A Survey on Cauchy-Buniakowsky-Schwartz Type Discrete Inequalities

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January 20, 2003

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Preface

The Cauchy-Buniakowski-Schwartz inequality, or for short, the $(CBS)-$ inequality, plays an important role in different branches of Modern Mathematics including Hilbert Spaces Theory, Probability & Statistics, Classical Real and Complex Analysis, Numerical Analysis, Qualitative Theory of Differential Equations and their applications.

The main purpose of this survey is to identify and highlight the discrete inequalities that are connected with $(CBS)-$ inequality and provide refinements, counterparts and reverse results as well as to study some functional properties of certain mappings that can be naturally associated with this inequality such as superadditivity, supermultiplicity, the strong versions of these and the corresponding monotonicity properties. Many companions and related results both for real and complex numbers are also presented.

The first chapter is devoted to a number of $(CBS)-$ type inequalities that provides not only natural generalizations but also several extensions for different classes of analytic functions of a real variable. A generalization of the Wagner inequality for complex numbers is obtained. Several results discovered by the author in the late eighties and published in different journals of lesser circulation are also surveyed.

The second chapter contains different refinements of the $(CBS)-$ inequality including de Bruijn's inequality, McLaughlin's inequality, the Daykin-Eliezer-Carlitz result in the version presented by Mitrinović-Pečarić and Fink as well as the refinements of a particular version obtained by Alzer and Zheng. A number of new results obtained by the author, which are connected with the above ones, are also presented.

Chapter 3 is devoted to the study of functional properties of different mappings naturally associated to the $(CBS)-$ inequality. Properties such as superadditivity, strong superadditivity, monotonicity and supermultiplicity and the corresponding inequalities are mentioned.

In the next chapter, Chapter 4, counterpart results for the $(CBS)-$ inequality are surveyed. The results of Cassels, Pólya-Szegö, Greub-Rheinbold, Shisha-Mond and Zagier are presented with their original proofs. New results and versions for complex numbers are also obtained. Counterparts in terms of $p-$norms of the forward difference recently discovered by the author and some refinements of Cassels and Pólya-Szegö results obtained via Andrica-Badea inequality are mentioned. Some new facts derived from Grüss type inequalities are also pointed out.

Chapter 5 is devoted to various inequalities related to the $(CBS)-$ inequality. The two inequalities obtained by Ostrowski and Fan-Todd results are presented. New inequalities obtained via Jensen type inequality for convex functions are derived, some inequalities for the Čebyşev functionals are pointed out. Versions for complex numbers that generalize Ostrowski results are also emphasised.

It was one of the main aims of the survey to provide complete proofs for the results considered. We also note that in most cases only the original references are mentioned.

Being self contained, the survey may be used by both postgraduate students and researchers interested in Theory of Inequalities & Applications as well as by Mathematicians and other Scientists dealing with numerical computations, bounds and estimates where the $(CBS)-$ inequality may be used as a powerful tool.

The author intends to continue this survey with another one devoted to the functional and integral versions of the $(CBS)-$ inequality. The corresponding results holding in inner-product and normed spaces will be considered as well.

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December 2002.
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Chapter 1

(CBS) – Type Inequalities

1.1 (CBS) –Inequality for Real Numbers

The following inequality is known in the literature as *Cauchy's* or *Cauchy-Schwartz's* or *Cauchy-Buniakowski-Schwartz's* inequality. For simplicity, we shall refer to it throughout this work as the (CBS) –inequality.

Theorem 1 *If* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ *and* $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *are sequences of real numbers, then*

$$\left(\sum_{k=1}^{n} a_k b_k \right)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \tag{1.1}$$

with equality if and only if the sequences $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *are proportional, i.e., there is a* $r \in \mathbb{R}$ *such that* $a_k = r b_k$ *for each* $k \in \{1, \dots, n\}$.

Proof.

1. Consider the quadratic polynomial $P : \mathbb{R} \to \mathbb{R}$,

$$P(t) = \sum_{k=1}^{n} (a_k t - b_k)^2 . \tag{1.2}$$

It is obvious that

$$P(t) = \left(\sum_{k=1}^{n} a_k^2 \right) t^2 - 2 \left(\sum_{k=1}^{n} a_k b_k \right) t + \sum_{k=1}^{n} b_k^2$$

for any $t \in \mathbb{R}$.

