Advances in Inequalities of the Schwarz, Triangle and Heisenberg Type in Inner Product Spaces

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Abstract. The purpose of this book is to give a comprehensive introduction to several inequalities in Inner Product Spaces that have important applications in various topics of Contemporary Mathematics such as: Linear Operators Theory, Partial Differential Equations, Nonlinear Analysis, Approximation Theory, Optimization Theory, Numerical Analysis, Probability Theory, Statistics and other fields.

Contents

Preface

The purpose of this book, that can be seen as a continuation of the previous one entitled "*Advances on Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*" (Nova Science Publishers, NY, 2005), is to give a comprehensive introduction to other classes of inequalities in Inner Product Spaces that have important applications in various topics of Contemporary Mathematics such as: Linear Operators Theory, Partial Differential Equations, Nonlinear Analysis, Approximation Theory, Optimization Theory, Numerical Analysis, Probability Theory, Statistics and other fields.

The monograph is intended for use by both researchers in various fields of Mathematical Inequalities, domains which have grown exponentially in the last decade, as well as by postgraduate students and scientists applying inequalities in their specific areas.

The aim of Chapter 1 is to present some fundamental analytic properties concerning Hermitian forms defined on real or complex linear spaces. The basic inequalities as well as various properties of superadditivity and monotonicity for the diverse functionals that can be naturally associated with the quantities involved in the Schwarz inequality are given. Applications for orthonormal families, Gram determinants, linear operators defined on Hilbert spaces and sequences of vectors are also pointed out.

In Chapter 2, classical and recent refinements and reverse inequalities for the Schwarz and the triangle inequalities are presented. Further on, the inequalities obtained by Buzano, Richards, Precupanu and Moore and their extensions and generalizations for orthonormal families of vectors in both real and complex inner product spaces are outlined. Recent results concerning the classical refinement of Schwarz inequality due to Kurepa for the complexification of real inner product spaces are also reviewed. Various applications for integral inequalities including a version of Heisenberg inequality for vector valued functions in Hilbert spaces are provided as well.

The aim of Chapter 3 is to survey various recent reverses for the generalised triangle inequality in both its simple form, that are closely related to the Diaz-Metcalf results, or in the equivalent quadratic form that maybe be of interest in the Geometry of Inner product Spaces. Applications for vector valued integral inequalities and for complex numbers are given as well.

Further on, in Chapter 4, some recent reverses of the continuous triangle inequality for Bochner integrable functions with values in Hilbert spaces and defined on a compact interval $[a, b] \subset \mathbb{R}$ are surveyed. Applications for Lebesgue integrable complex-valued functions that generalise and extend the classical result of Karamata are provided as well.

In Chapter 5 some reverses of the Cauchy-Buniakovsky-Schwarz vector-valued integral inequalities under various assumptions of boundedness for the functions involved are given. Natural applications for the Heisenberg inequality for vector-valued functions in Hilbert spaces are also provided.

The last chapter, Chapter 6, is a potpourri of other inequalities in inner product spaces. The aim of the first section is to point out some upper bounds for the distance $d(x, M)$ from a vector x to a finite dimensional subspace M in terms of the linearly independent vectors $\{x_1, \dots, x_n\}$ that span M. As a by-product of this endeavour, some refinements of the generalisations for Bessel's inequality due to several authors including: Boas, Bellman and Bombieri are obtained. Refinements for the well known Hadamard's inequality for Gram determinants are also derived.

In the second and third sections of this last chapter, several reverses for the Cauchy-Bunyakovsky-Schwarz (CBS) inequality for sequences of vectors in Hilbert spaces are obtained. Applications for bounding the distance to a finite-dimensional subspace and in reversing the generalised triangle inequality are also given.

For the sake of completeness, all the results presented are completely proved and the original references where they have been firstly obtained are mentioned. The chapters are relatively independent and can be read separately.

The Author,
March, 2005.

CHAPTER 1

Inequalities for Hermitian Forms

1.1. Introduction

Let $\mathbb{K}$ be the field of real or complex numbers, i.e., $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and X be a linear space over $\mathbb{K}$.

Definition 1. *A functional* $(\cdot, \cdot) : X \times X \to \mathbb{K}$ *is said to be a Hermitian form on* X *if*

(H1) $(ax + by, z) = a(x, z) + b(y, z)$ *for* $a, b \in \mathbb{K}$ *and* $x, y, z \in X$;
(H2) $(x, y) = \overline{(y, x)}$ *for all* $x, y \in X$.

The functional $(\cdot, \cdot)$ is said to be *positive semi-definite* on a subspace Y of X if

(H3) $(y, y) \geq 0$ for every $y \in Y$,

and *positive definite* on Y if it is positive semi-definite on Y and

(H4) $(y, y) = 0$, $y \in Y$ implies $y = 0$.

The functional $(\cdot, \cdot)$ is said to be *definite* on Y provided that either $(\cdot, \cdot)$ or $-(\cdot, \cdot)$ is positive semi-definite on Y.

When a Hermitian functional $(\cdot, \cdot)$ is positive-definite on the whole space X, then, as usual, we will call it an *inner product* on X and will denote it by $\langle \cdot, \cdot \rangle$.

The aim of this chapter is to present some fundamental analytic properties concerning Hermitian forms defined on real or complex linear spaces. The basic inequalities as well as various properties of superadditivity and monotonicity for diverse functionals that can be naturally associated with the quantities involved in the Schwarz inequality are given. Applications for orthonormal families, Gram determinants, linear operators defined on Hilbert spaces and sequences of vectors are also pointed out. The results are completely proved and the original references where they have been firstly obtained are mentioned.

1.2. Hermitian Forms, Fundamental Properties

1.2.1. Schwarz's Inequality. We use the following notations related to a given Hermitian form $(\cdot,\cdot)$ on X :

$$X_0 := \{x \in X | \, (x,x) = 0\}\,,$$
$$K := \{x \in X | \, (x,x) < 0\}$$

and, for a given $z \in X$,

$$X^{(z)} := \{x \in X | \, (x,z) = 0\} \quad \text{and} \quad L(z) := \{az | a \in \mathbb{K}\}\,.$$

The following fundamental facts concerning Hermitian forms hold [**5**]:

THEOREM 1 (Kurepa, 1968). *Let X and $(\cdot,\cdot)$ be as above.*

(1) *If $e \in X$ is such that $(e,e) \neq 0$, then we have the decomposition*

$$X = L(e) \bigoplus X^{(e)}, \tag{1.1}$$

where $\bigoplus$ denotes the direct sum of the linear subspaces $X^{(e)}$ and $L(e)$;

(2) *If the functional $(\cdot,\cdot)$ is positive semi-definite on $X^{(e)}$ for at least one $e \in K$, then $(\cdot,\cdot)$ is positive semi-definite on $X^{(f)}$ for each $f \in K$;*

(3) *The functional $(\cdot,\cdot)$ is positive semi-definite on $X^{(e)}$ with $e \in K$ if and only if the inequality*

$$|(x,y)|^2 \geq (x,x)(y,y) \tag{1.2}$$

holds for all $x \in K$ and all $y \in X$;

(4) *The functional $(\cdot,\cdot)$ is semi-definite on X if and only if the Schwarz's inequality*

$$|(x,y)|^2 \leq (x,x)(y,y) \tag{1.3}$$

holds for all $x, y \in X$;

(5) *The case of equality holds in (1.3) for $x, y \in X$ and in (1.2), for $x \in K$, $y \in X$, respectively; if and only if there exists a scalar $a \in \mathbb{K}$ such that*

$$y - ax \in X_0^{(x)} := X_0 \cap X^{(x)}.$$

PROOF. We follow the argument in [**5**].

If $(e,e) \neq 0$, then the element

$$x := y - \frac{(y,e)}{(e,e)} e$$

has the property that $(x, e) = 0$, i.e., $x \in X^{(e)}$. This proves that X is a sum of the subspaces $L(e)$ and $X^{(s)}$. The fact that the sum is direct is obvious.

Suppose that $(e, e) \neq 0$ and that $(\cdot, \cdot)$ is positive semi-definite on X. Then for each $y \in X$ we have $y = ae + z$ with $a \in \mathbb{K}$ and $z \in X^{(e)}$, from where we get

$$|(e, y)|^2 - (e, e)(y, y) = -(e, e)(z, z). \tag{1.4}$$

From (1.4) we get the inequality (1.3), with $x = e$, in the case that $(e, e) > 0$ and (1.2) in the case that $(e, e) < 0$. In addition to this, from (1.4) we observe that the case of equality holds in (1.2) or in (1.3) if and only if $(z, z) = 0$, i.e., if and only if $y - ae \in X_0^{(e)}$.

Conversely, if (1.3) holds for all $x, y \in X$, then (x, x) has the same sign over the whole of X, i.e., $(\cdot, \cdot)$ is semi-definite on X. In the same manner, from (1.2), for $y \in X^{(e)}$, we get $(e, e) \cdot (y, y) \leq 0$, which implies $(y, y) \geq 0$, i.e., $(\cdot, \cdot)$ is positive semi-definite on $X^{(e)}$.

Now, suppose that $(\cdot, \cdot)$ is positive semi-definite on $X^{(e)}$ for at least one $e \in K$. Let us prove that $(\cdot, \cdot)$ is positive semi-definite on $X^{(f)}$ for each $f \in K$.

For a given $f \in K$, consider the vector

$$e' := e - \frac{(e, f)}{(f, f)} f. \tag{1.5}$$

Now,

$$(e', e') = (e', e) = \frac{(e, e)(f, f) - |(e, f)|^2}{(f, f)}, \quad (e', f) = 0$$

and together with

$$|(e, y)|^2 \geq (e, e)(y, y) \quad \text{for any} \quad y \in X$$

imply $(e', e') \geq 0$.

There are two cases to be considered: $(e', e') > 0$ and $(e', e') = 0$.

If $(e', e') > 0$, then for any $x \in X^{(f)}$, the vector

$$x' := x - ae' \quad \text{with} \quad a = \frac{(x, e')}{(e', e')}$$

satisfies the conditions

$$(x', e) = 0 \quad \text{and} \quad (x', f) = 0$$

which implies

$$x' \in X^{(e)} \quad \text{and} \quad (x, x) = |a|^2 (e', e') + (x', x') \geq 0.$$

Therefore $(\cdot, \cdot)$ is a positive semi-definite functional on $X^{(f)}$.

From the parallelogram identity:

$$(x+y,x+y)+(x-y,x-y)=2\left[(x,x)+(y,y)\right],\quad x,y\in X \tag{1.6}$$

we conclude that the set $X_0^{(e)}=X_0\cap X^{(e)}$ is a linear subspace of X.

Since

$$(x,y)=\frac{1}{4}\left[(x+y,x+y)+(x-y,x-y)\right],\quad x,y\in X \tag{1.7}$$

in the case of real spaces, and

$$\begin{aligned}(x,y)=\frac{1}{4}&\left[(x+y,x+y)+(x-y,x-y)\right]\\ &+\frac{i}{4}\left[(x+iy,x+iy)-(x-iy,x-iy)\right],\quad x,y\in X\end{aligned} \tag{1.8}$$

in the case of complex spaces, hence $(x,y)=0$ provided that x and y belong to $X_0^{(e)}$.

If $(e',e')=0$, then $(e',e)=(e',e')=0$ and then we can conclude that $e'\in X_0^{(e)}$. Also, since $(e',e')=0$ implies $(e,f)\neq 0$, hence we have

$$f=b\left(e-e'\right)\quad\text{with}\quad b=\frac{(f,f)}{(e,f)}.$$

Now write

$$X^{(e)}=X_0^{(e)}\bigoplus X_+^{(e)},$$

where $X_+^{(e)}$ is any direct complement of $X_0^{(e)}$ in the space $X^{(e)}$. If $y\neq 0$, then $y\in X_+^{(e)}$ implies $(y,y)>0$. For such a vector y, the vector

$$y':=e'-\frac{(e',y)}{(y,y)}\cdot y.$$

is in $X^{(e)}$ and therefore $(y',y')\geq 0$.

On the other hand

$$(y',y')=(e',y')=-\frac{\left|(e',y)\right|^2}{(y,y)}.$$

Hence $y\in X_+^{(e)}$ implies that $(e',y)=0$, i.e.,

$$(e,y)=\frac{(e,f)}{(f,f)}\left(f,y\right),$$

which together with $y\in X^{(e)}$ leads to $(f,y)=0$. Thus $y\in X_+^{(e)}$ implies $y\in X^{(f)}$.

On the other hand $x\in X_0^{(e)}$ and $f=b\left(e-e'\right)$ imply $(f,x)=-b\left(e',x\right)=0$ due to the fact that $e',x\in X_0^{(e)}$.

Hence $x\in X_0^{(e)}$ implies $(x,f)=0$, i.e., $x\in X^{(f)}$.

From $X_0^{(e)} \subseteq X^{(f)}$ and $X_+^{(e)} \subseteq X^{(f)}$ we get $X^{(e)} \subseteq X^{(f)}$. Since $e \notin X^{(f)}$ and $X = L(e) \bigoplus X^{(e)}$, we deduce $X^{(e)} = X^{(f)}$ and then $(\cdot,\cdot)$ is positive semi-definite on $X^{(f)}$.

The theorem is completely proved. ∎

In the case of complex linear spaces we may state the following result as well [**5**]:

Theorem 2 (Kurepa, 1968). *Let X be a complex linear space and $(\cdot,\cdot)$ a hermitian functional on X.*

(1) *The functional $(\cdot,\cdot)$ is semi-definite on X if and only if there exists at least one vector $e \in X$ with $(e,e) \neq 0$ such that*

$$[\operatorname{Re}(e,y)]^2 \leq (e,e)\,(y,y)\,, \tag{1.9}$$

for all $y \in X$;

(2) *There is no nonzero Hermitian functional $(\cdot,\cdot)$ such that the inequality*

$$[\operatorname{Re}(e,y)]^2 \geq (e,e)\,(y,y)\,, \quad (e,e) \neq 0, \tag{1.10}$$

holds for all $y \in X$ and for an $e \in X$.

Proof. We follow the proof in [**5**].

Let σ and τ be real numbers and $x \in X^{(e)}$ a given vector. For $y := (\sigma + i\tau)\,e + x$ we get

$$[\operatorname{Re}(e,y)]^2 - (e,e)\,(y,y) = -\tau^2\,(e,e)^2 - (e,e)\,(x,x)\,. \tag{1.11}$$

If $(\cdot,\cdot)$ is semi-definite on X, then (1.11) implies (1.9).

Conversely, if (1.9) holds for all $y \in X$ and for at least one $e \in X$, then $(\cdot,\cdot)$ is semi-definite on $X^{(e)}$. But (1.9) and (1.11) for $\tau = 0$ lead to $-(e,e)\,(x,x) \leq 0$ from which it follows that (e,e) and (x,x) are of the same sign so that $(\cdot,\cdot)$ is semi-definite on X.

Suppose that $(\cdot,\cdot) \neq 0$ and that (1.10) holds. We can assume that $(e,e) < 0$. Then (1.10) implies that $(\cdot,\cdot)$ is positive semi-definite on $X^{(e)}$. On the other hand, if τ is such that

$$\tau^2 > -\frac{(x,x)}{(e,e)},$$

then (1.11) leads to $[\operatorname{Re}(e,y)]^2 < (e,e)\,(y,y)$, contradicting (1.10).

Hence, if a Hermitian functional $(\cdot,\cdot)$ is not semi-definite and if $-(e,e) \neq 0$, then the function $y \longmapsto [\operatorname{Re}(e,y)]^2 - (e,e)\,(y,y)$ takes both positive and negative values.

The theorem is completely proved. ∎

1.2.2. Schwarz's Inequality for the Complexification of a Real Space. Let X be a real linear space. The *complexification* $X_{\mathbb{C}}$ of X is defined as a complex linear space $X \times X$ of all ordered pairs $\{x, y\}$ $(x, y \in X)$ endowed with the operations:

$$\{x, y\} + \{x', y'\} := \{x + x', y + y'\},$$
$$(\sigma + i\tau) \cdot \{x, y\} := \{\sigma x - \tau y, \sigma x + \tau y\},$$

where $x, y, x', y' \in X$ and $\sigma, \tau \in \mathbb{R}$ (see for instance [**6**]).

If $z = \{x, y\}$, then we can define the conjugate vector $\bar{z}$ of z by $\bar{z} := \{x, -y\}$. Similarly, with the scalar case, we denote

$$\operatorname{Re} z = \{x, 0\} \quad \text{and} \quad \operatorname{Im} z := \{0, y\}.$$

Formally, we can write $z = x + iy = \operatorname{Re} z + i \operatorname{Im} z$ and $\bar{z} = x - iy = \operatorname{Re} z - i \operatorname{Im} z$.

Now, let $(\cdot, \cdot)$ be a Hermitian functional on X. We may define on the complexification $X_{\mathbb{C}}$ of X, the *complexification* of $(\cdot, \cdot)$, denoted by $(\cdot, \cdot)_{\mathbb{C}}$ and defined by:

$$(x + iy, x' + iy')_{\mathbb{C}} := (x, x') + (y, y') + i\left[(y, x') - (x, y')\right],$$

for $x, y, x', y' \in X$.

The following result may be stated [**5**]:

Theorem 3 (Kurepa, 1968). *Let X, $X_{\mathbb{C}}$, $(\cdot, \cdot)$ and $(\cdot, \cdot)_{\mathbb{C}}$ be as above. An inequality of type (1.2) and (1.3) holds for the functional $(\cdot, \cdot)_{\mathbb{C}}$ in the space $X_{\mathbb{C}}$ if and only if the same type of inequality holds for the functional $(\cdot, \cdot)$ in the space X.*

Proof. We follow the proof in [**5**].

Firstly, observe that $(\cdot, \cdot)$ is semi-definite if and only if $(\cdot, \cdot)_{\mathbb{C}}$ is semi-definite.

Now, suppose that $e \in X$ is such that

$$|(e, y)|^2 \geq (e, e)\,(y, y), \quad (e, e) < 0$$

for all $y \in X$. Then for $x, y \in X$ we have

$$\begin{aligned} |(e, x + iy)_{\mathbb{C}}|^2 &= [(e, x)]^2 + [(e, y)]^2 \\ &\geq (e, e)\left[(x, x) + (y, y)\right] \\ &= (e, e)\,(x + iy, x + iy)_{\mathbb{C}}. \end{aligned}$$

Hence, if for the functional $(\cdot, \cdot)$ on X an inequality of type (1.2) holds, then the same type of inequality holds in $X_{\mathbb{C}}$ for the corresponding functional $(\cdot, \cdot)_{\mathbb{C}}$.

Conversely, suppose that $e, f \in X$ are such that

$$|(e + if, x + iy)_{\mathbb{C}}|^2 \geq (e + if, e + if)_{\mathbb{C}}\,(x + iy, x + iy)_{\mathbb{C}} \tag{1.12}$$

holds for all $x, y \in X$ and that

$$(e + if, e + if)_{\mathbb{C}} = (e, e) + (f, f) < 0. \tag{1.13}$$

If $e = af$ with a real number a, then (1.13) implies that $(f, f) < 0$ and (1.12) for $y = 0$ leads to

$$[(f, x)]^2 \geq (f, f)\,(x, x)\,,$$

for all $x \in X$. Hence, in this case, we have an inequality of type (1.2) for the functional $(\cdot, \cdot)$ in X.

Suppose that e and g are linearly independent and by $Y = L\,(e, f)$ let us denote the subspace of X consisting of all linear combinations of e and f. On Y we define a hermitian functional D by setting $D\,(x, y) = (x, y)$ for $x, y \in Y$. Let $D_{\mathbb{C}}$ be the complexification of D. Then (1.12) implies:

$$\begin{aligned} &|D_{\mathbb{C}}\,(e + if, x + iy)|^2 \\ &\qquad \geq D_{\mathbb{C}}\,(e + if, e + if)\,D_{\mathbb{C}}\,(x + iy, x + iy)\,, \quad x, y \in X \end{aligned} \tag{1.14}$$

and (1.13) implies

$$D\,(e, e) + D\,(f, f) < 0. \tag{1.15}$$

Further, consider in Y a base consisting of the two vectors $\{u_1, u_2\}$ on which D is diagonal, i.e., D satisfies

$$D\,(x, y) = \lambda_1 x_1 y_1 + \lambda_2 x_2 y_2,$$

where

$$x = x_1 u_1 + x_2 u_2, \quad y = y_1 u_1 + y_2 u_2,$$

and

$$\lambda_1 = D\,(u_1, u_1)\,, \quad \lambda_2 = D\,(u_2, u_2)\,.$$

Since for the functional D we have the relations (1.15) and (1.14), we conclude that D is not a semi-definite functional on Y. Hence $\lambda_1 \cdot \lambda_2 < 0$, so we can take $\lambda_1 < 0$ and $\lambda_2 > 0$.

Set

$$X^+ := \{x|\,(x, e) = (x, f) = 0,\ x \in X\}\,.$$

Obviously, $(x, e) = (x, f) = 0$ if and only if $(x_1 u_1) = (x_2 u_2) = 0$.

Now, if $y \in X$, then the vector

$$x := y - \frac{(y, u_1)}{(u_1, u_1)} u_1 - \frac{(y, u_2)}{(u_2, u_2)} u_2 \tag{1.16}$$

belongs to X^+. From this it follows that

$$X = L\,(e, f) \bigoplus X^+.$$

Now, replacing in (1.12) the vector $x+iy$ with $z\in X^{+}$, we get from (1.13) that

$$[(e,e)+(f,f)]\,(z,z)\leq 0,$$

which, together with (1.13) leads to $(z,z)\geq 0$. Therefore the functional $(\cdot,\cdot)$ is positive semi-definite on X^{+}.

Now, since any $y\in X$ is of the form (1.16), hence for $y\in X^{(u_1)}$ we get

$$(y,y)=(x,x)+\frac{[(y,u_2)]^2}{\lambda_2},$$

which is a nonnegative number. Thus, $(\cdot,\cdot)$ is positive semi-definite on the space $X^{(u_1)}$. Since $(u_1,u_1)<0$ we have $[(u_1,y)]^2\geq (u_1,u_1)\,(y,y)$ for any $y\in X$ and the theorem is completely proved. ∎

1.3. Superadditivity and Monotonicity

1.3.1. The Convex Cone of Nonnegative Hermitian Forms. Let X be a linear space over the real or complex number field $\mathbb{K}$ and let us denote by $\mathcal{H}(X)$ the class of all positive semi-definite Hermitian forms on X, or, for simplicity, *nonnegative* forms on X, i.e., the mapping $(\cdot,\cdot):X\times X\to\mathbb{K}$ belongs to $\mathcal{H}(X)$ if it satisfies the conditions

(i) $(x,x)\geq 0$ for all x in X;
(ii) $(\alpha x+\beta y,z)=\alpha\,(x,z)+\beta\,(y,z)$ for all $x,y\in X$ and $\alpha,\beta\in\mathbb{K}$
(iii) $(y,x)=\overline{(x,y)}$ for all $x,y\in X$.

If $(\cdot,\cdot)\in\mathcal{H}(X)$, then the functional $\|\cdot\|=(\cdot,\cdot)^{\frac{1}{2}}$ is a *semi-norm* on X and the following equivalent versions of Schwarz's inequality hold:

$$\|x\|^2\,\|y\|^2\geq |(x,y)|^2 \quad\text{or}\quad \|x\|\,\|y\|\geq |(x,y)| \tag{1.17}$$

for any $x,y\in X$.

Now, let us observe that $\mathcal{H}(X)$ is a *convex cone* in the linear space of all mappings defined on X^2 with values in $\mathbb{K}$, i.e.,

(e) $(\cdot,\cdot)_1,(\cdot,\cdot)_2\in\mathcal{H}(X)$ implies that $(\cdot,\cdot)_1+(\cdot,\cdot)_2\in\mathcal{H}(X)$;
(ee) $\alpha\geq 0$ and $(\cdot,\cdot)\in\mathcal{H}(X)$ implies that $\alpha\,(\cdot,\cdot)\in\mathcal{H}(X)$.

We can introduce on $\mathcal{H}(X)$ the following binary relation [**1**]:

$$(\cdot,\cdot)_2\geq(\cdot,\cdot)_1 \quad\text{if and only if}\quad \|x\|_2\geq\|x\|_1 \quad\text{for all}\ \ x\in X. \tag{1.18}$$

We observe that the following properties hold:

(b) $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$ for all $(\cdot,\cdot)\in\mathcal{H}(X)$;
(bb) $(\cdot,\cdot)_3\geq(\cdot,\cdot)_2$ and $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$ implies that $(\cdot,\cdot)_3\geq(\cdot,\cdot)_1$;
(bbb) $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$ and $(\cdot,\cdot)_1\geq(\cdot,\cdot)_2$ implies that $(\cdot,\cdot)_2=(\cdot,\cdot)_1$;

i.e., the binary relation defined by (1.18) is *an order relation* on $\mathcal{H}(X)$.

While (b) and (bb) are obvious from the definition, we should remark, for (bbb), that if $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1$ and $(\cdot,\cdot)_1 \geq (\cdot,\cdot)_2$, then obviously $\|x\|_2 = \|x\|_1$ for all $x \in X$, which implies, by the following well known identity:

$$(x,y)_k := \frac{1}{4}\left[\|x+y\|_k^2 - \|x-y\|_k^2 + i\left(\|x+iy\|_k^2 - \|x-iy\|_k^2\right)\right] \tag{1.19}$$

with $x, y \in X$ and $k \in \{1,2\}$, that $(x,y)_2 = (x,y)_1$ for all $x, y \in X$.

1.3.2. The Superadditivity and Monotonicity of $\sigma-$Mapping. Let us consider the following mapping [**1**]:

$$\sigma : \mathcal{H}(X) \times X^2 \to \mathbb{R}_+, \quad \sigma((\cdot,\cdot); x, y) := \|x\|\,\|y\| - |(x,y)|,$$

which is closely related to Schwarz's inequality (1.17).

The following simple properties of σ are obvious:

(s) $\sigma(\alpha(\cdot,\cdot); x, y) = \alpha\sigma((\cdot,\cdot); x, y)$;

(ss) $\sigma((\cdot,\cdot); y, x) = \sigma((\cdot,\cdot); x, y)$;

(sss) $\sigma((\cdot,\cdot); x, y) \geq 0$ (Schwarz's inequality);

for any $\alpha \geq 0$, $(\cdot,\cdot) \in \mathcal{H}(X)$ and $x, y \in X$.

The following result concerning the functional properties of σ as a function depending on the nonnegative hermitian form $(\cdot,\cdot)$ has been obtained in [**1**]:

Theorem 4 (Dragomir-Mond, 1994). *The mapping σ satisfies the following statements:*

(i) *For every $(\cdot,\cdot)_i \in \mathcal{H}(X)$ $(i = 1,2)$ one has the inequality*

$$\sigma((\cdot,\cdot)_1 + (\cdot,\cdot)_2; x, y) \geq \sigma((\cdot,\cdot)_1; x, y) + \sigma((\cdot,\cdot)_2; x, y) \qquad (\geq 0) \tag{1.20}$$

for all $x, y \in X$, i.e., the mapping $\sigma(\cdot; x, y)$ is superadditive on $\mathcal{H}(X)$;

(ii) *For every $(\cdot,\cdot)_i \in \mathcal{H}(X)$ $(i = 1,2)$ with $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1$ one has*

$$\sigma((\cdot,\cdot)_2; x, y) \geq \sigma((\cdot,\cdot)_1; x, y) \qquad (\geq 0) \tag{1.21}$$

for all $x, y \in X$, i.e., the mapping $\sigma(\cdot; x, y)$ is nondecreasing on $\mathcal{H}(X)$.

Proof. We follow the proof in [**1**].

(i) By the Cauchy-Bunyakovsky-Schwarz inequality for real numbers , we have

$$\left(a^2+b^2\right)^{\frac{1}{2}}\left(c^2+d^2\right)^{\frac{1}{2}} \geq ac+bd; \quad a,b,c,d \geq 0.$$

Therefore,

$$\begin{aligned}
&\sigma\left((\cdot,\cdot)_1+(\cdot,\cdot)_2\,;x,y\right)\\
&=\left(\|x\|_1^2+\|x\|_2^2\right)^{\frac{1}{2}}\left(\|y\|_1^2+\|y\|_2^2\right)^{\frac{1}{2}}-\left|(x,y)_1+(x,y)_2\right|\\
&\geq \|x\|_1\|y\|_1+\|x\|_2\|y\|_2-\left|(x,y)_1\right|-\left|(x,y)_2\right|\\
&=\sigma\left((\cdot,\cdot)_1\,;x,y\right)+\sigma\left((\cdot,\cdot)_2\,;x,y\right),
\end{aligned}$$

for all $(\cdot,\cdot)_i \in \mathcal{H}(X)$ $(i=1,2)$ and $x,y\in X$, and the statement is proved.

(ii) Suppose that $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1$ and define $(\cdot,\cdot)_{2,1} := (\cdot,\cdot)_2-(\cdot,\cdot)_1$. It is obvious that $(\cdot,\cdot)_{2,1}$ is a nonnegative hermitian form and thus, by the above property one has,

$$\begin{aligned}
\sigma\left((\cdot,\cdot)_2\,;x,y\right) &\geq \sigma\Big((\cdot,\cdot)_{2,1}+(\cdot,\cdot)_1\,;x,y\Big)\\
&\geq \sigma\Big((\cdot,\cdot)_{2,1}\,;x,y\Big)+\sigma\left((\cdot,\cdot)_1\,;x,y\right)
\end{aligned}$$

from where we get:

$$\sigma\left((\cdot,\cdot)_2\,;x,y\right)-\sigma\left((\cdot,\cdot)_1\,;x,y\right) \geq \sigma\Big((\cdot,\cdot)_{2,1}\,;x,y\Big) \geq 0$$

and the proof of the theorem is completed.

∎

Remark 1. *If we consider the related mapping* [**1**]

$$\sigma_r\left((\cdot,\cdot)\,;x,y\right) := \|x\|\,\|y\|-\operatorname{Re}(x,y),$$

then we can show, as above, that $\sigma(\cdot;x,y)$ *is* ***superadditive and nondecreasing*** *on* $\mathcal{H}(X)$.

Moreover, if we introduce another mapping, namely, [**1**]

$$\tau:\mathcal{H}(X)\times X^2\to\mathbb{R}_+, \quad \tau\left((\cdot,\cdot)\,;x,y\right) := \left(\|x\|+\|y\|\right)^2-\|x+y\|^2,$$

which is connected with the ***triangle inequality***

$$\|x+y\| \leq \|x\|+\|y\| \quad \text{for any} \quad x,y\in X \tag{1.22}$$

then we observe that

$$\tau\left((\cdot,\cdot)\,;x,y\right)=2\sigma_r\left((\cdot,\cdot)\,;x,y\right) \tag{1.23}$$

for all $(\cdot,\cdot)\in\mathcal{H}(X)$ *and* $x,y\in X$, *therefore* $\sigma(\cdot;x,y)$ *is in its turn a* ***superadditive*** *and* ***nondecreasing*** *functional on* $\mathcal{H}(X)$.

1.3.3. The Superadditivity and Monotonicity of $\delta-$Mapping. Now consider another mapping naturally associated to Schwarz's inequality, namely [**1**]

$$\delta : \mathcal{H}(X) \times X^2 \to \mathbb{R}_+, \quad \delta((\cdot,\cdot);x,y) := \|x\|^2 \|y\|^2 - |(x,y)|^2 .$$

It is obvious that the following properties are valid:

(i) $\delta((\cdot,\cdot);x,y) \geq 0$ (Schwarz's inequality);
(ii) $\delta((\cdot,\cdot);x,y) = \delta((\cdot,\cdot);y,x)$;
(iii) $\delta(\alpha(\cdot,\cdot);x,y) = \alpha^2\delta((\cdot,\cdot);x,y)$

for all $x,y \in X$, $\alpha \geq 0$ and $(\cdot,\cdot) \in \mathcal{H}(X)$.

The following theorem incorporates some further properties of this functional [**1**]:

Theorem 5 (Dragomir-Mond, 1994). *With the above assumptions, we have:*

(i) *If* $(\cdot,\cdot)_i \in \mathcal{H}(X)$ $(i=1,2)$, *then*

$$\begin{aligned} (1.24)\quad & \delta((\cdot,\cdot)_1 + (\cdot,\cdot)_2;x,y) - \delta((\cdot,\cdot)_1;x,y) - \delta((\cdot,\cdot)_2;x,y) \\ & \geq \left(\det\begin{bmatrix} \|x\|_1 & \|y\|_1 \\ \|x\|_2 & \|y\|_2 \end{bmatrix}\right)^2 \qquad (\geq 0); \end{aligned}$$

i.e., the mapping $\delta(\cdot;x,y)$ *is strong superadditive on* $\mathcal{H}(X)$.

(ii) *If* $(\cdot,\cdot)_i \in \mathcal{H}(X)$ $(i=1,2)$, *with* $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1$, *then*

$$\begin{aligned} (1.25)\quad & \delta((\cdot,\cdot)_2;x,y) - \delta((\cdot,\cdot)_1;x,y) \\ & \geq \left(\det\begin{bmatrix} \|x\|_1 & \|y\|_1 \\ \left(\|x\|_2^2 - \|x\|_1^2\right)^{\frac{1}{2}} & \left(\|y\|_2^2 - \|y\|_1^2\right)^{\frac{1}{2}} \end{bmatrix}\right)^2 \qquad (\geq 0); \end{aligned}$$

i.e., the mapping $\delta(\cdot;x,y)$ *is strong nondecreasing on* $\mathcal{H}(X)$.

Proof. (i) For all $(\cdot,\cdot)_i \in \mathcal{H}(X)$ $(i=1,2)$ and $x,y \in X$ we have

$$\begin{aligned} (1.26)\qquad & \delta((\cdot,\cdot)_1 + (\cdot,\cdot)_2;x,y) \\ & = \left(\|x\|_2^2 - \|x\|_1^2\right)\left(\|y\|_2^2 - \|y\|_1^2\right) - |(x,y)_2 + (x,y)_1|^2 \\ & \geq \|x\|_2^2\|y\|_2^2 + \|x\|_1^2\|y\|_1^2 + \|x\|_1^2\|y\|_2^2 + \|x\|_2^2\|y\|_1^2 \\ & \qquad - \left(|(x,y)_2| + |(x,y)_1|\right)^2 \\ & = \delta((\cdot,\cdot)_2;x,y) + \delta((\cdot,\cdot)_1;x,y) \\ & \qquad + \|x\|_1^2\|y\|_2^2 + \|x\|_2^2\|y\|_1^2 - 2|(x,y)_2(x,y)_1| . \end{aligned}$$

By Schwarz's inequality we have

$$(1.27)\qquad |(x,y)_2(x,y)_1| \leq \|x\|_1\|y\|_1\|x\|_2\|y\|_2 ,$$

therefore, by (1.26) and (1.27), we can state that

$$\begin{aligned}
&\delta\left((\cdot,\cdot)_1+(\cdot,\cdot)_2\,;x,y\right)-\delta\left((\cdot,\cdot)_1\,;x,y\right)-\delta\left((\cdot,\cdot)_2\,;x,y\right)\\
&\geq \|x\|_1^2\,\|y\|_2^2+\|x\|_2^2\,\|y\|_1^2-2\,\|x\|_1\,\|y\|_1\,\|x\|_2\,\|y\|_2\\
&=\left(\|x\|_1\,\|y\|_2-\|x\|_2\,\|y\|_1\right)^2
\end{aligned}$$

and the inequality (1.24) is proved.

(ii) Suppose that $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1$ and, as in Theorem 4, define $(\cdot,\cdot)_{2,1} := (\cdot,\cdot)_2 - (\cdot,\cdot)_1$. Then $(\cdot,\cdot)_{2,1}$ is a nonnegative hermitian form and by (i) we have

$$\begin{aligned}
&\delta\left((\cdot,\cdot)_{2,1}\,;x,y\right)-\delta\left((\cdot,\cdot)_1\,;x,y\right)\\
&=\delta\left((\cdot,\cdot)_{2,1}+(\cdot,\cdot)_1\,;x,y\right)-\delta\left((\cdot,\cdot)_1\,;x,y\right)\\
&\geq \delta\left((\cdot,\cdot)_{2,1}\,;x,y\right)+\left(\det\begin{bmatrix}\|x\|_1 & \|y\|_1\\ \|x\|_{2,1} & \|y\|_{2,1}\end{bmatrix}\right)^2\\
&\geq \left(\det\begin{bmatrix}\|x\|_1 & \|y\|_1\\ \|x\|_{2,1} & \|y\|_{2,1}\end{bmatrix}\right)^2.
\end{aligned}$$

Since $\|z\|_{2,1} = \left(\|z\|_2^2 - \|z\|_1^2\right)^{\frac{1}{2}}$ for $z \in X$, hence the inequality (1.25) is proved. ∎

Remark 2. *If we consider the functional* $\delta_r\left((\cdot,\cdot)\,;x,y\right) := \|x\|^2\,\|y\|^2 - \left[\operatorname{Re}(x,y)\right]^2$, *then we can state similar properties for it. We omit the details.*

1.3.4. Superadditivity and Monotonicity of $\beta-$Mapping. Consider the functional $\beta : \mathcal{H}(X) \times X^2 \to \mathbb{R}$ defined by [**2**]

$$\beta\left((\cdot,\cdot)\,;x,y\right) = \left(\|x\|^2\,\|y\|^2 - |(x,y)|^2\right)^{\frac{1}{2}}. \tag{1.28}$$

It is obvious that $\beta\left((\cdot,\cdot)\,;x,y\right) = \left[\delta\left((\cdot,\cdot)\,;x,y\right)\right]^{\frac{1}{2}}$ and thus it is monotonic nondecreasing on $\mathcal{H}(X)$. Before we prove that $\beta(\cdot;x,y)$ is also superadditive, which apparently does not follow from the properties of δ pointed out in the subsection above, we need the following simple lemma:

Lemma 1. *If* $(\cdot,\cdot)$ *is a nonnegative Hermitian form on* X, $x,y \in X$ *and* $\|y\| \neq 0$, *then*

$$\inf_{\lambda\in\mathbb{K}} \|x-\lambda y\|^2 = \frac{\|x\|^2\,\|y\|^2 - |(x,y)|^2}{\|y\|^2}. \tag{1.29}$$

PROOF. Observe that

$$\|x-\lambda y\|^2 = \|x\|^2 - 2\operatorname{Re}\left[\lambda\left(x,y\right)\right] + |\lambda|^2 \|y\|^2$$

and, for $\|y\| \neq 0$,

$$\frac{\|x\|^2 \|y\|^2 - |(x,y)|^2 + \left|\mu \|y\|^2 - (x,y)\right|^2}{\|y\|^2} = \|x\|^2 - 2\operatorname{Re}\left[\mu\overline{(x,y)}\right] + |\mu|^2 \|y\|^2,$$

and since $\operatorname{Re}\left[\bar{\lambda}\left(x,y\right)\right] = \operatorname{Re}\overline{\left[\bar{\lambda}\left(x,y\right)\right]} = \operatorname{Re}\left[\lambda\overline{(x,y)}\right]$, we deduce the equality

$$\|x-\lambda y\|^2 = \frac{\|x\|^2 \|y\|^2 - |(x,y)|^2 + \left|\mu \|y\|^2 - (x,y)\right|^2}{\|y\|^2}, \tag{1.30}$$

for any $x, y \in X$ with $\|y\| \neq 0$.

Taking the infimum over $\lambda \in \mathbb{K}$ in (1.30), we deduce the desired result (1.29). ∎

For the subclass $\mathcal{IP}(X)$, of all inner products defined on X, of $\mathcal{H}(X)$ and $y \neq 0$, we may define

$$\begin{aligned}\gamma\left(\left(\cdot,\cdot\right);x,y\right) &= \frac{\|x\|^2 \|y\|^2 - |(x,y)|^2}{\|y\|^2} \\ &= \frac{\delta\left(\left(\cdot,\cdot\right);x,y\right)}{\|y\|^2}.\end{aligned}$$

The following result may be stated (see also [**2**]):

THEOREM 6 (Dragomir-Mond, 1996). *The functional $\gamma(\cdot;x,y)$ is superadditive and monotonic nondecreasing on $\mathcal{IP}(X)$ for any $x, y \in X$ with $y \neq 0$.*

PROOF. Let $(\cdot,\cdot)_1, (\cdot,\cdot)_2 \in \mathcal{IP}(X)$. Then

$$\begin{aligned}&\gamma\left(\left(\cdot,\cdot\right)_1 + \left(\cdot,\cdot\right)_2; x,y\right) \\ &= \frac{\left(\|x\|_1^2 + \|x\|_2^2\right)\left(\|y\|_1^2 + \|y\|_2^2\right) - \left|(x,y)_1 + (x,y)_2\right|^2}{\|y\|_1^2 \|y\|_2^2} \\ &= \inf_{\lambda\in\mathbb{K}} \left[\|x-\lambda y\|_1^2 + \|x-\lambda y\|_2^2\right],\end{aligned} \tag{1.31}$$

and for the last equality we have used Lemma 1.

Also,

$$\gamma\left((\cdot,\cdot)_i;x,y\right)=\frac{\|x\|_i^2\,\|y\|_i^2-|(x,y)_i|^2}{\|y\|_i^2}=\inf_{\lambda\in\mathbb{K}}\|x-\lambda y\|_i^2,\qquad i=1,2. \tag{1.32}$$

Utilising the infimum property that

$$\inf_{\lambda\in\mathbb{K}}\left(f(\lambda)+g(\lambda)\right)\geq\inf_{\lambda\in\mathbb{K}}f(\lambda)+\inf_{\lambda\in\mathbb{K}}g(\lambda),$$

we can write that

$$\inf_{\lambda\in\mathbb{K}}\left[\|x-\lambda y\|_1^2+\|x-\lambda y\|_2^2\right]\geq\inf_{\lambda\in\mathbb{K}}\|x-\lambda y\|_1^2+\inf_{\lambda\in\mathbb{K}}\|x-\lambda y\|_2^2,$$

which proves the superadditivity of $\gamma(\cdot;x,y)$.

The monotonicity follows by the superadditivity property and the theorem is completely proved. ∎

Corollary 1. *If $(\cdot,\cdot)_i\in\mathcal{JP}(X)$ with $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$ and $x,y\in X$ are such that $x,y\neq0$, then:*

$$\delta\left((\cdot,\cdot)_2;x,y\right)\geq\max\left\{\frac{\|y\|_2^2}{\|y\|_1^2},\frac{\|x\|_2^2}{\|x\|_1^2}\right\}\delta\left((\cdot,\cdot)_1;x,y\right)\quad\left(\geq\delta\left((\cdot,\cdot)_1;x,y\right)\right) \tag{1.33}$$

or equivalently, [**2**]

$$\delta\left((\cdot,\cdot)_2;x,y\right)-\delta\left((\cdot,\cdot)_1;x,y\right)\geq\max\left\{\frac{\|y\|_2^2-\|y\|_1^2}{\|y\|_1^2},\frac{\|x\|_2^2-\|x\|_1^2}{\|x\|_1^2}\right\}\delta\left((\cdot,\cdot)_1;x,y\right). \tag{1.34}$$

The following strong superadditivity property of $\delta(\cdot;x,y)$ that is different from the one in Subsection 1.3.2 holds [**2**]:

Corollary 2 (Dragomir-Mond, 1996). *If $(\cdot,\cdot)_i\in\mathcal{JP}(X)$ and $x,y\in X$ with $x,y\neq0$, then*

$$\begin{aligned}&\delta\left((\cdot,\cdot)_1+(\cdot,\cdot)_2;x,y\right)-\delta\left((\cdot,\cdot)_1;x,y\right)-\delta\left((\cdot,\cdot)_2;x,y\right)\\&\geq\max\left\{\left(\frac{\|y\|_2}{\|y\|_1}\right)^2\delta\left((\cdot,\cdot)_1;x,y\right)+\left(\frac{\|y\|_1}{\|y\|_2}\right)^2\delta\left((\cdot,\cdot)_2;x,y\right);\right.\\&\left.\left(\frac{\|x\|_2}{\|x\|_1}\right)^2\delta\left((\cdot,\cdot)_1;x,y\right)+\left(\frac{\|x\|_1}{\|x\|_2}\right)^2\delta\left((\cdot,\cdot)_2;x,y\right)\right\}\qquad(\geq0).\end{aligned} \tag{1.35}$$

PROOF. Utilising the identities (1.31) and (1.32) and taking into account that $\gamma(\cdot;x,y)$ is superadditive, we can state that

$$\begin{aligned}
(1.36)\quad & \delta\left((\cdot,\cdot)_1+(\cdot,\cdot)_2;x,y\right)\\
&\geq \frac{\|y\|_1^2+\|y\|_2^2}{\|y\|_1^2}\delta\left((\cdot,\cdot)_1;x,y\right)+\frac{\|y\|_1^2+\|y\|_2^2}{\|y\|_2^2}\delta\left((\cdot,\cdot)_2;x,y\right)\\
&=\delta\left((\cdot,\cdot)_1;x,y\right)+\delta\left((\cdot,\cdot)_2;x,y\right)\\
&\qquad+\left(\frac{\|y\|_2}{\|y\|_1}\right)^2\delta\left((\cdot,\cdot)_1;x,y\right)+\left(\frac{\|y\|_1}{\|y\|_2}\right)^2\delta\left((\cdot,\cdot)_2;x,y\right)
\end{aligned}$$

and a similar inequality with x instead of y. These show that the desired inequality (1.35) holds true. ∎

REMARK 3. *Obviously, all the inequalities above remain true if* $(\cdot,\cdot)_i$, $i=1,2$ *are nonnegative Hermitian forms for which we have* $\|x\|_i$, $\|y\|_i\neq 0$.

Finally, we may state and prove the superadditivity result for the mapping β (see [**2**]):

THEOREM 7 (Dragomir-Mond, 1996). *The mapping* β *defined by (1.28) is superadditive on* $\mathcal{H}(X)$.

PROOF. Without loss of generality, if $(\cdot,\cdot)_i\in\mathcal{H}(X)$ and $x,y\in X$, we may assume, for instance, that $\|y\|_i\neq 0$, $i=1,2$.

If so, then

$$\begin{aligned}
&\left(\frac{\|y\|_2}{\|y\|_1}\right)^2\delta\left((\cdot,\cdot)_1;x,y\right)+\left(\frac{\|y\|_1}{\|y\|_2}\right)^2\delta\left((\cdot,\cdot)_2;x,y\right)\\
&\qquad\qquad\geq 2\left[\delta\left((\cdot,\cdot)_1;x,y\right)\delta\left((\cdot,\cdot)_2;x,y\right)\right]^{\frac{1}{2}},
\end{aligned}$$

and by making use of (1.36) we get:

$$\delta\left((\cdot,\cdot)_1+(\cdot,\cdot)_2;x,y\right)\geq\left\{\left[\delta\left((\cdot,\cdot)_1;x,y\right)\right]^{\frac{1}{2}}+\left[\delta\left((\cdot,\cdot)_2;x,y\right)\right]^{\frac{1}{2}}\right\}^2,$$

which is exactly the superadditivity property for β. ∎

1.4. Applications for General Inner Product Spaces

1.4.1. Inequalities for Orthonormal Families. Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. The family of vectors $E:=\{e_i\}_{i\in I}$ (I is a finite or infinite) is an *orthonormal family* of vectors if $\langle e_i,e_j\rangle=\delta_{ij}$ for $i,j\in I$, where δ_{ij} is Kronecker's delta.

The following inequality is well known in the literature as Bessel's inequality:

$$\sum_{i\in F} |\langle x, e_i\rangle|^2 \leq \|x\|^2 \tag{1.37}$$

for any F a finite part of I and x a vector in H.

If by $\mathcal{F}(I)$ we denote the family of all finite parts of I (including the empty set $\varnothing$), then for any $F \in \mathcal{F}(I) \setminus \{\varnothing\}$ the functional $(\cdot,\cdot)_F : H \times H \to \mathbb{K}$ given by

$$(x, y)_F := \sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle \tag{1.38}$$

is a Hermitian form on H.

It is obvious that if $F_1, F_2 \in \mathcal{F}(I) \setminus \{\varnothing\}$ and $F_1 \cap F_2 = \varnothing$, then $(\cdot,\cdot)_{F_1\cup F_2} = (\cdot,\cdot)_{F_1} + (\cdot,\cdot)_{F_2}$.

We can define the functional $\sigma : \mathcal{F}(I) \times H^2 \to \mathbb{R}_+$ by

$$\sigma(F; x, y) := \|x\|_F \|y\|_F - |(x, y)_F|, \tag{1.39}$$

where

$$\|x\|_F := \left(\sum_{i\in F} |\langle x, e_i\rangle|^2\right)^{\frac{1}{2}} = [(x, x)_F]^{\frac{1}{2}}, \qquad x \in H.$$

The following proposition may be stated (see also [**2**]):

Proposition 1 (Dragomir-Mond, 1995). *The mapping σ satisfies the following*

(i) *If $F_1, F_2 \in \mathcal{F}(I) \setminus \{\varnothing\}$ with $F_1 \cap F_2 = \varnothing$, then*

$$\sigma(F_1 \cup F_2; x, y) \geq \sigma(F_1; x, y) + \sigma(F_2; x, y) \qquad (\geq 0)$$

for any $x, y \in H$, i.e., the mapping $\sigma(\cdot; x, y)$ is an index set superadditive mapping on $\mathcal{F}(I)$;

(ii) *If $\varnothing \neq F_1 \subseteq F_2$, $F_1, F_2 \in \mathcal{F}(I)$, then*

$$\sigma(F_2; x, y) \geq \sigma(F_1; x, y) \qquad (\geq 0),$$

i.e., the mapping $\sigma(\cdot; x, y)$ is an index set monotonic mapping on $\mathcal{F}(I)$.

The proof is obvious by Theorem 4 and we omit the details.

We can also define the mapping $\sigma_r(\cdot;\cdot,\cdot) : \mathcal{F}(I) \times H^2 \to \mathbb{R}_+$ by

$$\sigma_r(F; x, y) := \|x\|_F \|y\|_F - \operatorname{Re}(x, y)_F,$$

which also has the properties (i) and (ii) of Proposition 1.

Since, by Bessel's inequality the hermitian form $(\cdot,\cdot)_F \le \langle\cdot,\cdot\rangle$ in the sense of definition (1.18) then by Theorem 4 we may state the following *refinements* of Schwarz's inequality [**1**]:

PROPOSITION 2 (Dragomir-Mond, 1994). *For any $F \in \mathcal{F}(I)\setminus\{0\}$, we have the inequalities*

$$\begin{aligned}\|x\|\,\|y\| - |\langle x,y\rangle| \\ \ge \left(\sum_{i\in F}|\langle x,e_i\rangle|^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}|\langle y,e_i\rangle|^2\right)^{\frac{1}{2}} - \left|\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right|\end{aligned}\tag{1.40}$$

and

$$\begin{aligned}\|x\|\,\|y\| - |\langle x,y\rangle| \\ \ge \left(\|x\|^2 - \sum_{i\in F}|\langle x,e_i\rangle|^2\right)^{\frac{1}{2}}\left(\|y\|^2 - \sum_{i\in F}|\langle y,e_i\rangle|^2\right)^{\frac{1}{2}} \\ - \left|\langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right|\end{aligned}\tag{1.41}$$

and the corresponding versions on replacing $|\cdot|$ by $\operatorname{Re}(\cdot)$, where x,y are vectors in H.

REMARK 4. *Note that the inequality (1.40) and its version for* $\operatorname{Re}(\cdot)$ *has been established for the first time and utilising a different argument by Dragomir and Sándor in 1994 (see* [**3**, Theorem 5 and Remark 2]*).*

If we now define the mapping $\delta : \mathcal{F}(I)\times H^2 \to \mathbb{R}_+$ by

$$\delta(F;x,y) := \|x\|_F^2\,\|y\|_F^2 - |(x,y)_F|^2$$

and making use of Theorem 5, we may state the following result [**2**].

PROPOSITION 3 (Dragomir-Mond, 1995). *The mapping δ satisfies the following properties:*

(i) *If $F_1, F_2 \in \mathcal{F}(I)\setminus\{\varnothing\}$ with $F_1\cap F_2=\varnothing$, then*

$$\begin{aligned}\delta(F_1\cup F_2;x,y) - \delta(F_1;x,y) - \delta(F_2;x,y) \\ \ge \left(\det\begin{bmatrix}\|x\|_{F_1} & \|y\|_{F_1}\\ \|x\|_{F_2} & \|y\|_{F_2}\end{bmatrix}\right)^2 \qquad (\ge 0),\end{aligned}\tag{1.42}$$

i.e., the mapping $\delta(\cdot;x,y)$ is strong superadditive as an index set mapping;

(ii) *If* $\varnothing \neq F_1 \subseteq F_2,\ F_1, F_2 \in \mathcal{F}(I)$, *then*

$$\begin{aligned}(1.43)\quad & \delta(F_2; x, y) - \delta(F_1; x, y) \\ & \geq \left(\det \begin{bmatrix} \|x\|_{F_1} & \|y\|_{F_1} \\ \left(\|x\|_{F_2}^2 - \|x\|_{F_1}^2\right)^{\frac{1}{2}} & \left(\|y\|_{F_2}^2 - \|y\|_{F_1}^2\right)^{\frac{1}{2}} \end{bmatrix}\right)^2 \qquad (\geq 0),\end{aligned}$$

i.e., the mapping $\delta(\cdot; x, y)$ is strong nondecreasing as an index set mapping.

On applying the same general result in Theorem 5, (ii) for the hermitian functionals $(\cdot,\cdot)_F$ $(F \in \mathcal{F}(I) \setminus \{\varnothing\})$ and $\langle \cdot,\cdot \rangle$ for which, by Bessel's inequality we know that $(\cdot,\cdot)_F \leq \langle \cdot,\cdot \rangle$, we may state the following result as well, which provides refinements for the Schwarz inequality.

Proposition 4 (Dragomir-Mond, 1994). *For any $F \in \mathcal{F}(I) \setminus \{\varnothing\}$, we have the inequalities:*

$$\begin{aligned}(1.44)\quad & \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \\ & \geq \sum_{i \in F} |\langle x, e_i\rangle|^2 \sum_{i \in F} |\langle y, e_i\rangle|^2 - \left|\sum_{i \in F} \langle x, e_i\rangle \langle e_i, y\rangle\right|^2 \qquad (\geq 0)\end{aligned}$$

and

$$\begin{aligned}(1.45)\quad & \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \\ & \geq \left(\|x\|^2 - \sum_{i \in F} |\langle x, e_i\rangle|^2\right)\left(\|y\|^2 - \sum_{i \in F} |\langle y, e_i\rangle|^2\right) \\ & \qquad - \left|\langle x, y\rangle - \sum_{i \in F} \langle x, e_i\rangle \langle e_i, y\rangle\right|^2 \qquad (\geq 0),\end{aligned}$$

for any $x, y \in H$.

On utilising Corollary 2 we may state the following different superadditivity property for the mapping $\delta(\cdot; x, y)$.

PROPOSITION 5. *If $F_1, F_2 \in \mathcal{F}(I) \setminus \{\varnothing\}$ with $F_1 \cap F_2 = \varnothing$, then*

$$\begin{aligned} &\delta(F_1 \cup F_2; x, y) - \delta(F_1; x, y) - \delta(F_2; x, y) \\ &\geq \max\left\{\left(\frac{\|y\|_{F_2}}{\|y\|_{F_1}}\right)^2 \delta(F_1; x, y) + \left(\frac{\|y\|_{F_1}}{\|y\|_{F_2}}\right)^2 \delta(F_2; x, y);\right. \\ &\left.\left(\frac{\|x\|_{F_2}}{\|x\|_{F_1}}\right)^2 \delta(F_1; x, y) + \left(\frac{\|x\|_{F_1}}{\|x\|_{F_2}}\right)^2 \delta(F_2; x, y)\right\} \qquad (\geq 0) \end{aligned} \tag{1.46}$$

for any $x, y \in H \setminus \{0\}$.

Further, for $y \notin M^{\perp}$ where $M = Sp\{e_i\}_{i \in I}$ is the linear space spanned by $E = \{e_i\}_{i \in I}$, we can also consider the functional $\gamma : \mathcal{F}(I) \times H^2 \to \mathbb{R}_+$ defined by

$$\gamma(F; x, y) := \frac{\delta(F; x, y)}{\|y\|_F^2} = \frac{\|x\|_F^2 \|y\|_F^2 - |(x, y)_F|^2}{\|y\|_F^2},$$

where $x \in H$ and $F \neq \varnothing$.

Utilising Theorem 6, we may state the following result concerning the properties of the functional $\gamma(\cdot; x, y)$ with x and y as above.

PROPOSITION 6. *For any $x \in H$ and $y \in H \setminus M^{\perp}$, the functional $\gamma(\cdot; x, y)$ is superadditive and monotonic nondecreasing as an index set mapping on $\mathcal{F}(I)$.*

Since $\langle \cdot, \cdot \rangle \geq (\cdot, \cdot)_F$ for any $F \in \mathcal{F}(I)$, on making use of Corollary 1, we may state the following refinement of Schwarz's inequality:

PROPOSITION 7. *Let $x \in H$ and $y \in H \setminus M_F^{\perp}$, where $M_F := Sp\{e_i\}_{i \in I}$ and $F \in \mathcal{F}(I) \setminus \{\varnothing\}$ is given. Then*

$$\begin{aligned} &\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \geq \max\left\{\frac{\|y\|^2}{\sum_{i \in F} |\langle y, e_i \rangle|^2}, \frac{\|x\|^2}{\sum_{i \in F} |\langle x, e_i \rangle|^2}\right\} \\ &\times \left(\sum_{i \in F} |\langle x, e_i \rangle|^2 \sum_{i \in F} |\langle y, e_i \rangle|^2 - \left|\sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle\right|^2\right) \\ &\left(\geq \sum_{i \in F} |\langle x, e_i \rangle|^2 \sum_{i \in F} |\langle y, e_i \rangle|^2 - \left|\sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle\right|^2\right), \end{aligned} \tag{1.47}$$

which is a refinement of (1.45) in the case that $y \in H \setminus M_F^{\perp}$.

Finally, consider the functional $\beta : \mathcal{F}(I) \times H^2 \to \mathbb{R}_+$ given by

$$\beta(F; x, y) := [\delta(F; x, y)]^{\frac{1}{2}} = \left(\|x\|_F^2 \|y\|_F^2 - |(x, y)_F|^2\right)^{\frac{1}{2}}.$$

Utilising Theorem 7, we may state the following:

PROPOSITION 8. *The functional $\beta(\cdot; x, y)$ is superadditive as an index set mapping on $\mathcal{F}(I)$ for each $x, y \in H$.*

As a dual approach, one may also consider the following form $(\cdot,\cdot)_{C,F} : H \times H \to \mathbb{R}$ given by:

$$(x, y)_{C,F} := \langle x, y\rangle - (x, y)_F = \langle x, y\rangle - \sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle . \tag{1.48}$$

By Bessel's inequality, we observe that $(\cdot,\cdot)_{C,F}$ is a nonnegative hermitian form and, obviously

$$(\cdot,\cdot)_I + (\cdot,\cdot)_{C,F} = \langle\cdot,\cdot\rangle .$$

Utilising the superadditivity properties from Section 1.3, one may state the following refinement of the Schwarz inequality:

$$\begin{aligned} &\|x\| \|y\| - |\langle x, y\rangle| \\ &\quad\geq \left(\sum_{i\in F} |\langle x, e_i\rangle|^2 \sum_{i\in F} |\langle y, e_i\rangle|^2\right)^{\frac{1}{2}} - \left|\sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle\right| \\ &\quad+ \left(\|x\|^2 - \sum_{i\in F} |\langle x, e_i\rangle|^2\right)^{\frac{1}{2}} \left(\|y\|^2 - \sum_{i\in F} |\langle y, e_i\rangle|^2\right)^{\frac{1}{2}} \\ &\qquad\qquad - \left|\langle x, y\rangle - \sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle\right| \qquad (\geq 0), \end{aligned} \tag{1.49}$$

$$\begin{aligned} &\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \\ &\quad\geq \sum_{i\in F} |\langle x, e_i\rangle|^2 \sum_{i\in F} |\langle y, e_i\rangle|^2 - \left|\sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle\right|^2 \\ &\quad+ \left(\|x\|^2 - \sum_{i\in F} |\langle x, e_i\rangle|^2\right) \left(\|y\|^2 - \sum_{i\in F} |\langle y, e_i\rangle|^2\right) \\ &\qquad\qquad - \left|\langle x, y\rangle - \sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle\right|^2 \qquad (\geq 0) \end{aligned} \tag{1.50}$$

and

$$(1.51)\quad \left(\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\right)^{\frac{1}{2}}$$
$$\geq\left[\sum_{i\in F}|\langle x,e_i\rangle|^2\sum_{i\in F}|\langle y,e_i\rangle|^2-\left|\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right|^2\right]^{\frac{1}{2}}$$
$$+\left[\left(\|x\|^2-\sum_{i\in F}|\langle x,e_i\rangle|^2\right)\left(\|y\|^2-\sum_{i\in F}|\langle y,e_i\rangle|^2\right)\right.$$
$$\left.-\left|\langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right|^2\right]^{\frac{1}{2}}\qquad(\geq 0),$$

for any $x,y\in H$ and $F\in\mathcal{F}(I)\setminus\{\varnothing\}$.

1.4.2. Inequalities for Gram Determinants. Let $\{x_1,\dots,x_n\}$ be vectors in the inner product space $(H,\langle\cdot,\cdot\rangle)$ over the real or complex number field $\mathbb{K}$. Consider the *gram matrix* associated to the above vectors:

$$G(x_1,\dots,x_n):=\begin{bmatrix}\langle x_1,x_1\rangle & \langle x_1,x_2\rangle & \cdots & \langle x_1,x_n\rangle\\ \langle x_2,x_1\rangle & & \cdots & \langle x_2,x_n\rangle\\ \cdots & & \cdots & \cdots\\ \langle x_n,x_1\rangle & \langle x_n,x_2\rangle & \cdots & \langle x_n,x_n\rangle\end{bmatrix}.$$

The determinant

$$\Gamma(x_1,\dots,x_n):=\det G(x_1,\dots,x_n)$$

is called the Gram determinant associated to the system $\{x_1,\dots,x_n\}$.

If $\{x_1,\dots,x_n\}$ does not contain the null vector 0, then [**4**]

$$(1.52)\qquad 0\leq\Gamma(x_1,\dots,x_n)\leq\|x_1\|^2\|x_2\|^2\cdots\|x_n\|^2.$$

The equality holds on the left (respectively right) side of (1.52) if and only if $\{x_1,\dots,x_n\}$ is linearly dependent (respectively orthogonal). The first inequality in (1.52) is known in the literature as *Gram's inequality* while the second one is known as *Hadamard's inequality.*

The following result obtained in [**3**] may be regarded as a refinement of Gram's inequality:

Theorem 8 (Dragomir-Sándor, 1994). *Let $\{x_1,\dots,x_n\}$ be a system of nonzero vectors in H. Then for any $x,y\in H$ one has:*

$$(1.53)\qquad \Gamma(x,x_1,\dots,x_n)\Gamma(y,x_1,\dots,x_n)\geq|\Gamma(x_1,\dots,x_n)(x,y)|^2,$$

where $\Gamma(x_1,\dots,x_n)(x,y)$ *is defined by:*

$$\Gamma(x_1,\dots,x_n)(x,y) := \det\begin{bmatrix} \langle x,y\rangle & \langle x,x_1\rangle & \cdots & \langle x,x_n\rangle \\ \langle x_1,y\rangle & & & \\ \cdots & & G(x_1,\dots,x_n) & \\ \langle x_n,y\rangle & & & \end{bmatrix}.$$

Proof. We follow the proof from [**3**].

Let us consider the mapping $p: H\times H\to\mathbb{K}$ given by

$$p(x,y)=\Gamma(x_1,\dots,x_n)(x,y).$$

Utilising the properties of determinants, we notice that

$$\begin{aligned} p(x,y) &= \Gamma(x,x_1,\dots,x_n)\geq 0,\\ p(x+y,z) &= \Gamma(x_1,\dots,x_n)(x+y,z)\\ &= \Gamma(x_1,\dots,x_n)(x,z)+\Gamma(x_1,\dots,x_n)(y,z)\\ &= p(x,z)+p(y,z),\\ p(\alpha x,y) &= \alpha p(x,y),\\ p(y,x) &= \overline{p(x,y)}, \end{aligned}$$

for any $x,y,z\in H$ and $\alpha\in\mathbb{K}$, showing that $p(\cdot,\cdot)$ is a nonnegative hermitian from on X. Writing Schwarz's inequality for $p(\cdot,\cdot)$ we deduce the desired result (1.53). ∎

In a similar manner, if we define $q: H\times H\to\mathbb{K}$ by

$$\begin{aligned} q(x,y) &:= (x,y)\prod_{i=1}^{n}\|x_i\|^2 - p(x,y)\\ &= (x,y)\prod_{i=1}^{n}\|x_i\|^2-\Gamma(x_1,\dots,x_n)(x,y), \end{aligned}$$

then, using Hadamard's inequality, we conclude that $q(\cdot,\cdot)$ is also a nonnegative hermitian form. Therefore, by Schwarz's inequality applied for $q(\cdot,\cdot)$, we can state the following result as well [**3**]:

Theorem 9 (Dragomir-Sándor, 1994). *With the assumptions of Theorem 8, we have:*

$$\left[\|x\|^2 \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(x, x_1, \dots, x_n)\right] \times \left[\|y\|^2 \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(y, x_1, \dots, x_n)\right] \geq \left| \langle x, y\rangle \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(x_1, \dots, x_n)(x, y)\right|^2, \tag{1.54}$$

for each $x, y \in H$.

Observing that, for a given set of nonzero vectors $\{x_1, \dots, x_n\}$,

$$p(x, y) + q(x, y) = (x, y) \prod_{i=1}^{n} \|x_i\|^2,$$

for any $x, y \in H$, then, on making use of the superadditivity properties of the various functionals defined in Section 1.3, we can state the following refinements of the Schwarz inequality in inner product spaces:

$$\begin{aligned} &[\|x\| \|y\| - |\langle x, y\rangle|] \prod_{i=1}^{n} \|x_i\|^2 \\ &\geq [\Gamma(x, x_1, \dots, x_n) \Gamma(y, x_1, \dots, x_n)]^{\frac{1}{2}} - |\Gamma(x_1, \dots, x_n)(x, y)| \\ &\quad + \left[\|x\|^2 \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(x, x_1, \dots, x_n)\right]^{\frac{1}{2}} \\ &\quad \times \left[\|y\|^2 \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(y, x_1, \dots, x_n)\right]^{\frac{1}{2}} \\ &\quad - \left| \langle x, y\rangle \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(x_1, \dots, x_n)(x, y)\right| \qquad (\geq 0), \end{aligned} \tag{1.55}$$

$$\begin{aligned} &\left[\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2\right] \prod_{i=1}^{n} \|x_i\|^4 \\ &\Gamma(x, x_1, \dots, x_n) \Gamma(y, x_1, \dots, x_n) - |\Gamma(x_1, \dots, x_n)(x, y)|^2 \\ &\quad + \left[\|x\|^2 \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(x, x_1, \dots, x_n)\right] \end{aligned} \tag{1.56}$$

$$\times\left[\|y\|^2\prod_{i=1}^{n}\|x_i\|^2-\Gamma(y,x_1,\dots,x_n)\right]$$
$$-\left|\langle x,y\rangle\prod_{i=1}^{n}\|x_i\|^2-\Gamma(x_1,\dots,x_n)(x,y)\right|^2 \qquad (\geq 0),$$

and

$$[\|x\|\,\|y\|-|\langle x,y\rangle|]^{\frac{1}{2}}\prod_{i=1}^{n}\|x_i\|^2 \tag{1.57}$$
$$\geq\left[\Gamma(x,x_1,\dots,x_n)\,\Gamma(y,x_1,\dots,x_n)-|\Gamma(x_1,\dots,x_n)(x,y)|^2\right]^{\frac{1}{2}}$$
$$+\left\{\left[\|x\|^2\prod_{i=1}^{n}\|x_i\|^2-\Gamma(x,x_1,\dots,x_n)\right]\right.$$
$$\times\left[\|y\|^2\prod_{i=1}^{n}\|x_i\|^2-\Gamma(y,x_1,\dots,x_n)\right]$$
$$\left.-\left|\langle x,y\rangle\prod_{i=1}^{n}\|x_i\|^2-\Gamma(x_1,\dots,x_n)(x,y)\right|^2\right\}^{\frac{1}{2}} \qquad (\geq 0).$$

1.4.3. Inequalities for Linear Operators. Let $A:H\to H$ be a linear bounded operator and

$$\|A\|:=\sup\{\|Ax\|,\|x\|<1\}$$

its norm.

If we consider the hermitian forms $(\cdot,\cdot)_2$, $(\cdot,\cdot)_1 : H\to H$ defined by

$$(x,y)_1:=\langle Ax,Ay\rangle, \qquad (x,y)_2:=\|A\|^2\langle x,y\rangle$$

then obviously $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$ in the sense of definition (1.18) and utilising the monotonicity properties of the functional considered in Section 1.3, we may state the following inequalities:

$$\|A\|^2[\|x\|\,\|y\|-|\langle x,y\rangle|]\geq\|Ax\|\,\|Ay\|-|\langle Ax,Ay\rangle| \qquad (\geq 0), \tag{1.58}$$

$$\|A\|^4\left[\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\right] \geq\|Ax\|^2\|Ay\|^2-|\langle Ax,Ay\rangle|^2 \qquad (\geq 0) \tag{1.59}$$

for any $x,y\in H$, and the corresponding versions on replacing $|\cdot|$ by $\mathrm{Re}(\cdot)$.

The results (1.58) and (1.59) have been obtained by Dragomir and Mond in [**1**].

On using Corollary 1, we may deduce the following inequality as well:

$$\|A\|^2 \left[\|x\|^2 \|y\|^2 - |\langle x,y\rangle|^2\right] \geq \max\left\{\frac{\|x\|^2}{\|Ax\|^2}, \frac{\|y\|^2}{\|Ay\|^2}\right\} \left[\|Ax\|^2 \|Ay\|^2 - |\langle Ax,Ay\rangle|^2\right] \qquad (\geq 0) \tag{1.60}$$

for any $x, y \in H$ with $Ax, Ay \neq 0$; which improves (1.59) for x, y specified before.

Similarly, if $B : H \to H$ is a linear operator satisfying the condition

$$\|Bx\| \geq m\|x\| \quad \text{for any} \quad x \in H, \tag{1.61}$$

where $m > 0$ is given, then the hermitian forms $[x,y]_2 := \langle Bx, By\rangle$, $[x,y]_1 := m^2 \langle x,y\rangle$, have the property that $[\cdot,\cdot]_2 \geq [\cdot,\cdot]_1$. Therefore, from the monotonicity results established in Section 1.3, we can state that

$$\|Bx\| \|By\| - |\langle Bx,By\rangle| \geq m^2 [\|x\| \|y\| - |\langle x,y\rangle|] \qquad (\geq 0), \tag{1.62}$$

$$\|Bx\|^2 \|By\|^2 - |\langle Bx,By\rangle|^2 \geq m^4 \left[\|x\|^2 \|y\|^2 - |\langle x,y\rangle|^2\right] \quad (\geq 0) \tag{1.63}$$

for any $x, y \in H$, and the corresponding results on replacing $|\cdot|$ by $\operatorname{Re}(\cdot)$.

The same Corollary 1, would give the inequality

$$\|Bx\|^2 \|By\|^2 - |\langle Bx,By\rangle|^2 \geq m^2 \max\left\{\frac{\|Bx\|^2}{\|x\|^2}, \frac{\|By\|^2}{\|y\|^2}\right\} \left[\|x\|^2 \|y\|^2 - |\langle x,y\rangle|^2\right] \tag{1.64}$$

for $x, y \neq 0$, which is an improvement of (1.63).

We recall that a linear self-adjoint operator $P : H \to H$ is *nonnegative* if $\langle Px, x\rangle \geq 0$ for any $x \in H$. P is called *positive* if $\langle Px, x\rangle = 0$ and *positive definite with the constant* $\gamma > 0$ if $\langle Px, x\rangle \geq \gamma \|x\|^2$ for any $x \in H$.

If $A, B : H \to H$ are two linear self-adjoint operators such that $A \geq B$ (this means that $A - B$ is nonnegative), then the corresponding hermitian forms $(x,y)_A := \langle Ax, y\rangle$ and $(x,y)_B := \langle Bx, y\rangle$ satisfies the property that $(\cdot,\cdot)_A \geq (\cdot,\cdot)_B$.

If by $\mathcal{P}(H)$ we denote the *cone* of all linear self-adjoint and nonnegative operators defined in the Hilbert space H, then, on utilising the results of Section 1.3, we may state that the functionals $\sigma_0, \delta_0, \beta_0 : \mathcal{P}(H) \times H^2 \to [0, \infty]$ given by

$$\sigma_0 (P; x, y) := \langle Ax, x\rangle^{\frac{1}{2}} \langle Py, y\rangle^{\frac{1}{2}} - |\langle Px, y\rangle| ,$$
$$\delta_0 (P; x, y) := \langle Px, x\rangle \langle Py, y\rangle - |\langle Px, y\rangle|^2 ,$$
$$\beta_0 (P; x, y) := \left[\langle Px, x\rangle \langle Py, y\rangle - |\langle Px, y\rangle|^2\right]^{\frac{1}{2}} ,$$

are *superadditive* and *monotonic decreasing* on $\mathcal{P}(H)$, i.e.,

$$\gamma_0 (P + Q; x, y) \geq \gamma_0 (P; x, y) + \gamma_0 (Q; x, y) \qquad (\geq 0)$$

for any $P, Q \in \mathcal{P}(H)$ and $x, y \in H$, and

$$\gamma_0 (P; x, y) \geq \gamma_0 (Q; x, y) \qquad (\geq 0)$$

for any P, Q with $P \geq Q \geq 0$ and $x, y \in H$, where $\gamma \in \{\sigma, \delta, \beta\}$.

The superadditivity and monotonicity properties of σ_0 and δ_0 have been noted by Dragomir and Mond in [**1**].

If $u \in \mathcal{P}(H)$ is such that $I \geq U \geq 0$, where I is the identity operator, then on using the superadditivity property of the functionals σ_0, δ_0 and β_0 one may state the following refinements for the Schwarz inequality:

$$\begin{aligned} \|x\| \|y\| - |\langle x, y\rangle| &\geq \langle Ux, x\rangle^{\frac{1}{2}} \langle Uy, y\rangle^{\frac{1}{2}} - |\langle Ux, y\rangle| \\ &+ \langle (I - U) x, x\rangle^{\frac{1}{2}} \langle (I - U) y, y\rangle^{\frac{1}{2}} - |\langle (I - U) x, y\rangle| \qquad (\geq 0) , \end{aligned} \tag{1.65}$$

$$\begin{aligned} \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 &\geq \langle Ux, x\rangle \langle Uy, y\rangle - |\langle Ux, y\rangle|^2 \\ &+ \langle (I - U) x, x\rangle \langle (I - U) y, y\rangle - |\langle (I - U) x, y\rangle|^2 \qquad (\geq 0) , \end{aligned} \tag{1.66}$$

and

$$\begin{aligned} \left(\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2\right)^{\frac{1}{2}} &\geq \left(\langle Ux, x\rangle \langle Uy, y\rangle - |\langle Ux, y\rangle|^2\right)^{\frac{1}{2}} \\ &+ \left(\langle (I - U) x, x\rangle \langle (I - U) y, y\rangle - |\langle (I - U) x, y\rangle|^2\right)^{\frac{1}{2}} \qquad (\geq 0) \end{aligned} \tag{1.67}$$

for any $x, y \in H$.

Note that (1.67) is a better result than (1.66).

Finally, if we assume that $D \in \mathcal{P}(H)$ with $D \geq \gamma I$, where $\gamma > 0$, i.e., D is positive definite on H, then we may state the following inequalities

$$\langle Dx, x\rangle^{\frac{1}{2}} \langle Dy, y\rangle^{\frac{1}{2}} - |\langle Dx, y\rangle| \geq \gamma \left[\|x\| \|y\| - |\langle x, y\rangle|\right] \quad (\geq 0) , \tag{1.68}$$

$$\begin{aligned} &\langle Dx, x\rangle \langle Dy, y\rangle - |\langle Dx, y\rangle|^2 \\ &\qquad \geq \gamma^2 \left[\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2\right] \quad (\geq 0) , \end{aligned} \tag{1.69}$$

for any $x, y \in H$ and

$$(1.70)\quad \langle Dx, x\rangle \langle Dy, y\rangle - |\langle Dx, y\rangle|^2 \geq \gamma \max\left\{\frac{\langle Dx, x\rangle}{\|x\|^2}, \frac{\langle Dy, y\rangle}{\|y\|^2}\right\}\left[\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2\right] \quad (\geq 0)$$

for any $x, y \in H \backslash \{0\}$.

The results (1.68) and (1.69) have been obtained by Dragomir and Mond in [**1**].

Note that (1.70) is a better result than (1.69).

The above results (1.65) – (1.70) also hold for $\mathrm{Re}(\cdot)$ instead of $|\cdot|$.

1.5. Applications for Sequences of Vectors

1.5.1. The Case of Mapping σ. Let $\mathcal{P}_f(\mathbb{N})$ be the family of finite parts of the natural number set $\mathbb{N}$, $\mathcal{S}_+(\mathbb{R})$ the cone of nonnegative real sequences and for a given inner product space $(H; \langle\cdot,\cdot\rangle)$ over the real or complex number field $\mathbb{K}$, $\mathcal{S}(H)$ the linear space of all sequences of vectors from H, i.e.,

$$\mathcal{S}(H) := \left\{\mathbf{x} | \mathbf{x} = (x_i)_{i\in\mathbb{N}},\ x_i \in H,\ i \in \mathbb{N}\right\}.$$

Consider $\langle\cdot,\cdot\rangle_{\mathbf{p},I} : \mathcal{S}(H) \times \mathcal{S}(H) \to \mathbb{R}$ defined by

$$\langle \mathbf{x}, \mathbf{y}\rangle_{\mathbf{p},I} := \sum_{i\in I} p_i \langle x_i, y_i\rangle .$$

We may define the mapping σ by

$$(1.71)\quad \sigma(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) := \left(\sum_{i\in I} p_i \|x_i\|^2 \sum_{i\in I} p_i \|y_i\|^2\right)^{\frac{1}{2}} - \left|\sum_{i\in I} p_i \langle x_i, y_i\rangle\right|,$$

where $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$, $I \in \mathcal{P}_f(\mathbb{N})$ and $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$.

We observe that, for a $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\varnothing\}$, the functional $\langle\cdot,\cdot\rangle_{\mathbf{p},I} \geq \langle\cdot,\cdot\rangle_{\mathbf{q},I}$, provided $\mathbf{p} \geq \mathbf{q} \geq \mathbf{0}$.

Using Theorem 4, we may state the following result.

Proposition 9. *Let $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\varnothing\}$, $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$. Then the functional $\sigma(\cdot, I, \mathbf{x}, \mathbf{y})$ is superadditive and monotonic nondecreasing on $\mathcal{S}_+(\mathbb{R})$.*

If $I, J \in \mathcal{P}_f(\mathbb{N}) \setminus \{\varnothing\}$, with $I \cap J = \varnothing$, for a given $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$, we observe that

$$(1.72)\quad \langle\cdot,\cdot\rangle_{\mathbf{p},I\cup J} = \langle\cdot,\cdot\rangle_{\mathbf{p},I} + \langle\cdot,\cdot\rangle_{\mathbf{p},J}.$$

Taking into account this property and on making use of Theorem 4, we may state the following result.

