mathbb{K})$, *with* $\mathbf{\Gamma}\neq-\boldsymbol{\gamma}$, $\mathbf{\Phi}\neq-\boldsymbol{\phi}$, *and* $x,y\in H$ *are such that, either*

$$\left\|x-\sum_{i=1}^{\infty}\frac{\Gamma_i+\gamma_i}{2}\cdot e_i\right\|\leq\frac{1}{2}\left(\sum_{i=1}^{\infty}\left|\Gamma_i-\gamma_i\right|^2\right)^{\frac{1}{2}},$$

$$\left\|y-\sum_{i=1}^{\infty}\frac{\Phi_i+\phi_i}{2}\cdot e_i\right\|\leq\frac{1}{2}\left(\sum_{i=1}^{\infty}\left|\Phi_i-\phi_i\right|^2\right)^{\frac{1}{2}},$$

or, equivalently,

$$\operatorname{Re}\left\langle\sum_{i=1}^{\infty}\Gamma_ie_i-x,x-\sum_{i=1}^{\infty}\gamma_ie_i\right\rangle\geq0,$$

$$\operatorname{Re}\left\langle\sum_{i=1}^{\infty}\Phi_ie_i-y,y-\sum_{i=1}^{\infty}\phi_ie_i\right\rangle\geq0,$$

holds, then we have the inequality

$$\begin{aligned}&\left|\langle x,y\rangle-\sum_{i=1}^{\infty}\langle x,e_i\rangle\langle e_i,y\rangle\right|\\&\leq\frac{1}{4}\cdot\left(\sum_{i=1}^{\infty}\left|\Phi_i-\phi_i\right|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{\infty}\left|\Gamma_i-\gamma_i\right|^2\right)^{\frac{1}{2}}\\&\qquad\times\frac{\left[\|x\|+\left(\sum_{i=1}^{\infty}\left|\langle x,e_i\rangle\right|^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}\left[\|y\|+\left(\sum_{i=1}^{\infty}\left|\langle y,e_i\rangle\right|^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}}{\left(\sum_{i=1}^{\infty}\left|\Phi_i+\phi_i\right|^2\right)^{\frac{1}{4}}\left(\sum_{i=1}^{\infty}\left|\Gamma_i+\gamma_i\right|^2\right)^{\frac{1}{4}}}\\&\leq\frac{1}{2}\cdot\frac{\left(\sum_{i=1}^{\infty}\left|\Phi_i-\phi_i\right|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{\infty}\left|\Gamma_i-\gamma_i\right|^2\right)^{\frac{1}{2}}}{\left(\sum_{i=1}^{\infty}\left|\Phi_i+\phi_i\right|^2\right)^{\frac{1}{4}}\left(\sum_{i=1}^{\infty}\left|\Gamma_i+\gamma_i\right|^2\right)^{\frac{1}{4}}}\|x\|^{\frac{1}{2}}\|y\|^{\frac{1}{2}}.\end{aligned}$$

The proof follow by (5.16) applied for x and y. We omit the details.

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Part 2

Other Inequalities in Inner Product Spaces

CHAPTER 4

Generalisations of Bessel's Inequality

1. Boas-Bellman Type Inequalities

1.1. Introduction. Let $(H;(\cdot,\cdot))$ be an inner product space over the real or complex number field $\mathbb{K}$. If $(e_i)_{1\leq i\leq n}$ are orthonormal vectors in the inner product space H, i.e., $(e_i,e_j)=\delta_{ij}$ for all $i,j\in\{1,\dots,n\}$, where δ_{ij} is the Kronecker delta, then we have the following inequality that is well known in the literature as Bessel's inequality (see for example [**11**, p. 391]):

$$\sum_{i=1}^{n}|(x,e_i)|^2\leq\|x\|^2\quad\text{for any}\quad x\in H.$$

For other results related to Bessel's inequality, see [**8**]-[**9**]and Chapter XV in the book [**11**].

In 1941, R.P. Boas [**2**] and in 1944, independently, R. Bellman [**1**] proved the following generalisation of Bessel's inequality (see also [**11**, p. 392]).

Theorem 47. *If $x,y_1,\dots,y_n$ are vectors in an inner product space $(H;(\cdot,\cdot))$, then the following inequality:*

$$\sum_{i=1}^{n}|(x,y_i)|^2\leq\|x\|^2\left[\max_{1\leq i\leq n}\|y_i\|^2+\left(\sum_{1\leq i\neq j\leq n}|(y_i,y_j)|^2\right)^{\frac{1}{2}}\right],\tag{1.1}$$

holds.

A recent generalisation of the Boas-Bellman result was given in Mitrinović-Pečarić-Fink [**11**, p. 392] where they proved the following:

Theorem 48. *If $x,y_1,\dots,y_n$ are as in Theorem 47 and $c_1,\dots,c_n\in\mathbb{K}$, then one has the inequality:*

$$\begin{aligned}&\left|\sum_{i=1}^{n}c_i\,(x,y_i)\right|^2\\&\qquad\leq\|x\|^2\sum_{i=1}^{n}|c_i|^2\left[\max_{1\leq i\leq n}\|y_i\|^2+\left(\sum_{1\leq i\neq j\leq n}|(y_i,y_j)|^2\right)^{\frac{1}{2}}\right].\end{aligned}\tag{1.2}$$

They also noted that if in (1.2) one chooses $c_i = \overline{(x, y_i)}$, then this inequality becomes (1.1).

For other results related to the Boas-Bellman inequality, see [**7**].

In this section, by following [**5**], we point out some new results that may be related to both the Mitrinović-Pečarić-Fink and Boas-Bellman inequalities.

1.2. Preliminary Results. We start with the following lemma which is also interesting in itself [**5**].

LEMMA 6. *Let $z_1, \dots, z_n \in H$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. Then one has the inequality:*

$$\left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \tag{1.3}$$

$$\leq \begin{cases} \max\limits_{1\leq i\leq n} |\alpha_i|^2 \sum\limits_{i=1}^{n} \|z_i\|^2 ; \\ \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2\alpha} \right)^{\frac{1}{\alpha}} \left(\sum\limits_{i=1}^{n} \|z_i\|^{2\beta} \right)^{\frac{1}{\beta}}, \quad \text{where } \alpha > 1, \\ \qquad\qquad\qquad\qquad\qquad\qquad \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum\limits_{i=1}^{n} |\alpha_i|^2 \max\limits_{1\leq i\leq n} \|z_i\|^2 , \end{cases}$$

$$+ \begin{cases} \max\limits_{1\leq i\neq j\leq n} \{|\alpha_i \alpha_j|\} \sum\limits_{1\leq i\neq j\leq n} |(z_i, z_j)| ; \\ \left[\left(\sum\limits_{i=1}^{n} |\alpha_i|^{\gamma} \right)^2 - \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2\gamma} \right) \right]^{\frac{1}{\gamma}} \left(\sum\limits_{1\leq i\neq j\leq n} |(z_i, z_j)|^{\delta} \right)^{\frac{1}{\delta}}, \\ \qquad\qquad\qquad\qquad \text{where } \gamma > 1, \quad \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left[\left(\sum\limits_{i=1}^{n} |\alpha_i| \right)^2 - \sum\limits_{i=1}^{n} |\alpha_i|^2 \right] \max\limits_{1\leq i\neq j\leq n} |(z_i, z_j)| . \end{cases}$$

PROOF. We observe that

$$\begin{aligned} \left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 &= \left(\sum_{i=1}^{n} \alpha_i z_i, \sum_{j=1}^{n} \alpha_j z_j \right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} (z_i, z_j) = \left| \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} (z_i, z_j) \right| \\ &\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_i| |\overline{\alpha_j}| |(z_i, z_j)| \end{aligned} \tag{1.4}$$

$$= \sum_{i=1}^{n} |\alpha_i|^2 \|z_i\|^2 + \sum_{1\le i\ne j\le n} |\alpha_i| |\alpha_j| |(z_i, z_j)| .$$

Using Hölder's inequality, we may write that

$$\sum_{i=1}^{n} |\alpha_i|^2 \|z_i\|^2 \tag{1.5}$$
$$\le \begin{cases} \max\limits_{1\le i\le n} |\alpha_i|^2 \sum\limits_{i=1}^{n} \|z_i\|^2 ; \\[2ex] \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2\alpha}\right)^{\frac{1}{\alpha}} \left(\sum\limits_{i=1}^{n} \|z_i\|^{2\beta}\right)^{\frac{1}{\beta}}, \quad \text{where } \ \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\[2ex] \sum\limits_{i=1}^{n} |\alpha_i|^2 \max\limits_{1\le i\le n} \|z_i\|^2 . \end{cases}$$

By Hölder's inequality for double sums we also have

$$\sum_{1\le i\ne j\le n} |\alpha_i| |\alpha_j| |(z_i, z_j)| \tag{1.6}$$
$$\le \begin{cases} \max\limits_{1\le i\ne j\le n} |\alpha_i\alpha_j| \sum\limits_{1\le i\ne j\le n} |(z_i, z_j)| ; \\[2ex] \left(\sum\limits_{1\le i\ne j\le n} |\alpha_i|^{\gamma} |\alpha_j|^{\gamma}\right)^{\frac{1}{\gamma}} \left(\sum\limits_{1\le i\ne j\le n} |(z_i, z_j)|^{\delta}\right)^{\frac{1}{\delta}}, \\ \qquad\qquad \text{where } \ \gamma > 1, \quad \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\[2ex] \sum\limits_{1\le i\ne j\le n} |\alpha_i| |\alpha_j| \max\limits_{1\le i\ne j\le n} |(z_i, z_j)| , \end{cases}$$
$$= \begin{cases} \max\limits_{1\le i\ne j\le n} \{|\alpha_i\alpha_j|\} \sum\limits_{1\le i\ne j\le n} |(z_i, z_j)| ; \\[2ex] \left[\left(\sum\limits_{i=1}^{n} |\alpha_i|^{\gamma}\right)^2 - \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2\gamma}\right)\right]^{\frac{1}{\gamma}} \left(\sum\limits_{1\le i\ne j\le n} |(z_i, z_j)|^{\delta}\right)^{\frac{1}{\delta}}, \\ \qquad\qquad \text{where } \ \gamma > 1, \quad \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\[2ex] \left[\left(\sum\limits_{i=1}^{n} |\alpha_i|\right)^2 - \sum\limits_{i=1}^{n} |\alpha_i|^2\right] \max\limits_{1\le i\ne j\le n} |(z_i, z_j)| . \end{cases}$$

Utilising (1.5) and (1.6) in (1.4), we may deduce the desired result (1.3). ∎

REMARK 44. *Inequality (1.3) contains in fact 9 different inequalities which may be obtained by combining the first 3 ones with the last 3 ones.*

A particular case that may be related to the Boas-Bellman result is embodied in the following inequality [**5**].

COROLLARY 36. *With the assumptions in Lemma 6, we have*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \tag{1.7}$$

$$\leq \sum_{i=1}^{n}|\alpha_i|^2$$

$$\cdot\left\{\max_{1\leq i\leq n}\|z_i\|^2+\frac{\left[\left(\sum_{i=1}^{n}|\alpha_i|^2\right)^2-\sum_{i=1}^{n}|\alpha_i|^4\right]^{\frac{1}{2}}}{\sum_{i=1}^{n}|\alpha_i|^2}\left(\sum_{1\leq i\neq j\leq n}|(z_i,z_j)|^2\right)^{\frac{1}{2}}\right\}$$

$$\leq \sum_{i=1}^{n}|\alpha_i|^2\left\{\max_{1\leq i\leq n}\|z_i\|^2+\left(\sum_{1\leq i\neq j\leq n}|(z_i,z_j)|^2\right)^{\frac{1}{2}}\right\}.$$

The first inequality follows by taking the third branch in the first curly bracket with the second branch in the second curly bracket for $\gamma=\delta=2$.

The second inequality in (1.7) follows by the fact that

$$\left[\left(\sum_{i=1}^{n}|\alpha_i|^2\right)^2-\sum_{i=1}^{n}|\alpha_i|^4\right]^{\frac{1}{2}}\leq\sum_{i=1}^{n}|\alpha_i|^2.$$

Applying the following Cauchy-Bunyakovsky-Schwarz type inequality

$$\left(\sum_{i=1}^{n}a_i\right)^2\leq n\sum_{i=1}^{n}a_i^2,\quad a_i\in\mathbb{R}_+,\quad 1\leq i\leq n,$$

we may write that

$$\left(\sum_{i=1}^{n}|\alpha_i|^{\gamma}\right)^2-\sum_{i=1}^{n}|\alpha_i|^{2\gamma}\leq(n-1)\sum_{i=1}^{n}|\alpha_i|^{2\gamma}\qquad(n\geq1) \tag{1.8}$$

and

$$\left(\sum_{i=1}^{n}|\alpha_i|\right)^2-\sum_{i=1}^{n}|\alpha_i|^2\leq(n-1)\sum_{i=1}^{n}|\alpha_i|^2\qquad(n\geq1)\,. \tag{1.9}$$

Also, it is obvious that:

$$\max_{1\leq i\neq j\leq n}\{|\alpha_i\alpha_j|\}\leq\max_{1\leq i\leq n}|\alpha_i|^2. \tag{1.10}$$

Consequently, we may state the following coarser upper bounds for $\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2$ that may be useful in applications [**5**].

Corollary 37. *With the assumptions in Lemma 6, we have the inequalities:*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \tag{1.11}$$
$$\leq\begin{cases}\max\limits_{1\leq i\leq n}|\alpha_i|^2\sum\limits_{i=1}^{n}\|z_i\|^2;\\[2ex]\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\alpha}\right)^{\frac{1}{\alpha}}\left(\sum\limits_{i=1}^{n}\|z_i\|^{2\beta}\right)^{\frac{1}{\beta}}, \quad \textit{where}\ \ \alpha>1,\frac{1}{\alpha}+\frac{1}{\beta}=1;\\[2ex]\sum\limits_{i=1}^{n}|\alpha_i|^2\max\limits_{1\leq i\leq n}\|z_i\|^2,\end{cases}$$
$$+\begin{cases}\max\limits_{1\leq i\leq n}|\alpha_i|^2\sum\limits_{1\leq i\neq j\leq n}|(z_i,z_j)|;\\[2ex](n-1)^{\frac{1}{\gamma}}\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\gamma}\right)^{\frac{1}{\gamma}}\left(\sum\limits_{1\leq i\neq j\leq n}|(z_i,z_j)|^{\delta}\right)^{\frac{1}{\delta}},\\ \qquad\qquad\qquad\qquad \textit{where}\ \ \gamma>1,\quad\frac{1}{\gamma}+\frac{1}{\delta}=1;\\[2ex](n-1)\sum\limits_{i=1}^{n}|\alpha_i|^2\max\limits_{1\leq i\neq j\leq n}|(z_i,z_j)|.\end{cases}$$

The proof is obvious by Lemma 6 in applying the inequalities (1.8) – (1.10).

Remark 45. *The following inequalities which are incorporated in (1.11) are of special interest:*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2\leq\max_{1\leq i\leq n}|\alpha_i|^2\left[\sum_{i=1}^{n}\|z_i\|^2+\sum_{1\leq i\neq j\leq n}|(z_i,z_j)|\right]; \tag{1.12}$$

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \leq \left(\sum_{i=1}^{n}|\alpha_i|^{2p}\right)^{\frac{1}{p}}\left[\left(\sum_{i=1}^{n}\|z_i\|^{2q}\right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}}\left(\sum_{1\leq i\neq j\leq n}|(z_i,z_j)|^q\right)^{\frac{1}{q}}\right], \tag{1.13}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$; *and*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \leq \sum_{i=1}^{n}|\alpha_i|^2\left[\max_{1\leq i\leq n}\|z_i\|^2+(n-1)\max_{1\leq i\neq j\leq n}|(z_i,z_j)|\right]. \tag{1.14}$$

1.3. Mitrinović-Pečarić-Fink Type Inequalities. We are now able to present the following result obtained in [**5**], which complements the inequality (1.2) due to Mitrinović, Pečarić and Fink [**11**, p. 392].

Theorem 49. *Let* $x, y_1, \dots, y_n$ *be vectors of an inner product space* $(H;(\cdot,\cdot))$ *and* $c_1,\dots,c_n\in\mathbb{K}\ (\mathbb{K}=\mathbb{C},\mathbb{R})$. *Then one has the inequalities:*

$$\left|\sum_{i=1}^{n}c_i(x,y_i)\right|^2 \leq \|x\|^2\times\begin{cases}\max\limits_{1\leq i\leq n}|c_i|^2\sum_{i=1}^{n}\|y_i\|^2; \\ \left(\sum_{i=1}^{n}|c_i|^{2\alpha}\right)^{\frac{1}{\alpha}}\left(\sum_{i=1}^{n}\|y_i\|^{2\beta}\right)^{\frac{1}{\beta}}, \quad \text{where } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \sum_{i=1}^{n}|c_i|^2\max\limits_{1\leq i\leq n}\|y_i\|^2,\end{cases}$$

$$+\|x\|^2\times\begin{cases}\max\limits_{1\leq i\neq j\leq n}\{|c_ic_j|\}\sum_{1\leq i\neq j\leq n}|(y_i,y_j)|; \\ \left[\left(\sum_{i=1}^{n}|c_i|^{\gamma}\right)^2-\left(\sum_{i=1}^{n}|c_i|^{2\gamma}\right)\right]^{\frac{1}{\gamma}} \\ \quad\times\left(\sum_{1\leq i\neq j\leq n}|(y_i,y_j)|^{\delta}\right)^{\frac{1}{\delta}}, \\ \qquad\text{where } \gamma>1,\quad \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \left[\left(\sum_{i=1}^{n}|c_i|\right)^2-\sum_{i=1}^{n}|c_i|^2\right]\max\limits_{1\leq i\neq j\leq n}|(y_i,y_j)|.\end{cases} \tag{1.15}$$

PROOF. We note that

$$\sum_{i=1}^{n} c_i (x, y_i) = \left(x, \sum_{i=1}^{n} \overline{c_i} y_i \right).$$

Using Schwarz's inequality in inner product spaces, we have:

$$\left| \sum_{i=1}^{n} c_i (x, y_i) \right|^2 \leq \|x\|^2 \left\| \sum_{i=1}^{n} \overline{c_i} y_i \right\|^2.$$

Now using Lemma 6 with $\alpha_i = \overline{c_i}$, $z_i = y_i$ $(i = 1, \dots, n)$, we deduce the desired inequality (1.15). ∎

The following particular inequalities that may be obtained by the Corollaries 36 and 37 and Remark 45 hold [**5**].

COROLLARY 38. *With the assumptions in Theorem 49, one has the inequalities:*

$$(1.16) \quad \left| \sum_{i=1}^{n} c_i (x, y_i) \right|^2$$

$$\leq \|x\|^2 \times \begin{cases} \sum_{i=1}^{n} |c_i|^2 \left\{ \max\limits_{1 \leq i \leq n} \|y_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^2 \right)^{\frac{1}{2}} \right\}; \\[2ex] \max\limits_{1 \leq i \leq n} |c_i|^2 \left\{ \sum_{i=1}^{n} \|y_i\|^2 + \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)| \right\}; \\[2ex] \left(\sum_{i=1}^{n} |c_i|^{2p} \right)^{\frac{1}{p}} \left\{ \left(\sum_{i=1}^{n} \|y_i\|^{2q} \right)^{\frac{1}{q}} \right. \\ \qquad \left. + (n-1)^{\frac{1}{p}} \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^q \right)^{\frac{1}{q}} \right\}, \\ \hfill \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\[2ex] \sum_{i=1}^{n} |c_i|^2 \left\{ \max\limits_{1 \leq i \leq n} \|y_i\|^2 + (n-1) \max\limits_{1 \leq i \neq j \leq n} |(y_i, y_j)| \right\}. \end{cases}$$

REMARK 46. *Note that the first inequality in (1.16) is the result obtained by Mitrinović-Pečarić-Fink in* [**11**]. *The other 3 provide similar bounds in terms of the* $p-$*norms of the vector* $\left(|c_1|^2, \dots, |c_n|^2 \right)$.

1.4. Boas-Bellman Type Inequalities. If one chooses $c_i = \overline{(x, y_i)}$ $(i = 1, \dots, n)$ in (1.15), then it is possible to obtain 9 different inequalities between the Fourier coefficients (x, y_i) and the norms and inner products of the vectors y_i $(i = 1, \dots, n)$. We restrict ourselves only to those inequalities that may be obtained from (1.16).

As Mitrinović, Pečarić and Fink noted in [**11**, p. 392], the first inequality in (1.16) for the above selection of c_i will produce the Boas-Bellman inequality (1.1).

From the second inequality in (1.16) for $c_i = \overline{(x, y_i)}$ we get

$$\left(\sum_{i=1}^{n} |(x, y_i)|^2\right)^2 \leq \|x\|^2 \max_{1\leq i\leq n} |(x, y_i)|^2 \left\{\sum_{i=1}^{n} \|y_i\|^2 + \sum_{1\leq i\neq j\leq n} |(y_i, y_j)|\right\}.$$

Taking the square root in this inequality we obtain:

$$\begin{aligned}(1.17)\quad & \sum_{i=1}^{n} |(x, y_i)|^2 \\ & \leq \|x\| \max_{1\leq i\leq n} |(x, y_i)| \left\{\sum_{i=1}^{n} \|y_i\|^2 + \sum_{1\leq i\neq j\leq n} |(y_i, y_j)|\right\}^{\frac{1}{2}},\end{aligned}$$

for any $x, y_1, \dots, y_n$ vectors in the inner product space $(H; (\cdot, \cdot))$.

If we assume that $(e_i)_{1\leq i\leq n}$ is an orthonormal family in H, then by (1.17) we have

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \sqrt{n}\,\|x\| \max_{1\leq i\leq n} |(x, e_i)|, \quad x \in H.$$

From the third inequality in (1.16) for $c_i = \overline{(x, y_i)}$ we deduce

$$\begin{aligned}\left(\sum_{i=1}^{n} |(x, y_i)|^2\right)^2 &\leq \|x\|^2 \left(\sum_{i=1}^{n} |(x, y_i)|^{2p}\right)^{\frac{1}{p}} \\ &\times \left\{\left(\sum_{i=1}^{n} \|y_i\|^{2q}\right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left(\sum_{1\leq i\neq j\leq n} |(y_i, y_j)|^q\right)^{\frac{1}{q}}\right\},\end{aligned}$$

for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Taking the square root in this inequality we get

$$\begin{aligned}(1.18)\quad \sum_{i=1}^{n} |(x, y_i)|^2 &\leq \|x\| \left(\sum_{i=1}^{n} |(x, y_i)|^{2p}\right)^{\frac{1}{2p}} \\ &\times \left\{\left(\sum_{i=1}^{n} \|y_i\|^{2q}\right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left(\sum_{1\leq i\neq j\leq n} |(y_i, y_j)|^q\right)^{\frac{1}{q}}\right\}^{\frac{1}{2}},\end{aligned}$$

for any $x, y_1, \dots, y_n \in H$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The above inequality (1.18) becomes, for an orthornormal family $(e_i)_{1\leq i\leq n}$,

$$\sum_{i=1}^{n}\left|(x,e_i)\right|^2 \leq n^{\frac{1}{q}}\left\|x\right\|\left(\sum_{i=1}^{n}\left|(x,e_i)\right|^{2p}\right)^{\frac{1}{2p}}, \quad x\in H.$$

Finally, the choice $c_i = \overline{(x,y_i)}$ $(i=1,\dots,n)$ will produce in the last inequality in (1.16)

$$\left(\sum_{i=1}^{n}\left|(x,y_i)\right|^2\right)^2 \leq \left\|x\right\|^2\sum_{i=1}^{n}\left|(x,y_i)\right|^2\left\{\max_{1\leq i\leq n}\left\|y_i\right\|^2+(n-1)\max_{1\leq i\neq j\leq n}\left|(y_i,y_j)\right|\right\}$$

giving the following Boas-Bellman type inequality

$$\sum_{i=1}^{n}\left|(x,y_i)\right|^2 \leq \left\|x\right\|^2\left\{\max_{1\leq i\leq n}\left\|y_i\right\|^2+(n-1)\max_{1\leq i\neq j\leq n}\left|(y_i,y_j)\right|\right\}, \tag{1.19}$$

for any $x,y_1,\dots,y_n\in H$.

It is obvious that (1.19) will give for orthonormal families the well known Bessel inequality.

Remark 47. *In order the compare the Boas-Bellman result with our result (1.19), it is enough to compare the quantities*

$$A := \left(\sum_{1\leq i\neq j\leq n}\left|(y_i,y_j)\right|^2\right)^{\frac{1}{2}}$$

and

$$B := (n-1)\max_{1\leq i\neq j\leq n}\left|(y_i,y_j)\right|.$$

Consider the inner product space $H=\mathbb{R}$ with $(x,y)=xy$, and choose $n=3$, $y_1=a>0$, $y_2=b>0$, $y_3=c>0$. Then

$$A=\sqrt{2}\left(a^2b^2+b^2c^2+c^2a^2\right)^{\frac{1}{2}}, \qquad B=2\max\left(ab,ac,bc\right).$$

Denote $ab=p$, $bc=q$, $ca=r$. Then

$$A=\sqrt{2}\left(p^2+q^2+r^2\right)^{\frac{1}{2}}, \qquad B=2\max\left(p,q,r\right).$$

Firstly, if we assume that $p=q=r$, then $A=\sqrt{6}p$, $B=2p$ which shows that $A>B$.

Now choose $r=1$ and $p,q=\frac{1}{2}$. Then $A=\sqrt{3}$ and $B=2$ showing that $B>A$.

Consequently, in general, the Boas-Bellman inequality and our inequality (1.19) cannot be compared.

2. Bombieri Type Inequalities

2.1. Introduction. In 1971, E. Bombieri [**3**] (see also [**11**, p. 394]) gave the following generalisation of Bessel's inequality.

THEOREM 50. *If $x, y_1, \dots, y_n$ are vectors in the inner product space $(H; (\cdot,\cdot))$, then the following inequality:*

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\|^2 \max_{1\leq i\leq n} \left\{ \sum_{j=1}^{n} |(y_i, y_j)| \right\}, \tag{2.1}$$

holds.

It is obvious that if $(y_i)_{1\leq i\leq n}$ are orthonormal, then from (2.1) one can deduce Bessel's inequality.

Another generalisation of Bessel's inequality was obtained by A. Selberg (see for example [**11**, p. 394]):

THEOREM 51. *Let $x, y_1, \dots, y_n$ be vectors in H with $y_i \neq 0$ $(i = 1, \dots, n)$. Then one has the inequality:*

$$\sum_{i=1}^{n} \frac{|(x, y_i)|^2}{\sum_{j=1}^{n} |(y_i, y_j)|} \leq \|x\|^2. \tag{2.2}$$

In this case, also, if $(y_i)_{1\leq i\leq n}$ are orthonormal, then from (2.2) one may deduce Bessel's inequality.

Another type of inequality related to Bessel's result, was discovered in 1958 by H. Heilbronn [**10**] (see also [**11**, p. 395]).

THEOREM 52. *With the assumptions in Theorem 50, one has*

$$\sum_{i=1}^{n} |(x, y_i)| \leq \|x\| \left(\sum_{i,j=1}^{n} |(y_i, y_j)| \right)^{\frac{1}{2}}. \tag{2.3}$$

If in (2.3) one chooses $y_i = e_i$ $(i = 1, \dots, n)$, where $(e_i)_{1\leq i\leq n}$ are orthonormal vectors in H, then

$$\sum_{i=1}^{n} |(x, e_i)| \leq \sqrt{n}\, \|x\|, \quad \text{for any} \quad x \in H.$$

In 1992, J.E. Pečarić [**12**] (see also [**11**, p. 394]) proved the following general inequality in inner product spaces.

THEOREM 53. *Let $x, y_1, \dots, y_n \in H$ and $c_1, \dots, c_n \in \mathbb{K}$. Then*

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \le \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \left(\sum_{j=1}^{n} |(y_i, y_j)|\right) \le \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \max_{1\le i\le n} \left\{\sum_{j=1}^{n} |(y_i, y_j)|\right\}. \tag{2.4}$$

He showed that the Bombieri inequality (2.1) may be obtained from (2.4) for the choice $c_i = \overline{(x, y_i)}$ (using the second inequality), the Selberg inequality (2.2) may be obtained from the first part of (2.4) for the choice

$$c_i = \frac{\overline{(x, y_i)}}{\sum_{j=1}^{n} |(y_i, y_j)|}, \quad i \in \{1, \dots, n\};$$

while the Heilbronn inequality (2.3) may be obtained from the first part of (2.4) if one chooses $c_i = \frac{\overline{(x,y_i)}}{|(x,y_i)|}$, for any $i \in \{1, \dots, n\}$.

For other results connected with the above ones, see [**7**] and [**8**].

2.2. Some Norm Inequalities. We start with the following lemma which is also interesting in itself [**6**].

LEMMA 7. *Let $z_1, \dots, z_n \in H$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. Then one has the inequality:*

$$\left\|\sum_{i=1}^{n} \alpha_i z_i\right\|^2 \le \begin{cases} A \\ B \\ C \end{cases}, \tag{2.5}$$

where

$$A := \begin{cases} \max\limits_{1\le k\le n} |\alpha_k|^2 \sum\limits_{i,j=1}^{n} |(z_i, z_j)|; \\ \max\limits_{1\le k\le n} |\alpha_k| \left(\sum\limits_{i=1}^{n} |\alpha_i|^r\right)^{\frac{1}{r}} \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(z_i, z_j)|\right)^s\right)^{\frac{1}{s}}, \\ \qquad\qquad r > 1,\ \frac{1}{r} + \frac{1}{s} = 1; \\ \max\limits_{1\le k\le n} |\alpha_k| \sum\limits_{k=1}^{n} |\alpha_k| \max\limits_{1\le i\le n} \left(\sum\limits_{j=1}^{n} |(z_i, z_j)|\right); \end{cases}$$

$$B := \begin{cases} \left(\sum\limits_{k=1}^{n} |\alpha_k|^p\right)^{\frac{1}{p}} \max\limits_{1\le i\le n} |\alpha_i| \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(z_i,z_j)|\right)^q\right)^{\frac{1}{q}}, \\ \hfill p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left(\sum\limits_{k=1}^{n} |\alpha_k|^p\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^{n} |\alpha_i|^t\right)^{\frac{1}{t}} \left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(z_i,z_j)|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, \\ \hfill p>1,\ \frac{1}{p}+\frac{1}{q}=1;\ t>1,\ \frac{1}{t}+\frac{1}{u}=1; \\ \\ \left(\sum\limits_{k=1}^{n} |\alpha_k|^p\right)^{\frac{1}{p}} \sum\limits_{i=1}^{n} |\alpha_i| \max\limits_{1\le i\le n}\left\{\left(\sum\limits_{j=1}^{n} |(z_i,z_j)|^q\right)^{\frac{1}{q}}\right\}, \\ \hfill p>1,\ \frac{1}{p}+\frac{1}{q}=1; \end{cases}$$

and

$$C := \begin{cases} \sum\limits_{k=1}^{n} |\alpha_k| \ \max\limits_{1\le i\le n} |\alpha_i| \ \sum\limits_{i=1}^{n}\left[\max\limits_{1\le j\le n} |(z_i,z_j)|\right]; \\ \sum\limits_{k=1}^{n} |\alpha_k| \ \left(\sum\limits_{i=1}^{n} |\alpha_i|^m\right)^{\frac{1}{m}} \left(\sum\limits_{i=1}^{n}\left[\max\limits_{1\le j\le n} |(z_i,z_j)|\right]^l\right)^{\frac{1}{l}}, \\ \hfill m>1,\ \frac{1}{m}+\frac{1}{l}=1; \\ \\ \left(\sum\limits_{k=1}^{n} |\alpha_k|\right)^2 \max\limits_{1\le j\le n} |(z_i,z_j)|. \end{cases}$$

Proof. We observe that

$$\begin{aligned} \left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 &= \left(\sum_{i=1}^{n}\alpha_i z_i, \sum_{j=1}^{n}\alpha_j z_j\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}\,(z_i,z_j) = \left|\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}\,(z_i,z_j)\right| \\ &\le \sum_{i=1}^{n}\sum_{j=1}^{n}|\alpha_i|\,|\alpha_j|\,|(z_i,z_j)| = \sum_{i=1}^{n}|\alpha_i|\left(\sum_{j=1}^{n}|\alpha_j|\,|(z_i,z_j)|\right) \\ &:= M. \end{aligned}$$

Using Hölder's inequality, we may write that

$$\sum_{j=1}^{n}|\alpha_j|\,|(z_i,z_j)| \leq \begin{cases} \max\limits_{1\leq k\leq n}|\alpha_k|\sum\limits_{j=1}^{n}|(z_i,z_j)| \\ \left(\sum\limits_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|^q\right)^{\frac{1}{q}}, \\ \hfill p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum\limits_{k=1}^{n}|\alpha_k|\ \max\limits_{1\leq j\leq n}|(z_i,z_j)| \end{cases}$$

for any $i\in\{1,\ldots,n\}$, giving

$$M \leq \begin{cases} \max\limits_{1\leq k\leq n}|\alpha_k|\sum\limits_{i=1}^{n}|\alpha_i|\sum\limits_{j=1}^{n}|(z_i,z_j)| =: M_1; \\ \left(\sum\limits_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}}\sum\limits_{i=1}^{n}|\alpha_i|\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|^q\right)^{\frac{1}{q}} := M_p, \\ \hfill p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum\limits_{k=1}^{n}|\alpha_k|\sum\limits_{i=1}^{n}|\alpha_i|\ \max\limits_{1\leq j\leq n}|(z_i,z_j)| =: M_\infty. \end{cases}$$

By Hölder's inequality we also have:

$$\sum_{i=1}^{n}|\alpha_i|\left(\sum_{j=1}^{n}|(z_i,z_j)|\right) \leq \begin{cases} \max\limits_{1\leq i\leq n}|\alpha_i|\sum\limits_{i,j=1}^{n}|(z_i,z_j)|; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^r\right)^{\frac{1}{r}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^s\right)^{\frac{1}{s}}, \\ \hfill r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\ \sum\limits_{i=1}^{n}|\alpha_i|\ \max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right); \end{cases}$$

and thus

$$M_1 \leq \begin{cases} \max\limits_{1\leq k\leq n}|\alpha_k|^2\sum\limits_{i,j=1}^{n}|(z_i,z_j)|; \\ \max\limits_{1\leq k\leq n}|\alpha_k|\left(\sum\limits_{i=1}^{n}|\alpha_i|^r\right)^{\frac{1}{r}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^s\right)^{\frac{1}{s}}, \\ \hfill r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\ \max\limits_{1\leq k\leq n}|\alpha_k|\sum\limits_{i=1}^{n}|\alpha_i|\ \max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right); \end{cases}$$

and the first 3 inequalities in (2.5) are obtained.

By Hölder's inequality we also have:

$$M_p \leq \left(\sum_{k=1}^{n} |\alpha_k|^p\right)^{\frac{1}{p}} \times \begin{cases} \max\limits_{1\leq i\leq n} |\alpha_i| \sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(z_i, z_j)|^q\right)^{\frac{1}{q}}; \\ \left(\sum\limits_{i=1}^{n} |\alpha_i|^t\right)^{\frac{1}{t}} \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(z_i, z_j)|^q\right)^{\frac{u}{q}}\right)^{\frac{1}{u}}, \\ \qquad\qquad t > 1,\ \frac{1}{t} + \frac{1}{u} = 1; \\ \sum\limits_{i=1}^{n} |\alpha_i| \max\limits_{1\leq i\leq n} \left\{\left(\sum\limits_{j=1}^{n} |(z_i, z_j)|^q\right)^{\frac{1}{q}}\right\}; \end{cases}$$

and the next 3 inequalities in (2.5) are proved.