Since $P(t) \geq 0$ for any $t \in \mathbb{R}$ it follows that the discriminant Δ of P is negative, i.e.,

$$0 \geq \frac{1}{4}\Delta = \left(\sum_{k=1}^{n} a_k b_k\right)^2 - \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2$$

and the inequality (1.1) is proved.

2. If we use Lagrange's identity

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \frac{1}{2}\sum_{i,j=1}^{n} (a_i b_j - a_j b_i)^2 = \sum_{1\leq i<j\leq n} (a_i b_j - a_j b_i)^2 \tag{1.3}$$

then (1.1) obviously holds.

The equality holds in (1.1) iff

$$(a_i b_j - a_j b_i)^2 = 0$$

for any $i, j \in \{1, \ldots, n\}$ which is equivalent with the fact that $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ are proportional. ■

Remark 2 *The inequality (1.1) apparently was firstly mentioned in the work [2] of A.L. Cauchy in 1821. The integral form was obtained in 1859 by V. Buniakowski [1]. The corresponding version for inner-product spaces is mainly known as Schwartz's inequality.*

1.2 (CBS) −Inequality for Complex Numbers

The following version of the (CBS) −inequality for complex numbers holds [3, p. 84].

Theorem 3 *If* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$ *and* $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ *are sequences of complex numbers, then*

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 \leq \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2, \tag{1.4}$$

with equality if and only if there is a complex number $c \in \mathbb{C}$ such that $a_k = c\bar{b}_k$ for any $k \in \{1, \dots, n\}$.

Proof.

1. For any complex number $\lambda \in \mathbb{C}$ one has the equality

$$\sum_{k=1}^{n} \left|a_k - \lambda \bar{b}_k\right|^2 = \sum_{k=1}^{n} \left(a_k - \lambda \bar{b}_k\right)\left(\bar{a}_k - \bar{\lambda} b_k\right) \tag{1.5}$$
$$= \sum_{k=1}^{n} |a_k|^2 + |\lambda|^2 \sum_{k=1}^{n} |b_k|^2 - 2\,\mathrm{Re}\left(\bar{\lambda} \sum_{k=1}^{n} a_k b_k\right).$$

If in (1.5) we choose $\lambda_0 \in \mathbb{C}$,

$$\lambda_0 := \frac{\sum_{k=1}^{n} a_k b_k}{\sum_{k=1}^{n} |b_k|^2}, \qquad \bar{\mathbf{b}} \neq \mathbf{0}$$

then we get the identity

$$0 \le \sum_{k=1}^{n} \left|a_k - \lambda_0 \bar{b}_k\right|^2 = \sum_{k=1}^{n} |a_k|^2 - \frac{\left|\sum_{k=1}^{n} a_k b_k\right|^2}{\sum_{k=1}^{n} |b_k|^2}, \tag{1.6}$$

which proves (1.4).

By virtue of (1.6), we conclude that equality holds in (1.4) if and only if $a_k = \lambda_0 \bar{b}_k$ for any $k \in \{1, \dots, n\}$.

2. Using Binet-Cauchy's identity for complex numbers

$$\sum_{i=1}^{n} x_i y_i \sum_{i=1}^{n} z_i t_i - \sum_{i=1}^{n} x_i t_i \sum_{i=1}^{n} z_i y_i \tag{1.7}$$
$$= \frac{1}{2} \sum_{i,j=1}^{n} (x_i z_j - x_j z_i)(y_i t_j - y_j t_i)$$
$$= \sum_{1 \le i < j \le n} (x_i z_j - x_j z_i)(y_i t_j - y_j t_i)$$

for the choices $x_i = \bar{a}_i$, $z_i = b_i$, $y_i = a_i$, $t_i = \bar{b}_i$, $i = \{1, \dots, n\}$, we get

$$\sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \left| \sum_{i=1}^{n} a_i b_i \right|^2 = \frac{1}{2} \sum_{i,j=1}^{n} |\bar{a}_i b_j - \bar{a}_j b_i|^2 = \sum_{1 \le i < j \le n} |\bar{a}_i b_j - \bar{a}_j b_i|^2 . \tag{1.8}$$

Now the inequality (1.4) is a simple consequence of (1.8).

The case of equality is obvious by the identity (1.8) as well.