Proposition 10. *Let* $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$ *and* $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$.

(i) *For any* $I, J \in \mathcal{P}_f(\mathbb{N}) \setminus \{\varnothing\}$, *with* $I \cap J = \varnothing$, *we have*

$$\sigma(\mathbf{p}, I \cup J, \mathbf{x}, \mathbf{y}) \geq \sigma(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) + \sigma(\mathbf{p}, J, \mathbf{x}, \mathbf{y}) \qquad (\geq 0), \tag{1.73}$$

i.e., $\sigma(\mathbf{p}, \cdot, \mathbf{x}, \mathbf{y})$ *is superadditive as an index set mapping on* $\mathcal{P}_f(\mathbb{N})$.

(ii) *If* $\varnothing \neq J \subseteq I$, $I, J \in \mathcal{P}_f(\mathbb{N})$, *then*

$$\sigma(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) \geq \sigma(\mathbf{p}, J, \mathbf{x}, \mathbf{y}) \qquad (\geq 0), \tag{1.74}$$

i.e., $\sigma(\mathbf{p}, \cdot, \mathbf{x}, \mathbf{y})$ *is monotonic nondecreasing as an index set mapping on* $\mathcal{S}_+(\mathbb{R})$.

It is well known that the following Cauchy-Bunyakovsky-Schwarz (CBS) type inequality for sequences of vectors in an inner product space holds true:

$$\sum_{i \in I} p_i \|x_i\|^2 \sum_{i \in I} p_i \|y_i\|^2 \geq \left| \sum_{i \in I} p_i \langle x_i, y_i \rangle \right|^2 \tag{1.75}$$

for $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\varnothing\}$, $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$ and $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$.

If $p_i > 0$ for all $i \in I$, then equality holds in (1.75) if and only if there exists a scalar $\lambda \in \mathbb{K}$ such that $x_i = \lambda y_i$, $i \in I$.

Utilising the above results for the functional σ, we may state the following inequalities related to the (CBS)-inequality (1.75).

(1) Let $\alpha_i \in \mathbb{R}$, $x_i, y_i \in H$, $i \in \{1, \dots, n\}$. Then one has the inequality:

$$\begin{aligned} &\sum_{i=1}^n \|x_i\|^2 \sum_{i=1}^n \|y_i\|^2 - \left| \sum_{i=1}^n \langle x_i, y_i \rangle \right| \\ &\geq \left(\sum_{i=1}^n \|x_i\|^2 \sin^2 \alpha_i \sum_{i=1}^n \|y_i\|^2 \sin^2 \alpha_i \right)^{\frac{1}{2}} - \left| \sum_{i=1}^n \langle x_i, y_i \rangle \sin^2 \alpha_i \right| \\ &\quad + \left(\sum_{i=1}^n \|x_i\|^2 \cos^2 \alpha_i \sum_{i=1}^n \|y_i\|^2 \cos^2 \alpha_i \right)^{\frac{1}{2}} \\ &\quad - \left| \sum_{i=1}^n \langle x_i, y_i \rangle \cos^2 \alpha_i \right| \geq 0. \end{aligned} \tag{1.76}$$

(2) Denote $S_n(\mathbf{1}) := \{\mathbf{p} \in \mathcal{S}_+(\mathbb{R}) \,|\, p_i \le 1 \text{ for all } i \in \{1,\dots,n\}\}$. Then for all $x_i, y_i \in H$, $i \in \{1,\dots,n\}$, we have the bound:

$$\left(\sum_{i=1}^{n} \|x_i\|^2 \sum_{i=1}^{n} \|y_i\|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} \langle x_i, y_i\rangle\right| = \sup_{\mathbf{p} \in S_n(\mathbf{1})} \left[\left(\sum_{i=1}^{n} p_i \|x_i\|^2 \sum_{i=1}^{n} p_i \|y_i\|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right|\right] \ge 0. \tag{1.77}$$

(3) Let $p_i \ge 0$, $x_i, y_i \in H$, $i \in \{1,\dots,n\}$. Then we have the inequality:

$$\begin{aligned}\left(\sum_{i=1}^{2n} p_i \|x_i\|^2 \sum_{i=1}^{2n} p_i \|y_i\|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{2n} p_i \langle x_i, y_i\rangle\right| &\\ \ge \left(\sum_{k=1}^{n} p_{2k} \|x_{2k}\|^2 \sum_{k=1}^{n} p_{2k} \|y_{2k}\|^2\right)^{\frac{1}{2}} - \left|\sum_{k=1}^{n} p_{2k} \langle x_{2k}, y_{2k}\rangle\right| &\\ + \left(\sum_{k=1}^{n} p_{2k-1} \|x_{2k-1}\|^2 \sum_{k=1}^{n} p_{2k-1} \|y_{2k-1}\|^2\right)^{\frac{1}{2}} &\\ - \left|\sum_{k=1}^{n} p_{2k-1} \langle x_{2k-1}, y_{2k-1}\rangle\right| \qquad (\ge 0)&.\end{aligned} \tag{1.78}$$

(4) We have the bound:

$$\left[\sum_{i=1}^{n} p_i \|x_i\|^2 \sum_{i=1}^{n} p_i \|y_i\|^2\right]^{\frac{1}{2}} - \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right| = \sup_{\varnothing \ne I \subseteq \{1,\dots,n\}} \left(\left[\sum_{i\in I} p_i \|x_i\|^2 \sum_{i\in I} p_i \|y_i\|^2\right]^{\frac{1}{2}} - \left|\sum_{i\in I} p_i \langle x_i, y_i\rangle\right|\right) \ge 0. \tag{1.79}$$

(5) The sequence S_n given by

$$S_n := \left(\sum_{i=1}^{n} p_i \|x_i\|^2 \sum_{i=1}^{n} p_i \|y_i\|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right|$$

is nondecreasing, i.e.,

$$S_{k+1} \ge S_k, \quad k \ge 2 \tag{1.80}$$

and we have the bound

$$S_n \geq \max_{1\leq i<j\leq n} \Big\{ \left(p_i \|x_i\|^2 + p_j \|x_j\|^2\right)^{\frac{1}{2}} \left(p_i \|y_i\|^2 + p_j \|y_j\|^2\right)^{\frac{1}{2}} - |p_i \langle x_i, y_i\rangle + p_j \langle x_j, y_j\rangle| \Big\} \geq 0, \tag{1.81}$$

for $n \geq 2$ and $x_i, y_i \in H$, $i \in \{1, \dots, n\}$.

Remark 5. *The results in this subsection have been obtained by Dragomir and Mond in* [**1**] *for the particular case of scalar sequences* $\mathbf{x}$ *and* $\mathbf{y}$.

1.5.2. The Case of Mapping δ. Under the assumptions of the above subsection, we can define the following functional

$$\delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) := \sum_{i\in I} p_i \|x_i\|^2 \sum_{i\in I} p_i \|y_i\|^2 - \left|\sum_{i\in I} p_i \langle x_i, y_i\rangle\right|^2,$$

where $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$, $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\varnothing\}$ and $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$.

Utilising Theorem 5, we may state the following results.

Proposition 11. *We have*

(i) *For any* $\mathbf{p}, \mathbf{q} \in \mathcal{S}_+(\mathbb{R})$, $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\varnothing\}$ *and* $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$ *we have*

$$\begin{aligned} &\delta(\mathbf{p}+\mathbf{q}, I, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{q}, I, \mathbf{x}, \mathbf{y}) \\ &\geq \left(\det \begin{bmatrix} \left(\sum_{i\in I} p_i \|x_i\|^2\right)^{\frac{1}{2}} & \left(\sum_{i\in I} p_i \|y_i\|^2\right)^{\frac{1}{2}} \\ \left(\sum_{i\in I} q_i \|x_i\|^2\right)^{\frac{1}{2}} & \left(\sum_{i\in I} q_i \|y_i\|^2\right)^{\frac{1}{2}} \end{bmatrix}\right)^2 \geq 0. \end{aligned} \tag{1.82}$$

(ii) *If* $\mathbf{p} \geq \mathbf{q} \geq \mathbf{0}$, *then*

$$\begin{aligned} &\delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{q}, I, \mathbf{x}, \mathbf{y}) \\ &\geq \left(\det \begin{bmatrix} \left(\sum_{i\in I} p_i \|x_i\|^2\right)^{\frac{1}{2}} & \left(\sum_{i\in I} p_i \|y_i\|^2\right)^{\frac{1}{2}} \\ \left(\sum_{i\in I} (p_i - q_i) \|x_i\|^2\right)^{\frac{1}{2}} & \left(\sum_{i\in I} (p_i - q_i) \|y_i\|^2\right)^{\frac{1}{2}} \end{bmatrix}\right)^2 \geq 0. \end{aligned} \tag{1.83}$$

Proposition 12. *We have*

(i) *For any* $I, J \in \mathcal{P}_f(\mathbb{N})$, *with* $I \cap J = \varnothing$ *and* $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$, $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$, *we have*

$$\delta(\mathbf{p}, I \cup J, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{p}, J, \mathbf{x}, \mathbf{y}) \\ \geq \left(\det\begin{bmatrix} \left(\sum_{i\in I} p_i \|x_i\|^2\right)^{\frac{1}{2}} & \left(\sum_{i\in I} p_i \|y_i\|^2\right)^{\frac{1}{2}} \\ \left(\sum_{i\in J} p_i \|x_i\|^2\right)^{\frac{1}{2}} & \left(\sum_{i\in J} p_i \|y_i\|^2\right)^{\frac{1}{2}} \end{bmatrix}\right)^2 \geq 0. \tag{1.84}$$

(ii) *If* $\varnothing \neq J \subseteq I$, $I \neq J$, $I, J \in \mathcal{P}_f(\mathbb{N})$, *then we have*

$$\delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{p}, J, \mathbf{x}, \mathbf{y}) \\ \geq \left(\det\begin{bmatrix} \left(\sum_{i\in I} p_i \|x_i\|^2\right)^{\frac{1}{2}} & \left(\sum_{i\in I} p_i \|y_i\|^2\right)^{\frac{1}{2}} \\ \left(\sum_{i\in I\setminus J} p_i \|x_i\|^2\right)^{\frac{1}{2}} & \left(\sum_{i\in I\setminus J} p_i \|y_i\|^2\right)^{\frac{1}{2}} \end{bmatrix}\right)^2 \geq 0. \tag{1.85}$$

The following particular instances that provide refinements for the (CBS)-inequality may be stated as well:

$$\begin{aligned} &\sum_{i\in I} \|x_i\|^2 \sum_{i\in I} \|y_i\|^2 - \left|\sum_{i\in I} \langle x_i, y_i\rangle\right|^2 \\ &\geq \sum_{i\in I} \|x_i\|^2 \sin^2\alpha_i \sum_{i\in I} \|y_i\|^2 \sin^2\alpha_i - \left|\sum_{i\in I} \langle x_i, y_i\rangle \sin^2\alpha_i\right|^2 \\ &\quad + \sum_{i\in I} \|x_i\|^2 \cos^2\alpha_i \sum_{i\in I} \|y_i\|^2 \cos^2\alpha_i \\ &\quad - \left|\sum_{i\in I} \langle x_i, y_i\rangle \cos^2\alpha_i\right|^2 \\ &\geq \left(\det\begin{bmatrix} \left(\sum_{i\in I} \|x_i\|^2 \sin^2\alpha_i\right)^{\frac{1}{2}} & \left(\sum_{i\in I} \|y_i\|^2 \sin^2\alpha_i\right)^{\frac{1}{2}} \\ \left(\sum_{i\in I} \|x_i\|^2 \cos^2\alpha_i\right)^{\frac{1}{2}} & \left(\sum_{i\in I} \|y_i\|^2 \cos^2\alpha_i\right)^{\frac{1}{2}} \end{bmatrix}\right)^2 \\ &\geq 0, \end{aligned} \tag{1.86}$$

where $x_i, y_i \in H$, $\alpha_i \in \mathbb{R}$, $i \in I$ and $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\varnothing\}$.

Suppose that $p_i \geq 0$, $x_i, y_i \in H$, $i \in \{1, \ldots, 2n\}$. Then

$$\begin{aligned}
(1.87)\quad & \sum_{i=1}^{2n} p_i \|x_i\|^2 \sum_{i=1}^{2n} p_i \|y_i\|^2 - \left| \sum_{i=1}^{2n} p_i \langle x_i, y_i \rangle \right|^2 \\
& \geq \sum_{k=1}^{n} p_{2k} \|x_{2k}\|^2 \sum_{k=1}^{n} p_{2k} \|y_{2k}\|^2 - \left| \sum_{k=1}^{n} p_{2k} \langle x_{2k}, y_{2k} \rangle \right|^2 \\
& \quad + \sum_{k=1}^{n} p_{2k-1} \|x_{2k-1}\|^2 \sum_{k=1}^{n} p_{2k-1} \|y_{2k-1}\|^2 \\
& \quad - \left| \sum_{k=1}^{n} p_{2k-1} \langle x_{2k-1}, y_{2k-1} \rangle \right|^2 \\
& \geq \left(\det \begin{bmatrix} \left(\sum_{k=1}^{n} p_{2k} \|x_{2k}\|^2 \right)^{\frac{1}{2}} & \left(\sum_{k=1}^{n} p_{2k} \|y_{2k}\|^2 \right)^{\frac{1}{2}} \\ \left(\sum_{k=1}^{n} p_{2k-1} \|x_{2k-1}\|^2 \right)^{\frac{1}{2}} & \left(\sum_{k=1}^{n} p_{2k-1} \|y_{2k-1}\|^2 \right)^{\frac{1}{2}} \end{bmatrix} \right)^2 \\
& \geq 0.
\end{aligned}$$

Remark 6. *The above results (1.82) – (1.87) have been obtained for the case where* $\mathbf{x}$ *and* $\mathbf{y}$ *are real or complex numbers by Dragomir and Mond* [**1**].

Further, if we use Corollaries 2 and 1, then we can state the following propositions as well.

Proposition 13. *We have*

(i) *For any* $\mathbf{p}, \mathbf{q} \in \mathcal{S}_+(\mathbb{R})$, $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\varnothing\}$ *and* $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H) \setminus \{0\}$ *we have*

$$\begin{aligned}
(1.88)\quad & \delta(\mathbf{p}+\mathbf{q}, I, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{q}, I, \mathbf{x}, \mathbf{y}) \\
& \geq \max \left\{ \frac{\sum_{i \in I} p_i \|x_i\|^2}{\sum_{i \in I} q_i \|x_i\|^2} \delta(\mathbf{q}, I, \mathbf{x}, \mathbf{y}) + \frac{\sum_{i \in I} q_i \|x_i\|^2}{\sum_{i \in I} p_i \|x_i\|^2} \delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}), \right. \\
& \left. \frac{\sum_{i \in I} p_i \|y_i\|^2}{\sum_{i \in I} q_i \|y_i\|^2} \delta(\mathbf{q}, I, \mathbf{x}, \mathbf{y}) + \frac{\sum_{i \in I} q_i \|y_i\|^2}{\sum_{i \in I} p_i \|y_i\|^2} \delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) \right\} \geq 0.
\end{aligned}$$

(ii) *If* $\mathbf{p} \geq \mathbf{q} \geq \mathbf{0}$ *and* $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\varnothing\}$, $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H) \setminus \{0\}$, *then:*

$$(1.89) \quad \delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{q}, I, \mathbf{x}, \mathbf{y}) \\ \geq \max \left\{ \frac{\sum_{i \in I} (p_i - q_i) \|x_i\|^2}{\sum_{i \in I} p_i \|x_i\|^2}, \frac{\sum_{i \in I} (p_i - q_i) \|y_i\|^2}{\sum_{i \in I} p_i \|y_i\|^2} \right\} \delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) \geq 0.$$

Proposition 14. *We have*

(i) *For any* $I, J \in \mathcal{P}_f(\mathbb{N}) \setminus \{\varnothing\}$, *with* $I \cap J = \varnothing$ *and* $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$, $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H) \setminus \{0\}$, *we have*

$$(1.90) \quad \delta(\mathbf{p}, I \cup J, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{p}, J, \mathbf{x}, \mathbf{y}) \\ \geq \max \left\{ \frac{\sum_{i \in I} p_i \|x_i\|^2}{\sum_{j \in J} p_j \|x_j\|^2} \delta(\mathbf{p}, J, \mathbf{x}, \mathbf{y}) + \frac{\sum_{j \in J} p_j \|x_j\|^2}{\sum_{i \in I} p_i \|x_i\|^2} \delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}), \right. \\ \left. \frac{\sum_{i \in I} p_i \|y_i\|^2}{\sum_{j \in J} p_j \|y_j\|^2} \delta(\mathbf{p}, J, \mathbf{x}, \mathbf{y}) + \frac{\sum_{j \in J} p_j \|y_j\|^2}{\sum_{i \in I} p_i \|y_i\|^2} \delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) \right\} \geq 0.$$

(ii) *If* $\varnothing \neq J \subseteq I$, $I \neq J$, $I, J \in \mathcal{P}_f(\mathbb{N}) \setminus \{\varnothing\}$ *and* $\mathbf{p} \in \mathcal{S}_+(\mathbb{R}) \setminus \{0\}$, $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H) \setminus \{0\}$, *then*

$$(1.91) \quad \delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{p}, J, \mathbf{x}, \mathbf{y}) \\ \geq \max \left\{ \frac{\sum_{k \in I \setminus J} p_k \|x_k\|^2}{\sum_{i \in I} p_i \|x_i\|^2}, \frac{\sum_{k \in I \setminus J} p_k \|y_k\|^2}{\sum_{i \in I} p_i \|y_i\|^2} \right\} \delta(\mathbf{p}, J, \mathbf{x}, \mathbf{y}) \geq 0.$$

Remark 7. *The results in Proposition 13 have been obtained by Dragomir and Mond in* [**2**] *for the case of scalar sequences* $\mathbf{x}$ *and* $\mathbf{y}$.

1.5.3. The Case of Mapping β**.** With the assumptions in the first subsections, we can define the following functional

$$\beta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) := [\delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y})]^{\frac{1}{2}} \\ = \left[\sum_{i \in I} p_i \|x_i\|^2 \sum_{i \in I} p_i \|y_i\|^2 - \left| \sum_{i \in I} p_i \langle x_i, y_i \rangle \right|^2 \right]^{\frac{1}{2}},$$

where $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$, $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\varnothing\}$ and $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$.

Utilising Theorem 7, we can state the following results:

Proposition 15. *We have*

(i) *The functional* $\beta(\cdot, I, \mathbf{x}, \mathbf{y})$ *is superadditive on* $\mathcal{S}_+(\mathbb{R})$ *for any* $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\varnothing\}$ *and* $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$.

(ii) *The functional* $\beta(\mathbf{p}, \cdot, \mathbf{x}, \mathbf{y})$ *is superadditive as an index set mapping on* $\mathcal{P}_f(\mathbb{N})$ *and* $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$.

As simple consequences of the above proposition, we may state the following refinements of the (CBS)-inequality.

(a) If $\mathbf{x},\mathbf{y}\in\mathcal{S}(H)$ and $\alpha_i\in\mathbb{R}$, $i\in I$ with $I\in\mathcal{P}_f(\mathbb{N})\setminus\{\varnothing\}$, then

$$\begin{aligned}(1.92)\quad &\left(\sum_{i\in I}\|x_i\|^2\sum_{i\in I}\|y_i\|^2-\left|\sum_{i\in I}\langle x_i,y_i\rangle\right|^2\right)^{\frac{1}{2}}\\ &\geq\left(\sum_{i\in I}\|x_i\|^2\sin^2\alpha_i\sum_{i\in I}\|y_i\|^2\sin^2\alpha_i-\left|\sum_{i\in I}\langle x_i,y_i\rangle\sin^2\alpha_i\right|^2\right)^{\frac{1}{2}}\\ &+\left(\sum_{i\in I}\|x_i\|^2\cos^2\alpha_i\sum_{i\in I}\|y_i\|^2\cos^2\alpha_i-\left|\sum_{i\in I}\langle x_i,y_i\rangle\cos^2\alpha_i\right|^2\right)^{\frac{1}{2}}\geq 0.\end{aligned}$$

(b) If $x_i,y_i\in H$, $p_i>0$, $i\in\{1,\dots,2n\}$, then

$$\begin{aligned}(1.93)\quad &\left(\sum_{i=1}^{2n}p_i\|x_i\|^2\sum_{i=1}^{2n}p_i\|y_i\|^2-\left|\sum_{i=1}^{2n}p_i\langle x_i,y_i\rangle\right|^2\right)^{\frac{1}{2}}\\ &\geq\left(\sum_{k=1}^{n}p_{2k}\|x_{2k}\|^2\sum_{k=1}^{n}p_{2k}\|y_{2k}\|^2-\left|\sum_{k=1}^{n}p_{2k}\langle x_{2k},y_{2k}\rangle\right|^2\right)^{\frac{1}{2}}\\ &\quad+\left(\sum_{k=1}^{n}p_{2k-1}\|x_{2k-1}\|^2\sum_{k=1}^{n}p_{2k-1}\|y_{2k-1}\|^2\right.\\ &\qquad\left.-\left|\sum_{k=1}^{n}p_{2k-1}\langle x_{2k-1},y_{2k-1}\rangle\right|^2\right)^{\frac{1}{2}}\qquad(\geq 0).\end{aligned}$$

Remark 8. *Part (i) of Proposition 15 and the inequality (1.91) have been obtained by Dragomir and Mond in* [**2**] *for the case of scalar sequences* $\mathbf{x}$ *and* $\mathbf{y}$.

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CHAPTER 2

Schwarz Related Inequalities

2.1. Introduction

Let H be a linear space over the real or complex number field $\mathbb{K}$. The functional $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{K}$ is called an *inner product* on H if it satisfies the conditions

(i) $\langle x, x \rangle \geq 0$ for any $x \in H$ and $\langle x, x \rangle = 0$ iff $x = 0$;
(ii) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for any $\alpha, \beta \in \mathbb{K}$ and $x, y, z \in H$;
(iii) $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for any $x, y \in H$.

A first fundamental consequence of the properties (i)-(iii) above, is the *Schwarz inequality:*

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle , \tag{2.1}$$

for any $x, y \in H$. The equality holds in (2.1) if and only if the vectors x and y are *linearly dependent,* i.e., there exists a nonzero constant $\alpha \in \mathbb{K}$ so that $x = \alpha y$.

If we denote $\|x\| := \sqrt{\langle x, x \rangle}, x \in H$, then one may state the following properties

(n) $\|x\| \geq 0$ for any $x \in H$ and $\|x\| = 0$ iff $x = 0$;
(nn) $\|\alpha x\| = |\alpha| \|x\|$ for any $\alpha \in \mathbb{K}$ and $x \in H$;
(nnn) $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in H$ (the triangle inequality);

i.e., $\|\cdot\|$ is a *norm* on H.

In this chapter we present some classical and recent refinements and reverse inequalities for the Schwarz and the triangle inequalities. More precisely, we point out upper bounds or positive lower bounds for the nonnegative quantities

$$\|x\| \|y\| - |\langle x, y \rangle| , \quad \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2$$

and

$$\|x\| + \|y\| - \|x + y\|$$

under various assumptions for the vectors $x, y \in H$.

If the vectors $x, y \in H$ are not *orthogonal*, i.e., $\langle x, y\rangle \neq 0$, then some upper and lower bounds for the supra-unitary quantities

$$\frac{\|x\|\,\|y\|}{|\langle x, y\rangle|}, \quad \frac{\|x\|^2\,\|y\|^2}{|\langle x, y\rangle|^2}$$

under appropriate restrictions for the vectors x and y are provided as well.

The inequalities obtained by Buzano, Richards, Precupanu and Moore and their extensions and generalizations for orthonormal families of vectors in both real and complex inner product spaces are presented. Recent results concerning the classical refinement of Schwarz inequality due to Kurepa for the complexification of real inner product spaces are also reviewed. Various applications for integral inequalities including a version of Heisenberg inequality for vector valued functions in Hilbert spaces are provided as well.

2.2. Inequalities Related to Schwarz's One

2.2.1. Some Refinements. The following result holds [**15**, Theorem 1] (see also [**18**, Theorem 2]).

THEOREM 10 (Dragomir, 1985). *Let $(H, \langle\cdot,\cdot\rangle)$ be a real or complex inner product space. Then*

$$\left(\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2\right)\left(\|y\|^2\|z\|^2 - |\langle y, z\rangle|^2\right) \geq \left|\langle x, z\rangle\|y\|^2 - \langle x, y\rangle\langle y, z\rangle\right|^2 \tag{2.2}$$

for any $x, y, z \in H$.

PROOF. We follow the proof in [**15**].

Let us consider the mapping

$$p_y : H \times H \to \mathbb{K}, \qquad p_y(x, z) = \langle x, z\rangle\|y\|^2 - \langle x, y\rangle\langle y, z\rangle$$

for each $y \in H\setminus\{0\}$.

It is easy to see that $p_y(\cdot,\cdot)$ is a nonnegative Hermitian form and then on writing Schwarz's inequality

$$|p_y(x, z)|^2 \leq p_y(x, x)\,p_y(z, z), \qquad x, z \in H$$

we obtain the desired inequality (2.2). ∎

REMARK 9. *From (2.2) it follows that* [**15**, Corollary 1] *(see also* [**18**, Corollary 2.1]*)*

$$\left(\|x+z\|^2\|y\|^2 - |\langle x+z, y\rangle|^2\right)^{\frac{1}{2}} \leq \left(\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2\right)^{\frac{1}{2}} + \left(\|y\|^2\|z\|^2 - |\langle y, z\rangle|^2\right)^{\frac{1}{2}} \tag{2.3}$$

for every $x, y, z \in H$.

Putting $z = \lambda y$ *in (2.3), we get:*

$$0 \leq \|x + \lambda y\|^2 \|y\|^2 - |\langle x + \lambda y, y\rangle|^2 \leq \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \tag{2.4}$$

and, in particular,

$$0 \leq \|x \pm y\|^2 \|y\|^2 - |\langle x \pm y, y\rangle|^2 \leq \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \tag{2.5}$$

for every $x, y \in H$.

Both inequalities (2.4) and (2.5) have been obtained in [**15**].

We note here that the inequality (2.4) is in fact equivalent to the following statement

$$\sup_{\lambda \in \mathbb{K}} \left[\|x + \lambda y\|^2 \|y\|^2 - |\langle x + \lambda y, y\rangle|^2\right] = \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \tag{2.6}$$

for each $x, y \in H$.

The following corollary may be stated [**15**, Corollary 2] (see also [**18**, Corollary 2.2]):

Corollary 3 (Dragomir, 1985). *For any* $x, y, z \in H \backslash \{0\}$ *we have the inequality*

$$\left|\frac{\langle x, y\rangle}{\|x\| \|y\|}\right|^2 + \left|\frac{\langle y, z\rangle}{\|y\| \|z\|}\right|^2 + \left|\frac{\langle z, x\rangle}{\|z\| \|x\|}\right|^2 \leq 1 + 2\left|\frac{\langle x, y\rangle \langle y, z\rangle \langle z, x\rangle}{\|x\|^2 \|y\|^2 \|z\|^2}\right|. \tag{2.7}$$

Proof. By the modulus properties we obviously have

$$\left|\langle x, z\rangle \|y\|^2 - \langle x, y\rangle \langle y, z\rangle\right| \geq \left||\langle x, z\rangle| \|y\|^2 - |\langle x, y\rangle| |\langle y, z\rangle|\right|.$$

Therefore, by (2.2) we may state that

$$\begin{aligned} &\left(\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2\right)\left(\|y\|^2 \|z\|^2 - |\langle y, z\rangle|^2\right) \\ &\qquad \geq |\langle x, z\rangle|^2 \|y\|^4 - 2|\langle x, y\rangle \langle y, z\rangle \langle z, x\rangle| \|y\|^2 + |\langle x, y\rangle|^2 |\langle y, z\rangle|^2, \end{aligned}$$

which, upon elementary calculation, is equivalent to (2.7). ∎

Remark 10. *If we utilise the elementary inequality* $a^2 + b^2 + c^2 \geq 3abc$ *when* $a, b, c \geq 0$, *then one can state the following inequality*

$$3\left|\frac{\langle x, y\rangle \langle y, z\rangle \langle z, x\rangle}{\|x\|^2 \|y\|^2 \|z\|^2}\right| \leq \left|\frac{\langle x, y\rangle}{\|x\| \|y\|}\right|^2 + \left|\frac{\langle y, z\rangle}{\|y\| \|z\|}\right|^2 + \left|\frac{\langle z, x\rangle}{\|z\| \|x\|}\right|^2 \tag{2.8}$$

for any $x, y, z \in H \backslash \{0\}$. *Therefore, the inequality (2.7) may be regarded as a reverse inequality of (2.8).*

The following refinement of the Schwarz inequality holds [**15**, Theorem 2] (see also [**18**, Corollary 1.1]):

THEOREM 11 (Dragomir, 1985). *For any $x, y \in H$ and $e \in H$ with $\|e\| = 1$, the following refinement of the Schwarz inequality holds:*

$$\|x\| \|y\| \geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| + |\langle x, e\rangle \langle e, y\rangle| \geq |\langle x, y\rangle| . \tag{2.9}$$

PROOF. We follow the proof in [**15**].

Applying the inequality (2.2), we can state that

$$\left(\|x\|^2 - |\langle x, e\rangle|^2\right) \left(\|y\|^2 - |\langle y, e\rangle|^2\right) \geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle|^2 . \tag{2.10}$$

Utilising the elementary inequality for real numbers

$$\left(m^2 - n^2\right) \left(p^2 - q^2\right) \leq (mp - nq)^2 , \tag{2.11}$$

we can easily see that

$$\begin{aligned} (\|x\| \|y\| - |\langle x, e\rangle \langle e, y\rangle|)^2 \\ \geq \left(\|x\|^2 - |\langle x, e\rangle|^2\right) \left(\|y\|^2 - |\langle y, e\rangle|^2\right) \end{aligned} \tag{2.12}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Since, by Schwarz's inequality

$$\|x\| \|y\| \geq |\langle x, e\rangle \langle e, y\rangle| \tag{2.13}$$

hence, by (2.10) and (2.12) we deduce the first part of (2.10).

The second part of (2.10) is obvious. ∎

COROLLARY 4 (Dragomir, 1985). *If $x, y, e \in H$ are such that $\|e\| = 1$ and $x \perp y$, then*

$$\|x\| \|y\| \geq 2 |\langle x, e\rangle \langle e, y\rangle| . \tag{2.14}$$

REMARK 11. *Assume that $A : H \to H$ is a bounded linear operator on H. For $x, e \in H$ with $\|x\| = \|e\| = 1$, we have by (2.9) that*

$$\|Ay\| \geq |\langle x, Ay\rangle - \langle x, e\rangle \langle e, Ay\rangle| + |\langle x, e\rangle \langle e, Ay\rangle| \geq |\langle x, Ay\rangle| \tag{2.15}$$

for any $y \in H$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.15) and noting that $\|Ay\| = \sup_{\|x\|=1} |\langle x, Ay\rangle|$, we deduce the representation

$$\|Ay\| = \sup_{\|x\|=1} \{|\langle x, Ay\rangle - \langle x, e\rangle \langle e, Ay\rangle| + |\langle x, e\rangle \langle e, Ay\rangle|\} \tag{2.16}$$

for any $y \in H$. Finally, on taking the supremum over $y \in H$, $\|y\| = 1$ in (2.16) we get

$$\|A\| = \sup_{\|y\|=1, \|x\|=1} \{|\langle x, Ay\rangle - \langle x, e\rangle \langle e, Ay\rangle| + |\langle x, e\rangle \langle e, Ay\rangle|\} \tag{2.17}$$

for any $e \in H$, $\|e\| = 1$, a representation that has been obtained in [**15**, Eq. 9].

REMARK 12. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space. Then for any continuous linear functional $f:H\to\mathbb{K}$, $f\neq 0$, there exists, by the Riesz representation theorem a unique vector $e\in H\backslash\{0\}$ such that $f(x)=\langle x,e\rangle$ for $x\in H$ and $\|f\|=\|e\|$.*

If E is a nonzero linear subspace of H and if we denote by $E^{\perp}$ its orthogonal complement, i.e., we recall that $E^{\perp}:=\{y\in H|y\perp x\}$ then for any $x\in E$ and $y\in E^{\perp}$, by (2.14) we may state that

$$\|x\|\,\|y\|\geq 2\left|\left\langle x,\frac{e}{\|x\|}\right\rangle\left\langle y,\frac{e}{\|y\|}\right\rangle\right|,$$

giving, for $x,y\neq 0$, that

$$\|f\|^2\geq 2\left|\langle x,e\rangle\langle y,e\rangle\right|=2\left|f(x)\right|\left|f(y)\right| \tag{2.18}$$

for any $x\in E$ and $y\in E^{\perp}$.

If by $\|f\|_E$ we denote the norm of the functional f restricted to E, i.e., $\|f\|_E=\sup_{x\in E\backslash\{0\}}\frac{|f(x)|}{\|x\|}$, then, on taking the supremum over $x\in E$ and $y\in E^{\perp}$ in (2.18) we deduce

$$\|f\|^2\geq 2\|f\|_E\cdot\|f\|_{E^{\perp}} \tag{2.19}$$

for any E a nonzero linear subspace of the Hilbert space H and a given functional $f\in H^{*}\backslash\{0\}$.

We note that the inequality (2.19) has been obtained in [**15**, Eq. 10].

2.2.2. A Conditional Inequality. The following result providing a lower bound for the norm product under suitable conditions holds [**19**] (see also [**18**, Theorem 1]):

THEOREM 12 (Dragomir-Sándor, 1986). *Let $x,y,a,b\in H$, where $(H;\langle\cdot,\cdot\rangle)$ is an inner product space, be such that*

$$\|a\|^2\leq 2\operatorname{Re}\langle x,a\rangle \quad \text{and} \quad \|y\|^2\leq 2\operatorname{Re}\langle y,b\rangle \tag{2.20}$$

holds true. Then

$$\|x\|\,\|y\|\geq\left(2\operatorname{Re}\langle x,a\rangle-\|a\|^2\right)^{\frac{1}{2}}\left(2\operatorname{Re}\langle y,b\rangle-\|b\|^2\right)^{\frac{1}{2}} + \left|\langle x,y\rangle-\langle x,b\rangle-\langle a,y\rangle+\langle a,b\rangle\right|. \tag{2.21}$$

PROOF. We follow the proof in [**19**].

Observe that

$$\begin{aligned}(2.22)\quad & |\langle x,y\rangle-\langle x,b\rangle-\langle a,y\rangle+\langle a,b\rangle| \\ & =|\langle x-a,y-b\rangle|^2 \\ & \le \|x-a\|^2\|y-b\|^2 \\ & =\left[\|x\|^2-\left(2\operatorname{Re}\langle x,a\rangle-\|a\|^2\right)\right]\left[\|y\|^2-\left(2\operatorname{Re}\langle y,b\rangle-\|b\|^2\right)\right].\end{aligned}$$

Applying the elementary inequality (2.11) we have

$$\begin{aligned}(2.23)\quad & \left\{\|x\|^2-\left[\left(2\operatorname{Re}\langle x,a\rangle-\|a\|^2\right)^{\frac{1}{2}}\right]^2\right\} \\ & \qquad\times\left\{\|y\|^2-\left[\left(2\operatorname{Re}\langle y,b\rangle-\|b\|^2\right)^{\frac{1}{2}}\right]^2\right\} \\ & \le\left[\|x\|\|y\|-\left(2\operatorname{Re}\langle x,a\rangle-\|a\|^2\right)^{\frac{1}{2}}\left(2\operatorname{Re}\langle y,b\rangle-\|b\|^2\right)^{\frac{1}{2}}\right].\end{aligned}$$

Since

$$0\le\left(2\operatorname{Re}\langle x,a\rangle-\|a\|^2\right)^{\frac{1}{2}}\le\|x\|\quad\text{and}$$
$$0\le\left(2\operatorname{Re}\langle y,b\rangle-\|b\|^2\right)^{\frac{1}{2}}\le\|y\|$$

hence

$$\|x\|\|y\|\ge\left(2\operatorname{Re}\langle x,a\rangle-\|a\|^2\right)^{\frac{1}{2}}\left(2\operatorname{Re}\langle y,b\rangle-\|b\|^2\right)^{\frac{1}{2}}$$

and by (2.22) and (2.23) we deduce the desired result (2.21). ∎

Remark 13. *As pointed out in* [**19**], *if we consider* $a=\langle x,e\rangle e$, $b=\langle y,e\rangle e$ *with* $e\in H$, $\|e\|=1$, *then the condition (2.20) is obviously satisfied and the inequality (2.21) becomes*

$$\begin{aligned}(2.24)\qquad \|x\|\|y\| & \ge|\langle x,e\rangle\langle e,y\rangle|+|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle| \\ & \quad(\ge|\langle x,y\rangle|),\end{aligned}$$

which is the refinement of the Schwarz inequality incorporated in (2.9).

For vectors located in a closed ball centered at 0 and of radius $\sqrt{2}$, one can state the following corollary as well [**18**, Corollary 1.2].

Corollary 5. *Let* $x,y\in H$ *such that* $\|x\|,\|y\|\le\sqrt{2}$. *Then*

$$\begin{aligned}(2.25)\quad \|x\|\|y\| & \ge|\langle x,y\rangle|^2\left(2-\|x\|^2\right)^{\frac{1}{2}}\left(2-\|y\|^2\right)^{\frac{1}{2}} \\ & \qquad+|\langle x,y\rangle|\left|1-\|x\|^2-\|y\|^2+|\langle x,y\rangle|^2\right|.\end{aligned}$$

Proof. Follows by Theorem 12 on choosing $a=\langle x,y\rangle y$, $b=\langle y,x\rangle x$. We omit the details. ∎

2.2.3. A Refinement for Orthonormal Families. The following result provides a generalisation for a refinement of the Schwarz inequality incorporated in (2.9) [**15**, Theorem 3] (see also [**8**, Theorem] or [**18**, Theorem 3]):

THEOREM 13 (Dragomir, 1985). *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $\{e_i\}_{i \in I}$ an orthonormal family in I. For any F a nonempty finite part of I we have the following refinement of Schwarz's inequality:*

$$\begin{aligned} \|x\| \|y\| &\geq \left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right| + \sum_{i \in F} |\langle x, e_i \rangle \langle e_i, y \rangle| \\ &\geq \left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right| + \left| \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right| \\ &\geq |\langle x, y \rangle| , \end{aligned} \tag{2.26}$$

where $x, y \in H$.

PROOF. We follow the proof in [**15**].
We apply the Schwarz inequality to obtain

$$\begin{aligned} &\left| \left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, \quad y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle \right|^2 \\ &\qquad \leq \left\| x - \sum_{i \in F} \langle x, e_i \rangle e_i \right\|^2 \left\| y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\|^2 . \end{aligned} \tag{2.27}$$

Since a simple calculation with orthonormal vectors shows that

$$\left\| x - \sum_{i \in F} \langle x, e_i \rangle e_i \right\|^2 = \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 ,$$
$$\left\| y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\|^2 = \|y\|^2 - \sum_{i \in F} |\langle y, e_i \rangle|^2 ,$$

and

$$\left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, \quad y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle = \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle ,$$

hence (2.27) is equivalent to

$$\left|\langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right|^2 \leq \left(\|x\|^2-\sum_{i\in F}|\langle x,e_i\rangle|^2\right)\left(\|y\|^2-\sum_{i\in F}|\langle y,e_i\rangle|^2\right) \tag{2.28}$$

for any $x,y\in H$.

Further, we need the following Aczél type inequality

$$\left(\alpha^2-\sum_{i\in F}\alpha_i^2\right)\left(\beta^2-\sum_{i\in F}\beta_i^2\right)\leq\left(\alpha\beta-\sum_{i\in F}\alpha_i\beta_i\right)^2, \tag{2.29}$$

provided that $\alpha^2\geq\sum_{i\in F}\alpha_i^2$ and $\beta^2\geq\sum_{i\in F}\beta_i^2$, where $\alpha,\beta,\alpha_i,\beta_i\in\mathbb{R}$, $i\in F$.

For an Aczél inequality that holds under slightly weaker conditions and a different proof based on polynomials, see [**26**, p. 57].

For the sake of completeness, we give here a direct proof of (2.29).

Utilising the elementary inequality (2.11), we can write

$$\left(\alpha^2-\left[\left(\sum_{i\in F}\alpha_i^2\right)^{\frac{1}{2}}\right]^2\right)\left(\beta^2-\left[\left(\sum_{i\in F}\beta_i^2\right)^{\frac{1}{2}}\right]^2\right) \leq\left[|\alpha\beta|-\left(\sum_{i\in F}\alpha_i^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}\beta_i^2\right)^{\frac{1}{2}}\right]^2. \tag{2.30}$$

Since $|\alpha|\geq\left(\sum_{i\in F}\alpha_i^2\right)^{\frac{1}{2}}$ and $|\beta|\geq\left(\sum_{i\in F}\beta_i^2\right)^{\frac{1}{2}}$, then

$$|\alpha\beta|\geq\left(\sum_{i\in F}\alpha_i^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}\beta_i^2\right)^{\frac{1}{2}}.$$

Therefore, by the Cauchy-Bunyakovsky-Schwarz inequality, we have that

$$\begin{aligned}\left| |\alpha\beta| - \left(\sum_{i\in F}\alpha_i^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}\beta_i^2\right)^{\frac{1}{2}}\right| &= |\alpha\beta| - \left(\sum_{i\in F}\alpha_i^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}\beta_i^2\right)^{\frac{1}{2}} \\ &\leq |\alpha\beta| - \left|\sum_{i\in F}\alpha_i\beta_i\right| \\ &= \left| |\alpha\beta| - \left|\sum_{i\in F}\alpha_i\beta_i\right| \right| \\ &\leq \left|\alpha\beta - \sum_{i\in F}\alpha_i\beta_i\right|,\end{aligned}$$

showing that

$$\left[|\alpha\beta| - \left(\sum_{i\in F}\alpha_i^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}\beta_i^2\right)^{\frac{1}{2}}\right]^2 \leq \left(\alpha\beta - \sum_{i\in F}\alpha_i\beta_i\right)^2 \tag{2.31}$$

and then, by (2.30) and (2.31) we deduce the desired result (2.29).

By Bessel's inequality we obviously have that

$$\|x\|^2 \geq \sum_{i\in F}|\langle x, e_i\rangle|^2 \qquad \text{and} \qquad \|y\|^2 \geq \sum_{i\in F}|\langle y, e_i\rangle|^2,$$

therefore, on applying the inequality (2.29) we deduce that

$$\begin{aligned}&\left(\|x\|^2 - \sum_{i\in F}|\langle x, e_i\rangle|^2\right)\left(\|y\|^2 - \sum_{i\in F}|\langle y, e_i\rangle|^2\right) \\ &\qquad\leq \left(\|x\|\,\|y\| - \sum_{i\in F}|\langle x, e_i\rangle\langle e_i, y\rangle|\right)^2.\end{aligned} \tag{2.32}$$

Since $\|x\|\,\|y\| - \sum_{i\in F}|\langle x, e_i\rangle\langle e_i, y\rangle| \geq 0$, hence by (2.28) and (2.32) we deduce the first part of (2.26).

The second and third parts are obvious. ∎

When the vectors are orthogonal, the following result may be stated [**8**] (see also [**18**, Corollary 3.1]).

COROLLARY 6. *If* $\{e_i\}_{i\in I}$ *is an orthonormal family in* $(H,\langle\cdot,\cdot\rangle)$ *and* $x,y\in H$ *with* $x\perp y$, *then we have the inequality:*

$$\|x\|\,\|y\| \geq \left|\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right| + \sum_{i\in F}|\langle x,e_i\rangle\langle e_i,y\rangle| \geq 2\left|\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right|, \tag{2.33}$$

for any nonempty finite part of I.

2.3. Kurepa Type Refinements for the Schwarz Inequality

2.3.1. Kurepa's Inequality. In 1960, N.G. de Bruijn proved the following refinement of the celebrated Cauchy-Bunyakovsky-Schwarz (CBS) inequality for a sequence of real numbers and the second of complex numbers, see [**2**] or [**9**, p. 48]:

THEOREM 14 (de Bruijn, 1960). *Let* $(a_1,\dots,a_n)$ *be an* $n-$*tuple of real numbers and* $(z_1,\dots,z_n)$ *an* $n-$*tuple of complex numbers. Then*

$$\left|\sum_{k=1}^n a_k z_k\right|^2 \leq \frac{1}{2}\sum_{k=1}^n a_k^2\left[\sum_{k=1}^n |z_k|^2 + \left|\sum_{k=1}^n z_k^2\right|\right] \left(\leq \sum_{k=1}^n a_k^2\cdot\sum_{k=1}^n |z_k|^2\right). \tag{2.34}$$

Equality holds in (2.34) if and only if, for $k\in\{1,\dots,n\}$, $a_k = \operatorname{Re}(\lambda z_k)$, *where* λ *is a complex number such that* $\lambda^2\sum_{k=1}^n z_n^2$ *is a non-negative real number.*

In 1966, in an effort to extend this result to inner products, Kurepa [**25**] obtained the following refinement for the complexification of a real inner product space $(H;\langle\cdot,\cdot\rangle)$:

THEOREM 15 (Kurepa, 1966). *Let* $(H;\langle\cdot,\cdot\rangle)$ *be a real inner product space and* $(H_{\mathbb{C}},\langle\cdot,\cdot\rangle_{\mathbb{C}})$ *its complexification. For any* $a\in H$ *and* $z\in H_{\mathbb{C}}$ *we have the inequality:*

$$|\langle z,a\rangle_{\mathbb{C}}|^2 \leq \frac{1}{2}\|a\|^2\left[\|z\|_{\mathbb{C}}^2 + |\langle z,\bar{z}\rangle_{\mathbb{C}}|\right] \left(\leq \|a\|^2\,\|z\|_{\mathbb{C}}^2\right). \tag{2.35}$$

To be comprehensive, we define in the following the concept of complexification for a real inner product space.

Let H be a real inner product space with the scalar product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. The *complexification* $H_{\mathbb{C}}$ of H is defined as a complex linear space $H \times H$ of all ordered pairs (x, y) $(x, y \in H)$ endowed with the operations

$$(x, y) + (x', y') := (x + x', y + y'), \qquad x, x', y, y' \in H;$$
$$(\sigma + i\tau) \cdot (x, y) := (\sigma x - \tau y, \tau x + \sigma y), \qquad x, y \in H \text{ and } \sigma, \tau \in \mathbb{R}.$$

On $H_{\mathbb{C}}$ one can canonically consider the *scalar product* $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ defined by:

$$\langle z, z' \rangle_{\mathbb{C}} := \langle x, x' \rangle + \langle y, y' \rangle + i \left[\langle y, x' \rangle - \langle x, y' \rangle \right]$$

where $z = (x, y)$, $z' = (x', y') \in H_{\mathbb{C}}$. Obviously,

$$\|z\|_{\mathbb{C}}^2 = \|x\|^2 + \|y\|^2,$$

where $z = (x, y)$.

The conjugate of a vector $z = (x, y) \in H_{\mathbb{C}}$ is defined by $\bar{z} := (x, -y)$.

It is easy to see that the elements of $H_{\mathbb{C}}$ under defined operations behave as formal "complex" combinations $x + iy$ with $x, y \in H$. Because of this, we may write $z = x + iy$ instead of $z = (x, y)$. Thus, $\bar{z} = x - iy$.

2.3.2. A Generalisation of Kurepa's Inequality. The following lemma is of interest [**6**].

LEMMA 2. *Let $f : [0, 2\pi] \to \mathbb{R}$ given by*

$$f(\alpha) = \lambda \sin^2 \alpha + 2\beta \sin \alpha \cos \alpha + \alpha \cos^2 \alpha, \tag{2.36}$$

where $\lambda, \beta, \gamma \in \mathbb{R}$. Then

$$\sup_{\alpha \in [0, 2\pi]} f(\alpha) = \frac{1}{2}(\lambda + \gamma) + \frac{1}{2}\left[(\gamma - \lambda)^2 + 4\beta^2\right]^{\frac{1}{2}}. \tag{2.37}$$

PROOF. Since

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}, \quad \cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}, \quad 2 \sin \alpha \cos \alpha = \sin 2\alpha,$$

hence f may be written as

$$f(\alpha) = \frac{1}{2}(\lambda + \gamma) + \frac{1}{2}(\gamma - \lambda) \cos 2\alpha + \beta \sin 2\alpha. \tag{2.38}$$

If $\beta = 0$, then (2.38) becomes

$$f(\alpha) = \frac{1}{2}(\lambda + \gamma) + \frac{1}{2}(\gamma - \lambda) \cos 2\alpha.$$

Obviously, in this case

$$\sup_{\alpha \in [0, 2\pi]} f(\alpha) = \frac{1}{2}(\lambda + \gamma) + \frac{1}{2}|\gamma - \lambda| = \max\{\gamma, \lambda\}.$$

If $\beta \neq 0$, then (2.38) becomes

$$f(\alpha) = \frac{1}{2}(\lambda + \gamma) + \beta \left[\sin 2\alpha + \frac{(\gamma - \lambda)}{\beta} \cos 2\alpha \right].$$

Let $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which $\tan \varphi = \frac{\gamma - \lambda}{2\beta}$. Then f can be written as

$$f(\alpha) = \frac{1}{2}(\lambda + \gamma) + \frac{\beta}{\cos \varphi} \sin(2\alpha + \varphi).$$

For this function, obviously

$$\sup_{\alpha \in [0,2\pi]} f(\alpha) = \frac{1}{2}(\lambda + \gamma) + \frac{|\beta|}{|\cos \varphi|}. \tag{2.39}$$

Since

$$\frac{\sin^2 \varphi}{\cos^2 \varphi} = \frac{(\gamma - \lambda)^2}{4\beta^2},$$

hence,

$$\frac{1}{|\cos \varphi|} = \frac{\left[(\gamma - \lambda)^2 + 4\beta^2\right]^{\frac{1}{2}}}{2|\beta|},$$

and from (2.39) we deduce the desired result (2.37). ∎

The following result holds [**6**].

Theorem 16 (Dragomir, 2004). *Let $(H; \langle \cdot, \cdot \rangle)$ be a complex inner product space. If $x, y, z \in H$ are such that*

$$\operatorname{Im} \langle x, z \rangle = \operatorname{Im} \langle y, z \rangle = 0, \tag{2.40}$$

then we have the inequality:

$$\begin{aligned} \operatorname{Re}^2 \langle x, z \rangle + \operatorname{Re}^2 \langle y, z \rangle &= |\langle x + iy, z \rangle|^2 \\ &\leq \frac{1}{2} \left\{ \|x\|^2 + \|y\|^2 + \left[\left(\|x\|^2 - \|y\|^2 \right)^2 - 4 \operatorname{Re}^2 \langle x, y \rangle \right]^{\frac{1}{2}} \right\} \|z\|^2 \\ &\leq \left(\|x\|^2 + \|y\|^2 \right) \|z\|^2 . \end{aligned} \tag{2.41}$$

Proof. Obviously, by (2.40), we have

$$\langle x + iy, z \rangle = \operatorname{Re} \langle x, z \rangle + i \operatorname{Re} \langle y, z \rangle$$

and the first part of (2.41) holds true.

Now, let $\varphi \in [0, 2\pi]$ be such that $\langle x + iy, z \rangle = e^{i\varphi} |\langle x + iy, z \rangle|$. Then

$$|\langle x + iy, z \rangle| = e^{-i\varphi} \langle x + iy, z \rangle = \left\langle e^{-i\varphi}(x + iy), z \right\rangle.$$

Utilising the above identity, we can write:

$$\begin{aligned}
|\langle x+iy,z\rangle| &= \operatorname{Re}\left\langle e^{-i\varphi}(x+iy),z\right\rangle \\
&= \operatorname{Re}\left\langle (\cos\varphi - i\sin\varphi)(x+iy),z\right\rangle \\
&= \operatorname{Re}\langle \cos\varphi\cdot x+\sin\varphi\cdot y - i\sin\varphi\cdot x + i\cos\varphi\cdot y, z\rangle \\
&= \operatorname{Re}\langle \cos\varphi\cdot x+\sin\varphi\cdot y, z\rangle + \operatorname{Im}\langle \sin\varphi\cdot x-\cos\varphi\cdot y, z\rangle \\
&= \operatorname{Re}\langle \cos\varphi\cdot x+\sin\varphi\cdot y, z\rangle + \sin\varphi\operatorname{Im}\langle x,z\rangle - \cos\varphi\operatorname{Im}\langle y,z\rangle \\
&= \operatorname{Re}\langle \cos\varphi\cdot x+\sin\varphi\cdot y, z\rangle,
\end{aligned}$$

and for the last equality we have used the assumption (2.40).

Taking the square and using the Schwarz inequality for the inner product $\langle\cdot,\cdot\rangle$, we have

$$\begin{aligned}
|\langle x+iy,z\rangle|^2 &= [\operatorname{Re}\langle \cos\varphi\cdot x+\sin\varphi\cdot y, z\rangle]^2 \\
&\le \|\cos\varphi\cdot x+\sin\varphi\cdot y\|^2\|z\|^2.
\end{aligned} \tag{2.42}$$

On making use of Lemma 2, we have

$$\begin{aligned}
&\sup_{\alpha\in[0,2\pi]}\|\cos\varphi\cdot x+\sin\varphi\cdot y\|^2 \\
&= \sup_{\alpha\in[0,2\pi]}\left[\|x\|^2\cos^2\varphi + 2\operatorname{Re}\langle x,y\rangle\sin\varphi\cos\varphi + \|y\|^2\sin^2\varphi\right] \\
&= \frac{1}{2}\left\{\|x\|^2+\|y\|^2+\left[\left(\|x\|^2-\|y\|^2\right)^2+4\operatorname{Re}^2\langle x,y\rangle\right]^{\frac{1}{2}}\right\}
\end{aligned}$$

and the first inequality in (2.41) is proved.

Observe that

$$\begin{aligned}
&\left(\|x\|^2-\|y\|^2\right)^2+4\operatorname{Re}^2\langle x,y\rangle \\
&= \left(\|x\|^2+\|y\|^2\right)^2 - 4\left[\|x\|^2\|y\|^2-\operatorname{Re}^2\langle x,y\rangle\right] \\
&\le \left(\|x\|^2+\|y\|^2\right)^2
\end{aligned}$$

and the last part of (2.41) is proved. ∎

Remark 14. *Observe that if $(H,\langle\cdot,\cdot\rangle)$ is a real inner product space, then for any $x,y,z\in H$ one has:*

$$\begin{aligned}
&\langle x,z\rangle^2+\langle y,z\rangle^2 \\
&\le \frac{1}{2}\left\{\|x\|^2+\|y\|^2+\left[\left(\|x\|^2-\|y\|^2\right)^2+4\langle x,y\rangle^2\right]\right\}^{\frac{1}{2}}\|z\|^2 \\
&\le \left(\|x\|^2+\|y\|^2\right)\|z\|^2.
\end{aligned} \tag{2.43}$$

Remark 15. *If H is a real space, $\langle\cdot,\cdot\rangle$ the real inner product, $H_{\mathbb{C}}$ its complexification and $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ the corresponding complexification for $\langle\cdot,\cdot\rangle$, then for $x,y\in H$ and $w:=x+iy\in H_{\mathbb{C}}$ and for $e\in H$ we have*

$$\operatorname{Im}\langle x,e\rangle_{\mathbb{C}}=\operatorname{Im}\langle y,e\rangle_{\mathbb{C}}=0,$$

$$\|w\|_{\mathbb{C}}^2=\|x\|^2+\|y\|^2, \qquad |\langle w,\bar{w}\rangle_{\mathbb{C}}|=\left(\|x\|^2-\|y\|^2\right)^2+4\langle x,y\rangle^2,$$

where $\bar{w}=x-iy\in H_{\mathbb{C}}$.

Applying Theorem 16 for the complex space $H_{\mathbb{C}}$ and complex inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}}$, we deduce

$$|\langle w,e\rangle_{\mathbb{C}}|^2\le\frac{1}{2}\|e\|^2\left[\|w\|_{\mathbb{C}}^2+|\langle w,\bar{w}\rangle_{\mathbb{C}}|\right]\le\|e\|^2\|w\|_{\mathbb{C}}^2, \tag{2.44}$$

which is Kurepa's inequality (2.35).

Corollary 7. *Let x,y,z be as in Theorem 16. In addition, if $\operatorname{Re}\langle x,y\rangle=0$, then*

$$\left[\operatorname{Re}^2\langle x,z\rangle+\operatorname{Re}^2\langle y,z\rangle\right]^{\frac{1}{2}}\le\|z\|\cdot\max\{\|x\|,\|y\|\}. \tag{2.45}$$

Remark 16. *If H is a real space and $\langle\cdot,\cdot\rangle$ a real inner product on H, then for any $x,y,z\in H$ with $\langle x,y\rangle=0$ we have*

$$\left[\langle x,z\rangle^2+\langle y,z\rangle^2\right]^{\frac{1}{2}}\le\|z\|\cdot\max\{\|x\|,\|y\|\}. \tag{2.46}$$

2.3.3. A Related Result. Utilising Lemma 2, we may state and prove the following result as well.

Theorem 17 (Dragomir, 2004). *Let $(H,\langle\cdot,\cdot\rangle)$ be a real or complex inner product space. Then we have the inequalities:*

$$\begin{aligned}&\frac{1}{2}\Big\{|\langle v,t\rangle|^2+|\langle w,t\rangle|^2+\Big[\left(|\langle v,t\rangle|^2-|\langle w,t\rangle|^2\right)^2\\&\qquad+4\left(\operatorname{Re}\langle v,t\rangle\operatorname{Re}\langle w,t\rangle+\operatorname{Im}\langle v,t\rangle\operatorname{Im}\langle w,t\rangle\right)^2\Big]^{\frac{1}{2}}\Big\}\\&\le\frac{1}{2}\|t\|^2\left\{\|v\|^2+\|w\|^2+\left[\left(\|v\|^2-\|w\|^2\right)^2+4\operatorname{Re}^2(v,w)\right]^{\frac{1}{2}}\right\}\\&\le\left(\|v\|^2+\|w\|^2\right)\|t\|^2,\end{aligned} \tag{2.47}$$

for all $v,w,t\in H$.

Proof. Observe that, by Schwarz's inequality

$$|(\cos\varphi\cdot v+\sin\varphi\cdot w,z)|^2\le\|\cos\varphi\cdot v+\sin\varphi\cdot w\|^2\|z\|^2 \tag{2.48}$$

for any $\varphi\in[0,2\pi]$.

Since

$$\begin{aligned} I(\varphi) &:= \|\cos\varphi \cdot v + \sin\varphi \cdot w\|^2 \\ &= \cos^2\varphi \|v\|^2 + 2\operatorname{Re}(v,w)\sin\varphi\cos\varphi + \|w\|^2 \sin^2\varphi, \end{aligned}$$

hence, as in Theorem 16,

$$\sup_{\varphi\in[0,2\pi]} I(\varphi) = \frac{1}{2}\left\{\|v\|^2 + \|w\|^2 + \left[\left(\|v\|^2 - \|w\|^2\right)^2 + 4\operatorname{Re}^2(v,w)\right]^{\frac{1}{2}}\right\}.$$

Also, denoting

$$\begin{aligned} J(\varphi) &:= |\cos\varphi \langle v,z\rangle + \sin\varphi \langle w,z\rangle| \\ &= \cos^2\varphi |\langle v,z\rangle|^2 + 2\sin\varphi\cos\varphi \operatorname{Re}\left[\langle v,z\rangle \overline{\langle w,z\rangle}\right] + \sin^2\varphi |\langle w,z\rangle|^2, \end{aligned}$$

then, on applying Lemma 2, we deduce that

$$\begin{aligned} \sup_{\varphi\in[0,2\pi]} J(\varphi) = \frac{1}{2}\Bigg\{ & |\langle v,t\rangle|^2 + |\langle w,t\rangle|^2 \\ & + \left[\left(|\langle v,t\rangle|^2 - |\langle w,t\rangle|^2\right)^2 + 4\operatorname{Re}^2\left[\langle v,z\rangle \overline{\langle w,z\rangle}\right]\right]^{\frac{1}{2}}\Bigg\} \end{aligned}$$

and, since

$$\operatorname{Re}\left[\langle v,z\rangle \overline{\langle w,z\rangle}\right] = \operatorname{Re}\langle v,t\rangle \operatorname{Re}\langle w,t\rangle + \operatorname{Im}\langle v,t\rangle \operatorname{Im}\langle w,t\rangle,$$

hence, on taking the supremum in the inequality (2.48), we deduce the desired inequality (2.47). ∎

REMARK 17. *In the real case, (2.47) provides the same inequality we obtained in (2.43).*

In the complex case, if we assume that $v, w, t \in H$ *are such that*

$$\operatorname{Re}\langle v,t\rangle \operatorname{Re}\langle w,t\rangle = -\operatorname{Im}\langle v,t\rangle \operatorname{Im}\langle w,t\rangle,$$

then (2.47) becomes:

$$\begin{aligned} &\max\left\{|\langle v,t\rangle|^2, |\langle w,t\rangle|^2\right\} \\ &\leq \frac{1}{2}\|t\|^2 \left\{\|v\|^2 + \|w\|^2 + \left[\left(\|v\|^2 - \|w\|^2\right)^2 + 4\operatorname{Re}^2(v,w)\right]^{\frac{1}{2}}\right\}. \end{aligned} \tag{2.49}$$

2.4. Refinements of Buzano's and Kurepa's Inequalities

2.4.1. Introduction. In [**3**], M.L. Buzano obtained the following extension of the celebrated Schwarz's inequality in a real or complex inner product space $(H; \langle\cdot,\cdot\rangle)$:

$$|\langle a,x\rangle \langle x,b\rangle| \leq \frac{1}{2}\left[\|a\| \cdot \|b\| + |\langle a,b\rangle|\right] \|x\|^2, \tag{2.50}$$

for any $a, b, x \in H$.

It is clear that for $a = b$, the above inequality becomes the standard Schwarz inequality

$$|\langle a, x\rangle|^2 \le \|a\|^2 \|x\|^2, \qquad a, x \in H; \tag{2.51}$$

with equality if and only if there exists a scalar $\lambda \in \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) such that $x = \lambda a$.

As noted by M. Fujii and F. Kubo in [**21**], where they provided a simple proof of (2.50) by utilising orthogonal projection arguments, the case of equality holds in (2.50) if

$$x = \begin{cases} \alpha\left(\frac{a}{\|a\|} + \frac{\langle a,b\rangle}{|\langle a,b\rangle|} \cdot \frac{b}{\|b\|}\right), & \text{when } \langle a, b\rangle \neq 0 \\ \\ \alpha\left(\frac{a}{\|a\|} + \beta \cdot \frac{b}{\|b\|}\right), & \text{when } \langle a, b\rangle = 0, \end{cases}$$

where $\alpha, \beta \in \mathbb{K}$.

It might be useful to observe that, out of (2.50), one may get the following discrete inequality:

$$\begin{aligned} &\left|\sum_{i=1}^{n} p_i a_i \overline{x_i} \sum_{i=1}^{n} p_i x_i \overline{b_i}\right| \\ &\le \frac{1}{2}\left[\left(\sum_{i=1}^{n} p_i |a_i|^2 \sum_{i=1}^{n} p_i |b_i|^2\right)^{\frac{1}{2}} + \left|\sum_{i=1}^{n} p_i a_i \overline{b_i}\right|\right] \sum_{i=1}^{n} p_i |x_i|^2, \end{aligned} \tag{2.52}$$

where $p_i \ge 0$, $a_i, x_i, b_i \in \mathbb{C}$, $i \in \{1, \dots, n\}$.

If one takes in (2.52) $b_i = \overline{a_i}$ for $i \in \{1, \dots, n\}$, then one obtains

$$\left|\sum_{i=1}^{n} p_i a_i \overline{x_i} \sum_{i=1}^{n} p_i a_i x_i\right| \le \frac{1}{2}\left[\sum_{i=1}^{n} p_i |a_i|^2 + \left|\sum_{i=1}^{n} p_i a_i^2\right|\right] \sum_{i=1}^{n} p_i |x_i|^2, \tag{2.53}$$

for any $p_i \ge 0$, $a_i, x_i, b_i \in \mathbb{C}$, $i \in \{1, \dots, n\}$.

Note that, if x_i, $i \in \{1, \dots, n\}$ are real numbers, then out of (2.53), we may deduce the de Bruijn refinement of the celebrated Cauchy-Bunyakovsky-Schwarz inequality [**2**]

$$\left|\sum_{i=1}^{n} p_i x_i z_i\right|^2 \le \frac{1}{2}\sum_{i=1}^{n} p_i x_i^2 \left[\sum_{i=1}^{n} p_i |z_i|^2 + \left|\sum_{i=1}^{n} p_i z_i^2\right|\right], \tag{2.54}$$

where $z_i \in \mathbb{C}$, $i \in \{1, \dots, n\}$. In this way, Buzano's result may be regarded as a generalisation of de Bruijn's inequality.

Similar comments obviously apply for integrals, but, for the sake of brevity we do not mention them here.

The aim of the present section is to establish some related results as well as a refinement of Buzano's inequality for real or complex inner product spaces. An improvement of Kurepa's inequality for the complexification of a real inner product and the corresponding applications for discrete and integral inequalities are also provided.

2.4.2. Some Buzano Type Inequalities. The following result may be stated [**16**].

THEOREM 18 (Dragomir, 2004). *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. For all $\alpha \in \mathbb{K} \setminus \{0\}$ and $x, a, b \in H$, $\alpha \neq 0$, one has the inequality*

$$\left| \frac{\langle a, x \rangle \langle x, b \rangle}{\|x\|^2} - \frac{\langle a, b \rangle}{\alpha} \right| \leq \frac{\|b\|}{|\alpha| \|x\|} \left[|\alpha - 1|^2 |\langle a, x \rangle|^2 + \|x\|^2 \|a\|^2 - |\langle a, x \rangle|^2 \right]. \tag{2.55}$$

The case of equality holds in (2.55) if and only if there exists a scalar $\lambda \in \mathbb{K}$ so that

$$\alpha \cdot \frac{\langle a, x \rangle}{\|x\|^2} x = a + \lambda b. \tag{2.56}$$

PROOF. We follow the proof in [**16**].

Using Schwarz's inequality, we have that

$$\left| \left\langle \alpha \cdot \frac{\langle a, x \rangle}{\|x\|^2} x - a, b \right\rangle \right|^2 \leq \left\| \alpha \cdot \frac{\langle a, x \rangle}{\|x\|^2} x - a \right\|^2 \|b\|^2 \tag{2.57}$$

and since

$$\begin{aligned} \left\| \alpha \cdot \frac{\langle a, x \rangle}{\|x\|^2} x - a \right\|^2 &= |\alpha|^2 \frac{|\langle a, x \rangle|^2}{\|x\|^2} - 2 \frac{|\langle a, x \rangle|^2}{\|x\|^2} \operatorname{Re} \alpha + \|a\|^2 \\ &= \frac{|\alpha - 1|^2 |\langle a, x \rangle|^2 + \|x\|^2 \|a\|^2 - |\langle a, x \rangle|^2}{\|x\|^2} \end{aligned}$$

and

$$\left\langle \alpha \cdot \frac{\langle a, x \rangle}{\|x\|^2} x - a, b \right\rangle = \alpha \left[\frac{\langle a, x \rangle \langle x, b \rangle}{\|x\|^2} - \frac{\langle a, b \rangle}{\alpha} \right],$$

hence by (2.55) we deduce the desired inequality (2.55).

The case of equality is obvious from the above considerations related to the Schwarz's inequality (2.51). ∎

Remark 18. *Using the continuity property of the modulus, i.e.,* $||z|-|u|| \leq |z-u|$, $z,u\in\mathbb{K}$, *we have:*

$$\left|\frac{|\langle a,x\rangle\langle x,b\rangle|}{\|x\|^2}-\frac{|\langle a,b\rangle|}{|\alpha|}\right| \leq \left|\frac{\langle a,x\rangle\langle x,b\rangle}{\|x\|^2}-\frac{\langle a,b\rangle}{\alpha}\right|. \tag{2.58}$$

Therefore, by (2.55) and (2.58), one may deduce the following double inequality:

$$\begin{aligned}\frac{1}{|\alpha|}\Bigg[|\langle a,b\rangle| - \frac{\|b\|}{\|x\|} \\ \times\left[\left(|\alpha-1|^2|\langle x,a\rangle|^2+\|x\|^2\|a\|^2-|\langle a,x\rangle|^2\right)^{\frac{1}{2}}\right]\Bigg] \\ \leq \frac{|\langle a,x\rangle\langle x,b\rangle|}{\|x\|^2} \\ \leq \frac{1}{|\alpha|}\left[|\langle a,b\rangle|+\frac{\|b\|}{\|x\|}\right] \\ \times\left[\left(|\alpha-1|^2|\langle x,a\rangle|^2+\|x\|^2\|a\|^2-|\langle x,a\rangle|^2\right)^{\frac{1}{2}}\right],\end{aligned} \tag{2.59}$$

for each $\alpha\in\mathbb{K}\backslash\{0\}$, $a,b,x\in H$ *and* $x\neq 0$.

It is obvious that, out of (2.55), we can obtain various particular inequalities. We mention in the following a class of these which is related to Buzano's result (2.50) [**16**].

Corollary 8 (Dragomir, 2004). *Let* $a,b,x\in H$, $x\neq 0$ *and* $\eta\in\mathbb{K}$ *with* $|\eta|=1$, $\operatorname{Re}\eta\neq -1$. *Then we have the inequality:*

$$\left|\frac{\langle a,x\rangle\langle x,b\rangle}{\|x\|^2}-\frac{\langle a,b\rangle}{1+\eta}\right| \leq \frac{\|a\|\,\|b\|}{\sqrt{2}\sqrt{1+\operatorname{Re}\eta}}, \tag{2.60}$$

and, in particular, for $\eta=1$, *the inequality:*

$$\left|\frac{\langle a,x\rangle\langle x,b\rangle}{\|x\|^2}-\frac{\langle a,b\rangle}{2}\right| \leq \frac{\|a\|\,\|b\|}{2}. \tag{2.61}$$

Proof. It follows by Theorem 18 on choosing $\alpha=1+\eta$ and we omit the details. ∎

Remark 19. *Using the continuity property of modulus, we get from (2.60) that:*

$$\frac{|\langle a,x\rangle\langle x,b\rangle|}{\|x\|^2} \leq \frac{|\langle a,b\rangle|+\|a\|\,\|b\|}{\sqrt{2}\sqrt{1+\operatorname{Re}\eta}}, \qquad |\eta|=1,\quad \operatorname{Re}\eta\neq -1,$$

which provides, as the best possible inequality, the above result due to Buzano (2.50).

REMARK 20. *If the space is real, then the inequality (2.55) is obviously equivalent to:*

$$\begin{aligned}(2.62)\qquad &\frac{\langle a,b\rangle}{\alpha}-\frac{\|b\|}{|\alpha|\,\|x\|}\left[(\alpha-1)^2\langle a,x\rangle^2+\|x\|^2\|a\|^2-\langle a,x\rangle^2\right]^{\frac{1}{2}}\\ &\le\frac{\langle a,x\rangle\langle x,b\rangle}{\|x\|^2}\\ &\le\frac{\langle a,b\rangle}{\alpha}+\frac{\|b\|}{|\alpha|\,\|x\|}\left[(\alpha-1)^2\langle a,x\rangle^2+\|x\|^2\|a\|^2-\langle a,x\rangle^2\right]^{\frac{1}{2}}\end{aligned}$$

for any $\alpha\in\mathbb{R}\setminus\{0\}$ and $a,b,x\in H$, $x\neq 0$.

If in (2.62) we take $\alpha=2$, then we get

$$\begin{aligned}(2.63)\qquad \frac{1}{2}\left[\langle a,b\rangle-\|a\|\,\|b\|\right]\|x\|^2&\le\langle a,x\rangle\langle x,b\rangle\\ &\le\frac{1}{2}\left[\langle a,b\rangle+\|a\|\,\|y\|\right]\|x\|^2,\end{aligned}$$

which apparently, as mentioned by T. Precupanu in [**29**]*, has been obtained independently of Buzano, by U. Richard in* [**30**].

In [**28**]*, Pečarić gave a simple direct proof of (2.63) without mentioning the work of either Buzano or Richard, but tracked down the result, in a particular form, to an earlier paper due to C. Blatter* [**1**].

Obviously, the following refinement of Buzano's result may be stated [**16**].

COROLLARY 9 (Dragomir, 2004). *Let $(H;\langle\cdot,\cdot\rangle)$ be a real or complex inner product space and $a,b,x\in H$. Then*

$$\begin{aligned}(2.64)\qquad |\langle a,x\rangle\langle x,b\rangle|&\le\left|\langle a,x\rangle\langle x,b\rangle-\frac{1}{2}\langle a,b\rangle\|x\|^2\right|+\frac{1}{2}|\langle a,b\rangle|\,\|x\|^2\\ &\le\frac{1}{2}\left[\|a\|\,\|b\|+|\langle a,b\rangle|\right]\|x\|^2.\end{aligned}$$

PROOF. The first inequality in (2.64) follows by the triangle inequality for the modulus $|\cdot|$. The second inequality is merely (2.61) in which we added the same quantity to both sides. ∎

REMARK 21. *For $\alpha=1$, we deduce from (2.55) the following inequality:*

$$(2.65)\qquad \left|\frac{\langle a,x\rangle\langle x,b\rangle}{\|x\|^2}-\langle a,b\rangle\right|\le\frac{\|b\|}{\|x\|}\left[\|x\|^2\|a\|^2-|\langle a,x\rangle|^2\right]^{\frac{1}{2}}$$

for any $a,b,x\in H$ with $x\neq 0$.

If the space is real, then (2.65) is equivalent to

$$\begin{aligned}(2.66)\qquad &\langle a,b\rangle-\frac{\|b\|}{\|x\|}\left[\|x\|^2\|a\|^2-|\langle a,x\rangle|^2\right]^{\frac{1}{2}}\\ &\leq\frac{\langle a,x\rangle\langle x,b\rangle}{\|x\|^2}\\ &\leq\frac{\|b\|}{\|x\|}\left[\|x\|^2\|a\|^2-|\langle a,x\rangle|^2\right]^{\frac{1}{2}}+\langle a,b\rangle,\end{aligned}$$

which is similar to Richard's inequality (2.63).

2.4.3. Applications to Kurepa's Inequality. In 1960, N.G. de Bruijn [**2**] obtained the following refinement of the Cauchy-Bunyakovsky-Schwarz inequality:

$$(2.67)\qquad \left|\sum_{i=1}^n a_i z_i\right|^2\leq\frac{1}{2}\sum_{i=1}^n a_i^2\left[\sum_{i=1}^n|z_i|^2+\left|\sum_{i=1}^n z_i^2\right|\right],$$

provided that a_i are real numbers while z_i are complex for each $i\in\{1,...,n\}$.

In [**25**], S. Kurepa proved the following generalisation of the de Bruijn result:

THEOREM 19 (Kurepa, 1966). *Let $(H;\langle\cdot,\cdot\rangle)$ be a real inner product space and $(H_{\mathbb{C}},\langle\cdot,\cdot\rangle_{\mathbb{C}})$ its complexification. Then for any $a\in H$ and $z\in H_{\mathbb{C}}$, one has the following refinement of Schwarz's inequality*

$$(2.68)\qquad |\langle a,z\rangle_{\mathbb{C}}|^2\leq\frac{1}{2}\|a\|^2\left[\|z\|_{\mathbb{C}}^2+|\langle z,\bar{z}\rangle_{\mathbb{C}}|\right]\leq\|a\|^2\|z\|_{\mathbb{C}}^2,$$

where $\bar{z}$ denotes the conjugate of $z\in H_{\mathbb{C}}$.

As consequences of this general result, Kurepa noted the following integral, respectively, discrete inequality:

COROLLARY 10 (Kurepa, 1966). *Let (S,Σ,μ) be a positive measure space and $a,z\in L_2(S,\Sigma,\mu)$, the Hilbert space of complex-valued $2-\mu-$integrable functions defined on S. If a is a real function and z is a complex function, then*

$$\begin{aligned}(2.69)\qquad &\left|\int_S a(t)\,z(t)\,d\mu(t)\right|^2\\ &\leq\frac{1}{2}\cdot\int_S a^2(t)\,d\mu(t)\left[\int_S|z(t)|^2\,d\mu(t)+\left|\int_S z^2(t)\,d\mu(t)\right|\right].\end{aligned}$$

COROLLARY 11 (Kurepa, 1966). *If $a_1, \dots, a_n$ are real numbers, $z_1, \dots, z_n$ are complex numbers and (A_{ij}) is a positive definite real matrix of dimension $n \times n$, then*

$$\left| \sum_{i,j=1}^{n} A_{ij} a_i z_j \right|^2 \leq \frac{1}{2} \sum_{i,j=1}^{n} A_{ij} a_i a_j \left[\sum_{i,j=1}^{n} A_{ij} z_i \overline{z_j} + \left| \sum_{i,j=1}^{n} A_{ij} z_i \overline{z_j} \right| \right]. \tag{2.70}$$

The following refinement of Kurepa's result may be stated [**16**].

THEOREM 20 (Dragomir, 2004). *Let $(H; \langle \cdot, \cdot \rangle)$ be a real inner product space and $(H_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathbb{C}})$ its complexification. Then for any $e \in H$ and $w \in H_{\mathbb{C}}$, one has the inequality:*

$$\begin{aligned} |\langle w, e \rangle_{\mathbb{C}}|^2 &\leq \left| \langle w, e \rangle_{\mathbb{C}}^2 - \frac{1}{2} \langle w, \bar{w} \rangle_{\mathbb{C}} \|e\|^2 \right| + \frac{1}{2} |\langle w, \bar{w} \rangle_{\mathbb{C}}| \|e\|^2 \\ &\leq \frac{1}{2} \|e\|^2 \left[\|w\|_{\mathbb{C}}^2 + |\langle w, \bar{w} \rangle_{\mathbb{C}}| \right]. \end{aligned} \tag{2.71}$$

PROOF. We follow the proof in [**16**].