Finally, by the same Hölder inequality we may state that:

$$M_\infty \leq \sum_{k=1}^{n} |\alpha_k| \times \begin{cases} \max\limits_{1\leq i\leq n} |\alpha_i| \sum\limits_{i=1}^{n} \left(\max\limits_{1\leq j\leq n} |(z_i, z_j)|\right); \\ \left(\sum\limits_{i=1}^{n} |\alpha_i|^m\right)^{\frac{1}{m}} \left(\sum\limits_{i=1}^{n} \left(\max\limits_{1\leq j\leq n} |(z_i, z_j)|\right)^l\right)^{\frac{1}{l}}, \\ \qquad\qquad m > 1,\ \frac{1}{m} + \frac{1}{l} = 1; \\ \sum\limits_{i=1}^{n} |\alpha_i| \max\limits_{1\leq i,j\leq n} |(z_i, z_j)|; \end{cases}$$

and the last 3 inequalities in (2.5) are proved. ∎

If we would like to have some bounds for $\left\|\sum_{i=1}^{n} \alpha_i z_i\right\|^2$ in terms of $\sum_{i=1}^{n} |\alpha_i|^2$, then the following corollaries may be used.

Corollary 39. *Let $z_1, \dots, z_n$ and $\alpha_1, \dots, \alpha_n$ be as in Lemma 7. If $1 < p \leq 2$, $1 < t \leq 2$, then one has the inequality*

$$\left\|\sum_{i=1}^{n} \alpha_i z_i\right\|^2 \leq n^{\frac{1}{p}+\frac{1}{t}-1} \sum_{k=1}^{n} |\alpha_k|^2 \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(z_i, z_j)|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, \tag{2.6}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{t} + \frac{1}{u} = 1$.

PROOF. Observe, by the monotonicity of power means, we may write that

$$\left(\frac{\sum_{k=1}^{n}|\alpha_k|^p}{n}\right)^{\frac{1}{p}} \le \left(\frac{\sum_{k=1}^{n}|\alpha_k|^2}{n}\right)^{\frac{1}{2}}; \quad 1<p\le 2,$$

$$\left(\frac{\sum_{k=1}^{n}|\alpha_k|^t}{n}\right)^{\frac{1}{t}} \le \left(\frac{\sum_{k=1}^{n}|\alpha_k|^2}{n}\right)^{\frac{1}{2}}; \quad 1<t\le 2,$$

from where we get

$$\left(\sum_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}} \le n^{\frac{1}{p}-\frac{1}{2}}\left(\sum_{k=1}^{n}|\alpha_k|^2\right)^{\frac{1}{2}},$$

$$\left(\sum_{k=1}^{n}|\alpha_k|^t\right)^{\frac{1}{t}} \le n^{\frac{1}{t}-\frac{1}{2}}\left(\sum_{k=1}^{n}|\alpha_k|^2\right)^{\frac{1}{2}}.$$

Using the fifth inequality in (2.5), we then deduce (2.6). ∎

REMARK 48. *An interesting particular case is the one for* $p=q=t=u=2$, *giving*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \le \sum_{k=1}^{n}|\alpha_k|^2\left(\sum_{i,j=1}^{n}|(z_i,z_j)|^2\right)^{\frac{1}{2}}.$$

COROLLARY 40. *With the assumptions of Lemma 7 and if* $1<p\le 2$, *then*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \le n^{\frac{1}{p}}\sum_{k=1}^{n}|\alpha_k|^2 \max_{1\le i\le n}\left[\left(\sum_{j=1}^{n}|(z_i,z_j)|^q\right)^{\frac{1}{q}}\right], \tag{2.7}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

PROOF. Since

$$\left(\sum_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}} \le n^{\frac{1}{p}-\frac{1}{2}}\left(\sum_{k=1}^{n}|\alpha_k|^2\right)^{\frac{1}{2}},$$

and

$$\sum_{k=1}^{n}|\alpha_k| \le n^{\frac{1}{2}}\left(\sum_{k=1}^{n}|\alpha_k|^2\right)^{\frac{1}{2}},$$

then by the sixth inequality in (2.5) we deduce (2.7). ∎

In a similar fashion, one may prove the following two corollaries.

Corollary 41. *With the assumptions of Lemma 7 and if* $1 < m \leq 2$, *then*

$$\left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \leq n^{\frac{1}{m}} \sum_{k=1}^{n} |\alpha_k|^2 \left(\sum_{i=1}^{n} \left[\max_{1 \leq j \leq n} |(z_i, z_j)| \right]^l \right)^{\frac{1}{l}},$$

where $\frac{1}{m} + \frac{1}{l} = 1$.

Corollary 42. *With the assumptions of Lemma 7, we have:*

$$\left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \leq n \sum_{k=1}^{n} |\alpha_k|^2 \max_{1 \leq i,j \leq n} |(z_i, z_j)|.$$

The following lemma may be of interest as well [**6**].

Lemma 8. *With the assumptions of Lemma 7, one has the inequalities*

$$\left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \leq \sum_{i=1}^{n} |\alpha_i|^2 \sum_{j=1}^{n} |(z_i, z_j)| \tag{2.8}$$

$$\leq \begin{cases} \sum_{i=1}^{n} |\alpha_i|^2 \max_{1 \leq i \leq n} \left[\sum_{j=1}^{n} |(z_i, z_j)| \right]; \\ \left(\sum_{i=1}^{n} |\alpha_i|^{2p} \right)^{\frac{1}{p}} \left(\left(\sum_{j=1}^{n} |(z_i, z_j)| \right)^q \right)^{\frac{1}{q}}, \\ \qquad\qquad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i,j=1}^{n} |(z_i, z_j)|. \end{cases}$$

Proof. As in Lemma 7, we know that

$$\left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_i| |\alpha_j| |(z_i, z_j)|.$$

Using the simple observation that (see also [**11**, p. 394])

$$|\alpha_i| |\alpha_j| \leq \frac{1}{2} \left(|\alpha_i|^2 + |\alpha_j|^2 \right), \quad i, j \in \{1, \dots, n\},$$

we have

$$\begin{aligned} \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_i| |\alpha_j| |(z_i, z_j)| &\leq \frac{1}{2} \sum_{i,j=1}^{n} \left(|\alpha_i|^2 + |\alpha_j|^2 \right) |(z_i, z_j)| \\ &= \frac{1}{2} \left[\sum_{i,j=1}^{n} |\alpha_i|^2 |(z_i, z_j)| + \sum_{i,j=1}^{n} |\alpha_j|^2 |(z_i, z_j)| \right] \end{aligned}$$

$$= \sum_{i,j=1}^{n} |\alpha_i|^2 |(z_i, z_j)|,$$

which proves the first inequality in (2.8).

The second part follows by Hölder's inequality and we omit the details. ∎

Remark 49. *The first part in (2.8) is the inequality obtained by Pečarić in* [**12**].

2.3. Pečarić Type Inequalities. We are now able to present the following result obtainend in [**6**], which complements the inequality (2.4) due to J.E. Pečarić [**12**] (see also [**11**, p. 394]).

Theorem 54. *Let $x, y_1, \dots, y_n$ be vectors of an inner product space $(H; (\cdot, \cdot))$ and $c_1, \dots, c_n \in \mathbb{K}$. Then one has the inequalities:*

$$\left| \sum_{i=1}^{n} c_i (x, y_i) \right|^2 \le \|x\|^2 \times \begin{cases} D \\ E \\ F \end{cases}, \tag{2.9}$$

where

$$D := \begin{cases} \max_{1 \le k \le n} |c_k|^2 \sum_{i,j=1}^{n} |(y_i, y_j)|; \\[2mm] \max_{1 \le k \le n} |c_k| \left(\sum_{i=1}^{n} |c_i|^r\right)^{\frac{1}{r}} \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^s\right]^{\frac{1}{s}}, \\ \hfill r > 1,\ \frac{1}{r} + \frac{1}{s} = 1; \\[2mm] \max_{1 \le k \le n} |c_k| \sum_{i=1}^{n} |c_i| \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right); \end{cases}$$

$$E := \begin{cases} \left(\sum_{k=1}^{n} |c_k|^p\right)^{\frac{1}{p}} \max_{1 \le i \le n} |c_i| \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^q\right)^{\frac{1}{q}}, \\ \hfill p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\[2mm] \left(\sum_{k=1}^{n} |c_k|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |c_i|^t\right)^{\frac{1}{t}} \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, \\ \hfill p > 1,\ \frac{1}{p} + \frac{1}{q} = 1;\ t > 1,\ \frac{1}{t} + \frac{1}{u} = 1; \\[2mm] \left(\sum_{k=1}^{n} |c_k|^p\right)^{\frac{1}{p}} \sum_{i=1}^{n} |c_i| \max_{1 \le i \le n} \left\{\left(\sum_{j=1}^{n} |(y_i, y_j)|^q\right)^{\frac{1}{q}}\right\}, \\ \hfill p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \end{cases}$$

and

$$F := \begin{cases} \sum_{k=1}^{n} |c_k| \max\limits_{1 \le i \le n} |c_i| \sum_{i=1}^{n} \left[\max\limits_{1 \le j \le n} |(y_i, y_j)| \right]; \\ \sum_{k=1}^{n} |c_k| \left(\sum_{i=1}^{n} |c_i|^m\right)^{\frac{1}{m}} \left(\sum_{i=1}^{n} \left[\max\limits_{1 \le j \le n} |(y_i, y_j)| \right]^l \right)^{\frac{1}{l}}, \\ \qquad\qquad m > 1, \ \frac{1}{m} + \frac{1}{l} = 1; \\ \left(\sum_{k=1}^{n} |c_k|\right)^2 \max\limits_{1 \le j \le n} |(y_i, y_j)|. \end{cases}$$

PROOF. We note that

$$\sum_{i=1}^{n} c_i (x, y_i) = \left(x, \sum_{i=1}^{n} \overline{c_i} y_i \right).$$

Using Schwarz's inequality in inner product spaces, we have

$$\left| \sum_{i=1}^{n} c_i (x, y_i) \right|^2 \le \|x\|^2 \left\| \sum_{i=1}^{n} \overline{c_i} y_i \right\|^2.$$

Finally, using Lemma 7 with $\alpha_i = \overline{c_i}$, $z_i = y_i$ $(i = 1, \dots, n)$, we deduce the desired inequality (2.9). We omit the details. ∎

The following corollaries may be useful if one needs bounds in terms of $\sum_{i=1}^{n} |c_i|^2$.

COROLLARY 43. *With the assumptions in Theorem 54 and if* $1 < p \le 2$, $1 < t \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{t} + \frac{1}{u} = 1$, *then one has the inequality:*

$$\left| \sum_{i=1}^{n} c_i (x, y_i) \right|^2 \le \|x\|^2 n^{\frac{1}{p} + \frac{1}{t} - 1} \sum_{i=1}^{n} |c_i|^2 \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}}, \tag{2.10}$$

and, in particular, for $p = q = t = u = 2$,

$$\left| \sum_{i=1}^{n} c_i (x, y_i) \right|^2 \le \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \left(\sum_{i,j=1}^{n} |(y_i, y_j)|^2 \right)^{\frac{1}{2}}.$$

The proof is similar to the one in Corollary 39.

Corollary 44. *With the assumptions in Theorem 54 and if* $1 < p \leq 2$, *then*

$$\left| \sum_{i=1}^{n} c_i \left(x, y_i \right) \right|^2 \leq \left\| x \right\|^2 n^{\frac{1}{p}} \sum_{k=1}^{n} \left| c_k \right|^2 \max_{1 \leq i \leq n} \left[\sum_{j=1}^{n} \left| \left(y_i, y_j \right) \right|^q \right]^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

The proof is similar to the one in Corollary 40.

The following two inequalities also hold.

Corollary 45. *With the above assumptions for* x, y_i, c_i *and if* $1 < m \leq 2$, *then*

$$\left| \sum_{i=1}^{n} c_i \left(x, y_i \right) \right|^2 \leq \left\| x \right\|^2 n^{\frac{1}{m}} \sum_{k=1}^{n} \left| c_k \right|^2 \left(\sum_{i=1}^{n} \left[\max_{1 \leq j \leq n} \left| \left(y_i, y_j \right) \right| \right]^l \right)^{\frac{1}{l}}, \tag{2.11}$$

where $\frac{1}{m} + \frac{1}{l} = 1$.

Corollary 46. *With the above assumptions for* x, y_i, c_i, *one has*

$$\left| \sum_{i=1}^{n} c_i \left(x, y_i \right) \right|^2 \leq \left\| x \right\|^2 n \sum_{k=1}^{n} \left| c_k \right|^2 \max_{1 \leq j \leq n} \left| \left(y_i, y_j \right) \right|. \tag{2.12}$$

Using Lemma 8, we may state the following result as well.

Remark 50. *With the assumptions of Theorem 54, one has the inequalities:*

$$\left| \sum_{i=1}^{n} c_i \left(x, y_i \right) \right|^2 \leq \left\| x \right\|^2 \sum_{i=1}^{n} \left| c_i \right|^2 \sum_{j=1}^{n} \left| \left(y_i, y_j \right) \right|$$

$$\leq \left\| x \right\|^2 \times \begin{cases} \sum_{i=1}^{n} \left| c_i \right|^2 \max_{1 \leq i \leq n} \left[\sum_{j=1}^{n} \left| \left(y_i, y_j \right) \right| \right]; \\ \left(\sum_{i=1}^{n} \left| c_i \right|^{2p} \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left| \left(y_i, y_j \right) \right| \right)^q \right)^{\frac{1}{q}}, \\ \hfill p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \max_{1 \leq i \leq n} \left| c_i \right|^2 \ \sum_{i,j=1}^{n} \left| \left(y_i, y_j \right) \right|; \end{cases}$$

that provide some alternatives to Pečarić's result (2.4).

2.4. Inequalities of Bombieri Type. In this section we point out some inequalities of Bombieri type that may be obtained from (2.9) on choosing $c_i = \overline{(x, y_i)}$ $(i = 1, \dots, n)$.

If the above choice was made in the first inequality in (2.9), then one can obtain:

$$\left(\sum_{i=1}^{n}|(x,y_i)|^2\right)^2 \leq \|x\|^2 \max_{1\leq i\leq n}|(x,y_i)|^2 \sum_{i,j=1}^{n}|(y_i,y_j)|$$

giving, by taking the square root,

$$\sum_{i=1}^{n}|(x,y_i)|^2 \leq \|x\| \max_{1\leq i\leq n}|(x,y_i)| \left(\sum_{i,j=1}^{n}|(y_i,y_j)|\right)^{\frac{1}{2}}, \quad x\in H. \tag{2.13}$$

If the same choice for c_i is made in the second inequality in (2.9), then one can get

$$\left(\sum_{i=1}^{n}|(x,y_i)|^2\right)^2 \leq \|x\|^2 \max_{1\leq i\leq n}|(x,y_i)| \left(\sum_{i=1}^{n}|(x,y_i)|^r\right)^{\frac{1}{r}} \times \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|(y_i,y_j)|\right)^s\right]^{\frac{1}{s}},$$

implying

$$\sum_{i=1}^{n}|(x,y_i)|^2 \leq \|x\| \max_{1\leq i\leq n}|(x,y_i)|^{\frac{1}{2}} \left(\sum_{i=1}^{n}|(x,y_i)|^r\right)^{\frac{1}{2r}} \times \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|(y_i,y_j)|\right)^s\right]^{\frac{1}{2s}}, \tag{2.14}$$

where $\frac{1}{r}+\frac{1}{s}=1$, $s>1$.

The other inequalities in (2.9) will produce the following results, respectively

$$\sum_{i=1}^{n}|(x,y_i)|^2 \leq \|x\| \max_{1\leq i\leq n}|(x,y_i)|^{\frac{1}{2}} \left(\sum_{i=1}^{n}|(x,y_i)|\right)^{\frac{1}{2}} \times \left[\max_{1\leq i\leq n}\left(\sum_{j=1}^{n}|(y_i,y_j)|\right)\right]; \tag{2.15}$$

$$\sum_{i=1}^{n} |(x, y_i)|^2 \le \|x\| \max_{1\le i\le n} |(x, y_i)|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(x, y_i)|^p \right)^{\frac{1}{2p}} \times \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|^q \right)^{\frac{1}{q}} \right]^{\frac{1}{2}}, \tag{2.16}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$;

$$\sum_{i=1}^{n} |(x, y_i)|^2 \le \|x\| \left(\sum_{i=1}^{n} |(x, y_i)|^p \right)^{\frac{1}{2p}} \left(\sum_{i=1}^{n} |(x, y_i)|^t \right)^{\frac{1}{2t}} \times \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{2u}}, \tag{2.17}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $t > 1$, $\frac{1}{t} + \frac{1}{u} = 1$;

$$\sum_{i=1}^{n} |(x, y_i)|^2 \le \|x\| \left(\sum_{i=1}^{n} |(x, y_i)|^p \right)^{\frac{1}{2p}} \left(\sum_{i=1}^{n} |(x, y_i)| \right)^{\frac{1}{2}} \times \max_{1\le i\le n} \left\{ \left(\sum_{j=1}^{n} |(y_i, y_j)|^q \right)^{\frac{1}{2q}} \right\}, \tag{2.18}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$;

$$\sum_{i=1}^{n} |(x, y_i)|^2 \le \|x\| \left[\sum_{i=1}^{n} |(x, y_i)| \right]^{\frac{1}{2}} \max_{1\le i\le n} |(x, y_i)|^{\frac{1}{2}} \times \left(\sum_{i=1}^{n} \left[\max_{1\le j\le n} |(y_i, y_j)| \right] \right)^{\frac{1}{2}}; \tag{2.19}$$

$$\sum_{i=1}^{n} |(x, y_i)|^2 \le \|x\| \left[\sum_{i=1}^{n} |(x, y_i)|^m \right]^{\frac{1}{2m}} \left[\sum_{i=1}^{n} \left[\max_{1\le j\le n} |(y_i, y_j)|^l \right] \right]^{\frac{1}{2l}}, \tag{2.20}$$

where $m > 1$, $\frac{1}{m} + \frac{1}{l} = 1$; and

$$\sum_{i=1}^{n} \left|(x, y_i)\right|^2 \le \|x\| \sum_{i=1}^{n} \left|(x, y_i)\right| \max_{1 \le j \le n} \left|(y_i, y_j)\right|^{\frac{1}{2}}. \tag{2.21}$$

If in the above inequalities we assume that $(y_i)_{1 \le i \le n} = (e_i)_{1 \le i \le n}$, where $(e_i)_{1 \le i \le n}$ are orthonormal vectors in the inner product space $(H, (\cdot, \cdot))$, then from (2.13) – (2.21) we may deduce the following inequalities similar in a sense to Bessel's inequality:

$$\sum_{i=1}^{n} \left|(x, e_i)\right|^2 \le \sqrt{n} \|x\| \max_{1 \le i \le n} \left\{\left|(x, e_i)\right|\right\};$$

$$\sum_{i=1}^{n} \left|(x, e_i)\right|^2 \le n^{\frac{1}{2s}} \|x\| \max_{1 \le i \le n} \left\{\left|(x, e_i)\right|^{\frac{1}{2}}\right\} \left(\sum_{i=1}^{n} \left|(x, e_i)\right|^r\right)^{\frac{1}{2r}},$$

where $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$;

$$\sum_{i=1}^{n} \left|(x, e_i)\right|^2 \le \|x\| \max_{1 \le i \le n} \left\{\left|(x, e_i)\right|^{\frac{1}{2}}\right\} \left(\sum_{i=1}^{n} \left|(x, e_i)\right|\right)^{\frac{1}{2}},$$

$$\sum_{i=1}^{n} \left|(x, e_i)\right|^2 \le \sqrt{n} \|x\| \max_{1 \le i \le n} \left\{\left|(x, e_i)\right|^{\frac{1}{2}}\right\} \left(\sum_{i=1}^{n} \left|(x, e_i)\right|^p\right)^{\frac{1}{2p}},$$

where $p > 1$;

$$\sum_{i=1}^{n} \left|(x, e_i)\right|^2 \le n^{\frac{1}{2u}} \|x\| \left(\sum_{i=1}^{n} \left|(x, e_i)\right|^p\right)^{\frac{1}{2p}} \left(\sum_{i=1}^{n} \left|(x, e_i)\right|^t\right)^{\frac{1}{2t}},$$

where $p > 1$, $t > 1$, $\frac{1}{t} + \frac{1}{u} = 1$;

$$\sum_{i=1}^{n} \left|(x, e_i)\right|^2 \le \|x\| \left(\sum_{i=1}^{n} \left|(x, e_i)\right|^p\right)^{\frac{1}{2p}} \left(\sum_{i=1}^{n} \left|(x, e_i)\right|\right)^{\frac{1}{2}}, \quad p > 1;$$

$$\sum_{i=1}^{n} \left|(x, e_i)\right|^2 \le \sqrt{n} \|x\| \left(\sum_{i=1}^{n} \left|(x, e_i)\right|\right)^{\frac{1}{2}} \max_{1 \le i \le n} \left\{\left|(x, e_i)\right|^{\frac{1}{2}}\right\};$$

$$\sum_{i=1}^{n} \left|(x, e_i)\right|^2 \le n^{\frac{1}{2l}} \|x\| \left[\sum_{i=1}^{n} \left|(x, e_i)\right|^m\right]^{\frac{1}{m}}, \quad m > 1, \ \frac{1}{m} + \frac{1}{l} = 1;$$

$$\sum_{i=1}^{n} \left|(x, e_i)\right|^2 \le \|x\| \sum_{i=1}^{n} \left|(x, e_i)\right|.$$

Corollaries 43 – 46 will produce the following results which do not contain the Fourier coefficients in the right side of the inequality.

Indeed, if one chooses $c_i = \overline{(x, y_i)}$ in (2.10), then

$$\left(\sum_{i=1}^{n} |(x, y_i)|^2\right)^2 \leq \|x\|^2 \, n^{\frac{1}{p}+\frac{1}{t}-1} \sum_{i=1}^{n} |(x, y_i)|^2 \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}},$$

giving the following Bombieri type inequality:

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq n^{\frac{1}{p}+\frac{1}{t}-1} \|x\|^2 \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}},$$

where $1 < p \leq 2$, $1 < t \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{t} + \frac{1}{u} = 1$.

If in this inequality we consider $p = q = t = u = 2$, then

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\|^2 \left(\sum_{i,j=1}^{n} |(y_i, y_j)|^2\right)^{\frac{1}{2}}.$$

For a different proof of this result see also [**8**].

In a similar way, if $c_i = \overline{(x, y_i)}$ in (2.11), then

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq n^{\frac{1}{m}} \|x\|^2 \left(\sum_{i=1}^{n} \left[\max_{1\leq j\leq n} |(y_i, y_j)|\right]^l\right)^{\frac{1}{l}},$$

where $m > 1$, $\frac{1}{m} + \frac{1}{l} = 1$.

Finally, if $c_i = \overline{(x, y_i)}$ $(i = 1, \dots, n)$, is taken in (2.12), then

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq n \|x\|^2 \max_{1\leq i,j\leq n} |(y_i, y_j)|.$$

Remark 51. *Let us compare Bombieri's result*

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\|^2 \max_{1\leq i\leq n} \left\{\sum_{j=1}^{n} |(y_i, y_j)|\right\} \tag{2.22}$$

with our result

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\|^2 \left\{\sum_{i,j=1}^{n} |(y_i, y_j)|^2\right\}^{\frac{1}{2}}. \tag{2.23}$$

Denote

$$M_1 := \max_{1\leq i\leq n} \left\{\sum_{j=1}^{n} |(y_i, y_j)|\right\}$$

and

$$M_2 := \left[\sum_{i,j=1}^{n} |(y_i, y_j)|^2\right]^{\frac{1}{2}}.$$

If we choose the inner product space $H = \mathbb{R}$, $(x, y) := xy$ and $n = 2$, then for $y_1 = a$, $y_2 = b$, $a, b > 0$, we have

$$M_1 = \max\left\{a^2 + ab, ab + b^2\right\} = (a + b)\max(a, b),$$

$$M_2 = \left(a^4 + a^2b^2 + a^2b^2 + b^4\right)^{\frac{1}{2}} = a^2 + b^2.$$

Assume that $a \geq b$. Then $M_1 = a^2 + ab \geq a^2 + b^2 = M_2$, showing that, in this case, the bound provided by (2.23) is better than the bound provided by (2.22). If $(y_i)_{1\leq i\leq n}$ are orthonormal vectors, then $M_1 = 1$, $M_2 = \sqrt{n}$, showing that in this case the Bombieri inequality (which becomes Bessel's inequality) provides a better bound than (2.23).

3. Pečarić Type Inequalities

3.1. Introduction. In 1992, J.E. Pečarić [**12**] proved the following inequality for vectors in complex inner product spaces $(H; (\cdot, \cdot))$.

Theorem 55. *Suppose that $x, y_1, \ldots, y_n$ are vectors in H and $c_1, \ldots, c_n$ are complex numbers. Then the following inequalities*

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \left(\sum_{j=1}^{n} |(y_i, y_j)|\right) \leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \max_{1\leq i\leq n}\left(\sum_{j=1}^{n} |(y_i, y_j)|\right), \tag{3.1}$$

hold.

He also showed that for $c_i = \overline{(x, y_i)}$, $i \in \{1, \ldots, n\}$, one gets

$$\left(\sum_{i=1}^{n} |(x, y_i)|^2\right)^2 \leq \|x\|^2 \sum_{i=1}^{n} |(x, y_i)|^2 \left(\sum_{j=1}^{n} |(y_i, y_j)|\right) \leq \|x\|^2 \sum_{i=1}^{n} |(x, y_i)|^2 \max_{1\leq i\leq n}\left(\sum_{j=1}^{n} |(y_i, y_j)|\right),$$

which improves Bombieri's result [**3**] (see also [**11**, p. 394])

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\|^2 \max_{1\leq i\leq n}\left(\sum_{j=1}^{n} |(y_i, y_j)|\right). \tag{3.2}$$

Note that (3.2) is in its turn a natural generalisation of *Bessel's inequality*

$$\sum_{i=1}^{n} |(x, e_i)|^2 \le \|x\|^2, \quad x \in H, \tag{3.3}$$

which holds for the orthornormal vectors $(e_i)_{1 \le i \le n}$.

In this section, by following [**4**], we point out other related results to Pečarić's inequality (3.1) than the ones stated in the previous sections. Some results of Bombieri type are also mentioned.

3.2. Some Norm Inequalities. We start with the following lemma that is interesting in its own right [**4**].

LEMMA 9. *Let $z_1, \dots, z_n \in H$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. Then one has the inequalities:*

$$\begin{aligned} \left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 &\le \left(\sum_{i=1}^{n} |\alpha_i|^p \left(\sum_{j=1}^{n} |(z_i, z_j)| \right) \right)^{\frac{1}{p}} \\ &\quad \times \left(\sum_{i=1}^{n} |\alpha_i|^q \left(\sum_{j=1}^{n} |(z_i, z_j)| \right) \right)^{\frac{1}{q}} \\ &\le \begin{cases} A \\ B \\ C \end{cases}, \end{aligned} \tag{3.4}$$

where

$$A := \begin{cases} \max\limits_{1 \le i \le n} |\alpha_i|^2 \sum_{i,j=1}^{n} |(z_i, z_j)|; \\ \max\limits_{1 \le i \le n} |\alpha_i| \left(\sum_{i=1}^{n} |\alpha_i|^{\gamma q} \right)^{\frac{1}{\gamma q}} \left(\sum_{i,j=1}^{n} |(z_i, z_j)| \right)^{\frac{1}{p}} \\ \qquad \times \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(z_i, z_j)| \right)^{\delta} \right)^{\frac{1}{\delta q}}, \\ \qquad\qquad \text{if } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \max\limits_{1 \le i \le n} |\alpha_i| \left(\sum_{i=1}^{n} |\alpha_i|^q \right)^{\frac{1}{q}} \left(\sum_{i,j=1}^{n} |(z_i, z_j)| \right)^{\frac{1}{p}} \\ \qquad \times \max\limits_{1 \le i \le n} \left(\sum_{j=1}^{n} |(z_i, z_j)| \right)^{\frac{1}{q}}; \end{cases}$$

$$B := \begin{cases} \max\limits_{1\le i\le n} |\alpha_i| \left(\sum_{i=1}^n |\alpha_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \\ \qquad \times \left(\sum_{i,j=1}^n |(z_i, z_j)|\right)^{\frac{1}{q}} \left(\sum_{i=1}^n \left(\sum_{j=1}^n |(z_i, z_j)|\right)^{\beta}\right)^{\frac{1}{\beta q}}, \\ \hfill \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \left(\sum_{i=1}^n |\alpha_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \left(\sum_{i=1}^n |\alpha_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \\ \qquad \times \left(\sum_{i=1}^n \left(\sum_{j=1}^n |(z_i, z_j)|\right)^{\beta}\right)^{\frac{1}{p\beta}} \left(\sum_{i=1}^n \left(\sum_{j=1}^n |(z_i, z_j)|\right)^{\delta}\right)^{\frac{1}{\delta q}} \\ \hfill \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1 \text{ and } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left(\sum_{i=1}^n |\alpha_i|^{q}\right)^{\frac{1}{q}} \left(\sum_{i=1}^n |\alpha_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \max\limits_{1\le i\le n} \left(\sum_{j=1}^n |(z_i, z_j)|\right)^{\frac{1}{q}} \\ \qquad \times \left(\sum_{i=1}^n \left(\sum_{j=1}^n |(z_i, z_j)|\right)^{\beta}\right)^{\frac{1}{p\beta}}, \qquad \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \end{cases}$$

and

$$C := \begin{cases} \max\limits_{1\le i\le n} |\alpha_i| \left(\sum_{i=1}^n |\alpha_i|^{p}\right)^{\frac{1}{p}} \max\limits_{1\le i\le n} \left(\sum_{j=1}^n |(z_i, z_j)|\right)^{\frac{1}{p}} \\ \qquad \times \left(\sum_{i,j=1}^n |(z_i, z_j)|\right)^{\frac{1}{q}}; \\ \left(\sum_{i=1}^n |\alpha_i|^{p}\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |\alpha_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \max\limits_{1\le i\le n} \left(\sum_{j=1}^n |(z_i, z_j)|\right)^{\frac{1}{p}} \\ \qquad \times \left(\sum_{i=1}^n \left(\sum_{j=1}^n |(z_i, z_j)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \quad \text{if } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left(\sum_{i=1}^n |\alpha_i|^{p}\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |\alpha_i|^{q}\right)^{\frac{1}{q}} \max\limits_{1\le i\le n} \left(\sum_{j=1}^n |(z_i, z_j)|\right), \end{cases}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

PROOF. We observe that

$$\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 = \left(\sum_{i=1}^n \alpha_i z_i, \sum_{j=1}^n \alpha_j z_j \right) \tag{3.5}$$
$$= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} (z_i, z_j) = \left| \sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} (z_i, z_j) \right|$$
$$\le \sum_{i=1}^n \sum_{j=1}^n |\alpha_i| |\alpha_j| |(z_i, z_j)| =: M.$$

If one uses the Hölder inequality for double sums, i.e., we recall it

$$\sum_{i,j=1}^{n} m_{ij} a_{ij} b_{ij} \leq \left(\sum_{i,j=1}^{n} m_{ij} a_{ij}^{p}\right)^{\frac{1}{p}} \left(\sum_{i,j=1}^{n} m_{ij} b_{ij}^{q}\right)^{\frac{1}{q}},$$

where $m_{ij}, a_{ij}, b_{ij} \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$; then

$$\begin{aligned} M &\leq \left(\sum_{i,j=1}^{n} |(z_i, z_j)| \, |\alpha_i|^{p}\right)^{\frac{1}{p}} \left(\sum_{i,j=1}^{n} |(z_i, z_j)| \, |\alpha_i|^{q}\right)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^{n} |\alpha_i|^{p} \left(\sum_{j=1}^{n} |(z_i, z_j)|\right)\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |\alpha_i|^{q} \left(\sum_{j=1}^{n} |(z_i, z_j)|\right)\right)^{\frac{1}{q}}, \end{aligned} \tag{3.6}$$

and the first inequality in (3.4) is proved.

Observe that

$$\sum_{i=1}^{n} |\alpha_i|^{p} \left(\sum_{j=1}^{n} |(z_i, z_j)|\right) \leq \begin{cases} \max\limits_{1 \leq i \leq n} |\alpha_i|^{p} \sum_{i,j=1}^{n} |(z_i, z_j)| ; \\ \left(\sum_{i=1}^{n} |\alpha_i|^{\alpha p}\right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(z_i, z_j)|\right)^{\beta}\right)^{\frac{1}{\beta}} \\ \qquad\qquad \text{if } \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum_{i=1}^{n} |\alpha_i|^{p} \max\limits_{1 \leq i \leq n} \left(\sum_{j=1}^{n} |(z_i, z_j)|\right); \end{cases}$$

giving

$$\left(\sum_{i=1}^{n} |\alpha_i|^{p} \left(\sum_{j=1}^{n} |(z_i, z_j)|\right)\right)^{\frac{1}{p}} \leq \begin{cases} \max\limits_{1 \leq i \leq n} |\alpha_i| \left(\sum_{i,j=1}^{n} |(z_i, z_j)|\right)^{\frac{1}{p}} ; \\ \left(\sum_{i=1}^{n} |\alpha_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(z_i, z_j)|\right)^{\beta}\right)^{\frac{1}{\beta p}} \\ \qquad\qquad \text{if } \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \left(\sum_{i=1}^{n} |\alpha_i|^{p}\right)^{\frac{1}{p}} \max\limits_{1 \leq i \leq n} \left(\sum_{j=1}^{n} |(z_i, z_j)|\right)^{\frac{1}{p}} . \end{cases} \tag{3.7}$$

Similarly, we have

$$\left(\sum_{i=1}^{n}|\alpha_i|^q\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)\right)^{\frac{1}{q}} \tag{3.8}$$
$$\leq \begin{cases} \max\limits_{1\leq i\leq n}|\alpha_i|\left(\sum_{i,j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{q}} \\ \left(\sum_{i=1}^{n}|\alpha_i|^{\gamma q}\right)^{\frac{1}{\gamma q}}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)^{\delta}\right)^{\frac{1}{\delta q}} \\ \qquad\qquad\qquad\text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \left(\sum_{i=1}^{n}|\alpha_i|^q\right)^{\frac{1}{q}}\max\limits_{1\leq i\leq n}\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{q}}. \end{cases}$$

Using (3.5) and (3.7) – (3.8), we deduce the 9 inequalities in the second part of (3.4). ∎

If we choose $p=q=2$, then the following result holds [**4**].