■

Remark 4 *By the (CBS) −inequality for real numbers and the generalised triangle inequality for complex numbers*

$$\sum_{i=1}^{n} |z_i| \ge \left| \sum_{i=1}^{n} z_i \right|, \quad z_i \in \mathbb{C}, \ \ i \in \{1, \dots, n\}$$

we also have

$$\left| \sum_{k=1}^{n} a_k b_k \right|^2 \le \left(\sum_{k=1}^{n} |a_k b_k| \right)^2 \le \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 .$$

Remark 5 *The Lagrange identity for complex numbers stated in [3, p. 85] is wrong. It should be corrected as in (1.8).*

1.3 An Additive Generalisation

The following generalisation of the (CBS) −inequality was obtained in [4, p. 5].

Theorem 6 *If $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$, $\bar{\mathbf{c}} = (c_1, \dots, c_n)$ and $\bar{\mathbf{d}} = (d_1, \dots, d_n)$ are sequences of real numbers and $\bar{\mathbf{p}} = (p_1, \dots, p_n)$, $\bar{\mathbf{q}} = (q_1, \dots, q_n)$ are nonnegative, then*

$$\sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} q_i b_i^2 + \sum_{i=1}^{n} p_i c_i^2 \sum_{i=1}^{n} q_i d_i^2 \ge 2 \sum_{i=1}^{n} p_i a_i c_i \sum_{i=1}^{n} q_i b_i d_i. \tag{1.9}$$

If $\bar{\mathbf{p}}$ and $\bar{\mathbf{q}}$ are sequences of positive numbers, then the equality holds in (1.9) iff $a_i b_j = c_i d_j$ for any $i, j \in \{1, \dots, n\}$.

Proof. We will follow the proof from [4].

From the elementary inequality

$$a^2 + b^2 \geq 2ab \text{ for any } a, b \in \mathbb{R} \tag{1.10}$$

with equality iff $a = b$, we have

$$a_i^2 b_j^2 + c_i^2 d_j^2 \geq 2a_i c_i b_j d_j \quad \text{for any} \quad i, j \in \{1, \dots, n\}. \tag{1.11}$$

Multiplying (1.11) by $p_i q_j \geq 0$, $i, j \in \{1, \dots, n\}$ and summing over i and j from 1 to n, we deduce (1.9).

If $p_i, q_j > 0$ $(i = 1, \dots, n)$, then the equality holds in (1.9) iff $a_i b_j = c_i d_j$ for any $i, j \in \{1, \dots, n\}$. ■

Remark 7 *The condition $a_i b_j = c_i d_j$ for $c_i \neq 0$, $b_j \neq 0$ $(i, j = 1, \dots, n)$ is equivalent with $\frac{a_i}{c_i} = \frac{d_j}{b_j}$ $(i, j = 1, \dots, n)$, i.e., $\bar{\mathbf{a}}$, $\bar{\mathbf{c}}$ and $\bar{\mathbf{b}}, \bar{\mathbf{d}}$ are proportional with the same constant k.*

Remark 8 *If in (1.9) we choose $p_i = q_i = 1$ $(i = 1, \dots, n)$, $c_i = b_i$, and $d_i = a_i$ $(i = 1, \dots, n)$, then we recapture the $(CBS)-$inequality.*

The following corollary holds [4, p. 6].

Corollary 9 *If $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, $\bar{\mathbf{c}}$ and $\bar{\mathbf{d}}$ are nonnegative, then*

$$\frac{1}{2}\left[\sum_{i=1}^{n} a_i^3 c_i \sum_{i=1}^{n} b_i^3 d_i + \sum_{i=1}^{n} c_i^3 a_i \sum_{i=1}^{n} d_i^3 b_i\right] \geq \sum_{i=1}^{n} a_i^2 c_i^2 \sum_{i=1}^{n} b_i^2 d_i^2, \tag{1.12}$$

$$\frac{1}{2}\left[\sum_{i=1}^{n} a_i^2 b_i d_i \cdot \sum_{i=1}^{n} b_i^2 a_i c_i + \sum_{i=1}^{n} c_i^2 b_i d_i \cdot \sum_{i=1}^{n} d_i^2 a_i c_i\right] \geq \left(\sum_{i=1}^{n} a_i b_i c_i d_i\right)^2. \tag{1.13}$$

Another result is embodied in the following corollary [4, p. 6].