If we apply Corollary 11 for $(H_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathbb{C}})$ and $x = e \in H$, $a = w$ and $b = \bar{w}$, then we have

$$\begin{aligned} &|\langle w, e \rangle_{\mathbb{C}} \langle e, \bar{w} \rangle_{\mathbb{C}}| \\ &\leq \left| \langle w, e \rangle_{\mathbb{C}} \langle e, \bar{w} \rangle_{\mathbb{C}} - \frac{1}{2} \langle w, \bar{w} \rangle_{\mathbb{C}} \|e\|^2 \right| + \frac{1}{2} |\langle w, \bar{w} \rangle_{\mathbb{C}}| \|e\|^2 \\ &\leq \frac{1}{2} \|e\|^2 \left[\|w\|_{\mathbb{C}} \|\bar{w}\|_{\mathbb{C}} + |\langle w, \bar{w} \rangle_{\mathbb{C}}| \right]. \end{aligned} \tag{2.72}$$

Now, if we assume that $w = (x, y) \in H_{\mathbb{C}}$, then, by the definition of $\langle \cdot, \cdot \rangle_{\mathbb{C}}$, we have

$$\begin{aligned} \langle w, e \rangle_{\mathbb{C}} &= \langle (x, y), (e, 0) \rangle_{\mathbb{C}} \\ &= \langle x, e \rangle + \langle y, 0 \rangle + i [\langle y, e \rangle - \langle x, 0 \rangle] \\ &= \langle e, x \rangle + i \langle e, y \rangle, \end{aligned}$$

$$\begin{aligned} \langle e, \bar{w} \rangle_{\mathbb{C}} &= \langle (e, 0), (x, -y) \rangle_{\mathbb{C}} \\ &= \langle e, x \rangle + \langle 0, -y \rangle + i [\langle 0, x \rangle - \langle e, -y \rangle] \\ &= \langle e, x \rangle + i \langle e, y \rangle = \langle w, e \rangle_{\mathbb{C}} \end{aligned}$$

and

$$\|\bar{w}\|_{\mathbb{C}}^2 = \|x\|^2 + \|y\|^2 = \|w\|_{\mathbb{C}}^2.$$

Therefore, by (2.72), we deduce the desired result (2.71). ∎

Denote by $\ell_{\rho}^{2}(\mathbb{C})$ the Hilbert space of all complex sequences $z = (z_i)_{i\in\mathbb{N}}$ with the property that for $\rho_i \geq 0$ with $\sum_{i=1}^{\infty}\rho_i = 1$ we have $\sum_{i=1}^{\infty}\rho_i |z_i|^2 < \infty$. If $a = (a_i)_{i\in\mathbb{N}}$ is a sequence of real numbers such that $a \in \ell_{\rho}^{2}(\mathbb{C})$, then for any $z \in \ell_{\rho}^{2}(\mathbb{C})$ we have the inequality:

$$
\begin{aligned}
&\left|\sum_{i=1}^{\infty}\rho_i a_i z_i\right|^2 \qquad (2.73)\\
&\leq \left|\left(\sum_{i=1}^{\infty}\rho_i a_i z_i\right)^2 - \frac{1}{2}\sum_{i=1}^{\infty}\rho_i a_i^2 \sum_{i=1}^{\infty}\rho_i z_i^2\right| + \frac{1}{2}\sum_{i=1}^{\infty}\rho_i a_i^2 \left|\sum_{i=1}^{\infty}\rho_i z_i^2\right|\\
&\leq \frac{1}{2}\sum_{i=1}^{\infty}\rho_i a_i^2 \left[\sum_{i=1}^{\infty}\rho_i |z_i|^2 + \left|\sum_{i=1}^{\infty}\rho_i z_i^2\right|\right].
\end{aligned}
$$

Similarly, if by $L_{\rho}^{2}(S,\Sigma,\mu)$ we understand the Hilbert space of all complex-valued functions $f : S \to \mathbb{C}$ with the property that for the $\mu-$measurable function $\rho \geq 0$ with $\int_S \rho(t)\,d\mu(t) = 1$ we have

$$\int_S \rho(t)\,|f(t)|^2\,d\mu(t) < \infty,$$

then for a real function $a \in L_{\rho}^{2}(S,\Sigma,\mu)$ and any $f \in L_{\rho}^{2}(S,\Sigma,\mu)$, we have the inequalities

$$
\begin{aligned}
&\left|\int_S \rho(t)\,a(t)\,f(t)\,d\mu(t)\right|^2 \qquad (2.74)\\
&\leq \left|\left(\int_S \rho(t)\,a(t)\,f(t)\,d\mu(t)\right)^2\right.\\
&\qquad\qquad \left. - \frac{1}{2}\int_S \rho(t)\,f^2(t)\,d\mu(t)\int_S \rho(t)\,a^2(t)\,d\mu(t)\right|\\
&\qquad\qquad + \frac{1}{2}\left|\int_S \rho(t)\,f^2(t)\,d\mu(t)\right|\int_S \rho(t)\,a^2(t)\,d\mu(t)\\
&\leq \frac{1}{2}\int_S \rho(t)\,a^2(t)\,d\mu(t)\\
&\qquad\qquad \times\left[\int_S \rho(t)\,|f(t)|^2\,d\mu(t) + \left|\int_S \rho(t)\,f^2(t)\,d\mu(t)\right|\right].
\end{aligned}
$$

2.5. Inequalities for Orthornormal Families

2.5.1. Introduction. In [**3**], M.L. Buzano obtained the following extension of the celebrated Schwarz's inequality in a real or complex inner product space $(H;\langle\cdot,\cdot\rangle)$:

$$|\langle a,x\rangle\langle x,b\rangle|\leq\frac{1}{2}\left[\|a\|\|b\|+|\langle a,b\rangle|\right]\|x\|^{2}, \tag{2.75}$$

for any $a,b,x\in H$.

It is clear that the above inequality becomes, for $a=b$, the Schwarz's inequality

$$|\langle a,x\rangle|^{2}\leq\|a\|^{2}\|x\|^{2},\quad a,x\in H; \tag{2.76}$$

in which the equality holds if and only if there exists a scalar $\lambda\in\mathbb{K}$ $(\mathbb{R},\mathbb{C})$ so that $x=\lambda a$.

As noted by T. Precupanu in [**29**], independently of Buzano, U. Richard [**30**] obtained the following similar inequality holding in real inner product spaces:

$$\begin{aligned}\frac{1}{2}\|x\|^{2}\left[\langle a,b\rangle-\|a\|\|b\|\right]&\leq\langle a,x\rangle\langle x,b\rangle\\&\leq\frac{1}{2}\|x\|^{2}\left[\langle a,b\rangle+\|a\|\|b\|\right].\end{aligned} \tag{2.77}$$

The main aim of the present section is to obtain similar results for families of orthonormal vectors in $(H;\langle\cdot,\cdot\rangle)$, real or complex space, that are naturally connected with the celebrated Bessel inequality and improve the results of Busano, Richard and Kurepa.

2.5.2. A Generalisation for Orthonormal Families. We say that the finite family $\{e_i\}_{i\in I}$ (I is finite) of vectors is *orthonormal* if $\langle e_i,e_j\rangle=0$ if $i,j\in I$ with $i\neq j$ and $\|e_i\|=1$ for each $i\in I$. The following result may be stated [**11**]:

Theorem 21 (Dragomir, 2004). *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $\{e_i\}_{i\in I}$ a finite orthonormal family in H. Then for any $a,b\in H$, one has the inequality:*

$$\left|\sum_{i\in I}\langle a,e_i\rangle\langle e_i,b\rangle-\frac{1}{2}\langle a,b\rangle\right|\leq\frac{1}{2}\|a\|\|b\|. \tag{2.78}$$

The case of equality holds in (2.78) if and only if

$$\sum_{i\in I}\langle a,e_i\rangle e_i=\frac{1}{2}a+\left(\sum_{i\in I}\langle a,e_i\rangle\langle e_i,b\rangle-\frac{1}{2}\langle a,b\rangle\right)\cdot\frac{b}{\|b\|^{2}}. \tag{2.79}$$

Proof. We follow the proof in [**11**].

It is well known that, for $e \neq 0$ and $f \in H$, the following identity holds:

$$\frac{\|f\|^2 \|e\|^2 - |\langle f, e\rangle|^2}{\|e\|^2} = \left\| f - \frac{\langle f, e\rangle e}{\|e\|^2} \right\|^2 . \tag{2.80}$$

Therefore, in Schwarz's inequality

$$|\langle f, e\rangle|^2 \leq \|f\|^2 \|e\|^2 , \quad f, e \in H; \tag{2.81}$$

the case of equality, for $e \neq 0$, holds if and only if

$$f = \frac{\langle f, e\rangle e}{\|e\|^2}.$$

Let $f := 2\sum_{i\in I} \langle a, e_i\rangle e_i - a$ and $e := b$. Then, by Schwarz's inequality (2.81), we may state that

$$\left| \left\langle 2\sum_{i\in I} \langle a, e_i\rangle e_i - a, b \right\rangle \right|^2 \leq \left\| 2\sum_{i\in I} \langle a, e_i\rangle e_i - a \right\|^2 \|b\|^2 \tag{2.82}$$

with equality, for $b \neq 0$, if and only if

$$2\sum_{i\in I} \langle a, e_i\rangle e_i - a = \left\langle 2\sum_{i\in I} \langle a, e_i\rangle e_i - a, b \right\rangle \frac{b}{\|b\|^2}. \tag{2.83}$$

Since

$$\left\langle 2\sum_{i\in I} \langle a, e_i\rangle e_i - a, b \right\rangle = 2\sum_{i\in I} \langle a, e_i\rangle \langle e_i, b\rangle - \langle a, b\rangle$$

and

$$\begin{aligned}
&\left\| 2\sum_{i\in I} \langle a, e_i\rangle e_i - a \right\|^2 \\
&= 4\left\| \sum_{i\in I} \langle a, e_i\rangle e_i \right\|^2 - 4\operatorname{Re}\left\langle \sum_{i\in I} \langle a, e_i\rangle e_i, a \right\rangle + \|a\|^2 \\
&= 4\sum_{i\in I} |\langle a, e_i\rangle|^2 - 4\sum_{i\in I} |\langle a, e_i\rangle|^2 + \|a\|^2 \\
&= \|a\|^2 ,
\end{aligned}$$

hence by (2.82) we deduce the desired inequality (2.78).

Finally, as (2.79) is equivalent to

$$\sum_{i\in I} \langle a, e_i\rangle e_i - \frac{a}{2} = \left(\sum_{i\in I} \langle a, e_i\rangle \langle e_i, b\rangle - \frac{1}{2}\langle a, b\rangle \right) \frac{b}{\|b\|^2},$$

hence the equality holds in (2.78) if and only if (2.79) is valid. ∎

The following result is well known in the literature as Bessel's inequality

$$\sum_{i\in I}|\langle x,e_i\rangle|^2\le\|x\|^2,\quad x\in H, \tag{2.84}$$

where, as above, $\{e_i\}_{i\in I}$ is a finite orthonormal family in the inner product space $(H;\langle\cdot,\cdot\rangle)$.

If one chooses $a=b=x$ in (2.78), then one gets the inequality

$$\left|\sum_{i\in I}|\langle x,e_i\rangle|^2-\frac{1}{2}\|x\|^2\right|\le\frac{1}{2}\|x\|^2,$$

which is obviously equivalent to Bessel's inequality (2.84). Therefore, the inequality (2.78) may be regarded as a generalisation of Bessel's inequality as well.

Utilising the Bessel and Cauchy-Bunyakovsky-Schwarz inequalities, one may state that

$$\left|\sum_{i\in I}\langle a,e_i\rangle\langle e_i,b\rangle\right|\le\left[\sum_{i\in I}|\langle a,e_i\rangle|^2\sum_{i\in I}|\langle b,e_i\rangle|^2\right]^{\frac{1}{2}}\le\|a\|\,\|b\| \tag{2.85}$$

A different refinement of the inequality between the first and the last term in (2.85) is incorporated in the following [**11**]:

Corollary 12 (Dragomir, 2004). *With the assumption of Theorem 21, we have*

$$\begin{aligned}\left|\sum_{i\in I}\langle a,e_i\rangle\langle e_i,b\rangle\right|&\le\left|\sum_{i\in I}\langle a,e_i\rangle\langle e_i,b\rangle-\frac{1}{2}\langle a,b\rangle\right|+\frac{1}{2}|\langle a,b\rangle|\\&\le\frac{1}{2}\left[\|a\|\,\|b\|+|\langle a,b\rangle|\right]\\&\le\|a\|\,\|b\|.\end{aligned} \tag{2.86}$$

Remark 22. *If the space $(H;\langle\cdot,\cdot\rangle)$ is real, then, obviously, (2.78) is equivalent to:*

$$\frac{1}{2}\left(\langle a,b\rangle-\|a\|\,\|b\|\right)\le\sum_{i\in I}\langle a,e_i\rangle\langle e_i,b\rangle\le\frac{1}{2}\left[\|a\|\,\|b\|+\langle a,b\rangle\right]. \tag{2.87}$$

Remark 23. *It is obvious that if the family comprises of only a single element $e=\frac{x}{\|x\|}$, $x\in H$, $x\neq0$, then from (2.86) we recapture the refinement of Buzano's inequality incorporated in (2.75) while from (2.87) we deduce Richard's result from (2.77).*

The following corollary of Theorem 21 is of interest as well [**11**]:

Corollary 13 (Dragomir, 2004). *Let $\{e_i\}_{i\in I}$ be a finite orthonormal family in $(H;\langle\cdot,\cdot\rangle)$. If $x,y\in H\backslash\{0\}$ are such that there exists the constants $m_i,n_i,\ M_i,\ N_i\in\mathbb{R},\ i\in I$ such that:*

$$-1\leq m_i\leq\frac{\operatorname{Re}\langle x,e_i\rangle}{\|x\|}\cdot\frac{\operatorname{Re}\langle y,e_i\rangle}{\|y\|}\leq M_i\leq 1,\quad i\in I \tag{2.88}$$

and

$$-1\leq n_i\leq\frac{\operatorname{Im}\langle x,e_i\rangle}{\|x\|}\cdot\frac{\operatorname{Im}\langle y,e_i\rangle}{\|y\|}\leq N_i\leq 1,\quad i\in I \tag{2.89}$$

then

$$2\sum_{i\in I}(m_i+n_i)-1\leq\frac{\operatorname{Re}\langle x,y\rangle}{\|x\|\,\|y\|}\leq 1+2\sum_{i\in I}(M_i+N_i). \tag{2.90}$$

Proof. We follow the proof in [**11**].

Using Theorem 21 and the fact that for any complex number z, $|z|\geq|\operatorname{Re}z|$, we have

$$\begin{aligned}\left|\sum_{i\in I}\operatorname{Re}\left[\langle x,e_i\rangle\langle e_i,y\rangle\right]-\frac{1}{2}\operatorname{Re}\langle x,y\rangle\right| &\\ \leq\left|\sum_{i\in I}\langle x,e_i\rangle\langle e_i,y\rangle-\frac{1}{2}\langle x,y\rangle\right| &\\ \leq\frac{1}{2}\|x\|\,\|y\|. &\end{aligned} \tag{2.91}$$

Since

$$\operatorname{Re}\left[\langle x,e_i\rangle\langle e_i,y\rangle\right]=\operatorname{Re}\langle x,e_i\rangle\operatorname{Re}\langle y,e_i\rangle+\operatorname{Im}\langle x,e_i\rangle\operatorname{Im}\langle y,e_i\rangle,$$

hence by (2.91) we have:

$$\begin{aligned}&-\frac{1}{2}\|x\|\,\|y\|+\frac{1}{2}\operatorname{Re}\langle x,y\rangle\\ &\leq\sum_{i\in I}\operatorname{Re}\langle x,e_i\rangle\operatorname{Re}\langle y,e_i\rangle+\sum_{i\in I}\operatorname{Im}\langle x,e_i\rangle\operatorname{Im}\langle y,e_i\rangle\\ &\leq\frac{1}{2}\|x\|\,\|y\|+\frac{1}{2}\operatorname{Re}\langle x,y\rangle.\end{aligned} \tag{2.92}$$

Utilising the assumptions (2.88) and (2.89), we have

$$\sum_{i\in I}m_i\leq\sum_{i\in I}\frac{\operatorname{Re}\langle x,e_i\rangle\operatorname{Re}\langle y,e_i\rangle}{\|x\|\,\|y\|}\leq\sum_{i\in I}M_i \tag{2.93}$$

and

$$\sum_{i\in I} n_i \leq \sum_{i\in I} \frac{\operatorname{Im}\langle x, e_i\rangle \operatorname{Im}\langle y, e_i\rangle}{\|x\|\,\|y\|} \leq \sum_{i\in I} N_i. \tag{2.94}$$

Finally, on making use of (2.92) – (2.94), we deduce the desired result (2.90). ∎

Remark 24. *By Schwarz's inequality, is it obvious that, in general,*

$$-1 \leq \frac{\operatorname{Re}\langle x, y\rangle}{\|x\|\,\|y\|} \leq 1.$$

Consequently, the left inequality in (2.90) is of interest when $\sum_{i\in I}(m_i + n_i) > 0$, *while the right inequality in (2.90) is of interest when* $\sum_{i\in I}(M_i + N_i) < 0$.

2.5.3. Refinements of Kurepa's Inequality. The following result holds [**11**].

Theorem 22 (Dragomir, 2004). *Let* $\{e_j\}_{j\in I}$ *be a finite orthonormal family in the real inner product space* $(H; \langle\cdot,\cdot\rangle)$. *Then for any* $w \in H_{\mathbb{C}}$, *where* $(H_{\mathbb{C}}; \langle\cdot,\cdot\rangle_{\mathbb{C}})$ *is the complexification of* $(H; \langle\cdot,\cdot\rangle)$, *one has the following Bessel's type inequality:*

$$\left|\sum_{j\in I}\langle w, e_j\rangle_{\mathbb{C}}^2\right| \leq \left|\sum_{j\in I}\langle w, e_j\rangle_{\mathbb{C}}^2 - \frac{1}{2}\langle w, \bar{w}\rangle_{\mathbb{C}}\right| + \frac{1}{2}\left|\langle w, \bar{w}\rangle_{\mathbb{C}}\right| \tag{2.95}$$
$$\leq \frac{1}{2}\left[\|w\|_{\mathbb{C}}^2 + \left|\langle w, \bar{w}\rangle_{\mathbb{C}}\right|\right] \leq \|w\|_{\mathbb{C}}^2 .$$

Proof. We follow the proof in [**11**].

Define $f_j \in H_{\mathbb{C}}$, $f_j := (e_j, 0)$, $j \in I$. For any $k, j \in I$ we have

$$\langle f_i, f_j\rangle_{\mathbb{C}} = \langle (e_k, 0), (e_j, 0)\rangle_{\mathbb{C}} = \langle e_k, e_j\rangle = \delta_{kj},$$

therefore $\{f_j\}_{j\in I}$ is an orthonormal family in $(H_{\mathbb{C}}; \langle\cdot,\cdot\rangle_{\mathbb{C}})$.

If we apply Theorem 21 for $(H_{\mathbb{C}}; \langle\cdot,\cdot\rangle_{\mathbb{C}})$, $a = w$, $b = \bar{w}$, we may write:

$$\left|\sum_{j\in I}\langle w, e_j\rangle_{\mathbb{C}}\langle e_j, \bar{w}\rangle_{\mathbb{C}} - \frac{1}{2}\langle w, \bar{w}\rangle_{\mathbb{C}}\right| \leq \frac{1}{2}\|w\|_{\mathbb{C}}\|\bar{w}\|_{\mathbb{C}} . \tag{2.96}$$

However, for $w := (x, y) \in H_{\mathbb{C}}$, we have $\bar{w} = (x, -y)$ and

$$\langle e_j, \bar{w}\rangle_{\mathbb{C}} = \langle (e_j, 0), (x, -y)\rangle_{\mathbb{C}} = \langle e_j, x\rangle - i\langle e_j, -y\rangle = \langle e_j, x\rangle + i\langle e_j, y\rangle$$

and

$$\langle w, e_j\rangle_{\mathbb{C}} = \langle (x, y), (e_j, 0)\rangle_{\mathbb{C}} = \langle e_j, x\rangle - i\langle e_j, -y\rangle = \langle x, e_j\rangle + i\langle e_j, y\rangle$$

for any $j \in I$. Thus $\langle e_j, \bar{w} \rangle = \langle w, e_j \rangle$ for each $j \in I$ and since

$$\|w\|_{\mathbb{C}} = \|\bar{w}\|_{\mathbb{C}} = \left(\|x\|^2 + \|y\|^2\right)^{\frac{1}{2}},$$

we get from (2.96) that

$$\left| \sum_{j\in I} \langle w, e_j \rangle_{\mathbb{C}}^2 - \frac{1}{2} \langle w, \bar{w} \rangle_{\mathbb{C}} \right| \leq \frac{1}{2} \|w\|_{\mathbb{C}}^2 . \tag{2.97}$$

Now, observe that the first inequality in (2.95) follows by the triangle inequality, the second is an obvious consequence of (2.97) and the last one is derived from Schwarz's result. ∎

REMARK 25. *If the family* $\{e_j\}_{j\in I}$ *contains only a single element* $e = \frac{x}{\|x\|}$, $x \in H$, $x \neq 0$, *then from (2.95) we deduce (2.72), which, in its turn, provides a refinement of Kurepa's inequality (2.68).*

2.5.4. An Application for $L_2[-\pi, \pi]$. It is well known that in the Hilbert space $L_2[-\pi, \pi]$ of all functions $f : [-\pi, \pi] \to \mathbb{C}$ with the property that f is Lebesgue measurable on $[-\pi, \pi]$ and $\int_{-\pi}^{\pi} |f(t)|^2 dt < \infty$, the set of functions

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos t, \frac{1}{\sqrt{\pi}} \sin t, \dots, \frac{1}{\sqrt{\pi}} \cos nt, \frac{1}{\sqrt{\pi}} \sin nt, \dots \right\}$$

is orthonormal.

If by $\operatorname{trig} t$, we denote either $\sin t$ or $\cos t$, $t \in [-\pi, \pi]$, then on using the results from Sections 2.5.2 and 2.5.3, we may state the following inequality:

$$\begin{aligned} &\left| \frac{1}{\pi} \sum_{k=1}^{n} \int_{-\pi}^{\pi} f(t) \operatorname{trig}(kt)\, dt \cdot \int_{-\pi}^{\pi} \overline{g(t)} \operatorname{trig}(kt)\, dt \right. \\ &\qquad \left. - \frac{1}{2} \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt \right|^2 \\ &\qquad\qquad \leq \frac{1}{4} \int_{-\pi}^{\pi} |f(t)|^2 dt \int_{-\pi}^{\pi} |g(t)|^2 dt, \end{aligned} \tag{2.98}$$

where all $\operatorname{trig}(kt)$ is either $\sin kt$ or $\cos kt$, $k \in \{1, \dots, n\}$ and $f \in L_2[-\pi, \pi]$.

This follows by Theorem 21.

If one uses Corollary 12, then one can state the following chain of inequalities

$$\begin{aligned}
(2.99)\quad & \left|\frac{1}{\pi}\sum_{k=1}^{n}\int_{-\pi}^{\pi} f(t)\operatorname{trig}(kt)\,dt\cdot\int_{-\pi}^{\pi}\overline{g(t)}\operatorname{trig}(kt)\,dt\right| \\
& \leq \left|\frac{1}{\pi}\sum_{k=1}^{n}\int_{-\pi}^{\pi} f(t)\operatorname{trig}(kt)\,dt\cdot\int_{-\pi}^{\pi}\overline{g(t)}\operatorname{trig}(kt)\,dt\right. \\
& \qquad \left. -\frac{1}{2}\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt\right| + \frac{1}{2}\left|\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt\right| \\
& \leq \frac{1}{2}\left[\left(\int_{-\pi}^{\pi}|f(t)|^2\,dt\int_{-\pi}^{\pi}|g(t)|^2\,dt\right)^{\frac{1}{2}} + \left|\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt\right|\right] \\
& \leq \left(\int_{-\pi}^{\pi}|f(t)|^2\,dt\int_{-\pi}^{\pi}|g(t)|^2\,dt\right)^{\frac{1}{2}},
\end{aligned}$$

where $f \in L_2[-\pi,\pi]$.

Finally, by employing Theorem 22, we may state:

$$\begin{aligned}
& \frac{1}{\pi}\left|\sum_{k=1}^{n}\left[\int_{-\pi}^{\pi} f(t)\operatorname{trig}(kt)\,dt\right]^2\right| \\
& \leq \left|\frac{1}{\pi}\sum_{k=1}^{n}\left[\int_{-\pi}^{\pi} f(t)\operatorname{trig}(kt)\,dt\right]^2 - \frac{1}{2}\int_{-\pi}^{\pi} f^2(t)\,dt\right| + \frac{1}{2}\left|\int_{-\pi}^{\pi} f^2(t)\,dt\right| \\
& \leq \frac{1}{2}\left[\int_{-\pi}^{\pi}|f(t)|^2\,dt + \left|\int_{-\pi}^{\pi} f^2(t)\,dt\right|\right] \leq \int_{-\pi}^{\pi}|f(t)|^2\,dt,
\end{aligned}$$

where $f \in L_2[-\pi,\pi]$.

2.6. Generalizations of Precupanu 's Inequality

2.6.1. Introduction. In 1976, T. Precupanu [**29**] obtained the following result related to the Schwarz inequality in a real inner product space $(H;\langle\cdot,\cdot\rangle)$:

THEOREM 23 (Precupanu , 1976). *For any $a \in H$, $x, y \in H \setminus \{0\}$, we have the inequality:*

$$\begin{aligned} &\frac{-\|a\|\|b\| + \langle a, b\rangle}{2} \\ &\leq \frac{\langle x, a\rangle \langle x, b\rangle}{\|x\|^2} + \frac{\langle y, a\rangle \langle y, b\rangle}{\|y\|^2} - 2 \cdot \frac{\langle x, a\rangle \langle y, b\rangle \langle x, y\rangle}{\|x\|^2 \|y\|^2} \\ &\leq \frac{\|a\|\|b\| + \langle a, b\rangle}{2}. \end{aligned} \tag{2.100}$$

In the right-hand side or in the left-hand side of (2.100) we have equality if and only if there are $\lambda, \mu \in \mathbb{R}$ such that

$$\lambda \frac{\langle x, a\rangle}{\|x\|^2} \cdot x + \mu \frac{\langle y, b\rangle}{\|y\|^2} \cdot y = \frac{1}{2} (\lambda a + \mu b). \tag{2.101}$$

Note for instance that [**29**], if $y \perp b$, i.e., $\langle y, b\rangle = 0$, then by (2.100) one may deduce:

$$\frac{-\|a\|\|b\| + \langle a, b\rangle}{2} \|x\|^2 \leq \langle x, a\rangle \langle x, b\rangle \leq \frac{\|a\|\|b\| + \langle a, b\rangle}{2} \|x\|^2 \tag{2.102}$$

for any $a, b, x \in H$, an inequality that has been obtained previously by U. Richard [**30**]. The case of equality in the right-hand side or in the left-hand side of (2.102) holds if and only if there are $\lambda, \mu \in \mathbb{R}$ with

$$2\lambda \langle x, a\rangle x = (\lambda a + \mu b) \|x\|^2. \tag{2.103}$$

For $a = b$, we may obtain from (2.100) the following inequality [**29**]

$$0 \leq \frac{\langle x, a\rangle^2}{\|x\|^2} + \frac{\langle y, a\rangle^2}{\|y\|^2} - 2 \cdot \frac{\langle x, a\rangle \langle y, a\rangle \langle x, y\rangle}{\|x\|^2 \|y\|^2} \leq \|a\|^2. \tag{2.104}$$

This inequality implies [**29**]:

$$\frac{\langle x, y\rangle}{\|x\|\|y\|} \geq \frac{1}{2} \left[\frac{\langle x, a\rangle}{\|x\|\|a\|} + \frac{\langle y, a\rangle}{\|y\|\|a\|} \right]^2 - \frac{3}{2}. \tag{2.105}$$

In [**27**], M.H. Moore pointed out the following reverse of the Schwarz inequality

$$|\langle y, z\rangle| \leq \|y\|\|z\|, \qquad y, z \in H, \tag{2.106}$$

where some information about a third vector x is known:

THEOREM 24 (Moore, 1973). *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real field $\mathbb{R}$ and $x, y, z \in H$ such that:*

$$|\langle x, y\rangle| \geq (1 - \varepsilon) \|x\|\|y\|, \qquad |\langle x, z\rangle| \geq (1 - \varepsilon) \|x\|\|z\|, \tag{2.107}$$

where ε is a positive real number, reasonably small. Then

$$|\langle y, z\rangle| \geq \max\left\{1-\varepsilon-\sqrt{2\varepsilon}, 1-4\varepsilon, 0\right\}\|y\|\|z\|. \tag{2.108}$$

Utilising Richard's inequality (2.102) written in the following equivalent form:

$$2\cdot\frac{\langle x, a\rangle\langle x, b\rangle}{\|x\|^2}-\|a\|\|b\| \leq \langle a, b\rangle \leq 2\cdot\frac{\langle x, a\rangle\langle x, b\rangle}{\|x\|^2}+\|a\|\|b\| \tag{2.109}$$

for any $a, b \in H$ and $a \in H\backslash\{0\}$, Precupanu has obtained the following Moore's type result:

Theorem 25 (Precupanu , 1976). *Let $(H; \langle\cdot,\cdot\rangle)$ be a real inner product space. If $a, b, x \in H$ and $0 < \varepsilon_1 < \varepsilon_2$ are such that:*

$$\begin{aligned}\varepsilon_1\|x\|\|a\| &\leq \langle x, a\rangle \leq \varepsilon_2\|x\|\|a\|,\\ \varepsilon_1\|x\|\|b\| &\leq \langle x, b\rangle \leq \varepsilon_2\|x\|\|b\|,\end{aligned} \tag{2.110}$$

then

$$\left(2\varepsilon_1^2-1\right)\|a\|\|b\| \leq \langle a, b\rangle \leq \left(2\varepsilon_1^2+1\right)\|a\|\|b\|. \tag{2.111}$$

Remark that the right inequality is always satisfied, since by Schwarz's inequality, we have $\langle a, b\rangle \leq \|a\|\|b\|$. The left inequality may be useful when one assumes that $\varepsilon_1 \in (0, 1]$. In that case, from (2.111), we obtain

$$-\|a\|\|b\| \leq \left(2\varepsilon_1^2-1\right)\|a\|\|b\| \leq \langle a, b\rangle \tag{2.112}$$

provided $\varepsilon_1\|x\|\|a\| \leq \langle x, a\rangle$ and $\varepsilon_1\|x\|\|b\| \leq \langle x, b\rangle$, which is a refinement of Schwarz's inequality

$$-\|a\|\|b\| \leq \langle a, b\rangle.$$

In the complex case, apparently independent of Richard, M.L. Buzano obtained in [**3**] the following inequality

$$|\langle x, a\rangle\langle x, b\rangle| \leq \frac{\|a\|\|b\|+|\langle a, b\rangle|}{2}\cdot\|x\|^2, \tag{2.113}$$

provided x, a, b are vectors in the complex inner product space $(H; \langle\cdot,\cdot\rangle)$.

In the same paper [**29**], Precupanu , without mentioning Buzano's name in relation to the inequality (2.113), observed that, on utilising (2.113), one may obtain the following result of Moore type:

Theorem 26 (Precupanu , 1976). *Let $(H; \langle\cdot,\cdot\rangle)$ be a (real or) complex inner product space. If $x, a, b \in H$ are such that*

$$|\langle x, a\rangle| \geq (1-\varepsilon)\|x\|\|a\|, \qquad |\langle x, b\rangle| \geq (1-\varepsilon)\|x\|\|b\|, \tag{2.114}$$

then

$$|\langle a, b\rangle| \geq \left(1-4\varepsilon+2\varepsilon^2\right)\|a\|\|b\|. \tag{2.115}$$

Note that the above theorem is useful when, for $\varepsilon \in (0,1]$, the quantity $1-4\varepsilon+2\varepsilon^2>0$, i.e., $\varepsilon \in \left(0, 1-\frac{\sqrt{2}}{2}\right]$.

Remark 26. *When the space is real, the inequality (2.115) provides a better lower bound for $|\langle a,b\rangle|$ than the second bound in Moore's result (2.108). However, it is not known if the first bound in (2.108) remains valid for the case of complex spaces. From Moore's original proof, apparently, the fact that the space $(H;\langle\cdot,\cdot\rangle)$ is real plays an essential role.*

Before we point out some new results for orthonormal families of vectors in real or complex inner product spaces, we state the following result that complements the Moore type results outlined above for real spaces [**10**]:

Theorem 27 (Dragomir, 2004). *Let $(H;\langle\cdot,\cdot\rangle)$ be a real inner product space and $a,b,x,y \in H\backslash\{0\}$.*

(i) *If there exist $\delta_1,\delta_2 \in (0,1]$ such that*

$$\frac{\langle x,a\rangle}{\|x\|\|a\|} \geq \delta_1, \qquad \frac{\langle y,a\rangle}{\|y\|\|a\|} \geq \delta_2$$

and $\delta_1+\delta_2 \geq 1$, then

$$\frac{\langle x,y\rangle}{\|x\|\|y\|} \geq \frac{1}{2}(\delta_1+\delta_2)^2-\frac{3}{2} \qquad (\geq -1). \tag{2.116}$$

(ii) *If there exist $\mu_1\,(\mu_2) \in \mathbb{R}$ such that*

$$\mu_1\|a\|\|b\| \leq \frac{\langle x,a\rangle\langle x,b\rangle}{\|x\|^2}\,(\leq \mu_2\|a\|\|b\|)$$

and $1 \geq \mu_1 \geq 0\ (-1 \leq \mu_2 \leq 0)$, then

$$[-1 \leq]\,2\mu_1-1 \leq \frac{\langle a,b\rangle}{\|a\|\|b\|}\,(\leq 2\mu_2+1\,[\leq 1]). \tag{2.117}$$

The proof is obvious by the inequalities (2.105) and (2.109). We omit the details.

2.6.2. Inequalities for Orthonormal Families.

The following result may be stated [**10**].

Theorem 28 (Dragomir, 2004). *Let $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ be two finite families of orthonormal vectors in $(H;\langle\cdot,\cdot\rangle)$. For any $x,y \in$*

$H\backslash\{0\}$ *one has the inequality*

$$\left|\sum_{i\in I}\langle x,e_i\rangle\langle e_i,y\rangle+\sum_{j\in J}\langle x,f_j\rangle\langle f_j,y\rangle\right.\\ \left.-2\sum_{i\in I,j\in J}\langle x,e_i\rangle\langle f_j,y\rangle\langle e_i,f_j\rangle-\frac{1}{2}\langle x,y\rangle\right|\leq\frac{1}{2}\|x\|\|y\|. \tag{2.118}$$

The case of equality holds in (2.118) if and only if there exists a $\lambda\in\mathbb{K}$ *such that*

$$x-\lambda y=2\left(\sum_{i\in I}\langle x,e_i\rangle e_i-\lambda\sum_{j\in J}\langle y,f_j\rangle f_j\right). \tag{2.119}$$

Proof. We follow the proof in [**10**].

We know that, if $u,v\in H$, $v\neq 0$, then

$$\left\|u-\frac{\langle u,v\rangle}{\|v\|^2}\cdot v\right\|^2=\frac{\|u\|^2\|v\|^2-|\langle u,v\rangle|^2}{\|v\|^2} \tag{2.120}$$

showing that, in Schwarz's inequality

$$|\langle u,v\rangle|^2\leq\|u\|^2\|v\|^2, \tag{2.121}$$

the case of equality, for $v\neq 0$, holds if and only if

$$u=\frac{\langle u,v\rangle}{\|v\|^2}\cdot v, \tag{2.122}$$

i.e. there exists a $\lambda\in\mathbb{R}$ such that $u=\lambda v$.

Now, let $u:=2\sum_{i\in I}\langle x,e_i\rangle e_i-x$ and $v:=2\sum_{j\in J}\langle y,f_j\rangle f_j-y$.

Observe that

$$\begin{aligned}\|u\|^2&=\left\|2\sum_{i\in I}\langle x,e_i\rangle e_i\right\|^2-4\operatorname{Re}\left\langle\sum_{i\in I}\langle x,e_i\rangle e_i,x\right\rangle+\|x\|^2\\&=4\sum_{i\in I}|\langle x,e_i\rangle|^2-4\sum_{i\in I}|\langle x,e_i\rangle|^2+\|x\|^2=\|x\|^2,\end{aligned}$$

and, similarly

$$\|v\|^2=\|y\|^2.$$

Also,

$$\langle u,v\rangle=4\sum_{i\in I,j\in J}\langle x,e_i\rangle\langle f_j,y\rangle\langle e_i,f_j\rangle+\langle x,y\rangle\\-2\sum_{i\in I}\langle x,e_i\rangle\langle e_i,y\rangle-2\sum_{j\in J}\langle x,f_j\rangle\langle f_j,y\rangle.$$

Therefore, by Schwarz's inequality (2.121) we deduce the desired inequality (2.118). By (2.122), the case of equality holds in (2.118) if and only if there exists a $\lambda \in \mathbb{K}$ such that

$$2\sum_{i\in I}\langle x,e_i\rangle e_i - x = \lambda\left(2\sum_{j\in J}\langle y,f_j\rangle f_j - y\right),$$

which is equivalent to (2.119). ∎

Remark 27. *If in (2.119) we choose $x = y$, then we get the inequality:*

$$\left|\sum_{i\in I}|\langle x,e_i\rangle|^2 + \sum_{j\in J}|\langle x,f_j\rangle|^2 - 2\sum_{i\in I,j\in J}\langle x,e_i\rangle\langle f_j,x\rangle\langle e_i,f_j\rangle - \frac{1}{2}\|x\|^2\right| \le \frac{1}{2}\|x\|^2 \tag{2.123}$$

for any $x \in H$.

If in the above theorem we assume that $I = J$ and $f_i = e_i$, $i \in I$, then we get from (2.118) the Schwarz inequality $|\langle x,y\rangle| \le \|x\|\|y\|$.

If $I \cap J = \varnothing$, $I \cup J = K$, $g_k = e_k$, $k \in I$, $g_k = f_k$, $k \in J$ and $\{g_k\}_{k\in K}$ is orthonormal, then from (2.118) we get:

$$\left|\sum_{k\in K}\langle x,g_k\rangle\langle g_k,y\rangle - \frac{1}{2}\langle x,y\rangle\right| \le \frac{1}{2}\|x\|\|y\|, \qquad x,y \in H \tag{2.124}$$

which has been obtained earlier by the author in [**16**].

If I and J reduce to one element, namely $e_1 = \frac{e}{\|e\|}$, $f_1 = \frac{f}{\|f\|}$ with $e,f \neq 0$, then from (2.118) we get

$$\left|\frac{\langle x,e\rangle\langle e,y\rangle}{\|e\|^2} + \frac{\langle x,f\rangle\langle f,y\rangle}{\|f\|^2} - 2\cdot\frac{\langle x,e\rangle\langle f,y\rangle\langle e,f\rangle}{\|e\|^2\|f\|^2} - \frac{1}{2}\langle x,y\rangle\right| \le \frac{1}{2}\|x\|\|y\|, \qquad x,y \in H \tag{2.125}$$

which is the corresponding complex version of Precupanu 's inequality (2.100).

If in (2.125) we assume that $x = y$, then we get

$$\left|\frac{|\langle x,e\rangle|^2}{\|e\|^2} + \frac{|\langle x,f\rangle|^2}{\|f\|^2} - 2\cdot\frac{\langle x,e\rangle\langle f,e\rangle\langle e,f\rangle}{\|e\|^2\|f\|^2} - \frac{1}{2}\|x\|^2\right| \le \frac{1}{2}\|x\|^2. \tag{2.126}$$

The following corollary may be stated [**10**]:

COROLLARY 14 (Dragomir, 2004). *With the assumptions of Theorem 28, we have:*

$$\left|\sum_{i\in I}\langle x,e_i\rangle\langle e_i,y\rangle+\sum_{j\in J}\langle x,f_j\rangle\langle f_j,y\rangle-2\sum_{i\in I,j\in J}\langle x,e_i\rangle\langle f_j,y\rangle\langle e_i,f_j\rangle\right|$$
$$\le\frac{1}{2}\left|\langle x,y\rangle\right|+\left|\sum_{i\in I}\langle x,e_i\rangle\langle e_i,y\rangle+\sum_{j\in J}\langle x,f_j\rangle\langle f_j,y\rangle-2\sum_{i\in I,j\in J}\langle x,e_i\rangle\langle f_j,y\rangle\langle e_i,f_j\rangle-\frac{1}{2}\left|\langle x,y\rangle\right|\right|$$
$$\le\frac{1}{2}\left[\left|\langle x,y\rangle\right|+\|x\|\,\|y\|\right]. \tag{2.127}$$

PROOF. The first inequality follows by the triangle inequality for the modulus. The second inequality follows by (2.118) on adding the quantity $\frac{1}{2}\left|\langle x,y\rangle\right|$ on both sides. ∎

REMARK 28. (1) *If we choose in (2.127), $x=y$, then we get:*

$$\left|\sum_{i\in I}\left|\langle x,e_i\rangle\right|^2+\sum_{j\in J}\left|\langle x,f_j\rangle\right|^2-2\sum_{i\in I,j\in J}\langle x,e_i\rangle\langle f_j,x\rangle\langle e_i,f_j\rangle\right|$$
$$\le\left|\sum_{i\in I}\left|\langle x,e_i\rangle\right|^2+\sum_{j\in J}\left|\langle x,f_j\rangle\right|^2-2\sum_{i\in I,j\in J}\langle x,e_i\rangle\langle f_j,x\rangle\langle e_i,f_j\rangle-\frac{1}{2}\|x\|^2\right|+\frac{1}{2}\|x\|^2$$
$$\le\|x\|^2. \tag{2.128}$$

We observe that (2.128) will generate Bessel's inequality if $\{e_i\}_{i\in I}$, $\{f_j\}_{j\in J}$ are disjoint parts of a larger orthonormal family.

(2) *From (2.125) one can obtain:*

$$\left|\frac{\langle x,e\rangle\langle e,y\rangle}{\|e\|^2}+\frac{\langle x,f\rangle\langle f,y\rangle}{\|f\|^2}-2\cdot\frac{\langle x,e\rangle\langle f,y\rangle\langle e,f\rangle}{\|e\|^2\|f\|^2}\right| \leq \frac{1}{2}\left[\|x\|\|y\|+|\langle x,y\rangle|\right] \tag{2.129}$$

and in particular

$$\left|\frac{|\langle x,e\rangle|^2}{\|e\|^2}+\frac{|\langle x,f\rangle|^2}{\|f\|^2}-2\cdot\frac{\langle x,e\rangle\langle f,e\rangle\langle e,f\rangle}{\|e\|^2\|f\|^2}\right|\leq\|x\|^2, \tag{2.130}$$

for any $x,y\in H$.

The case of real inner products will provide a natural genearlization for Precupanu 's inequality (2.100) [**10**]:

Corollary 15 (Dragomir, 2004). *Let $(H;\langle\cdot,\cdot\rangle)$ be a real inner product space and $\{e_i\}_{i\in I}$, $\{f_j\}_{j\in J}$ two finite families of orthonormal vectors in $(H;\langle\cdot,\cdot\rangle)$. For any $x,y\in H\setminus\{0\}$ one has the double inequality:*

$$\begin{aligned}\frac{1}{2}\left[|\langle x,y\rangle|-\|x\|\|y\|\right]&\leq\sum_{i\in I}\langle x,e_i\rangle\langle y,e_i\rangle+\sum_{j\in J}\langle x,f_j\rangle\langle y,f_j\rangle\\&\qquad-2\sum_{i\in I,j\in J}\langle x,e_i\rangle\langle y,f_j\rangle\langle e_i,f_j\rangle\\&\leq\frac{1}{2}\left[\|x\|\|y\|+|\langle x,y\rangle|\right].\end{aligned} \tag{2.131}$$

In particular, we have

$$\begin{aligned}0&\leq\sum_{i\in I}\langle x,e_i\rangle^2+\sum_{j\in J}\langle x,f_j\rangle^2-2\sum_{i\in I,j\in J}\langle x,e_i\rangle\langle x,f_j\rangle\langle e_i,f_j\rangle\\&\leq\|x\|^2,\end{aligned} \tag{2.132}$$

for any $x\in H$.

Remark 29. *Similar particular inequalities to those incorporated in (2.124) – (2.130) may be stated, but we omit them.*

2.6.3. Refinements of Kurepa's Inequality. The following result may be stated [**10**].

Theorem 29 (Dragomir, 2004). *Let $(H;\langle\cdot,\cdot\rangle)$ be a real inner product space and $\{e_i\}_{i\in I}$, $\{f_j\}_{j\in J}$ two finite families in H. If $(H_{\mathbb{C}};\langle\cdot,\cdot\rangle_{\mathbb{C}})$*

is the complexification of $(H;\langle\cdot,\cdot\rangle)$ *, then for any* $w\in H_{\mathbb{C}}$*, we have the inequalities*

$$\left|\sum_{i\in I}\langle w,e_i\rangle_{\mathbb{C}}^2+\sum_{j\in J}\langle w,f_j\rangle_{\mathbb{C}}^2-2\sum_{i\in I,j\in J}\langle w,e_i\rangle_{\mathbb{C}}\langle w,f_j\rangle_{\mathbb{C}}\langle e_i,f_j\rangle\right| \tag{2.133}$$
$$\leq\frac{1}{2}\left|\langle w,\bar{w}\rangle_{\mathbb{C}}\right|+\left|\sum_{i\in I}\langle w,e_i\rangle_{\mathbb{C}}^2+\sum_{j\in J}\langle w,f_j\rangle_{\mathbb{C}}^2\right.$$
$$\left.-2\sum_{i\in I,j\in J}\langle w,e_i\rangle_{\mathbb{C}}\langle w,f_j\rangle_{\mathbb{C}}\langle e_i,f_j\rangle-\frac{1}{2}\langle w,\bar{w}\rangle_{\mathbb{C}}\right|$$
$$\leq\frac{1}{2}\left[\|w\|_{\mathbb{C}}^2+\left|\langle w,\bar{w}\rangle_{\mathbb{C}}\right|\right]\leq\|w\|_{\mathbb{C}}^2 .$$

Proof. Define $g_j\in H_{\mathbb{C}}$, $g_j:=(e_j,0)$, $j\in I$. For any $k,j\in I$ we have

$$\langle g_k,g_j\rangle_{\mathbb{C}}=\langle(e_k,0),(e_j,0)\rangle_{\mathbb{C}}=\langle e_k,e_j\rangle=\delta_{kj},$$

therefore $\{g_j\}_{j\in I}$ is an orthonormal family in $(H_{\mathbb{C}};\langle\cdot,\cdot\rangle_{\mathbb{C}})$.

If we apply Corollary 14 for $(H_{\mathbb{C}};\langle\cdot,\cdot\rangle_{\mathbb{C}})$, $x=w$, $y=\bar{w}$, we may write:

$$\left|\sum_{i\in I}\langle w,e_i\rangle_{\mathbb{C}}\langle e_i,\bar{w}\rangle_{\mathbb{C}}+\sum_{j\in J}\langle w,f_j\rangle\langle f_j,\bar{w}\rangle\right. \tag{2.134}$$
$$\left.-2\sum_{i\in I,j\in J}\langle w,e_i\rangle_{\mathbb{C}}\langle f_j,\overline{w}\rangle_{\mathbb{C}}\langle e_i,f_j\rangle\right|$$
$$\leq\frac{1}{2}\|w\|_{\mathbb{C}}\|\bar{w}\|_{\mathbb{C}}+\left|\sum_{i\in I}\langle w,e_i\rangle_{\mathbb{C}}\langle e_i,\bar{w}\rangle_{\mathbb{C}}+\sum_{j\in J}\langle w,f_j\rangle\langle f_j,\bar{w}\rangle\right.$$
$$\left.-2\sum_{i\in I,j\in J}\langle w,e_i\rangle_{\mathbb{C}}\langle f_j,\overline{w}\rangle_{\mathbb{C}}\langle e_i,f_j\rangle-\frac{1}{2}\langle w,\bar{w}\rangle_{\mathbb{C}}\right|$$
$$\leq\frac{1}{2}\left[\left|\langle w,\bar{w}\rangle_{\mathbb{C}}\right|+\|w\|_{\mathbb{C}}\|\bar{w}\|_{\mathbb{C}}\right].$$

However, for $w:=(x,y)\in H_{\mathbb{C}}$, we have $\bar{w}=(x,-y)$ and

$$\langle e_j,\bar{w}\rangle_{\mathbb{C}}=\langle(e_j,0),(x,-y)\rangle_{\mathbb{C}}=\langle e_j,x\rangle+i\langle e_j,y\rangle$$

and

$$\langle w,e_j\rangle_{\mathbb{C}}=\langle(x,y),(e_j,0)\rangle_{\mathbb{C}}=\langle x,e_j\rangle+i\langle e_j,y\rangle$$

showing that $\langle e_j, \bar{w}\rangle = \langle w, e_j\rangle$ for any $j \in I$. A similar relation is true for f_j and since

$$\|w\|_{\mathbb{C}} = \|\bar{w}\|_{\mathbb{C}} = \left(\|x\|^2 + \|y\|^2\right)^{\frac{1}{2}},$$

hence from (2.134) we deduce the desired inequality (2.133). ∎

Remark 30. *It is obvious that, if one family, say $\{f_j\}_{j\in J}$ is empty, then, on observing that all sums $\sum_{j\in J}$ should be zero, from (2.133) one would get* [**16**]

$$\begin{aligned}
(2.135) \qquad & \left|\sum_{i\in I} \langle w, e_i\rangle_{\mathbb{C}}^2\right| \\
& \leq \frac{1}{2}\left|\langle w, \bar{w}\rangle_{\mathbb{C}}\right| + \left|\sum_{i\in I} \langle w, e_i\rangle_{\mathbb{C}}^2 - \frac{1}{2}\langle w, \bar{w}\rangle_{\mathbb{C}}\right| \\
& \leq \frac{1}{2}\left[\|w\|_{\mathbb{C}}^2 + \left|\langle w, \bar{w}\rangle_{\mathbb{C}}\right|\right] \leq \|w\|_{\mathbb{C}}^2 .
\end{aligned}$$

If in (2.135) one assumes that the family $\{e_i\}_{i\in I}$ contains only one element $e = \frac{a}{\|a\|}, a \neq 0$, then by selecting $w = z$, one would deduce (2.71), which is a refinement for Kurepa's inequality.

2.7. Some New Refinements of the Schwarz Inequality

2.7.1. Refinements. The following result holds [**12**].

Theorem 30 (Dragomir, 2004). *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $r_1, r_2 > 0$. If $x, y \in H$ are with the property that*

$$\|x - y\| \geq r_2 \geq r_1 \geq \left|\|x\| - \|y\|\right|, \tag{2.136}$$

then we have the following refinement of Schwarz's inequality

$$\|x\|\,\|y\| - \operatorname{Re}\langle x, y\rangle \geq \frac{1}{2}\left(r_2^2 - r_1^2\right) (\geq 0) . \tag{2.137}$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a larger quantity.

Proof. From the first inequality in (2.136) we have

$$\|x\|^2 + \|y\|^2 \geq r_2^2 + 2\operatorname{Re}\langle x, y\rangle . \tag{2.138}$$

Subtracting in (2.138) the quantity $2\|x\|\,\|y\|$, we get

$$\left(\|x\| - \|y\|\right)^2 \geq r_2^2 - 2\left(\|x\|\,\|y\| - \operatorname{Re}\langle x, y\rangle\right) . \tag{2.139}$$

Since, by the second inequality in (2.136) we have

$$r_1^2 \geq (\|x\| - \|y\|)^2, \tag{2.140}$$

hence from (2.139) and (2.140) we deduce the desired inequality (2.137).

To prove the sharpness of the constant $\frac{1}{2}$ in (2.137), let us assume that there is a constant $C > 0$ such that

$$\|x\| \|y\| - \operatorname{Re} \langle x, y \rangle \geq C \left(r_2^2 - r_1^2 \right), \tag{2.141}$$

provided that x and y satisfy (2.136).

Let $e \in H$ with $\|e\| = 1$ and for $r_2 > r_1 > 0$, define

$$x = \frac{r_2 + r_1}{2} \cdot e \quad \text{and} \quad y = \frac{r_1 - r_2}{2} \cdot e. \tag{2.142}$$

Then

$$\|x - y\| = r_2 \text{ and } |\|x\| - \|y\|| = r_1,$$

showing that the condition (2.136) is fulfilled with equality.

If we replace x and y as defined in (2.142) into the inequality (2.141), then we get

$$\frac{r_2^2 - r_1^2}{2} \geq C \left(r_2^2 - r_1^2 \right),$$

which implies that $C \leq \frac{1}{2}$, and the theorem is completely proved. ∎

The following corollary holds.

Corollary 16. *With the assumptions of Theorem 30, we have the inequality:*

$$\|x\| + \|y\| - \frac{\sqrt{2}}{2} \|x + y\| \geq \frac{\sqrt{2}}{2} \sqrt{r_2^2 - r_1^2}. \tag{2.143}$$

Proof. We have, by (2.137), that

$$(\|x\| + \|y\|)^2 - \|x + y\|^2 = 2 (\|x\| \|y\| - \operatorname{Re} \langle x, y \rangle) \geq r_2^2 - r_1^2 \geq 0$$

which gives

$$(\|x\| + \|y\|)^2 \geq \|x + y\|^2 + \left(\sqrt{r_2^2 - r_1^2} \right)^2. \tag{2.144}$$

By making use of the elementary inequality

$$2 \left(\alpha^2 + \beta^2 \right) \geq (\alpha + \beta)^2, \qquad \alpha, \beta \geq 0;$$

we get

$$\|x + y\|^2 + \left(\sqrt{r_2^2 - r_1^2} \right)^2 \geq \frac{1}{2} \left(\|x + y\| + \sqrt{r_2^2 - r_1^2} \right)^2. \tag{2.145}$$

Utilising (2.144) and (2.145), we deduce the desired inequality (2.143). ∎

If $(H;\langle\cdot,\cdot\rangle)$ is a Hilbert space and $\{e_i\}_{i\in I}$ is an orthornormal family in H, i.e., we recall that $\langle e_i, e_j\rangle = \delta_{ij}$ for any $i,j \in I$, where δ_{ij} is Kronecker's delta, then we have the following inequality which is well known in the literature as *Bessel's inequality*

$$\sum_{i\in I} |\langle x, e_i\rangle|^2 \le \|x\|^2 \quad \text{for each} \quad x \in H. \tag{2.146}$$

Here, the meaning of the sum is

$$\sum_{i\in I} |\langle x, e_i\rangle|^2 = \sup_{F\subset I}\left\{\sum_{i\in F} |\langle x, e_i\rangle|^2,\ F \text{ is a finite part of } I\right\}.$$

The following result providing a refinement of the Bessel inequality (2.146) holds [**12**].

Theorem 31 (Dragomir, 2004). *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space and $\{e_i\}_{i\in I}$ an orthonormal family in H. If $x \in H$, $x \neq 0$, and $r_2, r_1 > 0$ are such that:*

$$\begin{aligned} &\left\| x - \sum_{i\in I} \langle x, e_i\rangle e_i \right\| \\ &\qquad \ge r_2 \ge r_1 \ge \|x\| - \left(\sum_{i\in I} |\langle x, e_i\rangle|^2\right)^{\frac{1}{2}} (\ge 0), \end{aligned} \tag{2.147}$$

then we have the inequality

$$\|x\| - \left(\sum_{i\in I} |\langle x, e_i\rangle|^2\right)^{\frac{1}{2}} \ge \frac{1}{2}\cdot\frac{r_2^2 - r_1^2}{\left(\sum_{i\in I} |\langle x, e_i\rangle|^2\right)^{\frac{1}{2}}} (\ge 0). \tag{2.148}$$

The constant $\frac{1}{2}$ is best possible.

Proof. Consider $y := \sum_{i\in I} \langle x, e_i\rangle e_i$. Obviously, since H is a Hilbert space, $y \in H$. We also note that

$$\|y\| = \left\|\sum_{i\in I} \langle x, e_i\rangle e_i\right\| = \sqrt{\left\|\sum_{i\in I} \langle x, e_i\rangle e_i\right\|^2} = \sqrt{\sum_{i\in I} |\langle x, e_i\rangle|^2},$$

and thus (2.147) is in fact (2.136) of Theorem 30.

Since

$$\|x\|\,\|y\| - \operatorname{Re}\langle x, y\rangle = \|x\| \left(\sum_{i\in I} |\langle x, e_i\rangle|^2\right)^{\frac{1}{2}} - \operatorname{Re}\left\langle x, \sum_{i\in I} \langle x, e_i\rangle e_i \right\rangle$$
$$= \left(\sum_{i\in I} |\langle x, e_i\rangle|^2\right)^{\frac{1}{2}} \left[\|x\| - \left(\sum_{i\in I} |\langle x, e_i\rangle|^2\right)^{\frac{1}{2}}\right],$$

hence, by (2.137), we deduce the desired result (2.148).

We will prove the sharpness of the constant for the case of one element, i.e., $I = \{1\}$, $e_1 = e \in H$, $\|e\| = 1$. For this, assume that there exists a constant $D > 0$ such that

$$\|x\| - |\langle x, e\rangle| \geq D \cdot \frac{r_2^2 - r_1^2}{|\langle x, e\rangle|} \tag{2.149}$$

provided $x \in H \setminus \{0\}$ satisfies the condition

$$\|x - \langle x, e\rangle e\| \geq r_2 \geq r_1 \geq \|x\| - |\langle x, e\rangle| . \tag{2.150}$$

Assume that $x = \lambda e + \mu f$ with $e, f \in H$, $\|e\| = \|f\| = 1$ and $e \perp f$. We wish to see if there exists positive numbers λ, μ such that

$$\|x - \langle x, e\rangle e\| = r_2 > r_1 = \|x\| - |\langle x, e\rangle| . \tag{2.151}$$

Since (for $\lambda, \mu > 0$)

$$\|x - \langle x, e\rangle e\| = \mu$$

and

$$\|x\| - |\langle x, e\rangle| = \sqrt{\lambda^2 + \mu^2} - \lambda$$

hence, by (2.151), we get $\mu = r_2$ and

$$\sqrt{\lambda^2 + r_2^2} - \lambda = r_1$$

giving

$$\lambda^2 + r_2^2 = \lambda^2 + 2\lambda r_1 + r_1^2$$

from where we get

$$\lambda = \frac{r_2^2 - r_1^2}{2r_1} > 0.$$

With these values for λ and μ, we have

$$\|x\| - |\langle x, e\rangle| = r_1, \qquad |\langle x, e\rangle| = \frac{r_2^2 - r_1^2}{2r_1}$$

and thus, from (2.149), we deduce

$$r_1 \geq D \cdot \frac{r_2^2 - r_1^2}{\frac{r_2^2 - r_1^2}{2r_1}},$$

giving $D \leq \frac{1}{2}$. This proves the theorem. ∎

The following corollary is obvious.

Corollary 17. *Let $x, y \in H$ with $\langle x, y\rangle \neq 0$ and $r_2 \geq r_1 > 0$ such that*

$$\left\| \|y\| x - \frac{\langle x, y\rangle}{\|y\|} \cdot y \right\| \geq r_2 \|y\| \geq r_1 \|y\| \geq \|x\| \|y\| - |\langle x, y\rangle| \, (\geq 0) . \tag{2.152}$$

Then we have the following refinement of the Schwarz's inequality:

$$\|x\| \|y\| - |\langle x, y\rangle| \geq \frac{1}{2} \left(r_2^2 - r_1^2\right) \frac{\|y\|^2}{|\langle x, y\rangle|} \, (\geq 0) . \tag{2.153}$$

The constant $\frac{1}{2}$ is best possible.

The following lemma holds [**12**].

Lemma 3 (Dragomir, 2004). *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space and $R \geq 1$. For $x, y \in H$, the subsequent statements are equivalent:*

(i) *The following refinement of the triangle inequality holds:*

$$\|x\| + \|y\| \geq R \|x + y\| ; \tag{2.154}$$

(ii) *The following refinement of the Schwarz inequality holds:*

$$\|x\| \|y\| - \operatorname{Re} \langle x, y\rangle \geq \frac{1}{2} \left(R^2 - 1\right) \|x + y\|^2 . \tag{2.155}$$

Proof. Taking the square in (2.154), we have

$$2 \|x\| \|y\| \geq \left(R^2 - 1\right) \|x\|^2 + 2R^2 \operatorname{Re} \langle x, y\rangle + \left(R^2 - 1\right) \|y\|^2 . \tag{2.156}$$

Subtracting from both sides of (2.156) the quantity $2 \operatorname{Re} \langle x, y\rangle$, we obtain

$$\begin{aligned} 2 (\|x\| \|y\| - \operatorname{Re} \langle x, y\rangle) &\geq \left(R^2 - 1\right) \left[\|x\|^2 + 2 \operatorname{Re} \langle x, y\rangle + \|y\|^2\right] \\ &= \left(R^2 - 1\right) \|x + y\|^2 , \end{aligned}$$

which is clearly equivalent to (2.155). ∎

By the use of the above lemma, we may now state the following theorem concerning another refinement of the Schwarz inequality [**12**].

Theorem 32 (Dragomir, 2004). *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field and $R \geq 1$, $r \geq 0$. If $x, y \in H$ are such that*

$$\frac{1}{R} (\|x\| + \|y\|) \geq \|x + y\| \geq r, \tag{2.157}$$

then we have the following refinement of the Schwarz inequality

$$\|x\|\,\|y\| - \operatorname{Re}\langle x,y\rangle \geq \frac{1}{2}\left(R^2-1\right)r^2. \tag{2.158}$$

The constant $\frac{1}{2}$ *is best possible in the sense that it cannot be replaced by a larger quantity.*

Proof. The inequality (2.158) follows easily from Lemma 3. We need only prove that $\frac{1}{2}$ is the best possible constant in (2.158).

Assume that there exists a $C>0$ such that

$$\|x\|\,\|y\| - \operatorname{Re}\langle x,y\rangle \geq C\left(R^2-1\right)r^2 \tag{2.159}$$

provided x,y, R and r satisfy (2.157).

Consider $r=1$, $R>1$ and choose $x=\frac{1-R}{2}e$, $y=\frac{1+R}{2}e$ with $e\in H$, $\|e\|=1$. Then

$$x+y=e,\quad \frac{\|x\|+\|y\|}{R}=1$$

and thus (2.157) holds with equality on both sides.

From (2.159), for the above choices, we have $\frac{1}{2}\left(R^2-1\right)\geq C\left(R^2-1\right)$, which shows that $C\leq\frac{1}{2}$. ∎

Finally, the following result also holds [**12**].

Theorem 33 (Dragomir, 2004). *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$ *and* $r\in(0,1]$. *For* $x,y\in H$, *the following statements are equivalent:*

(i) *We have the inequality*

$$|\|x\|-\|y\||\leq r\|x-y\|; \tag{2.160}$$

(ii) *We have the following refinement of the Schwarz inequality*

$$\|x\|\,\|y\| - \operatorname{Re}\langle x,y\rangle \geq \frac{1}{2}\left(1-r^2\right)\|x-y\|^2. \tag{2.161}$$

The constant $\frac{1}{2}$ *in (2.161) is best possible.*

Proof. Taking the square in (2.160), we have

$$\|x\|^2 - 2\|x\|\,\|y\| + \|y\|^2 \leq r^2\left(\|x\|^2 - 2\operatorname{Re}\langle x,y\rangle + \|y\|^2\right)$$

which is clearly equivalent to

$$\left(1-r^2\right)\left[\|x\|^2 - 2\operatorname{Re}\langle x,y\rangle + \|y\|^2\right] \leq 2\left(\|x\|\,\|y\| - \operatorname{Re}\langle x,y\rangle\right)$$

or with (2.161).

Now, assume that (2.161) holds with a constant $E>0$, i.e.,

$$\|x\|\,\|y\| - \operatorname{Re}\langle x,y\rangle \geq E\left(1-r^2\right)\|x-y\|^2, \tag{2.162}$$

provided (2.160) holds.

Define $x = \frac{r+1}{2}e$, $y = \frac{r-1}{2}e$ with $e \in H$, $\|e\| = 1$. Then

$$|\|x\| - \|y\|| = r, \quad \|x - y\| = 1$$

showing that (2.160) holds with equality.

If we replace x and y in (2.162), then we get $E\left(1 - r^2\right) \leq \frac{1}{2}\left(1 - r^2\right)$, implying that $E \leq \frac{1}{2}$. ∎

2.7.2. Discrete Inequalities. Assume that $(K; (\cdot, \cdot))$ is a Hilbert space over the real or complex number field . Assume also that $p_i \geq 0$, $i \in H$ with $\sum_{i=1}^{\infty} p_i = 1$ and define

$$\ell_p^2(K) := \left\{ \mathbf{x} := (x_i)_{i \in \mathbb{N}} \middle| x_i \in \mathbb{K}, \ i \in \mathbb{N} \text{ and } \sum_{i=1}^{\infty} p_i \|x_i\|^2 < \infty \right\}.$$

It is well known that $\ell_p^2(K)$ endowed with the inner product $\langle \cdot, \cdot \rangle_p$ defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle_p := \sum_{i=1}^{\infty} p_i (x_i, y_i)$$

and generating the norm

$$\|\mathbf{x}\|_p := \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}}$$

is a Hilbert space over $\mathbb{K}$.

We may state the following discrete inequality improving the Cauchy-Bunyakovsky-Schwarz classical result [**12**].

Proposition 16. *Let $(K; (\cdot, \cdot))$ be a Hilbert space and $p_i \geq 0$ $(i \in \mathbb{N})$ with $\sum_{i=1}^{\infty} p_i = 1$. Assume that $\mathbf{x}, \mathbf{y} \in \ell_p^2(K)$ and $r_1, r_2 > 0$ satisfy the condition*

$$\|x_i - y_i\| \geq r_2 \geq r_1 \geq |\|x_i\| - \|y_i\|| \tag{2.163}$$

for each $i \in \mathbb{N}$. Then we have the following refinement of the Cauchy-Bunyakovsky-Schwarz inequality

$$\left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 \right)^{\frac{1}{2}} - \sum_{i=1}^{\infty} p_i \operatorname{Re}(x_i, y_i) \geq \frac{1}{2}\left(r_2^2 - r_1^2\right) \geq 0. \tag{2.164}$$

The constant $\frac{1}{2}$ is best possible.

PROOF. From the condition (2.163) we simply deduce

$$\sum_{i=1}^{\infty} p_i \|x_i - y_i\|^2 \geq r_2^2 \geq r_1^2 \geq \sum_{i=1}^{\infty} p_i \left(\|x_i\| - \|y_i\|\right)^2 \tag{2.165}$$
$$\geq \left[\left(\sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \left(\sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}}\right]^2.$$

In terms of the norm $\|\cdot\|_p$, the inequality (2.165) may be written as

$$\|\mathbf{x} - \mathbf{y}\|_p \geq r_2 \geq r_1 \geq \left|\|\mathbf{x}\|_p - \|\mathbf{y}\|_p\right|. \tag{2.166}$$

Utilising Theorem 30 for the Hilbert space $\left(\ell_p^2(K), \langle\cdot,\cdot\rangle_p\right)$, we deduce the desired inequality (2.164).

For $n = 1$ $(p_1 = 1)$, the inequality (2.164) reduces to (2.137) for which we have shown that $\frac{1}{2}$ is the best possible constant. ∎

By the use of Corollary 16, we may state the following result as well.

COROLLARY 18. *With the assumptions of Proposition 16, we have the inequality*

$$\left(\sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}} \tag{2.167}$$
$$- \frac{\sqrt{2}}{2}\left(\sum_{i=1}^{\infty} p_i \|x_i + y_i\|^2\right)^{\frac{1}{2}} \geq \frac{\sqrt{2}}{2}\sqrt{r_2^2 - r_1^2}.$$

The following proposition also holds [**12**].

PROPOSITION 17. *Let $(K; (\cdot,\cdot))$ be a Hilbert space and $p_i \geq 0$ $(i \in \mathbb{N})$ with $\sum_{i=1}^{\infty} p_i = 1$. Assume that $\mathbf{x}, \mathbf{y} \in \ell_p^2(K)$ and $R \geq 1$, $r \geq 0$ satisfy the condition*

$$\frac{1}{R}\left(\|x_i\| + \|y_i\|\right) \geq \|x_i + y_i\| \geq r \tag{2.168}$$

for each $i \in \mathbb{N}$. Then we have the following refinement of the Schwarz inequality

$$\left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}} - \sum_{i=1}^{\infty} p_i \operatorname{Re}(x_i, y_i) \tag{2.169}$$
$$\geq \frac{1}{2}\left(R^2 - 1\right) r^2.$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a larger quantity.

PROOF. By (2.168) we deduce

$$\frac{1}{R}\left[\sum_{i=1}^{\infty} p_i\left(\|x_i\|+\|y_i\|\right)^2\right]^{\frac{1}{2}} \geq \left(\sum_{i=1}^{\infty} p_i\left\|x_i+y_i\right\|^2\right)^{\frac{1}{2}} \geq r. \tag{2.170}$$

By the classical Minkowsky inequality for nonnegative numbers, we have

$$\left(\sum_{i=1}^{\infty} p_i\left\|x_i\right\|^2\right)^{\frac{1}{2}}+\left(\sum_{i=1}^{\infty} p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}} \geq \left[\sum_{i=1}^{\infty} p_i\left(\|x_i\|+\|y_i\|\right)^2\right]^{\frac{1}{2}}, \tag{2.171}$$

and thus, by utilising (2.170) and (2.171), we may state in terms of $\|\cdot\|_p$ the following inequality

$$\frac{1}{R}\left(\|\mathbf{x}\|_p+\|\mathbf{y}\|_p\right) \geq \|\mathbf{x}+\mathbf{y}\|_p \geq r. \tag{2.172}$$

Employing Theorem 32 for the Hilbert space $\ell_p^2(K)$ and the inequality (2.172), we deduce the desired result (2.169).

Since, for $p=1$, $n=1$, (2.169) reduced to (2.158) for which we have shown that $\frac{1}{2}$ is the best constant, we conclude that $\frac{1}{2}$ is the best constant in (2.169) as well. ∎

Finally, we may state and prove the following result [**12**] incorporated in

PROPOSITION 18. *Let $(K;(\cdot,\cdot))$ be a Hilbert space and $p_i \geq 0$ $(i \in \mathbb{N})$ with $\sum_{i=1}^{\infty} p_i=1$. Assume that $\mathbf{x},\mathbf{y} \in \ell_p^2(K)$ and $r \in (0,1]$ such that*

$$\left|\|x_i\|-\|y_i\|\right| \leq r\left\|x_i-y_i\right\| \quad \text{for each } i \in \mathbb{N}, \tag{2.173}$$

holds true. Then we have the following refinement of the Schwarz inequality

$$\left(\sum_{i=1}^{\infty} p_i\left\|x_i\right\|^2 \sum_{i=1}^{\infty} p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}-\sum_{i=1}^{\infty} p_i \operatorname{Re}\left(x_i, y_i\right) \geq \frac{1}{2}\left(1-r^2\right) \sum_{i=1}^{\infty} p_i\left\|x_i-y_i\right\|^2. \tag{2.174}$$

The constant $\frac{1}{2}$ *is best possible in (2.174).*

Proof. From (2.173) we have

$$\left[\sum_{i=1}^{\infty} p_i \left(\|x_i\| - \|y_i\|\right)^2\right]^{\frac{1}{2}} \leq r \left[\sum_{i=1}^{\infty} p_i \|x_i - y_i\|^2\right]^{\frac{1}{2}}.$$

Utilising the following elementary result

$$\left|\left(\sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \left(\sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}}\right| \leq \left(\sum_{i=1}^{\infty} p_i \left(\|x_i\| - \|y_i\|\right)^2\right)^{\frac{1}{2}},$$

we may state that

$$\left|\|\mathbf{x}\|_p - \|\mathbf{y}\|_p\right| \leq r \|\mathbf{x} - \mathbf{y}\|_p .$$

Now, by making use of Theorem 33, we deduce the desired inequality (2.174) and the fact that $\frac{1}{2}$ is the best possible constant. We omit the details. ∎

2.7.3. Integral Inequalities. Assume that $(K; (\cdot, \cdot))$ is a Hilbert space over the real or complex number field $\mathbb{K}$. If $\rho : [a, b] \subset \mathbb{R} \to [0, \infty)$ is a Lebesgue integrable function with $\int_a^b \rho(t)\, dt = 1$, then we may consider the space $L_\rho^2([a, b]; K)$ of all functions $f : [a, b] \to K$, that are Bochner measurable and $\int_a^b \rho(t) \|f(t)\|^2 dt < \infty$. It is known that $L_\rho^2([a, b]; K)$ endowed with the inner product $\langle \cdot, \cdot \rangle_\rho$ defined by

$$\langle f, g \rangle_\rho := \int_a^b \rho(t) (f(t), g(t))\, dt$$

and generating the norm

$$\|f\|_\rho := \left(\int_a^b \rho(t) \|f(t)\|^2 dt\right)^{\frac{1}{2}}$$

is a Hilbert space over $\mathbb{K}$.

Now we may state and prove the first refinement of the Cauchy-Bunyakovsky-Schwarz integral inequality [**12**].

Proposition 19. *Assume that* $f, g \in L_\rho^2([a, b]; K)$ *and* $r_2, r_1 > 0$ *satisfy the condition*

$$\|f(t) - g(t)\| \geq r_2 \geq r_1 \geq \left|\|f(t)\| - \|g(t)\|\right| \tag{2.175}$$

for a.e. $t \in [a,b]$. Then we have the inequality

$$\left(\int_a^b \rho(t) \|f(t)\|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt\right)^{\frac{1}{2}} - \int_a^b \rho(t) \operatorname{Re}(f(t), g(t)) \, dt \geq \frac{1}{2}\left(r_2^2 - r_1^2\right) (\geq 0). \tag{2.176}$$

The constant $\frac{1}{2}$ is best possible in (2.176).

Proof. Integrating (2.175), we get

$$\left(\int_a^b \rho(t) (\|f(t) - g(t)\|)^2 dt\right)^{\frac{1}{2}} \geq r_2 \geq r_1 \geq \left(\int_a^b \rho(t) (\|f(t)\| - \|g(t)\|)^2 dt\right)^{\frac{1}{2}}. \tag{2.177}$$

Utilising the obvious fact

$$\left[\int_a^b \rho(t) (\|f(t)\| - \|g(t)\|)^2 dt\right]^{\frac{1}{2}} \geq \left|\left(\int_a^b \rho(t) \|f(t)\|^2 dt\right)^{\frac{1}{2}} - \left(\int_a^b \rho(t) \|g(t)\|^2 dt\right)^{\frac{1}{2}}\right|, \tag{2.178}$$

we can state the following inequality in terms of the $\|\cdot\|_\rho$ norm:

$$\|f - g\|_\rho \geq r_2 \geq r_1 \geq \left|\|f\|_\rho - \|g\|_\rho\right|. \tag{2.179}$$

Employing Theorem 30 for the Hilbert space $L_\rho^2([a,b]; K)$, we deduce the desired inequality (2.176).

To prove the sharpness of $\frac{1}{2}$ in (2.176), we choose $a = 0$, $b = 1$, $f(t) = 1$, $t \in [0,1]$ and $f(t) = x$, $g(t) = y$, $t \in [a,b]$, $x, y \in K$. Then (2.176) becomes

$$\|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \geq \frac{1}{2}\left(r_2^2 - r_1^2\right)$$

provided

$$\|x - y\| \geq r_2 \geq r_1 \geq |\|x\| - \|y\||,$$

which, by Theorem 30 has the quantity $\frac{1}{2}$ as the best possible constant. ∎

The following corollary holds.

Corollary 19. *With the assumptions of Proposition 19, we have the inequality*

$$\left(\int_a^b \rho(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}} + \left(\int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} - \frac{\sqrt{2}}{2}\left(\int_a^b \rho(t)\|f(t)+g(t)\|^2 dt\right)^{\frac{1}{2}} \geq \frac{\sqrt{2}}{2}\sqrt{r_2^2 - r_1^2}. \tag{2.180}$$

The following two refinements of the Cauchy-Bunyakovsky-Schwarz (CBS) integral inequality also hold.

Proposition 20. *If $f, g \in L_\rho^2([a,b];K)$ and $R \geq 1, r \geq 0$ satisfy the condition*

$$\frac{1}{R}(\|f(t)\| + \|g(t)\|) \geq \|f(t)+g(t)\| \geq r \tag{2.181}$$

for a.e. $t \in [a,b]$, then we have the inequality

$$\left(\int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} - \int_a^b \rho(t)\operatorname{Re}(f(t), g(t))\, dt \geq \frac{1}{2}\left(R^2 - 1\right) r^2. \tag{2.182}$$

The constant $\frac{1}{2}$ is best possible in (2.182).

The proof follows by Theorem 32 and we omit the details.

Proposition 21. *If $f, g \in L_\rho^2([a,b];K)$ and $\zeta \in (0,1]$ satisfy the condition*

$$|\|f(t)\| - \|g(t)\|| \leq \zeta\|f(t) - g(t)\| \tag{2.183}$$

for a.e. $t \in [a,b]$, then we have the inequality

$$\left(\int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} - \int_a^b \rho(t)\operatorname{Re}(f(t), g(t))\, dt \geq \frac{1}{2}\left(1 - \zeta^2\right)\int_a^b \rho(t)\|f(t) - g(t)\|^2 dt. \tag{2.184}$$

The constant $\frac{1}{2}$ is best possible in (2.184).

The proof follows by Theorem 33 and we omit the details.

2.7.4. Refinements of the Heisenberg Inequality. It is well known that if $(H;\langle\cdot,\cdot\rangle)$ is a real or complex Hilbert space and $f:[a,b]\subset\mathbb{R}\to H$ is an *absolutely continuous vector-valued* function, then f is differentiable almost everywhere on $[a,b]$, the derivative $f':[a,b]\to H$ is Bochner integrable on $[a,b]$ and

$$f(t)=\int_a^t f'(s)\,ds \qquad \text{for any } \ t\in[a,b]. \tag{2.185}$$

The following theorem provides a version of the Heisenberg inequalities in the general setting of Hilbert spaces [**12**].

Theorem 34 (Dragomir, 2004). *Let $\varphi:[a,b]\to H$ be an absolutely continuous function with the property that $b\|\varphi(b)\|^2=a\|\varphi(a)\|^2$. Then we have the inequality:*

$$\left(\int_a^b \|\varphi(t)\|^2\,dt\right)^2\le 4\int_a^b t^2\|\varphi(t)\|^2\,dt\cdot\int_a^b\|\varphi'(t)\|^2\,dt. \tag{2.186}$$

The constant 4 is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Integrating by parts, we have successively

$$\begin{aligned}
&\int_a^b \|\varphi(t)\|^2\,dt \\
&= t\|\varphi(t)\|^2\Big|_a^b-\int_a^b t\frac{d}{dt}\left(\|\varphi(t)\|^2\right)dt\\
&= b\|\varphi(b)\|^2-a\|\varphi(a)\|^2-\int_a^b t\frac{d}{dt}\langle\varphi(t),\varphi(t)\rangle\,dt\\
&= -\int_a^b t\left[\langle\varphi'(t),\varphi(t)\rangle+\langle\varphi(t),\varphi'(t)\rangle\right]dt\\
&= -2\int_a^b t\operatorname{Re}\langle\varphi'(t),\varphi(t)\rangle\,dt\\
&= 2\int_a^b \operatorname{Re}\langle\varphi'(t),(-t)\varphi(t)\rangle\,dt.
\end{aligned} \tag{2.187}$$

If we apply the Cauchy-Bunyakovsky-Schwarz integral inequality

$$\int_a^b \operatorname{Re}\langle g(t),h(t)\rangle\,dt\le\left(\int_a^b\|g(t)\|^2\,dt\int_a^b\|h(t)\|^2\,dt\right)^{\frac12}$$

for $g(t)=\varphi'(t)$, $h(t)=-t\varphi(t)$, $t\in[a,b]$, then we deduce the desired inequality (2.176).

The fact that 4 is the best constant in (2.176) follows from the fact that in the (CBS) inequality, the case of equality holds iff $g(t) = \lambda h(t)$ for a.e. $t \in [a,b]$ and λ a given scalar in $\mathbb{K}$. We omit the details. ∎

For details on the classical Heisenberg inequality, see, for instance, [**23**].