Corollary 47. *If $z_1,\dots,z_n\in H$ and $\alpha_1,\dots,\alpha_n\in\mathbb{K}$, then one has*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \leq \sum_{i=1}^{n}|\alpha_i|^2\left(\sum_{j=1}^{n}|(z_i,z_j)|\right) \leq \begin{cases} D \\ E \\ F \end{cases}, \tag{3.9}$$

where

$$D:=\begin{cases} \max\limits_{1\leq i\leq n}|\alpha_i|^2\sum_{i,j=1}^{n}|(z_i,z_j)|; \\ \max\limits_{1\leq i\leq n}|\alpha_i|\left(\sum_{i=1}^{n}|\alpha_i|^{2\gamma}\right)^{\frac{1}{2\gamma}}\left(\sum_{i,j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}} \\ \qquad\times\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)^{\delta}\right)^{\frac{1}{2\delta}}, \\ \qquad\qquad\qquad\textit{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \max\limits_{1\leq i\leq n}|\alpha_i|\left(\sum_{i=1}^{n}|\alpha_i|^2\right)^{\frac{1}{2}}\left(\sum_{i,j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}} \\ \qquad\times\max\limits_{1\leq i\leq n}\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}}; \end{cases}$$

$$E:=\begin{cases}\max\limits_{1\le i\le n}|\alpha_i|\left(\sum_{i=1}^n|\alpha_i|^{2\alpha}\right)^{\frac{1}{2\alpha}}\left(\sum_{i,j=1}^n|(z_i,z_j)|\right)^{\frac12}\\ \qquad\times\left(\sum_{i=1}^n\left(\sum_{j=1}^n|(z_i,z_j)|\right)^{\beta}\right)^{\frac{1}{2\beta}}, \quad \text{if } \alpha>1,\ \frac1\alpha+\frac1\beta=1;\\ \left(\sum_{i=1}^n|\alpha_i|^{2\alpha}\right)^{\frac{1}{2\alpha}}\left(\sum_{i=1}^n|\alpha_i|^{2\gamma}\right)^{\frac{1}{2\gamma}}\left(\sum_{i=1}^n\left(\sum_{j=1}^n|(z_i,z_j)|\right)^{\beta}\right)^{\frac{1}{2\beta}}\\ \qquad\times\left(\sum\limits_{i=1}^n\left(\sum_{j=1}^n|(z_i,z_j)|\right)^{\delta}\right)^{\frac{1}{2\delta}}\\ \qquad\text{if } \alpha>1,\ \frac1\alpha+\frac1\beta=1 \text{ and } \gamma>1,\ \frac1\gamma+\frac1\delta=1;\\ \left(\sum_{i=1}^n|\alpha_i|^{2}\right)^{\frac12}\left(\sum_{i=1}^n|\alpha_i|^{2\alpha}\right)^{\frac{1}{2\alpha}}\max\limits_{1\le i\le n}\left(\sum_{j=1}^n|(z_i,z_j)|\right)^{\frac12}\\ \qquad\times\left(\sum_{i=1}^n\left(\sum_{j=1}^n|(z_i,z_j)|\right)^{\beta}\right)^{\frac{1}{2\beta}}, \quad \text{if } \alpha>1,\ \frac1\alpha+\frac1\beta=1;\end{cases}$$

and

$$F:=\begin{cases}\max\limits_{1\le i\le n}|\alpha_i|\left(\sum_{i=1}^n|\alpha_i|^{2}\right)^{\frac12}\\ \qquad\times\max\limits_{1\le i\le n}\left(\sum_{j=1}^n|(z_i,z_j)|\right)^{\frac12}\left(\sum_{i,j=1}^n|(z_i,z_j)|\right)^{\frac12};\\ \left(\sum_{i=1}^n|\alpha_i|^{2}\right)^{\frac12}\left(\sum_{i=1}^n|\alpha_i|^{2\gamma}\right)^{\frac{1}{2\gamma}}\\ \qquad\times\max\limits_{1\le i\le n}\left(\sum_{j=1}^n|(z_i,z_j)|\right)^{\frac12}\left(\sum_{i=1}^n\left(\sum_{j=1}^n|(z_i,z_j)|\right)^{\delta}\right)^{\frac{1}{2\delta}},\\ \qquad\text{if } \gamma>1,\ \frac1\gamma+\frac1\delta=1;\\ \sum_{i=1}^n|\alpha_i|^2\max\limits_{1\le i\le n}\left(\sum_{j=1}^n|(z_i,z_j)|\right).\end{cases}$$

3.3. Some Pečarić Type Inequalities. We are now able to point out the following result obtained in [**4**], which complements and generalises the inequality (3.1) due to J. Pečarić.

THEOREM 56. *Let $x,y_1,\dots,y_n$ be vectors of an inner product space $(H;(\cdot,\cdot))$ and $c_1,\dots,c_n\in\mathbb{K}$. Then one has the inequalities:*

$$\left|\sum_{i=1}^n c_i\,(x,y_i)\right|^2 \tag{3.10}$$

$$\le\|x\|^2\left(\sum_{i=1}^n|c_i|^p\left(\sum_{j=1}^n|(y_i,y_j)|\right)\right)^{\frac1p}\left(\sum_{i=1}^n|c_i|^q\left(\sum_{j=1}^n|(y_i,y_j)|\right)\right)^{\frac1q}$$

$$\leq \|x\|^2 \times \begin{cases} G \\ H \\ I \end{cases},$$

where

$$G := \begin{cases} \max\limits_{1\leq i\leq n} |c_i|^2 \sum_{i,j=1}^n |(y_i, y_j)|; \\ \max\limits_{1\leq i\leq n} |c_i| \left(\sum_{i=1}^n |c_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \left(\sum_{i,j=1}^n |(y_i, y_j)|\right)^{\frac{1}{p}} \\ \qquad \times \left(\sum_{i=1}^n \left(\sum_{j=1}^n |(y_i, y_j)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \quad \textit{if } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \max\limits_{1\leq i\leq n} |c_i| \left(\sum_{i=1}^n |c_i|^{q}\right)^{\frac{1}{q}} \left(\sum_{i,j=1}^n |(y_i, y_j)|\right)^{\frac{1}{p}} \\ \qquad \times \max\limits_{1\leq i\leq n} \left(\sum_{j=1}^n |(y_i, y_j)|\right)^{\frac{1}{q}}; \end{cases}$$

$$H := \begin{cases} \max\limits_{1\leq i\leq n} |c_i| \left(\sum_{i=1}^n |c_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \left(\sum_{i,j=1}^n |(y_i, y_j)|\right)^{\frac{1}{q}} \\ \qquad \times \left(\sum_{i=1}^n \left(\sum_{j=1}^n |(y_i, y_j)|\right)^{\beta}\right)^{\frac{1}{p\beta}}, \quad \textit{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \left(\sum_{i=1}^n |c_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \left(\sum_{i=1}^n |c_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \left(\sum_{i=1}^n \left(\sum_{j=1}^n |(y_i, y_j)|\right)^{\beta}\right)^{\frac{1}{p\beta}} \\ \qquad \times \left(\sum_{i=1}^n \left(\sum_{j=1}^n |(y_i, y_j)|\right)^{\delta}\right)^{\frac{1}{\delta q}} \\ \qquad\qquad \textit{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1 \textit{ and } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left(\sum_{i=1}^n |c_i|^{q}\right)^{\frac{1}{q}} \left(\sum_{i=1}^n |c_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \max\limits_{1\leq i\leq n} \left(\sum_{j=1}^n |(y_i, y_j)|\right)^{\frac{1}{q}} \\ \qquad \times \left(\sum_{i=1}^n \left(\sum_{j=1}^n |(y_i, y_j)|\right)^{\beta}\right)^{\frac{1}{p\beta}}, \quad \textit{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \end{cases}$$

and

$$I := \begin{cases} \max\limits_{1\leq i\leq n} |c_i| \left(\sum_{i=1}^n |c_i|^{p}\right)^{\frac{1}{p}} \max\limits_{1\leq i\leq n} \left(\sum_{j=1}^n |(y_i, y_j)|\right)^{\frac{1}{p}} \\ \qquad \times \left(\sum_{i,j=1}^n |(y_i, y_j)|\right)^{\frac{1}{q}}; \\ \left(\sum_{i=1}^n |c_i|^{p}\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |c_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \max\limits_{1\leq i\leq n} \left(\sum_{j=1}^n |(y_i, y_j)|\right)^{\frac{1}{p}} \\ \qquad \times \left(\sum_{i=1}^n \left(\sum_{j=1}^n |(y_i, y_j)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \quad \textit{if } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left(\sum_{i=1}^n |c_i|^{p}\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |c_i|^{q}\right)^{\frac{1}{q}} \max\limits_{1\leq i\leq n} \left(\sum_{j=1}^n |(y_i, y_j)|\right); \end{cases}$$

for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

PROOF. We note that

$$\sum_{i=1}^{n} c_i\left(x,y_i\right)=\left(x,\sum_{i=1}^{n}\overline{c_i}y_i\right).$$

Using Schwarz's inequality in inner product spaces, we have

$$\left|\sum_{i=1}^{n} c_i\left(x,y_i\right)\right|^2\leq\|x\|^2\left\|\sum_{i=1}^{n}\overline{c_i}y_i\right\|^2.$$

Finally, using Lemma 9 with $\alpha_i=\overline{c_i}$, $z_i=y_i$ $(i=1,\dots,n)$, we deduce the desired inequality (3.10). ∎

REMARK 52. *If in (3.10) we choose* $p=q=2$, *we obtain amongst others, the result* (3.1) *due to J. Pečarić.*

3.4. More Results of Bombieri Type. The following results of Bombieri type hold [**4**].

THEOREM 57. *Let* $x,y_1,\dots,y_n\in H$. *Then one has the inequality:*

$$\begin{aligned}\sum_{i=1}^{n}\left|(x,y_i)\right|^2&\leq\|x\|\left[\sum_{i=1}^{n}\left|(x,y_i)\right|^p\left(\sum_{j=1}^{n}\left|(y_i,y_j)\right|\right)\right]^{\frac{1}{2p}}\\&\quad\times\left[\sum_{i=1}^{n}\left|(x,y_i)\right|^q\left(\sum_{j=1}^{n}\left|(y_i,y_j)\right|\right)\right]^{\frac{1}{2q}}\\&\leq\|x\|\times\begin{cases}J\\K\\L\end{cases},\end{aligned}\tag{3.11}$$

where

$$J:=\begin{cases}\max\limits_{1\leq i\leq n}\left|(x,y_i)\right|\left(\sum_{i,j=1}^{n}\left|(y_i,y_j)\right|\right)^{\frac{1}{2}};\\ \max\limits_{1\leq i\leq n}\left|(x,y_i)\right|^{\frac{1}{2}}\left(\sum_{i=1}^{n}\left|(x,y_i)\right|^{\gamma q}\right)^{\frac{1}{2\gamma q}}\left(\sum_{i,j=1}^{n}\left|(y_i,y_j)\right|\right)^{\frac{1}{2p}}\\ \qquad\times\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left|(y_i,y_j)\right|\right)^{\delta}\right)^{\frac{1}{2\delta q}},\quad \textit{if }\gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1;\\ \max\limits_{1\leq i\leq n}\left|(x,y_i)\right|^{\frac{1}{2}}\left(\sum_{i=1}^{n}\left|(x,y_i)\right|^{q}\right)^{\frac{1}{2q}}\\ \qquad\times\left(\sum_{i,j=1}^{n}\left|(y_i,y_j)\right|\right)^{\frac{1}{2p}}\max\limits_{1\leq i\leq n}\left(\sum_{j=1}^{n}\left|(y_i,y_j)\right|\right)^{\frac{1}{2q}};\end{cases}$$

$$K := \begin{cases} \max\limits_{1\le i\le n} |(x,y_i)|^{\frac{1}{2}} \\ \quad \times \left(\sum_{i=1}^n |(x,y_i)|^{\alpha p}\right)^{\frac{1}{2\alpha\beta}} \left(\sum_{i,j=1}^n |(y_i,y_j)|\right)^{\frac{1}{2q}} \\ \quad \times \left(\sum_{i=1}^n \left(\sum_{j=1}^n |(y_i,y_j)|\right)^{\beta}\right)^{\frac{1}{p\beta}}, \ \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \left(\sum_{i=1}^n |(x,y_i)|^{\alpha p}\right)^{\frac{1}{2\alpha p}} \left(\sum\limits_{i=1}^n |(x,y_i)|^{\gamma q}\right)^{\frac{1}{2\gamma q}} \\ \qquad \times \left(\sum_{i=1}^n \left(\sum_{j=1}^n |(y_i,y_j)|\right)^{\beta}\right)^{\frac{1}{2p\beta}} \\ \qquad \times \left(\sum_{i=1}^n \left(\sum_{j=1}^n |(y_i,y_j)|\right)^{\delta}\right)^{\frac{1}{2\delta q}} \\ \qquad\qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1 \text{ and } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \left(\sum_{i=1}^n |(x,y_i)|^{q}\right)^{\frac{1}{2q}} \left(\sum_{i=1}^n |(x,y_i)|^{\alpha p}\right)^{\frac{1}{2\alpha p}} \\ \quad \times \max\limits_{1\le i\le n} \left(\sum_{j=1}^n |(y_i,y_j)|\right)^{\frac{1}{2p}} \\ \quad \times \left(\sum_{i=1}^n \left(\sum_{j=1}^n |(y_i,y_j)|\right)^{\beta}\right)^{\frac{1}{2p\beta}}, \quad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \end{cases}$$

and

$$L := \begin{cases} \max\limits_{1\le i\le n} |(x,y_i)|^{\frac{1}{2}} \\ \qquad \times \left(\sum_{i=1}^n |(x,y_i)|^{p}\right)^{\frac{1}{2p}} \max\limits_{1\le i\le n} \left(\sum_{j=1}^n |(y_i,y_j)|\right)^{\frac{1}{2p}} \\ \qquad \times \left(\sum_{i,j=1}^n |(y_i,y_j)|\right)^{\frac{1}{2q}}; \\ \left(\sum_{i=1}^n |(x,y_i)|^{p}\right)^{\frac{1}{2p}} \left(\sum_{i=1}^n |(x,y_i)|^{\gamma q}\right)^{\frac{1}{2\gamma q}} \\ \quad \times \max\limits_{1\le i\le n} \left(\sum_{j=1}^n |(y_i,y_j)|\right)^{\frac{1}{2p}} \\ \quad \times \left(\sum_{i=1}^n \left(\sum_{j=1}^n |(y_i,y_j)|\right)^{\delta}\right)^{\frac{1}{2\delta q}}, \quad \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \left(\sum_{i=1}^n |(x,y_i)|^{p}\right)^{\frac{1}{2p}} \left(\sum_{i=1}^n |(x,y_i)|^{q}\right)^{\frac{1}{2q}} \\ \qquad \times \max\limits_{1\le i\le n} \left(\sum_{j=1}^n |(y_i,y_j)|\right)^{\frac{1}{2}}, \end{cases}$$

for $p>1, \frac{1}{p}+\frac{1}{q}=1.$

Proof. The proof follows by Theorem 56 on choosing $c_i = \overline{(x,y_i)}$, $i \in \{1,\dots,n\}$ and taking the square root in both sides of the inequalities involved. We omit the details. ∎

REMARK 53. *We observe, by the last inequality in* (3.11), *that*

$$\frac{\left(\sum_{i=1}^{n}\left|(x,y_i)\right|^2\right)^2}{\left(\sum_{i=1}^{n}\left|(x,y_i)\right|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}\left|(x,y_i)\right|^q\right)^{\frac{1}{q}}} \leq \|x\|^2 \max_{1\leq i\leq n}\left(\sum_{j=1}^{n}\left|(y_i,y_j)\right|\right),$$

where $p>1, \frac{1}{p}+\frac{1}{q}=1.$

If in this inequality we choose $p=q=2$, *then we recapture Bombieri's result* (3.2).

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CHAPTER 5

Some Grüss' Type Inequalities for n-Tuples of Vectors

1. Introduction

We start by recalling some of the most important Grüss type discrete inequalities for real numbers that are available in the literature.

1. (1950) *Biernacki, Pidek, Ryll-Nardzewski* [**2**].

   If $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ and $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ are n-tuples of real numbers such that there exists the real numbers a, A, b, B with

$$a \leq a_i \leq A, \quad b \leq b_i \leq B, \quad i \in \{1, \dots, n\}, \tag{1.1}$$

   then

$$\begin{aligned} \left|C_n\left(\bar{\mathbf{a}}, \bar{\mathbf{b}}\right)\right| &\leq \frac{1}{n}\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)(A-a)(B-b) \\ &= \frac{1}{n^2}\left[\frac{n^2}{4}\right](A-a)(B-b) \\ &\leq \frac{1}{4}(A-a)(B-b). \end{aligned}$$

2. (1988) *Andrica-Badea* [**1**].

   Let $\bar{\mathbf{a}}, \bar{\mathbf{b}}$ satisfy (1.1) and $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ be an $n-$tuple of nonnegative numbers with $P_n > 0$. If S is a subset of $\{1, \dots, n\}$ that minimises the expression

$$\left|\sum_{i \in S} q_i - \frac{1}{2} Q_n\right|,$$

   then

$$\begin{aligned} C_n\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) &\leq \frac{Q_S}{Q_n}\left(1 - \frac{Q_S}{Q_n}\right)(A-a)(B-b) \\ &\leq \frac{1}{4}(A-a)(B-b), \end{aligned}$$

where $Q_S := \sum_{i\in S} Q_i$.

3. (2000) *Dragomir-Booth* [**13**].

If $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ are real $n-$tuples and $\bar{\mathbf{p}}$ is nonnegative with $P_n > 0$, then

$$\left|C_n\left(\bar{\mathbf{p}},\bar{\mathbf{a}},\bar{\mathbf{b}}\right)\right| \le \max_{1\le j\le n-1}|\Delta a_j| \max_{1\le j\le n-1}|\Delta b_j| \, C_n\left(\bar{\mathbf{p}},\bar{\mathbf{e}},\bar{\mathbf{e}}\right),$$

where $\bar{\mathbf{e}} = (1,2,\ldots,n)$ and $\Delta a_j := a_{j+1} - a_j$ is the forward difference, $j = 1,\ldots,n-1$. Note that

$$C_n\left(\bar{\mathbf{p}},\bar{\mathbf{e}},\bar{\mathbf{e}}\right) = \frac{1}{P_n^2}\sum_{i=1}^n i^2 p_i - \left(\frac{1}{P_n}\sum_{i=1}^n i p_i\right)^2.$$

In particular, we have

$$\left|C_n\left(\bar{\mathbf{a}},\bar{\mathbf{b}}\right)\right| \le \frac{1}{12}\left(n^2-1\right) \max_{1\le j\le n-1}|\Delta a_j| \max_{1\le j\le n-1}|\Delta b_j|.$$

The constant $\frac{1}{12}$ is best possible.

4. (2002) *Dragomir* [**8**].

With the assumptions in 3, the following inequality holds

$$\left|C_n\left(\bar{\mathbf{p}},\bar{\mathbf{a}},\bar{\mathbf{b}}\right)\right| \le \frac{1}{P_n^2}\sum_{1\le j<i\le n}(i-j)\left(\sum_{k=1}^{n-1}|\Delta a_k|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n-1}|\Delta b_k|^q\right)^{\frac{1}{q}},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, we have

$$\left|C_n\left(\bar{\mathbf{a}},\bar{\mathbf{b}}\right)\right| \le \frac{1}{6}\cdot\frac{n^2-1}{n}\left(\sum_{k=1}^{n-1}|\Delta a_k|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n-1}|\Delta b_k|^q\right)^{\frac{1}{q}}.$$

The constant $\frac{1}{6}$ is best possible.

5. (2002) *Dragomir* [**6**].

The following inequality holds, where $\bar{\mathbf{p}},\bar{\mathbf{a}},\bar{\mathbf{b}}$ and P_n are as in assumption 3,

$$\left|C_n\left(\bar{\mathbf{p}},\bar{\mathbf{a}},\bar{\mathbf{b}}\right)\right| \le \frac{1}{2}\cdot\frac{1}{P_n^2}\sum_{i=1}^n p_i\left(P_n - p_i\right)\sum_{k=1}^{n-1}|\Delta a_k|\sum_{k=1}^{n-1}|\Delta b_k|.$$

In particular, we have

$$\left|C_n\left(\bar{\mathbf{a}},\bar{\mathbf{b}}\right)\right| \le \frac{1}{2}\left(1-\frac{1}{n}\right)\sum_{k=1}^{n-1}|\Delta a_k|\sum_{k=1}^{n-1}|\Delta b_k|.$$

The constant $\frac{1}{2}$ is sharp.

6. (2002) *Cerone-Dragomir* [**3**].

If $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ are real $n-$tuples and $\bar{\mathbf{p}}$ is a positive $n-$tuple and there exists $m, M \in \mathbb{R}$ such that

$$m \leq a_i \leq M,$$

then one has the inequality

$$\left|C_n\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right)\right| \leq \frac{1}{2}(M-m) \frac{1}{P_n} \sum_{i=1}^n p_i\left|b_i-\frac{1}{P_n} \sum_{j=1}^n p_j b_j\right|.$$

The constant $\frac{1}{2}$ is best possible.

In particular, we have

$$\left|C_n\left(\bar{\mathbf{a}}, \bar{\mathbf{b}}\right)\right| \leq \frac{1}{2}(M-m) \cdot \frac{1}{n} \sum_{i=1}^n\left|b_i-\frac{1}{n} \sum_{j=1}^n b_j\right|.$$

The constant $\frac{1}{2}$ is best possible.

The main aim of this chapter is to present some extensions of the above results holding in the general setting of n-tuples of vectors in an inner product space.

2. The Version for Norms

2.1. Preliminary Results. The following lemma is of interest in itself [**5**].

LEMMA 10. *Let $(H ; \langle\cdot, \cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x_i \in H$ and $p_i \geq 0$ $(i=1, \ldots, n)$ such that $\sum_{i=1}^n p_i=1$ $(n \geq 2)$. If $x, X \in H$ are such that*

$$\operatorname{Re}\left\langle X-x_i, x_i-x\right\rangle \geq 0 \text{ for all } i \in\{1, \ldots, n\}, \tag{2.1}$$

or, equivalently,

$$\left\|x_i-\frac{x+X}{2}\right\| \leq \frac{1}{2}\|X-x\| \text{ for all } i \in\{1, \ldots, n\},$$

then we have the inequality

$$0 \leq \sum_{i=1}^n p_i\left\|x_i\right\|^2-\left\|\sum_{i=1}^n p_i x_i\right\|^2 \leq \frac{1}{4}\|X-x\|^2. \tag{2.2}$$

The constant $\frac{1}{4}$ is sharp.

PROOF. Define

$$I_1:=\left\langle X-\sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i x_i-x\right\rangle$$

and

$$I_2 := \sum_{i=1}^{n} p_i \langle X - x_i, x_i - x \rangle .$$

Then

$$I_1 = \sum_{i=1}^{n} p_i \langle X, x_i \rangle - \langle X, x \rangle - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 + \sum_{i=1}^{n} p_i \langle x_i, x \rangle$$

and

$$I_2 = \sum_{i=1}^{n} p_i \langle X, x_i \rangle - \langle X, x \rangle - \sum_{i=1}^{n} p_i \|x_i\|^2 + \sum_{i=1}^{n} p_i \langle x_i, x \rangle .$$

Consequently,

$$I_1 - I_2 = \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 . \tag{2.3}$$

Taking the real value in (2.3), we can state that

$$\begin{aligned} &\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \\ &= \operatorname{Re} \left\langle X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x \right\rangle - \sum_{i=1}^{n} p_i \operatorname{Re} \langle X - x_i, x_i - x \rangle , \end{aligned} \tag{2.4}$$

which is also an identity of interest in itself.
Using the assumption (2.1), we can conclude, by (2.4), that

$$\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \leq \operatorname{Re} \left\langle X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x \right\rangle . \tag{2.5}$$

It is known that if $y, z \in H$, then

$$4 \operatorname{Re} \langle z, y \rangle \leq \|z + y\|^2 , \tag{2.6}$$

with equality iff $z = y$.

Now, by (2.6) we can state that

$$\begin{aligned} \operatorname{Re} \left\langle X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x \right\rangle &\leq \frac{1}{4} \left\| X - \sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} p_i x_i - x \right\|^2 \\ &= \frac{1}{4} \|X - x\|^2 . \end{aligned}$$

Using (2.5), we obtain (2.2).

To prove the sharpness of the constant $\frac{1}{4}$, let us assume that the inequality (2.2) holds with a constant $c > 0$, i.e.,

$$0 \leq \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \leq c \|X - x\|^2 \tag{2.7}$$

for all p_i, x_i and n as in the hypothesis of Lemma 10.

Assume that $n = 2$, $p_1 = p_2 = \frac{1}{2}$, $x_1 = x$ and $x_2 = X$ with $x, X \in H$ and $x \neq X$. Then, obviously,

$$\langle X - x_1, x_1 - x \rangle = \langle X - x_2, x_2 - x \rangle = 0,$$

which shows that the condition (2.1) holds.

If we replace n, p_1, p_2, x_1, x_2 in (2.7), we obtain

$$\begin{aligned} \sum_{i=1}^{2} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{2} p_i x_i \right\|^2 &= \frac{1}{2} \left(\|x\|^2 + \|X\|^2 \right) - \left\| \frac{x + X}{2} \right\|^2 \\ &= \frac{1}{4} \|x - X\|^2 \\ &\leq c \|x - X\|^2, \end{aligned}$$

from where we deduce that $c \geq \frac{1}{4}$, which proves the sharpness of the constant $\frac{1}{4}$. ∎

Remark 54. *The assumption* (2.1) *can be replaced by the more general condition*

$$\sum_{i=1}^{n} p_i \operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0, \tag{2.8}$$

and the conclusion (2.2) *will still remain valid.*

The following corollary is natural.

Corollary 48. *Let* $a_i \in \mathbb{K}$, $p_i \geq 0$, $(i = 1, \ldots, n)$ $(n \geq 2)$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $a, A \in \mathbb{K}$ *are such that*

$$\operatorname{Re} \left[(A - a_i) (\bar{a}_i - \bar{a}) \right] \geq 0 \text{ for all } i \in \{1, \ldots, n\}, \tag{2.9}$$

then we have the inequality

$$0 \leq \sum_{i=1}^{n} p_i |a_i|^2 - \left| \sum_{i=1}^{n} p_i a_i \right|^2 \leq \frac{1}{4} |A - a|^2 . \tag{2.10}$$

The constant $\frac{1}{4}$ *is sharp.*

The proof follows by the above lemma by choosing $H = \mathbb{K}$, $\langle x, y \rangle := x\bar{y}$, $x_i = a_i$, $x = a$ and $X = A$. We omit the details.

REMARK 55. *The condition* (2.9) *can be replaced by the more general assumption*

$$\sum_{i=1}^{n} p_i \operatorname{Re}\left[(A - a_i)(\bar{a}_i - \bar{a})\right] \geq 0.$$

2.2. A Discrete Inequality of Grüss' Type. The following Grüss type inequality holds [**5**].

THEOREM 58. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be an inner product space over* $\mathbb{K}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $x_i \in H$, $a_i \in \mathbb{K}$, $p_i \geq 0$ $(i = 1, \ldots, n)$ $(n \geq 2)$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $a, A \in \mathbb{K}$ *and* $x, X \in H$ *are such that*

$$\operatorname{Re}\left[(A - a_i)(\bar{a}_i - \bar{a})\right] \geq 0, \ \ \operatorname{Re}\langle X - x_i, x_i - x \rangle \geq 0 \tag{2.11}$$

for all $i \in \{1, \ldots, n\}$; *then we have the inequality*

$$0 \leq \left\| \sum_{i=1}^{n} p_i a_i x_i - \sum_{i=1}^{n} p_i a_i \cdot \sum_{i=1}^{n} p_i x_i \right\| \leq \frac{1}{4} |A - a| \, \|X - x\| . \tag{2.12}$$

The constant $\frac{1}{4}$ *is sharp.*

PROOF. A simple computation shows that

$$\sum_{i=1}^{n} p_i a_i x_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i x_i = \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j (a_i - a_j)(x_i - x_j). \tag{2.13}$$

Taking the norm in both parts of (2.13) and using the generalized triangle inequality, we obtain

$$\begin{aligned} \left\| \sum_{i=1}^{n} p_i a_i x_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i x_i \right\| \\ \leq \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j |a_i - a_j| \, \|x_i - x_j\| . \end{aligned} \tag{2.14}$$

By the Cauchy-Bunyakovsky-Schwarz discrete inequality for double sums, we obtain

$$\begin{aligned} &\left(\frac{1}{2} \sum_{i,j=1}^{n} p_i p_j |a_i - a_j| \, \|x_i - x_j\| \right)^2 \\ &\quad \leq \left(\frac{1}{2} \sum_{i,j=1}^{n} p_i p_j |a_i - a_j|^2 \right) \left(\frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^2 \right). \end{aligned} \tag{2.15}$$

As a simple calculation reveals that

$$\frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \left|a_i - a_j\right|^2 = \sum_{i=1}^{n} p_i \left|a_i\right|^2 - \left|\sum_{i=1}^{n} p_i a_i\right|^2$$

and

$$\frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \left\|x_i - x_j\right\|^2 = \sum_{i=1}^{n} p_i \left\|x_i\right\|^2 - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2,$$

then, by (2.14) and (2.15), we conclude that

$$\begin{aligned}(2.16)\quad & \left\|\sum_{i=1}^{n} p_i a_i x_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i x_i\right\| \\ & \leq \left(\sum_{i=1}^{n} p_i \left|a_i\right|^2 - \left|\sum_{i=1}^{n} p_i a_i\right|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} p_i \left\|x_i\right\|^2 - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2\right)^{\frac{1}{2}}.\end{aligned}$$

However, from Lemma 10 and Corollary 48, we know that

$$\left(\sum_{i=1}^{n} p_i \left\|x_i\right\|^2 - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2\right)^{\frac{1}{2}} \leq \frac{1}{2}\left\|X - x\right\| \tag{2.17}$$

and

$$\left(\sum_{i=1}^{n} p_i \left|a_i\right|^2 - \left|\sum_{i=1}^{n} p_i a_i\right|^2\right)^{\frac{1}{2}} \leq \frac{1}{2}\left|A - a\right|. \tag{2.18}$$

Consequently, by using $(2.16)-(2.18)$, we deduce the desired estimate (2.12).

To prove the sharpness of the constant $\frac{1}{4}$, assume that (2.12) holds with a constant $c > 0$, i.e.,

$$\left\|\sum_{i=1}^{n} p_i a_i x_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i x_i\right\| \leq c\left|A - a\right|\left\|X - x\right\| \tag{2.19}$$

for all $p_i, a_i, x_i, a, A, x, X$ and n as in the hypothesis of Theorem 58.

If we choose $n = 2$, $a_1 = a$, $a_2 = A$, $x_1 = x$, $x_2 = X$ $(a \neq A, x \neq X)$ and $p_1 = p_2 = \frac{1}{2}$, then

$$\begin{aligned}\sum_{i=1}^{2} p_i a_i x_i - \sum_{i=1}^{2} p_i a_i \sum_{i=1}^{2} p_i x_i &= \frac{1}{2}\sum_{i,j=1}^{2} p_i p_j \left(a_i - a_j\right)\left(x_i - x_j\right) \\ &= \frac{1}{4}\left(a - A\right)\left(x - X\right).\end{aligned}$$

Consequently, from (2.19), we deduce

$$\frac{1}{4}|a-A|\|X-x\| \leq c|A-a|\|X-x\|,$$

which implies that $c \geq \frac{1}{4}$, and the theorem is completely proved. ∎

REMARK 56. *The condition* (2.11) *can be replaced by the more general assumption*

$$\sum_{i=1}^{n} p_i \operatorname{Re}\left[(A-a_i)(\bar{a}_i-\bar{a})\right] \geq 0, \ \sum_{i=1}^{n} p_i \operatorname{Re}\langle X-x_i, x_i-x\rangle \geq 0$$

and the conclusion (2.12) *will still be valid.*

The following corollary for real or complex numbers holds.

COROLLARY 49. *Let* $a_i, b_i \in \mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$, $p_i \geq 0$ $(i=1,\dots,n)$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $a, A, b, B \in \mathbb{K}$ *are such that*

$$\operatorname{Re}\left[(A-a_i)(\bar{a}_i-\bar{a})\right] \geq 0, \quad \operatorname{Re}\left[(B-b_i)\left(\bar{b}_i-\bar{b}\right)\right] \geq 0, \tag{2.20}$$

then we have the inequality

$$0 \leq \left|\sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i\right| \leq \frac{1}{4}|A-a||B-b|, \tag{2.21}$$

where the constant $\frac{1}{4}$ *is sharp.*

REMARK 57. *If we assume that* a_i, b_i, a, A, b, B *are real numbers, then* (2.20) *is equivalent to*

$$a \leq a_i \leq A, b \leq b_i \leq B \text{ for all } i \in \{1,\dots,n\},$$

and (2.21) *becomes*

$$0 \leq \left|\sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i\right| \leq \frac{1}{4}(A-a)(B-b),$$

which is the classical Grüss inequality for sequences of real numbers.

2.3. Applications for Discrete Fourier Transforms. Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $\bar{\mathbf{x}} = (x_1,\dots,x_n)$ be a sequence of vectors in H.

For a given $w \in \mathbb{R}$, define the *discrete Fourier transform* as

$$\mathcal{F}_w(\bar{\mathbf{x}})(m) := \sum_{k=1}^{n} \exp(2wimk) \times x_k, \ m=1,\dots,n. \tag{2.22}$$

The following approximation result for the Fourier transform (2.22) holds [**5**].

THEOREM 59. *Let* $(H;\langle\cdot,\cdot\rangle)$ *and* $\bar{\mathbf{x}}\in H^n$ *be as above. If there exists the vectors* $x,X\in H$ *such that*

$$\operatorname{Re}\langle X-x_k,x_k-x\rangle\geq 0 \text{ for all } k\in\{1,\dots,n\}, \tag{2.23}$$

then we have the inequality

$$\left\|\mathcal{F}_w(\bar{\mathbf{x}})(m)-\frac{\sin(wmn)}{\sin(wm)}\exp[w(n+1)im]\times\frac{1}{n}\sum_{k=1}^{n}x_k\right\| \leq\frac{1}{2}\|X-x\|\left[n^2-\frac{\sin^2(wmn)}{\sin^2(wm)}\right]^{\frac{1}{2}}, \tag{2.24}$$

for all $m\in\{1,\dots,n\}$ *and* $w\in\mathbb{R},\ w\neq\frac{l}{m}\pi,\ l\in\mathbb{Z}$.

PROOF. From the inequality (2.16) in Theorem 58, we can state that

$$\left\|\frac{1}{n}\sum_{k=1}^{n}a_kx_k-\frac{1}{n}\sum_{k=1}^{n}a_k\cdot\frac{1}{n}\sum_{k=1}^{n}x_k\right\| \leq\left(\frac{1}{n}\sum_{k=1}^{n}|a_k|^2-\left|\frac{1}{n}\sum_{k=1}^{n}a_k\right|^2\right)^{\frac{1}{2}}\left(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|^2-\left\|\frac{1}{n}\sum_{k=1}^{n}x_k\right\|^2\right)^{\frac{1}{2}} \tag{2.25}$$

for all $a_k\in\mathbb{K},\ x_k\in H\ (k=1,\dots,n)$.

However, the $x_k\ (k=1,\dots,n)$ satisfy (2.23), and therefore, by Lemma 10, we have

$$0\leq\frac{1}{n}\sum_{k=1}^{n}\|x_k\|^2-\left\|\frac{1}{n}\sum_{k=1}^{n}x_k\right\|^2\leq\frac{1}{4}\|X-x\|^2. \tag{2.26}$$

Consequently, by (2.25) and (2.26), we conclude that

$$\left\|\sum_{k=1}^{n}a_kx_k-\sum_{k=1}^{n}a_k\cdot\frac{1}{n}\sum_{k=1}^{n}x_k\right\| \leq\frac{1}{2}\|X-x\|\left(n\sum_{k=1}^{n}|a_k|^2-\left|\sum_{k=1}^{n}a_k\right|^2\right)^{\frac{1}{2}} \tag{2.27}$$

for all $a_k\in\mathbb{K}\ (k=1,\dots,n)$.

We now choose in (2.27), $a_k = \exp(2wimk)$ to obtain

$$\left\|\mathcal{F}_w(\bar{\mathbf{x}})(m) - \sum_{k=1}^{n} \exp(2wimk) \times \frac{1}{n}\sum_{k=1}^{n} x_k\right\| \le \frac{1}{2}\|X - x\| \left(n\sum_{k=1}^{n} |\exp(2wimk)|^2 - \left|\sum_{k=1}^{n}\exp(2wimk)\right|^2\right)^{\frac{1}{2}} \tag{2.28}$$

for all $m \in \{1, \dots, n\}$.