Corollary 10 *If $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, $\bar{\mathbf{c}}$ and $\bar{\mathbf{d}}$ are sequences of positive and real numbers, then:*

$$\frac{1}{2}\left[\sum_{i=1}^{n} \frac{a_i^3}{c_i} \sum_{i=1}^{n} \frac{b_i^3}{d_i} + \sum_{i=1}^{n} a_i c_i \sum_{i=1}^{n} b_i d_i\right] \geq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2, \tag{1.14}$$

$$\frac{1}{2}\left[\sum_{i=1}^{n} \frac{a_i^2 b_i}{c_i} \sum_{i=1}^{n} \frac{b_i^2 a_i}{d_i} + \sum_{i=1}^{n} b_i c_i \sum_{i=1}^{n} a_i d_i\right] \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2. \tag{1.15}$$

Finally, we also have [4, p. 6].

Corollary 11 *If* $\bar{\mathbf{a}}$, *and* $\bar{\mathbf{b}}$ *are positive, then*

$$\frac{1}{2}\left[\sum_{i=1}^{n}\frac{a_i^3}{b_i}\sum_{i=1}^{n}\frac{b_i^3}{a_i}-\left(\sum_{i=1}^{n}a_ib_i\right)^2\right]\geq\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2-\left(\sum_{i=1}^{n}a_ib_i\right)^2\geq 0.$$

The following version for complex numbers also holds.

Theorem 12 *Let* $\bar{\mathbf{a}}=(a_1,\dots,a_n)$, $\bar{\mathbf{b}}=(b_1,\dots,b_n)$, $\bar{\mathbf{c}}=(c_1,\dots,c_n)$ *and* $\bar{\mathbf{d}}=(d_1,\dots,d_n)$ *be sequences of complex numbers and* $\bar{\mathbf{p}}=(p_1,\dots,p_n)$, $\bar{\mathbf{q}}=(q_1,\dots,q_n)$ *are nonnegative. Then one has the inequality*

$$\sum_{i=1}^{n}p_i|a_i|^2\sum_{i=1}^{n}q_i|b_i|^2+\sum_{i=1}^{n}p_i|c_i|^2\sum_{i=1}^{n}q_i|d_i|^2 \geq 2\operatorname{Re}\left[\sum_{i=1}^{n}p_ia_i\bar{c}_i\sum_{i=1}^{n}q_ib_i\bar{d}_i\right]. \quad (1.16)$$

The case of equality for $\bar{\mathbf{p}}$, $\bar{\mathbf{q}}$ *positive holds iff* $a_ib_j=c_id_j$ *for any* $i,j\in\{1,\dots,n\}$.

Proof. From the elementary inequality for complex numbers

$$|a|^2+|b|^2\geq 2\operatorname{Re}\left[a\bar{b}\right],\quad a,b\in\mathbb{C},$$

with equality iff $a=b$, we have

$$|a_i|^2|b_j|^2+|c_i|^2|d_j|^2\geq 2\operatorname{Re}\left[a_i\bar{c}_ib_j\bar{d}j\right] \quad (1.17)$$

for any $i,j\in\{1,\dots,n\}$. Multiplying (1.17) by $p_iq_j\geq 0$ and summing over i and j from 1 to n, we deduce (1.16).

The case of equality is obvious and we omit the details. ■

Remark 13 *Similar particular cases may be stated but we omit the details.*

1.4 A Related Additive Inequality

The following inequality was obtained in [4, Theorem 1.1].

Theorem 14 *If* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *are sequences of real numbers and* $\bar{\mathbf{c}} = (c_1, \dots, c_n)$, $\bar{\mathbf{d}} = (d_1, \dots, d_n)$ *are nonnegative, then*

$$\sum_{i=1}^{n} d_i \sum_{i=1}^{n} c_i a_i^2 + \sum_{i=1}^{n} c_i \sum_{i=1}^{n} d_i b_i^2 \geq 2 \sum_{i=1}^{n} c_i a_i \sum_{i=1}^{n} d_i b_i. \tag{1.18}$$

If c_i *and* d_i $(i = 1, \dots, n)$ *are positive, then equality holds in (1.18) iff* $\bar{\mathbf{a}} = \bar{\mathbf{b}} = \bar{\mathbf{k}}$ *where* $\bar{\mathbf{k}} = (k, k, \dots, k)$ *is a constant sequence.*

Proof. We will follow the proof from [4].