Utilising Proposition 19, we can state the following refinement [**12**] of the Heisenberg inequality obtained above in (2.186):

Proposition 22. *Assume that* $\varphi : [a,b] \to H$ *is as in the hypothesis of Theorem 34. In addition, if there exists* $r_2, r_1 > 0$ *so that*

$$\|\varphi'(t) + t\varphi(t)\| \geq r_2 \geq r_1 \geq |\|\varphi'(t)\| - |t| \|\varphi(t)\||$$

for a.e. $t \in [a,b]$, *then we have the inequality*

$$\left(\int_a^b t^2 \|\varphi(t)\|^2 dt \cdot \int_a^b \|\varphi'(t)\|^2 dt\right)^{\frac{1}{2}} - \frac{1}{2}\int_a^b \|\varphi(t)\|^2 dt \geq \frac{1}{2}(b-a)\left(r_2^2 - r_1^2\right)(\geq 0).$$

The proof follows by Proposition 19 on choosing $f(t) = \varphi'(t)$, $g(t) = -t\varphi(t)$ and $\rho(t) = \frac{1}{b-a}$, $t \in [a,b]$.

On utilising the Proposition 20 for the same choices of f, g and ρ, we may state the following results as well [**12**]:

Proposition 23. *Assume that* $\varphi : [a,b] \to H$ *is as in the hypothesis of Theorem 34. In addition, if there exists* $R \geq 1$ *and* $r > 0$ *so that*

$$\frac{1}{R}(\|\varphi'(t)\| + |t| \|\varphi(t)\|) \geq \|\varphi'(t) - t\varphi(t)\| \geq r$$

for a.e. $t \in [a,b]$, *then we have the inequality*

$$\left(\int_a^b t^2 \|\varphi(t)\|^2 dt \cdot \int_a^b \|\varphi'(t)\|^2 dt\right)^{\frac{1}{2}} - \frac{1}{2}\int_a^b \|\varphi(t)\|^2 dt \geq \frac{1}{2}(b-a)\left(R^2 - 1\right)r^2(\geq 0).$$

Finally, we can state

Proposition 24. *Let* $\varphi : [a,b] \to H$ *be as in the hypothesis of Theorem 34. In addition, if there exists* $\zeta \in (0,1]$ *so that*

$$|\|\varphi'(t)\| - |t| \|\varphi(t)\|| \leq \zeta \|\varphi'(t) + t\varphi(t)\|$$

for a.e. $t \in [a,b]$, *then we have the inequality*

$$\left(\int_a^b t^2 \|\varphi(t)\|^2 dt \cdot \int_a^b \|\varphi'(t)\|^2 dt\right)^{\frac{1}{2}} - \frac{1}{2}\int_a^b \|\varphi(t)\|^2 dt \geq \frac{1}{2}\left(1-\zeta^2\right)\int_a^b \|\varphi'(t) + t\varphi(t)\|^2 dt \,(\geq 0).$$

This follows by Proposition 21 and we omit the details.

2.8. More Schwarz Related Inequalities

2.8.1. Introduction. In practice, one may need reverses of the Schwarz inequality, namely, upper bounds for the quantities

$$\|x\|\|y\| - \operatorname{Re}\langle x,y\rangle, \qquad \|x\|^2\|y\|^2 - (\operatorname{Re}\langle x,y\rangle)^2$$

and

$$\frac{\|x\|\|y\|}{\operatorname{Re}\langle x,y\rangle}$$

or the corresponding expressions where $\operatorname{Re}\langle x,y\rangle$ is replaced by either $|\operatorname{Re}\langle x,y\rangle|$ or $|\langle x,y\rangle|$, under suitable assumptions for the vectors x,y in an inner product space $(H;\langle\cdot,\cdot\rangle)$ over the real or complex number field $\mathbb{K}$.

In this class of results, we mention the following recent reverses of the Schwarz inequality due to the present author, that can be found, for instance, in the survey work [**4**], where more specific references are provided:

Theorem 35 (Dragomir, 2004). *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$. *If* $a,A\in\mathbb{K}$ *and* $x,y\in H$ *are such that either*

$$\operatorname{Re}\langle Ay - x, x - ay\rangle \geq 0, \tag{2.188}$$

or, equivalently,

$$\left\|x - \frac{A+a}{2}y\right\| \leq \frac{1}{2}|A-a|\|y\|, \tag{2.189}$$

then the following reverse for the quadratic form of the Schwarz inequality

$$(0\leq)\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \leq \begin{cases} \frac{1}{4}|A-a|^2\|y\|^4 - \left|\frac{A+a}{2}\|y\|^2 - \langle x,y\rangle\right|^2 \\ \\ \frac{1}{4}|A-a|^2\|y\|^4 - \|y\|^2\operatorname{Re}\langle Ay-x,x-ay\rangle \end{cases} \leq \frac{1}{4}|A-a|^2\|y\|^4 \tag{2.190}$$

holds.

If in addition, we have $\operatorname{Re}(A\bar{a}) > 0$*, then*

$$\|x\| \|y\| \leq \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{A}+\bar{a}\right)\langle x, y\rangle\right]}{\sqrt{\operatorname{Re}(A\bar{a})}} \leq \frac{1}{2} \cdot \frac{|A+a|}{\sqrt{\operatorname{Re}(A\bar{a})}} |\langle x, y\rangle|, \tag{2.191}$$

and

$$(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4} \cdot \frac{|A-a|^2}{\operatorname{Re}(A\bar{a})} |\langle x, y\rangle|^2. \tag{2.192}$$

Also, if (2.188) or (2.189) are valid and $A \neq -a$*, then we have the reverse for the simple form of Schwarz inequality*

$$\begin{aligned}(0 \leq) \|x\| \|y\| - |\langle x, y\rangle| &\leq \|x\| \|y\| - \left|\operatorname{Re}\left[\frac{\bar{A}+\bar{a}}{|A+a|}\langle x, y\rangle\right]\right| \\ &\leq \|x\| \|y\| - \operatorname{Re}\left[\frac{\bar{A}+\bar{a}}{|A+a|}\langle x, y\rangle\right] \leq \frac{1}{4} \cdot \frac{|A-a|^2}{|A+a|} \|y\|^2.\end{aligned} \tag{2.193}$$

The multiplicative constants $\frac{1}{4}$ *and* $\frac{1}{2}$ *above are best possible in the sense that they cannot be replaced by a smaller quantity.*

For some classical results related to Schwarz inequality, see [**3**], [**21**], [**28**], [**29**], [**30**] and the references therein.

The main aim of the present section is to point out other results in connection with both the quadratic and simple forms of the Schwarz inequality. As applications, some reverse results for the generalised triangle inequality, i.e., upper bounds for the quantity

$$(0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\|\sum_{i=1}^{n} x_i\right\|$$

under various assumptions for the vectors $x_i \in H$, $i \in \{1, \ldots, n\}$, are established.

2.8.2. Refinements and Reverses. The following result holds [**7**].

Proposition 25. *Let* $(H; \langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$*. The subsequent statements are equivalent.*

(i) *The following inequality holds*

$$\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \leq (\geq)\, r; \tag{2.194}$$

(ii) *The following reverse (improvement) of Schwarz's inequality holds*

$$\|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \leq (\geq) \frac{1}{2} r^2 \|x\| \|y\| . \tag{2.195}$$

The constant $\frac{1}{2}$ is best possible in (2.195) in the sense that it cannot be replaced by a larger (smaller) quantity.

Remark 31. *Since*

$$\begin{aligned} \| \|y\| x - \|x\| y\| &= \| \|y\| (x-y) + (\|y\| - \|x\|) y\| \\ &\leq \|y\| \|x-y\| + |\|y\| - \|x\|| \|y\| \\ &\leq 2 \|y\| \|x-y\| \end{aligned}$$

hence a sufficient condition for (2.194) to hold is

$$\|x-y\| \leq \frac{r}{2} \|x\| . \tag{2.196}$$

Remark 32. *Utilising the Dunkl-Williams inequality* [**20**]

$$\|a-b\| \geq \frac{1}{2} (\|a\| + \|b\|) \left\| \frac{a}{\|a\|} - \frac{b}{\|b\|} \right\|, \quad a, b \in H \backslash \{0\} \tag{2.197}$$

with equality if and only if either $\|a\| = \|b\|$ or $\|a\| + \|b\| = \|a-b\|$, we can state the following inequality

$$\frac{\|x\| \|y\| - \operatorname{Re}\langle x, y\rangle}{\|x\| \|y\|} \leq 2 \left(\frac{\|x-y\|}{\|x\| + \|y\|} \right)^2, \quad x, y \in H \backslash \{0\} . \tag{2.198}$$

Obviously, if $x, y \in H \backslash \{0\}$ are such that

$$\|x-y\| \leq \eta (\|x\| + \|y\|), \tag{2.199}$$

with $\eta \in (0,1]$, then one has the following reverse of the Schwarz inequality

$$\|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \leq 2\eta^2 \|x\| \|y\| \tag{2.200}$$

that is similar to (2.195).

The following result may be stated as well [**7**].

Proposition 26. *If $x, y \in H \backslash \{0\}$ and $\rho > 0$ are such that*

$$\left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \leq \rho, \tag{2.201}$$

then we have the following reverse of Schwarz's inequality

$$\begin{aligned} (0 \leq) \|x\| \|y\| - |\langle x, y\rangle| &\leq \|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \\ &\leq \frac{1}{2} \rho^2 \|x\| \|y\| . \end{aligned} \tag{2.202}$$

The case of equality holds in the last inequality in (2.202) if and only if

$$\|x\| = \|y\| \qquad \text{and} \qquad \|x-y\| = \rho. \tag{2.203}$$

The constant $\frac{1}{2}$ *in (2.202) cannot be replaced by a smaller quantity.*

Proof. Taking the square in (2.201), we get

$$\frac{\|x\|^2}{\|y\|^2} - \frac{2\operatorname{Re}\langle x,y\rangle}{\|x\|\,\|y\|} + \frac{\|y\|^2}{\|x\|^2} \le \rho^2. \tag{2.204}$$

Since, obviously

$$2 \le \frac{\|x\|^2}{\|y\|^2} + \frac{\|y\|^2}{\|x\|^2} \tag{2.205}$$

with equality iff $\|x\| = \|y\|$, hence by (2.204) we deduce the second inequality in (2.202).

The case of equality and the best constant are obvious and we omit the details. ∎

Remark 33. *In* [**24**], *Hile obtained the following inequality*

$$\left\|\|x\|^v x - \|y\|^v y\right\| \le \frac{\|x\|^{v+1} - \|y\|^{v+1}}{\|x\| - \|y\|}\|x-y\| \tag{2.206}$$

provided $v > 0$ *and* $\|x\| \ne \|y\|$.

If in (2.206) we choose $v = 1$ *and take the square, then we get*

$$\|x\|^4 - 2\|x\|\,\|y\|\operatorname{Re}\langle x,y\rangle + \|y\|^4 \le (\|x\| + \|y\|)^2\|x-y\|^2. \tag{2.207}$$

Since,

$$\|x\|^4 + \|y\|^4 \ge 2\|x\|^2\|y\|^2,$$

hence, by (2.207) we deduce

$$(0 \le)\ \|x\|\,\|y\| - \operatorname{Re}\langle x,y\rangle \le \frac{1}{2}\cdot\frac{(\|x\| + \|y\|)^2\|x-y\|^2}{\|x\|\,\|y\|}, \tag{2.208}$$

provided $x, y \in H\backslash\{0\}$.

The following inequality is due to Goldstein, Ryff and Clarke [**22**, p. 309]:

$$\|x\|^{2r} + \|y\|^{2r} - 2\|x\|^r\|y\|^r\cdot\frac{\operatorname{Re}\langle x,y\rangle}{\|x\|\,\|y\|} \le \begin{cases} r^2\|x\|^{2r-2}\|x-y\|^2 & \text{if } r \ge 1 \\ \\ \|y\|^{2r-2}\|x-y\|^2 & \text{if } r < 1 \end{cases} \tag{2.209}$$

provided $r\in\mathbb{R}$ and $x,y\in H$ with $\|x\|\geq\|y\|$.

Utilising (2.209) we may state the following proposition containing a different reverse of the Schwarz inequality in inner product spaces [**7**].

Proposition 27. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. If $x,y\in H\setminus\{0\}$ and $\|x\|\geq\|y\|$, then we have*

$$
\begin{aligned}
(2.210)\qquad 0 &\leq \|x\|\,\|y\|-|\langle x,y\rangle| \leq \|x\|\,\|y\|-\operatorname{Re}\langle x,y\rangle \\
&\leq \begin{cases} \frac{1}{2}r^2\left(\frac{\|x\|}{\|y\|}\right)^{r-1}\|x-y\|^2 & \text{if } r\geq 1, \\ \\ \frac{1}{2}\left(\frac{\|x\|}{\|y\|}\right)^{1-r}\|x-y\|^2 & \text{if } r<1. \end{cases}
\end{aligned}
$$

Proof. It follows from (2.209), on dividing by $\|x\|^r\|y\|^r$, that

$$
\begin{aligned}
(2.211)\qquad &\left(\frac{\|x\|}{\|y\|}\right)^r+\left(\frac{\|y\|}{\|x\|}\right)^r-2\cdot\frac{\operatorname{Re}\langle x,y\rangle}{\|x\|\,\|y\|} \\
&\qquad\leq \begin{cases} r^2\cdot\frac{\|x\|^{r-2}}{\|y\|^r}\|x-y\|^2 & \text{if } r\geq 1, \\ \\ \frac{\|y\|^{r-2}}{\|x\|^r}\|x-y\|^2 & \text{if } r<1. \end{cases}
\end{aligned}
$$

Since

$$
\left(\frac{\|x\|}{\|y\|}\right)^r+\left(\frac{\|y\|}{\|x\|}\right)^r\geq 2,
$$

hence, by (2.211) one has

$$
2-2\cdot\frac{\operatorname{Re}\langle x,y\rangle}{\|x\|\,\|y\|}\leq \begin{cases} r^2\frac{\|x\|^{r-2}}{\|y\|^r}\|x-y\|^2 & \text{if } r\geq 1, \\ \\ \frac{\|y\|^{r-2}}{\|x\|^r}\|x-y\|^2 & \text{if } r<1. \end{cases}
$$

Dividing this inequality by 2 and multiplying with $\|x\|\,\|y\|$, we deduce the desired result in (2.210). ∎

Another result providing a different additive reverse (refinement) of the Schwarz inequality may be stated [**7**].

Proposition 28. *Let $x,y\in H$ with $y\neq 0$ and $r>0$. The subsequent statements are equivalent:*

(i) *The following inequality holds:*

$$
(2.212)\qquad \left\|x-\frac{\langle x,y\rangle}{\|y\|^2}\cdot y\right\|\leq(\geq)\,r;
$$

(ii) *The following reverse (refinement) of the quadratic Schwarz inequality holds:*

$$\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq (\geq)\, r^2 \|y\|^2 . \tag{2.213}$$

The proof is obvious on taking the square in (2.212) and performing the calculation.

Remark 34. *Since*

$$\begin{aligned}\left\|\|y\|^2 x - \langle x, y\rangle y\right\| &= \left\|\|y\|^2 (x-y) - \langle x-y, y\rangle y\right\| \\ &\leq \|y\|^2 \|x-y\| + |\langle x-y, y\rangle| \|y\| \\ &\leq 2\|x-y\| \|y\|^2 ,\end{aligned}$$

hence a sufficient condition for the inequality (2.212) to hold is that

$$\|x-y\| \leq \frac{r}{2}. \tag{2.214}$$

The following proposition may give a complementary approach [**7**]:

Proposition 29. *Let $x, y \in H$ with $\langle x, y\rangle \neq 0$ and $\rho > 0$. If*

$$\left\| x - \frac{\langle x, y\rangle}{|\langle x, y\rangle|} \cdot y \right\| \leq \rho, \tag{2.215}$$

then

$$(0 \leq)\, \|x\| \|y\| - |\langle x, y\rangle| \leq \frac{1}{2}\rho^2. \tag{2.216}$$

The multiplicative constant $\frac{1}{2}$ is best possible in (2.216).

The proof is similar to the ones outlined above and we omit it.

For the case of complex inner product spaces, we may state the following result [**7**].

Proposition 30. *Let $(H; \langle\cdot,\cdot\rangle)$ be a complex inner product space and $\alpha \in \mathbb{C}$ a given complex number with $\operatorname{Re}\alpha$, $\operatorname{Im}\alpha > 0$. If $x, y \in H$ are such that*

$$\left\| x - \frac{\operatorname{Im}\alpha}{\operatorname{Re}\alpha} \cdot y \right\| \leq r, \tag{2.217}$$

then we have the inequality

$$\begin{aligned}(0 \leq)\, \|x\| \|y\| - |\langle x, y\rangle| &\leq \|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \\ &\leq \frac{1}{2} \cdot \frac{\operatorname{Re}\alpha}{\operatorname{Im}\alpha} \cdot r^2.\end{aligned} \tag{2.218}$$

The equality holds in the second inequality in (2.218) if and only if the case of equality holds in (2.217) and $\operatorname{Re}\alpha \cdot \|x\| = \operatorname{Im}\alpha \cdot \|y\|$.

Proof. Observe that the condition (2.217) is equivalent to

$$[\operatorname{Re}\alpha]^2 \|x\|^2 + [\operatorname{Im}\alpha]^2 \|y\|^2 \le 2\operatorname{Re}\alpha \operatorname{Im}\alpha \operatorname{Re}\langle x,y\rangle + [\operatorname{Re}\alpha]^2 r^2. \tag{2.219}$$

On the other hand, on utilising the elementary inequality

$$2\operatorname{Re}\alpha \operatorname{Im}\alpha \|x\| \|y\| \le [\operatorname{Re}\alpha]^2 \|x\|^2 + [\operatorname{Im}\alpha]^2 \|y\|^2, \tag{2.220}$$

with equality if and only if $\operatorname{Re}\alpha \cdot \|x\| = \operatorname{Im}\alpha \cdot \|y\|$, we deduce from (2.219) that

$$2\operatorname{Re}\alpha \operatorname{Im}\alpha \|x\| \|y\| \le 2\operatorname{Re}\alpha \operatorname{Im}\alpha \operatorname{Re}\langle x,y\rangle + r^2 [\operatorname{Re}\alpha]^2 \tag{2.221}$$

giving the desired inequality (2.218).

The case of equality follows from the above and we omit the details. ∎

The following different reverse for the Schwarz inequality that holds for both real and complex inner product spaces may be stated as well [**7**].

Theorem 36 (Dragomir, 2004). *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$, $\mathbb{K}=\mathbb{C},\mathbb{R}$. If $\alpha \in \mathbb{K}\backslash\{0\}$, then*

$$\begin{aligned} 0 &\le \|x\|\|y\| - |\langle x,y\rangle| \le \|x\|\|y\| - \operatorname{Re}\left[\frac{\alpha^2}{|\alpha|^2}\langle x,y\rangle\right] \\ &\le \frac{1}{2}\cdot \frac{[|\operatorname{Re}\alpha|\,\|x-y\| + |\operatorname{Im}\alpha|\,\|x+y\|]^2}{|\alpha|^2} \le \frac{1}{2}\cdot I^2, \end{aligned} \tag{2.222}$$

where

$$I := \begin{cases} \max\{|\operatorname{Re}\alpha|, |\operatorname{Im}\alpha|\}(\|x-y\| + \|x+y\|); \\ \\ (|\operatorname{Re}\alpha|^p + |\operatorname{Im}\alpha|^p)^{\frac{1}{p}}(\|x-y\|^q + \|x+y\|^q)^{\frac{1}{q}}, \\ \qquad\qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \max\{\|x-y\|, \|x+y\|\}(|\operatorname{Re}\alpha| + |\operatorname{Im}\alpha|). \end{cases} \tag{2.223}$$

Proof. Observe, for $\alpha \in \mathbb{K}\backslash\{0\}$, that

$$\begin{aligned} \|\alpha x - \bar{\alpha} y\|^2 &= |\alpha|^2 \|x\|^2 - 2\operatorname{Re}\langle \alpha x, \bar{\alpha} y\rangle + |\alpha|^2\|y\|^2 \\ &= |\alpha|^2 \left(\|x\|^2 + \|y\|^2\right) - 2\operatorname{Re}\left[\alpha^2\langle x,y\rangle\right]. \end{aligned}$$

Since $\|x\|^2 + \|y\|^2 \ge 2\|x\|\|y\|$, hence

$$\|\alpha x - \bar{\alpha} y\|^2 \ge 2|\alpha|^2 \left\{\|x\|\|y\| - \operatorname{Re}\left[\frac{\alpha^2}{|\alpha|^2}\langle x,y\rangle\right]\right\}. \tag{2.224}$$

On the other hand, we have

$$\begin{aligned} \|\alpha x - \bar{\alpha} y\| &= \|(\operatorname{Re}\alpha + i\operatorname{Im}\alpha)\, x - (\operatorname{Re}\alpha - i\operatorname{Im}\alpha)\, y\| \\ &= \|\operatorname{Re}\alpha\,(x-y) + i\operatorname{Im}\alpha\,(x+y)\| \\ &\le |\operatorname{Re}\alpha|\,\|x-y\| + |\operatorname{Im}\alpha|\,\|x+y\|\,. \end{aligned} \tag{2.225}$$

Utilising (2.224) and (2.225) we deduce the third inequality in (2.222).

For the last inequality we use the following elementary inequality

$$\alpha a + \beta b \le \begin{cases} \max\{\alpha,\beta\}\,(a+b) \\ \\ (\alpha^p + \beta^p)^{\frac{1}{p}}\,(a^q + b^q)^{\frac{1}{q}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1, \end{cases} \tag{2.226}$$

provided $\alpha, \beta, a, b \ge 0$. ∎

The following result may be stated [**7**].

Proposition 31. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product over $\mathbb{K}$ and $e \in H$, $\|e\| = 1$. If $\lambda \in (0,1)$, then*

$$\begin{aligned} &\operatorname{Re}\left[\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\right] \\ &\qquad \le \frac{1}{4}\cdot\frac{1}{\lambda(1-\lambda)}\left[\|\lambda x + (1-\lambda)\, y\|^2 - |\langle \lambda x + (1-\lambda)\, y, e\rangle|^2\right]. \end{aligned} \tag{2.227}$$

The constant $\frac{1}{4}$ is best possible.

Proof. Firstly, note that the following equality holds true

$$\langle x - \langle x,e\rangle e, y - \langle y,e\rangle e\rangle = \langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\,.$$

Utilising the elementary inequality

$$\operatorname{Re}\langle z,w\rangle \le \frac{1}{4}\|z+w\|^2, \qquad z,w\in H$$

we have

$$\begin{aligned} &\operatorname{Re}\langle x - \langle x,e\rangle e, y - \langle y,e\rangle e\rangle \\ &= \frac{1}{\lambda(1-\lambda)}\operatorname{Re}\langle \lambda x - \langle \lambda x, e\rangle e, (1-\lambda)\, y - \langle (1-\lambda)\, y, e\rangle e\rangle \\ &\le \frac{1}{4}\cdot\frac{1}{\lambda(1-\lambda)}\left[\|\lambda x + (1-\lambda)\, y\|^2 - |\langle \lambda x + (1-\lambda)\, y, e\rangle|^2\right], \end{aligned}$$

proving the desired inequality (2.227). ∎

Remark 35. *For $\lambda = \frac{1}{2}$, we get the simpler inequality:*

$$\operatorname{Re}\left[\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\right] \le \left\|\frac{x+y}{2}\right\|^2 - \left|\left\langle\frac{x+y}{2}, e\right\rangle\right|^2, \tag{2.228}$$

that has been obtained in [**4**, p. 46]*, for which the sharpness of the inequality was established.*

The following result may be stated as well [**7**].

THEOREM 37 (Dragomir, 2004). *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$ *and* $p\geq 1$*. Then for any* $x,y\in H$ *we have*

$$0\leq\|x\|\,\|y\|-|\langle x,y\rangle|\leq\|x\|\,\|y\|-\operatorname{Re}\langle x,y\rangle \leq\frac{1}{2}\times\begin{cases}\left[(\|x\|+\|y\|)^{2p}-\|x+y\|^{2p}\right]^{\frac{1}{p}},\\[2ex] \left[\|x-y\|^{2p}-|\|x\|-\|y\||^{2p}\right]^{\frac{1}{p}}.\end{cases}\tag{2.229}$$

PROOF. Firstly, observe that

$$2\left(\|x\|\,\|y\|-\operatorname{Re}\langle x,y\rangle\right)=(\|x\|+\|y\|)^{2}-\|x+y\|^{2}.$$

Denoting $D:=\|x\|\,\|y\|-\operatorname{Re}\langle x,y\rangle$, then we have

$$2D+\|x+y\|^{2}=(\|x\|+\|y\|)^{2}.\tag{2.230}$$

Taking in (2.230) the power $p\geq 1$ and using the elementary inequality

$$(a+b)^{p}\geq a^{p}+b^{p};\,a,b\geq 0,\tag{2.231}$$

we have

$$(\|x\|+\|y\|)^{2p}=\left(2D+\|x+y\|^{2}\right)^{p}\geq 2^{p}D^{p}+\|x+y\|^{2p}$$

giving

$$D^{p}\leq\frac{1}{2^{p}}\left[(\|x\|+\|y\|)^{2p}-\|x+y\|^{2p}\right],$$

which is clearly equivalent to the first branch of the third inequality in (2.229).

With the above notation, we also have

$$2D+(\|x\|-\|y\|)^{2}=\|x-y\|^{2}.\tag{2.232}$$

Taking the power $p\geq 1$ in (2.232) and using the inequality (2.231) we deduce

$$\|x-y\|^{2p}\geq 2^{p}D^{p}+|\|x\|-\|y\||^{2p},$$

from where we get the last part of (2.229). ∎

2.8.3. More Schwarz Related Inequalities. Before we point out other inequalities related to the Schwarz inequality, we need the following identity that is interesting in itself [**7**].

LEMMA 4 (Dragomir, 2004). *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $e\in H$, $\|e\|=1$, $\alpha\in H$ and $\gamma,\Gamma\in\mathbb{K}$. Then we have the identity:*

$$\begin{aligned} \|x\|^2-|\langle x,e\rangle|^2 &= (\operatorname{Re}\Gamma-\operatorname{Re}\langle x,e\rangle)(\operatorname{Re}\langle x,e\rangle-\operatorname{Re}\gamma) \\ &+(\operatorname{Im}\Gamma-\operatorname{Im}\langle x,e\rangle)(\operatorname{Im}\langle x,e\rangle-\operatorname{Im}\gamma) \\ &+\left\|x-\frac{\gamma+\Gamma}{2}e\right\|^2-\frac{1}{4}|\Gamma-\gamma|^2 . \end{aligned} \tag{2.233}$$

PROOF. We start with the following known equality (see for instance [**5**, eq. (2.6)])

$$\|x\|^2-|\langle x,e\rangle|^2 = \operatorname{Re}\left[(\Gamma-\langle x,e\rangle)\left(\overline{\langle x,e\rangle}-\bar{\gamma}\right)\right]-\operatorname{Re}\langle\Gamma e-x,x-\gamma e\rangle \tag{2.234}$$

holding for $x\in H$, $e\in H$, $\|e\|=1$ and $\gamma,\Gamma\in\mathbb{K}$.

We also know that (see for instance [**14**])

$$-\operatorname{Re}\langle\Gamma e-x,x-\gamma e\rangle=\left\|x-\frac{\gamma+\Gamma}{2}e\right\|^2-\frac{1}{4}|\Gamma-\gamma|^2 . \tag{2.235}$$

Since

$$\begin{aligned} \operatorname{Re}\left[(\Gamma-\langle x,e\rangle)\left(\overline{\langle x,e\rangle}-\bar{\gamma}\right)\right] &= (\operatorname{Re}\Gamma-\operatorname{Re}\langle x,e\rangle)(\operatorname{Re}\langle x,e\rangle-\operatorname{Re}\gamma) \\ &+(\operatorname{Im}\Gamma-\operatorname{Im}\langle x,e\rangle)(\operatorname{Im}\langle x,e\rangle-\operatorname{Im}\gamma), \end{aligned} \tag{2.236}$$

hence, by (2.234) – (2.236), we deduce the desired identity (2.233). ∎

The following general result providing a reverse of the Schwarz inequality may be stated [**7**].

PROPOSITION 32. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$, $e\in H$, $\|e\|=1$, $x\in H$ and $\gamma,\Gamma\in\mathbb{K}$. Then we have the inequality:*

$$(0\le)\,\|x\|^2-|\langle x,e\rangle|^2\le\left\|x-\frac{\gamma+\Gamma}{2}\cdot e\right\|^2 . \tag{2.237}$$

The constant $\frac{1}{2}$ *is best possible in (2.237). The case of equality holds in (2.237) if and only if*

$$\operatorname{Re}\langle x,e\rangle=\operatorname{Re}\left(\frac{\gamma+\Gamma}{2}\right),\qquad \operatorname{Im}\langle x,e\rangle=\operatorname{Im}\left(\frac{\gamma+\Gamma}{2}\right). \tag{2.238}$$

PROOF. Utilising the elementary inequality for real numbers

$$\alpha\beta\le\frac{1}{4}(\alpha+\beta)^2,\qquad \alpha,\beta\in\mathbb{R};$$

with equality iff $\alpha=\beta$, we have

$$(\operatorname{Re}\Gamma-\operatorname{Re}\langle x,e\rangle)(\operatorname{Re}\langle x,e\rangle-\operatorname{Re}\gamma)\le\frac{1}{4}(\operatorname{Re}\Gamma-\operatorname{Re}\gamma)^2 \tag{2.239}$$

and

$$(\operatorname{Im}\Gamma-\operatorname{Im}\langle x,e\rangle)(\operatorname{Im}\langle x,e\rangle-\operatorname{Im}\gamma)\le\frac{1}{4}(\operatorname{Im}\Gamma-\operatorname{Im}\gamma)^2 \tag{2.240}$$

with equality if and only if

$$\operatorname{Re}\langle x,e\rangle=\frac{\operatorname{Re}\Gamma+\operatorname{Re}\gamma}{2}\qquad\text{and}\qquad\operatorname{Im}\langle x,e\rangle=\frac{\operatorname{Im}\Gamma+\operatorname{Im}\gamma}{2}.$$

Finally, on making use of (2.239), (2.240) and the identity (2.233), we deduce the desired result (2.237). ∎

The following result may be stated as well [**7**].

PROPOSITION 33. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$, $e\in H$, $\|e\|=1$, $x\in H$ *and* $\gamma,\Gamma\in\mathbb{K}$. *If* $x\in H$ *is such that*

$$\operatorname{Re}\gamma\le\operatorname{Re}\langle x,e\rangle\le\operatorname{Re}\Gamma\qquad\textit{and}\qquad\operatorname{Im}\gamma\le\operatorname{Im}\langle x,e\rangle\le\operatorname{Im}\Gamma, \tag{2.241}$$

then we have the inequality

$$\|x\|^2-|\langle x,e\rangle|^2\ge\left\|x-\frac{\gamma+\Gamma}{2}e\right\|^2-\frac{1}{4}|\Gamma-\gamma|^2. \tag{2.242}$$

The constant $\frac{1}{4}$ *is best possible in (2.242). The case of equality holds in (2.242) if and only if*

$$\operatorname{Re}\langle x,e\rangle=\operatorname{Re}\Gamma\ \textit{or}\ \operatorname{Re}\langle x,e\rangle=\operatorname{Re}\gamma$$

and

$$\operatorname{Im}\langle x,e\rangle=\operatorname{Im}\Gamma\ \textit{or}\ \operatorname{Im}\langle x,e\rangle=\operatorname{Im}\gamma.$$

PROOF. From the hypothesis we obviously have

$$(\operatorname{Re}\Gamma-\operatorname{Re}\langle x,e\rangle)(\operatorname{Re}\langle x,e\rangle-\operatorname{Re}\gamma)\ge 0$$

and

$$(\operatorname{Im}\Gamma-\operatorname{Im}\langle x,e\rangle)(\operatorname{Im}\langle x,e\rangle-\operatorname{Im}\gamma)\ge 0.$$

Utilising the identity (2.233) we deduce the desired result (2.242). The case of equality is obvious. ∎

Further on, we can state the following reverse of the quadratic Schwarz inequality [**7**]:

PROPOSITION 34. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$, $e \in H$, $\|e\| = 1$. If $\gamma, \Gamma \in \mathbb{K}$ and $x \in H$ are such that either*

$$\operatorname{Re}\langle \Gamma e - x, x - \gamma e\rangle \geq 0 \tag{2.243}$$

or, equivalently,

$$\left\| x - \frac{\gamma+\Gamma}{2} e \right\| \leq \frac{1}{2}\left|\Gamma - \gamma\right|, \tag{2.244}$$

then

$$\begin{aligned}(0 \leq)\ \|x\|^2 - |\langle x, e\rangle|^2 &\leq (\operatorname{Re}\Gamma - \operatorname{Re}\langle x, e\rangle)(\operatorname{Re}\langle x, e\rangle - \operatorname{Re}\gamma) \\ &\qquad + (\operatorname{Im}\Gamma - \operatorname{Im}\langle x, e\rangle)(\operatorname{Im}\langle x, e\rangle - \operatorname{Im}\gamma) \\ &\leq \frac{1}{4}\left|\Gamma - \gamma\right|^2.\end{aligned} \tag{2.245}$$

The case of equality holds in (2.245) if it holds either in (2.243) or (2.244).

The proof is obvious by Lemma 4 and we omit the details.

REMARK 36. *We remark that the inequality (2.245) may also be used to get, for instance, the following result*

$$\begin{aligned}\|x\|^2 - |\langle x, e\rangle|^2 &\leq \left[(\operatorname{Re}\Gamma - \operatorname{Re}\langle x, e\rangle)^2 + (\operatorname{Im}\Gamma - \operatorname{Im}\langle x, e\rangle)^2\right]^{\frac{1}{2}} \\ &\qquad \times \left[(\operatorname{Re}\langle x, e\rangle - \operatorname{Re}\gamma)^2 + (\operatorname{Im}\langle x, e\rangle - \operatorname{Im}\gamma)^2\right]^{\frac{1}{2}},\end{aligned} \tag{2.246}$$

that provides a different bound than $\frac{1}{4}\left|\Gamma - \gamma\right|^2$ for the quantity $\|x\|^2 - |\langle x, e\rangle|^2$.

The following result may be stated as well [**7**].

THEOREM 38 (Dragomir, 2004). *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $\alpha, \gamma > 0$, $\beta \in \mathbb{K}$ with $|\beta|^2 \geq \alpha\gamma$. If $x, a \in H$ are such that $a \neq 0$ and*

$$\left\| x - \frac{\beta}{\alpha} a \right\| \leq \frac{\left(|\beta|^2 - \alpha\gamma\right)^{\frac{1}{2}}}{\alpha}\|a\|, \tag{2.247}$$

then we have the following reverses of Schwarz's inequality

$$\|x\|\,\|a\| \leq \frac{\operatorname{Re}\beta\cdot\operatorname{Re}\langle x,a\rangle+\operatorname{Im}\beta\cdot\operatorname{Im}\langle x,a\rangle}{\sqrt{\alpha\gamma}} \leq \frac{|\beta|\,|\langle x,a\rangle|}{\sqrt{\alpha\gamma}} \tag{2.248}$$

and

$$(0\leq)\,\|x\|^2\|a\|^2-|\langle x,a\rangle|^2 \leq \frac{|\beta|^2-\alpha\gamma}{\alpha\gamma}\,|\langle x,a\rangle|^2 . \tag{2.249}$$

Proof. Taking the square in (2.247), it becomes equivalent to

$$\|x\|^2-\frac{2}{\alpha}\operatorname{Re}\left[\bar{\beta}\langle x,a\rangle\right]+\frac{|\beta|^2}{\alpha^2}\|a\|^2 \leq \frac{|\beta|^2-\alpha\gamma}{\alpha^2}\|a\|^2,$$

which is clearly equivalent to

$$\begin{aligned}\alpha\|x\|^2+\gamma\|a\|^2 &\leq 2\operatorname{Re}\left[\bar{\beta}\langle x,a\rangle\right]\\ &= 2\left[\operatorname{Re}\beta\cdot\operatorname{Re}\langle x,a\rangle+\operatorname{Im}\beta\cdot\operatorname{Im}\langle x,a\rangle\right].\end{aligned} \tag{2.250}$$

On the other hand, since

$$2\sqrt{\alpha\gamma}\,\|x\|\,\|a\| \leq \alpha\|x\|^2+\gamma\|a\|^2, \tag{2.251}$$

hence by (2.250) and (2.251) we deduce the first inequality in (2.248).

The other inequalities are obvious. ∎

Remark 37. *The above inequality (2.248) contains in particular the reverse (2.191) of the Schwarz inequality. Indeed, if we assume that* $\alpha=1$, $\beta=\frac{\delta+\Delta}{2}$, $\delta,\Delta\in\mathbb{K}$, *with* $\gamma=\operatorname{Re}(\Delta\bar{\gamma})>0$, *then the condition* $|\beta|^2\geq\alpha\gamma$ *is equivalent to* $|\delta+\Delta|^2\geq 4\operatorname{Re}(\Delta\bar{\gamma})$ *which is actually* $|\Delta-\delta|^2\geq 0$. *With this assumption, (2.247) becomes*

$$\left\|x-\frac{\delta+\Delta}{2}\cdot a\right\| \leq \frac{1}{2}|\Delta-\delta|\,\|a\|,$$

which implies the reverse of the Schwarz inequality

$$\|x\|\,\|a\| \leq \frac{\operatorname{Re}\left[\left(\bar{\Delta}+\bar{\delta}\right)\langle x,a\rangle\right]}{2\sqrt{\operatorname{Re}\left(\Delta\bar{\delta}\right)}} \leq \frac{|\Delta+\delta|}{2\sqrt{\operatorname{Re}\left(\Delta\bar{\delta}\right)}}\,|\langle x,a\rangle|,$$

which is (2.191).

The following particular case of Theorem 38 may be stated [**7**]:

COROLLARY 20. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$, $\varphi\in[0,2\pi)$, $\theta\in\left(0,\frac{\pi}{2}\right)$. *If* $x,a\in H$ *are such that* $a\neq 0$ *and*

$$\|x-(\cos\varphi+i\sin\varphi)\,a\|\leq\cos\theta\,\|a\|\,, \tag{2.252}$$

then we have the reverses of the Schwarz inequality

$$\|x\|\,\|a\|\leq\frac{\cos\varphi\operatorname{Re}\langle x,a\rangle+\sin\varphi\operatorname{Im}\langle x,a\rangle}{\sin\theta}. \tag{2.253}$$

In particular, if

$$\|x-a\|\leq\cos\theta\,\|a\|\,,$$

then

$$\|x\|\,\|a\|\leq\frac{1}{\cos\theta}\operatorname{Re}\langle x,a\rangle\,;$$

and if

$$\|x-ia\|\leq\cos\theta\,\|a\|\,,$$

then

$$\|x\|\,\|a\|\leq\frac{1}{\cos\theta}\operatorname{Im}\langle x,a\rangle\,.$$

2.8.4. Reverses of the Generalised Triangle Inequality. In [**13**], the author obtained the following reverse result for the generalised triangle inequality

$$\sum_{i=1}^{n}\|x_i\|\geq\left\|\sum_{i=1}^{n}x_i\right\|, \tag{2.254}$$

provided $x_i\in H$, $i\in\{1,\dots,n\}$ are vectors in a real or complex inner product $(H;\langle\cdot,\cdot\rangle)$:

THEOREM 39 (Dragomir, 2004). *Let* $e,x_i\in H$, $i\in\{1,\dots,n\}$ *with* $\|e\|=1$. *If* $k_i\geq 0$, $i\in\{1,\dots,n\}$ *are such that*

$$(0\leq)\,\|x_i\|-\operatorname{Re}\langle e,x_i\rangle\leq k_i\qquad\text{for each}\qquad i\in\{1,\dots,n\}\,, \tag{2.255}$$

then we have the inequality

$$(0\leq)\sum_{i=1}^{n}\|x_i\|-\left\|\sum_{i=1}^{n}x_i\right\|\leq\sum_{i=1}^{n}k_i. \tag{2.256}$$

The equality holds in (2.256) if and only if

$$\sum_{i=1}^{n}\|x_i\|\geq\sum_{i=1}^{n}k_i \tag{2.257}$$

and

$$\sum_{i=1}^{n}x_i=\left(\sum_{i=1}^{n}\|x_i\|-\sum_{i=1}^{n}k_i\right)e. \tag{2.258}$$

By utilising some of the results obtained in Section 2.8.2, we point out several reverses of the generalised triangle inequality (2.254) that are corollaries of the above Theorem 39 [**7**].

COROLLARY 21. *Let* $e,\ x_i \in H \backslash \{0\}$, $i \in \{1, \dots, n\}$ *with* $\|e\| = 1$. *If*

$$\left\| \frac{x_i}{\|x_i\|} - e \right\| \leq r_i \qquad \text{for each} \qquad i \in \{1, \dots, n\}, \tag{2.259}$$

then

$$\begin{aligned} (0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| &\leq \frac{1}{2} \sum_{i=1}^{n} r_i^2 \|x_i\| \\ &\leq \frac{1}{2} \times \begin{cases} \left(\max\limits_{1 \leq i \leq n} r_i \right)^2 \sum\limits_{i=1}^{n} \|x_i\|; \\ \\ \left(\sum\limits_{i=1}^{n} r_i^{2p} \right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^{n} \|x_i\|^q \right)^{\frac{1}{q}}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \max\limits_{1 \leq i \leq n} \|x_i\| \sum\limits_{i=1}^{n} r_i^2. \end{cases} \end{aligned} \tag{2.260}$$

PROOF. The first part follows from Proposition 25 on choosing $x = x_i$, $y = e$ and applying Theorem 39. The last part is obvious by Hölder's inequality. ∎

REMARK 38. *One would obtain the same reverse inequality (2.260) if one were to use Theorem 26. In this case, the assumption (2.259) should be replaced by*

$$\| \|x_i\| x_i - e\| \leq r_i \|x_i\| \qquad \text{for each} \qquad i \in \{1, \dots, n\}. \tag{2.261}$$

On utilising the inequalities (2.198) and (2.209) one may state the following corollary of Theorem 39 [**7**].

COROLLARY 22. *Let* $e,\ x_i \in H \backslash \{0\}$, $i \in \{1, \dots, n\}$ *with* $\|e\| = 1$. *Then we have the inequality*

$$(0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \min\{A, B\}, \tag{2.262}$$

where

$$A := 2\sum_{i=1}^{n} \|x_i\| \left(\frac{\|x_i - e\|}{\|x_i\| + 1} \right)^2,$$

and

$$B := \frac{1}{2}\sum_{i=1}^{n} \frac{(\|x_i\| + 1)^2 \|x_i - e\|^2}{\|x_i\|}.$$

For vectors located outside the closed unit ball $\bar{B}(0,1) := \{z \in H | \, \|z\| \le 1\}$, we may state the following result [**7**].

Corollary 23. *Assume that* $x_i \notin \bar{B}(0,1)$, $i \in \{1,\dots,n\}$ *and* $e \in H$, $\|e\| = 1$. *Then we have the inequality:*

$$(0 \le) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \le \begin{cases} \frac{1}{2} p^2 \sum_{i=1}^{n} \|x_i\|^{p-1} \|x_i - e\|^2, & \text{if } p \ge 1 \\ \\ \frac{1}{2} \sum_{i=1}^{n} \|x_i\|^{1-p} \|x_i - e\|^2, & \text{if } p < 1. \end{cases} \tag{2.263}$$

The proof follows by Proposition 27 and Theorem 39.

For complex spaces one may state the following result as well [**7**].

Corollary 24. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be a complex inner product space and* $\alpha_i \in \mathbb{C}$ *with* $\operatorname{Re}\alpha_i, \operatorname{Im}\alpha_i > 0$, $i \in \{1,\dots,n\}$. *If* $x_i, e \in H$, $i \in \{1,\dots,n\}$ *with* $\|e\| = 1$ *and*

$$\left\| x_i - \frac{\operatorname{Im}\alpha_i}{\operatorname{Re}\alpha_i} \cdot e \right\| \le d_i, \qquad i \in \{1,\dots,n\}, \tag{2.264}$$

then

$$(0 \le) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \le \frac{1}{2} \sum_{i=1}^{n} \frac{\operatorname{Re}\alpha_i}{\operatorname{Im}\alpha_i} \cdot d_i^2. \tag{2.265}$$

The proof follows by Theorems 30 and 39 and the details are omitted.

Finally, by the use of Theorem 37, we can state [**7**]:

COROLLARY 25. *If* $x_i, e \in H$, $i \in \{1, \dots, n\}$ *with* $\|e\| = 1$ *and* $p \geq 1$, *then we have the inequalities:*

$$(0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \frac{1}{2} \times \begin{cases} \sum_{i=1}^{n} \left[(\|x_i\| + 1)^{2p} - \|x_i + e\|^{2p} \right]^{\frac{1}{p}}, \\ \\ \sum_{i=1}^{n} \left[\|x_i - e\|^{2p} - \left| \|x_i\| - 1 \right|^{2p} \right]^{\frac{1}{p}}. \end{cases} \tag{2.266}$$

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CHAPTER 3

Reverses for the Triangle Inequality

3.1. Introduction

The following reverse of the generalised triangle inequality

$$\cos\theta \sum_{k=1}^{n} |z_k| \leq \left| \sum_{k=1}^{n} z_k \right|,$$

provided the complex numbers z_k, $k \in \{1, \dots, n\}$ satisfy the assumption

$$a - \theta \leq \arg(z_k) \leq a + \theta, \quad \text{for any} \quad k \in \{1, \dots, n\},$$

where $a \in \mathbb{R}$ and $\theta \in \left(0, \frac{\pi}{2}\right)$ was first discovered by M. Petrovich in 1917, [**11**] (see [**10**, p. 492]) and subsequently was rediscovered by other authors, including J. Karamata [**6**, p. 300 – 301], H.S. Wilf [**12**], and in an equivalent form by M. Marden [**8**].

In 1966, J.B. Diaz and F.T. Metcalf [**1**] proved the following reverse of the triangle inequality:

Theorem 40 (Diaz-Metcalf, 1966). *Let a be a unit vector in the inner product space $(H; \langle \cdot, \cdot \rangle)$ over the real or complex number field $\mathbb{K}$. Suppose that the vectors $x_i \in H \backslash \{0\}$, $i \in \{1, \dots, n\}$ satisfy*

$$0 \leq r \leq \frac{\operatorname{Re}\langle x_i, a \rangle}{\|x_i\|}, \qquad i \in \{1, \dots, n\}. \tag{3.1}$$

Then

$$r \sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\|, \tag{3.2}$$

where equality holds if and only if

$$\sum_{i=1}^{n} x_i = r \left(\sum_{i=1}^{n} \|x_i\| \right) a. \tag{3.3}$$

A generalisation of this result for orthonormal families is incorporated in the following result [**1**].

Theorem 41 (Diaz-Metcalf, 1966). *Let $a_1, \dots, a_n$ be orthonormal vectors in H. Suppose the vectors $x_1, \dots, x_n \in H \backslash \{0\}$ satisfy*

$$0 \leq r_k \leq \frac{\operatorname{Re}\langle x_i, a_k\rangle}{\|x_i\|}, \qquad i \in \{1, \dots, n\}, \; k \in \{1, \dots, m\}. \tag{3.4}$$

Then

$$\left(\sum_{k=1}^{m} r_k^2\right)^{\frac{1}{2}} \sum_{i=1}^{n} \|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|, \tag{3.5}$$

where equality holds if and only if

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\|\right) \sum_{k=1}^{m} r_k a_k. \tag{3.6}$$

Similar results valid for semi-inner products may be found in [**7**] and [**9**].

For other classical inequalities related to the triangle inequality, see Chapter XVII of the book [**10**] and the references therein.

The aim of the present chapter is to provide various recent reverses for the generalised triangle inequality in both its simple form that are closely related to the Diaz-Metcalf results mentioned above, or in the equivalent quadratic form, i.e., upper bounds for

$$\left(\sum_{i=1}^{n} \|x_i\|\right)^2 - \left\|\sum_{i=1}^{n} x_i\right\|^2$$

and

$$\frac{\left\|\sum_{i=1}^{n} x_i\right\|^2}{\left(\sum_{i=1}^{n} \|x_i\|\right)^2}.$$

Applications for vector valued integral inequalities and for complex numbers are given as well.

3.2. Some Inequalities of Diaz-Metcalf Type

3.2.1. The Case of One Vector. The following result with a natural geometrical meaning holds [**3**]:

Theorem 42 (Dragomir, 2004). *Let a be a unit vector in the inner product space $(H; \langle\cdot,\cdot\rangle)$ and $\rho \in (0,1)$. If $x_i \in H$, $i \in \{1, \dots, n\}$ are such that*

$$\|x_i - a\| \leq \rho \;\; \textit{for each} \;\; i \in \{1, \dots, n\}, \tag{3.7}$$

then we have the inequality

$$\sqrt{1-\rho^2}\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n} x_i\right\|, \tag{3.8}$$

with equality if and only if

$$\sum_{i=1}^{n} x_i = \sqrt{1-\rho^2}\left(\sum_{i=1}^{n}\|x_i\|\right) a. \tag{3.9}$$

PROOF. From (3.7) we have

$$\|x_i\|^2 - 2\operatorname{Re}\langle x_i, a\rangle + 1 \le \rho^2,$$

giving

$$\|x_i\|^2 + 1 - \rho^2 \le 2\operatorname{Re}\langle x_i, a\rangle, \tag{3.10}$$

for each $i \in \{1, \dots, n\}$.

Dividing by $\sqrt{1-\rho^2} > 0$, we deduce

$$\frac{\|x_i\|^2}{\sqrt{1-\rho^2}} + \sqrt{1-\rho^2} \le \frac{2\operatorname{Re}\langle x_i, a\rangle}{\sqrt{1-\rho^2}}, \tag{3.11}$$

for each $i \in \{1, \dots, n\}$.

On the other hand, by the elementary inequality

$$\frac{p}{\alpha} + q\alpha \ge 2\sqrt{pq}, \quad p, q \ge 0,\ \alpha > 0 \tag{3.12}$$

we have

$$2\|x_i\| \le \frac{\|x_i\|^2}{\sqrt{1-\rho^2}} + \sqrt{1-\rho^2} \tag{3.13}$$

and thus, by (3.11) and (3.13), we deduce

$$\frac{\operatorname{Re}\langle x_i, a\rangle}{\|x_i\|} \ge \sqrt{1-\rho^2},$$

for each $i \in \{1, \dots, n\}$. Applying Theorem 40 for $r = \sqrt{1-\rho^2}$, we deduce the desired inequality (3.8). ∎

The following results may be stated as well [**3**].

THEOREM 43 (Dragomir, 2004). *Let a be a unit vector in the inner product space $(H; \langle\cdot,\cdot\rangle)$ and $M \ge m > 0$. If $x_i \in H$, $i \in \{1, \dots, n\}$ are such that either*

$$\operatorname{Re}\langle Ma - x_i, x_i - ma\rangle \ge 0 \tag{3.14}$$

or, equivalently,

$$\left\| x_i - \frac{M+m}{2} \cdot a \right\| \le \frac{1}{2}(M-m) \tag{3.15}$$

holds for each $i \in \{1, \dots, n\}$, *then we have the inequality*

$$\frac{2\sqrt{mM}}{m+M} \sum_{i=1}^{n} \|x_i\| \le \left\| \sum_{i=1}^{n} x_i \right\|, \tag{3.16}$$

or, equivalently,

$$(0 \le) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \le \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \left\| \sum_{i=1}^{n} x_i \right\|. \tag{3.17}$$

The equality holds in (3.16) (or in (3.17)) if and only if

$$\sum_{i=1}^{n} x_i = \frac{2\sqrt{mM}}{m+M} \left(\sum_{i=1}^{n} \|x_i\| \right) a. \tag{3.18}$$

Proof. Firstly, we remark that if $x, z, Z \in H$, then the following statements are equivalent:

(i) $\operatorname{Re}\langle Z - x, x - z \rangle \ge 0$;
(ii) $\left\| x - \frac{Z+z}{2} \right\| \le \frac{1}{2} \|Z - z\|$.

Using this fact, one may simply realize that (3.14) and (3.15) are equivalent.

Now, from (3.14), we get

$$\|x_i\|^2 + mM \le (M+m) \operatorname{Re}\langle x_i, a \rangle,$$

for any $i \in \{1, \dots, n\}$. Dividing this inequality by $\sqrt{mM} > 0$, we deduce the following inequality that will be used in the sequel

$$\frac{\|x_i\|^2}{\sqrt{mM}} + \sqrt{mM} \le \frac{M+m}{\sqrt{mM}} \operatorname{Re}\langle x_i, a \rangle, \tag{3.19}$$

for each $i \in \{1, \dots, n\}$.

Using the inequality (3.12) from Theorem 42, we also have

$$2\|x_i\| \le \frac{\|x_i\|^2}{\sqrt{mM}} + \sqrt{mM}, \tag{3.20}$$

for each $i \in \{1, \dots, n\}$.

Utilizing (3.19) and (3.20), we may conclude with the following inequality

$$\|x_i\| \le \frac{M+m}{\sqrt{mM}} \operatorname{Re}\langle x_i, a \rangle,$$

which is equivalent to

$$\frac{2\sqrt{mM}}{m+M} \le \frac{\operatorname{Re}\langle x_i, a\rangle}{\|x_i\|} \tag{3.21}$$

for any $i \in \{1, \dots, n\}$.

Finally, on applying the Diaz-Metcalf result in Theorem 40 for $r = \frac{2\sqrt{mM}}{m+M}$, we deduce the desired conclusion.

The equivalence between (3.16) and (3.17) follows by simple calculation and we omit the details. ∎

3.2.2. The Case of m Vectors. In a similar manner to the one used in the proof of Theorem 42 and by the use of the Diaz-Metcalf inequality incorporated in Theorem 41, we can also prove the following result [**3**] :

PROPOSITION 35. *Let $a_1, \dots, a_n$ be orthonormal vectors in H. Suppose the vectors $x_1, \dots, x_n \in H \setminus \{0\}$ satisfy*

$$\|x_i - a_k\| \le \rho_k \quad \text{for each} \quad i \in \{1, \dots, n\}, \ k \in \{1, \dots, m\}, \tag{3.22}$$

where $\rho_k \in (0,1)$, $k \in \{1, \dots, m\}$. Then we have the following reverse of the triangle inequality

$$\left(m - \sum_{k=1}^{m} \rho_k^2\right)^{\frac{1}{2}} \sum_{i=1}^{n} \|x_i\| \le \left\| \sum_{i=1}^{n} x_i \right\|. \tag{3.23}$$

The equality holds in (3.23) if and only if

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\|\right) \sum_{k=1}^{m} \left(1 - \rho_k^2\right)^{\frac{1}{2}} a_k. \tag{3.24}$$

Finally, by the use of Theorem 41 and a similar technique to that employed in the proof of Theorem 43, we may state the following result [**3**]:

PROPOSITION 36. *Let $a_1, \dots, a_n$ be orthonormal vectors in H. Suppose the vectors $x_1, \dots, x_n \in H \setminus \{0\}$ satisfy*

$$\operatorname{Re}\langle M_k a_k - x_i, x_i - \mu_k a_k\rangle \ge 0, \tag{3.25}$$

or, equivalently,

$$\left\| x_i - \frac{M_k + \mu_k}{2} a_k \right\| \le \frac{1}{2}(M_k - \mu_k), \tag{3.26}$$

for any $i \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$, where $M_k \ge \mu_k > 0$ for each $k \in \{1, \dots, m\}$.

Then we have the inequality

$$2\left(\sum_{k=1}^{m}\frac{\mu_k M_k}{(\mu_k+M_k)^2}\right)^{\frac{1}{2}}\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n}x_i\right\|. \tag{3.27}$$

The equality holds in (3.27) iff

$$\sum_{i=1}^{n}x_i = 2\left(\sum_{i=1}^{n}\|x_i\|\right)\sum_{k=1}^{m}\frac{\sqrt{\mu_k M_k}}{\mu_k+M_k}a_k. \tag{3.28}$$

3.3. Additive Reverses for the Triangle Inequality

3.3.1. The Case of One Vector. In this section we establish some additive reverses of the generalised triangle inequality in real or complex inner product spaces.

The following result holds [**3**]:

THEOREM 44 (Dragomir, 2004). *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and e, $x_i\in H$, $i\in\{1,\dots,n\}$ with $\|e\|=1$. If $k_i\ge 0$, $i\in\{1,\dots,n\}$, are such that*

$$\|x_i\| - \operatorname{Re}\langle e,x_i\rangle \le k_i \quad \text{for each} \quad i\in\{1,\dots,n\}, \tag{3.29}$$

then we have the inequality

$$(0\le)\sum_{i=1}^{n}\|x_i\| - \left\|\sum_{i=1}^{n}x_i\right\| \le \sum_{i=1}^{n}k_i. \tag{3.30}$$

The equality holds in (3.30) if and only if

$$\sum_{i=1}^{n}\|x_i\| \ge \sum_{i=1}^{n}k_i \tag{3.31}$$

and

$$\sum_{i=1}^{n}x_i = \left(\sum_{i=1}^{n}\|x_i\| - \sum_{i=1}^{n}k_i\right)e. \tag{3.32}$$

PROOF. If we sum in (3.29) over i from 1 to n, then we get

$$\sum_{i=1}^{n}\|x_i\| \le \operatorname{Re}\left\langle e,\sum_{i=1}^{n}x_i\right\rangle + \sum_{i=1}^{n}k_i. \tag{3.33}$$

By Schwarz's inequality for e and $\sum_{i=1}^{n} x_i$, we have

$$\operatorname{Re}\left\langle e, \sum_{i=1}^{n} x_i \right\rangle \le \left| \operatorname{Re}\left\langle e, \sum_{i=1}^{n} x_i \right\rangle \right| \le \left| \left\langle e, \sum_{i=1}^{n} x_i \right\rangle \right| \le \|e\| \left\| \sum_{i=1}^{n} x_i \right\| = \left\| \sum_{i=1}^{n} x_i \right\|. \tag{3.34}$$

Making use of (3.33) and (3.34), we deduce the desired inequality (3.29).

If (3.31) and (3.32) hold, then

$$\left\| \sum_{i=1}^{n} x_i \right\| = \left| \sum_{i=1}^{n} \|x_i\| - \sum_{i=1}^{n} k_i \right| \|e\| = \sum_{i=1}^{n} \|x_i\| - \sum_{i=1}^{n} k_i,$$

and the equality in the second part of (3.30) holds true.

Conversely, if the equality holds in (3.30), then, obviously (3.31) is valid and we need only to prove (3.32).

Now, if the equality holds in (3.30) then it must hold in (3.29) for each $i \in \{1, \ldots, n\}$ and also must hold in any of the inequalities in (3.34).

It is well known that in Schwarz's inequality $|\langle u, v \rangle| \le \|u\| \|v\|$ ($u, v \in H$) the case of equality holds iff there exists a $\lambda \in \mathbb{K}$ such that $u = \lambda v$. We note that in the weaker inequality $\operatorname{Re}\langle u, v \rangle \le \|u\| \|v\|$ the case of equality holds iff $\lambda \ge 0$ and $u = \lambda v$.

Consequently, the equality holds in all inequalities (3.34) simultaneously iff there exists a $\mu \ge 0$ with

$$\mu e = \sum_{i=1}^{n} x_i. \tag{3.35}$$

If we sum the equalities in (3.29) over i from 1 to n, then we deduce

$$\sum_{i=1}^{n} \|x_i\| - \operatorname{Re}\left\langle e, \sum_{i=1}^{n} x_i \right\rangle = \sum_{i=1}^{n} k_i. \tag{3.36}$$

Replacing $\sum_{i=1}^{n} \|x_i\|$ from (3.35) into (3.36), we deduce

$$\sum_{i=1}^{n} \|x_i\| - \mu \|e\|^2 = \sum_{i=1}^{n} k_i,$$

from where we get $\mu = \sum_{i=1}^{n} \|x_i\| - \sum_{i=1}^{n} k_i$. Using (3.35), we deduce (3.32) and the theorem is proved. ∎

3.3.2. The Case of m Vectors. If we turn our attention to the case of orthogonal families, then we may state the following result as well [**3**].

THEOREM 45 (Dragomir, 2004). *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $\{e_k\}_{k\in\{1,\dots,m\}}$ a family of orthonormal vectors in H, $x_i \in H$, $M_{i,k} \geq 0$ for $i \in \{1,\dots,n\}$ and $k \in \{1,\dots,m\}$ such that*

$$\|x_i\| - \operatorname{Re}\langle e_k, x_i\rangle \leq M_{ik} \tag{3.37}$$

for each $i \in \{1,\dots,n\}$, $k \in \{1,\dots,m\}$. Then we have the inequality

$$\sum_{i=1}^{n}\|x_i\| \leq \frac{1}{\sqrt{m}}\left\|\sum_{i=1}^{n} x_i\right\| + \frac{1}{m}\sum_{i=1}^{n}\sum_{k=1}^{m} M_{ik}. \tag{3.38}$$

The equality holds true in (3.38) if and only if

$$\sum_{i=1}^{n}\|x_i\| \geq \frac{1}{m}\sum_{i=1}^{n}\sum_{k=1}^{m} M_{ik} \tag{3.39}$$

and

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n}\|x_i\| - \frac{1}{m}\sum_{i=1}^{n}\sum_{k=1}^{m} M_{ik}\right)\sum_{k=1}^{m} e_k. \tag{3.40}$$

PROOF. If we sum over i from 1 to n in (3.37), then we obtain

$$\sum_{i=1}^{n}\|x_i\| \leq \operatorname{Re}\left\langle e, \sum_{i=1}^{n} x_i\right\rangle + \sum_{i=1}^{n} M_{ik},$$

for each $k \in \{1,\dots,m\}$. Summing these inequalities over k from 1 to m, we deduce

$$\sum_{i=1}^{n}\|x_i\| \leq \frac{1}{m}\operatorname{Re}\left\langle \sum_{k=1}^{m} e_k, \sum_{i=1}^{n} x_i\right\rangle + \frac{1}{m}\sum_{i=1}^{n}\sum_{k=1}^{m} M_{ik}. \tag{3.41}$$

By Schwarz's inequality for $\sum_{k=1}^{m} e_k$ and $\sum_{i=1}^{n} x_i$ we have

$$\begin{aligned}
\operatorname{Re}\left\langle \sum_{k=1}^{m} e_k, \sum_{i=1}^{n} x_i \right\rangle &\le \left| \operatorname{Re}\left\langle \sum_{k=1}^{m} e_k, \sum_{i=1}^{n} x_i \right\rangle \right| \\
&\le \left| \left\langle \sum_{k=1}^{m} e_k, \sum_{i=1}^{n} x_i \right\rangle \right| \\
&\le \left\| \sum_{k=1}^{m} e_k \right\| \left\| \sum_{i=1}^{n} x_i \right\| \\
&= \sqrt{m} \left\| \sum_{i=1}^{n} x_i \right\|,
\end{aligned} \tag{3.42}$$

since, obviously,

$$\left\| \sum_{k=1}^{m} e_k \right\| = \sqrt{\left\| \sum_{k=1}^{m} e_k \right\|^2} = \sqrt{\sum_{k=1}^{m} \|e_k\|^2} = \sqrt{m}.$$

Making use of (3.41) and (3.42), we deduce the desired inequality (3.38).

If (3.39) and (3.40) hold, then

$$\begin{aligned}
\frac{1}{\sqrt{m}} \left\| \sum_{i=1}^{n} x_i \right\| &= \left| \sum_{i=1}^{n} \|x_i\| - \frac{1}{m} \sum_{i=1}^{n} \sum_{k=1}^{m} M_{ik} \right| \left\| \sum_{k=1}^{m} e_k \right\| \\
&= \frac{\sqrt{m}}{\sqrt{m}} \left(\sum_{i=1}^{n} \|x_i\| - \frac{1}{m} \sum_{i=1}^{n} \sum_{k=1}^{m} M_{ik} \right) \\
&= \sum_{i=1}^{n} \|x_i\| - \frac{1}{m} \sum_{i=1}^{n} \sum_{k=1}^{m} M_{ik},
\end{aligned}$$

and the equality in (3.38) holds true.

Conversely, if the equality holds in (3.38), then, obviously (3.39) is valid.

Now if the equality holds in (3.38), then it must hold in (3.37) for each $i \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$ and also must hold in any of the inequalities in (3.42).

It is well known that in Schwarz's inequality $\operatorname{Re}\langle u, v\rangle \le \|u\| \|v\|$, the equality occurs iff $u = \lambda v$ with $\lambda \ge 0$, consequently, the equality holds in all inequalities (3.42) simultaneously iff there exists a $\mu \ge 0$

with

$$\mu \sum_{k=1}^{m} e_k = \sum_{i=1}^{n} x_i. \tag{3.43}$$

If we sum the equality in (3.37) over i from 1 to n and k from 1 to m, then we deduce

$$m \sum_{i=1}^{n} \|x_i\| - \operatorname{Re} \left\langle \sum_{k=1}^{m} e_k, \sum_{i=1}^{n} x_i \right\rangle = \sum_{i=1}^{n} \sum_{k=1}^{m} M_{ik}. \tag{3.44}$$

Replacing $\sum_{i=1}^{n} x_i$ from (3.43) into (3.44), we deduce

$$m \sum_{i=1}^{n} \|x_i\| - \mu \sum_{k=1}^{m} \|e_k\|^2 = \sum_{i=1}^{n} \sum_{k=1}^{m} M_{ik}$$

giving

$$\mu = \sum_{i=1}^{n} \|x_i\| - \frac{1}{m} \sum_{i=1}^{n} \sum_{k=1}^{m} M_{ik}.$$

Using (3.43), we deduce (3.40) and the theorem is proved. ∎

3.4. Further Additive Reverses

3.4.1. The Case of Small Balls. In this section we point out different additive reverses of the generalised triangle inequality under simpler conditions for the vectors involved.

The following result holds [**3**]:

THEOREM 46 (Dragomir, 2004). *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $e, x_i \in H$, $i \in \{1, \dots, n\}$ with $\|e\| = 1$. If $\rho \in (0, 1)$ and x_i, $i \in \{1, \dots, n\}$ are such that*

$$\|x_i - e\| \leq \rho \quad \text{for each} \quad i \in \{1, \dots, n\}, \tag{3.45}$$

then we have the inequality

$$\begin{aligned} (0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| &\leq \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, e \right\rangle \\ &\left(\leq \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)} \left\| \sum_{i=1}^{n} x_i \right\| \right). \end{aligned} \tag{3.46}$$

The equality holds in (3.46) if and only if

$$\sum_{i=1}^{n}\|x_i\| \geq \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)}\operatorname{Re}\left\langle\sum_{i=1}^{n}x_i, e\right\rangle \tag{3.47}$$

and

$$\sum_{i=1}^{n}x_i = \left(\sum_{i=1}^{n}\|x_i\| - \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)}\operatorname{Re}\left\langle\sum_{i=1}^{n}x_i, e\right\rangle\right)e. \tag{3.48}$$

Proof. We know, from the proof of Theorem 44, that, if (3.45) is fulfilled, then we have the inequality

$$\|x_i\| \leq \frac{1}{\sqrt{1-\rho^2}}\operatorname{Re}\langle x_i, e\rangle$$

for each $i \in \{1, \dots, n\}$, implying

$$\begin{aligned}\|x_i\| - \operatorname{Re}\langle x_i, e\rangle &\leq \left(\frac{1}{\sqrt{1-\rho^2}} - 1\right)\operatorname{Re}\langle x_i, e\rangle \\ &= \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)}\operatorname{Re}\langle x_i, e\rangle\end{aligned} \tag{3.49}$$

for each $i \in \{1, \dots, n\}$.

Now, making use of Theorem 42, for

$$k_i := \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)}\operatorname{Re}\langle x_i, e\rangle, \quad i \in \{1, \dots, n\},$$

we easily deduce the conclusion of the theorem.

We omit the details. ∎

We may state the following result as well [**3**]:

Theorem 47 (Dragomir, 2004). *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space and $e \in H$, $M \geq m > 0$. If $x_i \in H$, $i \in \{1, \dots, n\}$ are such that either*

$$\operatorname{Re}\langle Me - x_i, x_i - me\rangle \geq 0, \tag{3.50}$$

or, equivalently,

$$\left\|x_i - \frac{M+m}{2}e\right\| \leq \frac{1}{2}(M-m) \tag{3.51}$$

holds for each $i \in \{1, \dots, n\}$, *then we have the inequality*

$$(0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, e \right\rangle \left(\leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \left\| \sum_{i=1}^{n} x_i \right\| \right). \tag{3.52}$$

The equality holds in (3.52) if and only if

$$\sum_{i=1}^{n} \|x_i\| \geq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, e \right\rangle \tag{3.53}$$

and

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\| - \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, e \right\rangle \right) e. \tag{3.54}$$

PROOF. We know, from the proof of Theorem 43, that if (3.50) is fulfilled, then we have the inequality

$$\|x_i\| \leq \frac{M + m}{2\sqrt{mM}} \operatorname{Re} \langle x_i, e \rangle$$

for each $i \in \{1, \dots, n\}$. This is equivalent to

$$\|x_i\| - \operatorname{Re} \langle x_i, e \rangle \leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re} \langle x_i, e \rangle$$

for each $i \in \{1, \dots, n\}$.

Now, making use of Theorem 44, we deduce the conclusion of the theorem. We omit the details. ∎

REMARK 39. *If one uses Theorem 45 instead of Theorem 44 above, then one can state the corresponding generalisation for families of orthonormal vectors of the inequalities (3.46) and (3.52) respectively. We do not provide them here.*

3.4.2. The Case of Arbitrary Balls. Now, on utilising a slightly different approach, we may point out the following result [**3**]:

THEOREM 48 (Dragomir, 2004). *Let* $(H; \langle \cdot, \cdot \rangle)$ *be an inner product space over* $\mathbb{K}$ *and* e, $x_i \in H$, $i \in \{1, \dots, n\}$ *with* $\|e\| = 1$. *If* $r_i > 0$,

$i \in \{1, \dots, n\}$ are such that

$$\|x_i - e\| \le r_i \quad \text{for each} \quad i \in \{1, \dots, n\}, \tag{3.55}$$

then we have the inequality

$$0 \le \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \le \frac{1}{2} \sum_{i=1}^{n} r_i^2. \tag{3.56}$$

The equality holds in (3.56) if and only if

$$\sum_{i=1}^{n} \|x_i\| \ge \frac{1}{2} \sum_{i=1}^{n} r_i^2 \tag{3.57}$$

and

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\| - \frac{1}{2} \sum_{i=1}^{n} r_i^2 \right) e. \tag{3.58}$$

PROOF. The condition (3.55) is clearly equivalent to

$$\|x_i\|^2 + 1 \le \operatorname{Re} \langle x_i, e \rangle + r_i^2 \tag{3.59}$$

for each $i \in \{1, \dots, n\}$.

Using the elementary inequality

$$2 \|x_i\| \le \|x_i\|^2 + 1, \tag{3.60}$$

for each $i \in \{1, \dots, n\}$, then, by (3.59) and (3.60), we deduce

$$2 \|x_i\| \le 2 \operatorname{Re} \langle x_i, e \rangle + r_i^2,$$

giving

$$\|x_i\| - \operatorname{Re} \langle x_i, e \rangle \le \frac{1}{2} r_i^2 \tag{3.61}$$

for each $i \in \{1, \dots, n\}$.

Now, utilising Theorem 44 for $k_i = \frac{1}{2} r_i^2$, $i \in \{1, \dots, n\}$, we deduce the desired result. We omit the details. ∎

Finally, we may state and prove the following result as well [**3**].

THEOREM 49 (Dragomir, 2004). *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ and e, $x_i \in H$, $i \in \{1, \dots, n\}$ with $\|e\| = 1$. If $M_i \ge m_i > 0$, $i \in \{1, \dots, n\}$, are such that*

$$\left\| x_i - \frac{M_i + m_i}{2} e \right\| \le \frac{1}{2} (M_i - m_i), \tag{3.62}$$

or, equivalently,

$$\operatorname{Re} \langle M_i e - x, x - m_i e \rangle \ge 0 \tag{3.63}$$

for each $i \in \{1, \dots, n\}$, then we have the inequality

$$(0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \frac{1}{4} \sum_{i=1}^{n} \frac{(M_i - m_i)^2}{M_i + m_i}. \tag{3.64}$$

The equality holds in (3.64) if and only if

$$\sum_{i=1}^{n} \|x_i\| \geq \frac{1}{4} \sum_{i=1}^{n} \frac{(M_i - m_i)^2}{M_i + m_i} \tag{3.65}$$

and

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\| - \frac{1}{4} \sum_{i=1}^{n} \frac{(M_i - m_i)^2}{M_i + m_i} \right) e. \tag{3.66}$$

Proof. The condition (3.62) is equivalent to:

$$\|x_i\|^2 + \left(\frac{M_i + m_i}{2} \right)^2 \leq 2 \operatorname{Re} \left\langle x_i, \frac{M_i + m_i}{2} e \right\rangle + \frac{1}{4} (M_i - m_i)^2$$

and since

$$2 \left(\frac{M_i + m_i}{2} \right) \|x_i\| \leq \|x_i\|^2 + \left(\frac{M_i + m_i}{2} \right)^2,$$

then we get

$$2 \left(\frac{M_i + m_i}{2} \right) \|x_i\| \leq 2 \cdot \frac{M_i + m_i}{2} \operatorname{Re} \langle x_i, e \rangle + \frac{1}{4} (M_i - m_i)^2,$$

or, equivalently,

$$\|x_i\| - \operatorname{Re} \langle x_i, e \rangle \leq \frac{1}{4} \cdot \frac{(M_i - m_i)^2}{M_i + m_i}$$

for each $i \in \{1, \dots, n\}$.

Now, making use of Theorem 44 for $k_i := \frac{1}{4} \cdot \frac{(M_i - m_i)^2}{M_i + m_i}$, $i \in \{1, \dots, n\}$, we deduce the desired result. ∎

Remark 40. *If one uses Theorem 45 instead of Theorem 44 above, then one can state the corresponding generalisation for families of orthonormal vectors of the inequalities in (3.56) and (3.64) respectively. We omit the details.*

3.5. Reverses of Schwarz Inequality

In this section we outline a procedure showing how some of the above results for triangle inequality may be employed to obtain reverses for the celebrated Schwarz inequality.

For $a \in H$, $\|a\| = 1$ and $r \in (0,1)$ define the closed ball

$$\overline{D}(a,r) := \{x \in H, \|x - a\| \le r\}.$$

The following reverse of the Schwarz inequality holds [**3**]:

Proposition 37. *If $x, y \in \overline{D}(a,r)$ with $a \in H$, $\|a\| = 1$ and $r \in (0,1)$, then we have the inequality*

$$(0 \le) \frac{\|x\| \|y\| - \operatorname{Re}\langle x, y\rangle}{(\|x\| + \|y\|)^2} \le \frac{1}{2} r^2. \tag{3.67}$$

The constant $\frac{1}{2}$ in (3.67) is best possible in the sense that it cannot be replaced by a smaller quantity.

Proof. Using Theorem 42 for $x_1 = x, x_2 = y, \rho = r$, we have

$$\sqrt{1 - r^2}\,(\|x\| + \|y\|) \le \|x + y\|. \tag{3.68}$$

Taking the square in (3.68) we deduce

$$\left(1 - r^2\right)\left(\|x\|^2 + 2\|x\|\|y\| + \|y\|^2\right) \le \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2$$

which is clearly equivalent to (3.67).

Now, assume that (3.67) holds with a constant $C > 0$ instead of $\frac{1}{2}$, *i.e.*,

$$\frac{\|x\| \|y\| - \operatorname{Re}\langle x, y\rangle}{(\|x\| + \|y\|)^2} \le C r^2 \tag{3.69}$$

provided $x, y \in \overline{D}(a,r)$ with $a \in H$, $\|a\| = 1$ and $r \in (0,1)$.

Let $e \in H$ with $\|e\| = 1$ and $e \perp a$. Define $x = a + re, y = a - re$. Then

$$\|x\| = \sqrt{1 + r^2} = \|y\|, \quad \operatorname{Re}\langle x, y\rangle = 1 - r^2$$

and thus, from (3.69), we have

$$\frac{1 + r^2 - (1 - r^2)}{\left(2\sqrt{1 + r^2}\right)^2} \le C r^2$$

giving

$$\frac{1}{2} \le \left(1 + r^2\right) C$$

for any $r \in (0,1)$. If in this inequality we let $r \to 0+$, then we get $C \ge \frac{1}{2}$ and the proposition is proved. ∎

In a similar way, by the use of Theorem 43, we may prove the following reverse of the Schwarz inequality as well [**3**]:

Proposition 38. *If $a \in H$, $\|a\| = 1$, $M \geq m > 0$ and $x, y \in H$ are so that either*

$$\operatorname{Re}\langle Ma - x, x - ma\rangle, \operatorname{Re}\langle Ma - y, y - ma\rangle \geq 0$$

or, equivalently,

$$\left\| x - \frac{m+M}{2}a \right\|, \left\| y - \frac{m+M}{2}a \right\| \leq \frac{1}{2}(M-m)$$

hold, then

$$(0 \leq) \frac{\|x\|\|y\| - \operatorname{Re}\langle x, y\rangle}{(\|x\| + \|y\|)^2} \leq \frac{1}{2}\left(\frac{M-m}{M+m}\right)^2.$$

The constant $\frac{1}{2}$ cannot be replaced by a smaller quantity.

Remark 41. *On utilising Theorem 35 and Theorem 36, we may deduce some similar reverses of Schwarz inequality provided $x, y \in \cap_{k=1}^{m} \overline{D}(a_k, \rho_k)$, assumed not to be empty, where $a_1, ..., a_n$ are orthonormal vectors in H and $\rho_k \in (0,1)$ for $k \in \{1, ..., m\}$. We omit the details.*

Remark 42. *For various different reverses of Schwarz inequality in inner product spaces, see the recent survey* [**2**].

3.6. Quadratic Reverses of the Triangle Inequality

3.6.1. The General Case. The following lemma holds [**4**]:

Lemma 5 (Dragomir, 2004). *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x_i \in H$, $i \in \{1, \ldots, n\}$ and $k_{ij} > 0$ for $1 \leq i < j \leq n$ such that*

$$0 \leq \|x_i\|\|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle \leq k_{ij} \tag{3.70}$$

for $1 \leq i < j \leq n$. Then we have the following quadratic reverse of the triangle inequality

$$\left(\sum_{i=1}^{n} \|x_i\|\right)^2 \leq \left\|\sum_{i=1}^{n} x_i\right\|^2 + 2\sum_{1 \leq i < j \leq n} k_{ij}. \tag{3.71}$$

The case of equality holds in (3.71) if and only if it holds in (3.70) for each i, j with $1 \leq i < j \leq n$.

PROOF. We observe that the following identity holds:

$$\begin{aligned}
&\left(\sum_{i=1}^{n}\|x_i\|\right)^2-\left\|\sum_{i=1}^{n}x_i\right\|^2 \\
&=\sum_{i,j=1}^{n}\|x_i\|\,\|x_j\|-\left\langle\sum_{i=1}^{n}x_i,\sum_{j=1}^{n}x_j\right\rangle \\
&=\sum_{i,j=1}^{n}\|x_i\|\,\|x_j\|-\sum_{i,j=1}^{n}\operatorname{Re}\langle x_i,x_j\rangle \\
&=\sum_{i,j=1}^{n}\left[\|x_i\|\,\|x_j\|-\operatorname{Re}\langle x_i,x_j\rangle\right] \\
&=\sum_{1\le i<j\le n}\left[\|x_i\|\,\|x_j\|-\operatorname{Re}\langle x_i,x_j\rangle\right] \\
&+\sum_{1\le j<i\le n}\left[\|x_i\|\,\|x_j\|-\operatorname{Re}\langle x_i,x_j\rangle\right] \\
&=2\sum_{1\le i<j\le n}\left[\|x_i\|\,\|x_j\|-\operatorname{Re}\langle x_i,x_j\rangle\right].
\end{aligned} \tag{3.72}$$

Using the condition (3.70), we deduce that

$$\sum_{1\le i<j\le n}\left[\|x_i\|\,\|x_j\|-\operatorname{Re}\langle x_i,x_j\rangle\right]\le\sum_{1\le i<j\le n}k_{ij},$$

and by (3.72), we get the desired inequality (3.71).