As a simple calculation reveals that

$$\begin{aligned}
\sum_{k=1}^{n} \exp(2wimk) &= \exp(2wim) \times \left[\frac{\exp(2wimn) - 1}{\exp(2wim) - 1}\right] \\
&= \exp(2wim) \times \left[\frac{\cos(2wmn) + i\sin(2wmn) - 1}{\cos(2wm) + i\sin(2wm) - 1}\right] \\
&= \exp(2wim) \times \frac{\sin(wmn)}{\sin(wm)}\left[\frac{\cos(wmn) + i\sin(wmn)}{\cos(wm) + i\sin(wm)}\right] \\
&= \frac{\sin(wmn)}{\sin(wm)} \times \exp(2wim)\left[\frac{\exp(iwmn)}{\exp(iwm)}\right] \\
&= \frac{\sin(wmn)}{\sin(wm)} \times \exp[w(n+1)im],
\end{aligned}$$

$$\sum_{k=1}^{n} |\exp(2wimk)|^2 = n$$

and

$$\left|\sum_{k=1}^{n}\exp(2wimk)\right|^2 = \frac{\sin^2(wmn)}{\sin^2(wm)}, \text{ for } w \ne \frac{l}{m}\pi,\ l \in \mathbb{Z},$$

thus, from (2.28), we deduce the desired inequality (2.24). ∎

Remark 58. *The assumption* (2.23) *can be replaced by the more general condition*

$$\sum_{i=1}^{n} \operatorname{Re}\langle X - x_i, x_i - x\rangle \ge 0,$$

and the conclusion (2.24) *will still remain valid.*

The following corollary is an obvious consequence of (2.24).

COROLLARY 50. *Let* $a_i \in \mathbb{K}$ $(i = 1, \ldots, n)$. *If* $a, A \in \mathbb{K}$ *are such that*

$$\operatorname{Re}\left[(A - a_i)(\bar{a}_i - \bar{a})\right] \geq 0 \text{ for all } i \in \{1, \ldots, n\}, \tag{2.29}$$

then we have an approximation of the Fourier transform for the vector $\overline{\mathbf{a}} = (a_1, \ldots, a_n) \in \mathbb{K}^n$:

$$\left\| \mathcal{F}_w(\overline{\mathbf{a}})(m) - \frac{\sin(wmn)}{\sin(wm)} \exp\left[w(n+1)im\right] \times \frac{1}{n} \sum_{k=1}^{n} a_k \right\| \leq \frac{1}{2} |A - a| \left[n^2 - \frac{\sin^2(wmn)}{\sin^2(wm)}\right]^{\frac{1}{2}}, \tag{2.30}$$

for all $m \in \{1, \ldots, n\}$ *and* $w \in \mathbb{R}$ *so that* $w \neq \frac{l}{m}\pi,\ l \in \mathbb{Z}$.

REMARK 59. *If we assume that* $\mathbb{K} = \mathbb{R}$, *then* (2.29) *is equivalent to*

$$a \leq a_i \leq A \text{ for all } i \in \{1, \ldots, n\}. \tag{2.31}$$

Consequently, with the assumption (2.31), *we obtain the following approximation of the Fourier transform*

$$\left\| \mathcal{F}_w(\overline{\mathbf{a}})(m) - \frac{\sin(wmn)}{\sin(wm)} \exp\left[w(n+1)im\right] \times \frac{1}{n} \sum_{k=1}^{n} a_k \right\| \leq \frac{1}{2}(A - a) \left[n^2 - \frac{\sin^2(wmn)}{\sin^2(wm)}\right]^{\frac{1}{2}},$$

for all $m \in \{1, \ldots, n\}$ *and* $w \neq \frac{l}{m}\pi,\ l \in \mathbb{Z}$.

2.4. Applications for the Discrete Mellin Transform. Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product over $\mathbb{R}$ and $\overline{\mathbf{x}} = (x_1, \ldots, x_n)$ be a sequence of vectors in H.

Define the *Mellin transform*:

$$\mathcal{M}(\overline{\mathbf{x}})(m) := \sum_{k=1}^{n} k^{m-1} x_k,\ m = 1, \ldots, n;$$

of the sequence $\overline{\mathbf{x}} \in H^n$.

The following approximation result holds [**5**].

THEOREM 60. *Let* H *and* $\overline{\mathbf{x}}$ *be as above. If there exist the vectors* $x, X \in H$ *such that*

$$\operatorname{Re}\langle X - x_k, x_k - x \rangle \geq 0 \text{ for all } k = 1, \ldots, n; \tag{2.32}$$

then we have the inequality

$$\left\|\mathcal{M}(\bar{\mathbf{x}})(m) - S_{m-1}(n) \cdot \frac{1}{n}\sum_{k=1}^{n} x_k\right\| \leq \frac{1}{2}\|X - x\| \left[nS_{2m-2}(n) - S_{m-1}^2(n)\right]^{\frac{1}{2}}, \ m \in \{1, \dots, n\}, \tag{2.33}$$

where $S_p(n)$, $p \in \mathbb{R}$, $n \in \mathbb{N}$ *is the* $p-$*powered sum of the first* n *natural numbers, i.e.,*

$$S_p(n) := \sum_{k=1}^{n} k^p.$$

Proof. We apply the inequality (2.27) to obtain

$$\begin{aligned}
&\left\|\sum_{k=1}^{n} k^{m-1} x_k - \sum_{k=1}^{n} k^{m-1} \cdot \frac{1}{n}\sum_{k=1}^{n} x_k\right\| \\
&\leq \frac{1}{2}\|X - x\| \left[n\sum_{k=1}^{n} k^{2(m-1)} - \left(\sum_{k=1}^{n} k^{m-1}\right)^2\right]^{\frac{1}{2}} \\
&= \frac{1}{2}\|X - x\| \left[nS_{2m-2}(n) - S_{m-1}^2(n)\right]^{\frac{1}{2}},
\end{aligned}$$

and the inequality (2.33) is proved. ∎

Consider the following particular values of Mellin Transform

$$\mu_1(\bar{\mathbf{x}}) := \sum_{k=1}^{n} k x_k$$

and

$$\mu_2(\bar{\mathbf{x}}) := \sum_{k=1}^{n} k^2 x_k.$$

The following corollary holds.

Corollary 51. *Let* H *and* $\bar{\mathbf{x}}$ *be as in Theorem 60. Then we have the inequalities:*

$$\left\|\mu_1(\bar{\mathbf{x}}) - \frac{n+1}{2} \cdot \sum_{k=1}^{n} x_k\right\| \leq \frac{1}{2}\|X - x\| \, n \left[\frac{n(n+1)}{2}\right]^{\frac{1}{2}} \tag{2.34}$$

and

$$\left\|\mu_2(\bar{x}) - \frac{(n+1)(2n+1)}{6} \cdot \sum_{k=1}^{n} x_k\right\| \leq \frac{1}{12\sqrt{5}} \|X - x\| n\sqrt{(n-1)(n+1)(2n+1)(8n+1)}. \tag{2.35}$$

REMARK 60. *If we assume that* $\overline{\mathbf{p}} = (p_1, \dots, p_n)$ *is a probability distribution, i.e.,* $p_k \geq 0$ $(k = 1, \dots, n)$ *and* $\sum_{k=1}^{n} p_k = 1$ *and* $p \leq p_k \leq P$ $(k = 1, \dots, n)$, *then by* (2.34) *and* (2.35), *we get the inequalities*

$$\left|\sum_{k=1}^{n} kp_k - \frac{n+1}{2}\right| \leq \frac{1}{2}(P - p) n \left[\frac{n(n+1)}{2}\right]^{\frac{1}{2}}$$

and

$$\left|\sum_{k=1}^{n} k^2 p_k - \frac{(n+1)(2n+1)}{6}\right| \leq \frac{1}{12\sqrt{5}} (P - p) n\sqrt{(n-1)(n+1)(2n+1)(8n+1)}.$$

2.5. Applications for Polynomials. Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ and $\bar{\mathbf{c}} = (c_0, \dots, c_n)$ be a sequence of vectors in H.

Define the polynomial $P : \mathbb{C} \to H$ with the coefficients $\bar{\mathbf{c}} = (c_0, \dots, c_n)$ by

$$P(z) = c_0 + zc_1 + \cdots + z^n c_n,\ z \in \mathbb{C},\ c_n \neq 0.$$

The following approximation result for the polynomial P holds [**5**].

THEOREM 61. *Let* $H, \bar{\mathbf{c}}$ *and* P *be as above. If there exist the vectors* $c, C \in H$ *such that*

$$\operatorname{Re}\langle C - c_k, c_k - c\rangle \geq 0 \text{ for all } k \in \{0, \dots, n\}, \tag{2.36}$$

then we have the inequality

$$\left\|P(z) - \frac{z^{n+1} - 1}{z - 1} \times \frac{c_0 + c_1 + \cdots + c_n}{n+1}\right\| \leq \frac{1}{2}\|C - c\| \left[(n+1)\frac{|z|^{2n+2} - 1}{|z|^2 - 1} - \frac{|z|^{2n+2} - 2\operatorname{Re}(z^{n+1}) + 1}{|z|^2 - 2\operatorname{Re}(z) + 1}\right]^{\frac{1}{2}} \tag{2.37}$$

for all $z \in \mathbb{C}$, $|z| \neq 1$.

PROOF. Using the inequality (2.27), we can state that

$$\begin{aligned}(2.38)\quad & \left\| \sum_{k=0}^{n} z^k c_k - \sum_{k=0}^{n} z^k \cdot \frac{1}{n+1} \sum_{k=0}^{n} c_k \right\| \\ & \leq \frac{1}{2} \|C - c\| \left((n+1) \sum_{k=0}^{n} |z|^{2k} - \left| \sum_{k=0}^{n} z^k \right|^2 \right)^{\frac{1}{2}} \\ & = \frac{1}{2} \|C - c\| \left[(n+1) \frac{|z|^{2n+2} - 1}{|z|^2 - 1} - \left| \frac{z^{n+1} - 1}{z - 1} \right|^2 \right]^{\frac{1}{2}} \\ & = \frac{1}{2} \|C - c\| \left[(n+1) \frac{|z|^{2n+2} - 1}{|z|^2 - 1} - \frac{|z|^{2n+2} - 2\operatorname{Re}(z^{n+1}) + 1}{|z|^2 - 2\operatorname{Re}(z) + 1} \right]^{\frac{1}{2}}\end{aligned}$$

and the inequality (2.37) is proved. ∎

The following result for the complex roots of the unity also holds [**5**].

THEOREM 62. *Let* $z_k := \cos\left(\frac{k\pi}{n+1}\right) + i \sin\left(\frac{k\pi}{n+1}\right),\ k \in \{0, \dots, n\}$ *be the complex* $(n+1)-$*roots of the unity. Then we have the inequality*

$$(2.39)\qquad \|P(z_k)\| \leq \frac{1}{2}(n+1)\|C - c\|,\ k \in \{1, \dots, n\};$$

where the coefficients $\bar{\mathbf{c}} = (c_0, \dots, c_n) \in H^{n+1}$ *satisfy the assumption* (2.36).

PROOF. From the inequality (2.38), we can state that

$$\begin{aligned} & \left\| P(z_k) - \frac{z^{n+1} - 1}{z - 1} \cdot \frac{1}{n+1} \sum_{k=0}^{n} c_k \right\| \\ & \qquad \leq \frac{1}{2} \|C - c\| \left[(n+1) \sum_{k=0}^{n} |z|^{2k} - \left| \frac{z^{n+1} - 1}{z - 1} \right|^2 \right]^{\frac{1}{2}}\end{aligned}$$

for all $z \in \mathbb{C},\ z \neq 1$.

Putting $z = z_k,\ k \in \{1, \dots, n\}$ and taking into account that $z_k^{n+1} = 1,\ |z_k| = 1$, we get the desired result (2.39). ∎

The following corollary is a natural consequence of Theorem 62.

COROLLARY 52. *Let* $P(z) := \sum_{k=0}^{n} a_k z^k$ *be a polynomial with real coefficients and* z_k *the* $(n+1)$*-roots of the unity as defined above. If*

$a \le a_k \le A,\ k = 0, \dots, n$, then we have the inequality:

$$|P(z_k)| \le \frac{1}{2}(n+1)(A-a).$$

2.6. Applications for Lipschitzian Mappings. Let $(H; \langle \cdot, \cdot \rangle)$ be as above and $F : H \to B$ a mapping defined on the inner product space H with values in the normed linear space B which satisfy the *Lipschitzian condition:*

$$|F(x) - F(y)| \le L \|x - y\|, \text{ for all } x, y \in H, \tag{2.40}$$

where $|\cdot|$ denotes the norm on B and $\|\cdot\|$ is the Euclidean norm on H.

The following theorem holds [**5**].

THEOREM 63. *Let $F : H \to B$ be as above and $x_i \in H$, $p_i \ge 0$ $(i = 1, \dots, n)$ with $P_n := \sum_{i=1}^n p_i > 0$. If there exists two vectors $x, X \in H$ such that*

$$\operatorname{Re}\langle X - x_i, x_i - x \rangle \ge 0 \textit{ for all } i \in \{1, \dots, n\}, \tag{2.41}$$

then we have the inequality

$$\left| \frac{1}{P_n} \sum_{i=1}^n p_i F(x_i) - F\left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i \right) \right| \le \frac{1}{2} \cdot L \|X - x\|. \tag{2.42}$$

PROOF. As F is Lipschitzian, we have (2.40) for all $x, y \in H$. Choose $x = \frac{1}{P_n} \sum_{i=1}^n p_i x_i$ and $y = x_j$ $(j = 1, \dots, n)$, to get

$$\left| F\left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i \right) - F(x_j) \right| \le L \left\| \frac{1}{P_n} \sum_{i=1}^n p_i x_i - x_j \right\|, \tag{2.43}$$

for all $j \in \{1, \dots, n\}$.

If we multiply (2.43) by $p_j \ge 0$ and sum over j from 1 to n, we obtain

$$\sum_{j=1}^n p_j \left| F\left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i \right) - F(x_j) \right| \le L \sum_{j=1}^n p_j \left\| \frac{1}{P_n} \sum_{i=1}^n p_i x_i - x_j \right\|. \tag{2.44}$$

Using the generalized triangle inequality, we have

$$\sum_{j=1}^{n} p_j \left| F\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - F(x_j)\right| \geq \left| P_n F\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - \sum_{j=1}^{n} p_j F(x_j)\right|. \tag{2.45}$$

By the Cauchy-Bunyakovsky-Schwarz inequality, we also have

$$\begin{aligned}
&\sum_{j=1}^{n} p_j \left\| \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i - x_j \right\| \\
&\leq \left[\sum_{j=1}^{n} p_j \left\| \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i - x_j \right\|^2\right]^{\frac{1}{2}} P_n^{\frac{1}{2}} \\
&= P_n^{\frac{1}{2}} \left[\sum_{j=1}^{n} p_j \left[\left\| \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \right\|^2 \right.\right. \\
&\qquad \left.\left. -2\operatorname{Re}\left\langle \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i, x_j \right\rangle + \|x_j\|^2 \right]\right]^{\frac{1}{2}} \\
&= P_n^{\frac{1}{2}} \left[P_n \left\| \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \right\|^2 \right. \\
&\qquad \left. -2\operatorname{Re}\left\langle \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} p_j x_j \right\rangle + \sum_{j=1}^{n} p_j \|x_j\|^2 \right]^{\frac{1}{2}} \\
&= P_n \left[\frac{1}{P_n}\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \right\|^2\right]^{\frac{1}{2}}.
\end{aligned} \tag{2.46}$$

Combining the above inequalities $(2.44)-(2.46)$ we deduce, by dividing with $P_n > 0$, that

$$\left| F\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - \frac{1}{P_n}\sum_{i=1}^{n} p_i F(x_i)\right| \leq L \cdot \left[\frac{1}{P_n}\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \right\|^2\right]^{\frac{1}{2}}. \tag{2.47}$$

Finally, using Lemma 10, we obtain the desired result. ∎

REMARK 61. *The condition* (2.41) *can be substituted by the more general condition*

$$\sum_{i=1}^{n} p_i \operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0,$$

and the conclusion (2.42) *will still remain valid.*

The following corollary is a natural consequence of the above findings.

COROLLARY 53. *Let* $x_i \in H$ $(i = 1, \dots, n)$ *and* $x, X \in H$ *be such that the condition* (2.41) *holds. Then we have the inequality*

$$0 \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i \|x_i\| - \left\| \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right\| \leq \frac{1}{2} \|X - x\|.$$

The proof follows by Theorem 63 by choosing $F : H \to \mathbb{R}$, $F(x) = \|x\|$ which is Lipschitzian with the constant $L = 1$, as

$$|F(x) - F(y)| = |\|x\| - \|y\|| \leq \|x - y\|,$$

for all $x, y \in H$.

3. The Version for Inner-Products

3.1. A Discrete Inequality of Grüss Type. The following Grüss type inequality holds [**7**].

THEOREM 64. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be an inner product space over* $\mathbb{K}$; $\mathbb{K} = \mathbb{C}, \mathbb{R}$, $x_i, y_i \in H$, $p_i \geq 0$ $(i = 0, \dots, n)$ $(n \geq 2)$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $x, X, y, Y \in H$ *are such that*

$$\operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \text{ and } \operatorname{Re} \langle Y - y_i, y_i - y \rangle \geq 0 \tag{3.1}$$

for all $i \in \{1, \dots, n\}$, *then we have the inequality*

$$\left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle \right| \leq \frac{1}{4} \|X - x\| \|Y - y\|. \tag{3.2}$$

The constant $\frac{1}{4}$ *is sharp.*

PROOF. A simple calculation shows that

$$\sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle = \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \langle x_i - x_j, y_i - y_j \rangle . \tag{3.3}$$

Taking the modulus in both parts of (3.3) and using the generalized triangle inequality, we obtain

$$\left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle \right| \leq \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \left| \langle x_i - x_j, y_i - y_j \rangle \right| .$$

By Schwarz's inequality in inner product spaces we have

$$\left| \langle x_i - x_j, y_i - y_j \rangle \right| \leq \| x_i - x_j \| \, \| y_i - y_j \| ,$$

for all $i, j \in \{1, \dots, n\}$, and therefore

$$\left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle \right| \leq \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \| x_i - x_j \| \, \| y_i - y_j \| .$$

Using the Cauchy-Bunyakovsky-Schwarz inequality for double sums, we can state that

$$\frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \| x_i - x_j \| \, \| y_i - y_j \| \leq \left(\frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \| x_i - x_j \|^2 \right)^{\frac{1}{2}} \left(\frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \| y_i - y_j \|^2 \right)^{\frac{1}{2}}$$

and, as a simple calculation shows that,

$$\frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \| x_i - x_j \|^2 = \sum_{i=1}^{n} p_i \| x_i \|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2$$

and

$$\frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \left\|y_i - y_j\right\|^2 = \sum_{i=1}^{n} p_i \left\|y_i\right\|^2 - \left\|\sum_{i=1}^{n} p_i y_i\right\|^2,$$

we obtain

$$\left|\sum_{i=1}^{n} p_i \left\langle x_i, y_i\right\rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i\right\rangle\right| \leq \left(\sum_{i=1}^{n} p_i \left\|x_i\right\|^2 - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} p_i \left\|y_i\right\|^2 - \left\|\sum_{i=1}^{n} p_i y_i\right\|^2\right)^{\frac{1}{2}}. \tag{3.4}$$

Using Lemma 10, we know that

$$\left(\sum_{i=1}^{n} p_i \left\|x_i\right\|^2 - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2\right)^{\frac{1}{2}} \leq \frac{1}{2}\left\|X - x\right\|$$

and

$$\left(\sum_{i=1}^{n} p_i \left\|y_i\right\|^2 - \left\|\sum_{i=1}^{n} p_i y_i\right\|^2\right)^{\frac{1}{2}} \leq \frac{1}{2}\left\|Y - y\right\|,$$

and then, by (3.4), we deduce the desired inequality (3.3).

To prove the sharpness of the constant $\frac{1}{4}$, let us assume that (3.2) holds with a constant $c > 0$, i.e.,

$$\left|\sum_{i=1}^{n} p_i \left\langle x_i, y_i\right\rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i\right\rangle\right| \leq c\left\|X - x\right\|\left\|Y - y\right\| \tag{3.5}$$

under the above assumptions for $p_i,\ x_i,\ y_i,\ x,\ X,\ y,\ Y$ and $n \geq 2$.

If we choose $n = 2$, $x_1 = x,\ x_2 = X$, $y_1 = y,\ y_2 = Y$ $(x \neq X,\ y \neq Y)$ and $p_1 = p_2 = \frac{1}{2}$, then

$$\begin{aligned}\sum_{i=1}^{2} p_i \left\langle x_i, y_i\right\rangle - \left\langle \sum_{i=1}^{2} p_i x_i, \sum_{i=1}^{2} p_i y_i\right\rangle &= \frac{1}{2}\sum_{i,j=1}^{2} p_i p_j \left\langle x_i - x_j, y_i - y_j\right\rangle \\ &= \sum_{1\leq i<j\leq 2} p_i p_j \left\langle x_i - x_j, y_i - y_j\right\rangle \\ &= \frac{1}{4}\left\langle x - X, y - Y\right\rangle\end{aligned}$$

and then

$$\left|\sum_{i=1}^{2} p_i \left\langle x_i, y_i\right\rangle - \left\langle \sum_{i=1}^{2} p_i x_i, \sum_{i=1}^{2} p_i y_i\right\rangle\right| = \frac{1}{4}\left|\left\langle x - X, y - Y\right\rangle\right|.$$

Choose $X - x = z$, $Y - y = z$, $z \neq 0$. Then using (3.5), we derive

$$\frac{1}{4}\|z\|^2 \leq c\|z\|^2, \quad z \neq 0$$

which implies that $c \geq \frac{1}{4}$, and the theorem is proved. ∎

REMARK 62. *The condition (3.1) can be replaced by the more general assumption*

$$\sum_{i=1}^{n} p_i \operatorname{Re}\langle X - x_i, x_i - x\rangle \geq 0, \quad \sum_{i=1}^{n} p_i \operatorname{Re}\langle Y - y_i, y_i - y\rangle \geq 0$$

and the conclusion (3.2) still remains valid.

The following corollary for real or complex numbers holds.

COROLLARY 54. *Let* $a_i, b_i \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$, $p_i \geq 0$ $(i = 1, \dots, n)$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $a, A, b, B \in \mathbb{K}$ *are such that*

$$\operatorname{Re}\left[(A - a_i)(\bar{a}_i - \bar{a})\right] \geq 0, \quad \operatorname{Re}\left[(B - b_i)\left(\bar{b}_i - \bar{b}\right)\right] \geq 0, \tag{3.6}$$

then we have the inequality

$$\left|\sum_{i=1}^{n} p_i a_i \bar{b}_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i \bar{b}_i\right| \leq \frac{1}{4}|A - a||B - b| \tag{3.7}$$

and the constant $\frac{1}{4}$ *is sharp.*

The proof is obvious by Theorem 64 applied for the inner product space $(\mathbb{C}, \langle\cdot,\cdot\rangle)$, where $\langle x, y\rangle = x \cdot \bar{y}$. We omit the details.

REMARK 63. *The condition (3.6) can be replaced by the more general condition*

$$\sum_{i=1}^{n} p_i \operatorname{Re}\left[(A - a_i)(\bar{a}_i - \bar{a})\right] \geq 0, \quad \sum_{i=1}^{n} p_i \operatorname{Re}\left[(B - b_i)\left(\bar{b}_i - \bar{b}\right)\right] \geq 0$$

and the conclusion of the above corollary will still remain valid.

REMARK 64. *If we assume that* a_i, b_i, a, b, A, B *are real numbers, then (3.6) is equivalent to*

$$a \leq a_i \leq A, \quad b \leq b_i \leq B \quad \text{for all } i \in \{1, \dots, n\}$$

and (3.7) becomes

$$0 \leq \left|\sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i\right| \leq \frac{1}{4}(A - a)(B - b),$$

which is the classical Grüss inequality for sequences of real numbers.

3.2. Applications for Convex Functions. Let $(H;\langle\cdot,\cdot\rangle)$ be a real inner product space and $F : H \to \mathbb{R}$ a Fréchet differentiable convex mapping on H. Then we have the "*gradient inequality*"

$$F(x) - F(y) \geq \langle \nabla F(y), x - y\rangle \tag{3.8}$$

for all $x, y \in H$, where $\nabla F : H \to H$ is the gradient operator associated to the differentiable convex function F.

The following theorem holds [**7**].

THEOREM 65. *Let $F : H \to \mathbb{R}$ be as above and $x_i \in H$ $(i = 1, \dots, n)$. Suppose that there exists the vectors $x, X \in H$ such that $\langle x_i - x, X - x_i\rangle \geq 0$ for all $i \in \{1, \dots, m\}$ and $y, Y \in H$ such that*

$$\langle \nabla F(x_i) - y,\ Y - \nabla F(x_i)\rangle \geq 0$$

for all $i \in \{1, \dots, m\}$. Then for all $p_i \geq 0$ $(i = 1, \dots, m)$ with $P_m := \sum_{i=1}^m p_i > 0$, we have the inequality

$$0 \leq \frac{1}{P_m}\sum_{i=1}^m p_i F(x_i) - F\left(\frac{1}{P_m}\sum_{i=1}^m p_i x_i\right) \leq \frac{1}{4}\|X - x\|\,\|Y - y\|. \tag{3.9}$$

PROOF. Choose in (3.8), $x = \frac{1}{P_M}\sum_{i=1}^m p_i x_i$ and $y = x_j$ to obtain

$$F\left(\frac{1}{P_m}\sum_{i=1}^m p_i x_i\right) - F(x_j) \geq \left\langle \nabla F(x_j), \frac{1}{P_m}\sum_{i=1}^m p_i x_i - x_j\right\rangle \tag{3.10}$$

for all $j \in \{1, \dots, n\}$.

If we multiply (3.10) by $p_j \geq 0$ and sum over j from 1 to m, we have

$$\begin{aligned}
P_m F\left(\frac{1}{P_m}\sum_{i=1}^m p_i x_i\right) &- \sum_{j=1}^m p_j F(x_j) \\
&\geq \frac{1}{P_m}\left\langle \sum_{j=1}^m \nabla F(x_j), \sum_{i=1}^m p_i x_i\right\rangle - \sum_{i=1}^m \langle \nabla F(x_j), x_j\rangle.
\end{aligned}$$

Dividing by $P_m > 0$, we obtain the inequality

$$\begin{aligned}
0 &\leq \frac{1}{P_m}\sum_{i=1}^m p_i F(x_i) - F\left(\frac{1}{P_m}\sum_{i=1}^m p_i x_i\right) \\
&\leq \frac{1}{P_m}\sum_{i=1}^m p_i \langle \nabla F(x_i), x_i\rangle \\
&\qquad - \left\langle \frac{1}{P_m}\sum_{i=1}^m p_i \nabla F(x_i), \frac{1}{P_m}\sum_{i=1}^m p_i x_i\right\rangle,
\end{aligned} \tag{3.11}$$

which is a generalisation for inner product spaces of the result by Dragomir-Goh established in 1996 for the case of differentiable mappings defined on $\mathbb{R}^n$ [**14**].

Applying Theorem 64 for real inner product spaces, and $y_i = \nabla F(x_i)$, we easily deduce

$$\frac{1}{P_m}\sum_{i=1}^{m} p_i \langle x_i, \nabla F(x_i)\rangle - \left\langle \frac{1}{P_m}\sum_{i=1}^{m} p_i x_i, \frac{1}{P_m}\sum_{i=1}^{m} p_i \nabla F(x_i)\right\rangle \leq \frac{1}{4}\|X-x\|\,\|Y-y\| \tag{3.12}$$

and then, by (3.11) and (3.12) we can conclude that the desired inequality (3.9) holds. ∎

3.3. Applications for Some Discrete Transforms. Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$, $\mathbb{K}=\mathbb{C}$, $\mathbb{R}$ and $\bar{\mathbf{x}} = (x_1,\dots,x_n)$ be a sequence of vectors in H.

For a given $m \in \mathbb{K}$, define the *discrete Fourier Transform*

$$\mathcal{F}_w(\bar{\mathbf{x}})(m) = \sum_{k=1}^{n} \exp(2wimk) \times x_k, \quad m = 1,\dots,n.$$

The complex number $\sum_{k=1}^{n} \exp(2wimk)\langle x_k, y_k\rangle$ is actually the usual Fourier transform of the vector $(\langle x_1, y_1\rangle, \dots, \langle x_n, y_n\rangle) \in \mathbb{K}^n$ and will be denoted by

$$\mathcal{F}_w(\bar{\mathbf{x}}\cdot\bar{\mathbf{y}})(m) = \sum_{k=1}^{n} \exp(2wimk)\langle x_k, y_k\rangle, \quad m = 1,\dots,n.$$

The following result holds [**7**].

Theorem 66. *Let $\bar{\mathbf{x}}$, $\bar{\mathbf{y}} \in H^n$ be sequences of vectors such that there exists the vectors $c, C, y, Y \in H$ with the properties*

$$\operatorname{Re}\langle C - \exp(2wimk)x_k, \exp(2wimk)x_k - c\rangle \geq 0, \quad k, m = 1,\dots,n$$

and

$$\operatorname{Re}\langle Y - y_k, y_k - y\rangle \geq 0, \quad k = 1,\dots,n. \tag{3.13}$$

Then we have the inequality

$$\left|\mathcal{F}_w(\bar{\mathbf{x}}\cdot\bar{\mathbf{y}})(m) - \left\langle \mathcal{F}_w(\bar{\mathbf{x}})(m), \frac{1}{n}\sum_{k=1}^{n} y_k\right\rangle\right| \leq \frac{n}{4}\|C-c\|\,\|Y-y\|,$$

for all $m \in \{1,\dots,n\}$.

The proof follows by Theorem 64 applied for $p_k = \frac{1}{n}$ and for the sequences $x_k \to c_k = \exp(2wimk)\, x_k$ and $y_k \quad (k = 1, \dots, n)$. We omit the details.

We can also consider the *Mellin transform*

$$\mathcal{M}(\bar{\mathbf{x}})(m) := \sum_{k=1}^{n} k^{m-1} x_k, \quad m = 1, \dots, n,$$

of the sequence $\bar{\mathbf{x}} = (x_1, \dots, x_n) \in H^n$.

We remark that the complex number $\sum_{k=1}^{n} k^{m-1} \langle x_k, y_k \rangle$ is actually the Mellin transform of the vector $(\langle x_1, y_1 \rangle, \dots, \langle x_n, y_n \rangle) \in \mathbb{K}^n$ and will be denoted by

$$\mathcal{M}(\bar{\mathbf{x}} \cdot \bar{\mathbf{y}})(m) := \sum_{k=1}^{n} k^{m-1} \langle x_k, y_k \rangle .$$

The following theorem holds [**7**].

Theorem 67. *Let $\bar{\mathbf{x}}$, $\bar{\mathbf{y}} \in H^n$ be sequences of vectors such that there exist the vectors $d, D, y, Y \in H$ with the properties*

$$\operatorname{Re}\left\langle D - k^{m-1} x_k, k^{m-1} x_k - d \right\rangle \geq 0$$

for all $k, m \in \{1, \dots, n\}$, and (3.13) is fulfilled.

Then we have the inequality

$$\left| \mathcal{M}(\bar{\mathbf{x}} \cdot \bar{\mathbf{y}})(m) - \left\langle \mathcal{M}(\bar{x})(m), \frac{1}{n} \sum_{k=1}^{n} y_k \right\rangle \right| \leq \frac{n}{4} \|D - d\| \|Y - y\|$$

for all $m \in \{1, \dots, n\}$.

The proof follows by Theorem 64 applied for $p_k = \frac{1}{n}$ and for the sequences $x_k \to d_k = k x_k$ and y_k $(k = 1, \dots, n)$. We omit the details.

Another result which connects the Fourier transforms for different parameters w also holds [**7**].

Theorem 68. *Let $\bar{\mathbf{x}}$, $\bar{\mathbf{y}} \in H^n$ and $w, z \in \mathbb{K}$. If there exist the vectors $e, E, f, F \in H$ such that*

$$\operatorname{Re}\langle E - \exp(2wimk)\, x_k, \exp(2wimk)\, x_k - e \rangle \geq 0, \quad k, m = 1, \dots, n$$

and

$$\operatorname{Re}\langle F - \exp(2zimk)\, y_k, \exp(2zimk)\, y_k - f \rangle \geq 0, \quad k, m = 1, \dots, n$$

then we have the inequality:

$$\left|\frac{1}{n}\mathcal{F}_{w+z}\left(\bar{\mathbf{x}}\cdot\bar{\mathbf{y}}\right)(m)-\left\langle\frac{1}{n}\mathcal{F}_{w}\left(\bar{\mathbf{x}}\right)(m),\frac{1}{n}\mathcal{F}_{z}\left(\bar{\mathbf{y}}\right)(m)\right\rangle\right| \leq\frac{1}{4}\left\|E-e\right\|\left\|F-f\right\|,$$

for all $m\in\{1,\dots,n\}$.

The proof follows by Theorem 64 for the sequences $\exp\left(2wimk\right)x_k$, $\exp\left(2zimk\right)y_k$ $(k=1,\dots,n)$. We omit the details.

4. More Grüss' Type Inequalities

4.1. Introduction. In the recent paper [**11**], the author has obtained the following Grüss type inequality for forward difference.

THEOREM 69. *Let* $\overline{\mathbf{x}}=(x_1,\dots,x_n)$, $\overline{\mathbf{y}}=(y_1,\dots,y_n)\in H^n$ *and* $\overline{\mathbf{p}}\in\mathbb{R}_+^n$ *be a probability sequence. Then one has the inequalities*

$$\left|\sum_{i=1}^{n}p_i\left\langle x_i,y_i\right\rangle-\left\langle\sum_{i=1}^{n}p_ix_i,\sum_{i=1}^{n}p_iy_i\right\rangle\right| \tag{4.1}$$

$$\leq\begin{cases}\left[\sum_{i=1}^{n}i^2p_i-\left(\sum_{i=1}^{n}ip_i\right)^2\right]\\ \qquad\times\max\limits_{k=1,\dots,n-1}\left\|\Delta x_k\right\|\max\limits_{k=1,\dots,n-1}\left\|\Delta y_k\right\|;\\ \sum_{1\leq j<i\leq n}p_ip_j\left(i-j\right)\left(\sum_{k=1}^{n-1}\left\|\Delta x_k\right\|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n-1}\left\|\Delta y_k\right\|^q\right)^{\frac{1}{q}}\\ \qquad\qquad\text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{2}\left[\sum_{i=1}^{n}p_i\left(1-p_i\right)\right]\sum_{k=1}^{n-1}\left\|\Delta x_k\right\|\sum_{k=1}^{n-1}\left\|\Delta y_k\right\|.\end{cases}$$

The constants 1, 1 *and* $\frac{1}{2}$ *in the right hand side of the inequality (4.1) are best in the sense that they cannot be replaced by smaller constants.*

If one chooses $p_i=\frac{1}{n}$ $(i=1,\dots,n)$ in (4.1), then the following unweighted inequalities hold:

$$\left|\frac{1}{n}\sum_{i=1}^{n}\left\langle x_i,y_i\right\rangle-\left\langle\frac{1}{n}\sum_{i=1}^{n}x_i,\frac{1}{n}\sum_{i=1}^{n}y_i\right\rangle\right| \tag{4.2}$$

$$\leq\begin{cases}\frac{n^2-1}{12}\max\limits_{k=1,\dots,n-1}\left\|\Delta x_k\right\|\max\limits_{k=1,\dots,n-1}\left\|\Delta y_k\right\|;\\ \frac{n^2-1}{6n}\left(\sum_{k=1}^{n-1}\left\|\Delta x_k\right\|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n-1}\left\|\Delta y_k\right\|^q\right)^{\frac{1}{q}}\\ \qquad\qquad\text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{n-1}{2n}\sum_{k=1}^{n-1}\left\|\Delta x_k\right\|\sum_{k=1}^{n-1}\left\|\Delta y_k\right\|.\end{cases}$$

Here, the constants $\frac{1}{12}$, $\frac{1}{6}$ and $\frac{1}{2}$ are also best possible in the above sense.