From the elementary inequality

$$a^2 + b^2 \geq 2ab \text{ for any } a, b \in \mathbb{R} \tag{1.19}$$

with equality iff $a = b$; we have

$$a_i^2 + b_j^2 \geq 2a_i b_j \text{ for any } i, j \in \{1, \dots, n\}. \tag{1.20}$$

Multiplying (1.20) by $c_i d_j \geq 0$, $i, j \in \{1, \dots, n\}$ and summing over i from 1 to n and over j from 1 to n, we deduce (1.18).

If $c_i, d_j > 0$ $(i = 1, \dots, n)$, then the equality holds in (1.18) iff $a_i = b_j$ for any $i, j \in \{1, \dots, n\}$ which is equivalent with the fact that $a_i = b_i = k$ for any $i \in \{1, \dots, n\}$. ■

The following corollary holds [4, p. 4].

Corollary 15 *If* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *are nonnegative sequences, then*

$$\frac{1}{2}\left[\sum_{i=1}^{n} a_i^3 \sum_{i=1}^{n} b_i + \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i^3\right] \geq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2; \tag{1.21}$$

$$\frac{1}{2}\left[\sum_{i=1}^{n} a_i \sum_{i=1}^{n} a_i^2 b_i + \sum_{i=1}^{n} b_i \sum_{i=1}^{n} b_i^2 a_i\right] \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2. \tag{1.22}$$

Another corollary that may be obtained is [4, p. 4 – 5].

Corollary 16 *If* $\mathbf{\bar{a}}$ *and* $\mathbf{\bar{b}}$ *are sequences of positive real numbers, then*

$$\sum_{i=1}^{n} \frac{a_i^2 + b_i^2}{2a_i b_i} \geq \frac{\sum_{i=1}^{n} \frac{1}{a_i} \sum_{i=1}^{n} \frac{1}{b_i}}{\sum_{i=1}^{n} \frac{1}{a_i b_i}}, \tag{1.23}$$

$$\sum_{i=1}^{n} a_i \sum_{i=1}^{n} \frac{1}{b_i} + \sum_{i=1}^{n} \frac{1}{a_i} \sum_{i=1}^{n} b_i \geq 2n^2, \tag{1.24}$$

and

$$n \sum_{i=1}^{n} \frac{a_i^2 + b_i^2}{2a_i^2 b_i^2} \geq \sum_{i=1}^{n} \frac{1}{a_i} \sum_{i=1}^{n} \frac{1}{b_i}. \tag{1.25}$$

The following version for complex numbers also holds.

Theorem 17 *If* $\mathbf{\bar{a}} = (a_1, \dots, a_n)$, $\mathbf{\bar{b}} = (b_1, \dots, b_n)$ *are sequences of complex numbers, then for* $\mathbf{\bar{p}} = (p_1, \dots, p_n)$ *and* $\mathbf{\bar{q}} = (q_1, \dots, q_n)$ *two sequences of nonnegative real numbers, one has the inequality*

$$\sum_{i=1}^{n} q_i \sum_{i=1}^{n} p_i |a_i|^2 + \sum_{i=1}^{n} p_i \sum_{i=1}^{n} q_i |b_i|^2 \geq 2 \operatorname{Re}\left[\sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} q_i \bar{b}_i\right]. \tag{1.26}$$

For $\mathbf{\bar{p}}, \mathbf{\bar{q}}$ *positive sequences, the equality holds in (1.26) iff* $\mathbf{\bar{a}} = \mathbf{\bar{b}} = \mathbf{\bar{k}} = (k, \dots, k)$.

The proof goes in a similar way with the one in Theorem 14 on making use of the following elementary inequality holding for complex numbers

$$|a|^2 + |b|^2 \geq 2 \operatorname{Re}\left[a\bar{b}\right], \quad a, b \in \mathbb{C}; \tag{1.27}$$

with equality iff $a = b$.

1.5 A Parameter Additive Inequality

The following inequality was obtained in [4, Theorem 4.1].

Theorem 18 *Let* $\mathbf{\bar{a}} = (a_1, \dots, a_n)$, $\mathbf{\bar{b}} = (b_1, \dots, b_n)$ *be sequences of real numbers and* $\mathbf{\bar{c}} = (c_1, \dots, c_n)$, $\mathbf{\bar{d}} = (d_1, \dots, d_n)$ *be nonnegative. If* $\alpha, \beta > 0$ *and* $\gamma \in \mathbb{R}$ *such that* $\gamma^2 \leq \alpha\beta$, *then*

$$\alpha \sum_{i=1}^{n} d_i \sum_{i=1}^{n} a_i^2 c_i + \beta \sum_{i=1}^{n} c_i \sum_{i=1}^{n} b_i^2 d_i \geq 2\gamma \sum_{i=1}^{n} c_i a_i \sum_{i=1}^{n} d_i b_i. \tag{1.28}$$

Proof. We will follow the proof from [4].