The case of equality is obvious by the identity (3.72) and we omit the details. ∎

REMARK 43. *From (3.71) one may deduce the coarser inequality that might be useful in some applications:*

$$\begin{aligned}
0&\le\sum_{i=1}^{n}\|x_i\|-\left\|\sum_{i=1}^{n}x_i\right\| \\
&\le\sqrt{2}\left(\sum_{1\le i<j\le n}k_{ij}\right)^{\frac{1}{2}}\qquad\left(\le\sqrt{2}\sum_{1\le i<j\le n}\sqrt{k_{ij}}\right).
\end{aligned}$$

REMARK 44. *If the condition (3.70) is replaced with the following refinement of Schwarz's inequality:*

$$(0\le)\,\delta_{ij}\le\|x_i\|\,\|x_j\|-\operatorname{Re}\langle x_i,x_j\rangle\ \ \text{for } 1\le i<j\le n, \tag{3.73}$$

then the following refinement of the quadratic generalised triangle inequality is valid:

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2 \geq \left\|\sum_{i=1}^{n}x_i\right\|^2 + 2\sum_{1\leq i<j\leq n}\delta_{ij} \quad \left(\geq \left\|\sum_{i=1}^{n}x_i\right\|^2\right). \tag{3.74}$$

The equality holds in the first part of (3.74) iff the case of equality holds in (3.73) for each $1 \leq i < j \leq n$.

The following result holds [**4**].

Proposition 39. *Let* $(H; \langle\cdot,\cdot\rangle)$ *be as above,* $x_i \in H$, $i \in \{1,\dots,n\}$ *and* $r > 0$ *such that*

$$\|x_i - x_j\| \leq r \tag{3.75}$$

for $1 \leq i < j \leq n$. *Then*

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2 \leq \left\|\sum_{i=1}^{n}x_i\right\|^2 + \frac{n(n-1)}{2}r^2. \tag{3.76}$$

The case of equality holds in (3.76) if and only if

$$\|x_i\|\,\|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle = \frac{1}{2}r^2 \tag{3.77}$$

for each i, j *with* $1 \leq i < j \leq n$.

Proof. The inequality (3.75) is obviously equivalent to

$$\|x_i\|^2 + \|x_j\|^2 \leq 2\operatorname{Re}\langle x_i, x_j\rangle + r^2$$

for $1 \leq i < j \leq n$. Since

$$2\|x_i\|\,\|x_j\| \leq \|x_i\|^2 + \|x_j\|^2, \quad 1 \leq i < j \leq n;$$

hence

$$\|x_i\|\,\|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle \leq \frac{1}{2}r^2 \tag{3.78}$$

for any i, j with $1 \leq i < j \leq n$.

Applying Lemma 5 for $k_{ij} := \frac{1}{2}r^2$ and taking into account that

$$\sum_{1\leq i<j\leq n} k_{ij} = \frac{n(n-1)}{4}r^2,$$

we deduce the desired inequality (3.76). The case of equality is also obvious by the above lemma and we omit the details. ∎

3.6.2. Inequalities in Terms of the Forward Difference. In the same spirit, and if some information about the forward difference $\Delta x_k := x_{k+1} - x_k$ $(1 \leq k \leq n-1)$ are available, then the following simple quadratic reverse of the generalised triangle inequality may be stated [**4**].

COROLLARY 26. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space and $x_i \in H$, $i \in \{1, \dots, n\}$. Then we have the inequality*

$$\left(\sum_{i=1}^{n} \|x_i\|\right)^2 \leq \left\|\sum_{i=1}^{n} x_i\right\|^2 + \frac{n(n-1)}{2} \sum_{k=1}^{n-1} \|\Delta x_k\|. \tag{3.79}$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced in general by a smaller quantity.

PROOF. Let $1 \leq i < j \leq n$. Then, obviously,

$$\|x_j - x_i\| = \left\|\sum_{k=i}^{j-1} \Delta x_k\right\| \leq \sum_{k=i}^{j-1} \|\Delta x_k\| \leq \sum_{k=1}^{n-1} \|\Delta x_k\|.$$

Applying Proposition 39 for $r := \sum_{k=1}^{n-1} \|\Delta x_k\|$, we deduce the desired result (3.79).

To prove the sharpness of the constant $\frac{1}{2}$, assume that the inequality (3.79) holds with a constant $c > 0$, i.e.,

$$\left(\sum_{i=1}^{n} \|x_i\|\right)^2 \leq \left\|\sum_{i=1}^{n} x_i\right\|^2 + cn(n-1) \sum_{k=1}^{n-1} \|\Delta x_k\| \tag{3.80}$$

for $n \geq 2$, $x_i \in H$, $i \in \{1, \dots, n\}$.

If we choose in (3.80), $n = 2$, $x_1 = -\frac{1}{2}e$, $x_2 = \frac{1}{2}e$, $e \in H$, $\|e\| = 1$, then we get $1 \leq 2c$, giving $c \geq \frac{1}{2}$. ∎

The following result providing a reverse of the quadratic generalised triangle inequality in terms of the sup-norm of the forward differences also holds [**4**].

PROPOSITION 40. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space and $x_i \in H$, $i \in \{1, \dots, n\}$. Then we have the inequality*

$$\left(\sum_{i=1}^{n} \|x_i\|\right)^2 \leq \left\|\sum_{i=1}^{n} x_i\right\|^2 + \frac{n^2(n^2-1)}{12} \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2. \tag{3.81}$$

The constant $\frac{1}{12}$ is best possible in (3.81).

Proof. As above, we have that

$$\|x_j - x_i\| \le \sum_{k=i}^{j-1} \|\Delta x_k\| \le (j-i) \max_{1\le k\le n-1} \|\Delta x_k\|,$$

for $1 \le i < j \le n$.

Squaring the above inequality, we get

$$\|x_j\|^2 + \|x_i\|^2 \le 2\operatorname{Re}\langle x_i, x_j\rangle + (j-i)^2 \max_{1\le k\le n-1} \|\Delta x_k\|^2$$

for any i, j with $1 \le i < j \le n$, and since

$$2\|x_i\|\|x_j\| \le \|x_j\|^2 + \|x_i\|^2,$$

hence

$$0 \le \|x_i\|\|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle \le \frac{1}{2}(j-i)^2 \max_{1\le k\le n-1} \|\Delta x_k\|^2 \tag{3.82}$$

for any i, j with $1 \le i < j \le n$.

Applying Lemma 5 for $k_{ij} := \frac{1}{2}(j-i)^2 \max_{1\le k\le n-1} \|\Delta x_k\|^2$, we can state that

$$\left(\sum_{i=1}^n \|x_i\|\right)^2 \le \left\|\sum_{i=1}^n x_i\right\|^2 + \sum_{1\le i<j\le n} (j-i)^2 \max_{1\le k\le n-1} \|\Delta x_k\|^2.$$

However,

$$\begin{aligned}\sum_{1\le i<j\le n} (j-i)^2 &= \frac{1}{2}\sum_{i,j=1}^n (j-i)^2 = n\sum_{k=1}^n k^2 - \left(\sum_{k=1}^n k\right)^2 \\ &= \frac{n^2(n^2-1)}{12}\end{aligned}$$

giving the desired inequality.

To prove the sharpness of the constant, assume that (3.81) holds with a constant $D > 0$, i.e.,

$$\left(\sum_{i=1}^n \|x_i\|\right)^2 \le \left\|\sum_{i=1}^n x_i\right\|^2 + Dn^2\left(n^2-1\right) \max_{1\le k\le n-1} \|\Delta x_k\|^2 \tag{3.83}$$

for $n \ge 2$, $x_i \in H$, $i \in \{1, \dots, n\}$.

If in (3.83) we choose $n = 2$, $x_1 = -\frac{1}{2}e$, $x_2 = \frac{1}{2}e$, $e \in H$, $\|e\| = 1$, then we get $1 \le 12D$ giving $D \ge \frac{1}{12}$. ∎

The following result may be stated as well [**4**].

Proposition 41. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space and $x_i \in H$, $i \in \{1,\dots,n\}$. Then we have the inequality:*

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2 \le \left\|\sum_{i=1}^{n}x_i\right\|^2 + \sum_{1\le i<j\le n}(j-i)^{\frac{2}{q}}\left(\sum_{k=1}^{n-1}\|\Delta x_k\|^p\right)^{\frac{2}{p}}, \tag{3.84}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

The constant $E=1$ in front of the double sum cannot generally be replaced by a smaller constant.

Proof. Using Hölder's inequality, we have

$$\begin{aligned}\|x_j - x_i\| &\le \sum_{k=i}^{j-1}\|\Delta x_k\| \le (j-i)^{\frac{1}{q}}\left(\sum_{k=i}^{j-1}\|\Delta x_k\|^p\right)^{\frac{1}{p}}\\ &\le (j-i)^{\frac{1}{q}}\left(\sum_{k=1}^{n-1}\|\Delta x_k\|^p\right)^{\frac{1}{p}},\end{aligned}$$

for $1\le i<j\le n$.

Squaring the previous inequality, we get

$$\|x_j\|^2 + \|x_i\|^2 \le 2\operatorname{Re}\langle x_i, x_j\rangle + (j-i)^{\frac{2}{q}}\left(\sum_{k=1}^{n-1}\|\Delta x_k\|^p\right)^{\frac{2}{p}},$$

for $1\le i<j\le n$.

Utilising the same argument from the proof of Proposition 40, we deduce the desired inequality (3.84).

Now assume that (3.84) holds with a constant $E>0$, i.e.,

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2 \le \left\|\sum_{i=1}^{n}x_i\right\|^2 + E\sum_{1\le i<j\le n}(j-i)^{\frac{2}{q}}\left(\sum_{k=1}^{n-1}\|\Delta x_k\|^p\right)^{\frac{2}{p}},$$

for $n\ge 2$ and $x_i\in H$, $i\in\{1,\dots,n\}$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

For $n=2$, $x_1=-\frac{1}{2}e$, $x_2=\frac{1}{2}e$, $\|e\|=1$, we get $1\le E$, showing the fact that the inequality (3.84) is sharp. ∎

The particular case $p=q=2$ is of interest [**4**].

Corollary 27. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space and $x_i \in H$, $i \in \{1,\dots,n\}$. Then we have the inequality:*

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2 \le \left\|\sum_{i=1}^{n}x_i\right\|^2 + \frac{(n^2-1)\,n}{6}\sum_{k=1}^{n-1}\|\Delta x_k\|^2. \tag{3.85}$$

The constant $\frac{1}{6}$ is best possible in (3.85).

Proof. For $p = q = 2$, Proposition 41 provides the inequality

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2 \le \left\|\sum_{i=1}^{n} x_i\right\|^2 + \sum_{1\le i<j\le n}(j-i)\sum_{k=1}^{n-1}\|\Delta x_k\|^2,$$

and since

$$\begin{aligned}&\sum_{1\le i<j\le n}(j-i)\\ &= 1 + (1+2) + (1+2+3) + \cdots + (1+2+\cdots+n-1)\\ &= \sum_{k=1}^{n-1}(1+2+\cdots+k) = \sum_{k=1}^{n-1}\frac{k(k+1)}{2} = \frac{n(n^2-1)}{6},\end{aligned}$$

hence the inequality (3.84) is proved. The best constant may be shown in the same way as above but we omit the details. ∎

3.6.3. A Different Quadratic Inequality. Finally, we may state and prove the following different result [**4**].

Theorem 50 (Dragomir, 2004). *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space, $y_i \in H$, $i \in \{1,\dots,n\}$ and $M \ge m > 0$ are such that either*

$$\operatorname{Re}\langle My_j - y_i, y_i - my_j\rangle \ge 0 \quad \text{for } 1 \le i < j \le n, \tag{3.86}$$

or, equivalently,

$$\left\|y_i - \frac{M+m}{2}y_j\right\| \le \frac{1}{2}(M-m)\|y_j\| \quad \text{for } 1 \le i < j \le n. \tag{3.87}$$

Then we have the inequality

$$\left(\sum_{i=1}^{n}\|y_i\|\right)^2 \le \left\|\sum_{i=1}^{n} y_i\right\|^2 + \frac{1}{2}\cdot\frac{(M-m)^2}{M+m}\sum_{k=1}^{n-1}k\|y_{k+1}\|^2. \tag{3.88}$$

The case of equality holds in (3.88) if and only if

$$\|y_i\|\|y_j\| - \operatorname{Re}\langle y_i, y_j\rangle = \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\|y_j\|^2 \tag{3.89}$$

for each i, j with $1 \le i < j \le n$.

Proof. Taking the square in (3.87), we get

$$\begin{aligned}&\|y_i\|^2 + \frac{(M-m)^2}{M+m}\|y_j\|^2\\ &\qquad\le 2\operatorname{Re}\left\langle y_i, \frac{M+m}{2}y_j\right\rangle + \frac{1}{n}(M-m)^2\|y_j\|^2\end{aligned}$$

for $1 \leq i < j \leq n$, and since, obviously,

$$2\left(\frac{M+m}{2}\right)\|y_i\|\|y_j\| \leq \|y_i\|^2 + \frac{(M-m)^2}{M+m}\|y_j\|^2,$$

hence

$$2\left(\frac{M+m}{2}\right)\|y_i\|\|y_j\| \leq 2\operatorname{Re}\left\langle y_i, \frac{M+m}{2}y_j\right\rangle + \frac{1}{n}(M-m)^2\|y_j\|^2,$$

giving the much simpler inequality

$$\|y_i\|\|y_j\| - \operatorname{Re}\langle y_i, y_j\rangle \leq \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\|y_j\|^2, \tag{3.90}$$

for $1 \leq i < j \leq n$.

Applying Lemma 5 for $k_{ij} := \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\|y_j\|^2$, we deduce

$$\left(\sum_{i=1}^{n}\|y_i\|\right)^2 \leq \left\|\sum_{i=1}^{n}y_i\right\|^2 + \frac{1}{2}\frac{(M-m)^2}{M+m}\sum_{1\leq i<j\leq n}\|y_j\|^2 \tag{3.91}$$

with equality if and only if (3.90) holds for each i, j with $1 \leq i < j \leq n$.

Since

$$\begin{aligned}\sum_{1\leq i<j\leq n}\|y_j\|^2 &= \sum_{1<j\leq n}\|y_j\|^2 + \sum_{2<j\leq n}\|y_j\|^2 + \cdots + \sum_{n-1<j\leq n}\|y_j\|^2 \\ &= \sum_{j=2}^{n}\|y_j\|^2 + \sum_{j=3}^{n}\|y_j\|^2 + \cdots + \sum_{j=n-1}^{n}\|y_j\|^2 + \|y_n\|^2 \\ &= \sum_{j=2}^{n}(j-1)\|y_j\|^2 = \sum_{k=1}^{n-1}k\|y_{k+1}\|^2,\end{aligned}$$

hence the inequality (3.88) is obtained. ∎

3.7. Further Quadratic Refinements

3.7.1. The General Case. The following lemma is of interest in itself as well [**4**].

LEMMA 6 (Dragomir, 2004). *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x_i \in H$, $i \in \{1,\dots,n\}$ and $k \geq 1$ with the property that:*

$$\|x_i\|\|x_j\| \leq k\operatorname{Re}\langle x_i, x_j\rangle, \tag{3.92}$$

for each i, j with $1 \leq i < j \leq n$. Then

$$\left(\sum_{i=1}^{n} \|x_i\|\right)^2 + (k-1)\sum_{i=1}^{n} \|x_i\|^2 \leq k \left\|\sum_{i=1}^{n} x_i\right\|^2 . \tag{3.93}$$

The equality holds in (3.93) if and only if it holds in (3.92) for each i, j with $1 \leq i < j \leq n$.

Proof. Firstly, let us observe that the following identity holds true:

$$\begin{aligned} \left(\sum_{i=1}^{n} \|x_i\|\right)^2 - k \left\|\sum_{i=1}^{n} x_i\right\|^2 &= \sum_{i,j=1}^{n} \|x_i\| \|x_j\| - k \left\langle \sum_{i=1}^{n} x_i, \sum_{j=1}^{n} x_j \right\rangle \\ &= \sum_{i,j=1}^{n} [\|x_i\| \|x_j\| - k \operatorname{Re}\langle x_i, x_j\rangle] \\ &= 2 \sum_{1 \leq i < j \leq n} [\|x_i\| \|x_j\| - k \operatorname{Re}\langle x_i, x_j\rangle] + (1-k)\sum_{i=1}^{n} \|x_i\|^2 , \end{aligned} \tag{3.94}$$

since, obviously, $\operatorname{Re}\langle x_i, x_j\rangle = \operatorname{Re}\langle x_j, x_i\rangle$ for any $i, j \in \{1, \ldots, n\}$.

Using the assumption (3.92), we obtain

$$\sum_{1 \leq i < j \leq n} [\|x_i\| \|x_j\| - k \operatorname{Re}\langle x_i, x_j\rangle] \leq 0$$

and thus, from (3.94), we deduce the desired inequality (3.93).

The case of equality is obvious by the identity (3.94) and we omit the details. ∎

Remark 45. *The inequality (3.93) provides the following reverse of the quadratic generalised triangle inequality:*

$$0 \leq \left(\sum_{i=1}^{n} \|x_i\|\right)^2 - \sum_{i=1}^{n} \|x_i\|^2 \leq k \left[\left\|\sum_{i=1}^{n} x_i\right\|^2 - \sum_{i=1}^{n} \|x_i\|^2\right] . \tag{3.95}$$

Remark 46. *Since $k = 1$ and $\sum_{i=1}^{n} \|x_i\|^2 \geq 0$, hence by (3.93) one may deduce the following reverse of the triangle inequality*

$$\sum_{i=1}^{n} \|x_i\| \leq \sqrt{k} \left\|\sum_{i=1}^{n} x_i\right\| , \tag{3.96}$$

provided (3.92) holds true for $1 \leq i < j \leq n$.

The following corollary providing a better bound for $\sum_{i=1}^{n}\|x_i\|,$ holds [**4**].

Corollary 28. *With the assumptions in Lemma 6, one has the inequality:*

$$\sum_{i=1}^{n}\|x_i\| \leq \sqrt{\frac{nk}{n+k-1}}\left\|\sum_{i=1}^{n}x_i\right\|. \tag{3.97}$$

Proof. Using the Cauchy-Bunyakovsky-Schwarz inequality

$$n\sum_{i=1}^{n}\|x_i\|^2 \geq \left(\sum_{i=1}^{n}\|x_i\|\right)^2$$

we get

$$(k-1)\sum_{i=1}^{n}\|x_i\|^2 + \left(\sum_{i=1}^{n}\|x_i\|\right)^2 \geq \left(\frac{k-1}{n}+1\right)\left(\sum_{i=1}^{n}\|x_i\|\right)^2. \tag{3.98}$$

Consequently, by (3.98) and (3.93) we deduce

$$k\left\|\sum_{i=1}^{n}x_i\right\|^2 \geq \frac{n+k-1}{n}\left(\sum_{i=1}^{n}\|x_i\|\right)^2$$

giving the desired inequality (3.97). ∎

3.7.2. Asymmetric Assumptions. The following result may be stated as well [**4**].

Theorem 51 (Dragomir, 2004). *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space and $x_i \in H\backslash\{0\}$, $i \in \{1,\ldots,n\}$, $\rho \in (0,1)$, such that*

$$\left\|x_i - \frac{x_j}{\|x_j\|}\right\| \leq \rho \quad \text{for } 1 \leq i < j \leq n. \tag{3.99}$$

Then we have the inequality

$$\sqrt{1-\rho^2}\left(\sum_{i=1}^{n}\|x_i\|\right)^2 + \left(1-\sqrt{1-\rho^2}\right)\sum_{i=1}^{n}\|x_i\|^2 \leq \left\|\sum_{i=1}^{n}x_i\right\|^2. \tag{3.100}$$

The case of equality holds in (3.100) iff

$$\|x_i\|\|x_j\| = \frac{1}{\sqrt{1-\rho^2}}\operatorname{Re}\langle x_i,x_j\rangle \tag{3.101}$$

for any $1 \leq i < j \leq n$.

Proof. The condition (3.92) is obviously equivalent to

$$\|x_i\|^2 + 1 - \rho^2 \le 2\operatorname{Re}\left\langle x_i, \frac{x_j}{\|x_j\|}\right\rangle$$

for each $1 \le i < j \le n$.

Dividing by $\sqrt{1-\rho^2} > 0$, we deduce

$$\frac{\|x_i\|^2}{\sqrt{1-\rho^2}} + \sqrt{1-\rho^2} \le \frac{2}{\sqrt{1-\rho^2}} \operatorname{Re}\left\langle x_i, \frac{x_j}{\|x_j\|}\right\rangle, \tag{3.102}$$

for $1 \le i < j \le n$.

On the other hand, by the elementary inequality

$$\frac{p}{\alpha} + q\alpha \ge 2\sqrt{pq}, \quad p, q \ge 0,\ \alpha > 0 \tag{3.103}$$

we have

$$2\|x_i\| \le \frac{\|x_i\|^2}{\sqrt{1-\rho^2}} + \sqrt{1-\rho^2}. \tag{3.104}$$

Making use of (3.102) and (3.104), we deduce that

$$\|x_i\|\,\|x_j\| \le \frac{1}{\sqrt{1-\rho^2}} \operatorname{Re}\langle x_i, x_j\rangle$$

for $1 \le i < j \le n$.

Now, applying Lemma 5 for $k = \frac{1}{\sqrt{1-\rho^2}}$, we deduce the desired result. ∎

Remark 47. *If we assume that* $\|x_i\| = 1$, $i \in \{1, \dots, n\}$, *satisfying the simpler condition*

$$\|x_j - x_i\| \le \rho \quad \text{for } 1 \le i < j \le n, \tag{3.105}$$

then, from (3.100), we deduce the following lower bound for $\|\sum_{i=1}^n x_i\|$, *namely*

$$\left[n + n(n-1)\sqrt{1-\rho^2}\right]^{\frac{1}{2}} \le \left\|\sum_{i=1}^n x_i\right\|. \tag{3.106}$$

The equality holds in (3.106) iff $\sqrt{1-\rho^2} = \operatorname{Re}\langle x_i, x_j\rangle$ *for* $1 \le i < j \le n$.

Remark 48. *Under the hypothesis of Proposition 41, we have the coarser but simpler reverse of the triangle inequality*

$$\sqrt[4]{1-\rho^2} \sum_{i=1}^n \|x_i\| \le \left\|\sum_{i=1}^n x_i\right\|. \tag{3.107}$$

Also, applying Corollary 28 for $k = \frac{1}{\sqrt{1-\rho^2}}$, we can state that

$$\sum_{i=1}^{n} \|x_i\| \leq \sqrt{\frac{n}{n\sqrt{1-\rho^2}+1-\sqrt{1-\rho^2}}} \left\| \sum_{i=1}^{n} x_i \right\|, \tag{3.108}$$

provided $x_i \in H$ satisfy (3.99) for $1 \leq i < j \leq n$.

In the same manner, we can state and prove the following reverse of the quadratic generalised triangle inequality [**4**].

THEOREM 52 (Dragomir, 2004). *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x_i \in H$, $i \in \{1, \dots, n\}$ and $M \geq m > 0$ such that either*

$$\operatorname{Re} \langle Mx_j - x_i, x_i - mx_j \rangle \geq 0 \quad \textit{for } 1 \leq i < j \leq n, \tag{3.109}$$

or, equivalently,

$$\left\| x_i - \frac{M+m}{2} x_j \right\| \leq \frac{1}{2} (M-m) \|x_j\| \quad \textit{for } 1 \leq i < j \leq n \tag{3.110}$$

hold. Then

$$\frac{2\sqrt{mM}}{M+m} \left(\sum_{i=1}^{n} \|x_i\| \right)^2 + \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{M+m} \sum_{i=1}^{n} \|x_i\|^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2. \tag{3.111}$$

The case of equality holds in (3.111) if and only if

$$\|x_i\| \|x_j\| = \frac{M+m}{2\sqrt{mM}} \operatorname{Re} \langle x_i, x_j \rangle \quad \textit{for } 1 \leq i < j \leq n. \tag{3.112}$$

PROOF. From (3.109), observe that

$$\|x_i\|^2 + Mm \|x_j\|^2 \leq (M+m) \operatorname{Re} \langle x_i, x_j \rangle, \tag{3.113}$$

for $1 \leq i < j \leq n$. Dividing (3.113) by $\sqrt{mM} > 0$, we deduce

$$\frac{\|x_i\|^2}{\sqrt{mM}} + \sqrt{mM} \|x_j\|^2 \leq \frac{M+m}{\sqrt{mM}} \operatorname{Re} \langle x_i, x_j \rangle,$$

and since, obviously

$$2 \|x_i\| \|x_j\| \leq \frac{\|x_i\|^2}{\sqrt{mM}} + \sqrt{mM} \|x_j\|^2$$

hence

$$\|x_i\| \|x_j\| \leq \frac{M+m}{2\sqrt{mM}} \operatorname{Re}\langle x_i, x_j\rangle, \quad \text{for } 1 \leq i < j \leq n.$$

Applying Lemma 6 for $k = \frac{M+m}{2\sqrt{mM}} \geq 1$, we deduce the desired result. ∎

Remark 49. *We also must note that a simpler but coarser inequality that can be obtained from (3.111) is*

$$\left(\frac{2\sqrt{mM}}{M+m}\right)^{\frac{1}{2}} \sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\|,$$

provided (3.109) holds true.

Finally, a different result related to the generalised triangle inequality is incorporated in the following theorem [**4**].

Theorem 53 (Dragomir, 2004). *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$, $\eta > 0$ and $x_i \in H$, $i \in \{1, \dots, n\}$ with the property that*

$$\|x_j - x_i\| \leq \eta < \|x_j\| \quad \textit{for each } \; i, j \in \{1, \dots, n\}. \tag{3.114}$$

Then we have the following reverse of the triangle inequality

$$\frac{\sum_{i=1}^{n} \sqrt{\|x_i\|^2 - \eta^2}}{\|\sum_{i=1}^{n} x_i\|} \leq \frac{\|\sum_{i=1}^{n} x_i\|}{\sum_{i=1}^{n} \|x_i\|}. \tag{3.115}$$

The equality holds in (3.115) iff

$$\|x_i\| \sqrt{\|x_j\|^2 - \eta^2} = \operatorname{Re}\langle x_i, x_j\rangle \quad \textit{for each } \; i, j \in \{1, \dots, n\}. \tag{3.116}$$

Proof. From (3.114), we have

$$\|x_i\|^2 + \|x_j\|^2 - \eta^2 \leq 2\operatorname{Re}\langle x_i, x_j\rangle, \quad i, j \in \{1, \dots, n\}.$$

On the other hand,

$$2\|x_i\| \sqrt{\|x_j\|^2 - \eta^2} \leq \|x_i\|^2 + \|x_j\|^2 - \eta^2, \quad i, j \in \{1, \dots, n\}$$

and thus

$$\|x_i\| \sqrt{\|x_j\|^2 - \eta^2} \leq \operatorname{Re}\langle x_i, x_j\rangle, \quad i, j \in \{1, \dots, n\}.$$

Summing over $i, j \in \{1, \dots, n\}$, we deduce the desired inequality (3.115).

The case of equality is also obvious from the above, and we omit the details. ∎

3.8. Reverses for Complex Spaces

3.8.1. The Case of One Vector. The following result holds [**5**].

THEOREM 54 (Dragomir, 2004). *Let $(H;\langle\cdot,\cdot\rangle)$ be a complex inner product space. Suppose that the vectors $x_k \in H$, $k \in \{1,\dots,n\}$ satisfy the condition*

$$0 \le r_1 \|x_k\| \le \operatorname{Re}\langle x_k, e\rangle, \quad 0 \le r_2 \|x_k\| \le \operatorname{Im}\langle x_k, e\rangle \tag{3.117}$$

for each $k \in \{1,\dots,n\}$, where $e \in H$ is such that $\|e\| = 1$ and $r_1, r_2 \ge 0$. Then we have the inequality

$$\sqrt{r_1^2 + r_2^2}\sum_{k=1}^{n}\|x_k\| \le \left\|\sum_{k=1}^{n} x_k\right\|, \tag{3.118}$$

where equality holds if and only if

$$\sum_{k=1}^{n} x_k = (r_1 + i r_2)\left(\sum_{k=1}^{n}\|x_k\|\right)e. \tag{3.119}$$

PROOF. In view of the Schwarz inequality in the complex inner product space $(H;\langle\cdot,\cdot\rangle)$, we have

$$\begin{aligned}\left\|\sum_{k=1}^{n} x_k\right\|^2 &= \left\|\sum_{k=1}^{n} x_k\right\|^2 \|e\|^2 \ge \left|\left\langle\sum_{k=1}^{n} x_k, e\right\rangle\right|^2 \\ &= \left|\left\langle\sum_{k=1}^{n} x_k, e\right\rangle\right|^2 \\ &= \left|\sum_{k=1}^{n}\operatorname{Re}\langle x_k, e\rangle + i\left(\sum_{k=1}^{n}\operatorname{Im}\langle x_k, e\rangle\right)\right|^2 \\ &= \left(\sum_{k=1}^{n}\operatorname{Re}\langle x_k, e\rangle\right)^2 + \left(\sum_{k=1}^{n}\operatorname{Im}\langle x_k, e\rangle\right)^2.\end{aligned} \tag{3.120}$$

Now, by hypothesis (3.117)

$$\left(\sum_{k=1}^{n}\operatorname{Re}\langle x_k, e\rangle\right)^2 \ge r_1^2\left(\sum_{k=1}^{n}\|x_k\|\right)^2 \tag{3.121}$$

and

$$\left(\sum_{k=1}^{n}\operatorname{Im}\langle x_k, e\rangle\right)^2 \ge r_2^2\left(\sum_{k=1}^{n}\|x_k\|\right)^2. \tag{3.122}$$

If we add (3.121) and (3.122) and use (3.120), then we deduce the desired inequality (3.118).

Now, if (3.119) holds, then

$$\left\|\sum_{k=1}^{n} x_k\right\| = |r_1 + ir_2| \left(\sum_{k=1}^{n} \|x_k\|\right) \|e\| = \sqrt{r_1^2 + r_2^2} \sum_{k=1}^{n} \|x_k\|$$

and the case of equality is valid in (3.118).

Before we prove the reverse implication, let us observe that for $x \in H$ and $e \in H$, $\|e\| = 1$, the following identity is true

$$\|x - \langle x, e\rangle e\|^2 = \|x\|^2 - |\langle x, e\rangle|^2,$$

therefore $\|x\| = |\langle x, e\rangle|$ if and only if $x = \langle x, e\rangle e$.

If we assume that equality holds in (3.118), then the case of equality must hold in all the inequalities required in the argument used to prove the inequality (3.118), and we may state that

$$\left\|\sum_{k=1}^{n} x_k\right\| = \left|\left\langle \sum_{k=1}^{n} x_k, e\right\rangle\right|, \tag{3.123}$$

and

$$r_1 \|x_k\| = \operatorname{Re}\langle x_k, e\rangle, \quad r_2 \|x_k\| = \operatorname{Im}\langle x_k, e\rangle \tag{3.124}$$

for each $k \in \{1, \dots, n\}$.

From (3.123) we deduce

$$\sum_{k=1}^{n} x_k = \left\langle \sum_{k=1}^{n} x_k, e\right\rangle e \tag{3.125}$$

and from (3.124), by multiplying the second equation with i and summing both equations over k from 1 to n, we deduce

$$(r_1 + ir_2) \sum_{k=1}^{n} \|x_k\| = \left\langle \sum_{k=1}^{n} x_k, e\right\rangle. \tag{3.126}$$

Finally, by (3.126) and (3.125), we get the desired equality (3.119). ∎

The following corollary is of interest [**5**].

Corollary 29. *Let e a unit vector in the complex inner product space $(H; \langle\cdot,\cdot\rangle)$ and $\rho_1, \rho_2 \in (0,1)$. If $x_k \in H$, $k \in \{1, \dots, n\}$ are such that*

$$\|x_k - e\| \le \rho_1, \quad \|x_k - ie\| \le \rho_2 \quad \text{for each } k \in \{1, \dots, n\}, \tag{3.127}$$

then we have the inequality

$$\sqrt{2-\rho_1^2-\rho_2^2}\sum_{k=1}^{n}\|x_k\| \leq \left\|\sum_{k=1}^{n}x_k\right\|, \tag{3.128}$$

with equality if and only if

$$\sum_{k=1}^{n}x_k = \left(\sqrt{1-\rho_1^2}+i\sqrt{1-\rho_2^2}\right)\left(\sum_{k=1}^{n}\|x_k\|\right)e. \tag{3.129}$$

Proof. From the first inequality in (3.127) we deduce

$$0 \leq \sqrt{1-\rho_1^2}\,\|x_k\| \leq \operatorname{Re}\langle x_k, e\rangle \tag{3.130}$$

for each $k \in \{1,\dots,n\}$.

From the second inequality in (3.127) we deduce

$$0 \leq \sqrt{1-\rho_2^2}\,\|x_k\| \leq \operatorname{Re}\langle x_k, ie\rangle$$

for each $k \in \{1,\dots,n\}$. Since

$$\operatorname{Re}\langle x_k, ie\rangle = \operatorname{Im}\langle x_k, e\rangle,$$

hence

$$0 \leq \sqrt{1-\rho_2^2}\,\|x_k\| \leq \operatorname{Im}\langle x_k, e\rangle \tag{3.131}$$

for each $k \in \{1,\dots,n\}$.

Now, observe from (3.130) and (3.131), that the condition (3.117) of Theorem 54 is satisfied for $r_1 = \sqrt{1-\rho_1^2}$, $r_2 = \sqrt{1-\rho_2^2} \in (0,1)$, and thus the corollary is proved. ∎

The following corollary may be stated as well [**5**].

Corollary 30. *Let e be a unit vector in the complex inner product space $(H;\langle\cdot,\cdot\rangle)$ and $M_1 \geq m_1 > 0$, $M_2 \geq m_2 > 0$. If $x_k \in H$, $k \in \{1,\dots,n\}$ are such that either*

$$\begin{aligned}\operatorname{Re}\langle M_1e-x_k, x_k-m_1e\rangle &\geq 0,\\ \operatorname{Re}\langle M_2ie-x_k, x_k-m_2ie\rangle &\geq 0\end{aligned} \tag{3.132}$$

or, equivalently,

$$\begin{aligned}\left\|x_k-\frac{M_1+m_1}{2}e\right\| &\leq \frac{1}{2}(M_1-m_1),\\ \left\|x_k-\frac{M_2+m_2}{2}ie\right\| &\leq \frac{1}{2}(M_2-m_2),\end{aligned} \tag{3.133}$$

for each $k \in \{1, \ldots, n\}$, *then we have the inequality*

$$2\left[\frac{m_1 M_1}{(M_1+m_1)^2} + \frac{m_2 M_2}{(M_2+m_2)^2}\right]^{\frac{1}{2}} \sum_{k=1}^{n} \|x_k\| \le \left\|\sum_{k=1}^{n} x_k\right\|. \tag{3.134}$$

The equality holds in (3.134) if and only if

$$\sum_{k=1}^{n} x_k = 2\left(\frac{\sqrt{m_1 M_1}}{M_1+m_1} + i\frac{\sqrt{m_2 M_2}}{M_2+m_2}\right)\left(\sum_{k=1}^{n} \|x_k\|\right) e. \tag{3.135}$$

Proof. From the first inequality in (3.132)

$$0 \le \frac{2\sqrt{m_1 M_1}}{M_1+m_1} \|x_k\| \le \operatorname{Re}\langle x_k, e\rangle \tag{3.136}$$

for each $k \in \{1, \ldots, n\}$.

Now, the proof follows the same path as the one of Corollary 29 and we omit the details. ∎

3.8.2. The Case of m Orthonormal Vectors. In [**1**], the authors have proved the following reverse of the generalised triangle inequality in terms of orthonormal vectors [**5**].

Theorem 55 (Diaz-Metcalf, 1966). *Let* $e_1, \ldots, e_m$ *be orthonormal vectors in* $(H; \langle\cdot,\cdot\rangle)$, *i.e., we recall that* $\langle e_i, e_j\rangle = 0$ *if* $i \neq j$ *and* $\|e_i\| = 1$, $i, j \in \{1, \ldots, m\}$. *Suppose that the vectors* $x_1, \ldots, x_n \in H$ *satisfy*

$$0 \le r_k \|x_j\| \le \operatorname{Re}\langle x_j, e_k\rangle,$$

$j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$. *Then*

$$\left(\sum_{k=1}^{m} r_k^2\right)^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\| \le \left\|\sum_{j=1}^{n} x_j\right\|, \tag{3.137}$$

where equality holds if and only if

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n} \|x_j\|\right) \sum_{k=1}^{m} r_k e_k. \tag{3.138}$$

If the space $(H; \langle\cdot,\cdot\rangle)$ is complex and more information is available for the imaginary part, then the following result may be stated as well [**5**].

Theorem 56 (Dragomir, 2004). *Let* $e_1, \ldots, e_m \in H$ *be an orthonormal family of vectors in the complex inner product space* H. *If the vectors* $x_1, \ldots, x_n \in H$ *satisfy the conditions*

$$0 \le r_k \|x_j\| \le \operatorname{Re}\langle x_j, e_k\rangle, \qquad 0 \le \rho_k \|x_j\| \le \operatorname{Im}\langle x_j, e_k\rangle \tag{3.139}$$

for each $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$, then we have the following reverse of the generalised triangle inequality;

$$\left[\sum_{k=1}^{m}\left(r_k^2+\rho_k^2\right)\right]^{\frac{1}{2}} \sum_{j=1}^{n}\|x_j\| \leq \left\|\sum_{j=1}^{n} x_j\right\|. \tag{3.140}$$

The equality holds in (3.140) if and only if

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n}\|x_j\|\right)\sum_{k=1}^{m}\left(r_k+i\rho_k\right)e_k. \tag{3.141}$$

Proof. Before we prove the theorem, let us recall that, if $x \in H$ and $e_1, \ldots, e_m$ are orthogonal vectors, then the following identity holds true:

$$\left\|x-\sum_{k=1}^{m}\langle x, e_k\rangle e_k\right\|^2 = \|x\|^2 - \sum_{k=1}^{n}|\langle x, e_k\rangle|^2. \tag{3.142}$$

As a consequence of this identity, we note the *Bessel inequality*

$$\sum_{k=1}^{m}|\langle x, e_k\rangle|^2 \leq \|x\|^2, x \in H. \tag{3.143}$$

The case of equality holds in (3.143) if and only if (see (3.142))

$$x = \sum_{k=1}^{m}\langle x, e_k\rangle e_k. \tag{3.144}$$

Applying Bessel's inequality for $x = \sum_{j=1}^{n} x_j$, we have

$$\begin{aligned} \left\|\sum_{j=1}^{n} x_j\right\|^2 &\geq \sum_{k=1}^{m}\left|\left\langle\sum_{j=1}^{n} x_j, e_k\right\rangle\right|^2 = \sum_{k=1}^{m}\left|\sum_{j=1}^{n}\langle x_j, e_k\rangle\right|^2 \\ &= \sum_{k=1}^{m}\left|\left(\sum_{j=1}^{n}\operatorname{Re}\langle x_j, e_k\rangle\right)+i\left(\sum_{j=1}^{n}\operatorname{Im}\langle x_j, e_k\rangle\right)\right|^2 \\ &= \sum_{k=1}^{m}\left[\left(\sum_{j=1}^{n}\operatorname{Re}\langle x_j, e_k\rangle\right)^2+\left(\sum_{j=1}^{n}\operatorname{Im}\langle x_j, e_k\rangle\right)^2\right]. \end{aligned} \tag{3.145}$$

Now, by the hypothesis (3.139) we have

$$\left(\sum_{j=1}^{n}\operatorname{Re}\langle x_j, e_k\rangle\right)^2 \geq r_k^2\left(\sum_{j=1}^{n}\|x_j\|\right)^2 \tag{3.146}$$

and

$$\left(\sum_{j=1}^{n} \operatorname{Im}\langle x_j, e_k\rangle\right)^2 \geq \rho_k^2 \left(\sum_{j=1}^{n} \|x_j\|\right)^2. \tag{3.147}$$

Further, on making use of (3.145) – (3.147), we deduce

$$\begin{aligned}\left\|\sum_{j=1}^{n} x_j\right\|^2 &\geq \sum_{k=1}^{m} \left[r_k^2 \left(\sum_{j=1}^{n} \|x_j\|\right)^2 + \rho_k^2 \left(\sum_{j=1}^{n} \|x_j\|\right)^2\right] \\ &= \left(\sum_{j=1}^{n} \|x_j\|\right)^2 \sum_{k=1}^{m} \left(r_k^2 + \rho_k^2\right),\end{aligned}$$

which is clearly equivalent to (3.140).

Now, if (3.141) holds, then

$$\begin{aligned}\left\|\sum_{j=1}^{n} x_j\right\|^2 &= \left(\sum_{j=1}^{n} \|x_j\|\right)^2 \left\|\sum_{k=1}^{m} (r_k + i\rho_k) e_k\right\|^2 \\ &= \left(\sum_{j=1}^{n} \|x_j\|\right)^2 \sum_{k=1}^{m} |r_k + i\rho_k|^2 \\ &= \left(\sum_{j=1}^{n} \|x_j\|\right)^2 \sum_{k=1}^{m} \left(r_k^2 + \rho_k^2\right),\end{aligned}$$

and the case of equality holds in (3.140).

Conversely, if the equality holds in (3.140), then it must hold in all the inequalities used to prove (3.140) and therefore we must have

$$\left\|\sum_{j=1}^{n} x_j\right\|^2 = \sum_{k=1}^{m} \left|\sum_{j=1}^{n} \langle x_j, e_k\rangle\right|^2 \tag{3.148}$$

and

$$r_k \|x_j\| = \operatorname{Re}\langle x_j, e_k\rangle, \qquad \rho_k \|x_j\| = \operatorname{Im}\langle x_j, e_k\rangle \tag{3.149}$$

for each $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$.

Using the identity (3.142), we deduce from (3.148) that

$$\sum_{j=1}^{n} x_j = \sum_{k=1}^{m} \left\langle \sum_{j=1}^{n} x_j, e_k \right\rangle e_k. \tag{3.150}$$

Multiplying the second equality in (3.149) with the imaginary unit i and summing the equality over j from 1 to n, we deduce

$$(r_k + i\rho_k)\sum_{j=1}^{n}\|x_j\| = \left\langle \sum_{j=1}^{n} x_j, e_k \right\rangle \tag{3.151}$$

for each $k \in \{1,\dots,n\}$.

Finally, utilising (3.150) and (3.151), we deduce (3.141) and the theorem is proved. ∎

The following corollaries are of interest [**5**].

Corollary 31. *Let $e_1,\dots,e_m$ be orthonormal vectors in the complex inner product space $(H;\langle\cdot,\cdot\rangle)$ and $\rho_k,\eta_k \in (0,1)$, $k \in \{1,\dots,n\}$. If $x_1,\dots,x_n \in H$ are such that*

$$\|x_j - e_k\| \le \rho_k, \qquad \|x_j - ie_k\| \le \eta_k$$

for each $j \in \{1,\dots,n\}$ and $k \in \{1,\dots,m\}$, then we have the inequality

$$\left[\sum_{k=1}^{m}\left(2-\rho_k^2-\eta_k^2\right)\right]^{\frac{1}{2}}\sum_{j=1}^{n}\|x_j\| \le \left\|\sum_{j=1}^{n}x_j\right\|. \tag{3.152}$$

The case of equality holds in (3.152) if and only if

$$\sum_{j=1}^{n}x_j = \left(\sum_{j=1}^{n}\|x_j\|\right)\sum_{k=1}^{m}\left(\sqrt{1-\rho_k^2}+i\sqrt{1-\eta_k^2}\right)e_k. \tag{3.153}$$

The proof employs Theorem 56 and is similar to the one from Corollary 29. We omit the details.

Corollary 32. *Let $e_1,\dots,e_m$ be as in Corollary 31 and $M_k \ge m_k > 0$, $N_k \ge n_k > 0$, $k \in \{1,\dots,m\}$. If $x_1,\dots,x_n \in H$ are such that either*

$$\operatorname{Re}\langle M_k e_k - x_j, x_j - m_k e_k\rangle \ge 0, \quad \operatorname{Re}\langle N_k ie_k - x_j, x_j - n_k ie_k\rangle \ge 0$$

or, equivalently,

$$\left\|x_j - \frac{M_k+m_k}{2}e_k\right\| \le \frac{1}{2}(M_k-m_k),$$
$$\left\|x_j - \frac{N_k+n_k}{2}ie_k\right\| \le \frac{1}{2}(N_k-n_k)$$

for each $j \in \{1,\dots,n\}$ and $k \in \{1,\dots,m\}$, then we have the inequality

$$2\left\{\sum_{k=1}^{m}\left[\frac{m_kM_k}{(M_k+m_k)^2}+\frac{n_kN_k}{(N_k+n_k)^2}\right]\right\}^{\frac{1}{2}}\sum_{j=1}^{n}\|x_j\| \le \left\|\sum_{j=1}^{n}x_j\right\|. \tag{3.154}$$

The case of equality holds in (3.154) if and only if

$$\sum_{j=1}^{n} x_j = 2\left(\sum_{j=1}^{n} \|x_j\|\right) \sum_{k=1}^{m} \left(\frac{\sqrt{m_k M_k}}{M_k + m_k} + i\frac{\sqrt{n_k N_k}}{N_k + n_k}\right) e_k. \tag{3.155}$$

The proof employs Theorem 56 and is similar to the one in Corollary 30. We omit the details.

3.9. Applications for Vector-Valued Integral Inequalities

Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space over the real or complex number field, $[a,b]$ a compact interval in $\mathbb{R}$ and $\eta : [a,b] \to [0,\infty)$ a Lebesgue integrable function on $[a,b]$ with the property that $\int_a^b \eta(t)\,dt = 1$. If, by $L_\eta([a,b];H)$ we denote the Hilbert space of all Bochner measurable functions $f : [a,b] \to H$ with the property that $\int_a^b \eta(t)\|f(t)\|^2\,dt < \infty$, then the norm $\|\cdot\|_\eta$ of this space is generated by the inner product $\langle\cdot,\cdot\rangle_\eta : H \times H \to \mathbb{K}$ defined by

$$\langle f, g\rangle_\eta := \int_a^b \eta(t)\langle f(t), g(t)\rangle\,dt.$$

The following proposition providing a reverse of the integral generalised triangle inequality may be stated [**3**].

Proposition 42. *Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\eta : [a,b] \to [0,\infty)$ as above. If $g \in L_\eta([a,b];H)$ is so that $\int_a^b \eta(t)\|g(t)\|^2\,dt = 1$ and $f_i \in L_\eta([a,b];H)$, $i \in \{1,\dots,n\}$, $\rho \in (0,1)$ are so that*

$$\|f_i(t) - g(t)\| \le \rho \tag{3.156}$$

for a.e. $t \in [a,b]$ and each $i \in \{1,\dots,n\}$, then we have the inequality

$$\sqrt{1-\rho^2}\sum_{i=1}^{n}\left(\int_a^b \eta(t)\|f_i(t)\|^2\,dt\right)^{\frac{1}{2}} \le \left(\int_a^b \eta(t)\left\|\sum_{i=1}^{n} f_i(t)\right\|^2 dt\right)^{\frac{1}{2}}. \tag{3.157}$$

The case of equality holds in (3.157) if and only if

$$\sum_{i=1}^{n} f_i(t) = \sqrt{1-\rho^2}\sum_{i=1}^{n}\left(\int_a^b \eta(t)\|f_i(t)\|^2\,dt\right)^{\frac{1}{2}} \cdot g(t)$$

for a.e. $t \in [a,b]$.

PROOF. Observe, by (3.157), that

$$\begin{aligned}\|f_i - g\|_\eta &= \left(\int_a^b \eta(t)\|f_i(t) - g(t)\|^2 dt\right)^{\frac{1}{2}} \\ &\le \left(\int_a^b \eta(t)\rho^2 dt\right)^{\frac{1}{2}} = \rho\end{aligned}$$

for each $i \in \{1, \ldots, n\}$. Applying Theorem 42 for the Hilbert space $L_\eta([a,b];H)$, we deduce the desired result. ∎

The following result may be stated as well [**3**].

PROPOSITION 43. *Let H, η, g be as in Proposition 42. If $f_i \in L_\eta([a,b];H), i \in \{1, \ldots, n\}$ and $M \ge m > 0$ are so that either*

$$\operatorname{Re}\langle Mg(t) - f_i(t), f_i(t) - mg(t)\rangle \ge 0$$

or, equivalently,

$$\left\| f_i(t) - \frac{m+M}{2} g(t) \right\| \le \frac{1}{2}(M-m)$$

for a.e. $t \in [a,b]$ and each $i \in \{1, \ldots, n\}$, then we have the inequality

$$\frac{2\sqrt{mM}}{m+M} \sum_{i=1}^n \left(\int_a^b \eta(t)\|f_i(t)\|^2 dt\right)^{\frac{1}{2}} \le \left(\int_a^b \eta(t)\left\|\sum_{i=1}^n f_i(t)\right\|^2 dt\right)^{\frac{1}{2}}. \tag{3.158}$$

The equality holds in (3.158) if and only if

$$\sum_{i=1}^n f_i(t) = \frac{2\sqrt{mM}}{m+M} \sum_{i=1}^n \left(\int_a^b \eta(t)\|f_i(t)\|^2 dt\right)^{\frac{1}{2}} \cdot g(t),$$

for a.e. $t \in [a,b]$.

The following proposition providing a reverse of the integral generalised triangle inequality may be stated [**4**].

PROPOSITION 44. *Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\eta : [a,b] \to [0,\infty)$ as above. If $g \in L_\eta([a,b];H)$ is so that $\int_a^b \eta(t)\|g(t)\|^2 dt = 1$ and $f_i \in L_\eta([a,b];H), i \in \{1, \ldots, n\}$, and $M \ge m > 0$ are so that either*

$$\operatorname{Re}\langle Mf_j(t) - f_i(t), f_i(t) - mf_j(t)\rangle \ge 0 \tag{3.159}$$

or, equivalently,

$$\left\|f_i(t)-\frac{m+M}{2}f_j(t)\right\|\le\frac{1}{2}(M-m)\|f_j(t)\|$$

for a.e. $t\in[a,b]$ *and* $1\le i<j\le n$, *then we have the inequality*

$$\left[\sum_{i=1}^{n}\left(\int_a^b\eta(t)\|f_i(t)\|^2dt\right)^{\frac{1}{2}}\right]^2 \le\int_a^b\eta(t)\left\|\sum_{i=1}^{n}f_i(t)\right\|^2dt +\frac{1}{2}\cdot\frac{(M-m)^2}{m+M}\int_a^b\eta(t)\left(\sum_{k=1}^{n-1}k\|f_{k+1}(t)\|^2\right)dt. \tag{3.160}$$

The case of equality holds in (3.160) if and only if

$$\left(\int_a^b\eta(t)\|f_i(t)\|^2dt\right)^{\frac{1}{2}}\left(\int_a^b\eta(t)\|f_j(t)\|^2dt\right)^{\frac{1}{2}} -\int_a^b\eta(t)\operatorname{Re}\langle f_i(t),f_j(t)\rangle dt =\frac{1}{4}\cdot\frac{(M-m)^2}{m+M}\int_a^b\eta(t)\|f_j(t)\|^2dt$$

for each i,j *with* $1\le i<j\le n$.

PROOF. We observe that

$$\operatorname{Re}\langle Mf_j-f_i,f_i-mf_j\rangle_\eta =\int_a^b\eta(t)\operatorname{Re}\langle Mf_j(t)-f_i(t),f_i(t)-mf_j(t)\rangle dt\ge0$$

for any i,j with $1\le i<j\le n$.

Applying Theorem 50 for the Hilbert space $L_\eta([a,b];H)$ and for $y_i=f_i, i\in\{1,\dots,n\}$, we deduce the desired result. ∎

Another integral inequality incorporated in the following proposition holds [**4**]:

PROPOSITION 45. *With the assumptions of Proposition 44, we have*

$$\frac{2\sqrt{mM}}{m+M}\left[\sum_{i=1}^{n}\left(\int_a^b \eta(t)\|f_i(t)\|^2\,dt\right)^{\frac{1}{2}}\right]^2 + \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{m+M}\sum_{i=1}^{n}\int_a^b \eta(t)\|f_i(t)\|^2\,dt \leq \int_a^b \eta(t)\left\|\sum_{i=1}^{n} f_i(t)\right\|^2 dt. \tag{3.161}$$

The case of equality holds in (3.161) if and only if

$$\left(\int_a^b \eta(t)\|f_i(t)\|^2\,dt\right)^{\frac{1}{2}}\left(\int_a^b \eta(t)\|f_j(t)\|^2\,dt\right)^{\frac{1}{2}} = \frac{M+m}{2\sqrt{mM}}\int_a^b \eta(t)\operatorname{Re}\langle f_i(t), f_j(t)\rangle\,dt$$

for any i,j with $1 \leq i < j \leq n$.

The proof is obvious by Theorem 52 and we omit the details.

3.10. Applications for Complex Numbers

The following reverse of the generalised triangle inequality with a clear geometric meaning may be stated [**5**].

PROPOSITION 46. *Let $z_1,\dots,z_n$ be complex numbers with the property that*

$$0 \leq \varphi_1 \leq \arg(z_k) \leq \varphi_2 < \frac{\pi}{2} \tag{3.162}$$

for each $k \in \{1,\dots,n\}$. Then we have the inequality

$$\sqrt{\sin^2\varphi_1+\cos^2\varphi_2}\sum_{k=1}^{n}|z_k| \leq \left|\sum_{k=1}^{n} z_k\right|. \tag{3.163}$$

The equality holds in (3.163) if and only if

$$\sum_{k=1}^{n} z_k = (\cos\varphi_2 + i\sin\varphi_1)\sum_{k=1}^{n}|z_k|. \tag{3.164}$$

Proof. Let $z_k = a_k + ib_k$. We may assume that $b_k \geq 0$, $a_k > 0$, $k \in \{1, \dots, n\}$, since, by (3.162), $\frac{b_k}{a_k} = \tan[\arg(z_k)] \in \left[0, \frac{\pi}{2}\right)$, $k \in \{1, \dots, n\}$. By (3.162), we obviously have

$$0 \leq \tan^2 \varphi_1 \leq \frac{b_k^2}{a_k^2} \leq \tan^2 \varphi_2, \qquad k \in \{1, \dots, n\}$$

from where we get

$$\frac{b_k^2 + a_k^2}{a_k^2} \leq \frac{1}{\cos^2 \varphi_2}, \qquad k \in \{1, \dots, n\}, \ \varphi_2 \in \left(0, \frac{\pi}{2}\right)$$

and

$$\frac{a_k^2 + b_k^2}{a_k^2} \leq \frac{1 + \tan^2 \varphi_1}{\tan^2 \varphi_1} = \frac{1}{\sin^2 \varphi_1}, \qquad k \in \{1, \dots, n\}, \ \varphi_1 \in \left(0, \frac{\pi}{2}\right)$$

giving the inequalities

$$|z_k| \cos \varphi_2 \leq \operatorname{Re}(z_k), \quad |z_k| \sin \varphi_1 \leq \operatorname{Im}(z_k)$$

for each $k \in \{1, \dots, n\}$.

Now, applying Theorem 54 for the complex inner product $\mathbb{C}$ endowed with the inner product $\langle z, w \rangle = z \cdot \bar{w}$ for $x_k = z_k$, $r_1 = \cos \varphi_2$, $r_2 = \sin \varphi_1$ and $e = 1$, we deduce the desired inequality (3.163). The case of equality is also obvious by Theorem 54 and the proposition is proven. ∎

Another result that has an obvious geometrical interpretation is the following one.

Proposition 47. *Let $c \in \mathbb{C}$ with $|z| = 1$ and $\rho_1, \rho_2 \in (0, 1)$. If $z_k \in \mathbb{C}$, $k \in \{1, \dots, n\}$ are such that*

$$|z_k - c| \leq \rho_1, \quad |z_k - ic| \leq \rho_2 \ \text{ for each } \ k \in \{1, \dots, n\}, \tag{3.165}$$

then we have the inequality

$$\sqrt{2 - \rho_1^2 - \rho_2^2} \sum_{k=1}^{n} |z_k| \leq \left| \sum_{k=1}^{n} z_k \right|, \tag{3.166}$$

with equality if and only if

$$\sum_{k=1}^{n} z_k = \left(\sqrt{1 - \rho_1^2} + i\sqrt{1 - \rho_2^2} \right) \left(\sum_{k=1}^{n} |z_k| \right) c. \tag{3.167}$$

The proof is obvious by Corollary 29 applied for $H = \mathbb{C}$.

REMARK 50. *If we choose* $e = 1$*, and for* $\rho_1, \rho_2 \in (0,1)$ *we define* $\bar{D}(1,\rho_1) := \{z \in \mathbb{C} \mid |z-1| \leq \rho_1\}$, $\bar{D}(i,\rho_2) := \{z \in \mathbb{C} \mid |z-i| \leq \rho_2\}$, *then obviously the intersection*

$$S_{\rho_1,\rho_2} := \bar{D}(1,\rho_1) \cap \bar{D}(i,\rho_2)$$

is nonempty if and only if $\rho_1 + \rho_2 \geq \sqrt{2}$.

If $z_k \in S_{\rho_1,\rho_2}$ *for* $k \in \{1,\dots,n\}$*, then (3.166) holds true. The equality holds in (3.166) if and only if*

$$\sum_{k=1}^{n} z_k = \left(\sqrt{1-\rho_1^2} + i\sqrt{1-\rho_2^2}\right) \sum_{k=1}^{n} |z_k| .$$

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CHAPTER 4

Reverses for the Continuous Triangle Inequality

4.1. Introduction

Let $f : [a, b] \to \mathbb{K}$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ be a Lebesgue integrable function. The following inequality, which is the continuous version of the *triangle inequality*

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx, \tag{4.1}$$

plays a fundamental role in Mathematical Analysis and its applications.

It appears, see [**8**, p. 492], that the first reverse inequality for (4.1) was obtained by J. Karamata in his book from 1949, [**6**]. It can be stated as

$$\cos\theta \int_a^b |f(x)|\, dx \leq \left| \int_a^b f(x)\, dx \right| \tag{4.2}$$

provided

$$-\theta \leq \arg f(x) \leq \theta, \quad x \in [a, b]$$

for given $\theta \in \left(0, \frac{\pi}{2}\right)$.

This integral inequality is the continuous version of a reverse inequality for the generalised triangle inequality

$$\cos\theta \sum_{i=1}^{n} |z_i| \leq \left| \sum_{i=1}^{n} z_i \right|, \tag{4.3}$$

provided

$$a - \theta \leq \arg(z_i) \leq a + \theta, \quad \text{for} \quad i \in \{1, \dots, n\},$$

where $a \in \mathbb{R}$ and $\theta \in \left(0, \frac{\pi}{2}\right)$, which, as pointed out in [**8**, p. 492], was first discovered by M. Petrovich in 1917, [**9**], and, subsequently rediscovered by other authors, including J. Karamata [**6**, p. 300 – 301], H.S. Wilf [**10**], and in an equivalent form, by M. Marden [**7**].

The first to consider the problem in the more general case of Hilbert and Banach spaces, were J.B. Diaz and F.T. Metcalf [**1**] who showed that, in an inner product space H over the real or complex number field , the following reverse of the triangle inequality holds

$$r\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n}x_i\right\|, \tag{4.4}$$

provided

$$0 \le r \le \frac{\operatorname{Re}\langle x_i, a\rangle}{\|x_i\|}, \qquad i \in \{1,\dots,n\},$$

and $a \in H$ is a unit vector, i.e., $\|a\| = 1$. The case of equality holds in (4.4) if and only if

$$\sum_{i=1}^{n}x_i = r\left(\sum_{i=1}^{n}\|x_i\|\right)a. \tag{4.5}$$

A generalisation of this result for orthonormal families is also well known [**1**]:

Let $a_1,\dots,a_m$ be m orthonormal vectors in H. Suppose the vectors $x_1,\dots,x_n \in H\backslash\{0\}$ satisfy

$$0 \le r_k \le \frac{\operatorname{Re}\langle x_i, a_k\rangle}{\|x_i\|}, \qquad i \in \{1,\dots,n\},\ k \in \{1,\dots,m\}.$$

Then

$$\left(\sum_{k=1}^{m}r_k^2\right)^{\frac{1}{2}}\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n}x_i\right\|,$$

where equality holds if and only if

$$\sum_{i=1}^{n}x_i = \left(\sum_{i=1}^{n}\|x_i\|\right)\sum_{k=1}^{m}r_k a_k.$$

The main aim of this chapter is to survey some recent reverses of the triangle inequality for Bochner integrable functions f with values in Hilbert spaces and defined on a compact interval $[a,b] \subset \mathbb{R}$. Applications for Lebesgue integrable complex-valued functions are provided as well.

4.2. Multiplicative Reverses

4.2.1. Reverses for a Unit Vector. We recall that $f \in L([a,b];H)$, the space of Bochner integrable functions with values in a Hilbert space H, if and only if $f:[a,b]\to H$ is Bochner measurable on $[a,b]$ and the Lebesgue integral $\int_a^b \|f(t)\|\,dt$ is finite.

The following result holds [**2**]:

THEOREM 57 (Dragomir, 2004). *If $f \in L([a,b];H)$ is such that there exists a constant $K \geq 1$ and a vector $e \in H$, $\|e\| = 1$ with*

$$\|f(t)\| \leq K \operatorname{Re}\langle f(t), e\rangle \quad \text{for a.e. } t \in [a,b], \tag{4.6}$$

then we have the inequality:

$$\int_a^b \|f(t)\|\, dt \leq K \left\| \int_a^b f(t)\, dt \right\|. \tag{4.7}$$

The case of equality holds in (4.7) if and only if

$$\int_a^b f(t)\, dt = \frac{1}{K}\left(\int_a^b \|f(t)\|\, dt\right) e. \tag{4.8}$$

PROOF. By the Schwarz inequality in inner product spaces, we have

$$\begin{aligned} \left\| \int_a^b f(t)\, dt \right\| &= \left\| \int_a^b f(t)\, dt \right\| \|e\| \\ &\geq \left| \left\langle \int_a^b f(t)\, dt, e \right\rangle \right| \geq \left| \operatorname{Re}\left\langle \int_a^b f(t)\, dt, e \right\rangle \right| \\ &\geq \operatorname{Re}\left\langle \int_a^b f(t)\, dt, e \right\rangle = \int_a^b \operatorname{Re}\langle f(t), e\rangle\, dt. \end{aligned} \tag{4.9}$$

From the condition (4.6), on integrating over $[a,b]$, we deduce

$$\int_a^b \operatorname{Re}\langle f(t), e\rangle\, dt \geq \frac{1}{K}\int_a^b \|f(t)\|\, dt, \tag{4.10}$$

and thus, on making use of (4.9) and (4.10), we obtain the desired inequality (4.7).

If (4.8) holds true, then, obviously

$$K \left\| \int_a^b f(t)\, dt \right\| = \|e\| \int_a^b \|f(t)\|\, dt = \int_a^b \|f(t)\|\, dt,$$

showing that (4.7) holds with equality.

If we assume that the equality holds in (4.7), then by the argument provided at the beginning of our proof, we must have equality in each of the inequalities from (4.9) and (4.10).

Observe that in Schwarz's inequality $\|x\|\,\|y\| \geq \operatorname{Re}\langle x, y\rangle$, $x, y \in H$, the case of equality holds if and only if there exists a positive scalar μ such that $x = \mu e$. Therefore, equality holds in the first inequality in (4.9) iff $\int_a^b f(t)\, dt = \lambda e$, with $\lambda \geq 0$.

If we assume that a strict inequality holds in (4.6) on a subset of nonzero Lebesgue measure in $[a,b]$, then

$$\int_a^b \|f(t)\|\,dt < K\int_a^b \operatorname{Re}\langle f(t), e\rangle\,dt,$$

and by (4.9) we deduce a strict inequality in (4.7), which contradicts the assumption. Thus, we must have $\|f(t)\| = K\operatorname{Re}\langle f(t), e\rangle$ for a.e. $t\in[a,b]$.

If we integrate this equality, we deduce

$$\begin{aligned}\int_a^b \|f(t)\|\,dt &= K\int_a^b \operatorname{Re}\langle f(t), e\rangle\,dt = K\operatorname{Re}\left\langle \int_a^b f(t)\,dt, e\right\rangle \\ &= K\operatorname{Re}\langle \lambda e, e\rangle = \lambda K\end{aligned}$$

giving

$$\lambda = \frac{1}{K}\int_a^b \|f(t)\|\,dt,$$

and thus the equality (4.8) is necessary.

This completes the proof. ∎

A more appropriate result from an applications point of view is perhaps the following result [**2**].

Corollary 33. *Let e be a unit vector in the Hilbert space $(H;\langle\cdot,\cdot\rangle)$, $\rho\in(0,1)$ and $f\in L([a,b];H)$ so that*

$$\|f(t)-e\|\le\rho \quad \text{for a.e. } t\in[a,b]. \tag{4.11}$$

Then we have the inequality

$$\sqrt{1-\rho^2}\int_a^b \|f(t)\|\,dt \le \left\|\int_a^b f(t)\,dt\right\|, \tag{4.12}$$

with equality if and only if

$$\int_a^b f(t)\,dt = \sqrt{1-\rho^2}\left(\int_a^b \|f(t)\|\,dt\right)e. \tag{4.13}$$

Proof. From (4.11), we have

$$\|f(t)\|^2 - 2\operatorname{Re}\langle f(t), e\rangle + 1 \le \rho^2,$$

giving

$$\|f(t)\|^2 + 1 - \rho^2 \le 2\operatorname{Re}\langle f(t), e\rangle$$

for a.e. $t\in[a,b]$.

Dividing by $\sqrt{1-\rho^2}>0$, we deduce

$$\frac{\|f(t)\|^2}{\sqrt{1-\rho^2}} + \sqrt{1-\rho^2} \le \frac{2\operatorname{Re}\langle f(t), e\rangle}{\sqrt{1-\rho^2}} \tag{4.14}$$

for a.e. $t \in [a,b]$.

On the other hand, by the elementary inequality

$$\frac{p}{\alpha} + q\alpha \geq 2\sqrt{pq}, \quad p,q \geq 0,\ \alpha > 0$$

we have

$$2\|f(t)\| \leq \frac{\|f(t)\|^2}{\sqrt{1-\rho^2}} + \sqrt{1-\rho^2} \tag{4.15}$$

for each $t \in [a,b]$.

Making use of (4.14) and (4.15), we deduce

$$\|f(t)\| \leq \frac{1}{\sqrt{1-\rho^2}} \operatorname{Re}\langle f(t), e\rangle$$

for a.e. $t \in [a,b]$.

Applying Theorem 57 for $K = \frac{1}{\sqrt{1-\rho^2}}$, we deduce the desired inequality (4.12). ∎

In the same spirit, we also have the following corollary [**2**].

Corollary 34. *Let e be a unit vector in H and $M \geq m > 0$. If $f \in L([a,b];H)$ is such that*

$$\operatorname{Re}\langle Me - f(t), f(t) - me\rangle \geq 0 \tag{4.16}$$

or, equivalently,

$$\left\| f(t) - \frac{M+m}{2} e \right\| \leq \frac{1}{2}(M-m) \tag{4.17}$$

for a.e. $t \in [a,b]$, then we have the inequality

$$\frac{2\sqrt{mM}}{M+m} \int_a^b \|f(t)\|\,dt \leq \left\| \int_a^b f(t)\,dt \right\|, \tag{4.18}$$

or, equivalently,

$$\begin{aligned}(0 \leq) \int_a^b \|f(t)\|\,dt - \left\| \int_a^b f(t)\,dt \right\| \\ \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{M+m} \left\| \int_a^b f(t)\,dt \right\|.\end{aligned} \tag{4.19}$$

The equality holds in (4.18) (or in the second part of (4.19)) if and only if

$$\int_a^b f(t)\,dt = \frac{2\sqrt{mM}}{M+m} \left(\int_a^b \|f(t)\|\,dt \right) e. \tag{4.20}$$

Proof. Firstly, we remark that if $x, z, Z \in H$, then the following statements are equivalent

(i) $\operatorname{Re}\langle Z - x, x - z\rangle \geq 0$

and

(ii) $\left\| x - \frac{Z+z}{2} \right\| \leq \frac{1}{2} \|Z - z\|$.

Using this fact, we may simply realise that (4.14) and (4.15) are equivalent.

Now, from (4.14), we obtain

$$\|f(t)\|^2 + mM \leq (M + m) \operatorname{Re}\langle f(t), e\rangle$$

for a.e. $t \in [a, b]$. Dividing this inequality with $\sqrt{mM} > 0$, we deduce the following inequality that will be used in the sequel

$$\frac{\|f(t)\|^2}{\sqrt{mM}} + \sqrt{mM} \leq \frac{M+m}{\sqrt{mM}} \operatorname{Re}\langle f(t), e\rangle \tag{4.21}$$

for a.e. $t \in [a, b]$.

On the other hand

$$2\|f(t)\| \leq \frac{\|f(t)\|^2}{\sqrt{mM}} + \sqrt{mM}, \tag{4.22}$$

for any $t \in [a, b]$.

Utilising (4.21) and (4.22), we may conclude with the following inequality

$$\|f(t)\| \leq \frac{M+m}{2\sqrt{mM}} \operatorname{Re}\langle f(t), e\rangle,$$

for a.e. $t \in [a, b]$.

Applying Theorem 57 for the constant $K := \frac{m+M}{2\sqrt{mM}} \geq 1$, we deduce the desired result. ∎

4.2.2. Reverses for Orthonormal Families of Vectors. The following result for orthonormal vectors in H holds [**2**].

Theorem 58 (Dragomir, 2004). *Let $\{e_1, \dots, e_n\}$ be a family of orthonormal vectors in H, $k_i \geq 0$, $i \in \{1, \dots, n\}$ and $f \in L([a, b]; H)$ such that*

$$k_i \|f(t)\| \leq \operatorname{Re}\langle f(t), e_i\rangle \tag{4.23}$$

for each $i \in \{1, \dots, n\}$ and for a.e. $t \in [a, b]$.

Then

$$\left(\sum_{i=1}^{n} k_i^2\right)^{\frac{1}{2}} \int_a^b \|f(t)\| \, dt \leq \left\| \int_a^b f(t) \, dt \right\|, \tag{4.24}$$

where equality holds if and only if

$$\int_a^b f(t)\,dt = \left(\int_a^b \|f(t)\|\,dt\right)\sum_{i=1}^n k_i e_i. \tag{4.25}$$

PROOF. By Bessel's inequality applied for $\int_a^b f(t)\,dt$ and the orthonormal vectors $\{e_1,\dots,e_n\}$, we have

$$\begin{aligned}\left\|\int_a^b f(t)\,dt\right\|^2 &\geq \sum_{i=1}^n \left|\left\langle \int_a^b f(t)\,dt, e_i\right\rangle\right|^2 \\ &\geq \sum_{i=1}^n \left[\operatorname{Re}\left\langle \int_a^b f(t)\,dt, e_i\right\rangle\right]^2 \\ &= \sum_{i=1}^n \left[\int_a^b \operatorname{Re}\langle f(t), e_i\rangle\,dt\right]^2.\end{aligned} \tag{4.26}$$

Integrating (4.23), we get for each $i \in \{1,\dots,n\}$

$$0 \leq k_i \int_a^b \|f(t)\|\,dt \leq \int_a^b \operatorname{Re}\langle f(t), e_i\rangle\,dt,$$

implying

$$\sum_{i=1}^n \left[\int_a^b \operatorname{Re}\langle f(t), e_i\rangle\,dt\right]^2 \geq \sum_{i=1}^n k_i^2 \left(\int_a^b \|f(t)\|\,dt\right)^2. \tag{4.27}$$

On making use of (4.26) and (4.27), we deduce

$$\left\|\int_a^b f(t)\,dt\right\|^2 \geq \sum_{i=1}^n k_i^2 \left(\int_a^b \|f(t)\|\,dt\right)^2,$$

which is clearly equivalent to (4.24).

If (4.25) holds true, then

$$\begin{aligned}\left\|\int_a^b f(t)\,dt\right\| &= \left(\int_a^b \|f(t)\|\,dt\right)\left\|\sum_{i=1}^n k_i e_i\right\| \\ &= \left(\int_a^b \|f(t)\|\,dt\right)\left[\sum_{i=1}^n k_i^2 \|e_i\|^2\right]^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^n k_i^2\right)^{\frac{1}{2}} \int_a^b \|f(t)\|\,dt,\end{aligned}$$

showing that (4.24) holds with equality.

Now, suppose that there is an $i_0 \in \{1, \dots, n\}$ for which

$$k_{i_0} \|f(t)\| < \operatorname{Re} \langle f(t), e_{i_0} \rangle$$

on a subset of nonzero Lebesgue measure in $[a, b]$. Then obviously

$$k_{i_0} \int_a^b \|f(t)\| \, dt < \int_a^b \operatorname{Re} \langle f(t), e_{i_0} \rangle \, dt,$$

and using the argument given above, we deduce

$$\left(\sum_{i=1}^n k_i^2 \right)^{\frac{1}{2}} \int_a^b \|f(t)\| \, dt < \left\| \int_a^b f(t) \, dt \right\|.$$

Therefore, if the equality holds in (4.24), we must have

$$k_i \|f(t)\| = \operatorname{Re} \langle f(t), e_i \rangle \tag{4.28}$$

for each $i \in \{1, \dots, n\}$ and a.e. $t \in [a, b]$.

Also, if the equality holds in (4.24), then we must have equality in all inequalities (4.26), this means that

$$\int_a^b f(t) \, dt = \sum_{i=1}^n \left\langle \int_a^b f(t) \, dt, e_i \right\rangle e_i \tag{4.29}$$

and

$$\operatorname{Im} \left\langle \int_a^b f(t) \, dt, e_i \right\rangle = 0 \quad \text{for each} \quad i \in \{1, \dots, n\}. \tag{4.30}$$

Using (4.28) and (4.30) in (4.29), we deduce

$$\begin{aligned} \int_a^b f(t) \, dt &= \sum_{i=1}^n \operatorname{Re} \left\langle \int_a^b f(t) \, dt, e_i \right\rangle e_i \\ &= \sum_{i=1}^n \int_a^b \operatorname{Re} \langle f(t), e_i \rangle \, e_i dt \\ &= \sum_{i=1}^n \left(\int_a^b \|f(t)\| \, dt \right) k_i e_i \\ &= \int_a^b \|f(t)\| \, dt \sum_{i=1}^n k_i e_i, \end{aligned}$$

and the condition (4.25) is necessary.

This completes the proof. ∎

The following two corollaries are of interest [**2**].

COROLLARY 35. *Let $\{e_1, \dots, e_n\}$ be a family of orthonormal vectors in H, $\rho_i \in (0,1)$, $i \in \{1, \dots, n\}$ and $f \in L([a,b];H)$ such that:*

$$\|f(t) - e_i\| \leq \rho_i \quad \text{for} \quad i \in \{1, \dots, n\} \quad \text{and a.e. } t \in [a,b]. \tag{4.31}$$

Then we have the inequality

$$\left(n - \sum_{i=1}^{n} \rho_i^2\right)^{\frac{1}{2}} \int_a^b \|f(t)\| \, dt \leq \left\| \int_a^b f(t) \, dt \right\|,$$

with equality if and only if

$$\int_a^b f(t) \, dt = \int_a^b \|f(t)\| \, dt \sum_{i=1}^{n} \left(1 - \rho_i^2\right)^{1/2} e_i.$$

PROOF. From the proof of Theorem 57, we know that (4.25) implies the inequality

$$\sqrt{1 - \rho_i^2} \, \|f(t)\| \leq \operatorname{Re} \langle f(t), e_i \rangle, \quad i \in \{1, \dots, n\}, \quad \text{for a.e. } t \in [a,b].$$

Now, applying Theorem 58 for $k_i := \sqrt{1 - \rho_i^2}$, $i \in \{1, \dots, n\}$, we deduce the desired result. ∎

A different results is incorporated in (see [**2**]):

COROLLARY 36. *Let $\{e_1, \dots, e_n\}$ be a family of orthonormal vectors in H, $M_i \geq m_i > 0$, $i \in \{1, \dots, n\}$ and $f \in L([a,b];H)$ such that*

$$\operatorname{Re} \langle M_i e_i - f(t), f(t) - m_i e_i \rangle \geq 0 \tag{4.32}$$

or, equivalently,

$$\left\| f(t) - \frac{M_i + m_i}{2} e_i \right\| \leq \frac{1}{2} (M_i - m_i)$$

for $i \in \{1, \dots, n\}$ and a.e. $t \in [a,b]$. Then we have the reverse of the continuous triangle inequality

$$\left[\sum_{i=1}^{n} \frac{4 m_i M_i}{(m_i + M_i)^2}\right]^{\frac{1}{2}} \int_a^b \|f(t)\| \, dt \leq \left\| \int_a^b f(t) \, dt \right\|,$$

with equality if and only if

$$\int_a^b f(t) \, dt = \int_a^b \|f(t)\| \, dt \left(\sum_{i=1}^{n} \frac{2\sqrt{m_i M_i}}{m_i + M_i} e_i \right).$$

Proof. From the proof of Corollary 35, we know (4.32) implies that

$$\frac{2\sqrt{m_i M_i}}{m_i + M_i} \|f(t)\| \le \operatorname{Re}\langle f(t), e_i\rangle, \quad i \in \{1, \dots, n\} \quad \text{and a.e. } t \in [a, b].$$

Now, applying Theorem 58 for $k_i := \frac{2\sqrt{m_i M_i}}{m_i + M_i}$, $i \in \{1, \dots, n\}$, we deduce the desired result. ∎

4.3. Some Additive Reverses

4.3.1. The Case of a Unit Vector. The following result holds [**3**].

Theorem 59 (Dragomir, 2004). *If $f \in L([a, b]; H)$ is such that there exists a vector $e \in H$, $\|e\| = 1$ and $k : [a, b] \to [0, \infty)$, a Lebesgue integrable function with*

$$\|f(t)\| - \operatorname{Re}\langle f(t), e\rangle \le k(t) \quad \textit{for a.e. } t \in [a, b], \tag{4.33}$$

then we have the inequality:

$$(0 \le) \int_a^b \|f(t)\| \, dt - \left\| \int_a^b f(t) \, dt \right\| \le \int_a^b k(t) \, dt. \tag{4.34}$$

The equality holds in (4.34) if and only if

$$\int_a^b \|f(t)\| \, dt \ge \int_a^b k(t) \, dt \tag{4.35}$$

and

$$\int_a^b f(t) \, dt = \left(\int_a^b \|f(t)\| \, dt - \int_a^b k(t) \, dt \right) e. \tag{4.36}$$

Proof. If we integrate the inequality (4.33), we get

$$\int_a^b \|f(t)\| \, dt \le \operatorname{Re}\left\langle \int_a^b f(t) \, dt, e \right\rangle + \int_a^b k(t) \, dt. \tag{4.37}$$

By Schwarz's inequality for e and $\int_a^b f(t) \, dt$, we have

$$\begin{aligned} &\operatorname{Re}\left\langle \int_a^b f(t) \, dt, e \right\rangle \\ &\le \left| \operatorname{Re}\left\langle \int_a^b f(t) \, dt, e \right\rangle \right| \le \left| \left\langle \int_a^b f(t) \, dt, e \right\rangle \right| \\ &\le \left\| \int_a^b f(t) \, dt \right\| \|e\| = \left\| \int_a^b f(t) \, dt \right\|. \end{aligned} \tag{4.38}$$

Making use of (4.37) and (4.38), we deduce the desired inequality (4.34).