The following reverse of the Cauchy-Bunyakovsky-Schwarz inequality for sequences of vectors in inner product spaces holds.

COROLLARY 55. *With the assumptions in Theorem 69 for* $\overline{\mathbf{x}}$ *and* $\overline{\mathbf{p}}$ *one has the inequalities*

$$0 \leq \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \tag{4.3}$$
$$\leq \begin{cases} \left[\sum\limits_{i=1}^{n} i^2 p_i - \left(\sum\limits_{i=1}^{n} i p_i \right)^2 \right] \max\limits_{k=\overline{1,n-1}} \|\Delta x_k\|^2 ; \\ \sum\limits_{1\leq j<i\leq n} p_i p_j (i-j) \left(\sum\limits_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left(\sum\limits_{k=1}^{n-1} \|\Delta x_k\|^q \right)^{\frac{1}{q}} \\ \qquad\qquad \textit{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2} \left[\sum\limits_{i=1}^{n} p_i (1-p_i) \right] \left(\sum\limits_{k=1}^{n-1} \|\Delta x_k\| \right)^2 . \end{cases}$$

The constants 1, 1 *and* $\frac{1}{2}$ *are best possible in the above sense.*

The following particular inequalities that may be deduced from (4.3) on choosing the equal weights $p_i = \frac{1}{n}$, $i = 1, \dots, n$ are also of interest

$$0 \leq \frac{1}{n} \sum_{i=1}^{n} \|x_i\|^2 - \left\| \frac{1}{n} \sum_{i=1}^{n} x_i \right\|^2 \tag{4.4}$$
$$\leq \begin{cases} \dfrac{n^2-1}{12} \max\limits_{k=\overline{1,n-1}} \|\Delta x_k\|^2 ; \\ \dfrac{n^2-1}{6n} \left(\sum\limits_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left(\sum\limits_{k=1}^{n-1} \|\Delta x_k\|^q \right)^{\frac{1}{q}} \\ \qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{n-1}{2n} \left(\sum\limits_{k=1}^{n-1} \|\Delta x_k\| \right)^2 . \end{cases}$$

Here the constants $\frac{1}{12}$, $\frac{1}{6}$ and $\frac{1}{2}$ are also best possible.

The main aim of this section is to present, by following [**10**], a different class of Grüss type inequalities for sequences of vectors in inner product spaces and to apply them for obtaining a reverse of Jensen's inequality for convex functions defined on such spaces.

4.2. More Grüss Type Inequalities. The following lemma holds (see also [**12**]).

Lemma 11. *Let a, x, A be vectors in the inner product space $(H; \langle\cdot,\cdot\rangle)$ over the real or complex number field $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ with $a \neq A$. The following statements are equivalent:*

(i) $\operatorname{Re}\langle A - x, x - a\rangle \geq 0$;
(ii) $\left\| x - \frac{a+A}{2} \right\| \leq \frac{1}{2}\|A - a\|$.

The following inequality of Grüss type for sequences of vectors in inner product spaces holds [**10**].

Theorem 70. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product over $\mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$, and $\overline{\mathbf{x}} = (x_1, \dots, x_n)$, $\overline{\mathbf{y}} = (y_1, \dots, y_n) \in H^n$, $\overline{\mathbf{p}} \in \mathbb{R}^n_+$ with $\sum_{i=1}^n p_i = 1$. If $x, X \in H$ are such that*

$$\operatorname{Re}\langle X - x_i, x_i - x\rangle \geq 0 \quad \text{for each} \quad i \in \{1, \dots, n\}, \tag{4.5}$$

or, equivalently,

$$\left\| x_i - \frac{x + X}{2} \right\| \leq \frac{1}{2}\|X - x\| \quad \text{for each} \quad i \in \{1, \dots, n\}, \tag{4.6}$$

then one has the inequality

$$\begin{aligned} &\left| \sum_{i=1}^n p_i \langle x_i, y_i\rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \\ &\qquad \leq \frac{1}{2}\|X - x\| \sum_{i=1}^n p_i \left\| y_i - \sum_{j=1}^n p_j y_j \right\| \\ &\qquad \leq \frac{1}{2}\|X - x\| \left[\sum_{i=1}^n p_i \|y_i\|^2 - \left\| \sum_{i=1}^n p_i y_i \right\|^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{4.7}$$

The constant $\frac{1}{2}$ is best possible in the first and second inequality in the sense that it cannot be replaced by a smaller constant.

Proof. It is easy to see that the following identity holds true

$$\begin{aligned} &\sum_{i=1}^n p_i \langle x_i, y_i\rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \\ &\qquad = \sum_{i=1}^n p_i \left\langle x_i - \frac{x + X}{2}, y_i - \sum_{j=1}^n p_j y_j \right\rangle. \end{aligned} \tag{4.8}$$

Taking the modulus in (4.8) and using the Schwarz inequality in the inner product space $(H;\langle\cdot,\cdot\rangle)$, we have

$$\begin{aligned}\left|\sum_{i=1}^{n}p_i\langle x_i,y_i\rangle-\left\langle\sum_{i=1}^{n}p_ix_i,\sum_{i=1}^{n}p_iy_i\right\rangle\right|&\\ \leq\sum_{i=1}^{n}p_i\left|\left\langle x_i-\frac{x+X}{2},y_i-\sum_{j=1}^{n}p_jy_j\right\rangle\right|&\\ \leq\sum_{i=1}^{n}p_i\left\|x_i-\frac{x+X}{2}\right\|\left\|y_i-\sum_{j=1}^{n}p_jy_j\right\|&\\ \leq\frac{1}{2}\|X-x\|\sum_{i=1}^{n}p_i\left\|y_i-\sum_{j=1}^{n}p_jy_j\right\|,&\end{aligned}$$

and the first inequality in (4.7) is proved.

Using the Cauchy-Bunyakovsky-Schwarz inequality for positive sequences and the calculation rules in inner product spaces, we have

$$\sum_{i=1}^{n}p_i\left\|y_i-\sum_{j=1}^{n}p_jy_j\right\|\leq\left[\sum_{i=1}^{n}p_i\left\|y_i-\sum_{j=1}^{n}p_jy_j\right\|^2\right]^{\frac{1}{2}}$$

and

$$\sum_{i=1}^{n}p_i\left\|y_i-\sum_{j=1}^{n}p_jy_j\right\|^2=\sum_{i=1}^{n}p_i\|y_i\|^2-\left\|\sum_{i=1}^{n}p_iy_i\right\|^2$$

giving the second part of (4.7).

To prove the sharpness of the constant $\frac{1}{2}$ in the first inequality in (4.7), let us assume that, under the assumptions of the theorem, the inequality holds with a constant $C>0$, i.e.,

$$\begin{aligned}\left|\sum_{i=1}^{n}p_i\langle x_i,y_i\rangle-\left\langle\sum_{i=1}^{n}p_ix_i,\sum_{i=1}^{n}p_iy_i\right\rangle\right|&\\ \leq C\|X-x\|\sum_{i=1}^{n}p_i\left\|y_i-\sum_{j=1}^{n}p_jy_j\right\|.&\end{aligned}\tag{4.9}$$

Consider $n=2$ and observe that

$$\sum_{i=1}^{2} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{2} p_i x_i, \sum_{i=1}^{2} p_i y_i \right\rangle = p_2 p_1 \langle x_2 - x_1, y_2 - y_1 \rangle,$$

$$\sum_{i=1}^{2} p_i \left\| y_i - \sum_{j=1}^{2} p_j y_j \right\| = 2 p_2 p_1 \| y_2 - y_1 \|$$

and then, by (4.9), we deduce

$$p_2 p_1 \left| \langle x_2 - x_1, y_2 - y_1 \rangle \right| \leq 2C \| X - x \| p_2 p_1 \| y_2 - y_1 \| . \tag{4.10}$$

If we choose $p_1, p_2 > 0$, $y_2 = x_2$, $y_1 = x_1$ and $x_2 = X$, $x_1 = x$ with $x \neq X$, then (4.6) holds and from (4.10) we deduce $C \geq \frac{1}{2}$.

The fact that $\frac{1}{2}$ is best possible in the second inequality may be proven in a similar manner and we omit the details. ∎

Remark 65. *If $\overline{\mathbf{x}}$ and $\overline{\mathbf{y}}$ satisfy the assumptions of Theorem 70, or, equivalently,*

$$\left\| x_i - \frac{x+X}{2} \right\| \leq \frac{1}{2} \| X - x \|, \qquad \left\| y_i - \frac{y+Y}{2} \right\| \leq \frac{1}{2} \| Y - y \|, \tag{4.11}$$

for each $i \in \{1, \dots, n\}$, then by Theorem 70 we may state the following sequence of inequalities improving the Grüss inequality (4.8)

$$\begin{aligned} 0 &\leq \left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle \right| \\ &\leq \frac{1}{2} \| X - x \| \sum_{i=1}^{n} p_i \left\| y_i - \sum_{j=1}^{n} p_j y_j \right\| \\ &\leq \frac{1}{2} \| X - x \| \left(\sum_{i=1}^{n} p_i \| y_i \|^2 - \left\| \sum_{i=1}^{n} p_i y_i \right\|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{4} \| X - x \| \| Y - y \| . \end{aligned} \tag{4.12}$$

In particular, for $x_i = y_i$ $(i = 1, \dots, n)$, one has

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} p_i \| x_i \|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \\ &\leq \frac{1}{2} \| X - x \| \sum_{i=1}^{n} p_i \left\| x_i - \sum_{j=1}^{n} p_j x_j \right\| \end{aligned} \tag{4.13}$$

and the constant $\frac{1}{2}$ is best possible.

The following result also holds [**10**].

THEOREM 71. *Let* $(H;\langle\cdot,\cdot\rangle)$ *and* $\mathbb{K}$ *be as above and* $\overline{\mathbf{x}}=(x_1,\dots,x_n)\in H^n$, $\overline{\boldsymbol{\alpha}}=(\alpha_1,\dots,\alpha_n)\in\mathbb{K}^n$ *and* $\overline{\mathbf{p}}$ *a probability vector. If* $x,X\in H$ *are such that (4.5) or, equivalently, (4.6) holds, then we have the inequality*

$$0\le\left\|\sum_{i=1}^n p_i\alpha_i x_i-\sum_{i=1}^n p_i\alpha_i\cdot\sum_{i=1}^n p_i x_i\right\| \le\frac{1}{2}\|X-x\|\sum_{i=1}^n p_i\left|\alpha_i-\sum_{j=1}^n p_j\alpha_j\right| \le\frac{1}{2}\|X-x\|\left[\sum_{i=1}^n p_i|\alpha_i|^2-\left|\sum_{i=1}^n p_i\alpha_i\right|^2\right]^{\frac{1}{2}}. \tag{4.14}$$

The constant $\frac{1}{2}$ *in the first and second inequalities is best possible in the sense that it cannot be replaced by a smaller constant.*

PROOF. We start with the following equality that may be easily verified by direct calculation

$$\sum_{i=1}^n p_i\alpha_i x_i-\sum_{i=1}^n p_i\alpha_i\cdot\sum_{i=1}^n p_i x_i =\sum_{i=1}^n p_i\left(\alpha_i-\sum_{j=1}^n p_j\alpha_j\right)\left(x_i-\frac{x+X}{2}\right). \tag{4.15}$$

If we take the norm in (4.15), we deduce

$$\begin{aligned}\left\|\sum_{i=1}^n p_i\alpha_i x_i-\sum_{i=1}^n p_i\alpha_i\cdot\sum_{i=1}^n p_i x_i\right\| &\le\sum_{i=1}^n p_i\left|\alpha_i-\sum_{j=1}^n p_j\alpha_j\right|\left\|x_i-\frac{x+X}{2}\right\|\\ &\le\frac{1}{2}\|X-x\|\sum_{i=1}^n p_i\left|\alpha_i-\sum_{j=1}^n p_j\alpha_j\right|\\ &\le\frac{1}{2}\|X-x\|\left(\sum_{i=1}^n p_i\left(\alpha_i-\sum_{j=1}^n p_j\alpha_j\right)^2\right)^{\frac{1}{2}}\end{aligned}$$

$$= \frac{1}{2} \|X - x\| \left(\sum_{i=1}^{n} p_i |\alpha_i|^2 - \left| \sum_{i=1}^{n} p_i \alpha_i \right|^2 \right)^{\frac{1}{2}},$$

proving the inequality (4.14).

The fact that the constant $\frac{1}{2}$ is sharp may be proven in a similar manner to the one embodied in the proof of Theorem 70. We omit the details. ∎

Remark 66. *If $\overline{\mathbf{x}}$ and $\overline{\boldsymbol{\alpha}}$ satisfy the assumption*

$$\left\| \alpha_i - \frac{a+A}{2} \right\| \le \frac{1}{2} |A - a|, \qquad \left\| x_i - \frac{x+X}{2} \right\| \le \frac{1}{2} \|X - x\|,$$

for each $i \in \{1, \dots, n\}$, then by Theorem 70 we may state the following sequence of inequalities improving the Grüss inequality

$$\begin{aligned} 0 &\le \left\| \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \cdot \sum_{i=1}^{n} p_i x_i \right\| \\ &\le \frac{1}{2} \|X - x\| \sum_{i=1}^{n} p_i \left| \alpha_i - \sum_{j=1}^{n} p_j \alpha_j \right| \\ &\le \frac{1}{2} \|X - x\| \left(\sum_{i=1}^{n} p_i |\alpha_i|^2 - \left| \sum_{i=1}^{n} p_i \alpha_i \right|^2 \right)^{\frac{1}{2}} \\ &\le \frac{1}{4} |A - a| \, \|X - x\|. \end{aligned}$$

Remark 67. *If in (4.14) we choose $x_i = \alpha_i \in \mathbb{C}$ and assume that $\left| \alpha_i - \frac{a+A}{2} \right| \le \frac{1}{2} |A - a|$, where $a, A \in \mathbb{C}$, then we get the following interesting inequality for complex numbers*

$$\begin{aligned} 0 &\le \left| \sum_{i=1}^{n} p_i \alpha_i^2 - \left(\sum_{i=1}^{n} p_i \alpha_i \right)^2 \right| \\ &\le \frac{1}{2} |A - a| \sum_{i=1}^{n} p_i \left| \alpha_i - \sum_{j=1}^{n} p_j \alpha_j \right| \\ &\le \frac{1}{2} |A - a| \left[\sum_{i=1}^{n} p_i |\alpha_i|^2 - \left| \sum_{i=1}^{n} p_i \alpha_i \right|^2 \right]^{\frac{1}{2}}. \end{aligned}$$

4.3. Applications for Convex Functions. Let $(H; \langle \cdot, \cdot \rangle)$ be a real inner product space and $F : H \to \mathbb{R}$ a Fréchet differentiable convex function on H. If $\nabla F : H \to H$ denotes the gradient operator associated to F, then we have the inequality

$$F(x) - F(y) \geq \langle \nabla F(y), x - y \rangle \tag{4.16}$$

for each $x, y \in H$.

The following result holds [**10**].

THEOREM 72. *Let $F : H \to \mathbb{R}$ be as above and $z_i \in H$, $i \in \{1, \dots, n\}$. Suppose that there exist the vectors $m, M \in H$ such that either*

$$\langle \nabla F(z_i) - m, M - \nabla F(z_i) \rangle \geq 0 \quad \textit{for each} \quad i \in \{1, \dots, n\};$$

or, equivalently,

$$\left\| \nabla F(z_i) - \frac{m+M}{2} \right\| \leq \frac{1}{2} \|M - m\| \quad \textit{for each} \quad i \in \{1, \dots, n\},$$

holds.

If $q_i \geq 0$ $(i \in \{1, \dots, n\})$ with $Q_n := \sum_{i=1}^{n} q_i > 0$, then we have the following reverse of Jensen's inequality

$$\begin{aligned} 0 &\leq \frac{1}{Q_n} \sum_{i=1}^{n} q_i F(z_i) - F\left(\frac{1}{Q_n} \sum_{i=1}^{n} q_i z_i \right) \\ &\leq \frac{1}{2} \|M - m\| \frac{1}{Q_n} \sum_{i=1}^{n} q_i \left\| z_i - \frac{1}{Q_n} \sum_{j=1}^{n} q_j z_j \right\| \\ &\leq \frac{1}{2} \|M - m\| \left[\frac{1}{Q_n} \sum_{i=1}^{n} q_i \|z_i\|^2 - \left\| \frac{1}{Q_n} \sum_{i=1}^{n} q_i z_i \right\|^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{4.17}$$

PROOF. We know, see for example [**7**, Eq. (4.4)], that the following reverse of Jensen's inequality for Fréchet differentiable convex functions

$$\begin{aligned} 0 &\leq \frac{1}{Q_n} \sum_{i=1}^{n} q_i F(z_i) - F\left(\frac{1}{Q_n} \sum_{i=1}^{n} q_i z_i \right) \\ &\leq \frac{1}{Q_n} \sum_{i=1}^{n} q_i \langle \nabla F(z_i), z_i \rangle - \left\langle \frac{1}{Q_n} \sum_{i=1}^{n} q_i \nabla F(z_i), \frac{1}{Q_n} \sum_{i=1}^{n} q_i z_i \right\rangle \end{aligned} \tag{4.18}$$

holds.

Now, if we use Theorem 70 for the choices $x_i = \nabla F(z_i)$, $y_i = z_i$ and $p_i = \frac{1}{Q_n} q_i$, $i \in \{1, \dots, n\}$, we can state the inequality

$$\begin{aligned}
(4.19)\qquad & \frac{1}{Q_n}\sum_{i=1}^{n} q_i \langle \nabla F(z_i), z_i\rangle - \left\langle \frac{1}{Q_n}\sum_{i=1}^{n} q_i \nabla F(z_i), \frac{1}{Q_n}\sum_{i=1}^{n} q_i z_i \right\rangle \\
& \leq \frac{1}{2}\|M-m\| \frac{1}{Q_n}\sum_{i=1}^{n} q_i \left\| z_i - \frac{1}{Q_n}\sum_{j=1}^{n} q_j z_j \right\| \\
& \leq \frac{1}{2}\|M-m\| \left[\frac{1}{Q_n}\sum_{i=1}^{n} q_i \|z_i\|^2 - \left\| \frac{1}{Q_n}\sum_{i=1}^{n} q_i z_i \right\|^2 \right]^{\frac{1}{2}}.
\end{aligned}$$

Utilizing (4.18) and (4.19), we deduce the desired result (4.17). ∎

If more information is available about the vector sequence $\overline{\mathbf{z}} = (z_1, \dots, z_n) \in H^n$, then we may state the following corollary.

Corollary 56. *With the assumptions in Theorem 72 and if there exist the vectors $z, Z \in H$ such that either*

$$(4.20)\qquad \langle z_i - z, Z - z_i\rangle \geq 0 \quad \text{for each} \quad i \in \{1, \dots, n\};$$

or, equivalently,

$$(4.21)\qquad \left\| z_i - \frac{z+Z}{2} \right\| \leq \frac{1}{2}\|Z - z\| \quad \text{for each} \quad i \in \{1, \dots, n\},$$

holds, then we have the inequality

$$\begin{aligned}
(4.22)\qquad 0 & \leq \frac{1}{Q_n}\sum_{i=1}^{n} q_i F(z_i) - F\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i z_i \right) \\
& \leq \frac{1}{2}\|M-m\| \frac{1}{Q_n}\sum_{i=1}^{n} q_i \left\| z_i - \frac{1}{Q_n}\sum_{j=1}^{n} q_j z_j \right\| \\
& \leq \frac{1}{2}\|M-m\| \left[\frac{1}{Q_n}\sum_{i=1}^{n} q_i \|z_i\|^2 - \left\| \frac{1}{Q_n}\sum_{i=1}^{n} q_i z_i \right\|^2 \right]^{\frac{1}{2}} \\
& \leq \frac{1}{4}\|M-m\| \|Z - z\|.
\end{aligned}$$

Remark 68. *Note that the inequality between the first term and the last term in (4.22) was first proved in* [**7**, Theorem 4.1]*. Consequently, the above corollary provides an improvement of the reverse of Jensen's inequality established in* [**7**].

5. Some Inequalities for Forward Difference

5.1. Introduction. In [**4**], we have proved the following generalisation of the Grüss inequality.

Theorem 73. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$ and $e \in H$, $\|e\| = 1$. If $\phi, \Phi, \gamma, \Gamma \in \mathbb{K}$ and $x, y \in H$ are such that*

$$\operatorname{Re}\langle \Phi e - x, x - \phi e\rangle \geq 0 \quad \text{and} \quad \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0$$

hold, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma| .$$

The constant $\frac{1}{4}$ is the best possible.

A Grüss type inequality for sequences of vectors in inner product spaces was pointed out in [**5**].

Theorem 74. *Let H and $\mathbb{K}$ be as in Theorem 73 and $x_i \in H$, $a_i \in \mathbb{K}$, $p_i \geq 0$ $(i = 1, \dots, n)$ $(n \geq 2)$ with $\sum_{i=1}^n p_i = 1$. If $a, A \in \mathbb{K}$ and $x, X \in H$ are such that:*

$$\operatorname{Re}\left[(A - a_i)\left(\overline{a_i} - \overline{a}\right)\right] \geq 0, \quad \operatorname{Re}\langle X - x_i, x_i - x\rangle \geq 0$$

for any $i \in \{1, \dots, n\}$, then we have the inequality

$$\begin{aligned} 0 &\leq \left\| \sum_{i=1}^n p_i a_i x_i - \sum_{i=1}^n p_i a_i \cdot \sum_{i=1}^n p_i x_i \right\| \\ &\leq \frac{1}{4} |A - a| \|X - x\| . \end{aligned}$$

The constant $\frac{1}{4}$ is best possible.

A complementary result for two sequences of vectors in inner product spaces is the following result that has been obtained in [**7**].

Theorem 75. *Let H and $\mathbb{K}$ be as above, $x_i, y_i \in H$, $p_i \geq 0$ $(i = 1, \dots, n)$ $(n \geq 2)$ with $\sum_{i=1}^n p_i = 1$. If $x, X, y, Y \in H$ are such that:*

$$\operatorname{Re}\langle X - x_i, x_i - x\rangle \geq 0 \quad \text{and} \quad \operatorname{Re}\langle Y - y_i, y_i - y\rangle \geq 0$$

for all $i \in \{1, \dots, n\}$, then we have the inequality

$$0 \leq \left| \sum_{i=1}^n p_i \langle x_i, y_i\rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \leq \frac{1}{4} \|X - x\| \|Y - y\| .$$

The constant $\frac{1}{4}$ is best possible.

In the general case of normed linear spaces, the following Grüss type inequality in terms of the forward difference is known, see [**13**].

THEOREM 76. *Let* $(E, \|\cdot\|)$ *be a normed linear space over* $\mathbb{K} = \mathbb{C}, \mathbb{R}$, $x_i \in E$, $\alpha_i \in \mathbb{K}$ *and* $p_i \geq 0$ $(i = 1, \dots, n)$ *such that* $\sum_{i=1}^n p_i = 1$. *Then we have the inequality*

$$
\begin{aligned}
(5.1)\quad 0 &\leq \left\| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \cdot \sum_{i=1}^n p_i x_i \right\| \\
&\leq \max_{1 \leq j \leq n-1} |\Delta \alpha_j| \max_{1 \leq j \leq n-1} \|\Delta x_j\| \left[\sum_{i=1}^n i^2 p_i - \left(\sum_{i=1}^n i p_i \right)^2 \right],
\end{aligned}
$$

where $\Delta \alpha_j = \alpha_{j+1} - \alpha_j$ *and* $\Delta x_j = x_{j+1} - x_j \, (j = 1, \dots, n-1)$ *are the forward differences of the vectors having the components* α_j *and* $x_j \, (j = 1, \dots, n-1)$, *respectively.*

The inequality (5.1) is sharp in the sense that the multiplicative constant $C = 1$ *in the right hand side cannot be replaced by a smaller one.*

An important particular case is the one where all the weights are equal, giving the following corollary [**13**].

COROLLARY 57. *Under the above assumptions for* α_i, x_i $(i = 1, \dots, n)$ *we have the inequality*

$$
\begin{aligned}
(5.2)\qquad 0 &\leq \left\| \frac{1}{n} \sum_{i=1}^n \alpha_i x_i - \frac{1}{n} \sum_{i=1}^n \alpha_i \cdot \frac{1}{n} \sum_{i=1}^n x_i \right\| \\
&\leq \frac{n^2 - 1}{12} \max_{1 \leq j \leq n-1} |\Delta \alpha_j| \max_{1 \leq j \leq n-1} \|\Delta x_j\|.
\end{aligned}
$$

The constant $\frac{1}{12}$ *is best possible.*

Another result of this type was proved in [**6**].

THEOREM 77. *With the assumptions of Theorem 76, one has the inequality*

$$
\begin{aligned}
(5.3)\qquad 0 &\leq \left\| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \cdot \sum_{i=1}^n p_i x_i \right\| \\
&\leq \frac{1}{2} \sum_{j=1}^{n-1} |\Delta \alpha_j| \sum_{j=1}^{n-1} \|\Delta x_j\| \sum_{i=1}^n p_i (1 - p_i).
\end{aligned}
$$

The constant $\frac{1}{2}$ *is best possible.*

As a useful particular case, we have the following corollary [**6**].

COROLLARY 58. *If α_i, x_i $(i = 1, \dots, n)$ are as in Theorem 76, then*

$$0 \leq \left\| \frac{1}{n} \sum_{i=1}^{n} \alpha_i x_i - \frac{1}{n} \sum_{i=1}^{n} \alpha_i \cdot \frac{1}{n} \sum_{i=1}^{n} x_i \right\| \leq \frac{1}{2} \left(1 - \frac{1}{n}\right) \sum_{i=1}^{n-1} |\Delta \alpha_i| \sum_{i=1}^{n-1} \|\Delta x_i\| .$$

The constant $\frac{1}{2}$ is the best possible.

Finally, the following result is also known [**8**].

THEOREM 78. *With the assumptions in Theorem 76, we have the inequality:*

$$0 \leq \left\| \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \cdot \sum_{i=1}^{n} p_i x_i \right\| \leq \left(\sum_{j=1}^{n-1} |\Delta \alpha_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^{n-1} \|\Delta x_j\|^q \right)^{\frac{1}{q}} \sum_{1 \leq i < j \leq n} (j - i)\, p_i p_j, \tag{5.4}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The constant $c = 1$ in the right hand side of (5.4) is sharp.

The case of equal weights is embodied in the following corollary [**8**].

COROLLARY 59. *With the above assumptions for α_i, x_i $(i = 1, \dots, n)$ one has*

$$0 \leq \left\| \frac{1}{n} \sum_{i=1}^{n} \alpha_i x_i - \frac{1}{n} \sum_{i=1}^{n} \alpha_i \cdot \frac{1}{n} \sum_{i=1}^{n} x_i \right\| \leq \frac{n^2 - 1}{6n} \left(\sum_{j=1}^{n-1} |\Delta \alpha_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^{n-1} \|\Delta x_j\|^q \right)^{\frac{1}{q}},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The constant $\frac{1}{6}$ is the best possible.

The main aim of this section is to establish some similar bounds for the absolute value of the difference

$$\sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle$$

provided that x_i, y_i $(i = 1, \dots, n)$ are vectors in an inner product space H, and $p_i \geq 0$ $(i = 1, \dots, n)$ with $\sum_{i=1}^{n} p_i = 1$.

5.2. The Main Results. We assume that $(H, \langle \cdot, \cdot \rangle)$ is an inner product space over $\mathbb{K}$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. The following discrete inequality of Grüss' type holds.

THEOREM 79. *If $x_i, y_i \in H$, $p_i \geq 0$ $(i = 1, \dots, n)$ with $\sum_{i=1}^{n} p_i = 1$, then one has the inequalities:*

$$\left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle \right| \tag{5.5}$$
$$\leq \begin{cases} \left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i \right)^2 \right] \max\limits_{k=1,\dots,n-1} \|\Delta x_k\| \max\limits_{k=1,\dots,n-1} \|\Delta y_k\| ; \\ \left[\sum_{1 \leq j < i \leq n} p_i p_j (i - j) \right] \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta y_k\|^q \right)^{\frac{1}{q}} \\ \hfill \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} \left[\sum_{i=1}^{n} p_i (1 - p_i) \right] \sum_{k=1}^{n-1} \|\Delta x_k\| \sum_{k=1}^{n-1} \|\Delta y_k\| . \end{cases}$$

All the inequalities in (5.5) are sharp.

The following particular case for equal vectors holds.

COROLLARY 60. *With the assumptions of Theorem 79, one has the inequalities*

$$0 \leq \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2$$
$$\leq \begin{cases} \left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i \right)^2 \right] \max\limits_{k=1,\dots,n-1} \|\Delta x_k\|^2 ; \\ \sum_{1 \leq j < i \leq n} p_i p_j (i - j) \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^q \right)^{\frac{1}{q}} \\ \hfill \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} \sum_{i=1}^{n} p_i (1 - p_i) \left(\sum_{k=1}^{n-1} \|\Delta x_k\| \right)^2 . \end{cases}$$

The following particular case for equal weights may be useful in practice.

COROLLARY 61. *If $x_i, y_i \in H$ $(i = 1, \dots, n)$, then one has the inequalities:*

$$\left| \frac{1}{n} \sum_{i=1}^{n} \langle x_i, y_i \rangle - \left\langle \frac{1}{n} \sum_{i=1}^{n} x_i, \frac{1}{n} \sum_{i=1}^{n} y_i \right\rangle \right|$$

$$\leq \begin{cases} \frac{n^2-1}{12} \max\limits_{k=1,\dots,n-1} \|\Delta x_k\| \max\limits_{k=1,\dots,n-1} \|\Delta y_k\|; \\ \frac{n^2-1}{6n} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta y_k\|^q\right)^{\frac{1}{q}} \\ \qquad\qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{n-1}{2n} \sum_{k=1}^{n-1} \|\Delta x_k\| \sum_{k=1}^{n-1} \|\Delta y_k\|. \end{cases}$$

The constants $\frac{1}{12}$, $\frac{1}{6}$ and $\frac{1}{2}$ are best possible.

In particular, the following corollary holds.

Corollary 62. *If $x_i \in H$ $(i = 1, \dots, n)$, then one has the inequality*

$$0 \leq \frac{1}{n}\sum_{i=1}^{n} \|x_i\|^2 - \left\|\frac{1}{n}\sum_{i=1}^{n} x_i\right\|^2$$
$$\leq \begin{cases} \frac{n^2-1}{12} \max\limits_{k=\overline{1,n}} \|\Delta x_k\|^2; \\ \frac{n^2-1}{6n} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^q\right)^{\frac{1}{q}} \\ \qquad\qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{n-1}{2n} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|\right)^2. \end{cases}$$

The constants $\frac{1}{12}$, $\frac{1}{6}$ and $\frac{1}{2}$ are best possible.

5.3. Proof of the Main Result. It is well known that, the following identity holds in inner product spaces:

$$\sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle \tag{5.6}$$
$$= \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \langle x_i - x_j, y_i - y_j \rangle$$
$$= \sum_{1 \leq j < i \leq n} p_i p_j \langle x_i - x_j, y_i - y_j \rangle.$$

We observe, for $i > j$, we can write that

$$x_i - x_j = \sum_{k=j}^{i-1} \Delta x_k, \qquad y_i - y_j = \sum_{k=j}^{i-1} \Delta y_k. \tag{5.7}$$

Taking the modulus in (5.6) and by the use of (5.7) and Schwarz's inequality in inner product spaces, i.e., we recall that $|\langle z, u \rangle| \leq \|z\| \|u\|,$

$z,u\in H$, we have:

$$\begin{aligned}
&\left|\sum_{i=1}^{n}p_i\left\langle x_i,y_i\right\rangle-\left\langle\sum_{i=1}^{n}p_ix_i,\sum_{i=1}^{n}p_iy_i\right\rangle\right|\\
&\leq\sum_{1\leq j<i\leq n}p_ip_j\left|\left\langle x_i-x_j,y_i-y_j\right\rangle\right|\\
&\leq\sum_{1\leq j<i\leq n}p_ip_j\left\|x_i-x_j\right\|\left\|y_i-y_j\right\|\\
&=\sum_{1\leq j<i\leq n}p_ip_j\left\|\sum_{k=j}^{i-1}\Delta x_k\right\|\left\|\sum_{l=j}^{i-1}\Delta y_l\right\|\\
&\leq\sum_{1\leq j<i\leq n}p_ip_j\sum_{k=j}^{i-1}\left\|\Delta x_k\right\|\sum_{l=j}^{i-1}\left\|\Delta y_l\right\|\\
&:=M.
\end{aligned}$$

It is obvious that

$$\sum_{k=j}^{i-1}\left\|\Delta x_k\right\|\leq(i-j)\max_{k=j,\dots,i-1}\left\|\Delta x_k\right\|\leq(i-j)\max_{k=1,\dots,n}\left\|\Delta x_k\right\|$$

and

$$\sum_{k=j}^{i-1}\left\|\Delta y_k\right\|\leq(i-j)\max_{k=j,\dots,i-1}\left\|\Delta y_k\right\|\leq(i-j)\max_{k=1,\dots,n}\left\|\Delta y_k\right\|,$$

giving that

$$M\leq\sum_{1\leq j<i\leq n}p_ip_j\left(i-j\right)^2\cdot\max_{k=1,\dots,n}\left\|\Delta x_k\right\|\max_{k=1,\dots,n}\left\|\Delta y_k\right\|,$$

and since

$$\sum_{1\leq j<i\leq n}p_ip_j\left(i-j\right)^2=\frac{1}{2}\sum_{i,j=1}^{n}p_ip_j\left(i-j\right)^2=\sum_{i=1}^{n}p_ii^2-\left(\sum_{i=1}^{n}ip_i\right)^2,$$

the first inequality in (5.5) is proved.

Using Hölder's discrete inequality, we can state that

$$\sum_{k=j}^{i-1}\left\|\Delta x_k\right\|\leq(i-j)^{\frac{1}{q}}\left(\sum_{k=j}^{i-1}\left\|\Delta x_k\right\|^p\right)^{\frac{1}{p}}\leq(i-j)^{\frac{1}{q}}\left(\sum_{k=1}^{n-1}\left\|\Delta x_k\right\|^p\right)^{\frac{1}{p}}$$

and

$$\sum_{k=j}^{i-1}\|\Delta y_k\| \le (i-j)^{\frac{1}{p}}\left(\sum_{k=j}^{i-1}\|\Delta y_k\|^q\right)^{\frac{1}{q}} \le (i-j)^{\frac{1}{p}}\left(\sum_{k=1}^{n-1}\|\Delta y_k\|^q\right)^{\frac{1}{q}},$$

for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, giving that:

$$M \le \sum_{1\le j<i\le n} p_i p_j (i-j)\cdot\left(\sum_{k=1}^{n-1}\|\Delta x_k\|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n-1}\|\Delta y_k\|^q\right)^{\frac{1}{q}}$$

and the second inequality in (5.5) is proved.