Since $\alpha, \beta > 0$ and $\gamma^2 \leq \alpha\beta$, it follows that for any $x, y \in \mathbb{R}$ one has

$$\alpha x^2 + \beta y^2 \geq 2\gamma xy. \tag{1.29}$$

Choosing in (1.29) $x = a_i$, $y = b_j$ $(i, j = 1, \dots, n)$, we get

$$\alpha a_i^2 + \beta b_j^2 \geq 2\gamma a_i b_j \text{ for any } i, j \in \{1, \dots, n\}. \tag{1.30}$$

If we multiply (1.30) by $c_i d_j \geq 0$ and sum over i and j from 1 to n, we deduce the desired inequality (1.28). ■

The following corollary holds.

Corollary 19 *If* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *are nonnegative sequences and* α, β, γ *are as in Theorem 18, then*

$$\alpha \sum_{i=1}^n b_i \sum_{i=1}^n a_i^3 + \beta \sum_{i=1}^n a_i \sum_{i=1}^n b_i^3 \geq 2\gamma \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2, \tag{1.31}$$

$$\alpha \sum_{i=1}^n a_i \sum_{i=1}^n a_i^2 b_i + \beta \sum_{i=1}^n b_i \sum_{i=1}^n b_i^2 a_i \geq 2\gamma \left(\sum_{i=1}^n a_i b_i \right)^2. \tag{1.32}$$

The following particular case is important [4, p. 8].

Theorem 20 *Let* $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ *be sequences of real numbers. If* $\bar{\mathbf{p}}$ *is a sequence of nonnegative real numbers with* $\sum_{i=1}^n p_i > 0$, *then:*

$$\sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 \geq \frac{\sum_{i=1}^n p_i a_i b_i \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i}{\sum_{i=1}^n p_i}. \tag{1.33}$$

In particular,

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \geq \frac{1}{n} \sum_{i=1}^n a_i b_i \sum_{i=1}^n a_i \sum_{i=1}^n b_i. \tag{1.34}$$

Proof. We will follow the proof from [4, p. 8].

If we choose in Theorem 18, $c_i = d_i = p_i$ $(i = 1, \dots, n)$ and $\alpha = \sum_{i=1}^n p_i b_i^2$, $\beta = \sum_{i=1}^n p_i a_i^2$, $\gamma = \sum_{i=1}^n p_i a_i b_i$, we observe, by the (CBS) −inequality with the weights p_i $(i = 1, \dots, n)$ one has $\gamma^2 \leq \alpha\beta$, and then by (1.28) we deduce (1.33). ■

Remark 21 *If we assume that* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *are asynchronous, i.e.,*

$$(a_i - a_j)(b_i - b_j) \leq 0 \text{ for any } i, j \in \{1, \dots, n\}$$

then by Čebyšev's inequality

$$\sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \geq \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i \tag{1.35}$$

respectively

$$\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \geq n \sum_{i=1}^{n} a_i b_i, \tag{1.36}$$

we have the following refinements of the $(CBS)-$*inequality*

$$\begin{aligned} \sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2 &\geq \frac{\sum_{i=1}^{n} p_i a_i b_i \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i}{\sum_{i=1}^{n} p_i} \\ &\geq \left(\sum_{i=1}^{n} p_i a_i b_i \right)^2 \end{aligned} \tag{1.37}$$

provided $\sum_{i=1}^{n} p_i a_i b_i \geq 0,$ *respectively*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \geq \frac{1}{n} \sum_{i=1}^{n} a_i b_i \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \geq \left(\sum_{i=1}^{n} a_i b_i \right)^2 \tag{1.38}$$

provided $\sum_{i=1}^{n} a_i b_i \geq 0.$

1.6 A Generalisation Provided by Young's Inequality

The following result was obtained in [4, Theorem 5.1].