If (4.35) and (4.36) hold true, then

$$\begin{aligned}\left\|\int_a^b f(t)\,dt\right\| &= \left|\int_a^b \|f(t)\|\,dt - \int_a^b k(t)\,dt\right| \|e\| \\ &= \int_a^b \|f(t)\|\,dt - \int_a^b k(t)\,dt\end{aligned}$$

and the equality holds true in (4.34).

Conversely, if the equality holds in (4.34), then, obviously (4.35) is valid and we need only to prove (4.36).

If $\|f(t)\| - \operatorname{Re}\langle f(t), e\rangle < k(t)$ on a subset of nonzero Lebesgue measure in $[a,b]$, then (4.37) holds as a strict inequality, implying that (4.34) also holds as a strict inequality. Therefore, if we assume that equality holds in (4.34), then we must have

$$\|f(t)\| = \operatorname{Re}\langle f(t), e\rangle + k(t) \quad \text{for a.e. } t \in [a,b]. \tag{4.39}$$

It is well known that in Schwarz's inequality $\|x\|\,\|y\| \geq \operatorname{Re}\langle x, y\rangle$ the equality holds iff there exists a $\lambda \geq 0$ such that $x = \lambda y$. Therefore, if we assume that the equality holds in all of (4.38), then there exists a $\lambda \geq 0$ such that

$$\int_a^b f(t)\,dt = \lambda e. \tag{4.40}$$

Integrating (4.39) on $[a,b]$, we deduce

$$\int_a^b \|f(t)\|\,dt = \operatorname{Re}\left\langle \int_a^b f(t)\,dt, e\right\rangle + \int_a^b k(t)\,dt,$$

and thus, by (4.40), we get

$$\int_a^b \|f(t)\|\,dt = \lambda\|e\|^2 + \int_a^b k(t)\,dt,$$

giving $\lambda = \int_a^b \|f(t)\|\,dt - \int_a^b k(t)\,dt$.

Using (4.40), we deduce (4.36) and the theorem is completely proved. ∎

The following corollary may be useful for applications [**3**].

Corollary 37. *If $f \in L([a,b]; H)$ is such that there exists a vector $e \in H$, $\|e\| = 1$ and $\rho \in (0,1)$ such that*

$$\|f(t) - e\| \leq \rho \quad \textit{for a.e. } t \in [a,b], \tag{4.41}$$

then we have the inequality

$$(0\leq)\int_a^b\|f(t)\|\,dt-\left\|\int_a^b f(t)\,dt\right\| \tag{4.42}$$
$$\leq\frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)}\operatorname{Re}\left\langle\int_a^b f(t)\,dt,e\right\rangle$$
$$\left(\leq\frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)}\left\|\int_a^b f(t)\,dt\right\|\right).$$

The equality holds in (4.42) if and only if

$$\int_a^b\|f(t)\|\,dt\geq\frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)}\operatorname{Re}\left\langle\int_a^b f(t)\,dt,e\right\rangle \tag{4.43}$$

and

$$\int_a^b f(t)\,dt \tag{4.44}$$
$$=\left(\int_a^b\|f(t)\|\,dt-\frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)}\operatorname{Re}\left\langle\int_a^b f(t)\,dt,e\right\rangle\right)e.$$

PROOF. Firstly, note that (4.35) is equivalent to

$$\|f(t)\|^2+1-\rho^2\leq 2\operatorname{Re}\langle f(t),e\rangle,$$

giving

$$\frac{\|f(t)\|^2}{\sqrt{1-\rho^2}}+\sqrt{1-\rho^2}\leq\frac{2\operatorname{Re}\langle f(t),e\rangle}{\sqrt{1-\rho^2}}$$

for a.e. $t\in[a,b]$.

Since, obviously

$$2\|f(t)\|\leq\frac{\|f(t)\|^2}{\sqrt{1-\rho^2}}+\sqrt{1-\rho^2}$$

for any $t\in[a,b]$, then we deduce the inequality

$$\|f(t)\|\leq\frac{\operatorname{Re}\langle f(t),e\rangle}{\sqrt{1-\rho^2}}\quad\text{for a.e. } t\in[a,b],$$

which is clearly equivalent to

$$\|f(t)\|-\operatorname{Re}\langle f(t),e\rangle\leq\frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)}\operatorname{Re}\langle f(t),e\rangle$$

for a.e. $t \in [a,b]$.

Applying Theorem 59 for $k(t) := \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)} \operatorname{Re}\langle f(t), e\rangle$, we deduce the desired result. ∎

In the same spirit, we also have the following corollary [**3**].

Corollary 38. *If $f \in L([a,b];H)$ is such that there exists a vector $e \in H$, $\|e\| = 1$ and $M \geq m > 0$ such that either*

$$\operatorname{Re}\langle Me - f(t), f(t) - me\rangle \geq 0 \tag{4.45}$$

or, equivalently,

$$\left\| f(t) - \frac{M+m}{2} e \right\| \leq \frac{1}{2}(M-m) \tag{4.46}$$

for a.e. $t \in [a,b]$, then we have the inequality

$$\begin{aligned} (0 \leq) \int_a^b \|f(t)\| \, dt - \left\| \int_a^b f(t) \, dt \right\| &\leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\left\langle \int_a^b f(t) \, dt, e \right\rangle \\ &\left(\leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \left\| \int_a^b f(t) \, dt \right\| \right). \end{aligned} \tag{4.47}$$

The equality holds in (4.47) if and only if

$$\int_a^b \|f(t)\| \, dt \geq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\left\langle \int_a^b f(t) \, dt, e \right\rangle$$

and

$$\int_a^b f(t) \, dt = \left(\int_a^b \|f(t)\| \, dt - \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\left\langle \int_a^b f(t) \, dt, e \right\rangle \right) e.$$

Proof. Observe that (4.45) is clearly equivalent to

$$\|f(t)\|^2 + mM \leq (M+m) \operatorname{Re}\langle f(t), e\rangle$$

for a.e. $t \in [a,b]$, giving the inequality

$$\frac{\|f(t)\|^2}{\sqrt{mM}} + \sqrt{mM} \leq \frac{M+m}{\sqrt{mM}} \operatorname{Re}\langle f(t), e\rangle$$

for a.e. $t \in [a,b]$.

Since, obviously,

$$2\|f(t)\| \leq \frac{\|f(t)\|^2}{\sqrt{mM}} + \sqrt{mM}$$

for any $t \in [a,b]$, hence we deduce the inequality

$$\|f(t)\| \leq \frac{M+m}{\sqrt{mM}} \operatorname{Re}\langle f(t), e\rangle \quad \text{for a.e. } t \in [a,b],$$

which is clearly equivalent to

$$\|f(t)\| - \operatorname{Re}\langle f(t), e\rangle \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\langle f(t), e\rangle$$

for a.e. $t \in [a,b]$.

Finally, applying Theorem 59, we obtain the desired result. ∎

We can state now (see also [**3**]):

Corollary 39. *If $f \in L([a,b];H)$ and $r \in L_2([a,b];H)$, $e \in H$, $\|e\| = 1$ are such that*

$$\|f(t) - e\| \leq r(t) \quad \text{for a.e. } t \in [a,b], \tag{4.48}$$

then we have the inequality

$$(0 \leq) \int_a^b \|f(t)\|\,dt - \left\|\int_a^b f(t)\,dt\right\| \leq \frac{1}{2}\int_a^b r^2(t)\,dt. \tag{4.49}$$

The equality holds in (4.49) if and only if

$$\int_a^b \|f(t)\|\,dt \geq \frac{1}{2}\int_a^b r^2(t)\,dt$$

and

$$\int_a^b f(t)\,dt = \left(\int_a^b \|f(t)\|\,dt - \frac{1}{2}\int_a^b r^2(t)\,dt\right) e.$$

Proof. The condition (4.48) is obviously equivalent to

$$\|f(t)\|^2 + 1 \leq 2\operatorname{Re}\langle f(t), e\rangle + r^2(t)$$

for a.e. $t \in [a,b]$.

Using the elementary inequality

$$2\|f(t)\| \leq \|f(t)\|^2 + 1, \quad t \in [a,b],$$

we deduce

$$\|f(t)\| - \operatorname{Re}\langle f(t), e\rangle \leq \frac{1}{2} r^2(t)$$

for a.e. $t \in [a,b]$.

Applying Theorem 59 for $k(t) := \frac{1}{2}r^2(t)$, $t \in [a,b]$, we deduce the desired result. ∎

Finally, we may state and prove the following result as well [**3**].

Corollary 40. *If $f \in L([a,b];H)$, $e \in H$, $\|e\| = 1$ and $M, m : [a,b] \to [0,\infty)$ with $M \geq m$ a.e. on $[a,b]$, are such that $\frac{(M-m)^2}{M+m} \in L[a,b]$ and either*

$$\left\| f(t) - \frac{M(t)+m(t)}{2} e \right\| \leq \frac{1}{2}[M(t) - m(t)] \tag{4.50}$$

or, equivalently,

$$\operatorname{Re}\langle M(t)e - f(t), f(t) - m(t)e\rangle \geq 0 \tag{4.51}$$

for a.e. $t \in [a,b]$, then we have the inequality

$$(0 \leq) \int_a^b \|f(t)\| \, dt - \left\| \int_a^b f(t)\, dt \right\| \leq \frac{1}{4} \int_a^b \frac{[M(t) - m(t)]^2}{M(t)+m(t)} dt. \tag{4.52}$$

The equality holds in (4.52) if and only if

$$\int_a^b \|f(t)\| \, dt \geq \frac{1}{4} \int_a^b \frac{[M(t) - m(t)]^2}{M(t)+m(t)} dt$$

and

$$\int_a^b f(t)\, dt = \left(\int_a^b \|f(t)\| \, dt - \frac{1}{4} \int_a^b \frac{[M(t) - m(t)]^2}{M(t)+m(t)} dt \right) e.$$

Proof. The condition (4.50) is equivalent to

$$\|f(t)\|^2 + \left(\frac{M(t)+m(t)}{2}\right)^2 \leq 2\left(\frac{M(t)+m(t)}{2}\right) \operatorname{Re}\langle f(t), e\rangle + \frac{1}{4}[M(t) - m(t)]^2$$

for a.e. $t \in [a,b]$, and since

$$2\left(\frac{M(t)+m(t)}{2}\right) \|f(t)\| \leq \|f(t)\|^2 + \left(\frac{M(t)+m(t)}{2}\right)^2, \quad t \in [a,b]$$

hence

$$\|f(t)\| - \operatorname{Re}\langle f(t), e\rangle \leq \frac{1}{4} \frac{[M(t) - m(t)]^2}{M(t)+m(t)}$$

for a.e. $t \in [a,b]$.

Now, applying Theorem 59 for $k(t) := \frac{1}{4}\frac{[M(t)-m(t)]^2}{M(t)+m(t)}$, $t \in [a,b]$, we deduce the desired inequality. ∎

4.3.2. Additive Reverses for Orthonormal Families. The following reverse of the continuous triangle inequality for vector valued integrals holds [**3**].

THEOREM 60 (Dragomir, 2004). *Let* $f \in L([a,b];H)$, *where* H *is a Hilbert space over the real or complex number field* $\mathbb{K}$, $\{e_i\}_{i\in\{1,\dots,n\}}$ *an orthonormal family in* H *and* $M_i \in L[a,b]$, $i \in \{1,\dots,n\}$. *If we assume that*

$$\|f(t)\| - \operatorname{Re}\langle f(t), e_i\rangle \le M_i(t) \quad \text{for a.e. } t \in [a,b], \tag{4.53}$$

then we have the inequality

$$\int_a^b \|f(t)\|\,dt \le \frac{1}{\sqrt{n}}\left\|\int_a^b f(t)\,dt\right\| + \frac{1}{n}\sum_{i=1}^n \int_a^b M_i(t)\,dt. \tag{4.54}$$

The equality holds in (4.54) if and only if

$$\int_a^b \|f(t)\|\,dt \ge \frac{1}{n}\sum_{i=1}^n \int_a^b M_i(t)\,dt \tag{4.55}$$

and

$$\int_a^b f(t)\,dt = \left(\int_a^b \|f(t)\|\,dt - \frac{1}{n}\sum_{i=1}^n \int_a^b M_i(t)\,dt\right)\sum_{i=1}^n e_i. \tag{4.56}$$

PROOF. If we integrate the inequality (4.53) on $[a,b]$, we get

$$\int_a^b \|f(t)\|\,dt \le \operatorname{Re}\left\langle \int_a^b f(t)\,dt, e_i\right\rangle + \int_a^b M_i(t)\,dt$$

for each $i \in \{1,\dots,n\}$. Summing these inequalities over i from 1 to n, we deduce

$$\int_a^b \|f(t)\|\,dt \le \frac{1}{n}\operatorname{Re}\left\langle \int_a^b f(t)\,dt, \sum_{i=1}^n e_i\right\rangle + \frac{1}{n}\sum_{i=1}^n \int_a^b M_i(t)\,dt. \tag{4.57}$$

By Schwarz's inequality for $\int_a^b f(t)\,dt$ and $\sum_{i=1}^n e_i$, we have

$$\begin{aligned}\operatorname{Re}&\left\langle \int_a^b f(t)\,dt, \sum_{i=1}^n e_i\right\rangle \\ &\le \left|\operatorname{Re}\left\langle \int_a^b f(t)\,dt, \sum_{i=1}^n e_i\right\rangle\right| \le \left|\left\langle \int_a^b f(t)\,dt, \sum_{i=1}^n e_i\right\rangle\right| \\ &\le \left\|\int_a^b f(t)\,dt\right\|\left\|\sum_{i=1}^n e_i\right\| = \sqrt{n}\left\|\int_a^b f(t)\,dt\right\|,\end{aligned} \tag{4.58}$$

since

$$\left\|\sum_{i=1}^{n} e_i\right\| = \sqrt{\left\|\sum_{i=1}^{n} e_i\right\|^2} = \sqrt{\sum_{i=1}^{n} \|e_i\|^2} = \sqrt{n}.$$

Making use of (4.57) and (4.58), we deduce the desired inequality (4.54).

If (4.55) and (4.56) hold, then

$$\begin{aligned}\frac{1}{\sqrt{n}}\left\|\int_a^b f(t)\,dt\right\| &= \frac{1}{\sqrt{n}}\left|\int_a^b \|f(t)\|\,dt - \frac{1}{n}\sum_{i=1}^{n}\int_a^b M_i(t)\,dt\right|\left\|\sum_{i=1}^{n} e_i\right\| \\ &= \left(\int_a^b \|f(t)\|\,dt - \frac{1}{n}\sum_{i=1}^{n}\int_a^b M_i(t)\,dt\right)\end{aligned}$$

and the equality in (4.54) holds true.

Conversely, if the equality holds in (4.54), then, obviously, (4.55) is valid.

Taking into account the argument presented above for the previous result (4.54), it is obvious that, if the equality holds in (4.54), then it must hold in (4.53) for a.e. $t \in [a, b]$ and for each $i \in \{1, \ldots, n\}$ and also the equality must hold in any of the inequalities in (4.58).

It is well known that in Schwarz's inequality $\operatorname{Re}\langle u, v\rangle \leq \|u\|\|v\|$, the equality occurs if and only if $u = \lambda v$ with $\lambda \geq 0$, consequently, the equality holds in all inequalities from (4.58) simultaneously iff there exists a $\mu \geq 0$ with

$$\mu\sum_{i=1}^{n} e_i = \int_a^b f(t)\,dt. \tag{4.59}$$

If we integrate the equality in (4.53) and sum over i, we deduce

$$n\int_a^b f(t)\,dt = \operatorname{Re}\left\langle \int_a^b f(t)\,dt, \sum_{i=1}^{n} e_i\right\rangle + \sum_{i=1}^{n}\int_a^b M_i(t)\,dt. \tag{4.60}$$

Replacing $\int_a^b f(t)\,dt$ from (4.59) into (4.60), we deduce

$$\begin{aligned}n\int_a^b f(t)\,dt &= \mu\left\|\sum_{i=1}^{n} e_i\right\|^2 + \sum_{i=1}^{n}\int_a^b M_i(t)\,dt \\ &= \mu n + \sum_{i=1}^{n}\int_a^b M_i(t)\,dt.\end{aligned} \tag{4.61}$$

Finally, we note that (4.59) and (4.61) will produce the required identity (4.56), and the proof is complete. ∎

The following corollaries may be of interest for applications [**3**].

COROLLARY 41. *Let $f \in L([a,b];H)$, $\{e_i\}_{i\in\{1,\dots,n\}}$ an orthonormal family in H and $\rho_i \in (0,1)$, $i \in \{1,\dots,n\}$ such that*

$$\|f(t) - e_i\| \leq \rho_i \quad \text{for a.e. } t \in [a,b]. \tag{4.62}$$

Then we have the inequalities:

$$\begin{aligned}\int_a^b \|f(t)\|\,dt &\leq \frac{1}{\sqrt{n}}\left\|\int_a^b f(t)\,dt\right\| \\ &\quad + \operatorname{Re}\left\langle \int_a^b f(t)\,dt, \frac{1}{n}\sum_{i=1}^n \frac{\rho_i^2}{\sqrt{1-\rho_i^2}\left(1+\sqrt{1-\rho_i^2}\right)} e_i\right\rangle \\ &\leq \frac{1}{\sqrt{n}}\left\|\int_a^b f(t)\,dt\right\| \\ &\quad \times \left[1 + \left(\frac{1}{n}\sum_{i=1}^n \frac{\rho_i^2}{\sqrt{1-\rho_i^2}\left(1+\sqrt{1-\rho_i^2}\right)}\right)^{\frac{1}{2}}\right].\end{aligned} \tag{4.63}$$

The equality holds in the first inequality in (4.63) if and only if

$$\int_a^b \|f(t)\|\,dt \geq \operatorname{Re}\left\langle \int_a^b f(t)\,dt, \frac{1}{n}\sum_{i=1}^n \frac{\rho_i^2}{\sqrt{1-\rho_i^2}\left(1+\sqrt{1-\rho_i^2}\right)} e_i\right\rangle$$

and

$$\begin{aligned}&\int_a^b f(t)\,dt \\ &= \left(\int_a^b \|f(t)\|\,dt - \operatorname{Re}\left\langle \int_a^b f(t)\,dt, \frac{1}{n}\sum_{i=1}^n \frac{\rho_i^2}{\sqrt{1-\rho_i^2}\left(1+\sqrt{1-\rho_i^2}\right)} e_i\right\rangle\right) \\ &\qquad \times \sum_{i=1}^n e_i.\end{aligned}$$

PROOF. As in the proof of Corollary 37, the assumption (4.62) implies

$$\|f(t)\| - \operatorname{Re}\langle f(t), e_i\rangle \leq \frac{\rho_i^2}{\sqrt{1-\rho_i^2}\left(\sqrt{1-\rho_i^2}+1\right)} \operatorname{Re}\langle f(t), e_i\rangle$$

for a.e. $t \in [a,b]$ and for each $i \in \{1,\dots,n\}$.

Now, if we apply Theorem 60 for

$$M_i(t) := \frac{\rho_i^2 \operatorname{Re}\langle f(t), e_i\rangle}{\sqrt{1-\rho_i^2}\left(\sqrt{1-\rho_i^2}+1\right)}, \quad i \in \{1,\dots,n\}, \quad t \in [a,b],$$

we deduce the first inequality in (4.63).

By Schwarz's inequality in H, we have

$$\begin{aligned}
&\operatorname{Re}\left\langle \int_a^b f(t)\,dt, \frac{1}{n}\sum_{i=1}^n \frac{\rho_i^2}{\sqrt{1-\rho_i^2}\left(1+\sqrt{1-\rho_i^2}\right)} e_i \right\rangle \\
&\leq \left\|\int_a^b f(t)\,dt\right\| \left\| \frac{1}{n}\sum_{i=1}^n \frac{\rho_i^2}{\sqrt{1-\rho_i^2}\left(1+\sqrt{1-\rho_i^2}\right)} e_i \right\| \\
&= \frac{1}{n}\left\|\int_a^b f(t)\,dt\right\| \left(\sum_{i=1}^n \left[\frac{\rho_i^2}{\sqrt{1-\rho_i^2}\left(1+\sqrt{1-\rho_i^2}\right)}\right]^2\right)^{\frac{1}{2}},
\end{aligned}$$

which implies the second inequality in (4.63). ∎

The second result is incorporated in [**3**]:

Corollary 42. *Let $f \in L([a,b];H)$, $\{e_i\}_{i\in\{1,\dots,n\}}$ an orthonormal family in H and $M_i \geq m_i > 0$ such that either*

$$\operatorname{Re}\langle M_i e_i - f(t), f(t) - m_i e_i\rangle \geq 0 \tag{4.64}$$

or, equivalently,

$$\left\| f(t) - \frac{M_i + m_i}{2}\cdot e_i \right\| \leq \frac{1}{2}(M_i - m_i)$$

for a.e. $t \in [a,b]$ and each $i \in \{1,\dots,n\}$. Then we have

$$\begin{aligned}
\int_a^b \|f(t)\|\,dt &\leq \frac{1}{\sqrt{n}}\left\|\int_a^b f(t)\,dt\right\| \\
&\quad + \operatorname{Re}\left\langle \int_a^b f(t)\,dt, \frac{1}{n}\sum_{i=1}^n \frac{\left(\sqrt{M_i}-\sqrt{m_i}\right)^2}{2\sqrt{m_i M_i}} e_i \right\rangle \\
&\leq \frac{1}{\sqrt{n}}\left\|\int_a^b f(t)\,dt\right\| \left[1 + \left(\frac{1}{n}\sum_{i=1}^n \frac{\left(\sqrt{M_i}-\sqrt{m_i}\right)^4}{4 m_i M_i}\right)^{\frac{1}{2}}\right].
\end{aligned} \tag{4.65}$$

The equality holds in the first inequality in (4.65) if and only if

$$\int_a^b \|f(t)\|\,dt \geq \operatorname{Re}\left\langle \int_a^b f(t)\,dt, \frac{1}{n}\sum_{i=1}^n \frac{\left(\sqrt{M_i}-\sqrt{m_i}\right)^2}{2\sqrt{m_i M_i}} e_i \right\rangle$$

and

$$\int_a^b f(t)\,dt$$
$$= \left(\int_a^b \|f(t)\|\,dt - \operatorname{Re}\left\langle \int_a^b f(t)\,dt, \frac{1}{n}\sum_{i=1}^n \frac{\left(\sqrt{M_i}-\sqrt{m_i}\right)^2}{2\sqrt{m_i M_i}} e_i \right\rangle \right)$$
$$\times \sum_{i=1}^n e_i.$$

Proof. As in the proof of Corollary 38, from (4.64), we have

$$\|f(t)\| - \operatorname{Re}\langle f(t), e_i\rangle \le \frac{\left(\sqrt{M_i}-\sqrt{m_i}\right)^2}{2\sqrt{m_i M_i}} \operatorname{Re}\langle f(t), e_i\rangle$$

for a.e. $t \in [a,b]$ and $i \in \{1,\dots,n\}$.

Applying Theorem 60 for

$$M_i(t) := \frac{\left(\sqrt{M_i}-\sqrt{m_i}\right)^2}{2\sqrt{m_i M_i}} \operatorname{Re}\langle f(t), e_i\rangle, \quad t \in [a,b], \ i \in \{1,\dots,n\},$$

we deduce the desired result. ∎

In a different direction, we may state the following result as well [**3**].

Corollary 43. *Let $f \in L([a,b];H)$, $\{e_i\}_{i\in\{1,\dots,n\}}$ an orthonormal family in H and $r_i \in L^2([a,b])$, $i \in \{1,\dots,n\}$ such that*

$$\|f(t) - e_i\| \le r_i(t) \quad \text{for a.e. } t \in [a,b] \quad \text{and} \quad i \in \{1,\dots,n\}.$$

Then we have the inequality

$$\int_a^b \|f(t)\|\,dt \le \frac{1}{\sqrt{n}} \left\| \int_a^b f(t)\,dt \right\| + \frac{1}{2n}\sum_{i=1}^n \left(\int_a^b r_i^2(t)\,dt \right). \tag{4.66}$$

The equality holds in (4.66) if and only if

$$\int_a^b \|f(t)\|\,dt \ge \frac{1}{2n}\sum_{i=1}^n \left(\int_a^b r_i^2(t)\,dt \right)$$

and

$$\int_a^b f(t)\,dt = \left[\int_a^b \|f(t)\|\,dt - \frac{1}{n}\sum_{i=1}^n \left(\int_a^b r_i^2(t)\,dt \right) \right] \sum_{i=1}^n e_i.$$

PROOF. As in the proof of Corollary 39, from (4.48), we deduce that

$$\|f(t)\| - \operatorname{Re}\langle f(t), e_i\rangle \le \frac{1}{2} r_i^2(t) \tag{4.67}$$

for a.e. $t \in [a,b]$ and $i \in \{1,\dots,n\}$.

Applying Theorem 60 for

$$M_i(t) := \frac{1}{2} r_i^2(t), \quad t \in [a,b], \ i \in \{1,\dots,n\},$$

we get the desired result. ∎

Finally, the following result holds [**3**].

COROLLARY 44. *Let $f \in L([a,b];H)$, $\{e_i\}_{i\in\{1,\dots,n\}}$ an orthonormal family in H, $M_i, m_i : [a,b] \to [0,\infty)$ with $M_i \ge m_i$ a.e. on $[a,b]$ and $\frac{(M_i-m_i)^2}{M_i+m_i} \in L[a,b]$, and either*

$$\left\| f(t) - \frac{M_i(t)+m_i(t)}{2} e_i \right\| \le \frac{1}{2}\left[M_i(t) - m_i(t)\right]^2 \tag{4.68}$$

or, equivalently,

$$\operatorname{Re}\langle M_i(t) e_i - f(t), f(t) - m_i(t) e_i\rangle \ge 0$$

for a.e. $t \in [a,b]$ and any $i \in \{1,\dots,n\}$, then we have the inequality

$$\int_a^b \|f(t)\|\, dt \le \frac{1}{\sqrt{n}} \left\| \int_a^b f(t)\, dt \right\| + \frac{1}{4n} \sum_{i=1}^n \left(\int_a^b \frac{\left[M_i(t) - m_i(t)\right]^2}{M_i(t)+m_i(t)} dt \right). \tag{4.69}$$

The equality holds in (4.69) if and only if

$$\int_a^b \|f(t)\|\, dt \ge \frac{1}{4n} \sum_{i=1}^n \left(\int_a^b \frac{\left[M_i(t) - m_i(t)\right]^2}{M_i(t)+m_i(t)} dt \right)$$

and

$$\int_a^b f(t)\, dt = \left(\int_a^b \|f(t)\|\, dt - \frac{1}{4n} \sum_{i=1}^n \left(\int_a^b \frac{\left[M_i(t) - m_i(t)\right]^2}{M_i(t)+m_i(t)} dt \right) \right) \sum_{i=1}^n e_i.$$

PROOF. As in the proof of Corollary 40, (4.68), implies that

$$\|f(t)\| - \operatorname{Re}\langle f(t), e_i\rangle \leq \frac{1}{4} \cdot \frac{[M_i(t) - m_i(t)]^2}{M_i(t) + m_i(t)}$$

for a.e. $t \in [a, b]$ and $i \in \{1, \ldots, n\}$.

Applying Theorem 60 for

$$M_i(t) := \frac{1}{4} \cdot \frac{[M_i(t) - m_i(t)]^2}{M_i(t) + m_i(t)}, \quad t \in [a, b], \ \ i \in \{1, \ldots, n\},$$

we deduce the desired result. ∎

4.4. Quadratic Reverses of the Triangle Inequality

4.4.1. Additive Reverses. The following lemma holds [**4**].

LEMMA 7 (Dragomir, 2004). *Let $f \in L([a, b]; H)$ be such that there exists a function $k : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$, $\Delta := \{(t, s) | a \leq t \leq s \leq b\}$ with the property that $k \in L(\Delta)$ and*

$$(0 \leq) \|f(t)\| \|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle \leq k(t, s), \tag{4.70}$$

for a.e. $(t, s) \in \Delta$. Then we have the following quadratic reverse of the continuous triangle inequality:

$$\left(\int_a^b \|f(t)\| \, dt\right)^2 \leq \left\|\int_a^b f(t) \, dt\right\|^2 + 2\iint_\Delta k(t, s) \, dtds. \tag{4.71}$$

The case of equality holds in (4.71) if and only if it holds in (4.70) for a.e. $(t, s) \in \Delta$.

PROOF. We observe that the following identity holds

$$\begin{aligned} &\left(\int_a^b \|f(t)\| \, dt\right)^2 - \left\|\int_a^b f(t) \, dt\right\|^2 \\ &= \int_a^b \int_a^b \|f(t)\| \|f(s)\| \, dtds - \left\langle \int_a^b f(t) \, dt, \int_a^b f(s) \, ds \right\rangle \\ &= \int_a^b \int_a^b \|f(t)\| \|f(s)\| \, dtds - \int_a^b \int_a^b \operatorname{Re}\langle f(t), f(s)\rangle \, dtds \\ &= \int_a^b \int_a^b [\|f(t)\| \|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle] \, dtds := I. \end{aligned} \tag{4.72}$$

Now, observe that for any $(t, s) \in [a, b] \times [a, b]$, we have

$$\begin{aligned} &\|f(t)\| \|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle \\ &\qquad = \|f(s)\| \|f(t)\| - \operatorname{Re}\langle f(s), f(t)\rangle \end{aligned}$$

and thus

$$I=2\iint_{\Delta}\left[\|f(t)\|\,\|f(s)\|-\operatorname{Re}\langle f(t),f(s)\rangle\right]dtds. \tag{4.73}$$

Using the assumption (4.70), we deduce

$$\iint_{\Delta}\left[\|f(t)\|\,\|f(s)\|-\operatorname{Re}\langle f(t),f(s)\rangle\right]dtds\leq\iint_{\Delta}k(t,s)\,dtds,$$

and, by the identities (4.72) and (4.73), we deduce the desired inequality (4.71).

The case of equality is obvious and we omit the details. ∎

Remark 51. *From (4.71) one may deduce a coarser inequality that can be useful in some applications. It is as follows:*

$$(0\leq)\int_a^b\|f(t)\|\,dt-\left\|\int_a^b f(t)\,dt\right\|\leq\sqrt{2}\left(\iint_{\Delta}k(t,s)\,dtds\right)^{\frac{1}{2}}.$$

Remark 52. *If the condition (4.70) is replaced with the following refinement of the Schwarz inequality*

$$(0\leq)\,k(t,s)\leq\|f(t)\|\,\|f(s)\|-\operatorname{Re}\langle f(t),f(s)\rangle \tag{4.74}$$

for a.e. $(t,s)\in\Delta$, then the following refinement of the quadratic triangle inequality is valid

$$\left(\int_a^b\|f(t)\|\,dt\right)^2\geq\left\|\int_a^b f(t)\,dt\right\|^2+2\iint_{\Delta}k(t,s)\,dtds \tag{4.75}$$
$$\left(\geq\left\|\int_a^b f(t)\,dt\right\|^2\right).$$

The equality holds in (4.75) iff the case of equality holds in (4.74) for a.e. $(t,s)\in\Delta$.

The following result holds [**4**].

Theorem 61 (Dragomir, 2004). *Let $f\in L([a,b];H)$ be such that there exists $M\geq 1\geq m\geq 0$ such that either*

$$\operatorname{Re}\langle Mf(s)-f(t),f(t)-mf(s)\rangle\geq 0 \tag{4.76}$$

or, equivalently,

$$\left\|f(t)-\frac{M+m}{2}f(s)\right\|\leq\frac{1}{2}(M-m)\,\|f(s)\| \tag{4.77}$$

for a.e. $(t,s) \in \Delta$. *Then we have the inequality:*

$$\left(\int_a^b \|f(t)\|\,dt\right)^2 \leq \left\|\int_a^b f(t)\,dt\right\|^2 + \frac{1}{2}\cdot\frac{(M-m)^2}{M+m}\int_a^b (s-a)\,\|f(s)\|^2\,ds. \tag{4.78}$$

The case of equality holds in (4.78) if and only if

$$\|f(t)\|\,\|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle = \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\|f(s)\|^2 \tag{4.79}$$

for a.e. $(t,s) \in \Delta$.

Proof. Taking the square in (4.77), we get

$$\|f(t)\|^2 + \left(\frac{M+m}{2}\right)^2\|f(s)\|^2 \leq 2\operatorname{Re}\left\langle f(t), \frac{M+m}{2}f(s)\right\rangle + \frac{1}{4}(M-m)^2\|f(s)\|^2,$$

for a.e. $(t,s)\in\Delta$, and obviously, since

$$2\left(\frac{M+m}{2}\right)\|f(t)\|\,\|f(s)\| \leq \|f(t)\|^2 + \left(\frac{M+m}{2}\right)^2\|f(s)\|^2,$$

we deduce that

$$2\left(\frac{M+m}{2}\right)\|f(t)\|\,\|f(s)\| \leq 2\operatorname{Re}\left\langle f(t), \frac{M+m}{2}f(s)\right\rangle + \frac{1}{4}(M-m)^2\|f(s)\|^2,$$

giving the much simpler inequality:

$$\|f(t)\|\,\|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle \leq \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\|f(s)\|^2 \tag{4.80}$$

for a.e. $(t,s)\in\Delta$.

Applying Lemma 7 for $k(t,s) := \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\|f(s)\|^2$, we deduce

$$\left(\int_a^b \|f(t)\|\,dt\right)^2 \leq \left\|\int_a^b f(t)\,dt\right\|^2 + \frac{1}{2}\cdot\frac{(M-m)^2}{M+m}\iint_\Delta \|f(s)\|^2\,ds \tag{4.81}$$

with equality if and only if (4.80) holds for a.e. $(t,s)\in\Delta$.

Since

$$\iint_{\Delta} \|f(s)\|^2 \, ds = \int_a^b \left(\int_a^s \|f(s)\|^2 \, dt \right) ds = \int_a^b (s-a) \|f(s)\|^2 \, ds,$$

then by (4.81) we deduce the desired result (4.78). ∎

Another result which is similar to the one above is incorporated in the following theorem [**4**].

Theorem 62 (Dragomir, 2004). *With the assumptions of Theorem 61, we have*

$$\left(\int_a^b \|f(t)\| \, dt \right)^2 - \left\| \int_a^b f(t) \, dt \right\|^2 \leq \frac{\left(\sqrt{M} - \sqrt{m} \right)^2}{2\sqrt{Mm}} \left\| \int_a^b f(t) \, dt \right\|^2 \tag{4.82}$$

or, equivalently,

$$\int_a^b \|f(t)\| \, dt \leq \left(\frac{M+m}{2\sqrt{Mm}} \right)^{\frac{1}{2}} \left\| \int_a^b f(t) \, dt \right\|. \tag{4.83}$$

The case of equality holds in (4.82) or (4.83) if and only if

$$\|f(t)\| \, \|f(s)\| = \frac{M+m}{2\sqrt{Mm}} \operatorname{Re} \langle f(t), f(s) \rangle, \tag{4.84}$$

for a.e. $(t,s) \in \Delta$.

Proof. From (4.76), we deduce

$$\|f(t)\|^2 + Mm \|f(s)\|^2 \leq (M+m) \operatorname{Re} \langle f(t), f(s) \rangle$$

for a.e. $(t,s) \in \Delta$. Dividing by $\sqrt{Mm} > 0$, we deduce

$$\frac{\|f(t)\|^2}{\sqrt{Mm}} + \sqrt{Mm} \|f(s)\|^2 \leq \frac{M+m}{\sqrt{Mm}} \operatorname{Re} \langle f(t), f(s) \rangle$$

and, obviously, since

$$2 \|f(t)\| \, \|f(s)\| \leq \frac{\|f(t)\|^2}{\sqrt{Mm}} + \sqrt{Mm} \|f(s)\|^2,$$

hence

$$\|f(t)\| \, \|f(s)\| \leq \frac{M+m}{\sqrt{Mm}} \operatorname{Re} \langle f(t), f(s) \rangle$$

for a.e. $(t,s)\in\Delta$, giving

$$\|f(t)\|\,\|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle \le \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{Mm}} \operatorname{Re}\langle f(t), f(s)\rangle .$$

Applying Lemma 7 for $k(t,s) := \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{\sqrt{Mm}} \operatorname{Re}\langle f(t), f(s)\rangle$, we deduce

$$\left(\int_a^b \|f(t)\|\,dt\right)^2 \le \left\|\int_a^b f(t)\,dt\right\|^2 + \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{Mm}} \operatorname{Re}\langle f(t), f(s)\rangle . \tag{4.85}$$

On the other hand, since

$$\operatorname{Re}\langle f(t), f(s)\rangle = \operatorname{Re}\langle f(s), f(t)\rangle \quad \text{for any} \quad (t,s)\in[a,b]^2 ,$$

hence

$$\begin{aligned}\iint_\Delta \operatorname{Re}\langle f(t), f(s)\rangle\,dtds &= \frac{1}{2}\int_a^b\int_a^b \operatorname{Re}\langle f(t), f(s)\rangle\,dtds \\ &= \frac{1}{2}\operatorname{Re}\left\langle \int_a^b f(t)\,dt, \int_a^b f(s)\,ds \right\rangle \\ &= \frac{1}{2}\left\|\int_a^b f(t)\,dt\right\|^2\end{aligned}$$

and thus, from (4.85), we get (4.82).

The equivalence between (4.82) and (4.83) is obvious and we omit the details. ∎

4.4.2. Related Results.

The following result also holds [**4**].

THEOREM 63 (Dragomir, 2004). *Let $f\in L([a,b];H)$ and $\gamma,\Gamma\in\mathbb{R}$ be such that either*

$$\operatorname{Re}\langle \Gamma f(s) - f(t), f(t) - \gamma f(s)\rangle \ge 0 \tag{4.86}$$

or, equivalently,

$$\left\| f(t) - \frac{\Gamma+\gamma}{2} f(s)\right\| \le \frac{1}{2}|\Gamma-\gamma|\,\|f(s)\| \tag{4.87}$$

for a.e. $(t,s)\in\Delta$. Then we have the inequality:

$$\int_a^b [(b-s)+\gamma\Gamma(s-a)]\,\|f(s)\|^2\,ds \le \frac{\Gamma+\gamma}{2}\left\|\int_a^b f(s)\,ds\right\|^2 . \tag{4.88}$$

The case of equality holds in (4.88) if and only if the case of equality holds in either (4.86) or (4.87) for a.e. $(t,s) \in \Delta$.

PROOF. The inequality (4.86) is obviously equivalent to

$$\|f(t)\|^2 + \gamma\Gamma \|f(s)\|^2 \le (\Gamma+\gamma) \operatorname{Re}\langle f(t), f(s)\rangle \tag{4.89}$$

for a.e. $(t,s) \in \Delta$.

Integrating (4.89) on Δ, we deduce

$$\begin{aligned}\int_a^b \left(\int_a^s \|f(t)\|^2 dt\right) ds + \gamma\Gamma \int_a^b \left(\|f(s)\|^2 \int_a^s dt\right) ds \\ = (\Gamma+\gamma) \int_a^b \left(\int_a^s \operatorname{Re}\langle f(t), f(s)\rangle dt\right) ds.\end{aligned} \tag{4.90}$$

It is easy to see, on integrating by parts, that

$$\begin{aligned}\int_a^b \left(\int_a^s \|f(t)\|^2 dt\right) ds &= s \int_a^s \|f(t)\|^2 dt \Big|_a^b - \int_a^b s\|f(s)\|^2 ds \\ &= b\int_a^s \|f(s)\|^2 ds - \int_a^b s\|f(s)\|^2 ds \\ &= \int_a^b (b-s)\|f(s)\|^2 ds\end{aligned}$$

and

$$\int_a^b \left(\|f(s)\|^2 \int_a^s dt\right) ds = \int_a^b (s-a)\|f(s)\|^2 ds.$$

Since

$$\begin{aligned}\frac{d}{ds}\left(\left\|\int_a^b f(t)\,dt\right\|^2\right) &= \frac{d}{ds}\left\langle \int_a^s f(t)\,dt, \int_a^s f(t)\,dt\right\rangle \\ &= \left\langle f(s), \int_a^s f(t)\,dt\right\rangle + \left\langle \int_a^s f(t)\,dt, f(s)\right\rangle \\ &= 2\operatorname{Re}\left\langle \int_a^s f(t)\,dt, f(s)\right\rangle,\end{aligned}$$

hence

$$\begin{aligned}\int_a^b \left(\int_a^s \operatorname{Re}\langle f(t), f(s)\rangle\, dt\right) ds &= \int_a^b \operatorname{Re}\left\langle \int_a^s f(t)\, dt, f(s)\right\rangle ds \\ &= \frac{1}{2}\int_a^b \frac{d}{ds}\left(\left\|\int_a^s f(t)\, dt\right\|^2\right) ds \\ &= \frac{1}{2}\left\|\int_a^b f(t)\, dt\right\|^2.\end{aligned}$$

Utilising (4.90), we deduce the desired inequality (4.88).

The case of equality is obvious and we omit the details. ∎

Remark 53. *Consider the function* $\varphi(s) := (b-s) + \gamma\Gamma(s-a)$, $s \in [a,b]$. *Obviously,*

$$\varphi(s) = (\Gamma\gamma - 1)s + b - \gamma\Gamma a.$$

Observe that, if $\Gamma\gamma \geq 1$, *then*

$$b - a = \varphi(a) \leq \varphi(s) \leq \varphi(b) = \gamma\Gamma(b-a), \qquad s \in [a,b]$$

and, if $\Gamma\gamma < 1$, *then*

$$\gamma\Gamma(b-a) \leq \varphi(s) \leq b-a, \qquad s \in [a,b].$$

Taking into account the above remark, we may state the following corollary [**4**].

Corollary 45. *Assume that* f, γ, Γ *are as in Theorem 63.*

a) *If* $\Gamma\gamma \geq 1$, *then we have the inequality*

$$(b-a)\int_a^b \|f(s)\|^2 ds \leq \frac{\Gamma+\gamma}{2}\left\|\int_a^b f(s)\, ds\right\|^2.$$

b) *If* $0 < \Gamma\gamma < 1$, *then we have the inequality*

$$\gamma\Gamma(b-a)\int_a^b \|f(s)\|^2 ds \leq \frac{\Gamma+\gamma}{2}\left\|\int_a^b f(s)\, ds\right\|^2.$$

4.5. Refinements for Complex Spaces

4.5.1. The Case of a Unit Vector. The following result holds [**5**].

Theorem 64 (Dragomir, 2004). *Let* $(H; \langle\cdot,\cdot\rangle)$ *be a complex Hilbert space. If* $f \in L([a,b]; H)$ *is such that there exists* $k_1, k_2 \geq 0$ *with*

$$k_1\|f(t)\| \leq \operatorname{Re}\langle f(t), e\rangle, \quad k_2\|f(t)\| \leq \operatorname{Im}\langle f(t), e\rangle \tag{4.91}$$

for a.e. $t \in [a,b]$, *where* $e \in H$, $\|e\| = 1$, *is given, then*

$$\sqrt{k_1^2 + k_2^2} \int_a^b \|f(t)\| \, dt \leq \left\| \int_a^b f(t) \, dt \right\| . \tag{4.92}$$

The case of equality holds in (4.92) if and only if

$$\int_a^b f(t) \, dt = (k_1 + ik_2) \left(\int_a^b \|f(t)\| \, dt \right) e. \tag{4.93}$$

Proof. Using the Schwarz inequality $\|u\| \, \|v\| \geq |\langle u, v \rangle|$, $u, v \in H$; in the complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$, we have

$$\begin{aligned} \left\| \int_a^b f(t) \, dt \right\|^2 &= \left\| \int_a^b f(t) \, dt \right\|^2 \|e\|^2 \\ &\geq \left| \left\langle \int_a^b f(t) \, dt, e \right\rangle \right|^2 = \left| \int_a^b \langle f(t), e \rangle \, dt \right|^2 \\ &= \left| \int_a^b \operatorname{Re} \langle f(t), e \rangle \, dt + i \left(\int_a^b \operatorname{Im} \langle f(t), e \rangle \, dt \right) \right|^2 \\ &= \left(\int_a^b \operatorname{Re} \langle f(t), e \rangle \, dt \right)^2 + \left(\int_a^b \operatorname{Im} \langle f(t), e \rangle \, dt \right)^2 . \end{aligned} \tag{4.94}$$

Now, on integrating (4.91), we deduce

$$\begin{aligned} k_1 \int_a^b \|f(t)\| \, dt &\leq \int_a^b \operatorname{Re} \langle f(t), e \rangle \, dt, \\ k_2 \int_a^b \|f(t)\| \, dt &\leq \int_a^b \operatorname{Im} \langle f(t), e \rangle \, dt \end{aligned} \tag{4.95}$$

implying

$$\left(\int_a^b \operatorname{Re} \langle f(t), e \rangle \, dt \right)^2 \geq k_1^2 \left(\int_a^b \|f(t)\| \, dt \right)^2 \tag{4.96}$$

and

$$\left(\int_a^b \operatorname{Im} \langle f(t), e \rangle \, dt \right)^2 \geq k_2^2 \left(\int_a^b \|f(t)\| \, dt \right)^2 . \tag{4.97}$$

If we add (4.96) and (4.97) and use (4.94), we deduce the desired inequality (4.92).

Further, if (4.93) holds, then obviously

$$\left\|\int_a^b f(t)\,dt\right\| = |k_1 + ik_2| \left(\int_a^b \|f(t)\|\,dt\right)\|e\| = \sqrt{k_1^2 + k_2^2}\int_a^b \|f(t)\|\,dt,$$

and the equality case holds in (4.92).

Before we prove the reverse implication, let us observe that, for $x \in H$ and $e \in H$, $\|e\| = 1$, the following identity is valid

$$\|x - \langle x, e\rangle e\|^2 = \|x\|^2 - |\langle x, e\rangle|^2,$$

therefore $\|x\| = |\langle x, e\rangle|$ if and only if $x = \langle x, e\rangle e$.

If we assume that equality holds in (4.92), then the case of equality must hold in all the inequalities required in the argument used to prove the inequality (4.92). Therefore, we must have

$$\left\|\int_a^b f(t)\,dt\right\| = \left|\left\langle \int_a^b f(t)\,dt, e\right\rangle\right| \tag{4.98}$$

and

$$k_1 \|f(t)\| = \operatorname{Re}\langle f(t), e\rangle, \quad k_2 \|f(t)\| = \operatorname{Im}\langle f(t), e\rangle \tag{4.99}$$

for a.e. $t \in [a, b]$.

From (4.98) we deduce

$$\int_a^b f(t)\,dt = \left\langle \int_a^b f(t)\,dt, e\right\rangle e, \tag{4.100}$$

and from (4.99), by multiplying the second equality with i, the imaginary unit, and integrating both equations on $[a, b]$, we deduce

$$(k_1 + ik_2)\int_a^b \|f(t)\|\,dt = \left\langle \int_a^b f(t)\,dt, e\right\rangle. \tag{4.101}$$

Finally, by (4.100) and (4.101), we deduce the desired equality (4.93). ∎

The following corollary is of interest [**5**].

Corollary 46. *Let e be a unit vector in the complex Hilbert space $(H; \langle\cdot,\cdot\rangle)$ and $\eta_1, \eta_2 \in (0, 1)$. If $f \in L([a, b]; H)$ is such that*

$$\|f(t) - e\| \leq \eta_1, \|f(t) - ie\| \leq \eta_2 \tag{4.102}$$

for a.e. $t \in [a, b]$, then we have the inequality

$$\sqrt{2 - \eta_1^2 - \eta_2^2}\int_a^b \|f(t)\|\,dt \leq \left\|\int_a^b f(t)\,dt\right\|. \tag{4.103}$$

The case of equality holds in (4.103) if and only if

$$\int_a^b f(t)\,dt = \left(\sqrt{1-\eta_1^2} + i\sqrt{1-\eta_2^2}\right)\left(\int_a^b \|f(t)\|\,dt\right)e. \tag{4.104}$$

PROOF. From the first inequality in (4.102) we deduce, by taking the square, that

$$\|f(t)\|^2 + 1 - \eta_1^2 \le 2\operatorname{Re}\langle f(t), e\rangle,$$

implying

$$\frac{\|f(t)\|^2}{\sqrt{1-\eta_1^2}} + \sqrt{1-\eta_1^2} \le \frac{2\operatorname{Re}\langle f(t), e\rangle}{\sqrt{1-\eta_1^2}} \tag{4.105}$$

for a.e. $t \in [a,b]$.

Since, obviously

$$2\|f(t)\| \le \frac{\|f(t)\|^2}{\sqrt{1-\eta_1^2}} + \sqrt{1-\eta_1^2}, \tag{4.106}$$

hence, by (4.105) and (4.106) we get

$$0 \le \sqrt{1-\eta_1^2}\,\|f(t)\| \le \operatorname{Re}\langle f(t), e\rangle \tag{4.107}$$

for a.e. $t \in [a,b]$.

From the second inequality in (4.102) we deduce

$$0 \le \sqrt{1-\eta_2^2}\,\|f(t)\| \le \operatorname{Re}\langle f(t), ie\rangle$$

for a.e. $t \in [a,b]$. Since

$$\operatorname{Re}\langle f(t), ie\rangle = \operatorname{Im}\langle f(t), e\rangle$$

hence

$$0 \le \sqrt{1-\eta_2^2}\,\|f(t)\| \le \operatorname{Im}\langle f(t), e\rangle \tag{4.108}$$

for a.e. $t \in [a,b]$.

Now, observe from (4.107) and (4.108), that the condition (4.91) of Theorem 64 is satisfied for $k_1 = \sqrt{1-\eta_1^2}$, $k_2 = \sqrt{1-\eta_2^2} \in (0,1)$, and thus the corollary is proved. ∎

The following corollary may be stated as well [**5**].

COROLLARY 47. *Let e be a unit vector in the complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$ and $M_1 \ge m_1 > 0$, $M_2 \ge m_2 > 0$. If $f \in L([a,b];H)$ is such that either*

$$\begin{aligned} \operatorname{Re}\langle M_1 e - f(t), f(t) - m_1 e\rangle &\ge 0, \\ \operatorname{Re}\langle M_2 ie - f(t), f(t) - m_2 ie\rangle &\ge 0 \end{aligned} \tag{4.109}$$

or, equivalently,

$$\left\| f(t) - \frac{M_1 + m_1}{2} e \right\| \leq \frac{1}{2}(M_1 - m_1), \tag{4.110}$$
$$\left\| f(t) - \frac{M_2 + m_2}{2} ie \right\| \leq \frac{1}{2}(M_2 - m_2),$$

for a.e. $t \in [a,b]$, *then we have the inequality*

$$2\left[\frac{m_1 M_1}{(M_1+m_1)^2} + \frac{m_2 M_2}{(M_2+m_2)^2}\right]^{\frac{1}{2}} \int_a^b \|f(t)\|\, dt \leq \left\| \int_a^b f(t)\, dt \right\|. \tag{4.111}$$

The equality holds in (4.111) if and only if

$$\int_a^b f(t)\, dt = 2\left(\frac{\sqrt{m_1 M_1}}{M_1+m_1} + i\frac{\sqrt{m_2 M_2}}{M_2+m_2}\right)\left(\int_a^b \|f(t)\|\, dt\right) e. \tag{4.112}$$

Proof. From the first inequality in (4.109), we get

$$\|f(t)\|^2 + m_1 M_1 \leq (M_1+m_1)\operatorname{Re}\langle f(t), e\rangle$$

implying

$$\frac{\|f(t)\|^2}{\sqrt{m_1 M_1}} + \sqrt{m_1 M_1} \leq \frac{M_1+m_1}{\sqrt{m_1 M_1}} \operatorname{Re}\langle f(t), e\rangle \tag{4.113}$$

for a.e. $t \in [a,b]$.

Since, obviously,

$$2\|f(t)\| \leq \frac{\|f(t)\|^2}{\sqrt{m_1 M_1}} + \sqrt{m_1 M_1}, \tag{4.114}$$

hence, by (4.113) and (4.114)

$$0 \leq \frac{2\sqrt{m_1 M_1}}{M_1+m_1}\|f(t)\| \leq \operatorname{Re}\langle f(t), e\rangle \tag{4.115}$$

for a.e. $t \in [a,b]$.

Using the same argument as in the proof of Corollary 46, we deduce the desired inequality. We omit the details. ∎

4.5.2. The Case of Orthonormal Vectors. The following result holds [**5**].

THEOREM 65 (Dragomir, 2004). *Let* $\{e_1, \ldots, e_n\}$ *be a family of orthonormal vectors in the complex Hilbert space* $(H; \langle\cdot,\cdot\rangle)$. *If* $k_j, h_j \geq 0$, $j \in \{1, \ldots, n\}$ *and* $f \in L([a,b]; H)$ *are such that*

$$k_j \|f(t)\| \leq \operatorname{Re}\langle f(t), e_j\rangle, \quad h_j \|f(t)\| \leq \operatorname{Im}\langle f(t), e_j\rangle \tag{4.116}$$

for each $j \in \{1, \ldots, n\}$ *and a.e.* $t \in [a,b]$, *then*

$$\left[\sum_{j=1}^{n}\left(k_j^2 + h_j^2\right)\right]^{\frac{1}{2}} \int_a^b \|f(t)\|\,dt \leq \left\|\int_a^b f(t)\,dt\right\|. \tag{4.117}$$

The case of equality holds in (4.117) if and only if

$$\int_a^b f(t)\,dt = \left(\int_a^b \|f(t)\|\,dt\right)\sum_{j=1}^{n}(k_j + ih_j)\,e_j. \tag{4.118}$$

PROOF. Before we prove the theorem, let us recall that, if $x \in H$ and $e_1, \ldots, e_n$ are orthonormal vectors, then the following identity holds true:

$$\left\|x - \sum_{j=1}^{n}\langle x, e_j\rangle e_j\right\|^2 = \|x\|^2 - \sum_{j=1}^{n}|\langle x, e_j\rangle|^2. \tag{4.119}$$

As a consequence of this identity, we have the *Bessel inequality*

$$\sum_{j=1}^{n}|\langle x, e_j\rangle|^2 \leq \|x\|^2, x \in H, \tag{4.120}$$

in which, the case of equality holds if and only if

$$x = \sum_{j=1}^{n}\langle x, e_j\rangle e_j. \tag{4.121}$$

Now, applying Bessel's inequality for $x = \int_a^b f(t)\,dt$, we have successively

$$\begin{aligned}\left\|\int_a^b f(t)\,dt\right\|^2 & \\ \geq \sum_{j=1}^{n}\left|\left\langle\int_a^b f(t)\,dt, e_j\right\rangle\right|^2 &= \sum_{j=1}^{n}\left|\int_a^b \langle f(t), e_j\rangle\,dt\right|^2\end{aligned} \tag{4.122}$$

$$= \sum_{j=1}^{n} \left| \int_a^b \operatorname{Re} \langle f(t), e_j \rangle \, dt + i \left(\int_a^b \operatorname{Im} \langle f(t), e_j \rangle \, dt \right) \right|^2$$

$$= \sum_{j=1}^{n} \left[\left(\int_a^b \operatorname{Re} \langle f(t), e_j \rangle \, dt \right)^2 + \left(\int_a^b \operatorname{Im} \langle f(t), e_j \rangle \, dt \right)^2 \right].$$

Integrating (4.116) on $[a, b]$, we get

$$\int_a^b \operatorname{Re} \langle f(t), e_j \rangle \, dt \geq k_j \int_a^b \|f(t)\| \, dt \tag{4.123}$$

and

$$\int_a^b \operatorname{Im} \langle f(t), e_j \rangle \, dt \geq h_j \int_a^b \|f(t)\| \, dt \tag{4.124}$$

for each $j \in \{1, \dots, n\}$.

Squaring and adding the above two inequalities (4.123) and (4.124), we deduce

$$\sum_{j=1}^{n} \left[\left(\int_a^b \operatorname{Re} \langle f(t), e_j \rangle \, dt \right)^2 + \left(\int_a^b \operatorname{Im} \langle f(t), e_j \rangle \, dt \right)^2 \right] \geq \sum_{j=1}^{n} \left(k_j^2 + h_j^2 \right) \left(\int_a^b \|f(t)\| \, dt \right)^2,$$

which combined with (4.122) will produce the desired inequality (4.117).

Now, if (4.118) holds true, then

$$\left\| \int_a^b f(t) \, dt \right\| = \left(\int_a^b \|f(t)\| \, dt \right) \left\| \sum_{j=1}^{n} (k_j + i h_j) e_j \right\|$$

$$= \left(\int_a^b \|f(t)\| \, dt \right) \left(\left\| \sum_{j=1}^{n} (k_j + i h_j) e_j \right\|^2 \right)^{\frac{1}{2}}$$

$$= \left(\int_a^b \|f(t)\| \, dt \right) \left[\sum_{j=1}^{n} \left(k_j^2 + h_j^2 \right) \right]^{\frac{1}{2}},$$

and the case of equality holds in (4.117).

Conversely, if the equality holds in (4.117), then it must hold in all the inequalities used to prove (4.117) and therefore we must have

$$\left\| \int_a^b f(t) \, dt \right\|^2 = \sum_{j=1}^{n} \left| \left\langle \int_a^b f(t) \, dt, e_j \right\rangle \right|^2 \tag{4.125}$$

and

$$k_j \|f(t)\| = \operatorname{Re}\langle f(t), e_j\rangle \quad \text{and} \quad h_j \|f(t)\| = \operatorname{Re}\langle f(t), e_j\rangle \tag{4.126}$$

for each $j \in \{1, \dots, n\}$ and a.e. $t \in [a, b]$.

From (4.125), on using the identity (4.121), we deduce that

$$\int_a^b f(t)\,dt = \sum_{j=1}^n \left\langle \int_a^b f(t)\,dt, e_j \right\rangle e_j. \tag{4.127}$$

Now, multiplying the second equality in (4.126) with the imaginary unit i, integrating both inequalities on $[a, b]$ and summing them up, we get

$$(k_j + ih_j) \int_a^b \|f(t)\|\,dt = \left\langle \int_a^b f(t)\,dt, e_j \right\rangle \tag{4.128}$$

for each $j \in \{1, \dots, n\}$.

Finally, utilising (4.127) and (4.128), we deduce (4.118) and the theorem is proved. ∎

The following corollaries are of interest [**5**].

Corollary 48. *Let $e_1, \dots, e_m$ be orthonormal vectors in the complex Hilbert space $(H; \langle\cdot,\cdot\rangle)$ and $\rho_k, \eta_k \in (0, 1)$, $k \in \{1, \dots, n\}$. If $f \in L([a, b]; H)$ is such that*

$$\|f(t) - e_k\| \le \rho_k, \qquad \|f(t) - ie_k\| \le \eta_k$$

for each $k \in \{1, \dots, n\}$ and for a.e. $t \in [a, b]$, then we have the inequality

$$\left[\sum_{k=1}^n \left(2 - \rho_k^2 - \eta_k^2\right)\right]^{\frac{1}{2}} \int_a^b \|f(t)\|\,dt \le \left\| \int_a^b f(t)\,dt \right\|. \tag{4.129}$$

The case of equality holds in (4.129) if and only if

$$\int_a^b f(t)\,dt = \left(\int_a^b \|f(t)\|\,dt\right) \sum_{k=1}^n \left(\sqrt{1-\rho_k^2} + i\sqrt{1-\eta_k^2}\right) e_k. \tag{4.130}$$

The proof follows by Theorem 65 and is similar to the one from Corollary 46. We omit the details.

Next, the following result may be stated [**5**]:

Corollary 49. *Let $e_1, \dots, e_m$ be as in Corollary 48 and $M_k \geq m_k > 0$, $N_k \geq n_k > 0$, $k \in \{1, \dots, n\}$. If $f \in L([a,b]; H)$ is such that either*

$$\operatorname{Re}\langle M_k e_k - f(t), f(t) - m_k e_k\rangle \geq 0,$$
$$\operatorname{Re}\langle N_k i e_k - f(t), f(t) - n_k i e_k\rangle \geq 0$$

or, equivalently,

$$\left\| f(t) - \frac{M_k + m_k}{2} e_k \right\| \leq \frac{1}{2}(M_k - m_k),$$
$$\left\| f(t) - \frac{N_k + n_k}{2} i e_k \right\| \leq \frac{1}{2}(N_k - n_k)$$

for each $k \in \{1, \dots, n\}$ and a.e. $t \in [a,b]$, then we have the inequality

$$2\left\{\sum_{k=1}^{m}\left[\frac{m_k M_k}{(M_k + m_k)^2} + \frac{n_k N_k}{(N_k + n_k)^2}\right]\right\}^{\frac{1}{2}} \int_a^b \|f(t)\|\, dt \leq \left\|\int_a^b f(t)\, dt\right\|. \tag{4.131}$$

The case of equality holds in (4.131) if and only if

$$\int_a^b f(t)\, dt = 2\left(\int_a^b \|f(t)\|\, dt\right) \times \sum_{k=1}^{n}\left(\frac{\sqrt{m_k M_k}}{M_k + m_k} + i\frac{\sqrt{n_k N_k}}{N_k + n_k}\right) e_k. \tag{4.132}$$

The proof employs Theorem 65 and is similar to the one in Corollary 47. We omit the details.

4.6. Applications for Complex-Valued Functions

The following proposition holds [**2**].

Proposition 48. *If $f : [a,b] \to \mathbb{C}$ is a Lebesgue integrable function with the property that there exists a constant $K \geq 1$ such that*

$$|f(t)| \leq K[\alpha \operatorname{Re} f(t) + \beta \operatorname{Im} f(t)] \tag{4.133}$$

for a.e. $t \in [a,b]$, where $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 = 1$ are given, then we have the following reverse of the continuous triangle inequality:

$$\int_a^b |f(t)|\, dt \leq K\left|\int_a^b f(t)\, dt\right|. \tag{4.134}$$

The case of equality holds in (4.134) if and only if

$$\int_a^b f(t)\,dt = \frac{1}{K}(\alpha + i\beta)\int_a^b |f(t)|\,dt.$$

The proof is obvious by Theorem 57, and we omit the details.

Remark 54. *If in the above Proposition 48 we choose* $\alpha = 1$, $\beta = 0$, *then the condition (4.133) for* $\operatorname{Re} f(t) > 0$ *is equivalent to*

$$[\operatorname{Re} f(t)]^2 + [\operatorname{Im} f(t)]^2 \le K^2 [\operatorname{Re} f(t)]^2$$

or with the inequality:

$$\frac{|\operatorname{Im} f(t)|}{\operatorname{Re} f(t)} \le \sqrt{K^2 - 1}.$$

Now, if we assume that

$$|\arg f(t)| \le \theta, \quad \theta \in \left(0, \frac{\pi}{2}\right), \tag{4.135}$$

then, for $\operatorname{Re} f(t) > 0$,

$$|\tan[\arg f(t)]| = \frac{|\operatorname{Im} f(t)|}{\operatorname{Re} f(t)} \le \tan\theta,$$

and if we choose $K = \frac{1}{\cos\theta} > 1$, *then*

$$\sqrt{K^2 - 1} = \tan\theta,$$

and by Proposition 48, we deduce

$$\cos\theta \int_a^b |f(t)|\,dt \le \left|\int_a^b f(t)\,dt\right|, \tag{4.136}$$

which is exactly the Karamata inequality (4.2) from the Introduction.

Obviously, the result from Proposition 48 is more comprehensive since for other values of $(\alpha, \beta) \in \mathbb{R}^2$ with $\alpha^2 + \beta^2 = 1$ we can get different sufficient conditions for the function f such that the inequality (4.134) holds true.

A different sufficient condition in terms of complex disks is incorporated in the following proposition [**2**].

Proposition 49. *Let* $e = \alpha + i\beta$ *with* $\alpha^2 + \beta^2 = 1$, $r \in (0, 1)$ *and* $f : [a, b] \to \mathbb{C}$ *a Lebesgue integrable function such that*

$$f(t) \in \bar{D}(e, r) := \{z \in \mathbb{C} \mid |z - e| \le r\} \quad \text{for a.e. } t \in [a, b]. \tag{4.137}$$

Then we have the inequality

$$\sqrt{1 - r^2} \int_a^b |f(t)|\,dt \le \left|\int_a^b f(t)\,dt\right|. \tag{4.138}$$

The case of equality holds in (4.138) if and only if

$$\int_a^b f(t)\,dt = \sqrt{1-r^2}\,(\alpha+i\beta)\int_a^b |f(t)|\,dt.$$

The proof follows by Corollary 33 and we omit the details.

Further, we may state the following proposition as well [**2**].

Proposition 50. *Let $e=\alpha+i\beta$ with $\alpha^2+\beta^2=1$ and $M\ge m>0$. If $f:[a,b]\to\mathbb{C}$ is such that*

$$\operatorname{Re}\left[(Me-f(t))\left(\overline{f(t)}-m\overline{e}\right)\right]\ge 0 \quad \text{for a.e. } t\in[a,b], \tag{4.139}$$

or, equivalently,

$$\left|f(t)-\frac{M+m}{2}e\right|\le\frac{1}{2}(M-m) \quad \text{for a.e. } t\in[a,b], \tag{4.140}$$

then we have the inequality

$$\frac{2\sqrt{mM}}{M+m}\int_a^b |f(t)|\,dt \le \left|\int_a^b f(t)\,dt\right|, \tag{4.141}$$

or, equivalently,

$$\begin{aligned}(0\le)\int_a^b |f(t)|\,dt-\left|\int_a^b f(t)\,dt\right| \\ \le\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{M+m}\left|\int_a^b f(t)\,dt\right|.\end{aligned} \tag{4.142}$$

The equality holds in (4.141) (or in the second part of (4.142)) if and only if

$$\int_a^b f(t)\,dt=\frac{2\sqrt{mM}}{M+m}(\alpha+i\beta)\int_a^b |f(t)|\,dt.$$

The proof follows by Corollary 34 and we omit the details.

Remark 55. *Since*

$$\begin{aligned}Me-f(t)&=M\alpha-\operatorname{Re}f(t)+i\left[M\beta-\operatorname{Im}f(t)\right],\\ \overline{f(t)}-m\overline{e}&=\operatorname{Re}f(t)-m\alpha-i\left[\operatorname{Im}f(t)-m\beta\right]\end{aligned}$$

hence

$$\begin{aligned}\operatorname{Re}&\left[(Me-f(t))\left(\overline{f(t)}-m\overline{e}\right)\right]\\ &=\left[M\alpha-\operatorname{Re}f(t)\right]\left[\operatorname{Re}f(t)-m\alpha\right]\\ &\quad+\left[M\beta-\operatorname{Im}f(t)\right]\left[\operatorname{Im}f(t)-m\beta\right].\end{aligned} \tag{4.143}$$

It is obvious that, if

$$m\alpha \leq \operatorname{Re} f(t) \leq M\alpha \quad \text{for a.e. } t \in [a,b], \tag{4.144}$$

and

$$m\beta \leq \operatorname{Im} f(t) \leq M\beta \quad \text{for a.e. } t \in [a,b], \tag{4.145}$$

then, by (4.143),

$$\operatorname{Re}\left[(Me - f(t))\left(\overline{f(t)} - m\bar{e}\right)\right] \geq 0 \quad \text{for a.e. } t \in [a,b],$$

and then either (4.141) or (4.144) hold true.

We observe that the conditions (4.144) and (4.145) are very easy to verify in practice and may be useful in various applications where reverses of the continuous triangle inequality are required.

Remark 56. *Similar results may be stated for functions $f : [a,b] \to \mathbb{R}^n$ or $f : [a,b] \to H$, with H particular instances of Hilbert spaces of significance in applications, but we leave them to the interested reader.*

Let $e = \alpha + i\beta$ $(\alpha, \beta \in \mathbb{R})$ be a complex number with the property that $|e| = 1$, i.e., $\alpha^2 + \beta^2 = 1$. The following proposition concerning a reverse of the continuous triangle inequality for complex-valued functions may be stated [**3**]:

Proposition 51. *Let $f : [a,b] \to \mathbb{C}$ be a Lebesgue integrable function with the property that there exists a constant $\rho \in (0,1)$ such that*

$$|f(t) - e| \leq \rho \text{ for a.e. } t \in [a,b], \tag{4.146}$$

where e has been defined above. Then we have the following reverse of the continuous triangle inequality

$$\begin{aligned}(0 \leq) \int_a^b |f(t)|\,dt - \left|\int_a^b f(t)\,dt\right| &\leq \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)} \\ &\quad \times \left[\alpha \int_a^b \operatorname{Re} f(t)\,dt + \beta \int_a^b \operatorname{Im} f(t)\,dt\right].\end{aligned} \tag{4.147}$$

The proof follows by Corollary 37, and the details are omitted.

On the other hand, the following result is perhaps more useful for applications [**3**]:

Proposition 52. *Assume that f and e are as in Proposition 51. If there exists the constants $M \geq m > 0$ such that either*

$$\operatorname{Re}\left[(Me - f(t))\left(\overline{f(t)} - m\bar{e}\right)\right] \geq 0 \tag{4.148}$$

or, equivalently,

$$\left|f(t) - \frac{M+m}{2}e\right| \leq \frac{1}{2}(M-m) \tag{4.149}$$

for a.e. $t \in [a,b]$, holds, then

$$\begin{aligned}(0 \leq) \int_a^b |f(t)|\,dt - \left|\int_a^b f(t)\,dt\right| \\ \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{Mm}}\left[\alpha \int_a^b \operatorname{Re} f(t)\,dt + \beta \int_a^b \operatorname{Im} f(t)\,dt\right].\end{aligned} \tag{4.150}$$

The proof may be done on utilising Corollary 38, but we omit the details

Subsequently, on making use of Corollary 40, one may state the following result as well [**3**]:

Proposition 53. *Let f be as in Proposition 51 and the measurable functions $K, k : [a,b] \to [0,\infty)$ with the property that*

$$\frac{(K-k)^2}{K+k} \in L[a,b]$$

and

$$\alpha k(t) \leq \operatorname{Re} f(t) \leq \alpha K(t) \text{ and } \beta k(t) \leq \operatorname{Im} f(t) \leq \beta K(t)$$

for a.e. $t \in [a,b]$, where α, β are assumed to be positive and satisfying the condition $\alpha^2 + \beta^2 = 1$. Then the following reverse of the continuous triangle inequality is valid:

$$\begin{aligned}(0 \leq) \int_a^b |f(t)|\,dt - \left|\int_a^b f(t)\,dt\right| \\ \leq \frac{1}{4}\int_a^b \frac{[K(t)-k(t)]^2}{K(t)+k(t)}dt.\end{aligned}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

Remark 57. *One may realise that similar results can be stated if the Corollaries 41-44 obtained above are used. For the sake of brevity, we do not mention them here.*

Let $f : [a, b] \to \mathbb{C}$ be a Lebesgue integrable function and $M \geq 1 \geq m \geq 0$. The condition (4.76) from Theorem 61, which plays a fundamental role in the results obtained above, can be translated in this case as

$$\operatorname{Re}\left[(Mf(s) - f(t))\left(\overline{f(t)} - m\overline{f(s)}\right)\right] \geq 0 \tag{4.151}$$

for a.e. $a \leq t \leq s \leq b$.

Since, obviously

$$\begin{aligned}
&\operatorname{Re}\left[(Mf(s) - f(t))\left(\overline{f(t)} - m\overline{f(s)}\right)\right] \\
&\quad = [(M \operatorname{Re} f(s) - \operatorname{Re} f(t))(\operatorname{Re} f(t) - m \operatorname{Re} f(s))] \\
&\qquad + [(M \operatorname{Im} f(s) - \operatorname{Im} f(t))(\operatorname{Im} f(t) - m \operatorname{Im} f(s))]
\end{aligned}$$

hence a sufficient condition for the inequality in (4.151) to hold is

$$m \operatorname{Re} f(s) \leq \operatorname{Re} f(t) \leq M \operatorname{Re} f(s) \tag{4.152}$$

and

$$m \operatorname{Im} f(s) \leq \operatorname{Im} f(t) \leq M \operatorname{Im} f(s)$$

for a.e. $a \leq t \leq s \leq b$.

Utilising Theorems 61, 62 and 63 we may state the following results incorporating quadratic reverses of the continuous triangle inequality [**4**]:

Proposition 54. *With the above assumptions for f, M and m, and if (4.151) holds true, then we have the inequalities*

$$\begin{aligned}
\left(\int_a^b |f(t)|\, dt\right)^2 &\leq \left|\int_a^b f(t)\, dt\right|^2 \\
&\qquad + \frac{1}{2} \cdot \frac{(M-m)^2}{M+m} \int_a^b (s-a)|f(s)|^2\, ds,
\end{aligned}$$

$$\int_a^b |f(t)|\, dt \leq \left(\frac{M+m}{2\sqrt{Mm}}\right)^{\frac{1}{2}} \left|\int_a^b f(t)\, dt\right|,$$

and

$$\int_a^b [(b-s) + mM(s-a)]\, |f(s)|^2\, ds \leq \frac{M+m}{2} \left|\int_a^b f(s)\, ds\right|^2.$$

Remark 58. *One may wonder if there are functions satisfying the condition (4.152) above. It suffices to find examples of real functions $\varphi : [a, b] \to \mathbb{R}$ verifying the following double inequality*

$$\gamma\varphi(s) \leq \varphi(t) \leq \Gamma\varphi(s) \tag{4.153}$$

for some given γ, Γ *with* $0 \le \gamma \le 1 \le \Gamma < \infty$ *for a.e.* $a \le t \le s \le b$.

For this purpose, consider $\psi : [a,b] \to \mathbb{R}$ *a differentiable function on* (a,b), *continuous on* $[a,b]$ *and with the property that there exists* $\Theta \ge 0 \ge \theta$ *such that*

$$\theta \le \psi'(u) \le \Theta \text{ for any } u \in (a,b). \tag{4.154}$$

By Lagrange's mean value theorem, we have, for any $a \le t \le s \le b$

$$\psi(s) - \psi(t) = \psi'(\xi)(s-t)$$

with $t \le \xi \le s$. *Therefore, for* $a \le t \le s \le b$, *by (4.154), we have the inequality*

$$\theta(b-a) \le \theta(s-t) \le \psi(s) - \psi(t) \le \Theta(s-t) \le \Theta(b-a).$$

If we choose the function $\varphi : [a,b] \to \mathbb{R}$ *given by*

$$\varphi(t) := \exp[-\psi(t)], \ t \in [a,b],$$

and $\gamma := \exp[\theta(b-a)] \le 1$, $\Gamma := \exp[\Theta(b-a)]$, *then (4.153) holds true for any* $a \le t \le s \le b$.

The following reverse of the continuous triangle inequality for complex-valued functions that improves Karamata's result (4.1) holds [**5**].

Proposition 55. *Let* $f \in L([a,b];\mathbb{C})$ *with the property that*

$$0 \le \varphi_1 \le \arg f(t) \le \varphi_2 < \frac{\pi}{2} \tag{4.155}$$

for a.e. $t \in [a,b]$. *Then we have the inequality*

$$\sqrt{\sin^2\varphi_1 + \cos^2\varphi_2} \int_a^b |f(t)|\, dt \le \left| \int_a^b f(t)\, dt \right|. \tag{4.156}$$

The equality holds in (4.156) if and only if

$$\int_a^b f(t)\, dt = (\cos\varphi_2 + i\sin\varphi_1) \int_a^b |f(t)|\, dt. \tag{4.157}$$

Proof. Let $f(t) = \operatorname{Re} f(t) + i \operatorname{Im} f(t)$. We may assume that $\operatorname{Re} f(t) \ge 0$, $\operatorname{Im} f(t) > 0$, for a.e. $t \in [a,b]$, since, by (4.155), $\frac{\operatorname{Im} f(t)}{\operatorname{Re} f(t)} = \tan[\arg f(t)] \in \left[0, \frac{\pi}{2}\right)$, for a.e. $t \in [a,b]$. By (4.155), we obviously have

$$0 \le \tan^2\varphi_1 \le \left[\frac{\operatorname{Im} f(t)}{\operatorname{Re} f(t)}\right]^2 \le \tan^2\varphi_2,$$

for a.e. $t \in [a,b]$, from where we get

$$\frac{[\operatorname{Im} f(t)]^2 + [\operatorname{Re} f(t)]^2}{[\operatorname{Re} f(t)]^2} \le \frac{1}{\cos^2\varphi_2},$$

for a.e. $t \in [a, b]$, and

$$\frac{[\operatorname{Im} f(t)]^2 + [\operatorname{Re} f(t)]^2}{[\operatorname{Im} f(t)]^2} \leq \frac{1 + \tan^2 \varphi_1}{\tan^2 \varphi_1} = \frac{1}{\sin \varphi_1},$$

for a.e. $t \in [a, b]$, giving the simpler inequalities

$$|f(t)| \cos \varphi_2 \leq \operatorname{Re}(f(t)), \quad |f(t)| \sin \varphi_1 \leq \operatorname{Im}(f(t))$$

for a.e. $t \in [a, b]$.

Now, applying Theorem 64 for the complex Hilbert space $\mathbb{C}$ endowed with the inner product $\langle z, w \rangle = z \cdot \bar{w}$ for $k_1 = \cos \varphi_2$, $k_2 = \sin \varphi_1$ and $e = 1$, we deduce the desired inequality (4.156). The case of equality is also obvious and we omit the details. ∎

Another result that has an obvious geometrical interpretation is the following one [**5**].

Proposition 56. *Let $e \in \mathbb{C}$ with $|e| = 1$ and $\rho_1, \rho_2 \in (0, 1)$. If $f(t) \in L([a, b]; \mathbb{C})$ such that*

$$|f(t) - e| \leq \rho_1, \quad |f(t) - ie| \leq \rho_2 \quad \text{for a.e. } t \in [a, b], \tag{4.158}$$

then we have the inequality

$$\sqrt{2 - \rho_1^2 - \rho_2^2} \int_a^b |f(t)|\, dt \leq \left| \int_a^b f(t)\, dt \right|, \tag{4.159}$$

with equality if and only if

$$\int_a^b f(t)\, dt = \left(\sqrt{1 - \rho_1^2} + i\sqrt{1 - \rho_2^2} \right) \int_a^b |f(t)|\, dt \cdot e. \tag{4.160}$$

The proof is obvious by Corollary 46 applied for $H = \mathbb{C}$ and we omit the details.

Remark 59. *If we choose $e = 1$, and for $\rho_1, \rho_2 \in (0, 1)$ we define*

$$\bar{D}(1, \rho_1) := \{z \in \mathbb{C} \mid |z - 1| \leq \rho_1\}, \quad \bar{D}(i, \rho_2) := \{z \in \mathbb{C} \mid |z - i| \leq \rho_2\},$$

then obviously the intersection domain

$$S_{\rho_1, \rho_2} := \bar{D}(1, \rho_1) \cap \bar{D}(i, \rho_2)$$

is nonempty if and only if $\rho_1 + \rho_2 > \sqrt{2}$.