Also, observe that

$$\sum_{k=j}^{i-1}\|\Delta x_k\| \le \sum_{k=1}^{n-1}\|\Delta x_k\| \quad\text{and}\quad \sum_{k=j}^{i-1}\|\Delta y_k\| \le \sum_{k=1}^{n-1}\|\Delta y_k\|$$

and thus

$$M \le \sum_{1\le j<i\le n} p_i p_j (i-j)\sum_{k=1}^{n-1}\|\Delta x_k\|\sum_{k=1}^{n-1}\|\Delta y_k\|.$$

Since

$$\begin{aligned}\sum_{1\le j<i\le n} p_i p_j &= \frac{1}{2}\left[\sum_{i,j=1}^{n} p_i p_j - \sum_{k=1}^{n} p_k^2\right]\\ &= \frac{1}{2}\left(1-\sum_{k=1}^{n} p_k^2\right)\\ &= \frac{1}{2}\sum_{i=1}^{n} p_i (1-p_i),\end{aligned}$$

the last part of (5.5) is also proved.

Now, assume that the first inequality in (5.5) holds with a constant $c>0$, i.e.,

$$\begin{aligned}&\sum_{i=1}^{n} p_i\langle x_i,y_i\rangle - \left\langle\sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i\right\rangle\\ &\qquad \le c\left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2\right]\max_{k=1,\dots,n-1}\|\Delta x_k\|\max_{k=1,\dots,n-1}\|\Delta y_k\|\end{aligned}$$

and choose $n=2$ to get

$$p_1 p_2\left|\langle x_2-x_1, y_2-y_1\rangle\right| \le c p_1 p_2 \|x_2-x_1\|\|y_2-y_1\| \tag{5.8}$$

for any $p_1, p_2 > 0$ and $x_1, x_2, y_1, y_2 \in H$.

If in (5.8) we choose $y_2 = x_2$, $y_1 = x_1$ and $x_2 \neq x_1$, then we deduce $c \geq 1$, which proves the sharpness of the constant in the first inequality in (5.5).

In a similar way one may show that the other two inequalities are sharp, and the theorem is completely proved.

5.4. A Reverse for Jensen's Inequality. Let $(H; \langle \cdot, \cdot \rangle)$ be a real inner product space and $F : H \to \mathbb{R}$ a Fréchet differentiable convex function on H. If $\nabla F : H \to H$ denotes the gradient operator associated to F, then we have the inequality

$$F(x) - F(y) \geq \langle \nabla F(y), x - y \rangle$$

for each $x, y \in H$.

The following result holds.

THEOREM 80. *Let $F : H \to \mathbb{R}$ be as above and $z_i \in H$, $i \in \{1, \dots, n\}$. If $q_i \geq 0 \quad (i \in \{1, \dots, n\})$ with $\sum_{i=1}^n q_i = 1$, then we have the following reverse of Jensen's inequality*

$$0 \leq \sum_{i=1}^n q_i F(z_i) - F\left(\sum_{i=1}^n q_i z_i\right) \tag{5.9}$$
$$\leq \begin{cases} \left[\sum_{i=1}^n i^2 q_i - \left(\sum_{i=1}^n i q_i\right)^2\right] \\ \qquad \times \max\limits_{k=1,\dots,n-1} \|\Delta(\nabla F(z_i))\| \max\limits_{k=1,\dots,n-1} \|\Delta z_i\|; \\ \left[\sum_{1 \leq j < i \leq n} q_i q_j (i - j)\right] \\ \qquad \times \left(\sum_{i=1}^{n-1} \|\Delta(\nabla F(z_i))\|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n-1} \|\Delta z_i\|^q\right)^{\frac{1}{q}} \\ \qquad\qquad \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2}\left[\sum_{i=1}^n q_i(1 - q_i)\right] \sum_{i=1}^{n-1} \|\Delta(\nabla F(z_i))\| \sum_{i=1}^{n-1} \|\Delta z_i\|. \end{cases}$$

PROOF. We know, see for example [**7**, Eq. (4.4)], that the following reverse of Jensen's inequality for Fréchet differentiable convex functions

$$\begin{aligned} 0 &\leq \sum_{i=1}^n q_i F(z_i) - F\left(\sum_{i=1}^n q_i z_i\right) \\ &\leq \sum_{i=1}^n q_i \langle \nabla F(z_i), z_i \rangle - \left\langle \sum_{i=1}^n q_i \nabla F(z_i), \sum_{i=1}^n q_i z_i \right\rangle \end{aligned} \tag{5.10}$$

holds.

Now, if we apply Theorem 79 for the choices $x_i = \nabla F(z_i), y_i = z_i$ and $p_i = q_i \, (i = 1, ..., n)$, then we may state

$$\left|\sum_{i=1}^{n} q_i \langle \nabla F(z_i), z_i \rangle - \left\langle \sum_{i=1}^{n} q_i \nabla F(z_i), \sum_{i=1}^{n} q_i z_i \right\rangle \right| \tag{5.11}$$
$$\leq \begin{cases} \left[\sum_{i=1}^{n} i^2 q_i - \left(\sum_{i=1}^{n} i q_i\right)^2\right] \\ \qquad \times \max\limits_{k=1,\dots,n-1} \|\Delta(\nabla F(z_k))\| \max\limits_{k=1,\dots,n-1} \|\Delta z_k\|; \\ \left[\sum_{1\leq j<i\leq n} q_i q_j (i-j)\right] \\ \qquad \times \left(\sum_{k=1}^{n-1} \|\Delta(\nabla F(z_k))\|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta z_k\|^q\right)^{\frac{1}{q}} \\ \qquad\qquad \text{if } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2}\left[\sum_{i=1}^{n} q_i (1 - p_i)\right] \sum_{k=1}^{n-1} \|\Delta(\nabla F(z_k))\| \sum_{k=1}^{n-1} \|\Delta z_k\|. \end{cases}$$

Finally, on making use of the inequalities (5.10) and (5.11), we deduce the desired result (5.9). ∎

The unweighted case may useful in application and is incorporated in the following corollary.

Corollary 63. *Let $F : H \to \mathbb{R}$ be as above and $z_i \in H$, $i \in \{1, \dots, n\}$. Then we have the inequalities*

$$0 \leq \frac{1}{n}\sum_{i=1}^{n} F(z_i) - F\left(\frac{1}{n}\sum_{i=1}^{n} z_i\right)$$
$$\leq \begin{cases} \frac{n^2-1}{12} \max\limits_{k=1,\dots,n-1} \|\Delta(\nabla F(z_k))\| \max\limits_{k=1,\dots,n-1} \|\Delta z_k\|; \\ \frac{n^2-1}{6n} \left(\sum\limits_{k=1}^{n-1} \|\Delta(\nabla F(z_k))\|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta z_k\|^q\right)^{\frac{1}{q}} \\ \qquad\qquad \textit{if } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{n-1}{2n} \sum_{k=1}^{n-1} \|\Delta(\nabla F(z_k))\| \sum_{k=1}^{n-1} \|\Delta z_k\|. \end{cases}$$

6. Bounds for a Pair of n-Tuples of Vectors

6.1. Introduction. Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product over the real or complex number field $\mathbb{K}$. For $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$ and $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in H^n$, define the *Čebyšev functional*

$$T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}, \bar{\mathbf{y}}) := P_n \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle, \tag{6.1}$$

where $P_n := \sum_{i=1}^{n} p_i$.

The following Grüss type inequality has been obtained in [**7**].

THEOREM 81. *Let H, $\mathbf{x}, \mathbf{y}$ be as above and $p_i \geq 0 \ \ (i \in \{1, \dots, n\})$ with $\sum_{i=1}^{n} p_i = 1$, i.e., $\mathbf{p}$ is a probability sequence. If $x, X, y, Y \in H$ are such that*

$$\operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0, \quad \operatorname{Re} \langle Y - y_i, y_i - y \rangle \geq 0 \tag{6.2}$$

for each $i \in \{1, \dots, n\}$, or, equivalently, (see [**10**]*)*

$$\left\| x_i - \frac{x + X}{2} \right\| \leq \frac{1}{2} \|X - x\|, \qquad \left\| y_i - \frac{y + Y}{2} \right\| \leq \frac{1}{2} \|Y - y\| \tag{6.3}$$

for each $i \in \{1, \dots, n\}$, then we have the inequality

$$\left| T_n \left(\bar{\mathbf{p}}; \bar{\mathbf{x}}, \bar{\mathbf{y}} \right) \right| \leq \frac{1}{4} \|X - x\| \|Y - y\|. \tag{6.4}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

In [**11**], the following Grüss type inequality for the forward difference of vectors was established.

THEOREM 82. *Let $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in H^n$ and $\mathbf{p} \in \mathbb{R}_+^n$ be a probability sequence. Then one has the inequality:*

$$\begin{aligned} & \left| T_n \left(\bar{\mathbf{p}}; \bar{\mathbf{x}}, \bar{\mathbf{y}} \right) \right| \\ & \leq \begin{cases} \left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i \right)^2 \right] \max\limits_{1 \leq k \leq n-1} \|\Delta x_k\| \\ \qquad \times \max\limits_{1 \leq k \leq n-1} \|\Delta y_k\|; \\ \sum_{1 \leq j < i \leq n} p_i p_j (i - j) \\ \qquad \times \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta y_k\|^q \right)^{\frac{1}{q}} \\ \qquad\qquad \text{if } \ p > 1, \ \frac{1}{p} + \frac{1}{q} + 1 \\ \frac{1}{2} \left[\sum_{i=1}^{n} p_i (1 - p_i) \right] \sum_{k=1}^{n-1} \|\Delta x_k\| \sum_{k=1}^{n-1} \|\Delta y_k\|. \end{cases} \end{aligned} \tag{6.5}$$

The constants $1, 1$ and $\frac{1}{2}$ in the right hand side of inequality (6.5) are best in the sense that they cannot be replaced by smaller constants.

Another result is incorporated in the following theorem (see [**10**]).

THEOREM 83. *Let $\mathbf{x}, \mathbf{y}$ and $\mathbf{p}$ be as in Theorem 82. If there exist $x, X \in H$ such that*

$$\operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \quad \text{for each } \ i \in \{1, \dots, n\}, \tag{6.6}$$

or, equivalently,

$$\left\| x_i - \frac{x+X}{2} \right\| \leq \frac{1}{2} \|X - x\| \quad \textit{for each} \ \ i \in \{1, \dots, n\}, \tag{6.7}$$

then one has the inequality

$$\begin{aligned} |T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}, \bar{\mathbf{y}})| &\leq \frac{1}{2} \|X - x\| \sum_{i=1}^{n} p_i \left\| y_i - \sum_{j=1}^{n} p_j y_j \right\| \\ &\leq \frac{1}{2} \|X - x\| \left[\sum_{i=1}^{n} p_i \|y_i\|^2 - \left\| \sum_{i=1}^{n} p_i y_i \right\|^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{6.8}$$

The constant $\frac{1}{2}$ is best possible in the first and second inequalities in the sense that it cannot be replaced by a smaller constant.

REMARK 69. *If $\mathbf{x}$ and $\mathbf{y}$ satisfy the assumptions of Theorem 81, then we have the following sequence of inequalities improving the Grüss inequality (6.4):*

$$\begin{aligned} |T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}, \bar{\mathbf{y}})| &\leq \frac{1}{2} \|X - x\| \sum_{i=1}^{n} p_i \left\| y_i - \sum_{j=1}^{n} p_j y_j \right\| \\ &\leq \frac{1}{2} \|X - x\| \left(\sum_{i=1}^{n} p_i \|y_i\|^2 - \left\| \sum_{i=1}^{n} p_i y_i \right\|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{4} \|X - x\| \|Y - y\|. \end{aligned} \tag{6.9}$$

Now, if we consider the Čebyšev functional for the uniform probability distribution $u = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$,

$$T_n(\bar{\mathbf{x}}, \bar{\mathbf{y}}) := \frac{1}{n} \sum_{i=1}^{n} \langle x_i, y_i \rangle - \left\langle \frac{1}{n} \sum_{i=1}^{n} x_i, \frac{1}{n} \sum_{i=1}^{n} y_i \right\rangle,$$

then, with the assumptions of Theorem 81, we have

$$|T_n(\bar{\mathbf{x}}, \bar{\mathbf{y}})| \leq \frac{1}{4} \|X - x\| \|Y - y\|. \tag{6.10}$$

Theorem 82 will provide the following inequalities

$$|T_n(\bar{\mathbf{x}}, \bar{\mathbf{y}})| \leq \begin{cases} \frac{1}{12}(n^2 - 1) \max\limits_{1 \leq k \leq n-1} \|\Delta x_k\| \max\limits_{1 \leq k \leq n-1} \|\Delta y_k\|; \\ \frac{1}{6}\left(n - \frac{1}{n}\right) \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta y_k\|^q\right)^{\frac{1}{q}} \\ \qquad\qquad \text{if} \ \ p > 1, \ \frac{1}{p} + \frac{1}{q} + 1; \\ \frac{1}{2}\left(1 - \frac{1}{n}\right) \sum_{k=1}^{n-1} \|\Delta x_k\| \sum_{k=1}^{n-1} \|\Delta y_k\|. \end{cases} \tag{6.11}$$

Here the constants $\frac{1}{12}$, $\frac{1}{6}$ and $\frac{1}{2}$ are best possible in the above sense.

Finally, from (6.9), we have

$$|T_n(\bar{\mathbf{x}},\bar{\mathbf{y}})| \le \frac{1}{2n}\|X-x\|\sum_{i=1}^{n}\left\|y_i-\frac{1}{n}\sum_{j=1}^{n}y_j\right\| \tag{6.12}$$
$$\le \frac{1}{2}\|X-x\|\left(\frac{1}{n}\sum_{i=1}^{n}\|y_i\|^2-\left\|\frac{1}{n}\sum_{i=1}^{n}y_i\right\|^2\right)^{\frac{1}{2}}$$
$$\le \frac{1}{4}\|X-x\|\,\|Y-y\|.$$

It is the main aim of this section to point out other bounds for the Čebyšev functionals $T_n(\mathbf{p},\mathbf{x},\mathbf{y})$ and $T_n(\mathbf{x},\mathbf{y})$.

6.2. Identities for Inner Products. For $\mathbf{p}=(p_1,\dots,p_n)\in\mathbb{R}^n$ and $\mathbf{a}=(a_1,\dots,a_n)\in H^n$ we define

$$P_i:=\sum_{k=1}^{i}p_k,\qquad \bar{P}_i=P_n-P_i,\qquad i\in\{1,\dots,n-1\}$$

and the vectors

$$A_i(\mathbf{p})=\sum_{k=1}^{i}p_ka_k,\qquad \bar{A}_i(\mathbf{p})=A_n(\mathbf{p})-A_i(\mathbf{p})$$

for $i\in\{1,\dots,n-1\}$.

The following result holds [**9**].

THEOREM 84. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$, $\mathbf{p}=(p_1,\dots,p_n)\in\mathbb{R}^n$ and $\mathbf{a}=(a_1,\dots,a_n),\mathbf{b}=(b_1,\dots,b_n)\in H^n$. Then we have the identities*

$$T_n(\mathbf{p};\mathbf{a},\mathbf{b})=\sum_{i=1}^{n-1}\langle P_iA_n(\mathbf{p})-P_nA_i(\mathbf{p}),\Delta b_i\rangle \tag{6.13}$$
$$=P_n\sum_{i=1}^{n-1}P_i\left\langle\frac{1}{P_n}A_n(\mathbf{p})-\frac{1}{P_i}A_i(\mathbf{p}),\Delta b_i\right\rangle$$
$$(\text{if } P_i\neq 0,\ i\in\{1,\dots,n\})$$
$$=\sum_{i=1}^{n-1}P_i\bar{P}_i\left\langle\frac{1}{\bar{P}_i}\bar{A}_i(\mathbf{p})-\frac{1}{P_i}A_i(\mathbf{p}),\Delta b_i\right\rangle$$
$$(\text{if } P_i,\bar{P}_i\neq 0,\ i\in\{1,\dots,n-1\}),$$

where $\Delta x_i=x_{i+1}-x_i$ $(i\in\{1,\dots,n-1\})$ is the forward difference.

PROOF. We use the following summation by parts formula for vectors in inner product spaces

$$\sum_{l=p}^{q-1}\langle d_l,\Delta v_l\rangle=\langle d_l,v_l\rangle\Big|_p^q-\sum_{l=p}^{q-1}\langle v_{l+1},\Delta d_l\rangle\,, \tag{6.14}$$

where d_l, v_l are vectors in H, $l=p,\dots,q$ ($q>p$; p,q are natural numbers).

If we choose in (6.14), $p=1$, $q=n$, $d_i=P_iA_n(\mathbf{p})-P_nA_i(\mathbf{p})$ and $v_i=b_i$ $(i\in\{1,\dots,n-1\})$, then we get

$$\begin{aligned}
&\sum_{i=1}^{n-1}\langle P_iA_n(\mathbf{p})-P_nA_i(\mathbf{p}),\Delta b_i\rangle\\
&=\langle P_iA_n(\mathbf{p})-P_nA_i(\mathbf{p}),b_i\rangle\Big|_1^n-\sum_{i=1}^{n-1}\langle\Delta(P_iA_n(\mathbf{p})-P_nA_i(\mathbf{p})),b_{i+1}\rangle\\
&=\langle P_nA_n(\mathbf{p})-P_nA_n(\mathbf{p}),b_n\rangle-\langle P_1A_n(\mathbf{p})-P_nA_1(\mathbf{p}),b_1\rangle\\
&\qquad-\sum_{i=1}^{n-1}\langle P_{i+1}A_n(\mathbf{p})-P_nA_{i+1}(\mathbf{p})-P_iA_n(\mathbf{p})+P_nA_i(\mathbf{p}),b_{i+1}\rangle\\
&=P_np_1\langle a_1,x_1\rangle-p_1\langle A_n(\mathbf{p}),b_1\rangle-\left\langle A_n(\mathbf{p}),\sum_{i=1}^{n-1}p_{i+1}b_{i+1}\right\rangle\\
&\qquad+P_n\sum_{i=1}^{n-1}p_{i+1}\langle a_{i+1},b_{i+1}\rangle\\
&=P_n\sum_{i=1}^{n}p_i\langle a_i,b_i\rangle-\left\langle\sum_{i=1}^{n}p_ia_i,\sum_{i=1}^{n}p_ib_i\right\rangle\\
&=T_n(\mathbf{p};\mathbf{a},\mathbf{b})\,,
\end{aligned}$$

proving the first identity in (6.13).

The second and third identities are obvious and we omit the details. ∎

The following lemma is of interest in itself [**9**].

LEMMA 12. *Let* $\mathbf{p}=(p_1,\dots,p_n)\in\mathbb{R}^n$ *and* $\mathbf{a}=(a_1,\dots,a_n)\in H$. *Then we have the equality*

$$P_iA_n(\mathbf{p})-P_nA_i(\mathbf{p})=\sum_{j=1}^{n-1}P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}}\Delta a_j \tag{6.15}$$

for each $i\in\{1,\dots,n-1\}$.

PROOF. Define, for $i \in \{1, \dots, n-1\}$, the vector

$$K(i) := \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j.$$

We have

$$K(i) = \sum_{j=1}^{i} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j + \sum_{j=i+1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j = \sum_{j=1}^{i} P_j \bar{P}_i \cdot \Delta a_j + \sum_{j=i+1}^{n-1} P_i \bar{P}_j \cdot \Delta a_j = \bar{P}_i \sum_{j=1}^{i} P_j \cdot \Delta a_j + P_i \sum_{j=i+1}^{n-1} \bar{P}_j \cdot \Delta a_j. \tag{6.16}$$

Using the summation by parts formula, we have

$$\sum_{j=1}^{i} P_j \cdot \Delta a_j = P_j a_j \Big|_1^{i+1} - \sum_{j=1}^{i} (P_{j+1} - P_j) a_{j+1} = P_{i+1} a_{i+1} - p_1 a_1 - \sum_{j=1}^{i} p_{j+1} a_{j+1} = P_{i+1} a_{i+1} - \sum_{j=1}^{i+1} p_j a_j \tag{6.17}$$

and

$$\sum_{j=i+1}^{n-1} \bar{P}_j \cdot \Delta a_j = \bar{P}_j a_j \Big|_{i+1}^{n} - \sum_{j=i+1}^{n-1} \left(\bar{P}_{j+1} - \bar{P}_j\right) a_{j+1} = \bar{P}_n a_n - \bar{P}_{i+1} a_{i+1} - \sum_{j=i+1}^{n-1} (P_n - P_{j+1} - P_n + P_j) a_{j+1} = -\bar{P}_{i+1} a_{i+1} + \sum_{j=i+1}^{n-1} p_{j+1} a_{j+1}. \tag{6.18}$$

Using (6.17) and (6.18), we have

$$
\begin{aligned}
K(i) &= \bar{P}_i \left(P_{i+1} a_{i+1} - \sum_{j=1}^{i+1} p_j a_j \right) + P_i \left(\sum_{j=i+1}^{n-1} p_{j+1} a_{j+1} - \bar{P}_{i+1} a_{i+1} \right) \\
&= \bar{P}_i P_{i+1} a_{i+1} - P_i \bar{P}_{i+1} a_{i+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j a_j + P_i \sum_{j=i+1}^{n-1} p_{j+1} a_{j+1} \\
&= \left[(P_n - P_i) P_{i+1} - P_i (P_n - P_{i+1}) \right] a_{i+1} \\
&\qquad + P_i \sum_{j=i+1}^{n-1} p_{j+1} a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j a_j \\
&= P_n p_{i+1} a_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1} a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j a_j \\
&= \left(P_i + \bar{P}_i \right) p_{i+1} a_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1} a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j a_j \\
&= P_i \sum_{j=i+1}^{n-1} p_j a_j - \bar{P}_i \sum_{j=1}^{i} p_j a_j \\
&= P_i \bar{A}_i(\mathbf{p}) - \bar{P}_i A_i(\mathbf{p}) \\
&= P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p}),
\end{aligned}
$$

and the identity is proved. ∎

We are able now to state and prove the second identity for the Čebyšev functional [**9**].

THEOREM 85. *With the assumptions of Theorem 84, we have the identity*

$$
T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \langle \Delta a_j, \Delta b_i \rangle . \tag{6.19}
$$

PROOF. Follows by Theorem 84 and Lemma 12 and we omit the details. ∎

6.3. New Inequalities. The following result holds [**9**].

THEOREM 86. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$; $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$ and $\mathbf{a} =$*

$(a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n) \in H^n$. *Then we have the inequalities*

$$|T_n(\mathbf{p};\mathbf{a},\mathbf{b})| \leq \begin{cases} \max\limits_{1\leq i\leq n-1} \|P_iA_n(\mathbf{p}) - P_nA_i(\mathbf{p})\| \sum_{j=1}^{n-1} \|\Delta b_j\|; \\ \left(\sum_{i=1}^{n-1} \|P_iA_n(\mathbf{p}) - P_nA_i(\mathbf{p})\|^q\right)^{\frac{1}{q}} \left(\sum_{j=1}^{n-1} \|\Delta b_j\|^p\right)^{\frac{1}{p}} \\ \qquad\qquad \text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum_{i=1}^{n-1} \|P_iA_n(\mathbf{p}) - P_nA_i(\mathbf{p})\| \cdot \max\limits_{1\leq j\leq n-1} \|\Delta b_j\|. \end{cases} \tag{6.20}$$

All the inequalities in (6.20) are sharp in the sense that the constants 1 *cannot be replaced by smaller constants.*

Proof. Using the first identity in (6.13) and Schwarz's inequality in H, i.e., $|\langle u,v\rangle| \leq \|u\|\|v\|$, $u, v \in H$, we have successively:

$$\begin{aligned} |T_n(\mathbf{p};\mathbf{a},\mathbf{b})| &\leq \sum_{i=1}^{n-1} |\langle P_iA_n(\mathbf{p}) - P_nA_i(\mathbf{p}), \Delta b_i\rangle| \\ &\leq \sum_{i=1}^{n-1} \|P_iA_n(\mathbf{p}) - P_nA_i(\mathbf{p})\| \|\Delta b_i\|. \end{aligned}$$

Using Hölder's inequality, we deduce the desired result (6.20).

Let us prove, for instance, that the constant 1 in the second inequality is best possible.

Assume, for $c > 0$, we have that

$$|T_n(\mathbf{p};\mathbf{a},\mathbf{b})| \leq c\left(\sum_{i=1}^{n-1} \|P_iA_n(\mathbf{p}) - P_nA_i(\mathbf{p})\|^q\right)^{\frac{1}{q}} \left(\sum_{j=1}^{n-1} \|\Delta b_j\|^p\right)^{\frac{1}{p}} \tag{6.21}$$

for $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$, $n \geq 2$.

If we choose $n = 2$, then we get

$$T_2(\mathbf{p};\mathbf{a},\mathbf{b}) = p_1p_2\langle a_2 - a_1, b_2 - b_1\rangle.$$

Also, for $n = 2$,

$$\left(\sum_{i=1}^{n-1} \|P_iA_n(\mathbf{p}) - P_nA_i(\mathbf{p})\|^q\right)^{\frac{1}{q}} = |p_1p_2| \|a_2 - a_1\|$$

and

$$\left(\sum_{j=1}^{n-1} \|\Delta b_j\|^p\right)^{\frac{1}{p}} = \|b_2 - b_1\|,$$

and then, from (6.21), for $n = 2$, we deduce

$$|p_1 p_2| |\langle a_2 - a_1, b_2 - b_1 \rangle| \leq c |p_1 p_2| \|a_2 - a_1\| \|b_2 - b_1\| . \tag{6.22}$$

If in (6.22) we choose $a_2 = b_2$, $a_2 = b_1$ and $b_2 \neq b_1$, $p_1, p_2 \neq 0$, we deduce $c \geq 1$, proving that 1 is the best possible constant in that inequality. ∎

The following corollary for the uniform distribution of the probability $\mathbf{p}$ holds.

Corollary 64. *With the assumptions of Theorem 86 for* $\mathbf{a}$ *and* $\mathbf{b}$, *we have the inequalities*

$$\begin{aligned} 0 &\leq |T_n(\mathbf{a}, \mathbf{b})| \\ &\leq \frac{1}{n^2} \begin{cases} \max\limits_{1 \leq i \leq n-1} \left\| i \sum_{k=1}^n a_k - n \sum_{k=1}^i a_k \right\| \sum_{j=1}^{n-1} \|\Delta b_j\| ; \\ \left(\sum_{i=1}^{n-1} \left\| i \sum_{k=1}^n a_k - n \sum_{k=1}^i a_k \right\|^q \right)^{\frac{1}{q}} \\ \qquad \times \left(\sum_{j=1}^{n-1} \|\Delta b_j\|^p \right)^{\frac{1}{p}} \quad \text{for } \; p > 1, \; \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} \left\| i \sum_{k=1}^n a_k - n \sum_{k=1}^i a_k \right\| \cdot \max\limits_{1 \leq j \leq n-1} \|\Delta b_j\| . \end{cases} \end{aligned} \tag{6.23}$$

The following result may be stated as well [**9**].

Theorem 87. *With the assumptions of Theorem 86 and if* $P_i \neq 0$ $(i = 1, \dots, n)$, *then we have the inequalities*

$$\begin{aligned} &|T_n(\mathbf{p}; \mathbf{a}, \mathbf{b})| \\ &\leq |P_n| \times \begin{cases} \max\limits_{1 \leq i \leq n-1} \left\| \frac{1}{P_n} A_n(\mathbf{p}) - \frac{1}{P_i} A_i(\mathbf{p}) \right\| \sum\limits_{i=1}^{n-1} |P_i| \|\Delta b_i\| ; \\ \left(\sum_{i=1}^{n-1} |P_i| \left\| \frac{1}{P_n} A_n(\mathbf{p}) - \frac{1}{P_i} A_i(\mathbf{p}) \right\|^q \right)^{\frac{1}{q}} \\ \qquad \times \left(\sum_{i=1}^{n} |P_i| \|\Delta b_i\|^p \right)^{\frac{1}{p}} \quad \text{for } \; p > 1, \; \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} |P_i| \left\| \frac{1}{P_n} A_n(\mathbf{p}) - \frac{1}{P_i} A_i(\mathbf{p}) \right\| \cdot \max\limits_{1 \leq i \leq n-1} \|\Delta b_i\| . \end{cases} \end{aligned} \tag{6.24}$$

All the inequalities in (6.24) are sharp in the sense that the constant 1 cannot be replaced by a smaller constant.

PROOF. Using the second equality in (6.13) and Schwarz's inequality, we have

$$\begin{aligned}|T_n(\mathbf{p};\mathbf{a},\mathbf{b})| &\le |P_n|\sum_{i=1}^{n-1}|P_i|\left|\left\langle \frac{1}{P_n}A_n(\mathbf{p})-\frac{1}{P_i}A_i(\mathbf{p}),\Delta b_i\right\rangle\right| \\ &\le |P_n|\sum_{i=1}^{n-1}|P_i|\left\|\frac{1}{P_n}A_n(\mathbf{p})-\frac{1}{P_i}A_i(\mathbf{p})\right\|\|\Delta b_i\|.\end{aligned}$$

Using Hölder's weighted inequality, we deduce (6.24).

The sharpness of the constant may be proven in a similar manner to the one in Theorem 86. We omit the details. ∎

The following corollary containing the unweighted inequalities holds.

COROLLARY 65. *With the above assumptions for* $\mathbf{a}$ *and* $\mathbf{b}$, *one has*

$$(6.25)\quad |T_n(\mathbf{a},\mathbf{b})| \le \frac{1}{n}\begin{cases}\max\limits_{1\le i\le n-1}\left\|\frac{1}{n}\sum\limits_{k=1}^{n}a_k-\frac{1}{i}\sum_{k=1}^{i}a_k\right\|\sum_{i=1}^{n-1}i\|\Delta b_i\|; \\ \left(\sum_{i=1}^{n-1}i\left\|\frac{1}{n}\sum\limits_{k=1}^{n}a_k-\frac{1}{i}\sum_{k=1}^{i}a_k\right\|^q\right)^{\frac{1}{q}} \\ \qquad\times\left(\sum_{i=1}^{n-1}i\|\Delta b_i\|^p\right)^{\frac{1}{p}}\quad \text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum_{i=1}^{n-1}i\left\|\frac{1}{n}\sum_{k=1}^{n}a_k-\frac{1}{i}\sum_{k=1}^{i}a_k\right\|\cdot\max\limits_{1\le i\le n-1}\|\Delta b_i\|.\end{cases}$$

The inequalities (6.25) are sharp in the sense mentioned above.

Another type of inequality may be stated if ones used the third identity in (6.13) and Hölder's weighted inequality with the weights: $|P_i|\left|\bar{P}_i\right|$, $i\in\{1,\dots,n-1\}$ [**9**].

THEOREM 88. *With the assumptions in Theorem 86 and if* P_i, $\bar{P}_i\neq 0$, $i\in\{1,\dots,n-1\}$, *then we have the inequalities*

$$(6.26)\quad |T_n(\mathbf{p};\mathbf{a},\mathbf{b})| \le |P_n|\times\begin{cases}\max\limits_{1\le i\le n-1}\left\|\frac{1}{\bar{P}_i}\bar{A}_i(\mathbf{p})-\frac{1}{P_i}A_i(\mathbf{p})\right\|\sum_{i=1}^{n-1}|P_i|\left|\bar{P}_i\right|\|\Delta b_i\|; \\ \left(\sum_{i=1}^{n-1}|P_i|\left|\bar{P}_i\right|\left\|\frac{1}{\bar{P}_i}\bar{A}_i(\mathbf{p})-\frac{1}{P_i}A_i(\mathbf{p})\right\|^q\right)^{\frac{1}{q}} \\ \times\left(\sum_{i=1}^{n-1}|P_i|\left|\bar{P}_i\right|\|\Delta b_i\|^p\right)^{\frac{1}{p}}\quad \text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum_{i=1}^{n-1}|P_i|\left|\bar{P}_i\right|\left\|\frac{1}{\bar{P}_i}\bar{A}_i(\mathbf{p})-\frac{1}{P_i}A_i(\mathbf{p})\right\|\cdot\max\limits_{1\le i\le n-1}\|\Delta b_i\|.\end{cases}$$

In particular, if $p_i = \frac{1}{n}$, $i \in \{1, \ldots, n\}$, *then we have*

$$|T_n(\mathbf{a}, \mathbf{b})| \tag{6.27}$$
$$\leq \frac{1}{n^2} \begin{cases} \max\limits_{1 \leq i \leq n-1} \left\| \frac{1}{n-i} \sum_{k=i+1}^{n} a_k - \frac{1}{i} \sum_{k=1}^{i} a_k \right\| \sum_{i=1}^{n-1} i(n-i) \|\Delta b_i\| ; \\ \left(\sum_{i=1}^{n-1} i(n-i) \left\| \frac{1}{n-i} \sum_{k=i+1}^{n} a_k - \frac{1}{i} \sum_{k=1}^{i} a_k \right\|^q \right)^{\frac{1}{q}} \\ \qquad \times \left(\sum_{i=1}^{n-1} i(n-i) \|\Delta b_i\|^p \right)^{\frac{1}{p}} \text{ for } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} i(n-i) \left\| \frac{1}{n-i} \sum_{k=i+1}^{n} a_k - \frac{1}{i} \sum_{k=1}^{i} a_k \right\| \cdot \max\limits_{1 \leq i \leq n-1} \|\Delta b_i\| . \end{cases}$$

The inequalities in (6.26) and (6.27) are sharp in the above mentioned sense.

A different approach may be considered if one uses the representation in terms of double sums for the Čebyšev functional provided by Theorem 85.

The following result holds [**9**].

Theorem 89. *With the above assumptions of Theorem 86, we have the inequalities*

$$|T_n(\mathbf{p}; \mathbf{a}, \mathbf{b})| \tag{6.28}$$
$$\leq |P_n| \times \begin{cases} \max\limits_{1 \leq i,j \leq n-1} \left\{ \left| P_{\min\{i,j\}} \right|, \left| \bar{P}_{\max\{i,j\}} \right| \right\} \\ \qquad \times \sum_{i=1}^{n-1} \|\Delta a_i\| \sum_{i=1}^{n-1} \|\Delta b_i\| ; \\ \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left| P_{\min\{i,j\}} \right|^q \left| \bar{P}_{\max\{i,j\}} \right|^q \right)^{\frac{1}{q}} \\ \qquad \times \left(\sum_{i=1}^{n-1} \|\Delta a_i\|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n-1} \|\Delta b_i\|^p \right)^{\frac{1}{p}} \\ \qquad \text{for } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left| P_{\min\{i,j\}} \right| \left| \bar{P}_{\max\{i,j\}} \right| \\ \qquad \times \max\limits_{1 \leq i \leq n-1} \|\Delta a_i\| \max\limits_{1 \leq i \leq n-1} \|\Delta b_i\| . \end{cases}$$

The inequalities are sharp in the sense mentioned above.

The proof follows by the identity (6.19) on using Hölder's inequality for double sums and we omit the details.

Now, define

$$k_\infty := \max_{1 \leq i,j \leq n-1} \left\{ \frac{\min\{i,j\}}{n} \left(1 - \frac{\max\{i,j\}}{n} \right) \right\}, \quad n \geq 2.$$

Using the elementary inequality

$$ab \leq \frac{1}{4}(a+b)^2, \qquad a, b \in \mathbb{R};$$

we deduce

$$\min\{i,j\}(n - \max\{i,j\}) \leq \frac{1}{4}(n - |i-j|)^2$$

for $1 \leq i, j \leq n-2$. Consequently, we deduce

$$k_\infty \leq \frac{1}{4n^2} \max_{1 \leq i,j \leq n-1} \left\{ (n - |i-j|)^2 \right\} = \frac{1}{4}.$$

We may now state the following corollary of Theorem 89 [**9**].