If $f(t) \in S_{\rho_1, \rho_2}$ for a.e. $t \in [a, b]$, then (4.159) holds true. The equality holds in (4.159) if and only if

$$\int_a^b f(t)\, dt = \left(\sqrt{1 - \rho_1^2} + i\sqrt{1 - \rho_2^2} \right) \int_a^b |f(t)|\, dt.$$

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CHAPTER 5

Reverses of the CBS and Heisenberg Inequalities

5.1. Introduction

Assume that $(K;\langle\cdot,\cdot\rangle)$ is a Hilbert space over the real or complex number field $\mathbb{K}$. If $\rho:[a,b]\subset\mathbb{R}\to[0,\infty)$ is a Lebesgue integrable function with $\int_a^b \rho(t)\,dt=1$, then we may consider the space $L_\rho^2([a,b];K)$ of all functions$f:[a,b]\to K$, that are Bochner measurable and $\int_a^b \rho(t)\|f(t)\|^2\,dt<\infty$. It is well known that $L_\rho^2([a,b];K)$ endowed with the inner product $\langle\cdot,\cdot\rangle_\rho$ defined by

$$\langle f,g\rangle_\rho := \int_a^b \rho(t)\langle f(t),g(t)\rangle\,dt$$

and generating the norm

$$\|f\|_\rho := \left(\int_a^b \rho(t)\|f(t)\|^2\,dt\right)^{\frac{1}{2}},$$

is a Hilbert space over $\mathbb{K}$.

The following integral inequality is known in the literature as the Cauchy-Bunyakovsky-Schwarz (CBS) inequality

$$\int_a^b \rho(t)\|f(t)\|^2\,dt\int_a^b \rho(t)\|g(t)\|^2\,dt \geq \left|\int_a^b \rho(t)\langle f(t),g(t)\rangle\,dt\right|^2, \tag{5.1}$$

provided $f,g\in L_\rho^2([a,b];K)$.

The case of equality holds in (5.1) iff there exists a constant $\lambda\in\mathbb{K}$ such that $f(t)=\lambda g(t)$ for a.e. $t\in[a,b]$.

Another version of the (CBS) inequality for one vector-valued and one scalar function is incorporated in:

$$\int_a^b \rho(t)\left|\alpha(t)\right|^2 dt \int_a^b \rho(t)\left\|f(t)\right\|^2 dt \geq \left\|\int_a^b \rho(t)\,\alpha(t)\,f(t)\,dt\right\|^2, \tag{5.2}$$

provided $\alpha \in L^2_\rho([a,b])$ and $f \in L^2_\rho([a,b];K)$, where $L^2_\rho([a,b])$ denotes the Hilbert space of scalar functions α for which $\int_a^b \rho(t)\left|\alpha(t)\right|^2 dt < \infty$. The equality holds in (5.2) iff there exists a vector $e \in K$ such that $f(t) = \overline{\alpha(t)}e$ for a.e. $t \in [a,b]$.

In this chapter some reverses of the inequalities (5.1) and (5.2) are given under various assumptions for the functions involved. Natural applications for the Heisenberg inequality for vector-valued functions in Hilbert spaces are also provided.

5.2. Some Reverse Inequalities

5.2.1. The General Case. The following result holds [**1**].

Theorem 66 (Dragomir, 2004). *Let $f, g \in L^2_\rho([a,b];K)$ and $r > 0$ be such that*

$$\left\|f(t) - g(t)\right\| \leq r \leq \left\|g(t)\right\| \tag{5.3}$$

for a.e. $t \in [a,b]$. Then we have the inequalities:

$$\begin{aligned} 0 &\leq \int_a^b \rho(t)\left\|f(t)\right\|^2 dt \int_a^b \rho(t)\left\|g(t)\right\|^2 dt \\ &\qquad - \left|\int_a^b \rho(t)\left\langle f(t), g(t)\right\rangle dt\right|^2 \\ &\leq \int_a^b \rho(t)\left\|f(t)\right\|^2 dt \int_a^b \rho(t)\left\|g(t)\right\|^2 dt \\ &\qquad - \left[\int_a^b \rho(t)\operatorname{Re}\left\langle f(t), g(t)\right\rangle dt\right]^2 \\ &\leq r^2 \int_a^b \rho(t)\left\|f(t)\right\|^2 dt. \end{aligned} \tag{5.4}$$

The constant $C = 1$ in front of r^2 is best possible in the sense that it cannot be replaced by a smaller quantity.

Proof. We will use the following result obtained in [**2**]:

In the inner product space $(H; \langle \cdot, \cdot \rangle)$, if $x, y \in H$ and $r > 0$ are such that $\|x - y\| \le r \le \|y\|$, then

$$0 \le \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \le \|x\|^2 \|y\|^2 - [\operatorname{Re} \langle x, y \rangle]^2 \le r^2 \|x\|^2 . \tag{5.5}$$

The constant $c = 1$ in front of r^2 is best possible in the sense that it cannot be replaced by a smaller quantity.

If (5.3) holds, true, then

$$\|f - g\|_\rho^2 = \int_a^b \rho(t) \|f(t) - g(t)\|^2 dt \le r^2 \int_a^b \rho(t) dt = r^2$$

and

$$\|g\|_\rho^2 = \int_a^b \rho(t) \|g(t)\|^2 dt \ge r^2 \int_a^b \rho(t) dt = r^2$$

and thus $\|f - g\|_\rho \le r \le \|g\|_\rho$. Applying the inequality (5.5) for $\left(L_\rho^2([a,b]; K), \langle \cdot, \cdot \rangle_p\right)$, we deduce the desired inequality (5.4).

If we choose $\rho(t) = \frac{1}{b-a}$, $f(t) = x$, $g(t) = y$, $x, y \in K$, $t \in [a,b]$, then from (5.4) we recapture (5.5) for which the constant $c = 1$ in front of r^2 is best possible. ∎

We next point out some general reverse inequalities for the second (CBS) inequality (5.2) [**1**].

Theorem 67 (Dragomir, 2004). *Let $\alpha \in L_\rho^2([a,b])$, $g \in L_\rho^2([a,b]; K)$ and $a \in K$, $r > 0$ such that $\|a\| > r$. If the following condition holds*

$$\|g(t) - \bar{\alpha}(t) a\| \le r |\alpha(t)| \tag{5.6}$$

for a.e. $t \in [a,b]$, (note that, if $\alpha(t) \ne 0$ for a.e. $t \in [a,b]$, then the condition (5.6) is equivalent to

$$\left\| \frac{g(t)}{\bar{\alpha}(t)} - a \right\| \le r \tag{5.7}$$

for a.e. $t \in [a,b]$), *then we have the following inequality*

$$(5.8)\qquad \left(\int_a^b \rho(t)\left|\alpha(t)\right|^2 dt \int_a^b \rho(t)\left\|g(t)\right\|^2 dt\right)^{\frac{1}{2}} \\ \leq \frac{1}{\sqrt{\left\|a\right\|^2 - r^2}} \operatorname{Re}\left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, a\right\rangle \\ \leq \frac{\left\|a\right\|}{\sqrt{\left\|a\right\|^2 - r^2}} \left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|;$$

$$(5.9)\qquad 0 \leq \left(\int_a^b \rho(t)\left|\alpha(t)\right|^2 dt \int_a^b \rho(t)\left\|g(t)\right\|^2 dt\right)^{\frac{1}{2}} - \left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\| \\ \leq \left(\int_a^b \rho(t)\left|\alpha(t)\right|^2 dt \int_a^b \rho(t)\left\|g(t)\right\|^2 dt\right)^{\frac{1}{2}} - \operatorname{Re}\left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, \frac{a}{\left\|a\right\|}\right\rangle \\ \leq \frac{r^2}{\sqrt{\left\|a\right\|^2 - r^2}\left(\left\|a\right\| + \sqrt{\left\|a\right\|^2 - r^2}\right)} \times \operatorname{Re}\left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, \frac{a}{\left\|a\right\|}\right\rangle \\ \leq \frac{r^2}{\sqrt{\left\|a\right\|^2 - r^2}\left(\left\|a\right\| + \sqrt{\left\|a\right\|^2 - r^2}\right)} \left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|;$$

$$(5.10)\qquad \int_a^b \rho(t)\left|\alpha(t)\right|^2 dt \int_a^b \rho(t)\left\|g(t)\right\|^2 dt \\ \leq \frac{1}{\left\|a\right\|^2 - r^2}\left[\operatorname{Re}\left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, a\right\rangle\right]^2 \\ \leq \frac{\left\|a\right\|^2}{\left\|a\right\|^2 - r^2}\left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|^2,$$

and

$$0 \leq \int_a^b \rho(t) |\alpha(t)|^2 \, dt \int_a^b \rho(t) \|g(t)\|^2 \, dt - \left\| \int_a^b \rho(t) \alpha(t) g(t) \, dt \right\|^2 \tag{5.11}$$
$$\leq \int_a^b \rho(t) |\alpha(t)|^2 \, dt \int_a^b \rho(t) \|g(t)\|^2 \, dt - \left[\operatorname{Re} \left\langle \int_a^b \rho(t) \alpha(t) g(t) \, dt, \frac{a}{\|a\|} \right\rangle \right]^2$$
$$\leq \frac{r^2}{\|a\|^2 \left(\|a\|^2 - r^2\right)} \left[\operatorname{Re} \left\langle \int_a^b \rho(t) \alpha(t) g(t) \, dt, a \right\rangle \right]^2$$
$$\leq \frac{r^2}{\|a\|^2 - r^2} \left\| \int_a^b \rho(t) \alpha(t) g(t) \, dt \right\|^2 .$$

All the inequalities (5.8) – (5.11) are sharp.

Proof. From (5.6) we deduce

$$\|g(t)\|^2 - 2 \operatorname{Re} \langle g(t), \bar{\alpha}(t) a \rangle + |\alpha(t)|^2 \|a\|^2 \leq |\alpha(t)|^2 r^2$$

for a.e. $t \in [a, b]$, which is clearly equivalent to:

$$\|g(t)\|^2 + \left(\|a\|^2 - r^2\right) |\alpha(t)|^2 \leq 2 \operatorname{Re} \langle \alpha(t) g(t), a \rangle \tag{5.12}$$

for a.e. $t \in [a, b]$.

If we multiply (5.12) by $\rho(t) \geq 0$ and integrate over $t \in [a, b]$, then we deduce

$$\int_a^b \rho(t) \|g(t)\|^2 \, dt + \left(\|a\|^2 - r^2\right) \int_a^b \rho(t) |\alpha(t)|^2 \, dt \leq 2 \operatorname{Re} \left\langle \int_a^b \rho(t) \alpha(t) g(t) \, dt, a \right\rangle . \tag{5.13}$$

Now, dividing (5.13) by $\sqrt{\|a\|^2 - r^2} > 0$, we get

$$\begin{aligned}(5.14)\quad & \frac{1}{\sqrt{\|a\|^2 - r^2}} \int_a^b \rho(t) \|g(t)\|^2 \, dt \\ & \quad + \sqrt{\|a\|^2 - r^2} \int_a^b \rho(t) |\alpha(t)|^2 \, dt \\ & \qquad \leq \frac{2}{\sqrt{\|a\|^2 - r^2}} \operatorname{Re} \left\langle \int_a^b \rho(t) \alpha(t) g(t) \, dt, a \right\rangle .\end{aligned}$$

On the other hand, by the elementary inequality

$$\frac{1}{\alpha} p + \alpha q \geq 2\sqrt{pq}, \qquad \alpha > 0,\ p, q \geq 0,$$

we can state that

$$\begin{aligned}(5.15)\quad & 2\sqrt{\int_a^b \rho(t) |\alpha(t)|^2 \, dt} \cdot \sqrt{\int_a^b \rho(t) \|g(t)\|^2 \, dt} \\ & \quad \leq \frac{1}{\sqrt{\|a\|^2 - r^2}} \int_a^b \rho(t) \|g(t)\|^2 \, dt \\ & \qquad + \sqrt{\|a\|^2 - r^2} \int_a^b \rho(t) |\alpha(t)|^2 \, dt.\end{aligned}$$

Making use of (5.14) and (5.15), we deduce the first part of (5.8).

The second part of (5.8) is obvious by Schwarz's inequality

$$\operatorname{Re} \left\langle \int_a^b \rho(t) \alpha(t) g(t) \, dt, a \right\rangle \leq \left\| \int_a^b \rho(t) \alpha(t) g(t) \, dt \right\| \|a\| .$$

If $\rho(t) = \frac{1}{b-a}$, $\alpha(t) = 1$, $g(t) = x \in K$, then, from (5.8) we get

$$\|x\| \leq \frac{1}{\sqrt{\|a\|^2 - r^2}} \operatorname{Re} \langle x, a \rangle \leq \frac{\|x\| \|a\|}{\sqrt{\|a\|^2 - r^2}},$$

provided $\|x - a\| \leq r < \|a\|$, $x, a \in K$. The sharpness of this inequality has been shown in [**2**], and we omit the details.

The other inequalities are obvious consequences of (5.8) and we omit the details. ∎

5.2.2. Some Particular Cases. It has been shown in [**2**] that, for $A, a \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) and $x, y \in H$, where $(H; \langle \cdot, \cdot \rangle)$ is an inner product over the real or complex number field $\mathbb{K}$, the following inequality holds

$$\|x\| \|y\| \leq \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{A} + \bar{a}\right) \langle x, y \rangle\right]}{\left[\operatorname{Re}(A\bar{a})\right]^{\frac{1}{2}}} \leq \frac{1}{2} \cdot \frac{|A + a|}{\left[\operatorname{Re}(A\bar{a})\right]^{\frac{1}{2}}} |\langle x, y \rangle| \tag{5.16}$$

provided $\operatorname{Re}(A\bar{a}) > 0$ and

$$\operatorname{Re}\langle Ay - x, x - ay \rangle \geq 0, \tag{5.17}$$

or, equivalently,

$$\left\| x - \frac{a + A}{2} \cdot y \right\| \leq \frac{1}{2} |A - a| \|y\|, \tag{5.18}$$

holds. The constant $\frac{1}{2}$ is best possible in (5.16).

From (5.16), we can deduce the following results

$$\begin{aligned} 0 &\leq \|x\| \|y\| - \operatorname{Re}\langle x, y \rangle \\ &\leq \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{A} + \bar{a} - 2\left[\operatorname{Re}(A\bar{a})\right]^{\frac{1}{2}}\right) \langle x, y \rangle\right]}{\left[\operatorname{Re}(A\bar{a})\right]^{\frac{1}{2}}} \\ &\leq \frac{1}{2} \cdot \frac{\left|\bar{A} + \bar{a} - 2\left[\operatorname{Re}(A\bar{a})\right]^{\frac{1}{2}}\right|}{\left[\operatorname{Re}(A\bar{a})\right]^{\frac{1}{2}}} |\langle x, y \rangle| \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} 0 &\leq \|x\| \|y\| - |\langle x, y \rangle| \\ &\leq \frac{1}{2} \cdot \frac{|A + a| - 2\left[\operatorname{Re}(A\bar{a})\right]^{\frac{1}{2}}}{\left[\operatorname{Re}(A\bar{a})\right]^{\frac{1}{2}}} |\langle x, y \rangle|. \end{aligned} \tag{5.20}$$

If one assumes that $A = M$, $a = m$, $M \geq m > 0$, then, from (5.16), (5.19) and (5.20) we deduce the much simpler and more useful results:

$$\|x\| \|y\| \leq \frac{1}{2} \cdot \frac{M + m}{\sqrt{Mm}} \operatorname{Re}\langle x, y \rangle, \tag{5.21}$$

$$0 \leq \|x\| \|y\| - \operatorname{Re}\langle x, y \rangle \leq \frac{1}{2} \cdot \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{\sqrt{Mm}} \operatorname{Re}\langle x, y \rangle \tag{5.22}$$

and

$$0 \leq \|x\| \|y\| - |\langle x, y \rangle| \leq \frac{1}{2} \cdot \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{\sqrt{Mm}} |\langle x, y \rangle| , \tag{5.23}$$

provided

$$\operatorname{Re} \langle My - x, x - my \rangle \geq 0$$

or, equivalently

$$\left\| x - \frac{M+m}{2} y \right\| \leq \frac{1}{2} (M - m) \|y\| . \tag{5.24}$$

Squaring the second inequality in (5.16), we can get the following results as well:

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{1}{4} \cdot \frac{|A - a|^2}{\operatorname{Re}(A\bar{a})} |\langle x, y \rangle|^2 , \tag{5.25}$$

provided (5.17) or (5.16) holds. Here the constant $\frac{1}{4}$ is also best possible.

Using the above inequalities for vectors in inner product spaces, we are able to state the following theorem concerning reverses of the (CBS) integral inequality for vector-valued functions in Hilbert spaces [**1**].

Theorem 68 (Dragomir, 2004). *Let $f, g \in L^2_\rho([a, b]; K)$ and $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$. If*

$$\operatorname{Re} \langle \Gamma g(t) - f(t), f(t) - \gamma g(t) \rangle \geq 0 \tag{5.26}$$

for a.e. $t \in [a, b]$, or, equivalently,

$$\left\| f(t) - \frac{\gamma + \Gamma}{2} \cdot g(t) \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|g(t)\| \tag{5.27}$$

for a.e. $t \in [a, b]$, then we have the inequalities

$$\begin{aligned} &\left(\int_a^b \rho(t) \|f(t)\|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{\Gamma} + \bar{\gamma}\right) \int_a^b \rho(t) \langle f(t), g(t) \rangle dt \right]}{[\operatorname{Re}(\Gamma\bar{\gamma})]^{\frac{1}{2}}} \\ &\leq \frac{1}{2} \cdot \frac{|\Gamma + \gamma|}{[\operatorname{Re}(\Gamma\bar{\gamma})]^{\frac{1}{2}}} \left| \int_a^b \rho(t) \langle f(t), g(t) \rangle dt \right| , \end{aligned} \tag{5.28}$$

$$
\begin{aligned}
(5.29)\quad 0 &\le \left(\int_a^b \rho(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}} \left(\int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} \\
&\quad - \int_a^b \rho(t) \operatorname{Re}\langle f(t), g(t)\rangle dt \\
&\le \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left\{\bar{\Gamma} + \bar{\gamma} - 2\left[\operatorname{Re}(\Gamma\bar{\gamma})\right]^{\frac{1}{2}}\right\} \int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right]}{\left[\operatorname{Re}(\Gamma\bar{\gamma})\right]^{\frac{1}{2}}} \\
&\le \frac{1}{2} \cdot \frac{\left|\bar{\Gamma} + \bar{\gamma} - 2\left[\operatorname{Re}(\Gamma\bar{\gamma})\right]^{\frac{1}{2}}\right|}{\left[\operatorname{Re}(\Gamma\bar{\gamma})\right]^{\frac{1}{2}}} \left|\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right|,
\end{aligned}
$$

$$
\begin{aligned}
(5.30)\quad 0 &\le \left(\int_a^b \rho(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}} \left(\int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} \\
&\quad - \left|\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right| \\
&\le \frac{1}{2} \cdot \frac{|\Gamma + \gamma| - 2\left[\operatorname{Re}(\Gamma\bar{\gamma})\right]^{\frac{1}{2}}}{\left[\operatorname{Re}(\Gamma\bar{\gamma})\right]^{\frac{1}{2}}} \left|\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right|,
\end{aligned}
$$

and

$$
\begin{aligned}
(5.31)\quad 0 &\le \int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt \\
&\quad - \left|\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right|^2 \\
&\le \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \left|\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right|^2.
\end{aligned}
$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ above are sharp.

In the case where Γ, γ are positive real numbers, the following corollary incorporating more convenient reverses for the (CBS) integral inequality, may be stated [**1**].

Corollary 50. *Let $f, g \in L^2_\rho([a,b]; K)$ and $M \ge m > 0$. If*

$$(5.32)\quad \operatorname{Re}\langle Mg(t) - f(t), f(t) - mg(t)\rangle \ge 0$$

for a.e. $t \in [a,b]$, or, equivalently,

$$(5.33)\quad \left\|f(t) - \frac{m+M}{2} \cdot g(t)\right\| \le \frac{1}{2}(M-m)\|g(t)\|$$

for a.e. $t \in [a,b]$, *then we have the inequalities*

$$\left(\int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} \leq \frac{1}{2}\cdot\frac{M+m}{\sqrt{mM}}\int_a^b \rho(t)\operatorname{Re}\langle f(t),g(t)\rangle dt, \tag{5.34}$$

$$\begin{aligned} 0 &\leq \left(\int_a^b \rho(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}}\left(\int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} \\ &\quad - \int_a^b \rho(t)\operatorname{Re}\langle f(t),g(t)\rangle dt \\ &\leq \frac{1}{2}\cdot\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{\sqrt{mM}}\int_a^b \rho(t)\operatorname{Re}\langle f(t),g(t)\rangle dt, \end{aligned} \tag{5.35}$$

$$\begin{aligned} 0 &\leq \left(\int_a^b \rho(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}}\left(\int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} \\ &\quad - \left|\int_a^b \rho(t)\langle f(t),g(t)\rangle dt\right| \\ &\leq \frac{1}{2}\cdot\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{\sqrt{mM}}\left|\int_a^b \rho(t)\langle f(t),g(t)\rangle dt\right|, \end{aligned} \tag{5.36}$$

and

$$\begin{aligned} 0 &\leq \int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt \\ &\quad - \left|\int_a^b \rho(t)\langle f(t),g(t)\rangle dt\right|^2 \\ &\leq \frac{1}{4}\cdot\frac{(M-m)^2}{mM}\left|\int_a^b \rho(t)\langle f(t),g(t)\rangle dt\right|^2. \end{aligned} \tag{5.37}$$

The constants $\frac{1}{2}$ *and* $\frac{1}{4}$ *above are best possible.*

On utilising the general result of Theorem 67, we are able to state a number of interesting reverses for the (CBS) inequality in the case when one function takes vector-values while the other is a scalar function [**1**].

Theorem 69 (Dragomir, 2004). *Let* $\alpha \in L_\rho^2([a,b])$, $g \in L_\rho^2([a,b];K)$, $e \in K$, $\|e\| = 1$, $\gamma, \Gamma \in \mathbb{K}$ *with* $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$. *If*

$$\left\| g(t) - \bar{\alpha}(t) \cdot \frac{\Gamma + \gamma}{2} e \right\| \le \frac{1}{2} |\Gamma - \gamma| |\alpha(t)| \tag{5.38}$$

for a.e. $t \in [a,b]$, *or, equivalently*

$$\operatorname{Re} \langle \Gamma \bar{\alpha}(t) e - g(t), g(t) - \gamma \bar{\alpha}(t) e \rangle \ge 0 \tag{5.39}$$

for a.e. $t \in [a,b]$, *(note that, if* $\alpha(t) \ne 0$ *for a.e.* $t \in [a,b]$, *then (5.38) is equivalent to*

$$\left\| \frac{g(t)}{\overline{\alpha(t)}} - \frac{\Gamma + \gamma}{2} e \right\| \le \frac{1}{2} |\Gamma - \gamma| \tag{5.40}$$

for a.e. $t \in [a,b]$, *and (5.39) is equivalent to*

$$\operatorname{Re} \left\langle \Gamma e - \frac{g(t)}{\overline{\alpha(t)}}, \frac{g(t)}{\overline{\alpha(t)}} - \gamma e \right\rangle \ge 0 \tag{5.41}$$

for a.e. $t \in [a,b]$*), then the following reverse inequalities are valid:*

$$\begin{aligned} &\left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} \\ &\le \frac{\operatorname{Re}\left[\left(\bar{\Gamma} + \bar{\gamma} \right) \left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, e \right\rangle \right]}{2 [\operatorname{Re}(\Gamma\bar{\gamma})]^{\frac{1}{2}}} \\ &\le \frac{1}{2} \cdot \frac{|\Gamma + \gamma|}{[\operatorname{Re}(\Gamma\bar{\gamma})]^{\frac{1}{2}}} \left\| \int_a^b \rho(t) \alpha(t) g(t) dt \right\|; \end{aligned} \tag{5.42}$$

$$\begin{aligned} 0 &\le \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} \\ &\qquad - \left\| \int_a^b \rho(t) \alpha(t) g(t) dt \right\| \\ &\le \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} \\ &\qquad - \operatorname{Re}\left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, e \right\rangle \right] \end{aligned} \tag{5.43}$$

$$\leq \frac{|\Gamma-\gamma|^2}{2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\left(|\Gamma+\gamma|+2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right)} \times \operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\left\langle\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, e\right\rangle\right]$$

$$\leq \frac{|\Gamma-\gamma|^2}{2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\left(|\Gamma+\gamma|+2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right)}\left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|;$$

$$\int_a^b \rho(t)\,|\alpha(t)|^2\,dt \int_a^b \rho(t)\,\|g(t)\|^2\,dt \leq \frac{1}{4}\cdot\frac{1}{\operatorname{Re}(\Gamma\bar{\gamma})}\left[\operatorname{Re}\left(\left(\overline{\Gamma}+\overline{\gamma}\right)\left\langle\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, e\right\rangle\right)\right]^2 \leq \frac{1}{4}\cdot\frac{|\Gamma+\gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})}\left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|^2 \tag{5.44}$$

and

$$0 \leq \int_a^b \rho(t)\,|\alpha(t)|^2\,dt \int_a^b \rho(t)\,\|g(t)\|^2\,dt - \left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|^2 \tag{5.45}$$
$$\leq \int_a^b \rho(t)\,|\alpha(t)|^2\,dt \int_a^b \rho(t)\,\|g(t)\|^2\,dt - \left[\operatorname{Re}\left(\frac{\overline{\Gamma}+\overline{\gamma}}{|\Gamma+\gamma|}\left\langle\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, e\right\rangle\right)\right]^2$$
$$\leq \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|^2\operatorname{Re}(\Gamma\bar{\gamma})} \times \left[\operatorname{Re}\left(\left(\overline{\Gamma}+\overline{\gamma}\right)\left\langle\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, e\right\rangle\right)\right]^2$$
$$\leq \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})}\left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|^2.$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ above are sharp.

In the particular case of positive constants, the following simpler version of the above inequalities may be stated.

Corollary 51. *Let* $\alpha \in L^2_\rho([a,b]) \setminus \{0\}$, $g \in L^2_\rho([a,b];K)$, $e \in K$, $\|e\| = 1$ *and* $M, m \in \mathbb{R}$ *with* $M \geq m > 0$. *If*

$$\left\| \frac{g(t)}{\bar{\alpha}(t)} - \frac{M+m}{2} \cdot e \right\| \leq \frac{1}{2}(M-m) \tag{5.46}$$

for a.e. $t \in [a,b]$, *or, equivalently,*

$$\operatorname{Re}\left\langle Me - \frac{g(t)}{\bar{\alpha}(t)}, \frac{g(t)}{\bar{\alpha}(t)} - me \right\rangle \geq 0 \tag{5.47}$$

for a.e. $t \in [a,b]$, *then we have*

$$\begin{aligned}
&\left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} \\
&\leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{Mm}} \operatorname{Re}\left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, e \right\rangle \\
&\leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{Mm}} \left\| \int_a^b \rho(t) \alpha(t) g(t) dt \right\|;
\end{aligned} \tag{5.48}$$

$$\begin{aligned}
0 &\leq \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} \\
&\qquad - \left\| \int_a^b \rho(t) \alpha(t) g(t) dt \right\| \\
&\leq \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} \\
&\qquad - \operatorname{Re}\left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, e \right\rangle \\
&\leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{Mm}} \operatorname{Re}\left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, e \right\rangle \\
&\leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{Mm}} \left\| \int_a^b \rho(t) \alpha(t) g(t) dt \right\|
\end{aligned} \tag{5.49}$$

$$
\begin{aligned}
(5.50) \qquad 0 &\leq \int_a^b \rho(t) |\alpha(t)|^2 \, dt \int_a^b \rho(t) \|g(t)\|^2 \, dt \\
&\leq \frac{1}{4} \cdot \frac{(M+m)^2}{Mm} \left[\operatorname{Re} \left\langle \int_a^b \rho(t) \alpha(t) g(t) \, dt, e \right\rangle \right]^2 \\
&\leq \frac{1}{4} \cdot \frac{(M+m)^2}{Mm} \left\| \int_a^b \rho(t) \alpha(t) g(t) \, dt \right\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
(5.51) \qquad 0 &\leq \int_a^b \rho(t) |\alpha(t)|^2 \, dt \int_a^b \rho(t) \|g(t)\|^2 \, dt \\
&\qquad - \left\| \int_a^b \rho(t) \alpha(t) g(t) \, dt \right\|^2 \\
&\leq \int_a^b \rho(t) |\alpha(t)|^2 \, dt \int_a^b \rho(t) \|g(t)\|^2 \, dt \\
&\qquad - \left[\operatorname{Re} \left\langle \int_a^b \rho(t) \alpha(t) g(t) \, dt, e \right\rangle \right]^2 \\
&\leq \frac{1}{4} \cdot \frac{(M-m)^2}{Mm} \left[\operatorname{Re} \left\langle \int_a^b \rho(t) \alpha(t) g(t) \, dt, e \right\rangle \right]^2 \\
&\leq \frac{1}{4} \cdot \frac{(M-m)^2}{Mm} \left\| \int_a^b \rho(t) \alpha(t) g(t) \, dt \right\|^2 .
\end{aligned}
$$

The constants $\frac{1}{2}$ *and* $\frac{1}{4}$ *above are sharp.*

5.2.3. Reverses of the Heisenberg Inequality. It is well known that if $(H; \langle\cdot,\cdot\rangle)$ is a real or complex Hilbert space and $f : [a,b] \subset \mathbb{R} \to H$ is an *absolutely continuous vector-valued* function, then f is differentiable almost everywhere on $[a,b]$, the derivative $f' : [a,b] \to H$ is Bochner integrable on $[a,b]$ and

$$
(5.52) \qquad f(t) = \int_a^t f'(s) \, ds \qquad \text{for any } \; t \in [a,b].
$$

The following theorem provides a version of the Heisenberg inequalities in the general setting of Hilbert spaces [**1**].

Theorem 70 (Dragomir, 2004). *Let* $\varphi : [a,b] \to H$ *be an absolutely continuous function with the property that* $b \|\varphi(b)\|^2 = a \|\varphi(a)\|^2$. *Then we have the inequality:*

$$
(5.53) \qquad \left(\int_a^b \|\varphi(t)\|^2 \, dt \right)^2 \leq 4 \int_a^b t^2 \|\varphi(t)\|^2 \, dt \cdot \int_a^b \|\varphi'(t)\|^2 \, dt.
$$

The constant 4 is best possible in the sense that it cannot be replaced by a smaller quantity.

PROOF. Integrating by parts, we have successively

$$\begin{aligned}
(5.54)\qquad & \int_a^b \|\varphi(t)\|^2 \, dt \\
&= t\|\varphi(t)\|^2 \Big|_a^b - \int_a^b t \frac{d}{dt}\left(\|\varphi(t)\|^2\right) dt \\
&= b\|\varphi(b)\|^2 - a\|\varphi(a)\|^2 - \int_a^b t \frac{d}{dt}\langle \varphi(t), \varphi(t)\rangle \, dt \\
&= -\int_a^b t\left[\langle \varphi'(t), \varphi(t)\rangle + \langle \varphi(t), \varphi'(t)\rangle\right] dt \\
&= -2\int_a^b t \operatorname{Re}\langle \varphi'(t), \varphi(t)\rangle \, dt \\
&= 2\int_a^b \operatorname{Re}\langle \varphi'(t), (-t)\varphi(t)\rangle \, dt.
\end{aligned}$$

If we apply the (CBS) integral inequality

$$\int_a^b \operatorname{Re}\langle g(t), h(t)\rangle \, dt \le \left(\int_a^b \|g(t)\|^2 \, dt \int_a^b \|h(t)\|^2 \, dt\right)^{\frac{1}{2}}$$

for $g(t) = \varphi'(t)$, $h(t) = -t\varphi(t)$, $t \in [a,b]$, then we deduce the desired inequality (5.53).

The fact that 4 is the best possible constant in (5.53) follows from the fact that in the (CBS) inequality, the case of equality holds iff $g(t) = \lambda h(t)$ for a.e. $t \in [a,b]$ and λ a given scalar in $\mathbb{K}$. We omit the details. ∎

For details on the classical Heisenberg inequality, see, for instance, [**4**].

The following reverse of the Heisenberg type inequality (5.53) holds [**1**].

THEOREM 71 (Dragomir, 2004). *Assume that* $\varphi : [a,b] \to H$ *is as in the hypothesis of Theorem 70. In addition, if there exists a* $r > 0$ *such that*

$$(5.55)\qquad \|\varphi'(t) - t\varphi(t)\| \le r \le \|\varphi'(t)\|$$

for a.e. $t \in [a,b]$, *then we have the inequalities*

$$0 \leq \int_a^b t^2 \|\varphi(t)\|^2 dt \int_a^b \|\varphi'(t)\|^2 dt - \frac{1}{4}\left(\int_a^b \|\varphi(t)\|^2 dt\right)^2 \leq r^2 \int_a^b t^2 \|\varphi(t)\|^2 dt. \tag{5.56}$$

PROOF. We observe, by the identity (5.54), that

$$\frac{1}{4}\left(\int_a^b \|\varphi(t)\|^2 dt\right)^2 = \left(\int_a^b \operatorname{Re}\langle \varphi'(t), t\varphi(t)\rangle dt\right)^2. \tag{5.57}$$

Now, if we apply Theorem 66 for the choices $f(t) = t\varphi(t)$, $g(t) = \varphi'(t)$, and $\rho(t) = \frac{1}{b-a}$, then by (5.4) and (5.57) we deduce the desired inequality (5.56). ∎

REMARK 60. *Interchanging the place of* $t\varphi(t)$ *with* $\varphi'(t)$ *in Theorem 71, we also have*

$$0 \leq \int_a^b t^2 \|\varphi(t)\|^2 dt \int_a^b \|\varphi'(t)\|^2 dt - \frac{1}{4}\left(\int_a^b \|\varphi(t)\|^2 dt\right)^2 \leq \rho^2 \int_a^b \|\varphi'(t)\|^2 dt, \tag{5.58}$$

provided

$$\|\varphi'(t) - t\varphi(t)\| \leq \rho \leq |t| \|\varphi(t)\|$$

for a.e. $t \in [a,b]$, *where* $\rho > 0$ *is a given positive number.*

The following result also holds [**1**].

THEOREM 72 (Dragomir, 2004). *Assume that* $\varphi : [a,b] \to H$ *is as in the hypothesis of Theorem 70. In addition, if there exists* $M \geq m > 0$ *such that*

$$\operatorname{Re}\langle Mt\varphi(t) - \varphi'(t), \varphi'(t) - mt\varphi(t)\rangle \geq 0 \tag{5.59}$$

for a.e. $t \in [a,b]$, *or, equivalently,*

$$\left\|\varphi'(t) - \frac{M+m}{2} t\varphi(t)\right\| \leq \frac{1}{2}(M-m)|t| \|\varphi(t)\| \tag{5.60}$$

for a.e. $t \in [a,b]$, *then we have the inequalities*

$$\int_a^b t^2 \|\varphi(t)\|^2 dt \int_a^b \|\varphi'(t)\|^2 dt \leq \frac{1}{16} \cdot \frac{(M+m)^2}{Mm}\left(\int_a^b \|\varphi(t)\|^2 dt\right)^2 \tag{5.61}$$

and

$$\int_a^b t^2 \|\varphi(t)\|^2 dt \int_a^b \|\varphi'(t)\|^2 dt - \frac{1}{4}\left(\int_a^b \|\varphi(t)\|^2 dt\right)^2 \leq \frac{1}{16}\cdot\frac{(M-m)^2}{Mm}\left(\int_a^b \|\varphi(t)\|^2 dt\right)^2 \tag{5.62}$$

respectively.

Proof. We use Corollary 50 for the choices $f(t) = \varphi'(t)$, $g(t) = t\varphi(t)$, $\rho(t) = \frac{1}{b-a}$, to get

$$\int_a^b \|\varphi'(t)\|^2 dt \int_a^b t^2 \|\varphi(t)\|^2 dt \leq \frac{(M+m)^2}{4Mm}\left(\int_a^b \operatorname{Re}\langle \varphi'(t), t\varphi(t)\rangle dt\right)^2.$$

Since, by (5.57)

$$\left(\int_a^b \operatorname{Re}\langle \varphi'(t), t\varphi(t)\rangle dt\right)^2 = \frac{1}{4}\left(\int_a^b \|\varphi(t)\|^2 dt\right)^2,$$

hence we deduce the desired result (5.61).

The inequality (5.62) follows from (5.61), and we omit the details. ∎

Remark 61. *If one is interested in reverses for the Heisenberg inequality for scalar valued functions, then all the other inequalities obtained above for one scalar function may be applied as well. For the sake of brevity, we do not list them here.*

5.3. Other Reverses

5.3.1. The General Case. The following result holds [**3**].

Theorem 73 (Dragomir, 2004). *Let $f, g \in L_\rho^2([a,b];K)$ and $r > 0$ be such that*

$$\|f(t) - g(t)\| \leq r \tag{5.63}$$

for a.e. $t \in [a,b]$. *Then we have the inequalities:*

$$0 \le \left(\int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} - \left|\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right| \tag{5.64}$$
$$\le \left(\int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} - \left|\int_a^b \rho(t)\operatorname{Re}\langle f(t), g(t)\rangle dt\right|$$
$$\le \left(\int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} - \int_a^b \rho(t)\operatorname{Re}\langle f(t), g(t)\rangle dt$$
$$\le \frac{1}{2}r^2.$$

The constant $\frac{1}{2}$ *in front of* r^2 *is best possible in the sense that it cannot be replaced by a smaller quantity.*

Proof. We will use the following result obtained in [**2**]:

In the inner product space $(H; \langle\cdot,\cdot\rangle)$, if $x, y \in H$ and $r > 0$ are such that $\|x - y\| \le r$, then

$$0 \le \|x\|\|y\| - |\langle x, y\rangle| \le \|x\|\|y\| - |\operatorname{Re}\langle x, y\rangle| \tag{5.65}$$
$$\le \|x\|\|y\| - \operatorname{Re}\langle x, y\rangle \le \frac{1}{2}r^2.$$

The constant $\frac{1}{2}$ in front of r^2 is best possible in the sense that it cannot be replaced by a smaller constant.

If (5.63) holds true, then

$$\|f - g\|_\rho^2 = \int_a^b \rho(t)\|f(t) - g(t)\|^2 dt \le r^2 \int_a^b \rho(t)\, dt = r^2$$

and thus $\|f - g\|_\rho \le r$.

Applying the inequality (5.65) for $\left(L_\rho^2([a,b];K), \langle\cdot,\cdot\rangle_p\right)$, we deduce the desired inequality (5.64).

If we choose $\rho(t) = \frac{1}{b-a}$, $f(t) = x$, $g(t) = y$, $x, y \in K$, $t \in [a,b]$, then from (5.64) we recapture (5.65) for which the constant $\frac{1}{2}$ in front of r^2 is best possible. ∎

We next point out some general reverse inequalities for the second CBS inequality (5.2)[**3**].

THEOREM 74 (Dragomir, 2004). *Let* $\alpha \in L^2_\rho([a,b])$, $g \in L^2_\rho([a,b];K)$ *and* $v \in K$, $r > 0$. *If*

$$\left\| \frac{g(t)}{\alpha(t)} - v \right\| \le r \tag{5.66}$$

for a.e. $t \in [a,b]$, *then we have the inequality*

$$\begin{aligned} 0 &\le \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} \\ &\qquad - \left\| \int_a^b \rho(t) \alpha(t) g(t) dt \right\| \\ &\le \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} \\ &\qquad - \left| \left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, \frac{v}{\|v\|} \right\rangle \right| \\ &\le \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} \\ &\qquad - \left| \operatorname{Re} \left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, \frac{v}{\|v\|} \right\rangle \right| \\ &\le \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} \\ &\qquad - \operatorname{Re} \left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, \frac{v}{\|v\|} \right\rangle \\ &\le \frac{1}{2} \cdot \frac{r^2}{\|v\|} \int_a^b \rho(t) |\alpha(t)|^2 dt. \end{aligned} \tag{5.67}$$

The constant $\frac{1}{2}$ *is best possible in the sense that it cannot be replaced by a smaller quantity.*

PROOF. From (5.66) we deduce

$$\|g(t)\|^2 - 2 \operatorname{Re} \langle \alpha(t) g(t), v \rangle + |\alpha(t)|^2 \|v\|^2 \le r^2 |\alpha(t)|^2$$

which is clearly equivalent to

$$\|g(t)\|^2 + |\alpha(t)|^2 \|v\|^2 \le 2 \operatorname{Re} \langle \alpha(t) g(t), v \rangle + r^2 |\alpha(t)|^2 . \tag{5.68}$$

If we multiply (5.68) by $\rho(t) \geq 0$ and integrate over $t \in [a, b]$, then we deduce

$$\begin{aligned} (5.69) \quad & \int_a^b \rho(t) \|g(t)\|^2 dt + \|v\|^2 \int_a^b \rho(t) |\alpha(t)|^2 dt \\ & \leq 2 \operatorname{Re} \left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, v \right\rangle + r^2 \int_a^b \rho(t) |\alpha(t)|^2 dt. \end{aligned}$$

Since, obviously

$$\begin{aligned} (5.70) \quad & 2\|v\| \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} \\ & \leq \int_a^b \rho(t) \|g(t)\|^2 dt + \|v\|^2 \int_a^b \rho(t) |\alpha(t)|^2 dt, \end{aligned}$$

hence, by (5.69) and (5.70), we deduce

$$\begin{aligned} & 2\|v\| \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} \\ & \leq 2 \operatorname{Re} \left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, v \right\rangle + r^2 \int_a^b \rho(t) |\alpha(t)|^2 dt, \end{aligned}$$

which is clearly equivalent with the last inequality in (5.67).

The other inequalities are obvious and we omit the details.

Now, if $\rho(t) = \frac{1}{b-a}$, $\alpha(t) = 1$, $g(t) = x$, $x \in K$, then, by the last inequality in (5.67) we get

$$\|x\| \|v\| - \operatorname{Re} \langle x, v \rangle \leq \frac{1}{2} r^2,$$

provided $\|x - v\| \leq r$, for which we know that (see [**2**]), the constant $\frac{1}{2}$ is best possible. ∎

5.3.2. Some Particular Cases of Interest. It has been shown in [**2**] that, for $\gamma, \Gamma \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$) with $\Gamma \neq -\gamma$ and $x, y \in H$, $(H; \langle \cdot, \cdot \rangle)$ is an inner product over the real or complex number field $\mathbb{K}$, such that either

$$(5.71) \qquad \operatorname{Re} \langle \Gamma y - x, x - \gamma y \rangle \geq 0,$$

or, equivalently,

$$(5.72) \qquad \left\| x - \frac{\gamma + \Gamma}{2} \cdot y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|y\|,$$

holds, then one has the following reverse of Schwarz's inequality

$$
\begin{aligned}
0 &\le \|x\| \|y\| - |\langle x, y\rangle| \\
&\le \|x\| \|y\| - \left|\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\langle x, y\rangle\right]\right| \\
&\le \|x\| \|y\| - \operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\langle x, y\rangle\right] \\
&\le \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\|y\|^2 .
\end{aligned} \tag{5.73}
$$

The constant $\frac{1}{4}$ is best possible in (5.73) in the sense that it cannot be replaced by a smaller constant.

If we assume that $\Gamma = M$, $\gamma = m$ with $M \ge m > 0$, then from (5.73) we deduce the much simpler and more useful result

$$
\begin{aligned}
0 &\le \|x\| \|y\| - |\langle x, y\rangle| \le \|x\| \|y\| - |\operatorname{Re}\langle x, y\rangle| \\
&\le \|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \le \frac{1}{4}\cdot\frac{(M-m)^2}{Mm}\|y\|^2 ,
\end{aligned} \tag{5.74}
$$

provided (5.71) or (5.72) holds true with M and m instead of Γ and γ.

Using the above inequalities for vectors in inner product spaces, we are able to state the following theorem concerning reverses of the CBS integral inequality for vector-valued functions in Hilbert spaces [**3**].

Theorem 75 (Dragomir, 2004). *Let $f, g \in L_\rho^2([a,b];K)$ and $\gamma, \Gamma \in \mathbb{K}$ with $\Gamma \ne -\gamma$. If*

$$
\operatorname{Re}\langle \Gamma g(t) - f(t), f(t) - \gamma g(t)\rangle \ge 0 \tag{5.75}
$$

for a.e. $t \in [a,b]$, or, equivalently,

$$
\left\| f(t) - \frac{\gamma+\Gamma}{2}\cdot g(t)\right\| \le \frac{1}{2}|\Gamma-\gamma| \|g(t)\| \tag{5.76}
$$

for a.e. $t \in [a,b]$, then we have the inequalities

$$
\begin{aligned}
0 \le &\left(\int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} \\
&- \left|\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right|
\end{aligned} \tag{5.77}
$$

$$\leq \left(\int_a^b \rho(t)\|f(t)\|^2\,dt \int_a^b \rho(t)\|g(t)\|^2\,dt\right)^{\frac{1}{2}} - \left|\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\int_a^b \rho(t)\langle f(t), g(t)\rangle\,dt\right]\right|$$

$$\leq \left(\int_a^b \rho(t)\|f(t)\|^2\,dt \int_a^b \rho(t)\|g(t)\|^2\,dt\right)^{\frac{1}{2}} - \operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\int_a^b \rho(t)\langle f(t), g(t)\rangle\,dt\right]$$

$$\leq \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\int_a^b \rho(t)\|g(t)\|^2\,dt.$$

The constant $\frac{1}{4}$ is best possible in (5.77).

Proof. Since, by (5.75),

$$\operatorname{Re}\langle \Gamma g - f, f-\gamma g\rangle_\rho = \int_a^b \rho(t)\operatorname{Re}\langle \Gamma g(t) - f(t), f(t)-\gamma g(t)\rangle\,dt \geq 0,$$

hence, by (5.73) applied for the Hilbert space $\left(L_\rho^2([a,b];K);\langle\cdot,\cdot\rangle_\rho\right)$, we deduce the desired inequality (5.77).

The best constant follows by the fact that $\frac{1}{4}$ is a best constant in (5.77) and we omit the details. ∎

Corollary 52. *Let $f,g\in L_\rho^2([a,b];K)$ and $M\geq m>0$. If*

$$\operatorname{Re}\langle Mg(t)-f(t), f(t)-mg(t)\rangle \geq 0 \tag{5.78}$$

for a.e. $t\in[a,b]$, or, equivalently,

$$\left\|f(t)-\frac{m+M}{2}\cdot g(t)\right\| \leq \frac{1}{2}(M-m)\|g(t)\| \tag{5.79}$$

for a.e. $t\in[a,b]$, then

$$0 \leq \left(\int_a^b \rho(t)\|f(t)\|^2\,dt \int_a^b \rho(t)\|g(t)\|^2\,dt\right)^{\frac{1}{2}} - \left|\int_a^b \rho(t)\langle f(t), g(t)\rangle\,dt\right| \tag{5.80}$$

$$\leq \left(\int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} - \left|\int_a^b \rho(t)\operatorname{Re}\langle f(t), g(t)\rangle dt\right|$$

$$\leq \left(\int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} - \int_a^b \rho(t)\operatorname{Re}\langle f(t), g(t)\rangle dt$$

$$\leq \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\int_a^b \rho(t)\|g(t)\|^2 dt.$$

The constant $\frac{1}{4}$ *is best possible.*

The case when a function is scalar is incorporated in the following theorem [**3**].

THEOREM 76 (Dragomir, 2004). *Let* $\alpha \in L_\rho^2([a,b])$, $g \in L_\rho^2([a,b];K)$, *and* $\gamma, \Gamma \in \mathbb{K}$ *with* $\Gamma \neq -\gamma$. *If* $e \in K$, $\|e\| = 1$ *and*

$$\left\|\frac{g(t)}{\overline{\alpha(t)}} - \frac{\Gamma+\gamma}{2}e\right\| \leq \frac{1}{2}|\Gamma-\gamma| \tag{5.81}$$

for a.e. $t \in [a,b]$, *or, equivalently,*

$$\operatorname{Re}\left\langle \Gamma e - \frac{g(t)}{\overline{\alpha(t)}}, \frac{g(t)}{\overline{\alpha(t)}} - \gamma e\right\rangle \geq 0 \tag{5.82}$$

for a.e. $t \in [a,b]$, *then we have the inequalities*

$$0 \leq \left(\int_a^b \rho(t)|\alpha(t)|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} - \left\|\int_a^b \rho(t)\alpha(t)g(t)dt\right\| \tag{5.83}$$

$$\leq \left(\int_a^b \rho(t)|\alpha(t)|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} - \left|\left\langle\int_a^b \rho(t)\alpha(t)g(t)dt, e\right\rangle\right|$$

$$\leq \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} - \left| \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, e \right\rangle \right] \right|$$

$$\leq \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} - \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, e \right\rangle \right]$$

$$\leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \int_a^b \rho(t) |\alpha(t)|^2 dt.$$

The constant $\frac{1}{4}$ is best possible in (5.83).

Proof. Follows by Theorem 74 on choosing

$$v := \frac{\Gamma + \gamma}{2} e \qquad \text{and} \qquad r := \frac{1}{2} |\Gamma - \gamma| .$$

We omit the details. ∎

Corollary 53. *Let $\alpha \in L_\rho^2([a,b])$, $g \in L_\rho^2([a,b];K)$, and $M \geq m > 0$. If $e \in K$, $\|e\| = 1$ and*

$$\left\| \frac{g(t)}{\overline{\alpha(t)}} - \frac{M+m}{2} \cdot e \right\| \leq \frac{1}{2}(M-m)$$

for a.e. $t \in [a,b]$, or, equivalently,

$$\operatorname{Re} \left\langle Me - \frac{g(t)}{\overline{\alpha(t)}}, \frac{g(t)}{\overline{\alpha(t)}} - me \right\rangle \geq 0$$

for a.e. $t \in [a,b]$, then we have the inequalities:

$$0 \leq \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} - \left\| \int_a^b \rho(t) \alpha(t) g(t) dt \right\| \tag{5.84}$$

$$\leq \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} - \left| \left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, e \right\rangle \right|$$

$$\leq \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} - \left| \operatorname{Re} \left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, e \right\rangle \right|$$

$$\leq \left(\int_a^b \rho(t) |\alpha(t)|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} - \operatorname{Re} \left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, e \right\rangle$$

$$\leq \frac{1}{4} \cdot \frac{(M-m)^2}{M+m} \int_a^b \rho(t) |\alpha(t)|^2 dt.$$

The constant $\frac{1}{4}$ is best possible in (5.84).

5.3.3. Applications for the Heisenberg Inequality. The following reverse of the Heisenberg type inequality (5.53) holds [**3**].

THEOREM 77 (Dragomir, 2004). *Assume that $\varphi : [a,b] \to H$ is as in the hypothesis of Theorem 70. In addition, if there exists a $r > 0$ such that*

$$\|\varphi'(t) + t\varphi(t)\| \leq r \tag{5.85}$$

for a.e. $t \in [a,b]$, then we have the inequalities

$$0 \leq \left(\int_a^b t^2 \|\varphi(t)\|^2 dt \int_a^b \|\varphi'(t)\|^2 dt \right)^{\frac{1}{2}} - \frac{1}{2} \int_a^b \|\varphi(t)\|^2 dt \leq \frac{1}{2} r^2 (b-a). \tag{5.86}$$

PROOF. We observe, by the identity (5.54), that

$$\int_a^b \operatorname{Re} \langle \varphi'(t), (-t)\varphi(t) \rangle dt = \frac{1}{2} \int_a^b \|\varphi(t)\|^2 dt. \tag{5.87}$$

Now, if we apply Theorem 73 for the choices $f(t) = t\varphi(t)$, $g(t) = -t\varphi'(t)$, $\rho(t) = \frac{1}{b-a}$, $t \in [a,b]$, then we deduce the desired inequality (5.86). ∎

REMARK 62. *It is interesting to remark that, from (5.87), we obviously have*

$$\frac{1}{2} \int_a^b \|\varphi(t)\|^2 dt = \left| \int_a^b \operatorname{Re} \langle \varphi'(t), t\varphi(t) \rangle dt \right|. \tag{5.88}$$

Now, if we apply the inequality (see (5.64))

$$\int_a^b \|f(t)\|^2 dt \int_a^b \|g(t)\|^2 dt - \left|\int_a^b \operatorname{Re}\langle f(t), g(t)\rangle dt\right| \le \frac{1}{2} r^2 (b-a),$$

for the choices $f(t) = \varphi'(t)$, $g(t) = t\varphi(t)$, $t \in [a,b]$, *then we get the same inequality (5.86), but under the condition*

$$\|\varphi'(t) - t\varphi(t)\| \le r \tag{5.89}$$

for a.e. $t \in [a,b]$.

The following result holds as well [**3**].

Theorem 78 (Dragomir, 2004). *Assume that* $\varphi : [a,b] \to H$ *is as in the hypothesis of Theorem 77. In addition, if there exists* $M \ge m > 0$ *such that*

$$\operatorname{Re}\langle Mt\varphi(t) - \varphi'(t), \varphi'(t) - mt\varphi(t)\rangle \ge 0 \tag{5.90}$$

for a.e. $t \in [a,b]$, *or, equivalently,*

$$\left\|\varphi'(t) - \frac{M+m}{2} t\varphi(t)\right\| \le \frac{1}{2}(M-m)|t| \|\varphi(t)\| \tag{5.91}$$

for a.e. $t \in [a,b]$, *then we have the inequalities*

$$\begin{aligned} 0 &\le \left(\int_a^b t^2 \|\varphi(t)\|^2 dt \int_a^b \|\varphi'(t)\|^2 dt\right)^{\frac{1}{2}} - \frac{1}{2}\int_a^b \|\varphi(t)\|^2 dt \\ &\le \frac{1}{4} \cdot \frac{(M-m)^2}{M+m} \int_a^b t^2 \|\varphi(t)\|^2 dt. \end{aligned} \tag{5.92}$$

Proof. The proof follows by Corollary 52 applied for the function $g(t) = t\varphi(t)$ and $f(t) = \varphi'(t)$, and on making use of the identity (5.88). We omit the details. ∎

Remark 63. *If one is interested in reverses for the Heisenberg inequality for real or complex valued functions, then all the other inequalities obtained above for one scalar and one vectorial function may be applied as well. For the sake of brevity, we do not list them here.*

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CHAPTER 6

Other Inequalities in Inner Product Spaces

6.1. Bounds for the Distance to Finite-Dimensional Subspaces

6.1.1. Introduction. Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $\{y_1,\dots,y_n\}$ a subset of H and $G(y_1,\dots,y_n)$ the *Gram matrix* of $\{y_1,\dots,y_n\}$ where $(i,j)-$entry is $\langle y_i,y_j\rangle$. The determinant of $G(y_1,\dots,y_n)$ is called the *Gram determinant* of $\{y_1,\dots,y_n\}$ and is denoted by $\Gamma(y_1,\dots,y_n)$. Thus,

$$\Gamma(y_1,\dots,y_n)=\begin{vmatrix}\langle y_1,y_1\rangle & \langle y_1,y_2\rangle & \cdots & \langle y_1,y_n\rangle\\ \langle y_2,y_1\rangle & \langle y_2,y_2\rangle & \cdots & \langle y_2,y_n\rangle\\ \cdots & \cdots & \cdots & \cdots\\ \langle y_n,y_1\rangle & \langle y_n,y_2\rangle & \cdots & \langle y_n,y_n\rangle\end{vmatrix}.$$

Following [**4**, p. 129 – 133], we state here some general results for the Gram determinant that will be used in the sequel.

(1) Let $\{x_1,\dots,x_n\}\subset H$. Then $\Gamma(x_1,\dots,x_n)\neq 0$ if and only if $\{x_1,\dots,x_n\}$ is linearly independent;
(2) Let $M=span\{x_1,\dots,x_n\}$ be $n-$dimensional in H, i.e., $\{x_1,\dots,x_n\}$ is linearly independent. Then for each $x\in H$, the distance $d(x,M)$ from x to the linear subspace H has the representations

$$d^2(x,M)=\frac{\Gamma(x_1,\dots,x_n,x)}{\Gamma(x_1,\dots,x_n)} \tag{6.1}$$

and

$$d^2(x,M)=\|x\|^2-\beta^T G^{-1}\beta, \tag{6.2}$$

where $G=G(x_1,\dots,x_n)$, G^{-1} is the inverse matrix of G and

$$\beta^T=(\langle x,x_1\rangle,\langle x,x_2\rangle,\dots,\langle x,x_n\rangle),$$

denotes the transpose of the column vector β.

Moreover, one has the simpler representation

$$d^2(x,M) = \begin{cases} \|x\|^2 - \dfrac{\left(\sum_{i=1}^n |\langle x, x_i\rangle|^2\right)^2}{\left\|\sum_{i=1}^n \langle x, x_i\rangle x_i\right\|^2} & \text{if } x \notin M^\perp, \\ \\ \|x\|^2 & \text{if } x \in M^\perp, \end{cases} \tag{6.3}$$

where $M^\perp$ denotes the orthogonal complement of M.

(3) Let $\{x_1, \dots, x_n\}$ be a set of nonzero vectors in H. Then

$$0 \le \Gamma(x_1, \dots, x_n) \le \|x_1\|^2 \|x_2\|^2 \cdots \|x_n\|^2 . \tag{6.4}$$

The equality holds on the left (respectively right) side of (6.4) if and only if $\{x_1, \dots, x_n\}$ is linearly dependent (respectively orthogonal). The first inequality in (6.4) is known in the literature as *Gram's inequality* while the second one is known as *Hadamard's inequality.*

(4) If $\{x_1, \dots, x_n\}$ is an orthonormal set in H, i.e., $\langle x_i, x_j\rangle = \delta_{ij}$, $i, j \in \{1, \dots, n\}$, where δ_{ij} is Kronecker's delta, then

$$d^2(x,M) = \|x\|^2 - \sum_{i=1}^n |\langle x, x_i\rangle|^2 . \tag{6.5}$$

The following inequalities which involve Gram determinants may be stated as well [**17**, p. 597]:

$$\frac{\Gamma(x_1, \dots, x_n)}{\Gamma(x_1, \dots, x_k)} \le \frac{\Gamma(x_2, \dots, x_n)}{\Gamma(x_1, \dots, x_k)} \le \cdots \le \Gamma(x_{k+1}, \dots, x_n), \tag{6.6}$$

$$\Gamma(x_1, \dots, x_n) \le \Gamma(x_1, \dots, x_k)\, \Gamma(x_{k+1}, \dots, x_n) \tag{6.7}$$

and

$$\begin{aligned} \Gamma^{\frac{1}{2}}(x_1 + y_1, x_2, \dots, x_n) \\ \le \Gamma^{\frac{1}{2}}(x_1, x_2, \dots, x_n) + \Gamma^{\frac{1}{2}}(y_1, x_2, \dots, x_n). \end{aligned} \tag{6.8}$$

The main aim of this section is to point out some upper bounds for the distance $d(x,M)$ in terms of the linearly independent vectors $\{x_1, \dots, x_n\}$ that span M and $x \notin M^\perp$, where $M^\perp$ is the orthogonal complement of M in the inner product space $(H; \langle\cdot,\cdot\rangle)$.

As a by-product of this endeavour, some refinements of the generalisations for Bessel's inequality due to several authors including: Boas, Bellman and Bombieri are obtained. Refinements for the well known Hadamard's inequality for Gram determinants are also derived.

6.1.2. Upper Bounds for $d(x,M)$. The following result may be stated [**16**].

THEOREM 79 (Dragomir, 2005). *Let $\{x_1,\dots,x_n\}$ be a linearly independent system of vectors in H and $M := span\{x_1,\dots,x_n\}$. If $x \notin M^{\perp}$, then*

$$d^2(x,M) < \frac{\|x\|^2 \sum_{i=1}^n \|x_i\|^2 - \sum_{i=1}^n |\langle x,x_i\rangle|^2}{\sum_{i=1}^n \|x_i\|^2} \tag{6.9}$$

or, equivalently,

$$\Gamma(x_1,\dots,x_n,x) < \frac{\|x\|^2 \sum_{i=1}^n \|x_i\|^2 - \sum_{i=1}^n |\langle x,x_i\rangle|^2}{\sum_{i=1}^n \|x_i\|^2} \cdot \Gamma(x_1,\dots,x_n). \tag{6.10}$$

PROOF. If we use the Cauchy-Bunyakovsky-Schwarz type inequality

$$\left\|\sum_{i=1}^n \alpha_i y_i\right\|^2 \le \sum_{i=1}^n |\alpha_i|^2 \sum_{i=1}^n \|y_i\|^2, \tag{6.11}$$

that can be easily deduced from the obvious identity

$$\sum_{i=1}^n |\alpha_i|^2 \sum_{i=1}^n \|y_i\|^2 - \left\|\sum_{i=1}^n \alpha_i y_i\right\|^2 = \frac{1}{2}\sum_{i,j=1}^n \|\overline{\alpha_i} x_j - \overline{\alpha_j} x_i\|^2, \tag{6.12}$$

we can state that

$$\left\|\sum_{i=1}^n \langle x,x_i\rangle x_i\right\|^2 \le \sum_{i=1}^n |\langle x,x_i\rangle|^2 \sum_{i=1}^n \|x_i\|^2. \tag{6.13}$$

Note that the equality case holds in (6.13) if and only if, by (6.12),

$$\overline{\langle x,x_i\rangle} x_j = \overline{\langle x,x_i\rangle} x_i \tag{6.14}$$

for each $i,j \in \{1,\dots,n\}$.

Utilising the expression (6.3) of the distance $d(x,M)$, we have

$$d^2(x,M) = \|x\|^2 - \frac{\sum_{i=1}^n |\langle x,x_i\rangle|^2 \sum_{i=1}^n \|x_i\|^2}{\left\|\sum_{i=1}^n \langle x,x_i\rangle x_i\right\|^2} \cdot \frac{\sum_{i=1}^n |\langle x,x_i\rangle|^2}{\sum_{i=1}^n \|x_i\|^2}. \tag{6.15}$$

Since $\{x_1,\dots,x_n\}$ are linearly independent, hence (6.14) cannot be achieved and then we have strict inequality in (6.13).

Finally, on using (6.13) and (6.15) we get the desired result (6.9). ∎

Remark 64. *It is known that (see (6.4)) if not all $\{x_1,\dots,x_n\}$ are orthogonal on each other, then the following result, which is well known in the literature as Hadamard's inequality holds:*

$$\Gamma\left(x_1,\dots,x_n\right) < \left\|x_1\right\|^2 \left\|x_2\right\|^2 \cdots \left\|x_n\right\|^2 . \tag{6.16}$$

Utilising the inequality (6.10), we may write successively:

$$\begin{aligned}
\Gamma\left(x_1,x_2\right) &\le \frac{\left\|x_1\right\|^2 \left\|x_2\right\|^2 - \left|\langle x_2,x_1\rangle\right|^2}{\left\|x_1\right\|^2}\left\|x_1\right\|^2 \le \left\|x_1\right\|^2 \left\|x_2\right\|^2 ,\\
\Gamma\left(x_1,x_2,x_3\right) &< \frac{\left\|x_3\right\|^2 \sum_{i=1}^{2}\left\|x_i\right\|^2 - \sum_{i=1}^{2}\left|\langle x_3,x_i\rangle\right|^2}{\sum_{i=1}^{2}\left\|x_i\right\|^2}\Gamma\left(x_1,x_2\right)\\
&\le \left\|x_3\right\|^2 \Gamma\left(x_1,x_2\right)\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
\Gamma\left(x_1,\dots,x_{n-1},x_n\right) &< \frac{\left\|x_n\right\|^2 \sum_{i=1}^{n-1}\left\|x_i\right\|^2 - \sum_{i=1}^{n-1}\left|\langle x_n,x_i\rangle\right|^2}{\sum_{i=1}^{n-1}\left\|x_i\right\|^2}\\
&\qquad\times \Gamma\left(x_1,\dots,x_{n-1}\right)\\
&\le \left\|x_n\right\|^2 \Gamma\left(x_1,\dots,x_{n-1}\right).
\end{aligned}$$

Multiplying the above inequalities, we deduce

$$\begin{aligned}
&\Gamma\left(x_1,\dots,x_{n-1},x_n\right)\\
&< \left\|x_1\right\|^2 \prod_{k=2}^{n}\left(\left\|x_k\right\|^2 - \frac{1}{\sum_{i=1}^{k-1}\left\|x_i\right\|^2}\sum_{i=1}^{k-1}\left|\langle x_k,x_i\rangle\right|^2\right)\\
&\le \prod_{j=1}^{n}\left\|x_j\right\|^2 ,
\end{aligned} \tag{6.17}$$

valid for a system of $n \ge 2$ linearly independent vectors which are not orthogonal on each other.

In [**15**], the author has obtained the following inequality.

Lemma 8 (Dragomir, 2004). *Let $z_1,\dots,z_n \in H$ and $\alpha_1,\dots,\alpha_n \in \mathbb{K}$. Then one has the inequalities:*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \tag{6.18}$$

$$
\leq \begin{cases}
\max\limits_{1\leq i\leq n} |\alpha_i|^2 \sum\limits_{i=1}^{n} \|z_i\|^2 ; \\[2ex]
\left(\sum\limits_{i=1}^{n} |\alpha_i|^{2\alpha}\right)^{\frac{1}{\alpha}} \left(\sum\limits_{i=1}^{n} \|z_i\|^{2\beta}\right)^{\frac{1}{p}} \\
\quad \textit{where} \quad \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\[2ex]
\sum\limits_{i=1}^{n} |\alpha_i|^2 \max\limits_{1\leq i\leq n} \|z_i\|^2 ;
\end{cases}
$$

$$
+ \begin{cases}
\max\limits_{1\leq i\neq j\leq n} \{|\alpha_i\alpha_j|\} \sum\limits_{1\leq i\neq j\leq n} |\langle z_i, z_j\rangle| ; \\[2ex]
\left[\left(\sum\limits_{i=1}^{n} |\alpha_i|^{\gamma}\right)^2 - \sum\limits_{i=1}^{n} |\alpha_i|^{2\gamma}\right]^{\frac{1}{\gamma}} \left(\sum\limits_{1\leq i\neq j\leq n} |\langle z_i, z_j\rangle|^{\delta}\right)^{\frac{1}{\delta}} \\
\qquad \textit{where} \quad \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\[2ex]
\left[\left(\sum\limits_{i=1}^{n} |\alpha_i|\right)^2 - \sum\limits_{i=1}^{n} |\alpha_i|^2\right] \max\limits_{1\leq i\neq j\leq n} |\langle z_i, z_j\rangle| ;
\end{cases}
$$

where any term in the first branch can be combined with each term from the second branch giving 9 possible combinations.

Out of these, we select the following ones that are of relevance for further consideration:

$$
\begin{aligned}
(6.19) \qquad & \left\|\sum_{i=1}^{n} \alpha_i z_i\right\|^2 \\
& \leq \max_{1\leq i\leq n} \|z_i\|^2 \sum_{i=1}^{n} |\alpha_i|^2 \\
& \qquad + \max_{1\leq i<j\leq n} |\langle z_i, z_j\rangle| \left[\left(\sum_{i=1}^{n} |\alpha_i|\right)^2 - \sum_{i=1}^{n} |\alpha_i|^2\right] \\
& \leq \sum_{i=1}^{n} |\alpha_i|^2 \left(\max_{1\leq i\leq n} \|z_i\|^2 + (n-1) \max_{1\leq i<j\leq n} |\langle z_i, z_j\rangle|\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(6.20)\qquad & \left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \\
& \leq \max_{1\leq i\leq n}\|z_i\|^2\sum_{i=1}^{n}|\alpha_i|^2+\left[\left(\sum_{i=1}^{n}|\alpha_i|^2\right)^2-\sum_{i=1}^{n}|\alpha_i|^4\right]^{1/2} \\
& \qquad\qquad \times\left(\sum_{1\leq i\neq j\leq n}|\langle z_i,z_j\rangle|^2\right)^{\frac{1}{2}} \\
& \leq \sum_{i=1}^{n}|\alpha_i|^2\left[\max_{1\leq i\leq n}\|z_i\|^2+\left(\sum_{1\leq i\neq j\leq n}|\langle z_i,z_j\rangle|^2\right)^{\frac{1}{2}}\right].
\end{aligned}
$$

Note that the last inequality in (6.19) follows by the fact that

$$
\left(\sum_{i=1}^{n}|\alpha_i|\right)^2\leq n\sum_{i=1}^{n}|\alpha_i|^2,
$$

while the last inequality in (6.20) is obvious.

Utilising the above inequalities (6.19) and (6.20) which provide alternatives to the Cauchy-Bunyakovsky-Schwarz inequality (6.11), we can state the following results [**16**].

Theorem 80 (Dragomir, 2005). *Let* $\{x_1,\dots,x_n\}$, M *and* x *be as in Theorem 79. Then*

$$
\begin{aligned}
(6.21)\quad & d^2(x,M) \\
& \leq \frac{\|x\|^2\left[\max\limits_{1\leq i\leq n}\|x_i\|^2+\left(\sum\limits_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|^2\right)^{\frac{1}{2}}\right]-\sum\limits_{i=1}^{n}|\langle x,x_i\rangle|^2}{\max\limits_{1\leq i\leq n}\|x_i\|^2+\left(\sum\limits_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|^2\right)^{\frac{1}{2}}}
\end{aligned}
$$

or, equivalently,

$$(6.22)\quad \Gamma(x_1,\dots,x_n,x) \le \frac{\|x\|^2\left[\max_{1\le i\le n}\|x_i\|^2+\left(\sum_{1\le i\ne j\le n}|\langle x_i,x_j\rangle|^2\right)^{\frac{1}{2}}\right]-\sum_{i=1}^n|\langle x,x_i\rangle|^2}{\max_{1\le i\le n}\|x_i\|^2+\left(\sum_{1\le i\ne j\le n}|\langle x_i,x_j\rangle|^2\right)^{\frac{1}{2}}} \times\Gamma(x_1,\dots,x_n).$$

PROOF. Utilising the inequality (6.20) for $\alpha_i=\langle x,x_i\rangle$ and $z_i=x_i$, $i\in\{1,\dots,n\}$, we can write:

$$(6.23)\quad \left\|\sum_{i=1}^n\langle x,x_i\rangle x_i\right\|^2 \le\sum_{i=1}^n|\langle x,x_i\rangle|^2\left[\max_{1\le i\le n}\|x_i\|^2+\left(\sum_{1\le i\ne j\le n}|\langle x_i,x_j\rangle|^2\right)^{\frac{1}{2}}\right]$$

for any $x\in H$.

Now, since, by the representation formula (6.3)

$$(6.24)\qquad d^2(x,M)=\|x\|^2-\frac{\sum_{i=1}^n|\langle x,x_i\rangle|^2}{\left\|\sum_{i=1}^n\langle x,x_i\rangle x_i\right\|^2}\cdot\sum_{i=1}^n|\langle x,x_i\rangle|^2,$$

for $x\notin M^\perp$, hence, by (6.23) and (6.24) we deduce the desired result (6.21). ∎

REMARK 65. *In 1941, R.P. Boas* [**2**] *and in 1944, R. Bellman* [**1**], *independent of each other, proved the following generalisation of Bessel's inequality:*

$$(6.25)\quad \sum_{i=1}^n|\langle y,y_i\rangle|^2\le\|y\|^2\left[\max_{1\le i\le n}\|y_i\|^2+\left(\sum_{1\le i\ne j\le n}|\langle y_i,y_j\rangle|^2\right)^{\frac{1}{2}}\right],$$

provided y and y_i $(i\in\{1,\dots,n\})$ are arbitrary vectors in the inner product space $(H;\langle\cdot,\cdot\rangle)$. If $\{y_i\}_{i\in\{1,\dots,n\}}$ are orthonormal, then (6.25) reduces to Bessel's inequality.

In this respect, one may see (6.21) as a refinement of the Boas-Bellman result (6.25).

Remark 66. *On making use of a similar argument to that utilised in Remark 64, one can obtain the following refinement of the Hadamard inequality:*

$$
\begin{aligned}
(6.26)\quad & \Gamma(x_1,\dots,x_n) \\
& \leq \|x_1\|^2 \\
& \times \prod_{k=2}^{n}\left(\|x_k\|^2-\frac{\sum_{i=1}^{k-1}|\langle x_k,x_i\rangle|^2}{\max\limits_{1\le i\le k-1}\|x_i\|^2+\left(\sum\limits_{1\le i\neq j\le k-1}|\langle x_i,x_j\rangle|^2\right)^{\frac{1}{2}}}\right) \\
& \leq \prod_{j=1}^{n}\|x_j\|^2 .
\end{aligned}
$$

Further on, if we choose $\alpha_i=\langle x,x_i\rangle$, $z_i=x_i$, $i\in\{1,\dots,n\}$ in (6.19), then we may state the inequality

$$
\begin{aligned}
(6.27)\quad & \left\|\sum_{i=1}^{n}\langle x,x_i\rangle x_i\right\|^2 \\
& \leq \sum_{i=1}^{n}|\langle x,x_i\rangle|^2\left(\max_{1\le i\le n}\|x_i\|^2+(n-1)\max_{1\le i\neq j\le n}|\langle x_i,x_j\rangle|\right).
\end{aligned}
$$

Utilising (6.27) and (6.24) we may state the following result as well [**16**]:

Theorem 81 (Dragomir, 2005). *Let $\{x_1,\dots,x_n\}$, M and x be as in Theorem 79. Then*

$$
\begin{aligned}
(6.28)\quad & d^2(x,M) \\
& \leq \frac{\|x\|^2\left[\max\limits_{1\le i\le n}\|x_i\|^2+(n-1)\max\limits_{1\le i\neq j\le n}|\langle x_i,x_j\rangle|\right]-\sum_{i=1}^{n}|\langle x,x_i\rangle|^2}{\max\limits_{1\le i\le n}\|x_i\|^2+(n-1)\max\limits_{1\le i\neq j\le n}|\langle x_i,x_j\rangle|}
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
(6.29)\quad & \Gamma\left(x_1,\dots,x_n,x\right) \\
& \leq \frac{\|x\|^2\left[\max\limits_{1\leq i\leq n}\|x_i\|^2+(n-1)\max\limits_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|\right]-\sum_{i=1}^n|\langle x,x_i\rangle|^2}{\max\limits_{1\leq i\leq n}\|x_i\|^2+(n-1)\max\limits_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\Gamma\left(x_1,\dots,x_n\right).
\end{aligned}
$$

REMARK 67. *The above result (6.28) provides a refinement for the following generalisation of Bessel's inequality:*

$$
(6.30)\quad \sum_{i=1}^n|\langle x,x_i\rangle|^2\leq\|x\|^2\left[\max_{1\leq i\leq n}\|x_i\|^2+(n-1)\max_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|\right],
$$

obtained by the author in [**15**].

One can also provide the corresponding refinement of Hadamard's inequality (6.4) on using (6.29), i.e.,

$$
\begin{aligned}
(6.31)\quad & \Gamma\left(x_1,\dots,x_n\right) \\
& \leq\|x_1\|^2 \\
& \times\prod_{k=2}^n\left(\|x_k\|^2-\frac{\sum\limits_{i=1}^{k-1}|\langle x_k,x_i\rangle|^2}{\max\limits_{1\leq i\leq k-1}\|x_i\|^2+(k-2)\max\limits_{1\leq i\neq j\leq k-1}|\langle x_i,x_j\rangle|}\right) \\
& \leq\prod_{j=1}^n\|x_j\|^2.
\end{aligned}
$$

6.1.3. Other Upper Bounds for $d(x,M)$. In [**7**, p. 140] the author obtained the following inequality that is similar to the Cauchy-Bunyakovsky-Schwarz result.

Lemma 9 (Dragomir, 2004). *Let $z_1, \dots, z_n \in H$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. Then one has the inequalities:*

$$\left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \le \sum_{i=1}^{n} |\alpha_i|^2 \sum_{j=1}^{n} |\langle z_i, z_j \rangle| \tag{6.32}$$

$$\le \begin{cases} \sum_{i=1}^{n} |\alpha_i|^2 \max_{1 \le i \le n} \left[\sum_{j=1}^{n} |\langle z_i, z_j \rangle| \right]; \\ \left(\sum_{i=1}^{n} |\alpha_i|^{2p} \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |\langle z_i, z_j \rangle| \right)^q \right)^{\frac{1}{q}} \\ \qquad\qquad \text{where } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \max_{1 \le i \le n} |\alpha_i|^2 \sum_{i,j=1}^{n} |\langle z_i, z_j \rangle|. \end{cases}$$

We can state and prove now another upper bound for the distance $d(x, M)$ as follows [**16**].

Theorem 82 (Dragomir, 2005). *Let $\{x_1, \dots, x_n\}$, M and x be as in Theorem 79. Then*

$$d^2(x, M) \le \frac{\|x\|^2 \max_{1 \le i \le n} \left[\sum_{j=1}^{n} |\langle x_i, x_j \rangle| \right] - \sum_{i=1}^{n} |\langle x, x_i \rangle|^2}{\max_{1 \le i \le n} \left[\sum_{j=1}^{n} |\langle x_i, x_j \rangle| \right]} \tag{6.33}$$

or, equivalently,

$$\Gamma(x_1, \dots, x_n, x) \le \frac{\|x\|^2 \max_{1 \le i \le n} \left[\sum_{j=1}^{n} |\langle x_i, x_j \rangle| \right] - \sum_{i=1}^{n} |\langle x, x_i \rangle|^2}{\max_{1 \le i \le n} \left[\sum_{j=1}^{n} |\langle x_i, x_j \rangle| \right]} \cdot \Gamma(x_1, \dots, x_n). \tag{6.34}$$

Proof. Utilising the first branch in (6.32) we may state that

$$\left\| \sum_{i=1}^{n} \langle x, x_i \rangle x_i \right\|^2 \le \sum_{i=1}^{n} |\langle x, x_i \rangle|^2 \max_{1 \le i \le n} \left[\sum_{j=1}^{n} |\langle x_i, x_j \rangle| \right] \tag{6.35}$$

for any $x \in H$.

Now, since, by the representation formula (6.3) we have

$$d^2(x,M) = \|x\|^2 - \frac{\sum_{i=1}^n |\langle x, x_i\rangle|^2}{\left\|\sum_{i=1}^n \langle x, x_i\rangle x_i\right\|^2} \cdot \sum_{i=1}^n |\langle x, x_i\rangle|^2, \tag{6.36}$$

for $x \notin M^{\perp}$, hence, by (6.35) and (6.36) we deduce the desired result (6.33). ∎

Remark 68. *In 1971, E. Bombieri* [**3**] *proved the following generalisation of Bessel's inequality, however not stated in the general form for inner products. The general version can be found for instance in* [**17**, p. 394]. *It reads as follows: if $y, y_1, \dots, y_n$ are vectors in the inner product space $(H; \langle\cdot,\cdot\rangle)$, then*

$$\sum_{i=1}^n |\langle y, y_i\rangle|^2 \le \|y\|^2 \max_{1\le i\le n} \left\{ \sum_{j=1}^n |\langle y_i, y_j\rangle| \right\}. \tag{6.37}$$

Obviously, when $\{y_1, \dots, y_n\}$ are orthonormal, the inequality (6.37) produces Bessel's inequality.

In this respect, we may regard our result (6.33) as a refinement of the Bombieri inequality (6.37).

Remark 69. *On making use of a similar argument to that in Remark 64, we obtain the following refinement for the Hadamard inequality:*

$$\begin{aligned} \Gamma(x_1, \dots, x_n) &\le \|x_1\|^2 \prod_{k=2}^n \left[\|x_k\|^2 - \frac{\sum_{i=1}^{k-1} |\langle x_k, x_i\rangle|^2}{\max_{1\le i\le k-1}\left[\sum_{j=1}^{k-1} |\langle x_i, x_j\rangle|\right]} \right] \\ &\le \prod_{j=1}^n \|x_j\|^2 . \end{aligned} \tag{6.38}$$

Another different Cauchy-Bunyakovsky-Schwarz type inequality is incorporated in the following lemma [**13**].