COROLLARY 66. *With the assumptions of Theorem 86 for* $\mathbf{a}$ *and* $\mathbf{b}$, *we have the inequality*

$$\begin{aligned} |T_n(\mathbf{a},\mathbf{b})| &\leq k_\infty \sum_{i=1}^{n-1} \|\Delta a_i\| \sum_{i=1}^{n-1} \|\Delta b_i\| \\ &\leq \frac{1}{4} \sum_{i=1}^{n-1} \|\Delta a_i\| \sum_{i=1}^{n-1} \|\Delta b_i\|. \end{aligned} \tag{6.29}$$

The constant $\frac{1}{4}$ *cannot be replaced in general by a smaller constant.*

REMARK 70. *The inequality (6.29) is better than the third inequality in (6.11).*

Consider now, for $q > 1$, the number

$$k_q := \frac{1}{n^2} \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} [\min\{i,j\}(n - \max\{i,j\})]^q \right)^{\frac{1}{q}}.$$

We observe, by symmetry of the terms under the summation symbol, that we have

$$k_q = \frac{1}{n^2} \left(2 \sum_{1 \leq i < j \leq n-1} i^q (n-j)^q + \sum_{i=1}^{n-1} i^q (n-i)^q \right)^{\frac{1}{q}}.$$

Note that the quantity k_q may be computed exactly if $q = 2$ or another natural number.

Since, as above,

$$[\min\{i,j\}(n - \max\{i,j\})]^q \leq \frac{1}{4^q}(n - |i-j|)^{2q},$$

we deduce

$$\begin{aligned} k_q &\leq \frac{1}{4n^2} \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (n - |i-j|)^{2q} \right)^{\frac{1}{q}} \\ &\leq \frac{1}{4n^2} \left[(n-1)^2 n^{2q} \right]^{\frac{1}{q}} \\ &= \frac{1}{4} (n-1)^{\frac{2}{q}} . \end{aligned}$$

Consequently, we may state the following corollary as well [**9**].

COROLLARY 67. *With the assumptions of Theorem 86 for* **a** *and* **b**, *we have the inequalities*

$$\begin{aligned} |T_n(\mathbf{a},\mathbf{b})| &\leq k_q \left(\sum_{i=1}^{n-1} \|\Delta a_i\|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n-1} \|\Delta b_i\|^p \right)^{\frac{1}{p}} \\ &\leq \frac{1}{4} (n-1)^{\frac{2}{q}} \left(\sum_{i=1}^{n-1} \|\Delta a_i\|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n-1} \|\Delta b_i\|^p \right)^{\frac{1}{p}} , \end{aligned} \tag{6.30}$$

provided $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *The constant* $\frac{1}{4}$ *cannot be replaced in general by a smaller constant.*

Finally, if we denote

$$k_1 := \frac{1}{n^2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left[\min\{i,j\} (n - \max\{i,j\}) \right],$$

then we observe, for $\mathbf{u} = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$, $\mathbf{e} = (1, 2, \dots, n)$, that

$$k_1 = |T_n(\mathbf{u}; \mathbf{e}, \mathbf{e})| = \frac{1}{n} \sum_{i=1}^{n} i^2 - \left(\frac{1}{n} \sum_{i=1}^{n} i \right)^2 = \frac{1}{12} \left(n^2 - 1 \right),$$

and by Theorem 89, we deduce the inequality

$$|T_n(\mathbf{a},\mathbf{b})| \leq \frac{1}{12} \left(n^2 - 1 \right) \max_{1 \leq j \leq n-1} \|\Delta a_j\| \max_{1 \leq j \leq n-1} \|\Delta b_j\| .$$

Note that, the above inequality has been discovered using a different method in [**10**]. The constant $\frac{1}{12}$ is best possible.

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CHAPTER 6

Other Inequalities in Inner Product Spaces

1. The Ostrowski Inequality

1.1. Introduction. In 1951, A.M. Ostrowski [**16**, p. 289] proved the following result (see also [**15**, p. 92]):

THEOREM 90. *Suppose that* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{x}$ *are real* n*-tuples such that* $\mathbf{a} \neq 0$ *and*

$$\sum_{i=1}^{n} a_i x_i = 0 \text{ and } \sum_{i=1}^{n} b_i x_i = 1.$$

Then

$$\sum_{i=1}^{n} x_i^2 \geq \frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2}$$

with equality if and only if

$$x_k = \frac{b_k \sum_{i=1}^{n} a_i^2 - a_k \sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2},$$

for $k \in \{1, \dots, n\}$.

An integral version of this inequality was obtained by Pearce, Pečarić and Varošanec in 1998, [**17**].

H. Šikić and T. Šikić in 2001, [**18**], by the use of an argument based on orthogonal projections in inner product spaces have observed that Ostrowski's inequality may be naturally stated in an abstract setting as follows:

THEOREM 91. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be a real or complex inner product space and* $a, b \in H$ *two linearly independent vectors. If* $x \in H$ *is such that*

$$\langle x, a \rangle = 0 \text{ and } \langle x, b \rangle = 1,$$

then one has the inequality

$$\|x\|^2 \geq \frac{\|a\|^2}{\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2}, \tag{1.1}$$

with equality if and only if

$$x=\frac{\|a\|^2 b-\overline{\langle a,b\rangle}\cdot a}{\|a\|^2\|b\|^2-|\langle a,b\rangle|^2}.$$

In the present section, by the use of elementary arguments and Schwarz's inequality in inner product spaces, we show that Ostrowski's inequality (1.1) holds true for a larger class of elements $x \in H$. The case of equality is analyzed. Applications for complex sequences and integrals are also provided.

1.2. The General Inequality. The following theorem holds [**6**].

THEOREM 92. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be a real or complex inner product space and* $a,b \in H$ *two linearly independent vectors. If* $x \in H$ *is such that*

$$\langle x,a\rangle = 0,\ \text{ and }\ |\langle x,b\rangle| = 1; \tag{1.2}$$

then one has the inequality

$$\|x\|^2 \geq \frac{\|a\|^2}{\|a\|^2\|b\|^2-|\langle a,b\rangle|^2}. \tag{1.3}$$

The equality holds in (1.3) *if and only if*

$$x=\mu\left(b-\frac{\overline{\langle a,b\rangle}}{\|a\|^2}\cdot a\right)$$

where $\mu \in \mathbb{K}\,(\mathbb{K}=\mathbb{R},\mathbb{C})$ *is such that*

$$|\mu| = \frac{\|a\|^2}{\|a\|^2\|b\|^2-|\langle a,b\rangle|^2}. \tag{1.4}$$

PROOF. We use Schwarz's inequality in the inner product space $(H;\langle\cdot,\cdot\rangle)$, i.e.,

$$\|u\|^2\|v\|^2 \geq |\langle u,v\rangle|^2; \qquad u,v \in H \tag{1.5}$$

with equality iff there exists a scalar $\alpha \in \mathbb{K}$ such that $u=\alpha v$.

If we apply (1.5) for

$$u = z-\frac{\langle z,c\rangle}{\|c\|^2}\cdot c, \qquad v = d-\frac{\langle d,c\rangle}{\|c\|^2}\cdot c,$$

where $c \neq 0$ and $c, d, z \in H$, then we have

$$\left\| z - \frac{\langle z, c\rangle}{\|c\|^2} \cdot c \right\|^2 \left\| d - \frac{\langle d, c\rangle}{\|c\|^2} \cdot c \right\|^2 \geq \left| \left\langle z - \frac{\langle z, c\rangle}{\|c\|^2} \cdot c, d - \frac{\langle d, c\rangle}{\|c\|^2} \cdot c \right\rangle \right|^2 \tag{1.6}$$

with equality iff there is a scalar $\beta \in \mathbb{K}$ such that

$$z = \frac{\langle z, c\rangle}{\|c\|^2} \cdot c + \beta \left(d - \frac{\langle d, c\rangle}{\|c\|^2} \cdot c \right). \tag{1.7}$$

Since simple calculations show that

$$\left\| z - \frac{\langle z, c\rangle}{\|c\|^2} \cdot c \right\|^2 = \frac{\|z\|^2 \|c\|^2 - |\langle z, c\rangle|^2}{\|c\|^2},$$

$$\left\| d - \frac{\langle d, c\rangle}{\|c\|^2} \cdot c \right\|^2 = \frac{\|d\|^2 \|c\|^2 - |\langle d, c\rangle|^2}{\|c\|^2},$$

and

$$\left\langle z - \frac{\langle z, c\rangle}{\|c\|^2} \cdot c, d - \frac{\langle d, c\rangle}{\|c\|^2} \cdot c \right\rangle = \frac{\langle z, d\rangle \|c\|^2 - \langle z, c\rangle \langle c, d\rangle}{\|c\|^2},$$

then, by (1.6), we deduce

$$\left[\|z\|^2 \|c\|^2 - |\langle z, c\rangle|^2 \right] \left[\|d\|^2 \|c\|^2 - |\langle d, c\rangle|^2 \right] \geq \left| \langle z, d\rangle \|c\|^2 - \langle z, c\rangle \langle c, d\rangle \right|^2, \tag{1.8}$$

with equality if and only if there is a $\beta \in \mathbb{K}$ such that (1.7) holds.

If a, x, b satisfy (1.2) then by (1.8) and (1.7) for the choices $z = x, c = a$ and $d = b$ we deduce the inequality (1.3) with equality iff there exists a $\mu \in \mathbb{K}$ such that

$$x = \mu \left(b - \frac{\overline{\langle a, b\rangle}}{\|a\|^2} \cdot a \right)$$

and, by the second condition in (1.2),

$$\left| \mu \left\langle b - \frac{\overline{\langle a, b\rangle}}{\|a\|^2} \cdot a, b \right\rangle \right| = 1. \tag{1.9}$$

Since (1.9) is clearly equivalent to (1.4), the theorem is completely proved. ∎

1.3. Applications for Sequences and Integrals. The following particular cases hold.

1. If $\mathbf{a},\mathbf{b},\mathbf{x}\in\ell^2(\mathbb{K})$, where $\ell^2(\mathbb{K}):=\left\{\mathbf{x}=(x_i)_{i\in\mathbb{N}},\sum_{i=1}^{\infty}|x_i|^2<\infty\right\}$, with $\mathbf{a},\mathbf{b}$ linearly independent and

$$\sum_{i=1}^{\infty}x_i\overline{a_i}=0 \text{ and } \left|\sum_{i=1}^{\infty}x_i\overline{b_i}\right|=1,$$

then one has the inequality

$$\sum_{i=1}^{\infty}|x_i|^2\geq\frac{\sum_{i=1}^{\infty}|a_i|^2}{\sum_{i=1}^{\infty}|a_i|^2\sum_{i=1}^{\infty}|b_i|^2-\left|\sum_{i=1}^{\infty}a_i\overline{b_i}\right|^2},$$

with equality iff

$$x_i=\mu\left[b_i-\frac{\sum_{k=1}^{\infty}\overline{a_k}b_k}{\sum_{k=1}^{\infty}|a_k|^2}\cdot a_i\right],\quad i\in\mathbb{N}$$

and $\mu\in\mathbb{K}$ with the property

$$|\mu|=\frac{\sum_{i=1}^{\infty}|a_i|^2}{\sum_{i=1}^{\infty}|a_i|^2\sum_{i=1}^{\infty}|b_i|^2-\left|\sum_{i=1}^{\infty}a_i\overline{b_i}\right|^2}.$$

2. If $f,g,h\in L^2(\Omega,m)$, where Ω is a measurable space and $L^2(\Omega,m):=\left\{f:\Omega\to\mathbb{K},\ \int_\Omega|f(x)|^2dm(x)<\infty\right\}$, with f,g linearly independent and

$$\int_\Omega h(x)\overline{f(x)}dm(x)=0,\quad\left|\int_\Omega h(x)\overline{g(x)}dm(x)\right|=1,$$

then one has the inequality

$$\int_\Omega|h(x)|^2dm(x)\geq\frac{\int_\Omega|f(x)|^2dm(x)}{\int_\Omega|f(x)|^2dm(x)\int_\Omega|g(x)|^2dm(x)-\left|\int_\Omega f(x)\overline{g(x)}dm(x)\right|^2}$$

with equality iff

$$h(x)=\nu\left[g(x)-\frac{\int_\Omega\overline{f(x)}g(x)\,dm(x)}{\int_\Omega|f(x)|^2dm(x)}\cdot f(x)\right]$$

for $m-$ a.e. $x\in\Omega$, and $\nu\in\mathbb{K}$ with

$$|\nu|=\frac{\int_\Omega|f(x)|^2dm(x)}{\int_\Omega|f(x)|^2dm(x)\int_\Omega|g(x)|^2dm(x)-\left|\int_\Omega f(x)\overline{g(x)}dm(x)\right|^2}.$$

2. Another Ostrowski Type Inequality

2.1. Introduction. Another result due to Ostrowski which is far less known than the one incorporated in Theorem 90 and obtained in the same work [**16**, p. 130] (see also [**15**, p. 94]), is the following one.

THEOREM 93. *Let* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{x}$ *be* $n-$*tuples of real numbers with* $\mathbf{a} \neq 0$ *and*

$$\sum_{i=1}^{n} a_i x_i = 0 \quad \textit{and} \quad \sum_{i=1}^{n} x_i^2 = 1.$$

Then

$$\frac{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2}{\sum_{i=1}^{n} a_i^2} \geq \left(\sum_{i=1}^{n} b_i x_i\right)^2. \tag{2.1}$$

If $\mathbf{a}$ *and* $\mathbf{b}$ *are not proportional, then the equality holds in (2.1) iff*

$$x_k = q \cdot \frac{b_k \sum_{i=1}^{n} a_i^2 - a_k \sum_{i=1}^{n} a_i b_i}{\left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}} \left[\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2\right]^{\frac{1}{2}}},$$

$k \in \{1, \dots, n\}$, *with* $q \in \{-1, 1,\}$.

The case of equality which was neither mentioned in [**16**] nor in [**15**] is considered in Remark 71.

In the present section, by the use of an elementary argument based on Schwarz's inequality, a natural generalisation in inner product spaces of (2.1) is given. The case of equality is analyzed. Applications for sequences and integrals are also provided.

2.2. The General Result. The following theorem holds [**9**].

THEOREM 94. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a real or complex inner product space and* $a, b \in H$ *two linearly independent vectors. If* $x \in H$ *is such that*

$$\langle x, a \rangle = 0 \ \textit{and} \ \|x\| = 1, \tag{i}$$

then

$$\frac{\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2}{\|a\|^2} \geq |\langle x, b \rangle|^2. \tag{2.2}$$

The equality holds in (2.2) iff

$$x = \nu \left(b - \frac{\overline{\langle a, b \rangle}}{\|a\|^2} \cdot a \right),$$

where $\nu \in \mathbb{K}$ $(\mathbb{C}, \mathbb{R})$ *is such that*

$$|\nu| = \frac{\|a\|}{\left[\|a\|^2 \|b\|^2 - |\langle a, b\rangle|^2\right]^{\frac{1}{2}}}.$$

Proof. We use Schwarz's inequality in the inner product space H, i.e.,

$$\|u\|^2 \|v\|^2 \geq |\langle u, v\rangle|^2, \quad u, v \in H \tag{2.3}$$

with equality iff there is a scalar $\alpha \in \mathbb{K}$ such that

$$u = \alpha v.$$

If we apply (2.3) for $u = z - \frac{\langle z,c\rangle}{\|c\|^2} \cdot c$, $v = d - \frac{\langle d,c\rangle}{\|c\|^2} \cdot c$, where $c \neq 0$ and $c, d, z \in H$, then we deduce the inequality

$$\left[\|z\|^2 \|c\|^2 - |\langle z, c\rangle|^2\right] \left[\|d\|^2 \|c\|^2 - |\langle d, c\rangle|^2\right] \geq \left|\langle z, d\rangle \|c\|^2 - \langle z, c\rangle \langle c, d\rangle\right|^2 \tag{2.4}$$

with equality iff there is a $\beta \in \mathbb{K}$ such that

$$z = \frac{\langle z, c\rangle}{\|c\|^2} \cdot c + \beta \left(d - \frac{\langle d, c\rangle}{\|c\|^2} \cdot c\right).$$

If in (2.4) we choose $z = x$, $c = a$ and $d = b$, where a and x statisfy (i), then we deduce

$$\|a\|^2 \left[\|a\|^2 \|b\|^2 - |\langle a, b\rangle|^2\right] \geq \left[\langle x, b\rangle \|a\|^2\right]^2$$

which is clearly equivalent to (2.2).

The equality holds in (2.2) iff

$$x = \nu \left(b - \frac{\overline{\langle a, b\rangle}}{\|a\|^2} \cdot a\right),$$

where $\nu \in \mathbb{K}$ satisfies the condition

$$1 = \|x\| = |\nu| \left\|b - \frac{\overline{\langle a, b\rangle}}{\|a\|^2} \cdot a\right\| = |\nu| \left[\frac{\|a\|^2 \|b\|^2 - |\langle a, b\rangle|^2}{\|a\|^2}\right]^{\frac{1}{2}},$$

and the theorem is thus proved. ∎

2.3. Applications for Sequences and Integrals. The following particular cases hold.

1. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{x} \in \ell^2(\mathbb{K})$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$, where

$$\ell^2(\mathbb{K}) := \left\{ \mathbf{x} = (x_i)_{i\in\mathbb{N}},\ \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}$$

with $\mathbf{a}$, $\mathbf{b}$ linearly independent and

$$\sum_{i=1}^{\infty} x_i \overline{a_i} = 0, \quad \sum_{i=1}^{\infty} |x_i|^2 = 1,$$

then

$$\frac{\sum_{i=1}^{\infty} |a_i|^2 \sum_{i=1}^{\infty} |b_i|^2 - \left|\sum_{i=1}^{\infty} a_i \overline{b_i}\right|^2}{\sum_{i=1}^{\infty} |a_i|^2} \geq \left| \sum_{i=1}^{\infty} x_i \overline{b_i} \right|^2. \tag{2.5}$$

The equality holds in (2.5) iff

$$x_i = \nu \left[b_i - \frac{\sum_{k=1}^{\infty} a_k \overline{b_k}}{\sum_{k=1}^{\infty} |a_k|^2} \cdot a_i \right], \quad i \in \{1, 2, \dots\}$$

with $\nu \in \mathbb{K}$ such that

$$|\nu| = \frac{\left(\sum_{k=1}^{\infty} |a_k|^2\right)^{\frac{1}{2}}}{\left[\sum_{k=1}^{\infty} |a_k|^2 \sum_{k=1}^{\infty} |b_k|^2 - \left|\sum_{k=1}^{\infty} a_k \overline{b_k}\right|^2\right]^{\frac{1}{2}}}.$$

Remark 71. *The case of equality in (2.1) is obviously a particular case of the above. We omit the details.*

2. If $f, g, h \in L^2(\Omega, m)$, where Ω is an $m-$measurable space and

$$L^2(\Omega, m) := \left\{ f : \Omega \to \mathbb{K},\ \int_\Omega |f(x)|^2 \, dm(x) < \infty \right\},$$

with f, g being linearly independent and

$$\int_\Omega h(x) \overline{f(x)} dm(x) = 0, \quad \int_\Omega |h(x)|^2 \, dm(x) = 1,$$

then

$$\frac{\int_\Omega |f(x)|^2 \, dm(x) \int_\Omega |g(x)|^2 \, dm(x) - \left| \int_\Omega f(x) \overline{g(x)} dm(x) \right|^2}{\int_\Omega |f(x)|^2 \, dm(x)} \geq \left| \int_\Omega h(x) \overline{g(x)} dm(x) \right|^2. \tag{2.6}$$

The equality holds in (2.6) iff

$$h(x) = \nu \left[g(x) - \frac{\int_\Omega g(x)\overline{f(x)}dm(x)}{\int_\Omega |f(x)|^2 dm(x)} f(x) \right] \quad \text{for a.e.} \quad x \in \Omega$$

and $\nu \in \mathbb{K}$ with

$$|\nu| = \frac{\left(\int_\Omega |f(x)|^2 dm(x)\right)^{\frac{1}{2}}}{\left[\int_\Omega |f(x)|^2 dm(x) \int_\Omega |g(x)|^2 dm(x) - \left|\int_\Omega f(x)\overline{g(x)}dm(x)\right|^2\right]^{\frac{1}{2}}}.$$

3. The Wagner Inequality in Inner Product Spaces

3.1. Introduction. In 1965, S.S. Wagner [**19**] (see also [**14**] or [**15**, p. 85]) pointed out the following generalisation of Cauchy-Bunyakovsky-Schwarz's inequality for real numbers.

THEOREM 95. *Let* $\mathbf{a} = (a_1, \dots, a_n)$ *and* $\mathbf{b} = (b_1, \dots, b_n)$ *be two n-tuples of real numbers. Then for any* $x \in [0,1]$, *one has the inequality*

$$\left(\sum_{k=1}^n a_k b_k + x \cdot \sum_{1 \le i \ne j \le n} a_i b_j\right)^2 \le \left(\sum_{k=1}^n a_k^2 + 2x \cdot \sum_{1 \le i < j \le n} a_i a_j\right) \cdot \left(\sum_{k=1}^n b_k^2 + 2x \cdot \sum_{1 \le i < j \le n} b_i b_j\right).$$

For $x = 0$, we recapture the Cauchy-Bunyakovsky-Schwarz's inequality, i.e., (see for example [**15**, p. 84])

$$\left(\sum_{k=1}^n a_k b_k\right)^2 \le \sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2,$$

with equality if and only if there exists a real number r such that $a_k = r b_k$ for each $k \in \{1, \dots, n\}$.

In this section we extend the above result for sequences of vectors in real or complex inner product spaces.

3.2. The Results. Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The following result holds [**3**].

THEOREM 96. *Let* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, \ldots, y_n)$ *be two* n*-tuples of vectors in* H. *Then for any* $\alpha \in [0,1]$ *one has the inequality*

$$\left[\sum_{k=1}^{n} \operatorname{Re}\langle x_k, y_k\rangle + \alpha \cdot \sum_{1\le i\ne j\le n} \operatorname{Re}\langle x_i, y_j\rangle\right]^2 \le \left[\sum_{k=1}^{n} \|x_k\|^2 + 2\alpha \cdot \sum_{1\le i<j\le n} \operatorname{Re}\langle x_i, x_j\rangle\right] \times \left[\sum_{k=1}^{n} \|y_k\|^2 + 2\alpha \cdot \sum_{1\le i<j\le n} \operatorname{Re}\langle y_i, y_j\rangle\right]. \tag{3.1}$$

PROOF. Following the proof by P. Flor [**14**], we may consider the function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(t) = (1-\alpha) \cdot \sum_{k=1}^{n} \|t x_k - y_k\|^2 + \alpha \cdot \left\|\sum_{k=1}^{n} (t x_k - y_k)\right\|^2. \tag{3.2}$$

Then

$$f(t) = \left[(1-\alpha) \cdot \sum_{k=1}^{n} \|x_k\|^2 + \alpha \cdot \left\|\sum_{k=1}^{n} x_k\right\|^2\right] t^2 + 2\left[(1-\alpha) \cdot \sum_{k=1}^{n} \operatorname{Re}\langle x_k, y_k\rangle + \alpha \cdot \operatorname{Re}\left\langle \sum_{k=1}^{n} x_k, \sum_{k=1}^{n} y_k\right\rangle\right] t + \left[(1-\alpha) \cdot \sum_{k=1}^{n} \|y_k\|^2 + \alpha \cdot \left\|\sum_{k=1}^{n} y_k\right\|^2\right]. \tag{3.3}$$

Observe that

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \sum_{k=1}^{n} \|x_k\|^2 + 2 \cdot \sum_{1\le i<j\le n} \operatorname{Re}\langle x_i, x_j\rangle \tag{3.4}$$

and

$$\left\|\sum_{k=1}^{n} y_k\right\|^2 = \sum_{k=1}^{n} \|y_k\|^2 + 2 \cdot \sum_{1\le i<j\le n} \operatorname{Re}\langle y_i, y_j\rangle. \tag{3.5}$$

Also

$$\operatorname{Re}\left\langle \sum_{k=1}^{n} x_k, \sum_{k=1}^{n} y_k\right\rangle = \sum_{k=1}^{n} \operatorname{Re}\langle x_k, y_k\rangle + \sum_{1\le i\ne j\le n} \operatorname{Re}\langle x_i, y_j\rangle. \tag{3.6}$$

Using $(3.3)-(3.6)$, we deduce that

$$f(t)=\left[\sum_{k=1}^{n}\|x_k\|^2+2\alpha\cdot\sum_{1\le i<j\le n}\operatorname{Re}\langle x_i,x_j\rangle\right]t^2 \tag{3.7}$$
$$+2\left[\sum_{k=1}^{n}\operatorname{Re}\langle x_k,y_k\rangle+\alpha\cdot\sum_{1\le i\neq j\le n}\operatorname{Re}\langle x_i,y_j\rangle\right]t$$
$$+\left[\sum_{k=1}^{n}\|y_k\|^2+2\alpha\cdot\sum_{1\le i<j\le n}\operatorname{Re}\langle y_i,y_j\rangle\right].$$

Since, by (3.2), $f(t)\ge 0$ for any $t\in\mathbb{R}$, it follows that the discriminant of the quadratic function given by (3.7) is negative, which is clearly equivalent with the desired inequality (3.1). ∎

One may obtain an interesting inequality if $\mathbf{x}$ and $\mathbf{y}$ are assumed to incorporate orthogonal vectors.

Corollary 68. *Assume that $\{x_i\}_{i=1,\dots,n}$ are orthogonal, i.e., $x_i\perp x_j$ for any $i,j\in\{1,\dots,n\}$, $i\neq j$; and $\{y_i\}_{i=1,\dots,n}$ are also orthogonal in the real inner product space $(H;\langle\cdot,\cdot\rangle)$. Then*

$$\sup_{\alpha\in[0,1]}\left[\sum_{k=1}^{n}\langle x_k,y_k\rangle+\alpha\cdot\sum_{1\le i\neq j\le n}\langle x_i,y_j\rangle\right]^2\le\sum_{k=1}^{n}\|x_k\|^2\sum_{k=1}^{n}\|y_k\|^2.$$

3.3. Applications.

1. If we assume that $H=\mathbb{C}$, with the inner product $\langle x,y\rangle=x\cdot\bar{y}$, then by (3.1) we may deduce the following Wagner type inequality for complex numbers

$$\left[\sum_{k=1}^{n}\operatorname{Re}\left(a_k\bar{b}_k\right)+\alpha\cdot\sum_{1\le i\neq j\le n}\operatorname{Re}\left(a_i\bar{b}_j\right)\right]^2$$
$$\le\left[\sum_{k=1}^{n}|a_k|^2+2\alpha\cdot\sum_{1\le i<j\le n}\operatorname{Re}\left(a_i\bar{a}_j\right)\right]$$
$$\times\left[\sum_{k=1}^{n}|b_k|^2+2\alpha\cdot\sum_{1\le i<j\le n}\operatorname{Re}\left(b_i\bar{b}_j\right)\right],$$

where $\alpha\in[0,1]$ and $\mathbf{a}=(a_1,\dots,a_n)$, $\mathbf{b}=(b_1,\dots,b_n)\in\mathbb{C}^n$.

2. Consider the Hilbert space

$$L_2(\Omega,\mu):=\left\{f:\Omega\to\mathbb{C},\ \int_{\Omega}|f(x)|^2\,d\mu(x)<\infty\right\},$$

where Ω is a μ-measurable space and $\mu : \Omega \to [0,\infty]$ is a positive measure on Ω. Then for $H = L_2(\Omega,\mu)$ and since the inner product generating the norm is given by

$$\langle f,g\rangle = \int_\Omega f(x)\,\bar{g}(x)\,d\mu(x),$$

we get the inequality

$$\begin{aligned}&\Bigg[\sum_{k=1}^{n}\int_\Omega \operatorname{Re}\left(f_k(x)\,\bar{g}_k(x)\right)d\mu(x)\\ &\qquad\qquad +\alpha\cdot\sum_{1\le i\neq j\le n}\int_\Omega \operatorname{Re}\left(f_i(x)\,\bar{g}_j(x)\right)d\mu(x)\Bigg]^2\\ &\le\left[\sum_{k=1}^{n}\int_\Omega |f_k(x)|^2\,d\mu(x)+2\alpha\cdot\sum_{1\le i<j\le n}\int_\Omega \operatorname{Re}\left(f_i(x)\,\bar{f}_j(x)\right)d\mu(x)\right]\\ &\times\left[\sum_{k=1}^{n}\int_\Omega |g_k(x)|^2\,d\mu(x)+2\alpha\cdot\sum_{1\le i<j\le n}\int_\Omega \operatorname{Re}\left(g_i(x)\,\bar{g}_j(x)\right)d\mu(x)\right],\end{aligned}$$

where $f_i, g_i \in L_2(\Omega,\mu)$, $i\in\{1,\dots,n\}$ and $\alpha\in[0,1]$.

4. A Monotoniciy Property of Bessel's Inequality

Let X be a linear space over the real or complex number field $\mathbb{K}$. A mapping $(\cdot,\cdot) : X\times X\to\mathbb{K}$ is said to be a *positive hermitian form* if the following conditions are satisfied:

(i) $(\alpha x+\beta y,z)=\alpha(x,z)+\beta(y,z)$ for all $x,y,z\in X$ and $\alpha,\beta\in\mathbb{K}$;
(ii) $(y,x)=\overline{(x,y)}$ for all $x,y\in X$;
(iii) $(x,x)\ge 0$ for all $x\in X$.

If $\|x\| := (x,x)^{\frac{1}{2}}$ denotes the semi-norm associated to this form and $(e_i)_{i\in I}$ is an orthonormal family of vectors in X, that is, $(e_i,e_j)=\delta_{ij}$ $(i,j\in I)$, then one has [**20**]:

$$\|x\|^2\ge\sum_{i\in I}|(x,e_i)|^2\quad\text{for all } x\in X, \tag{4.1}$$

which is well known in the literature as Bessel's inequality.

The main aim of the section is to point out an improvement for this result as follows [**4**].

THEOREM 97. *Let X be a linear space and $(\cdot,\cdot)_2,(\cdot,\cdot)_1$ two hermitian forms on X such that $\|\cdot\|_2$ is greater than or equal to $\|\cdot\|_1$, that* *is,* $\|x\|_2 \geq \|x\|_1$ *for all* $x \in X$*. Assume that* $(e_i)_{i\in I}$ *is an orthonormal family in* $(X;(\cdot,\cdot)_2)$ *and* $(f_j)_{j\in J}$ *is an orthonormal family in* $(X;(\cdot,\cdot)_1)$ *such that for any* $i \in I$ *there exists a finite* $K \subset J$ *such that*

$$e_i = \sum_{j\in K} \alpha_j f_j, \quad \alpha_j \in \mathbb{K}, \qquad (j \in K), \tag{4.2}$$

then one has the inequality:

$$\|x\|_2^2 - \sum_{i\in I} \left|(x,e_i)_2\right|^2 \geq \|x\|_1^2 - \sum_{j\in J} \left|(x,f_j)_1\right|^2 \geq 0, \text{ for all } x \in X. \tag{4.3}$$

In order to prove this, we require the following lemma.

Lemma 13. *Let* X *be a linear space endowed with a positive hermitian form* $(\cdot,\cdot)$ *and* (g_k), $k \in \{1,\dots,n\}$ *be an orthonormal family in* $(X;(\cdot,\cdot))$*. Then*

$$\left\|x - \sum_{k=1}^{n} \lambda_k g_k\right\|^2 \geq \|x\|^2 - \sum_{k=1}^{n} \left|(x,g_k)\right|^2 \geq 0,$$

for all $\lambda_k \in \mathbb{K}$, $k \in \{1\dots,n\}$ *and* $x \in X$.

The proof follows by mathematical induction.

Proof of Theorem 97. Let H be a finite subset of I. Since $\|\cdot\|_2$ is greater than $\|\cdot\|_1$, we have:

$$\begin{aligned}
\|x\|_2^2 - \sum_{i\in H} \left|(x,e_i)_2\right|^2 &= \left\|x - \sum_{i\in H} (x,e_i)_2\, e_i\right\|_2^2 \\
&\geq \left\|x - \sum_{i\in H} (x,e_i)_2\, e_i\right\|_1^2, \quad x \in X.
\end{aligned}$$

Since, by (4.2), we may state that for any $i \in H$ there exists a finite $K \subset J$ with

$$e_i = \sum_{j\in K} (e_i,f_j)_1\, f_j,$$

we have, for all $x \in X$

$$\begin{aligned}
\left\|x - \sum_{i\in H} (x,e_i)_2\, e_i\right\|_1^2 &= \left\|x - \sum_{i\in H} (x,e_i)_2 \sum_{j\in K} (e_i,f_j)_1\, f_j\right\|_1^2 \\
&= \left\|x - \sum_{j\in K} \left(\sum_{i\in H} (x,e_i)_2\, e_i, f_j\right)_1 f_j\right\|_1^2.
\end{aligned}$$

Applying Lemma 13, we can conclude that

$$\left\| x - \sum_{j \in K} \lambda_j f_j \right\|_1^2 \geq \|x\|_1^2 - \sum_{j \in K} \left| (x, f_j)_1 \right|^2, \quad x \in X,$$

where

$$\lambda_j = \left(\sum_{i \in H} (x, e_i)_2 \, e_i, f_j \right)_1 \in \mathbb{K}, (j \in K).$$

Consequently, we have

$$\begin{aligned} \|x\|_2^2 - \sum_{i \in H} |(x, e_i)_2|^2 &\geq \|x\|_1^2 - \sum_{j \in K} \left| (x, f_j)_1 \right|^2 \\ &\geq \|x\|_1^2 - \sum_{j \in J} \left| (x, f_j)_1 \right|^2 \end{aligned}$$

for all $x \in X$ and H a finite subset of I, from which (4.3) results. ∎

Corollary 69. *Let $\|\cdot\|_1, \|\cdot\|_2 : X \to \mathbb{R}_+$ be as above. Then for all $x, y \in X$, we have the inequality:*

$$\|x\|_2^2 \|y\|_2^2 - |(x, y)_2|^2 \geq \|x\|_1^2 \|y\|_1^2 - |(x, y)_1|^2 \geq 0, \tag{4.4}$$

which is an improvement of the well known Cauchy-Schwarz inequality.

Remark 72. *For a different proof of (4.4), see also* [**10**] *or* [**11**].

Now, we will give some natural applications of the above theorem.

(1) Let $(X; (\cdot, \cdot))$ be an inner product space and $(e_i)_{i \in I}$ an orthonormal family in X. Assume that $A : X \to X$ is a linear operator such that $\|Ax\| \leq \|x\|$ for all $x \in X$ and $(Ae_i, Ae_j) = \delta_{ij}$ for all $i, j \in I$. Then one has the inequality

$$\begin{aligned} \|x\|^2 - \sum_{i \in I} |(x, e_i)|^2 &\geq \|Ax\|^2 - \sum_{i \in I} |(Ax, Ae_i)|^2 \\ &\geq 0, \text{ for all } x \in X. \end{aligned}$$

(2) If $A : X \to X$ is such that $\|Ax\| \geq \|x\|$ for all $x \in X$, then, with the previous assumptions, we also have

$$\begin{aligned} 0 &\leq \|x\|^2 - \sum_{i \in I} |(x, e_i)|^2 \\ &\leq \|Ax\|^2 - \sum_{i \in I} |(Ax, Ae_i)|^2, \text{ for all } x \in X. \end{aligned}$$

(3) Suppose that $A : X \to X$ is a symmetric positive definite operator with $(Ax, x) \geq \|x\|^2$ for all $x \in X$. If $(e_i)_{i\in I}$ is an orthonormal family in X such that $(Ae_i, Ae_j) = \delta_{ij}$ for all $i, j \in I$, then one has the inequality

$$0 \leq \|x\|^2 - \sum_{i\in I} |(x, e_i)|^2 \leq (Ax, x) - \sum_{i\in I} |(Ax, e_i)|^2 ,$$

for any $x \in X$.