Lemma 10 (Dragomir, 2004). *Let $z_1, \dots, z_n \in H$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. Then*

$$\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \le \left(\sum_{i=1}^n |\alpha_i|^p \right)^{\frac{2}{p}} \left(\sum_{i,j=1}^n |\langle z_i, z_j\rangle|^q \right)^{\frac{1}{q}} \tag{6.39}$$

for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

If in (6.39) we choose $p = q = 2$, then we get

$$\left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \le \sum_{i=1}^{n} |\alpha_i|^2 \left(\sum_{i,j=1}^{n} |\langle z_i, z_j \rangle|^2 \right)^{\frac{1}{2}}. \tag{6.40}$$

Based on (6.40), we can state the following result that provides yet another upper bound for the distance $d(x, M)$ [**16**].

THEOREM 83 (Dragomir, 2005). *Let $\{x_1, \dots, x_n\}$, M and x be as in Theorem 79. Then*

$$d^2(x, M) \le \frac{\|x\|^2 \left(\sum_{i,j=1}^{n} |\langle x_i, x_j \rangle|^2 \right)^{\frac{1}{2}} - \sum_{i=1}^{n} |\langle x, x_i \rangle|^2}{\left(\sum_{i,j=1}^{n} |\langle x_i, x_j \rangle|^2 \right)^{\frac{1}{2}}} \tag{6.41}$$

or, equivalently,

$$\Gamma(x_1, \dots, x_n, x) \le \frac{\|x\|^2 \left(\sum_{i,j=1}^{n} |\langle x_i, x_j \rangle|^2 \right)^{\frac{1}{2}} - \sum_{i=1}^{n} |\langle x, x_i \rangle|^2}{\left(\sum_{i,j=1}^{n} |\langle x_i, x_j \rangle|^2 \right)^{\frac{1}{2}}} \cdot \Gamma(x_1, \dots, x_n). \tag{6.42}$$

Similar comments apply related to Hadamard's inequality. We omit the details.

6.1.4. Some Conditional Bounds. In the recent paper [**6**], the author has established the following reverse of the Bessel inequality.

Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $\{e_i\}_{i \in I}$ a finite family of orthonormal vectors in H, $\varphi_i, \phi_i \in \mathbb{K}$, $i \in I$ and $x \in H$. If

$$\operatorname{Re} \left\langle \sum_{i \in I} \phi_i e_i - x, x - \sum_{i \in I} \varphi_i e_i \right\rangle \ge 0 \tag{6.43}$$

or, equivalently,

$$\left\| x - \sum_{i \in I} \frac{\varphi_i + \phi_i}{2} e_i \right\| \le \frac{1}{2} \left(\sum_{i \in I} |\phi_i - \varphi_i|^2 \right)^{\frac{1}{2}}, \tag{6.44}$$

then

$$(0 \leq) \|x\|^2 - \sum_{i \in I} |\langle x, e_i \rangle|^2 \leq \frac{1}{4} \sum_{i \in I} |\phi_i - \varphi_i|^2 . \tag{6.45}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant [**16**].

THEOREM 84 (Dragomir, 2005). *Let $\{x_1, \ldots x_n\}$ be a linearly independent system of vectors in H and $M := span\{x_1, \ldots x_n\}$. If γ_i, $\Gamma_i \in \mathbb{K}$, $i \in \{1, \ldots, n\}$ and $x \in H \backslash M^{\perp}$ is such that*

$$\operatorname{Re} \left\langle \sum_{i=1}^{n} \Gamma_i x_i - x, x - \sum_{i=1}^{n} \gamma_i x_i \right\rangle \geq 0, \tag{6.46}$$

then we have the bound

$$d^2(x, M) \leq \frac{1}{4} \left\| \sum_{i=1}^{n} (\Gamma_i - \gamma_i) x_i \right\|^2 \tag{6.47}$$

or, equivalently,

$$\Gamma(x_1, \ldots, x_n, x) \leq \frac{1}{4} \left\| \sum_{i=1}^{n} (\Gamma_i - \gamma_i) x_i \right\|^2 \Gamma(x_1, \ldots, x_n) . \tag{6.48}$$

PROOF. It is easy to see that in an inner product space for any $x, z, Z \in H$ one has

$$\left\| x - \frac{z + Z}{2} \right\|^2 - \frac{1}{4} \|Z - z\|^2 = \operatorname{Re} \langle Z - x, x - z \rangle ,$$

therefore, the condition (6.46) is actually equivalent to

$$\left\| x - \sum_{i=1}^{n} \frac{\Gamma_i + \gamma_i}{2} x_i \right\|^2 \leq \frac{1}{4} \left\| \sum_{i=1}^{n} (\Gamma_i - \gamma_i) x_i \right\|^2 . \tag{6.49}$$

Now, obviously,

$$d^2(x, M) = \inf_{y \in M} \|x - y\|^2 \leq \left\| x - \sum_{i=1}^{n} \frac{\Gamma_i + \gamma_i}{2} x_i \right\|^2 \tag{6.50}$$

and thus, by (6.49) and (6.50) we deduce (6.47).

The last inequality is obvious by the representation (6.2). ∎

REMARK 70. *Utilising various Cauchy-Bunyakovsky-Schwarz type inequalities we may obtain more convenient (although coarser) bounds* *for* $d^2(x, M)$. *For instance, if we use the inequality (6.19) we can state the inequality:*

$$\left\| \sum_{i=1}^{n} (\Gamma_i - \gamma_i) x_i \right\|^2 \leq \sum_{i=1}^{n} |\Gamma_i - \gamma_i|^2 \left(\max_{1 \leq i \leq n} \|x_i\|^2 + (n-1) \max_{1 \leq i < j \leq n} |\langle x_i, x_j \rangle| \right),$$

giving the bound:

$$d^2(x, M) \leq \frac{1}{4} \sum_{i=1}^{n} |\Gamma_i - \gamma_i|^2 \times \left[\max_{1 \leq i \leq n} \|x_i\|^2 + (n-1) \max_{1 \leq i < j \leq n} |\langle x_i, x_j \rangle| \right], \tag{6.51}$$

provided (6.46) holds true.

Obviously, if $\{x_1, \dots, x_n\}$ *is an orthonormal family in* H, *then from (6.51) we deduce the reverse of Bessel's inequality incorporated in (6.45).*

If we use the inequality (6.20), then we can state the inequality

$$\left\| \sum_{i=1}^{n} (\Gamma_i - \gamma_i) x_i \right\|^2 \leq \sum_{i=1}^{n} |\Gamma_i - \gamma_i|^2 \left[\max_{1 \leq i \leq n} \|x_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle|^2 \right)^{\frac{1}{2}} \right],$$

giving the bound

$$d^2(x, M) \leq \frac{1}{4} \sum_{i=1}^{n} |\Gamma_i - \gamma_i|^2 \times \left[\max_{1 \leq i \leq n} \|x_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle|^2 \right)^{\frac{1}{2}} \right], \tag{6.52}$$

provided (6.46) holds true.

In this case, when one assumes that $\{x_1, \dots, x_n\}$ *is an orthonormal family of vectors, then (6.52) reduces to (6.45) as well.*

Finally, on utilising the first branch of the inequality (6.32), we can state that

$$d^2(x, M) \leq \frac{1}{4} \sum_{i=1}^{n} |\Gamma_i - \gamma_i|^2 \max_{1 \leq i \leq n} \left[\sum_{j=1}^{n} |\langle x_i, x_j \rangle| \right], \tag{6.53}$$

provided (6.46) holds true.

This inequality is also a generalisation of (6.45).

6.2. Reversing the CBS Inequality for Sequences

6.2.1. Introduction. Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. One of the most important inequalities in inner product spaces with numerous applications, is the *Schwarz inequality*

$$|\langle x, y \rangle|^2 \leq \|x\|^2 \|y\|^2, \quad x, y \in H \tag{6.54}$$

or, equivalently,

$$|\langle x, y \rangle| \leq \|x\| \|y\|, \quad x, y \in H. \tag{6.55}$$

The case of equality holds iff there exists a scalar $\alpha \in \mathbb{K}$ such that $x = \alpha y$.

By a *multiplicative reverse* of the Schwarz inequality we understand an inequality of the form

$$(1 \leq) \frac{\|x\| \|y\|}{|\langle x, y \rangle|} \leq k_1 \quad \text{or} \quad (1 \leq) \frac{\|x\|^2 \|y\|^2}{|\langle x, y \rangle|^2} \leq k_2 \tag{6.56}$$

with appropriate k_1 and k_2 and under various assumptions for the vectors x and y, while by an *additive reverse* we understand an inequality of the form

$$\begin{aligned} &(0 \leq) \|x\| \|y\| - |\langle x, y \rangle| \leq h_1 \quad \text{or} \\ &(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq h_2. \end{aligned} \tag{6.57}$$

Similar definition apply when $|\langle x, y \rangle|$ is replaced by $\operatorname{Re} \langle x, y \rangle$ or $|\operatorname{Re} \langle x, y \rangle|$.

The following recent reverses for the Schwarz inequality hold (see for instance the monograph on line [**7**, p. 20]).

Theorem 85 (Dragomir, 2004). *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. If $x, y \in H$ and $r > 0$ are such that*

$$\|x - y\| \leq r < \|y\|, \tag{6.58}$$

then we have the following multiplicative reverse of the Schwarz inequality

$$(1 \leq) \frac{\|x\| \|y\|}{|\langle x, y \rangle|} \leq \frac{\|x\| \|y\|}{\operatorname{Re} \langle x, y \rangle} \leq \frac{\|y\|}{\sqrt{\|y\|^2 - r^2}} \tag{6.59}$$

and the subsequent additive reverses

$$(0 \leq) \|x\| \|y\| - |\langle x, y\rangle| \leq \|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \leq \frac{r^2}{\sqrt{\|y\|^2 - r^2}\left(\|y\| + \sqrt{\|y\|^2 - r^2}\right)} \operatorname{Re}\langle x, y\rangle \tag{6.60}$$

and

$$\begin{aligned}(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 &\leq \|x\|^2 \|y\|^2 - [\operatorname{Re}\langle x, y\rangle]^2 \\ &\leq r^2 \|x\|^2 .\end{aligned} \tag{6.61}$$

All the above inequalities are sharp.

Other additive reverses of the quadratic Schwarz's inequality are incorporated in the following result [**7**, p. 18-19].

Theorem 86 (Dragomir, 2004). *Let $x, y \in H$ and $a, A \in \mathbb{K}$. If*

$$\operatorname{Re}\langle Ay - x, x - ay\rangle \geq 0 \tag{6.62}$$

or, equivalently,

$$\left\| x - \frac{a + A}{2} \cdot y \right\| \leq \frac{1}{2} |A - a| \|y\| , \tag{6.63}$$

then

$$\begin{aligned}(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 &\leq \frac{1}{4} |A - a|^2 \|y\|^4 - \begin{cases} \left| \frac{A+a}{2} \|y\|^2 - \langle x, y\rangle \right|^2 \\ \|y\|^2 \operatorname{Re}\langle Ay - x, x - ay\rangle \end{cases} \\ &\leq \frac{1}{4} |A - a|^2 \|y\|^4 .\end{aligned} \tag{6.64}$$

The constant $\frac{1}{4}$ is best possible in all inequalities.

If one were to assume more about the complex numbers A and a, then one may state the following result as well [**7**, p. 21-23].

Theorem 87 (Dragomir, 2004). *With the assumptions of Theorem 86 and, if in addition,* $\operatorname{Re}(A\bar{a}) > 0$, *then*

$$\|x\| \|y\| \leq \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{A} + \bar{a}\right) \langle x, y\rangle\right]}{\sqrt{\operatorname{Re}(A\bar{a})}} \leq \frac{1}{2} \cdot \frac{|A + a|}{\sqrt{\operatorname{Re}(A\bar{a})}} |\langle x, y\rangle| , \tag{6.65}$$

$$(0 \le) \|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \le \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{A} + \bar{a} - 2\sqrt{\operatorname{Re}(A\bar{a})}\right)\langle x, y\rangle\right]}{\sqrt{\operatorname{Re}(A\bar{a})}} \tag{6.66}$$

and

$$(0 \le) \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \le \frac{1}{4} \cdot \frac{|A - a|^2}{\operatorname{Re}(A\bar{a})} |\langle x, y\rangle|^2 . \tag{6.67}$$

The constants $\frac{1}{2}$ *and* $\frac{1}{4}$ *are best possible.*

Remark 71. *If* $A = M$, $a = m$ *and* $M \ge m > 0$, *then (6.65) and (6.66) may be written in a more convenient form as*

$$\|x\| \|y\| \le \frac{M + m}{2\sqrt{mM}} \operatorname{Re}\langle x, y\rangle \tag{6.68}$$

and

$$(0 \le) \|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \le \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\langle x, y\rangle . \tag{6.69}$$

Here the constant $\frac{1}{2}$ *is sharp in both inequalities.*

In this section several reverses for the Cauchy-Bunyakovsky-Schwarz (CBS) inequality for sequences of vectors in Hilbert spaces are obtained. Applications for bounding the distance to a finite-dimensional subspace and in reversing the generalised triangle inequality are also given.

6.2.2. Reverses of the (CBS) −Inequality for Two Sequences in $\ell^2_{\mathbf{p}}(K)$. Let $(K, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{K}$, $p_i \ge 0$, $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} p_i = 1$. Consider $\ell^2_{\mathbf{p}}(K)$ as the space

$$\ell^2_{\mathbf{p}}(K) := \left\{ x = (x_i)_{i\in\mathbb{N}} \,\middle|\, x_i \in K,\ i \in \mathbb{N} \ \text{ and } \ \sum_{i=1}^{\infty} p_i \|x_i\|^2 < \infty \right\}.$$

It is well known that $\ell^2_{\mathbf{p}}(K)$ endowed with the inner product

$$\langle x, y\rangle_{\mathbf{p}} := \sum_{i=1}^{\infty} p_i \langle x_i, y_i\rangle$$

is a Hilbert space over $\mathbb{K}$. The norm $\|\cdot\|_{\mathbf{p}}$ of $\ell^2_{\mathbf{p}}(K)$ is given by

$$\|x\|_{\mathbf{p}} := \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} .$$

If $x, y \in \ell_{\mathbf{p}}^2(K)$, then the following Cauchy-Bunyakovsky-Schwarz (CBS) inequality holds true

$$\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 \geq \left| \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right|^2 \tag{6.70}$$

with equality iff there exists a $\lambda \in \mathbb{K}$ such that $x_i = \lambda y_i$ for each $i \in \mathbb{N}$.

This is an obvious consequence of the Schwarz inequality (6.54) written for the inner product $\langle \cdot, \cdot \rangle_{\mathbf{p}}$ defined on $\ell_{\mathbf{p}}^2(K)$.

The following proposition may be stated [**11**].

Proposition 57. *Let $x, y \in \ell_{\mathbf{p}}^2(K)$ and $r > 0$. Assume that*

$$\|x_i - y_i\| \leq r < \|y_i\| \quad \text{for each} \quad i \in \mathbb{N}. \tag{6.71}$$

Then we have the inequality

$$\begin{aligned} (1 \leq) \frac{\left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}}}{\left|\sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle\right|} &\leq \frac{\left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}}}{\sum_{i=1}^{\infty} p_i \operatorname{Re} \langle x_i, y_i \rangle} \\ &\leq \frac{\left(\sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}}}{\sqrt{\sum_{i=1}^{\infty} p_i \|y_i\|^2 - r^2}}, \end{aligned} \tag{6.72}$$

$$\begin{aligned} (0 \leq) &\left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle\right| \\ &\leq \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}} - \sum_{i=1}^{\infty} p_i \operatorname{Re} \langle x_i, y_i \rangle \\ &\leq \frac{r^2 \cdot \sum_{i=1}^{\infty} p_i \operatorname{Re} \langle x_i, y_i \rangle}{\sqrt{\sum_{i=1}^{\infty} p_i \|y_i\|^2 - r^2} \left[\left(\sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}} + \sqrt{\sum_{i=1}^{\infty} p_i \|y_i\|^2 - r^2}\right]} \end{aligned} \tag{6.73}$$

and

$$(0 \le) \sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 - \left|\sum_{i=1}^{\infty} p_i \langle x_i, y_i\rangle\right|^2 \tag{6.74}$$
$$\le \sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 - \left[\sum_{i=1}^{\infty} p_i \operatorname{Re}\langle x_i, y_i\rangle\right]^2$$
$$\le r^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 .$$

Proof. From (6.71), we have

$$\|x-y\|_{\mathbf{p}}^2 = \sum_{i=1}^{\infty} p_i \|x_i - y_i\|^2 \le r^2 \sum_{i=1}^{\infty} p_i \le \sum_{i=1}^{\infty} p_i \|y_i\|^2 = \|y\|_{\mathbf{p}}^2 ,$$

giving $\|x-y\|_{\mathbf{p}} \le r \le \|y\|_{\mathbf{p}}$. Applying Theorem 85 for $\ell_{\mathbf{p}}^2(K)$ and $\langle \cdot, \cdot \rangle_{\mathbf{p}}$, we deduce the desired inequality. ∎

The following proposition holds [**11**].

Proposition 58. *Let $x, y \in \ell_{\mathbf{p}}^2(K)$ and $a, A \in \mathbb{K}$. If*

$$\operatorname{Re}\langle Ay_i - x_i, x_i - ay_i\rangle \ge 0 \quad \text{for each } i \in \mathbb{N} \tag{6.75}$$

or, equivalently,

$$\left\| x_i - \frac{a+A}{2} y_i \right\| \le \frac{1}{2} |A-a| \|y_i\| \quad \text{for each } i \in \mathbb{N} \tag{6.76}$$

then

$$(0 \le) \sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 - \left|\sum_{i=1}^{\infty} p_i \langle x_i, y_i\rangle\right|^2 \tag{6.77}$$
$$\le \frac{1}{4} |A-a|^2 \left(\sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^2$$
$$- \left\{ \begin{array}{c} \left|\frac{A+a}{2} \sum_{i=1}^{\infty} p_i \|y_i\|^2 - \sum_{i=1}^{\infty} p_i \langle x_i, y_i\rangle\right|^2 \\ \\ \sum_{i=1}^{\infty} p_i \|y_i\|^2 \sum_{i=1}^{\infty} p_i \operatorname{Re}\langle Ay_i - x_i, x_i - ay_i\rangle \end{array} \right.$$
$$\le \frac{1}{4} |A-a|^2 \left(\sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^2 .$$

The proof follows by Theorem 86, we omit the details.

Finally, on using Theorem 87, we may state [**11**]:

Proposition 59. *Assume that x, y, a and A are as in Proposition 58. Moreover, if* $\operatorname{Re}(A\bar{a}) > 0$, *then we have the inequality:*

$$\left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}} \leq \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{A}+\bar{a}\right) \sum_{i=1}^{\infty} p_i \langle x_i, y_i\rangle\right]}{\sqrt{\operatorname{Re}(A\bar{a})}} \leq \frac{1}{2} \cdot \frac{|A-a|}{\sqrt{\operatorname{Re}(A\bar{a})}} \left|\sum_{i=1}^{\infty} p_i \langle x_i, y_i\rangle\right|, \tag{6.78}$$

$$(0 \leq) \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}} - \sum_{i=1}^{\infty} p_i \operatorname{Re}\langle x_i, y_i\rangle \leq \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{A}+\bar{a}-2\sqrt{\operatorname{Re}(A\bar{a})}\right) \sum_{i=1}^{\infty} p_i \langle x_i, y_i\rangle\right]}{\sqrt{\operatorname{Re}(A\bar{a})}} \tag{6.79}$$

and

$$(0 \leq) \sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 - \left|\sum_{i=1}^{\infty} p_i \langle x_i, y_i\rangle\right|^2 \leq \frac{1}{4} \cdot \frac{|A-a|^2}{\operatorname{Re}(A\bar{a})} \left|\sum_{i=1}^{\infty} p_i \langle x_i, y_i\rangle\right|^2. \tag{6.80}$$

6.2.3. Reverses of the (CBS) −Inequality for Mixed Sequences. Let $(K, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{K}$ and for $p_i \geq 0$, $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} p_i = 1$, and $\ell_{\mathbf{p}}^2(K)$ the Hilbert space defined in the previous section.

If

$$\alpha \in \ell_{\mathbf{p}}^2(\mathbb{K}) := \left\{\alpha = (\alpha_i)_{i\in\mathbb{N}} \,\middle|\, \alpha_i \in \mathbb{K},\ i \in \mathbb{N} \text{ and } \sum_{i=1}^{\infty} p_i |\alpha_i|^2 < \infty\right\}$$

and $x \in \ell_{\mathbf{p}}^2(K)$, then the following Cauchy-Bunyakovsky-Schwarz (CBS) inequality holds true:

$$\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \geq \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|^2, \tag{6.81}$$

with equality if and only if there exists a vector $v \in K$ such that $x_i = \overline{\alpha_i} v$ for any $i \in \mathbb{N}$.

The inequality (6.81) follows by the obvious identity

$$\sum_{i=1}^{n} p_i |\alpha_i|^2 \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i \alpha_i x_i \right\|^2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \left\| \overline{\alpha_i} x_j - \overline{\alpha_j} x_i \right\|^2,$$

for any $n \in \mathbb{N}$, $n \geq 1$.

In the following we establish some reverses of the (CBS) $-$inequality in some of its various equivalent forms that will be specified where they occur [**11**].

THEOREM 88 (Dragomir, 2005). *Let $\alpha \in \ell_{\mathbf{p}}^2(\mathbb{K})$, $x \in \ell_{\mathbf{p}}^2(K)$ and $a \in K$, $r > 0$ such that $\|a\| > r$. If the following condition holds*

$$\|x_i - \overline{\alpha_i} a\| \leq r |\alpha_i| \quad \text{for each } i \in \mathbb{N}, \tag{6.82}$$

(note that if $\alpha_i \neq 0$ for any $i \in \mathbb{N}$, then the condition (6.82) is equivalent to

$$\left\| \frac{x_i}{\overline{\alpha_i}} - a \right\| \leq r \quad \text{for each } i \in \mathbb{N}), \tag{6.83}$$

then we have the following inequalities

$$\left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{\|a\|^2 - r^2}} \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, a \right\rangle \leq \frac{\|a\|}{\sqrt{\|a\|^2 - r^2}} \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|; \tag{6.84}$$

$$0 \leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\| \leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, \frac{a}{\|a\|} \right\rangle \tag{6.85}$$

$$\leq \frac{r^2}{\sqrt{\|a\|^2 - r^2}\left(\|a\| + \sqrt{\|a\|^2 - r^2}\right)} \operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, \frac{a}{\|a\|} \right\rangle$$

$$\leq \frac{r^2}{\sqrt{\|a\|^2 - r^2}\left(\|a\| + \sqrt{\|a\|^2 - r^2}\right)} \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|;$$

$$\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \leq \frac{1}{\|a\|^2 - r^2} \left[\operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, a \right\rangle \right]^2 \leq \frac{\|a\|^2}{\|a\|^2 - r^2} \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2 \tag{6.86}$$

and

$$\begin{aligned} 0 &\leq \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2 \\ &\leq \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 - \left[\operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, \frac{a}{\|a\|} \right\rangle \right]^2 \\ &\leq \frac{r^2}{\|a\|^2 \left(\|a\|^2 - r^2\right)} \left[\operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, a \right\rangle \right]^2 \\ &\leq \frac{r^2}{\|a\|^2 - r^2} \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2. \end{aligned} \tag{6.87}$$

All the inequalities in (6.84) – (6.87) are sharp.

Proof. From (6.82) we deduce

$$\|x_i\|^2 - 2\operatorname{Re}\langle x_i, \overline{\alpha_i} a\rangle + |\alpha_i|^2 \|a\|^2 \leq |\alpha_i|^2 r^2$$

for any $i \in \mathbb{N}$, which is clearly equivalent to

$$\|x_i\|^2 + \left(\|a\|^2 - r^2\right) |\alpha_i|^2 \leq 2\operatorname{Re}\langle \alpha_i x_i, a\rangle \tag{6.88}$$

for each $i \in \mathbb{N}$.

If we multiply (6.88) by $p_i \geq 0$ and sum over $i \in \mathbb{N}$, then we deduce

$$\sum_{i=1}^{\infty} p_i \|x_i\|^2 + \left(\|a\|^2 - r^2\right) \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \leq 2\operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, a \right\rangle. \tag{6.89}$$

Now, dividing (6.89) by $\sqrt{\|a\|^2 - r^2} > 0$ we get

$$\frac{1}{\sqrt{\|a\|^2 - r^2}} \sum_{i=1}^{\infty} p_i \|x_i\|^2 + \sqrt{\|a\|^2 - r^2} \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \leq \frac{2}{\sqrt{\|a\|^2 - r^2}} \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, a \right\rangle . \tag{6.90}$$

On the other hand, by the elementary inequality

$$\frac{1}{\alpha} p + \alpha q \geq 2\sqrt{pq}, \qquad \alpha > 0,\ p, q \geq 0,$$

we can state that:

$$2 \left[\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right]^{\frac{1}{2}} \leq \frac{1}{\sqrt{\|a\|^2 - r^2}} \sum_{i=1}^{\infty} p_i \|x_i\|^2 + \sqrt{\|a\|^2 - r^2} \sum_{i=1}^{\infty} p_i |\alpha_i|^2 . \tag{6.91}$$

Making use of (6.90) and (6.91), we deduce the first part of (6.84).

The second part is obvious by Schwarz's inequality

$$\operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, a \right\rangle \leq \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\| \|a\| .$$

If $p_1 = 1$, $x_1 = x$, $\alpha_1 = 1$ and $p_i = 0$, $\alpha_i = 0$, $x_i = 0$ for $i \geq 2$, then from (6.84) we deduce the inequality

$$\|x\| \leq \frac{1}{\sqrt{\|a\|^2 - r^2}} \operatorname{Re} \langle x, a \rangle \leq \frac{\|x\| \|a\|}{\sqrt{\|a\|^2 - r^2}}$$

provided $\|x - a\| \leq r < \|a\|$, $x, a \in K$. The sharpness of this inequality has been shown in [**7**, p. 20], and we omit the details.

The other inequalities are obvious consequences of (6.84) and we omit the details. ∎

The following corollary may be stated [**11**].

Corollary 54. *Let* $\alpha \in \ell_{\mathbf{p}}^2(\mathbb{K})$, $x \in \ell_{\mathbf{p}}^2(K)$, $e \in H$, $\|e\| = 1$ *and* $\varphi, \phi \in \mathbb{K}$ *with* $\operatorname{Re}(\phi\bar{\varphi}) > 0$. *If*

$$\left\| x_i - \overline{\alpha_i} \cdot \frac{\varphi + \phi}{2} \cdot e \right\| \leq \frac{1}{2} |\phi - \varphi| |\alpha_i| \tag{6.92}$$

for each $i \in \mathbb{N}$, or, equivalently

$$\operatorname{Re}\langle \phi\overline{\alpha_i}e - x_i, x_i - \varphi\overline{\alpha_i}e\rangle \geq 0 \tag{6.93}$$

for each $i \in \mathbb{N}$, (note that, if $\alpha_i \neq 0$ for any $i \in \mathbb{N}$, then (6.92) is equivalent to

$$\left\| \frac{x_i}{\overline{\alpha_i}} - \frac{\varphi + \phi}{2} \cdot e \right\| \leq \frac{1}{2}\left|\phi - \varphi\right| \tag{6.94}$$

for each $i \in \mathbb{N}$ and (6.93) is equivalent to

$$\operatorname{Re}\left\langle \phi e - \frac{x_i}{\overline{\alpha_i}}, \frac{x_i}{\overline{\alpha_i}} - \varphi e \right\rangle \geq 0$$

for each $i \in \mathbb{N}$), then the following reverses of the $(CBS)-$inequality are valid:

$$\left(\sum_{i=1}^{\infty} p_i \left|\alpha_i\right|^2 \sum_{i=1}^{\infty} p_i \left\|x_i\right\|^2\right)^{\frac{1}{2}} \leq \frac{\operatorname{Re}\left[\left(\bar{\phi} + \bar{\varphi}\right)\left\langle \sum_{i=1}^{\infty} p_i\alpha_i x_i, e\right\rangle\right]}{2\left[\operatorname{Re}\left(\phi\overline{\varphi}\right)\right]^{\frac{1}{2}}} \leq \frac{1}{2} \cdot \frac{\left|\varphi + \phi\right|}{\left[\operatorname{Re}\left(\phi\overline{\varphi}\right)\right]^{\frac{1}{2}}} \left\| \sum_{i=1}^{\infty} p_i\alpha_i x_i \right\|; \tag{6.95}$$

$$\begin{aligned} 0 &\leq \left(\sum_{i=1}^{\infty} p_i \left|\alpha_i\right|^2 \sum_{i=1}^{\infty} p_i \left\|x_i\right\|^2\right)^{\frac{1}{2}} - \left\| \sum_{i=1}^{\infty} p_i\alpha_i x_i \right\| \\ &\leq \left(\sum_{i=1}^{\infty} p_i \left|\alpha_i\right|^2 \sum_{i=1}^{\infty} p_i \left\|x_i\right\|^2\right)^{\frac{1}{2}} \\ &\qquad - \operatorname{Re}\left[\frac{\bar{\phi} + \bar{\varphi}}{\left|\varphi + \phi\right|}\left\langle \sum_{i=1}^{\infty} p_i\alpha_i x_i, e\right\rangle\right] \\ &\leq \frac{\left|\phi - \varphi\right|^2}{2\sqrt{\operatorname{Re}\left(\phi\varphi\right)}\left(\left|\varphi + \phi\right| + 2\sqrt{\operatorname{Re}\left(\phi\overline{\varphi}\right)}\right)} \\ &\qquad \times \operatorname{Re}\left[\frac{\bar{\phi} + \bar{\varphi}}{\left|\varphi + \phi\right|}\left\langle \sum_{i=1}^{\infty} p_i\alpha_i x_i, e\right\rangle\right] \\ &\leq \frac{\left|\phi - \varphi\right|^2}{2\sqrt{\operatorname{Re}\left(\phi\varphi\right)}\left(\left|\varphi + \phi\right| + 2\sqrt{\operatorname{Re}\left(\phi\overline{\varphi}\right)}\right)} \left\| \sum_{i=1}^{\infty} p_i\alpha_i x_i \right\|; \end{aligned} \tag{6.96}$$

$$\begin{aligned}(6.97)\qquad & \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \\ & \leq \frac{1}{4 \operatorname{Re}(\phi\bar{\varphi})} \left[\operatorname{Re}\left\{ (\bar{\phi}+\bar{\varphi}) \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e \right\rangle \right\} \right]^2 \\ & \leq \frac{1}{4} \cdot \frac{|\varphi+\phi|^2}{\operatorname{Re}(\phi\bar{\varphi})} \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2\end{aligned}$$

and

$$\begin{aligned}(6.98)\qquad 0 & \leq \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2 \\ & \leq \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \\ & \qquad - \left[\operatorname{Re}\left\{ \frac{\bar{\phi}+\bar{\varphi}}{|\varphi+\phi|} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e \right\rangle \right\} \right]^2 \\ & \leq \frac{|\phi-\varphi|^2}{4|\phi+\varphi|^2 \operatorname{Re}(\phi\bar{\varphi})} \left\{ \operatorname{Re}\left[(\bar{\phi}+\bar{\varphi}) \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e \right\rangle \right] \right\}^2 \\ & \leq \frac{|\phi-\varphi|^2}{4 \operatorname{Re}(\phi\bar{\varphi})} \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2 .\end{aligned}$$

All the inequalities in (6.95) – (6.98) are sharp.

Remark 72. *We remark that if $M \geq m > 0$ and for $\alpha \in \ell_{\mathbf{p}}^2(\mathbb{K})$, $x \in \ell_{\mathbf{p}}^2(K)$, $e \in H$ with $\|e\| = 1$, one would assume that either*

$$(6.99)\qquad \left\| \frac{x_i}{\overline{\alpha_i}} - \frac{M+m}{2} \cdot e \right\| \leq \frac{1}{2}(M-m)$$

for each $i \in \mathbb{N}$, or, equivalently

$$(6.100)\qquad \operatorname{Re}\left\langle Me - \frac{x_i}{\overline{\alpha_i}}, \frac{x_i}{\overline{\alpha_i}} - me \right\rangle \geq 0$$

for each $i \in \mathbb{N}$, then the following, much simpler reverses of the $(CBS)-$inequality may be stated:

$$\left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} \leq \frac{M+m}{2\sqrt{mM}} \operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle \leq \frac{M+m}{2\sqrt{mM}} \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|; \tag{6.101}$$

$$\begin{aligned} 0 &\leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\| \\ &\leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle \\ &\leq \frac{(M-m)^2}{2\left(\sqrt{M}+\sqrt{m}\right)^2 \sqrt{mM}} \operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle \\ &\leq \frac{(M-m)^2}{2\left(\sqrt{M}+\sqrt{m}\right)^2 \sqrt{mM}} \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|; \end{aligned} \tag{6.102}$$

$$\begin{aligned} &\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 - \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|^2 \\ &\leq \frac{(M+m)^2}{4mM} \left[\operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right]^2 \\ &\leq \frac{(M+m)^2}{4mM} \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|^2 \end{aligned} \tag{6.103}$$

and

$$0 \leq \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 - \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|^2 \tag{6.104}$$

$$\le \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 - \left[\operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right]^2$$
$$\le \frac{(M-m)^2}{4mM}\left[\operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right]^2$$
$$\le \frac{(M-m)^2}{4mM}\left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|^2.$$

6.2.4. Reverses for the Generalised Triangle Inequality. In 1966, J.B. Diaz and F.T. Metcalf [**5**] proved the following reverse of the generalised triangle inequality holding in an inner product space $(H; \langle\cdot,\cdot\rangle)$ over the real or complex number field $\mathbb{K}$:

$$r\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n} x_i\right\| \tag{6.105}$$

provided the vectors $x_1, \dots, x_n \in H\backslash\{0\}$ satisfy the assumption

$$0 \le r \le \frac{\operatorname{Re}\langle x_i, a\rangle}{\|x_i\|}, \tag{6.106}$$

where $a \in H$ and $\|a\| = 1$.

In an attempt to diversify the assumptions for which such reverse results hold, the author pointed out in [**10**] that

$$\sqrt{1-\rho^2}\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n} x_i\right\|, \tag{6.107}$$

where the vectors $x_{i,}i \in \{1,\dots,n\}$ satisfy the condition

$$\|x_i - a\| \le \rho, \qquad i \in \{1,\dots,n\} \tag{6.108}$$

where $a \in H$, $\|a\| = 1$ and $\rho \in (0,1)$.

If, for $M \ge m > 0$, the vectors $x_i \in H$, $i \in \{1,\dots,n\}$ verify either

$$\operatorname{Re}\langle Ma - x_i, x_i - ma\rangle \ge 0, \qquad i \in \{1,\dots,n\}, \tag{6.109}$$

or, equivalently,

$$\left\|x_i - \frac{M+m}{2}\cdot a\right\| \le \frac{1}{2}(M-m), \qquad i \in \{1,\dots,n\}, \tag{6.110}$$

where $a \in H$, $\|a\| = 1$, then the following reverse of the generalised triangle inequality may be stated as well [**10**]

$$\frac{2\sqrt{mM}}{M+m}\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n} x_i\right\|. \tag{6.111}$$

Note that the inequalities (6.105), (6.107), and (6.111) are sharp; necessary and sufficient equality conditions were provided (see [**5**] and [**10**]).

It is obvious, from Theorem 88, that, if

$$\|x_i - a\| \leq r, \quad \text{for} \quad i \in \{1, \dots, n\}, \tag{6.112}$$

where $\|a\| > r$, $a \in H$ and $x_i \in H$, $i \in \{1, \dots, n\}$, then one can state the inequalities

$$\begin{aligned} \sum_{i=1}^{n} \|x_i\| &\leq \sqrt{n} \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{\|a\|^2 - r^2}} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, a \right\rangle \\ &\leq \frac{\|a\|}{\sqrt{\|a\|^2 - r^2}} \left\| \sum_{i=1}^{n} x_i \right\| \end{aligned} \tag{6.113}$$

and

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \\ &\leq \sqrt{n} \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}} - \left\| \sum_{i=1}^{n} x_i \right\| \\ &\leq \sqrt{n} \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}} - \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, \frac{a}{\|a\|} \right\rangle \\ &\leq \frac{r^2}{\sqrt{\|a\|^2 - r^2} \left(\|a\| + \sqrt{\|a\|^2 - r^2} \right)} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, \frac{a}{\|a\|} \right\rangle \\ &\leq \frac{r^2}{\sqrt{\|a\|^2 - r^2} \left(\|a\| + \sqrt{\|a\|^2 - r^2} \right)} \left\| \sum_{i=1}^{n} x_i \right\|. \end{aligned} \tag{6.114}$$

We note that for $\|a\| = 1$ and $r \in (0, 1)$, the inequality (6.89) becomes

$$\sqrt{1 - r^2} \sum_{i=1}^{n} \|x_i\| \leq \sqrt{(1 - r^2) n} \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}} \tag{6.115}$$

$$\leq \operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, a \right\rangle \leq \left\| \sum_{i=1}^{n} x_i \right\|$$

which is a refinement of (6.107).

With the same assumptions for a and r, we have from (6.114) the following additive reverse of the generalised triangle inequality:

$$0 \leq \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \tag{6.116}$$
$$\leq \frac{r^2}{\sqrt{1-r^2}\left(1+\sqrt{1-r^2}\right)} \operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, a \right\rangle$$
$$\leq \frac{r^2}{\sqrt{1-r^2}\left(1+\sqrt{1-r^2}\right)} \left\| \sum_{i=1}^{n} x_i \right\|.$$

We can obtain the following reverses of the generalised triangle inequality from Corollary 54 when the assumptions are in terms of complex numbers ϕ and φ :

If $\varphi, \phi \in \mathbb{K}$ with $\operatorname{Re}(\phi\bar{\varphi}) > 0$ and $x_i \in H$, $i \in \{1, \dots, n\}$, $e \in H$, $\|e\| = 1$ are such that

$$\left\| x_i - \frac{\varphi + \phi}{2} e \right\| \leq \frac{1}{2} |\phi - \varphi| \quad \text{for each} \quad i \in \{1, \dots, n\}, \tag{6.117}$$

or, equivalently,

$$\operatorname{Re}\langle \phi e - x_i, x_i - \varphi e \rangle \geq 0 \quad \text{for each} \quad i \in \{1, \dots, n\},$$

then we have the following reverses of the generalised triangle inequality:

$$\sum_{i=1}^{n} \|x_i\| \leq \sqrt{n} \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}} \tag{6.118}$$
$$\leq \frac{\operatorname{Re}\left[\left(\bar{\phi} + \bar{\varphi}\right) \left\langle \sum_{i=1}^{n} x_i, e \right\rangle \right]}{2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}}$$
$$\leq \frac{1}{2} \cdot \frac{\left|\bar{\phi} + \bar{\varphi}\right|}{\sqrt{\operatorname{Re}(\phi\bar{\varphi})}} \left\| \sum_{i=1}^{n} x_i \right\|$$

and

$$
\begin{aligned}
(6.119)\qquad 0 &\le \sum_{i=1}^{n}\|x_i\| - \left\|\sum_{i=1}^{n} x_i\right\| \\
&\le \sqrt{n}\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} - \left\|\sum_{i=1}^{n} x_i\right\| \\
&\le \sqrt{n}\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} - \operatorname{Re}\left[\frac{|\bar{\phi}+\bar{\varphi}|}{\sqrt{\operatorname{Re}\left(\bar{\phi}\bar{\varphi}\right)}}\left\langle\sum_{i=1}^{n} x_i, e\right\rangle\right] \\
&\le \frac{|\phi-\varphi|^2}{2\sqrt{\operatorname{Re}\left(\phi\bar{\varphi}\right)}\left(|\phi+\varphi| + 2\sqrt{\operatorname{Re}\left(\phi\bar{\varphi}\right)}\right)} \\
&\qquad \times \operatorname{Re}\left[\frac{\bar{\phi}+\bar{\varphi}}{|\bar{\phi}+\bar{\varphi}|}\left\langle\sum_{i=1}^{n} x_i, e\right\rangle\right] \\
&\le \frac{|\phi-\varphi|^2}{2\sqrt{\operatorname{Re}\left(\phi\bar{\varphi}\right)}\left(|\phi+\varphi| + 2\sqrt{\operatorname{Re}\left(\phi\bar{\varphi}\right)}\right)}\left\|\sum_{i=1}^{n} x_i\right\|.
\end{aligned}
$$

Obviously (6.118) for $\phi = M$, $\varphi = m$, $M \ge m > 0$ provides a refinement for (6.111).

6.2.5. Lower Bounds for the Distance to Finite-Dimensional Subspaces. Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $\{y_1,\dots,y_n\}$ a subset of H and $G(y_1,\dots,y_n)$ the *Gram matrix* of $\{y_1,\dots,y_n\}$ where (i,j) −entry is $\langle y_i, y_j\rangle$. The determinant of $G(y_1,\dots,y_n)$ is called the *Gram determinant* of $\{y_1,\dots,y_n\}$ and is denoted by $\Gamma(y_1,\dots,y_n)$.

Following [**4**, p. 129 – 133], we state here some general results for the Gram determinant that will be used in the sequel:

(1) Let $\{x_1,\dots,x_n\}\subset H$. Then $\Gamma(x_1,\dots,x_n)\neq 0$ if and only if $\{x_1,\dots,x_n\}$ is linearly independent;
(2) Let $M = \mathit{span}\{x_1,\dots,x_n\}$ be n−dimensional in H, i.e., $\{x_1,\dots,x_n\}$ is linearly independent. Then for each $x\in H$, the distance $d(x,M)$ from x to the linear subspace H has the representations

$$
(6.120)\qquad d^2(x,M) = \frac{\Gamma(x_1,\dots,x_n,x)}{\Gamma(x_1,\dots,x_n)}
$$

and

$$d^2(x,M)=\begin{cases}\|x\|^2-\frac{\left(\sum_{i=1}^n|\langle x,x_i\rangle|^2\right)^2}{\left\|\sum_{i=1}^n\langle x,x_i\rangle x_i\right\|^2} & \text{if } x\notin M^{\perp},\\ \\ \|x\|^2 & \text{if } x\in M^{\perp},\end{cases} \tag{6.121}$$

where $M^{\perp}$ denotes the orthogonal complement of M.

The following result may be stated [**11**].

PROPOSITION 60. *Let* $\{x_1,\dots,x_n\}$ *be a system of linearly independent vectors,* $M=span\{x_1,\dots,x_n\}$, $x\in H\backslash M^{\perp}$, $a\in H$, $r>0$ *and* $\|a\|>r$. *If*

$$\left\|x_i-\overline{\langle x,x_i\rangle}a\right\|\le|\langle x,x_i\rangle|\,r\quad\text{for each}\quad i\in\{1,\dots,n\}, \tag{6.122}$$

(note that if $\langle x,x_i\rangle\neq0$ *for each* $i\in\{1,\dots,n\}$, *then (6.122) can be written as*

$$\left\|\frac{x_i}{\overline{\langle x,x_i\rangle}}-a\right\|\le r\quad\text{for each}\quad i\in\{1,\dots,n\}), \tag{6.123}$$

then we have the inequality

$$\begin{aligned}d^2(x,M)&\ge\|x\|^2-\frac{\|a\|^2}{\|a\|^2-r^2}\cdot\frac{\sum_{i=1}^n|\langle x,x_i\rangle|^2}{\sum_{i=1}^n\|x_i\|^2}\\&\ge0.\end{aligned} \tag{6.124}$$

PROOF. Utilising (6.121) we can state that

$$d^2(x,M)=\|x\|^2-\frac{\sum_{i=1}^n|\langle x,x_i\rangle|^2}{\left\|\sum_{i=1}^n\langle x,x_i\rangle x_i\right\|^2}\cdot\sum_{i=1}^n|\langle x,x_i\rangle|^2. \tag{6.125}$$

Also, by the inequality (6.86) applied for $\alpha_i=\langle x,x_i\rangle$, $p_i=\frac{1}{n}$, $i\in\{1,\dots,n\}$, we can state that

$$\frac{\sum_{i=1}^n|\langle x,x_i\rangle|^2}{\left\|\sum_{i=1}^n\langle x,x_i\rangle x_i\right\|^2}\le\frac{\|a\|^2}{\|a\|^2-r^2}\cdot\frac{1}{\sum_{i=1}^n\|x_i\|^2} \tag{6.126}$$

provided the condition (6.123) holds true.

Combining (6.125) with (6.126) we deduce the first inequality in (6.124).

The last inequality is obvious since, by Schwarz's inequality

$$\|x\|^2\sum_{i=1}^n\|x_i\|^2\ge\sum_{i=1}^n|\langle x,x_i\rangle|^2\ge\frac{\|a\|^2}{\|a\|^2-r^2}\sum_{i=1}^n|\langle x,x_i\rangle|^2.$$

∎

Remark 73. *Utilising (6.120), we can state the following result for Gram determinants*

$$\Gamma(x_1, \dots, x_n, x) \geq \left[\|x\|^2 - \frac{\|a\|^2}{\|a\|^2 - r^2} \cdot \frac{\sum_{i=1}^n |\langle x, x_i \rangle|^2}{\sum_{i=1}^n \|x_i\|^2} \right] \Gamma(x_1, \dots, x_n) \geq 0 \tag{6.127}$$

for $x \notin M^{\perp}$ and x, x_i, a and r are as in Proposition 60.

The following corollary of Proposition 60 may be stated as well [**11**].

Corollary 55. *Let $\{x_1, \dots, x_n\}$ be a system of linearly independent vectors, $M = span\{x_1, \dots, x_n\}$, $x \in H \backslash M^{\perp}$ and $\phi, \varphi \in K$ with* $\operatorname{Re}(\phi\bar{\varphi}) > 0$. *If $e \in H$, $\|e\| = 1$ and*

$$\left\| x_i - \overline{\langle x, x_i \rangle} \cdot \frac{\varphi + \phi}{2} e \right\| \leq \frac{1}{2} |\phi - \varphi| \, |\langle x, x_i \rangle| \tag{6.128}$$

or, equivalently,

$$\operatorname{Re} \left\langle \phi \cdot \overline{\langle x, x_i \rangle} e - x_i, x_i - \varphi \cdot \overline{\langle x, x_i \rangle} e \right\rangle \geq 0,$$

for each $i \in \{1, \dots, n\}$, then

$$d^2(x, M) \geq \|x\|^2 - \frac{1}{4} \cdot \frac{|\varphi + \phi|^2}{\operatorname{Re}(\phi\bar{\varphi})} \cdot \frac{\sum_{i=1}^n |\langle x, x_i \rangle|^2}{\sum_{i=1}^n \|x_i\|^2} \geq 0, \tag{6.129}$$

or, equivalently,

$$\Gamma(x_1, \dots, x_n, x) \geq \left[\|x\|^2 - \frac{1}{4} \cdot \frac{|\varphi + \phi|^2}{\operatorname{Re}(\phi\bar{\varphi})} \cdot \frac{\sum_{i=1}^n |\langle x, x_i \rangle|^2}{\sum_{i=1}^n \|x_i\|^2} \right] \Gamma(x_1, \dots, x_n) \geq 0. \tag{6.130}$$

6.2.6. Applications for Fourier Coefficients. Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over the real or complex number field $\mathbb{K}$ and $\{e_i\}_{i \in I}$ an *orthornormal basis* for H. Then (see for instance [**4**, p. 54 – 61])

(i) Every element $x \in H$ can be expanded in a *Fourier series,* i.e.,

$$x = \sum_{i \in I} \langle x, e_i \rangle e_i,$$

where $\langle x, e_i \rangle$, $i \in I$ are the *Fourier coefficients* of x;

(ii) (Parseval identity)

$$\|x\|^2 = \sum_{i \in I} \langle x, e_i \rangle e_i, \qquad x \in H;$$

(iii) (Extended Parseval identity)

$$\langle x,y\rangle = \sum_{i\in I}\langle x,e_i\rangle\langle e_i,y\rangle, \qquad x,y\in H;$$

(iv) (Elements are uniquely determined by their Fourier coefficients)

$$\langle x,e_i\rangle = \langle y,e_i\rangle \quad \text{for every } i\in I \quad \text{implies that } x=y.$$

Now, we must remark that all the results can be stated for $K=\mathbb{K}$ where $\mathbb{K}$ is the Hilbert space of complex (real) numbers endowed with the usual norm and inner product .

Therefore, we can state the following proposition [**11**].

Proposition 61. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{K}$ and $\{e_i\}_{i\in I}$ an orthornormal base for H. If $x,y\in H$ $(y\neq 0)$, $a\in\mathbb{K}$ $(\mathbb{C},\mathbb{R})$ and $r>0$ such that $|a|>r$ and*

$$\left|\frac{\langle x,e_i\rangle}{\langle y,e_i\rangle}-a\right|\leq r \quad \textit{for each} \quad i\in I, \tag{6.131}$$

then we have the following reverse of the Schwarz inequality

$$\begin{aligned}\|x\|\,\|y\| &\leq \frac{1}{\sqrt{|a|^2-r^2}}\operatorname{Re}\left[\bar{a}\cdot\langle x,y\rangle\right] \\ &\leq \frac{|a|}{\sqrt{|a|^2-r^2}}\left|\langle x,y\rangle\right|;\end{aligned} \tag{6.132}$$

$$\begin{aligned}&(0\leq)\ \|x\|\,\|y\|-\left|\langle x,y\rangle\right| \\ &\leq \|x\|\,\|y\|-\operatorname{Re}\left[\frac{\bar{a}}{|a|}\cdot\langle x,y\rangle\right] \\ &\leq \frac{r^2}{\sqrt{|a|^2-r^2}\left(|a|+\sqrt{|a|^2-r^2}\right)}\operatorname{Re}\left[\frac{\bar{a}}{|a|}\cdot\langle x,y\rangle\right] \\ &\leq \frac{r^2}{\sqrt{|a|^2-r^2}\left(|a|+\sqrt{|a|^2-r^2}\right)}\left|\langle x,y\rangle\right|;\end{aligned} \tag{6.133}$$

$$\begin{aligned}\|x\|^2\|y\|^2 &\leq \frac{1}{|a|^2-r^2}\left(\operatorname{Re}\left[\bar{a}\cdot\langle x,y\rangle\right]\right)^2 \\ &\leq \frac{|a|^2}{|a|^2-r^2}\left|\langle x,y\rangle\right|^2\end{aligned} \tag{6.134}$$

and

$$(0\leq)\left\|x\right\|^{2}\left\|y\right\|^{2}-\left|\left\langle x,y\right\rangle\right|^{2} \tag{6.135}$$
$$\leq\left\|x\right\|^{2}\left\|y\right\|^{2}-\left(\operatorname{Re}\left[\frac{\bar{a}}{|a|}\cdot\left\langle x,y\right\rangle\right]\right)^{2}$$
$$\leq\frac{r^{2}}{|a|^{2}\left(|a|^{2}-r^{2}\right)}-\left(\operatorname{Re}\left[\frac{\bar{a}}{|a|}\cdot\left\langle x,y\right\rangle\right]\right)^{2}$$
$$\leq\frac{r^{2}}{|a|^{2}-r^{2}}\left|\left\langle x,y\right\rangle\right|.$$

The proof is similar to the one in Theorem 88, when instead of x_i we take $\langle x,e_i\rangle$, instead of α_i we take $\langle e_i,y\rangle$, $\|\cdot\|=|\cdot|$, $p_i=1$, and we use the Parseval identities mentioned above in (ii) and (iii). We omit the details.

The following result may be stated as well [**11**].

Proposition 62. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{K}$ and $\{e_i\}_{i\in I}$ an orthornormal base for H. If $x,y\in H$ $(y\neq 0)$, $e,\varphi,\phi\in\mathbb{K}$ with* $\operatorname{Re}(\phi\bar{\varphi})>0$, $|e|=1$ *and, either*

$$\left|\frac{\langle x,e_i\rangle}{\langle y,e_i\rangle}-\frac{\varphi+\phi}{2}\cdot e\right|\leq\frac{1}{2}\left|\phi-\varphi\right| \tag{6.136}$$

or, equivalently,

$$\operatorname{Re}\left[\left(\phi e-\frac{\langle x,e_i\rangle}{\langle y,e_i\rangle}\right)\left(\frac{\langle e_i,x\rangle}{\langle e_i,y\rangle}-\bar{\varphi}\bar{e}\right)\right]\geq 0 \tag{6.137}$$

for each $i\in I$, then the following reverses of the Schwarz inequality hold:

$$\|x\|\,\|y\|\leq\frac{\operatorname{Re}\left[\left(\bar{\phi}+\bar{\varphi}\right)\bar{e}\left\langle x,y\right\rangle\right]}{2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}}\leq\frac{1}{2}\cdot\frac{|\varphi+\phi|}{\sqrt{\operatorname{Re}(\phi\bar{\varphi})}}\left|\left\langle x,y\right\rangle\right|, \tag{6.138}$$

$$(0\leq)\|x\|\,\|y\|-\left|\left\langle x,y\right\rangle\right| \tag{6.139}$$
$$\leq\|x\|\,\|y\|-\operatorname{Re}\left[\frac{\left(\bar{\phi}+\bar{\varphi}\right)\bar{e}}{|\varphi+\phi|}\left\langle x,y\right\rangle\right]$$

$$\leq \frac{|\phi-\varphi|^2}{2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}\left(|\varphi+\phi|+2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}\right)} \times \operatorname{Re}\left[\frac{(\bar{\phi}+\bar{\varphi})\,\bar{e}}{|\varphi+\phi|}\langle x,y\rangle\right]$$
$$\leq \frac{|\phi-\varphi|^2}{2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}\left(|\varphi+\phi|+2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}\right)}|\langle x,y\rangle|$$

and

$$(0\leq)\,\|x\|^2\|y\|^2-|\langle x,y\rangle|^2 \tag{6.140}$$
$$\leq \|x\|^2\|y\|^2-\left\{\operatorname{Re}\left[\frac{(\bar{\phi}+\bar{\varphi})\,\bar{e}}{|\varphi+\phi|}\langle x,y\rangle\right]\right\}^2$$
$$\leq \frac{|\phi-\varphi|^2}{4|\phi+\varphi|^2\operatorname{Re}(\phi\bar{\varphi})}\left\{\operatorname{Re}\left[(\bar{\phi}+\bar{\varphi})\,\bar{e}\langle x,y\rangle\right]\right\}^2$$
$$\leq \frac{|\phi-\varphi|^2}{4\operatorname{Re}(\phi\bar{\varphi})}|\langle x,y\rangle|^2.$$

REMARK 74. *If $\phi=M\geq m=\varphi>0$, then one may state simpler inequalities from (6.138) – (6.140). We omit the details.*

6.3. Other Reverses of the CBS Inequality

6.3.1. Introduction. Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$.

The following reverses for the Schwarz inequality hold (see [**8**], or the monograph on line [**7**, p. 27]).

THEOREM 89 (Dragomir, 2004). *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. If $x,a\in H$ and $r>0$ are such that*

$$x\in B(x,r):=\{z\in H|\,\|z-a\|\leq r\}, \tag{6.141}$$

then we have the inequalities

$$(0\leq)\,\|x\|\,\|a\|-|\langle x,a\rangle|\leq\|x\|\,\|a\|-|\operatorname{Re}\langle x,a\rangle| \leq \|x\|\,\|a\|-\operatorname{Re}\langle x,a\rangle\leq\frac{1}{2}r^2. \tag{6.142}$$

The constant $\frac{1}{2}$ is best possible in (6.141) in the sense that it cannot be replaced by a smaller quantity.

An additive version for the Schwarz inequality that may be more useful in applications is incorporated in [**8**] (see also [**7**, p. 28]).

THEOREM 90 (Dragomir, 2004). *Let* $(H; \langle \cdot, \cdot \rangle)$ *be an inner product space over* $\mathbb{K}$ *and* $x, y \in H$ *and* $\gamma, \Gamma \in \mathbb{K}$ *with* $\Gamma \neq -\gamma$ *and either*

$$\operatorname{Re} \langle \Gamma y - x, x - \gamma y \rangle \geq 0, \tag{6.143}$$

or, equivalently,

$$\left\| x - \frac{\gamma + \Gamma}{2} y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \, \|y\| \tag{6.144}$$

holds. Then we have the inequalities

$$\begin{aligned} 0 &\leq \|x\| \, \|y\| - |\langle x, y \rangle| \\ &\leq \|x\| \, \|y\| - \left| \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \cdot \langle x, y \rangle \right] \right| \\ &\leq \|x\| \, \|y\| - \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \cdot \langle x, y \rangle \right] \\ &\leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \|y\|^2 . \end{aligned} \tag{6.145}$$

The constant $\frac{1}{4}$ *in the last inequality is best possible.*

We remark that a simpler version of the above result may be stated if one assumed that the scalars are real:

COROLLARY 56. *If* $M \geq m > 0$, *and either*

$$\operatorname{Re} \langle My - x, x - my \rangle \geq 0, \tag{6.146}$$

or, equivalently,

$$\left\| x - \frac{m + M}{2} y \right\| \leq \frac{1}{2} (M - m) \|y\| \tag{6.147}$$

holds, then

$$\begin{aligned} 0 &\leq \|x\| \, \|y\| - |\langle x, y \rangle| \\ &\leq \|x\| \, \|y\| - |\operatorname{Re} \langle x, y \rangle| \\ &\leq \|x\| \, \|y\| - \operatorname{Re} \langle x, y \rangle \\ &\leq \frac{1}{4} \cdot \frac{(M - m)^2}{M + m} \|y\|^2 . \end{aligned} \tag{6.148}$$

The constant $\frac{1}{4}$ *is sharp.*

Now, let $(K, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{K}$, $p_i \geq 0$, $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} p_i = 1$. Consider $\ell_{\mathbf{p}}^2(K)$ as the space

$$\ell_{\mathbf{p}}^2(K) := \left\{ x = (x_i) \,|x_i \in K,\ i \in \mathbb{N} \ \text{ and } \ \sum_{i=1}^{\infty} p_i \|x_i\|^2 < \infty \right\}.$$

It is well known that $\ell_{\mathbf{p}}^2(K)$ endowed with the inner product

$$\langle x, y\rangle_{\mathbf{p}} := \sum_{i=1}^{\infty} p_i \langle x_i, y_i\rangle$$

is a Hilbert space over $\mathbb{K}$. The norm $\|\cdot\|_{\mathbf{p}}$ of $\ell_{\mathbf{p}}^2(K)$ is given by

$$\|x\|_{\mathbf{p}} := \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}}.$$

If $x, y \in \ell_{\mathbf{p}}^2(K)$, then the following Cauchy-Bunyakovsky-Schwarz (CBS) inequality holds true:

$$\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 \geq \left| \sum_{i=1}^{\infty} p_i \langle x_i, y_i\rangle \right|^2 \tag{6.149}$$

with equality iff there exists a $\lambda \in \mathbb{K}$ such that $x_i = \lambda y_i$ for each $i \in \mathbb{N}$.

If

$$\alpha \in \ell_{\mathbf{p}}^2(K) := \left\{ \alpha = (\alpha_i)_{i\in\mathbb{N}} \middle|\, \alpha_i \in \mathbb{K},\ i \in \mathbb{N} \ \text{ and } \ \sum_{i=1}^{\infty} p_i |\alpha_i|^2 < \infty \right\}$$

and $x \in \ell_{\mathbf{p}}^2(K)$, then the following (CBS)-type inequality is also valid:

$$\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \geq \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2 \tag{6.150}$$

with equality if and only if there exists a vector $v \in K$ such that $x_i = \overline{\alpha_i} v$ for each $i \in \mathbb{N}$.

In [**11**], by the use of some preliminary results obtained in [**9**], various reverses for the (CBS)-type inequalities (6.149) and (6.150) for sequences of vectors in Hilbert spaces were obtained. Applications for bounding the distance to a finite-dimensional subspace and in reversing the generalised triangle inequality have also been provided.

The aim of the present section is to provide different results by employing some inequalities discovered in [**8**]. Similar applications are pointed out.

6.3.2. Reverses of the (CBS)-Inequality for Two Sequences in $\ell_{\mathbf{p}}^2(K)$. The following proposition may be stated [**12**].

Proposition 63. *Let $x, y \in \ell_{\mathbf{p}}^2(K)$ and $r > 0$. If*

$$\|x_i - y_i\| \leq r \quad \text{for each} \quad i \in \mathbb{N}, \tag{6.151}$$

then

$$\begin{aligned}(0 \leq) & \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{\infty} p_i \langle x_i, y_i\rangle\right| \\ & \leq \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{\infty} p_i \operatorname{Re}\langle x_i, y_i\rangle\right| \\ & \leq \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}} - \sum_{i=1}^{\infty} p_i \operatorname{Re}\langle x_i, y_i\rangle \\ & \leq \frac{1}{2} r^2.\end{aligned} \tag{6.152}$$

The constant $\frac{1}{2}$ in front of r^2 is best possible in the sense that it cannot be replaced by a smaller quantity.

Proof. If (6.151) holds true, then

$$\|x - y\|_{\mathbf{p}}^2 = \sum_{i=1}^{\infty} p_i \|x_i - y_i\|^2 \leq r^2 \sum_{i=1}^{\infty} p_i = r^2$$

and thus $\|x - y\|_{\mathbf{p}} \leq r$.

Applying the inequality (6.142) for the inner product $\left(\ell_{\mathbf{p}}^2(K), \langle\cdot,\cdot\rangle_{\mathbf{p}}\right)$, we deduce the desired result (6.152).

The sharpness of the constant follows by Theorem 89 and we omit the details. ∎

The following result may be stated as well [**12**].

Proposition 64. *Let $x, y \in \ell_{\mathbf{p}}^2(K)$ and $\gamma, \Gamma \in \mathbb{K}$ with $\Gamma \neq -\gamma$. If either*

$$\operatorname{Re}\langle \Gamma y_i - x_i, x_i - \gamma y_i\rangle \geq 0 \quad \text{for each} \quad i \in \mathbb{N} \tag{6.153}$$

or, equivalently,

$$\left\|x_i - \frac{\gamma + \Gamma}{2} y_i\right\| \leq \frac{1}{2} |\Gamma - \gamma| \|y_i\| \quad \text{for each} \quad i \in \mathbb{N} \tag{6.154}$$

holds, then:

$$(0 \leq) \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 \right)^{\frac{1}{2}} - \left| \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right| \tag{6.155}$$
$$\leq \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 \right)^{\frac{1}{2}} - \left| \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right] \right|$$
$$\leq \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 \right)^{\frac{1}{2}} - \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right]$$
$$\leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \sum_{i=1}^{\infty} p_i \|y_i\|^2 .$$

The constant $\frac{1}{4}$ *is best possible in (6.155).*

Proof. Since, by (6.153),

$$\operatorname{Re} \langle \Gamma y - x, x - \gamma y \rangle_{\mathbf{p}} = \sum_{i=1}^{\infty} p_i \operatorname{Re} \langle \Gamma y_i - x_i, x_i - \gamma y_i \rangle \geq 0,$$

hence, on applying the inequality (6.145) for the Hilbert space $\left(\ell_{\mathbf{p}}^2 (K), \langle \cdot, \cdot \rangle_{\mathbf{p}} \right)$, we deduce the desired inequality (6.155).

The best constant follows by Theorem 90 and we omit the details. ∎

Corollary 57. *If the conditions (6.153) and (6.154) hold for* $\Gamma = M$, $\gamma = m$ *with* $M \geq m > 0$, *then*

$$(0 \leq) \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 \right)^{\frac{1}{2}} - \left| \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right| \tag{6.156}$$
$$\leq \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 \right)^{\frac{1}{2}} - \left| \sum_{i=1}^{\infty} p_i \operatorname{Re} \langle x_i, y_i \rangle \right|$$

$$\leq \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}} - \sum_{i=1}^{\infty} p_i \operatorname{Re}\langle x_i, y_i\rangle$$
$$\leq \frac{1}{4} \cdot \frac{(M-m)^2}{M+m} \sum_{i=1}^{\infty} p_i \|y_i\|^2 .$$

The constant $\frac{1}{4}$ *is best possible.*

6.3.3. Reverses of the (CBS)-Inequality for Mixed Sequences. The following result holds [**12**]:

Theorem 91 (Dragomir, 2005). *Let* $\alpha \in \ell_{\mathbf{p}}^2(K)$, $x \in \ell_{\mathbf{p}}^2(K)$ *and* $v \in K \setminus \{0\}$, $r > 0$. *If*

$$\|x_i - \overline{\alpha_i} v\| \leq r |\alpha_i| \quad \textit{for each } i \in \mathbb{N} \tag{6.157}$$

(note that if $\alpha_i \neq 0$ *for any* $i \in \mathbb{N}$*, then the condition (6.157) is equivalent to the simpler one*

$$\left\|\frac{x_i}{\overline{\alpha_i}} - v\right\| \leq r \quad \textit{for each } i \in \mathbb{N}), \tag{6.158}$$

then

$$(0 \leq) \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\| \tag{6.159}$$
$$\leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \left|\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, \frac{v}{\|v\|}\right\rangle\right|$$
$$\leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \left|\operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, \frac{v}{\|v\|}\right\rangle\right|$$
$$\leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, \frac{v}{\|v\|}\right\rangle$$
$$\leq \frac{1}{2} \cdot \frac{r^2}{\|v\|} \sum_{i=1}^{\infty} p_i |\alpha_i|^2 .$$

The constant $\frac{1}{2}$ *is best possible in (6.159).*

Proof. From (6.157) we deduce

$$\|x_i\|^2 - 2\operatorname{Re}\langle \alpha_i x_i, v\rangle + |\alpha_i|^2 \|v\|^2 \leq r^2 |\alpha_i|^2 ,$$

which is clearly equivalent to

$$\|x_i\|^2 + |\alpha_i|^2 \|v\|^2 \leq 2\operatorname{Re}\langle \alpha_i x_i, v\rangle + r^2 |\alpha_i|^2 \tag{6.160}$$

for each $i \in \mathbb{N}$.

If we multiply (6.160) by $p_i \geq 0$, $i \in \mathbb{N}$ and sum over $i \in \mathbb{N}$, then we deduce

$$\sum_{i=1}^{\infty} p_i \|x_i\|^2 + \|v\|^2 \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \leq 2 \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, v \right\rangle + r^2 \sum_{i=1}^{\infty} p_i |\alpha_i|^2 . \tag{6.161}$$

Since, obviously

$$2 \|v\| \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} \leq \sum_{i=1}^{\infty} p_i \|x_i\|^2 + \|v\|^2 \sum_{i=1}^{\infty} p_i |\alpha_i|^2 , \tag{6.162}$$

hence, by (6.161) and (6.162), we deduce

$$2 \|v\| \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} \leq 2 \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, v \right\rangle + r^2 \sum_{i=1}^{\infty} p_i |\alpha_i|^2 ,$$

which is clearly equivalent to the last inequality in (6.159).

The other inequalities are obvious.

The best constant follows by Theorem 89. ∎

The following corollary may be stated [**12**].

Corollary 58. *Let $\alpha \in \ell^2_{\mathbf{p}}(K)$, $x \in \ell^2_{\mathbf{p}}(K)$, $e \in H$, $\|e\| = 1$ and $\gamma, \Gamma \in \mathbb{K}$ with $\Gamma \neq -\gamma$. If*

$$\left\| x_i - \overline{\alpha_i} \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2} |\Gamma - \gamma| |\alpha_i| \tag{6.163}$$

for each $i \in \mathbb{N}$, or, equivalently,

$$\operatorname{Re} \langle \Gamma \overline{\alpha_i} e - x_i, x_i - \gamma \overline{\alpha_i} e \rangle \tag{6.164}$$

for each $i \in \mathbb{N}$ (note that, if $\alpha_i \neq 0$ for any $i \in \mathbb{N}$, then (6.163) is equivalent to

$$\left\| \frac{x_i}{\overline{\alpha_i}} - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma| \tag{6.165}$$

for each $i \in \mathbb{N}$ *and (6.164) is equivalent to*

$$\operatorname{Re}\left\langle \Gamma e - \frac{x_i}{\overline{\alpha_i}}, \frac{x_i}{\overline{\alpha_i}} - \gamma e \right\rangle \geq 0 \tag{6.166}$$

for each $i \in \mathbb{N}$*), then the following reverse of the (CBS)-inequality is valid:*

$$\begin{aligned}
(0 \leq) & \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\| \\
& \leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \left| \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e \right\rangle \right| \\
& \leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} \\
& \qquad - \left| \operatorname{Re}\left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e \right\rangle \right] \right| \\
& \leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} \\
& \qquad - \operatorname{Re}\left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e \right\rangle \right] \\
& \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \sum_{i=1}^{\infty} p_i |\alpha_i|^2 .
\end{aligned} \tag{6.167}$$

The constant $\frac{1}{4}$ *is best possible.*

Remark 75. *If* $M \geq m > 0$, $\alpha_i \neq 0$ *and for* e *as above, either*

$$\left\| \frac{x_i}{\overline{\alpha_i}} - \frac{M+m}{2} e \right\| \leq \frac{1}{2}(M-m) \quad \text{for each} \quad i \in \mathbb{N} \tag{6.168}$$

or, equivalently,

$$\operatorname{Re}\left\langle Me - \frac{x_i}{\overline{\alpha_i}}, \frac{x_i}{\overline{\alpha_i}} - me \right\rangle \geq 0 \quad \text{for each} \quad i \in \mathbb{N}$$

holds, then

$$(0 \leq) \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|$$

$$\leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \left|\left\langle \sum_{i=1}^{\infty} p_i\alpha_i x_i, e\right\rangle\right|$$
$$\leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \left|\operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i\alpha_i x_i, e\right\rangle\right|$$
$$\leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i\alpha_i x_i, e\right\rangle$$
$$\leq \frac{1}{4} \cdot \frac{(M-m)^2}{M+m} \sum_{i=1}^{\infty} p_i |\alpha_i|^2 .$$

The constant $\frac{1}{4}$ is best possible.

6.3.4. Reverses for the Generalised Triangle Inequality. In 1966, Diaz and Metcalf [**5**] proved the following interesting reverse of the generalised triangle inequality:

$$r \sum_{i=1}^{\infty} \|x_i\| \leq \left\| \sum_{i=1}^{\infty} x_i \right\|, \tag{6.169}$$

provided the vectors $x_1, \dots, x_n \in H \backslash \{0\}$ satisfy the assumption

$$0 \leq r \leq \frac{\operatorname{Re}\langle x_i, a\rangle}{\|x_i\|}, \qquad i \in \{1, \dots, n\}, \tag{6.170}$$

where $a \in H$, $\|a\| = 1$ and $(H; \langle\cdot,\cdot\rangle)$ is a real or complex inner product space.

In an attempt to provide other sufficient conditions for (6.169) to hold, the author pointed out in [**14**] that

$$\sqrt{1-\rho^2} \sum_{i=1}^{\infty} \|x_i\| \leq \left\| \sum_{i=1}^{\infty} x_i \right\| \tag{6.171}$$

where the vectors x_i, $i \in \{1, \dots, n\}$ satisfy the condition

$$\|x_i - a\| \leq \rho, \qquad i \in \{1, \dots, n\}, \tag{6.172}$$

where $r \in H$, $\|a\| = 1$ and $\rho \in (0, 1)$.

Following [**14**], if $M \geq m > 0$ and the vectors $x_i \in H$, $i \in \{1, \dots, n\}$ verify either

$$\operatorname{Re}\langle Ma - x_i, x_i - ma\rangle \geq 0, \qquad i \in \{1, \dots, n\}, \tag{6.173}$$

or, equivalently,

$$\left\| x_i - \frac{M+m}{2} \cdot a \right\| \leq \frac{1}{2}(M-m), \qquad i \in \{1, \dots, n\}, \tag{6.174}$$

where $a \in H$, $\|a\| = 1$, then

$$\frac{2\sqrt{mM}}{M+m}\sum_{i=1}^{n}\|x_i\| \leq \left\|\sum_{i=1}^{n}x_i\right\|. \tag{6.175}$$

It is obvious from Theorem 91, that, if

$$\|x_i - v\| \leq r, \qquad \text{for} \quad i \in \{1,\dots,n\}, \tag{6.176}$$

where $x_i \in H$, $i \in \{1,\dots,n\}$, $v \in H\setminus\{0\}$ and $r > 0$, then we can state the inequality

$$\begin{aligned}(0 \leq)\left(\frac{1}{n}\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} - \left\|\frac{1}{n}\sum_{i=1}^{n}x_i\right\| &\leq \left(\frac{1}{n}\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} - \left|\left\langle\frac{1}{n}\sum_{i=1}^{n}x_i, \frac{v}{\|v\|}\right\rangle\right| \\ &\leq \left(\frac{1}{n}\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} - \left|\operatorname{Re}\left\langle\frac{1}{n}\sum_{i=1}^{n}x_i, \frac{v}{\|v\|}\right\rangle\right| \\ &\leq \left(\frac{1}{n}\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} - \operatorname{Re}\left\langle\frac{1}{n}\sum_{i=1}^{n}x_i, \frac{v}{\|v\|}\right\rangle \\ &\leq \frac{1}{2}\cdot\frac{r^2}{\|v\|}.\end{aligned} \tag{6.177}$$

Since, by the (CBS)-inequality we have

$$\frac{1}{n}\sum_{i=1}^{n}\|x_i\| \leq \left(\frac{1}{n}\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}}, \tag{6.178}$$

hence, by (6.177) and (6.173) we have [**12**]:

$$(0 \leq)\sum_{i=1}^{n}\|x_i\| - \left\|\sum_{i=1}^{n}x_i\right\| \leq \frac{1}{2}n\cdot\frac{r^2}{\|v\|} \tag{6.179}$$

provided that (6.176) holds true.

Utilising Corollary 58, we may state that, if

$$\left\|x_i - \frac{\gamma+\Gamma}{2}\cdot e\right\| \leq \frac{1}{2}|\Gamma-\gamma|, \qquad i \in \{1,\dots,n\}, \tag{6.180}$$

or, equivalently,

$$\operatorname{Re}\langle\Gamma e - x_i, x_i - \gamma e\rangle \geq 0, \qquad i \in \{1,\dots,n\}, \tag{6.181}$$

where $e \in H$, $\|e\| = 1, \gamma, \Gamma \in \mathbb{K}$, $\Gamma \neq -\gamma$ and $x_i \in H$, $i \in \{1, \dots, n\}$, then

$$\begin{aligned}(6.182)\qquad (0 \leq) & \left(\frac{1}{n}\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} - \left\|\frac{1}{n}\sum_{i=1}^{n} x_i\right\| \\ & \leq \left(\frac{1}{n}\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} - \left|\left\langle \frac{1}{n}\sum_{i=1}^{n} x_i, e\right\rangle\right| \\ & \leq \left(\frac{1}{n}\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} - \left|\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\left\langle \frac{1}{n}\sum_{i=1}^{n} x_i, e\right\rangle\right]\right| \\ & \leq \left(\frac{1}{n}\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} - \operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\left\langle \frac{1}{n}\sum_{i=1}^{n} x_i, e\right\rangle\right] \\ & \leq \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}.\end{aligned}$$

Now, making use of (6.178) and (6.182) we can establish the following additive reverse of the generalised triangle inequality [**12**]

$$(6.183)\qquad (0 \leq) \sum_{i=1}^{n}\|x_i\| - \left\|\sum_{i=1}^{n} x_i\right\| \leq \frac{1}{4}n \cdot \frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|},$$

provided either (6.180) or (6.181) hold true.

6.3.5. Applications for Fourier Coefficients. Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space over the real or complex number field $\mathbb{K}$ and $\{e_i\}_{i\in I}$ an *orthonormal basis* for H. Then (see for instance [**4**, p. 54 – 61]):

(i) Every element $x \in H$ can be expanded in a *Fourier series*, i.e.,
$$x = \sum_{i\in I}\langle x, e_i\rangle e_i,$$
where $\langle x, e_i\rangle$, $i \in I$ are the *Fourier coefficients* of x;

(ii) (Parseval identity)
$$\|x\|^2 = \sum_{i\in I}\langle x, e_i\rangle e_i, \quad x \in H;$$

(iii) (Extended Parseval's identity)
$$\langle x, y\rangle = \sum_{i\in I}\langle x, e_i\rangle\langle e_i, y\rangle, \qquad x, y \in H;$$

(iv) (Elements are uniquely determined by their Fourier coefficients)

$$\langle x, e_i \rangle = \langle y, e_i \rangle \quad \text{for every } i \in I \text{ implies that } x = y.$$

We must remark that all the results from the second and third sections may be stated for $K = \mathbb{K}$ where $\mathbb{K}$ is the Hilbert space of complex (real) numbers endowed with the usual norm and inner product .

Therefore we can state the following reverses of the Schwarz inequality [**12**]:

PROPOSITION 65. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be a Hilbert space over* $\mathbb{K}$ *and* $\{e_i\}_{i\in I}$ *an orthonormal base for* H. *If* $x, y \in H$, $y \neq 0$, $a \in \mathbb{K}$ $(\mathbb{C}, \mathbb{R})$ *with* $r > 0$ *such that*

$$\left| \frac{\langle x, e_i \rangle}{\langle y, e_i \rangle} - a \right| \leq r \qquad \text{for each } i \in I, \tag{6.184}$$

then we have the following reverse of the Schwarz inequality:

$$\begin{aligned} (0 \leq) \|x\| \|y\| - |\langle x, y \rangle| &\\ \leq \|x\| \|y\| - \left| \operatorname{Re}\left[\langle x, y \rangle \cdot \frac{\bar{a}}{|a|} \right] \right| &\\ \leq \|x\| \|y\| - \operatorname{Re}\left[\langle x, y \rangle \cdot \frac{\bar{a}}{|a|} \right] &\\ \leq \frac{1}{2} \cdot \frac{r^2}{|a|} \|y\|^2 . & \end{aligned} \tag{6.185}$$

The constant $\frac{1}{2}$ *is best possible in (6.185).*

The proof is similar to the one in Theorem 91, where instead of x_i we take $\langle x, e_i \rangle$, instead of α_i we take $\langle e_i, y \rangle$, $\|\cdot\| = |\cdot|$, $p_i = 1$ and use the Parseval identities mentioned above in (ii) and (iii). We omit the details.

The following result may be stated as well [**12**].

PROPOSITION 66. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be a Hilbert space over* $\mathbb{K}$ *and* $\{e_i\}_{i\in I}$ *an orthonormal base for* H. *If* $x, y \in H$, $y \neq 0$, $e, \gamma, \Gamma \in \mathbb{K}$ *with* $|e| = 1$, $\Gamma \neq -\gamma$ *and*

$$\left| \frac{\langle x, e_i \rangle}{\langle y, e_i \rangle} - \frac{\gamma + \Gamma}{2} \cdot e \right| \leq \frac{1}{2} |\Gamma - \gamma| \tag{6.186}$$

or equivalently,

$$\operatorname{Re}\left[\left(\Gamma e - \frac{\langle x, e_i \rangle}{\langle y, e_i \rangle} \right) \left(\frac{\langle e_i, x \rangle}{\langle e_i, y \rangle} - \bar{\gamma} \bar{e} \right) \right] \geq 0 \tag{6.187}$$

for each $i \in I$, *then*

$$(0 \leq) \|x\| \|y\| - |\langle x, y \rangle| \leq \|x\| \|y\| - \left| \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \langle x, y \rangle \cdot \bar{e} \right] \right| \leq \|x\| \|y\| - \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \langle x, y \rangle \cdot \bar{e} \right] \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \|y\|^2 . \tag{6.188}$$

The constant $\frac{1}{4}$ *is best possible.*

REMARK 76. *If* $\Gamma = M \geq m = \gamma > 0$, *then one may state simpler inequalities from (6.188). We omit the details.*

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