5. Other Bombieri Type Inequalities

5.1. Introduction. In 1971, E. Bombieri [**1**] gave the following generalisation of Bessel's inequality:

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\|^2 \max_{1\leq i\leq n} \left\{ \sum_{j=1}^{n} |(y_i, y_j)| \right\}, \tag{5.1}$$

where $x, y_1, \dots, y_n$ are vectors in the inner product space $(H; (\cdot, \cdot))$.

It is obvious that if $(y_i)_{1\leq i\leq n} = (e_i)_{1\leq i\leq n}$, where $(e_i)_{1\leq i\leq n}$ are orthornormal vectors in H, i.e., $(e_i, e_j) = \delta_{ij}$ $(i, j = 1, \dots, n)$, where δ_{ij} is the Kronecker delta, then (5.1) provides Bessel's inequality

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2 , \quad x \in H.$$

In this section we point out some Bombieri type inequalities that complement the results obtained in Chapter 4.

5.2. The Results. The following lemma, which is of interest in itself, holds [**7**].

Lemma 14. *Let $z_1, \dots, z_n \in H$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. Then one has the inequalities:*

$$\left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \leq \begin{cases} \max\limits_{1\leq i\leq n} |\alpha_i|^2 \sum_{i,j=1}^{n} |(z_i, z_j)| ; \\ \left(\sum_{i=1}^{n} |\alpha_i|^p\right)^{\frac{2}{p}} \left(\sum_{i,j=1}^{n} |(z_i, z_j)|^q\right)^{\frac{1}{q}} , \\ \qquad \text{where } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \left(\sum_{i=1}^{n} |\alpha_i|\right)^2 \max\limits_{1\leq i,j\leq n} |(z_i, z_j)| ; \end{cases}$$

$$\leq \begin{cases} \max\limits_{1\leq i\leq n} |\alpha_i|^2 \left(\sum_{i=1}^{n} \|z_i\|\right)^2 ; \\ \left(\sum_{i=1}^{n} |\alpha_i|^p\right)^{\frac{2}{p}} \left(\sum_{i=1}^{n} \|z_i\|^q\right)^{\frac{2}{q}} , \\ \qquad \text{where } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \left(\sum_{i=1}^{n} |\alpha_i|\right)^2 \max\limits_{1\leq i\leq n} \|z_i\|^2 . \end{cases}$$

PROOF. We observe that

$$\begin{aligned}\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 &= \left(\sum_{i=1}^{n}\alpha_i z_i,\sum_{j=1}^{n}\alpha_j z_j\right)=\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}\left(z_i,z_j\right)\\ &=\left|\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}\left|(z_i,z_j)\right|\right|\\ &\le\sum_{i=1}^{n}\sum_{j=1}^{n}|\alpha_i|\,|\alpha_j|\,|(z_i,z_j)| =: M.\end{aligned}$$

Firstly, we have

$$\begin{aligned}M &\le \max_{1\le i,j\le n}\{|\alpha_i|\,|\alpha_j|\}\sum_{i,j=1}^{n}|(z_i,z_j)|\\ &=\max_{1\le i\le n}|\alpha_i|^2\sum_{i,j=1}^{n}|(z_i,z_j)|\,.\end{aligned}$$

Secondly, by the Hölder inequality for double sums, we obtain

$$\begin{aligned}M &\le \left[\sum_{i,j=1}^{n}(|\alpha_i|\,|\alpha_j|)^p\right]^{\frac{1}{p}}\left(\sum_{i,j=1}^{n}|(z_i,z_j)|^q\right)^{\frac{1}{q}}\\ &=\left(\sum_{i=1}^{n}|\alpha_i|^p\sum_{j=1}^{n}|\alpha_j|^p\right)^{\frac{1}{p}}\left(\sum_{i,j=1}^{n}|(z_i,z_j)|^q\right)^{\frac{1}{q}}\\ &=\left(\sum_{i=1}^{n}|\alpha_i|^p\right)^{\frac{2}{p}}\left(\sum_{i,j=1}^{n}|(z_i,z_j)|^q\right)^{\frac{1}{q}},\end{aligned}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Finally, we have

$$M\le\max_{1\le i,j\le n}|(z_i,z_j)|\sum_{i,j=1}^{n}|\alpha_i|\,|\alpha_j|=\left(\sum_{i=1}^{n}|\alpha_i|\right)^2\max_{1\le i,j\le n}|(z_i,z_j)|$$

and the first part of the lemma is proved.

The second part is obvious on taking into account, by Schwarz's inequality in H, that we have

$$|(z_i,z_j)|\le\|z_i\|\,\|z_j\|\,,$$

for any $i,j\in\{1,\dots,n\}$. We omit the details. ∎

Corollary 70. *With the assumptions in Lemma 14, one has*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \le \sum_{i=1}^{n}|\alpha_i|^2\left(\sum_{i,j=1}^{n}|(z_i,z_j)|^2\right)^{\frac{1}{2}} \le \sum_{i=1}^{n}|\alpha_i|^2\sum_{i=1}^{n}\|z_i\|^2 . \tag{5.2}$$

The proof follows by Lemma 14 on choosing $p=q=2$.

Note also that (5.2) provides a refinement of the well known Cauchy-Bunyakovsky-Schwarz inequality for sequences of vectors in inner product spaces, namely

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \le \sum_{i=1}^{n}|\alpha_i|^2\sum_{i=1}^{n}\|z_i\|^2 .$$

The following lemma also holds [**7**].

Lemma 15. *Let $x,y_1,\dots,y_n\in H$ and $c_1,\dots,c_n\in\mathbb{K}$. Then one has the inequalities:*

$$\left|\sum_{i=1}^{n}c_i(x,y_i)\right|^2 \le \|x\|^2\times\begin{cases}\max\limits_{1\le i\le n}|c_i|^2\sum_{i,j=1}^{n}|(y_i,y_j)|;\\[2mm] \left(\sum_{i=1}^{n}|c_i|^p\right)^{\frac{2}{p}}\left(\sum_{i,j=1}^{n}|(y_i,y_j)|^q\right)^{\frac{1}{q}},\\ \qquad\qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2mm] \left(\sum_{i=1}^{n}|c_i|\right)^2\max\limits_{1\le i,j\le n}|(y_i,y_j)|;\end{cases}$$
$$\le \|x\|^2\times\begin{cases}\max\limits_{1\le i\le n}|c_i|^2\left(\sum_{i=1}^{n}\|y_i\|\right)^2;\\[2mm] \left(\sum_{i=1}^{n}|c_i|^p\right)^{\frac{2}{p}}\left(\sum_{i=1}^{n}\|y_i\|^q\right)^{\frac{2}{q}},\\ \qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2mm] \left(\sum_{i=1}^{n}|c_i|\right)^2\max\limits_{1\le i\le n}\|y_i\|^2 .\end{cases} \tag{5.3}$$

Proof. We have, by Schwarz's inequality in the inner product $(H;(\cdot,\cdot))$, that

$$\left|\sum_{i=1}^{n}c_i(x,y_i)\right|^2 = \left|\left(x,\sum_{i=1}^{n}\overline{c_i}y_i\right)\right|^2 \le \|x\|^2\left\|\sum_{i=1}^{n}\overline{c_i}y_i\right\|^2 .$$

Now, applying Lemma 14 for $\alpha_i = \overline{c_i}$, $z_i = y_i$ $(i = 1, \ldots, n)$, the inequality (5.3) is proved. ∎

Corollary 71. *With the assumptions in Lemma 15, one has*

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \le \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \left(\sum_{i,j=1}^{n} |(y_i, y_j)|^2\right)^{\frac{1}{2}} \le \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \sum_{i=1}^{n} \|y_i\|^2 . \tag{5.4}$$

The proof follows by Lemma 15, on choosing $p = q = 2$.

Remark 73. *The inequality (5.4) was firstly obtained in* [**12**, Inequality (7)].

The following theorem incorporating three Bombieri type inequalities holds [**7**].

Theorem 98. *Let $x, y_1, \ldots, y_n \in H$. Then one has the inequalities:*

$$\sum_{i=1}^{n} |(x, y_i)|^2 \le \|x\| \times \begin{cases} \max\limits_{1\le i\le n} |(x, y_i)| \left(\sum_{i,j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{2}}; \\ \left(\sum_{i=1}^{n} |(x, y_i)|^p\right)^{\frac{1}{p}} \left(\sum_{i,j=1}^{n} |(y_i, y_j)|^q\right)^{\frac{1}{2q}}, \\ \qquad\qquad \textit{where } \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n} |(x, y_i)| \max\limits_{1\le i,j\le n} |(y_i, y_j)|^{\frac{1}{2}} . \end{cases} \tag{5.5}$$

Proof. Choosing $c_i = \overline{(x, y_i)}$ $(i = 1, \ldots, n)$ in (5.3) we deduce

$$\left(\sum_{i=1}^{n} |(x, y_i)|^2\right)^2 \le \|x\|^2 \times \begin{cases} \max\limits_{1\le i\le n} |(x, y_i)|^2 \left(\sum_{i,j=1}^{n} |(y_i, y_j)|\right); \\ \left(\sum_{i=1}^{n} |(x, y_i)|^p\right)^{\frac{2}{p}} \left(\sum_{i,j=1}^{n} |(y_i, y_j)|^q\right)^{\frac{1}{q}}, \\ \qquad\qquad \text{where } \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \left(\sum_{i=1}^{n} |(x, y_i)|\right)^2 \max\limits_{1\le i,j\le n} |(y_i, y_j)|; \end{cases} \tag{5.6}$$

which, by taking the square root, is clearly equivalent to (5.5). ∎

Remark 74. *If $(y_i)_{1\leq i\leq n}=(e_i)_{1\leq i\leq n}$, where $(e_i)_{1\leq i\leq n}$ are orthornormal vectors in H, then by (5.5) we deduce*

$$\sum_{i=1}^{n}\left|(x,e_i)\right|^2\leq\|x\|\begin{cases}\sqrt{n}\max\limits_{1\leq i\leq n}\left|(x,e_i)\right|; \\ n^{\frac{1}{2q}}\left(\sum_{i=1}^{n}\left|(x,e_i)\right|^p\right)^{\frac{1}{p}}, \\ \qquad\text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum_{i=1}^{n}\left|(x,e_i)\right|.\end{cases}\tag{5.7}$$

If in (5.6) we take $p=q=2$, then we obtain the following inequality which was formulated in [**12**, p. 81].

Corollary 72. *With the assumptions in Theorem 98, we have:*

$$\sum_{i=1}^{n}\left|(x,y_i)\right|^2\leq\|x\|^2\left(\sum_{i,j=1}^{n}\left|(y_i,y_j)\right|^2\right)^{\frac{1}{2}}.\tag{5.8}$$

Remark 75. *Observe, that by the monotonicity of power means, we may write*

$$\left(\frac{\sum_{i=1}^{n}\left|(x,y_i)\right|^p}{n}\right)^{\frac{1}{p}}\leq\left(\frac{\sum_{i=1}^{n}\left|(x,y_i)\right|^2}{n}\right)^{\frac{1}{2}},\quad 1<p\leq 2.$$

Taking the square in both sides, one has

$$\left(\frac{\sum_{i=1}^{n}\left|(x,y_i)\right|^p}{n}\right)^{\frac{2}{p}}\leq\frac{\sum_{i=1}^{n}\left|(x,y_i)\right|^2}{n},$$

giving

$$\left(\sum_{i=1}^{n}\left|(x,y_i)\right|^p\right)^{\frac{2}{p}}\leq n^{\frac{2}{p}-1}\sum_{i=1}^{n}\left|(x,y_i)\right|^2.\tag{5.9}$$

Using (5.9) and the second inequality in (5.6) we may deduce the following result

$$\sum_{i=1}^{n}\left|(x,y_i)\right|^2\leq n^{\frac{2}{p}-1}\|x\|^2\left(\sum_{i,j=1}^{n}\left|(y_i,y_j)\right|^q\right)^{\frac{1}{q}},\tag{5.10}$$

for $1<p\leq 2$, $\frac{1}{p}+\frac{1}{q}=1$.

Note that for $p=2$ $(q=2)$ we recapture (5.8).

Remark 76. *Let us compare Bombieri's result*

$$\sum_{i=1}^{n}\left|(x,y_i)\right|^2\leq\|x\|^2\max_{1\leq i\leq n}\left\{\sum_{j=1}^{n}\left|(y_i,y_j)\right|\right\}\tag{5.11}$$

with our general result (5.10).

To do that, denote

$$M_1 := \max_{1\le i\le n}\left\{\sum_{j=1}^{n}|(y_i,y_j)|\right\}$$

and

$$M_2 := n^{\frac{2}{p}-1}\left(\sum_{i,j=1}^{n}|(y_i,y_j)|^q\right)^{\frac{1}{q}},\quad 1<p\le 2,\ \frac{1}{p}+\frac{1}{q}=1.$$

Consider the inner product space $H=\mathbb{R}$, $(x,y)=x\cdot y$, $n=2$ *and* $y_1=a>0$, $y_2=b>0$. *Then*

$$M_1=\max\left\{a^2+ab,ab+b^2\right\}=(a+b)\max\{a,b\},$$

$$M_2=2^{\frac{2}{p}-1}\left(a^q+b^q\right)^{\frac{2}{q}}=2^{\frac{2}{p}-1}\left(a^{\frac{p}{p-1}}+b^{\frac{p}{p-1}}\right)^{\frac{2(p-1)}{p}},\ 1<p\le 2.$$

Assume that $a=1$, $b\in[0,1]$, $p\in(1,2]$. *Utilizing Maple 6, one may easily see by plotting the function*

$$f(b,p) := M_2-M_1 = 2^{\frac{2}{p}-1}\left(1+b^{\frac{p}{p-1}}\right)^{\frac{2(p-1)}{p}}-1-b$$

that it has positive and negative values in the box $[0,1]\times[1,2]$, *showing that the inequalities (5.10) and (5.11) cannot be compared. This means that one is not always better than the other.*

6. Some Pre-Grüss Inequalities

6.1. Introduction. Let f, g be two functions defined and integrable on $[a,b]$. Assume that

$$\varphi\le f(x)\le\Phi \quad\text{and}\quad \gamma\le g(x)\le\Gamma$$

for each $x\in[a,b]$, where φ, Φ, γ, Γ are given real constants. Then we have the following inequality which is well known in the literature as the Grüss inequality ([**15**, pp. 296])

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \le\frac{1}{4}|\Phi-\varphi|\cdot|\Gamma-\gamma|.$$

In this inequality, G. Grüss has proven that the constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one, and this is achieved when

$$f(x) = g(x) = \operatorname{sgn}\left(x - \frac{a+b}{2}\right).$$

Recently, S.S. Dragomir proved the following Grüss' type inequality in real or complex inner product spaces [**2**].

THEOREM 99. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ and $e \in H$, $\|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \geq 0 \quad \text{and} \quad \operatorname{Re}\langle \Gamma e - x, x - \gamma e\rangle \geq 0$$

hold, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \; \langle e, y\rangle| \leq \frac{1}{4}|\Phi - \varphi| \cdot |\Gamma - \gamma|. \tag{6.1}$$

The constant $\frac{1}{4}$ is best possible in sense that it cannot be replaced by a smaller constant.

In [**8**], by using the following lemmas

LEMMA 16. *Let $x, e \in H$ with $\|e\| = 1$ and δ, $\Delta \in \mathbb{K}$ with $\delta \neq \Delta$. Then*

$$\operatorname{Re}\langle \Delta e - x, x - \delta e\rangle \geq 0$$

if and only if

$$\left\| x - \frac{\delta + \Delta}{2} e \right\| \leq \frac{1}{2}|\Delta - \delta|.$$

and

LEMMA 17. *Let $x, e \in H$ with $\|e\| = 1$. Then one has the following representation*

$$0 \leq \|x\|^2 - |\langle x, e\rangle|^2 = \inf_{\lambda \in K} \|x - \lambda e\|^2.$$

the author gave an alternative proof for (6.1) and also obtained the following refinement of it, namely

THEOREM 100. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ $(\mathbb{K} = \mathbb{R},\mathbb{C})$ and $e \in H, \|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that either the conditions*

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \geq 0, \; \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0,$$

or, equivalently,

$$\left\| x - \frac{\varphi + \Phi}{2} \cdot e \right\| \leq \frac{1}{2}|\Phi - \varphi|, \quad \left\| y - \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2}|\Gamma - \gamma|,$$

hold, then we have the inequality

$$\begin{aligned}
&|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle| \\
&\leq \frac{1}{4}|\Phi-\varphi|\cdot|\Gamma-\gamma|-[\operatorname{Re}\langle\Phi e-x,x-\varphi e\rangle]^{\frac{1}{2}}[\operatorname{Re}\langle\Gamma e-y,y-\gamma e\rangle]^{\frac{1}{2}} \\
&\leq\left(\frac{1}{4}|\Phi-\varphi|\cdot|\Gamma-\gamma|\right).
\end{aligned}$$

The constant $\frac{1}{4}$ *is best possible.*

Further, as a generalization for orthonormal families of vectors in inner product spaces, S.S. Dragomir proved, in [**5**], the following reverse of Bessel's inequality:

THEOREM 101. *Let* $\{e_i\}$, $i\in I$ *be a family of orthonormal vectors in* H, F *a finite part of* I, φ_i, $\Phi_i\in\mathbb{K}$, $i\in F$ *and* x *a vector in* H *such that either the condition*

$$\operatorname{Re}\left\langle\sum_{i\in F}\Phi_i e_i-x,\,x-\sum_{i\in F}\varphi_i e_i\right\rangle\geq 0,$$

or, equivalently,

$$\left\|x-\sum_{i\in F}\frac{\Phi_i+\varphi_i}{2}e_i\right\|\leq\frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\varphi_i|^2\right)^{\frac{1}{2}},$$

holds, then we have the following reverse of Bessel's inequality

$$\begin{aligned}
(6.2)\qquad &\|x\|^2-\sum_{i\in F}|\langle x,\,e_i\rangle|^2 \\
&\qquad\leq\frac{1}{4}\sum_{i\in F}|\Phi_i-\varphi_i|^2-\sum_{i\in F}\left|\frac{\varphi_i+\Phi_i}{2}-\langle x,\,e_i\rangle\right|^2.
\end{aligned}$$

The constant $\frac{1}{4}$ *is best possible.*

The corresponding Grüss type inequality is embodied in the following theorem:

THEOREM 102. *Let* $\{e_i\}_{i\in I}$ *be a family of orthornormal vectors in* H, F *a finite part of* I, $\phi_i,\gamma_i,\Phi_i,\Gamma_i\in\mathbb{R}$ $(i\in F)$, *and* $x,y\in H$. *If either*

$$\operatorname{Re}\left\langle\sum_{i=1}^n\Phi_i e_i-x,x-\sum_{i=1}^n\phi_i e_i\right\rangle\geq 0,$$

$$\operatorname{Re}\left\langle\sum_{i=1}^n\Gamma_i e_i-y,y-\sum_{i=1}^n\gamma_i e_i\right\rangle\geq 0,$$

or, equivalently,

$$\left\| x - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} e_i \right\| \leq \frac{1}{2} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}},$$

$$\left\| y - \sum_{i \in F} \frac{\Gamma_i + \gamma_i}{2} e_i \right\| \leq \frac{1}{2} \left(\sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}},$$

hold true, then

$$\begin{aligned} 0 &\leq \left| \langle x, y \rangle - \sum_{i=1}^{n} \langle x, e_i \rangle \langle e_i, y \rangle \right| \\ &\leq \frac{1}{4} \left(\sum_{i=1}^{n} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{n} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \\ &\qquad - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle x, e_i \rangle \right| \left| \frac{\Gamma_i + \gamma_i}{2} - \langle y, e_i \rangle \right| \\ &\left(\leq \frac{1}{4} \left(\sum_{i=1}^{n} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{n} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \right). \end{aligned}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

The main aim here is to provide some similar inequalities which, giving refinements of the usual Grüss' inequality, are known in the literature as pre-Grüss type inequalities. Applications for Lebesgue integrals in general measure spaces are also given.

6.2. Pre-Grüss Inequalities in Inner Product Spaces. We start with the following result [**13**]:

Theorem 103. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$, $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ and $e \in H$, $\|e\| = 1$. If φ, Φ are real or complex numbers and x, y are vectors in H such that either the condition*

$$\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \geq 0,$$

or, equivalently,

$$\left\| x - \frac{\varphi + \Phi}{2} e \right\| \leq \frac{1}{2} |\Phi - \varphi|, \tag{6.3}$$

holds true, then we have the inequalities

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{2} |\Phi - \varphi| \cdot \sqrt{\|y\|^2 - |\langle y, e \rangle|^2} \tag{6.4}$$

and

$$|\langle x,y\rangle - \langle x,e\rangle \, \langle e,y\rangle| \leq \frac{1}{2}\,|\Phi-\varphi|\cdot\|y\| - (\operatorname{Re}\langle \Phi e - x, x-\varphi e\rangle)^{\frac{1}{2}}\cdot|\langle y,\, e\rangle| \,. \tag{6.5}$$

PROOF. It is obvious that:

$$\langle x,y\rangle - \langle x,e\rangle\,\langle e,y\rangle = \langle x-\langle x,e\rangle\, e, y - \langle y,e\rangle\, e\rangle\,.$$

Using Schwarz's inequality in inner product spaces $|\langle u,\, v\rangle| \leq \|u\|\cdot\|v\|$ for the vectors $x-\langle x,e\rangle\, e$ and $y-\langle y,e\rangle\, e$, we deduce:

$$|\langle x,y\rangle - \langle x,e\rangle\,\langle e,y\rangle|^2 \leq \left(\|x\|^2 - |\langle x,\, e\rangle|^2\right)\cdot\left(\|y\|^2 - |\langle y,\, e\rangle|^2\right). \tag{6.6}$$

Now, the inequality (6.4) is a simple consequence of (6.1) for $x=y$, or of Lemma 17 and (6.3).

Since (see for instance [**2**]),

$$\|x\|^2 - |\langle x,\, e\rangle|^2 = \operatorname{Re}\left((\Phi-\langle x,\, e\rangle)\cdot(\langle e,\, x\rangle - \bar{\varphi})\right) - \operatorname{Re}\langle \Phi e - x,\, x-\varphi e\rangle\,, \tag{6.7}$$

then making use of the elementary inequality $4\operatorname{Re}\left(a\bar{b}\right) \leq |a+b|^2$ with $a,b\in\mathbb{K}$ $(\mathbb{K}=\mathbb{R},\mathbb{C})$, we can state that

$$\operatorname{Re}\left((\Phi-\langle x,\, e\rangle)\cdot(\langle e,\, x\rangle-\bar{\varphi})\right) \leq \frac{1}{4}\,|\Phi-\varphi|^2\,. \tag{6.8}$$

Using (6.7) and (6.8) we have

$$\|x\|^2 - |\langle x,\, e\rangle|^2 \leq \left(\frac{1}{2}\,|\Phi-\varphi|\right)^2 - \left((\operatorname{Re}\langle \Phi e - x,\, x-\varphi e\rangle)^{\frac{1}{2}}\right)^2. \tag{6.9}$$

Taking into account the inequalities (6.6) and (6.9), we get that

$$\begin{aligned}
&|\langle x-\langle x,\, e\rangle\, e,\, y-\langle y,\, e\rangle\, e\rangle|^2 \\
&\leq \left(\left(\frac{1}{2}\,|\Phi-\varphi|\right)^2 - \left((\operatorname{Re}\langle \Phi e - x, x-\varphi e\rangle)^{\frac{1}{2}}\right)^2\right)\cdot\left(\|y\|^2 - |\langle y,e\rangle|^2\right).
\end{aligned}$$

Finally, using the elementary inequality for positive real numbers:

$$\left(m^2-n^2\right)\cdot\left(p^2-q^2\right) \leq (mp-nq)^2\,, \tag{6.10}$$

we have:

$$\begin{aligned}
&\left(\left(\frac{1}{2}\,|\Phi-\varphi|\right)^2 - \left((\operatorname{Re}\langle \Phi e - x,\, x-\varphi e\rangle)^{\frac{1}{2}}\right)^2\right)\cdot\left(\|y\|^2 - |\langle y,e\rangle|^2\right) \\
&\qquad \leq \left(\frac{1}{2}\,|\Phi-\varphi|\cdot\|y\| - (\operatorname{Re}\langle \Phi e - x,\, x-\varphi e\rangle)^{\frac{1}{2}}\cdot|\langle y,\, e\rangle|\right)^2,
\end{aligned}$$

giving the desired inequality (6.5). ∎

A similar version for Bessel's inequality is incorporated in the following theorem [**13**]:

THEOREM 104. *Let $\{e_i\}_{i\in I}$, be a family of orthonormal vectors in H, F a finite part of I, φ_i, $\Phi_i \in \mathbb{K}$, $i \in F$ and x, y vectors in H such that either the condition*

$$\operatorname{Re}\left\langle \sum_{i\in F} \Phi_i e_i - x,\, x - \sum_{i\in F} \varphi_i e_i \right\rangle \geq 0,$$

or, equivalently,

$$\left\| x - \sum_{i\in F} \frac{\Phi_i + \varphi_i}{2} e_i \right\| \leq \frac{1}{2}\left(\sum_{i\in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}}$$

holds. Then we have inequalities

$$\left| \langle x, y\rangle - \sum_{i\in F} \langle x, e_i\rangle \, \langle e_i, y\rangle \right| \leq \frac{1}{2}\left(\sum_{i\in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}} \sqrt{\|y\|^2 - \sum_{i\in F} |\langle y,\, e_i\rangle|^2} \tag{6.11}$$

and

$$\left| \langle x, y\rangle - \sum_{i\in F} \langle x, e_i\rangle \, \langle e_i, y\rangle \right| \leq \frac{1}{2}\left(\sum_{i\in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}} \cdot \|y\| - \left(\sum_{i\in F} \left| \frac{\Phi_i + \varphi_i}{2} - \langle x,\, e_i\rangle \right|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i\in F} |\langle y,\, e_i\rangle|^2 \right)^{\frac{1}{2}}. \tag{6.12}$$

PROOF. It is obvious (see for example [**5**]) that:

$$\langle x,\, y\rangle - \sum_{i\in F} \langle x,\, e_i\rangle \langle e_i,\, y\rangle = \left\langle x - \sum_{i\in F} \langle x,\, e_i\rangle e_i,\, y - \sum_{i\in F} \langle y,\, e_i\rangle e_i \right\rangle.$$

Using Schwarz's inequality in inner product spaces, we have:

$$\left| \left\langle x - \sum_{i\in F} \langle x,\, e_i\rangle e_i,\, y - \sum_{i\in F} \langle y,\, e_i\rangle e_i \right\rangle \right|^2 \tag{6.13}$$

$$\le \left\| x-\sum_{i\in F}\langle x,\, e_i\rangle\, e_i\right\|^2\cdot\left\| x-\sum_{i\in F}\langle y,\, e_i\rangle\, e_i\right\|^2$$
$$=\left(\|x\|^2-\sum_{i\in F}|\langle x,\, e_i\rangle|^2\right)\cdot\left(\|y\|^2-\sum_{i\in F}|\langle y,\, e_i\rangle|^2\right).$$

In a similar manner to the one in the proof of Theorem 103 we may conclude that (6.11) holds true.

Now, using (6.2) and (6.13) we also have:

$$\left|\left\langle x-\sum_{i\in F}\langle x,\, e_i\rangle\, e_i,\, y-\sum_{i\in F}\langle y,\, e_i\rangle\, e_i\right\rangle\right|^2$$
$$\le\left(\frac{1}{2}\left(\left(\sum_{i\in F}|\Phi_i-\varphi_i|^2\right)^{\frac{1}{2}}\right)^2-\left(\left(\sum_{i\in F}\left|\frac{\varphi_i+\Phi_i}{2}-\langle x,\, e_i\rangle\right|^2\right)^{\frac{1}{2}}\right)^2\right)$$
$$\times\left(\|y\|^2-\sum_{i\in F}|\langle y,\, e_i\rangle|^2\right).$$

Finally, utilizing the elementary inequality (6.10), we have

$$\left(\frac{1}{2}\left(\left(\sum_{i\in F}|\Phi_i-\varphi_i|^2\right)^{\frac{1}{2}}\right)^2-\left(\left(\sum_{i\in F}\frac{\varphi_i+\Phi_i}{2}-\langle x,\, e_i\rangle^2\right)^{\frac{1}{2}}\right)^2\right) \tag{6.14}$$
$$\times\left(\|y\|^2-\sum_{i\in F}|\langle y,\, e_i\rangle|^2\right)\le\left(\frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\varphi_i|^2\right)^{\frac{1}{2}}\cdot\|y\|^2\right.$$
$$\left.-\left(\sum_{i\in F}\left|\frac{\varphi_i+\Phi_i}{2}-\langle x,\, e_i\rangle\right|^2\right)^{\frac{1}{2}}\cdot\sum_{i\in F}|\langle y,\, e_i\rangle|^2\right)^2,$$

which gives the desired result (6.12). ∎

6.3. Applications for Integrals. Let $(\Omega,\,\Sigma,\,\mu)$ be a measure space consisting of a set Ω, Σ a $\sigma-$algebra of parts and μ a countably additive and positive measure on Σ with values in $\mathbb{R}\cup\{\infty\}$. Denote by $L^2(\Omega,\mathbb{K})$ the Hilbert space of all real or complex valued functions f defined on Ω and 2–integrable on Ω, i. e.

$$\int_\Omega|f(s)|^2\, d\mu(s)<\infty.$$

The following proposition holds [**13**].

Proposition 33. *If* $f, g, h \in L^2(\Omega,\mathbb{K})$ *and* $\varphi,\ \Phi \in \mathbb{K}$ *are such that* $\int_\Omega |h(s)|^2 d\mu(s) = 1$ *and, either*

$$\int_\Omega \operatorname{Re}\left((\Phi h(s) - f(s)) \left(\bar{f}(s) - \overline{\varphi} \bar{h}(s) \right) \right) d\mu(s) \geq 0, \tag{6.15}$$

or, equivalently,

$$\left(\int_\Omega \left| f(s) - \frac{\Phi + \varphi}{2} h(s) \right|^2 d\mu(s) \right)^{\frac{1}{2}} \leq \frac{1}{2} |\Phi - \varphi|$$

holds, then we have the inequalities

$$\begin{aligned} &\left| \int_\Omega f(s)\,\bar{g}(s)\,d\mu(s) - \int_\Omega f(s)\,\bar{h}(s)\,d\mu(s) \int_\Omega h(s)\,\bar{g}(s)\,d\mu(s) \right| \\ &\qquad \leq \frac{1}{2} |\Phi - \varphi| \cdot \sqrt{\left(\int_\Omega |g(s)|^2 d\mu(s) - \left| \int_\Omega h(s)\,\bar{g}(s)\,d\mu(s) \right|^2 \right)} \end{aligned}$$

and

$$\begin{aligned} &\left| \int_\Omega f(s)\,\bar{g}(s)\,d\mu(s) - \int_\Omega f(s)\,\bar{h}(s)\,d\mu(s) \int_\Omega h(s)\,\bar{g}(s)\,d\mu(s) \right| \\ &\quad \leq \frac{1}{2} |\Phi - \varphi| \cdot \left(\int_\Omega |g(s)|^2 d\mu(s) \right)^{\frac{1}{2}} \\ &\qquad - \left(\int_\Omega \operatorname{Re}\left((\Phi h(s) - f(s)) \left(h(s)\,\bar{f}(s) - \varphi h(s) \right) \right) d\mu(s) \right)^{\frac{1}{2}} \\ &\qquad\qquad \times \left| \int_\Omega h(s)\,\bar{g}(s)\,d\mu(s) \right|. \end{aligned}$$

Proof. The proof follows by Theorem 103 on choosing $H = L^2(\Omega,\ K)$ with the inner product

$$\langle f, g \rangle = \int_\Omega f(s)\,\bar{g}(s)\,d\mu(s).$$

∎

Remark 77. *We observe that a sufficient condition for (6.15) to hold is:*

$$\operatorname{Re}(\Phi h(s) - f(s)) \left(\bar{f}(s) - \overline{\varphi} \bar{h}(s) \right) \geq 0, \tag{6.16}$$

for $\mu-$*a.e.* $s \in \Omega$.

If the functions are real-valued, then, for Φ and φ real numbers, a sufficient condition for (6.16) to hold is

$$\Phi h(s) \geq f(s) \geq \varphi h(s)$$

for $\mu-$a.e. $s \in \Omega$.

In this way we can see the close connection that exists between the classical Grüss inequality and the results obtained above.

Now, consider the family $\{f_i\}_{i\in I}$ of functions in $L^2(\Omega, \mathbb{K})$ with the properties that

$$\int_\Omega f_i(s)\overline{f_j}(s)\,d\mu(s) = \delta_{ij}, \quad i, j \in I,$$

where δ_{ij} is 0 if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$. $\{f_i\}_{i\in I}$ is an orthornormal family in $L^2(\Omega, \mathbb{K})$.

The following proposition holds [**13**].

Proposition 34. *Let $\{f_i\}_{i\in I}$ be an orthornormal family of functions in $L^2(\Omega, \mathbb{K})$, F a finite subset of I, $\phi_i, \Phi_i \in \mathbb{K}$ $(i \in F)$ and $f \in L^2(\Omega, \mathbb{K})$, such that either*

$$\int_\Omega \operatorname{Re}\left[\left(\sum_{i\in F}\Phi_i f_i(s) - f(s)\right)\left(\overline{f}(s) - \sum_{i\in F}\overline{\phi_i}\;\overline{f_i}(s)\right)\right] d\mu(s) \geq 0 \tag{6.17}$$

or, equivalently,

$$\int_\Omega \left| f(s) - \sum_{i\in F}\frac{\Phi_i + \phi_i}{2} f_i(s)\right|^2 d\mu(s) \leq \frac{1}{4}\sum_{i\in F}|\Phi_i - \phi_i|^2.$$

holds. Then we have the inequalities

$$\begin{aligned}
&\left|\int_\Omega f(s)\overline{g(s)}d\mu(s) - \sum_{i\in F}\int_\Omega f(s)\overline{f_i}(s)\,d\mu(s)\int_\Omega f_i(s)\overline{g(s)}d\mu(s)\right| \\
&\qquad\qquad \leq \frac{1}{2}\left(\sum_{i\in F}|\Phi_i - \phi_i|^2\right)^{\frac{1}{2}} \\
&\qquad\qquad \times \left(\int_\Omega |g(s)|^2\,d\mu(s) - \sum_{i\in F}\left|\int_\Omega g(s)\,\overline{f_i(s)}d\mu(s)\right|^2\right)^{\frac{1}{2}}
\end{aligned}$$

and

$$\left|\int_{\Omega} f(s)\overline{g(s)}d\mu(s) - \sum_{i\in F}\int_{\Omega} f(s)\overline{f_i}(s)\,d\mu(s)\int_{\Omega} f_i(s)\overline{g(s)}d\mu(s)\right|$$
$$\leq \frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}\left(\int_{\Omega}|g(s)|^2\,d\mu(s)\right)^{\frac{1}{2}}$$
$$-\left(\sum_{i\in F}\left|\frac{\Phi_i+\phi_i}{2}-\int_{\Omega} f(s)\overline{f_i}(s)\,d\mu(s)\right|^2\right)^{\frac{1}{2}}$$
$$\times\left(\sum_{i\in F}\left|\int_{\Omega} f(s)\overline{f_i}(s)\,d\mu(s)\right|^2\right)^{\frac{1}{2}}.$$

The proof is obvious by Theorem 103 and we omit the details.

REMARK 78. *In the real case, we observe that a sufficient condition for (6.17) to hold is*

$$\sum_{i\in F}\Phi_i f_i(s) \geq f(s) \geq \sum_{i\in F}\varphi_i f_i(s)$$

for $\mu-$a.e. $s\in\Omega$.

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