Theorem 22 *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$, $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ *and* $\bar{\mathbf{q}} = (q_1, \dots, q_n)$ *be sequences of nonnegative real numbers and* $\alpha, \beta > 1$ *with* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. *Then one has the inequality*

$$\alpha \sum_{i=1}^{n} q_i \sum_{i=1}^{n} p_i b_i^{\beta} + \beta \sum_{i=1}^{n} p_i \sum_{i=1}^{n} q_i a_i^{\alpha} \geq \alpha\beta \sum_{i=1}^{n} p_i b_i \sum_{i=1}^{n} q_i a_i. \tag{1.39}$$

If $\bar{\mathbf{p}}$ and $\bar{\mathbf{q}}$ are sequences of positive real numbers, then the equality holds in (1.39) iff there exists a constant $k \geq 0$ such that $a_i^\alpha = b_i^\beta = k$ for each $i \in \{1, \dots, n\}$.

Proof. It is, by the Arithmetic-Geometric inequality [5, p. 15], well known that

$$\frac{1}{\alpha}x + \frac{1}{\beta}y \geq x^{\frac{1}{\alpha}} y^{\frac{1}{\beta}} \text{ for } x, y \geq 0, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \ \alpha, \beta > 1 \tag{1.40}$$

with equality iff $x = y$.

Applying (1.40) for $x = a_i^\alpha$, $y = b_j^\beta$ $(i, j = 1, \dots, n)$ we have

$$\alpha b_j^\beta + \beta a_i^\alpha \geq \alpha\beta a_i b_j \text{ for any } i, j \in \{1, \dots, n\} \tag{1.41}$$

with equality iff $a_i^\alpha = b_j^\beta$ for any $i, j \in \{1, \dots, n\}$.

If we multiply (1.41) by $q_i p_j \geq 0$ $(i, j \in \{1, \dots, n\})$ and sum over i and j from 1 to n we deduce (1.39).

The case of equality is obvious by the above considerations. ■

The following corollary is a natural consequence of the above theorem.

Corollary 23 *Let $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, α and β be as in Theorem 22. Then*

$$\frac{1}{\alpha}\sum_{i=1}^n b_i \sum_{i=1}^n a_i^{\alpha+1} + \frac{1}{\beta}\sum_{i=1}^n a_i \sum_{i=1}^n b_i^{\beta+1} \geq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2; \tag{1.42}$$

$$\frac{1}{\alpha}\sum_{i=1}^n a_i \sum_{i=1}^n b_i a_i^\alpha + \frac{1}{\beta}\sum_{i=1}^n b_i \sum_{i=1}^n a_i b_i^\beta \geq \left(\sum_{i=1}^n a_i b_i\right)^2. \tag{1.43}$$

The following result which provides a generalisation of the (CBS) −inequality may be obtained by Theorem 22 as well [4, Theorem 5.2].

Theorem 24 *Let $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ be sequences of positive real numbers. If α, β are as above, then*

$$\left(\frac{1}{\alpha}\sum_{i=1}^n x_i^\alpha y_i^{2-\alpha} + \frac{1}{\beta}\sum_{i=1}^n x_i^\beta y_i^{2-\beta}\right) \cdot \sum_{i=1}^n y_i^2 \geq \left(\sum_{i=1}^n x_i y_i\right)^2. \tag{1.44}$$

The equality holds iff $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are proportional.

Proof. Follows by Theorem 22 on choosing $p_i = q_i = y_i^2$, $a_i = \frac{x_i}{y_i}$, $b_i = \frac{x_i}{y_i}$, $i \in \{1, \ldots, n\}$. ■

Remark 25 *For $\alpha = \beta = 2$, we recapture the $(CBS)-$inequality.*

Remark 26 *For $a_i = |z_i|$, $b_i = |w_i|$, with $z_i, w_i \in \mathbb{C}$; $i = 1, \ldots, n$, we may obtain similar inequalities for complex numbers. We omit the details.*

1.7 Further Generalisations via Young's Inequality

The following inequality is known in the literature as Young's inequality

$$px^q + qy^p \geq pqxy, \quad x, y \geq 0 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \ p > 1 \tag{1.45}$$

with equality iff $x^q = y^p$.

The following result generalising the $(CBS)-$inequality was obtained in [6, Theorem 2.1] (see also [7, Theorem 1]).

Theorem 27 *Let $\bar{\mathbf{x}} = (x_1, \ldots, x_n)$, $\bar{\mathbf{y}} = (y_1, \ldots, y_n)$ be sequences of complex numbers and $\bar{\mathbf{p}} = (p_1, \ldots, p_n)$, $\bar{\mathbf{q}} = (q_1, \ldots, q_n)$ be two sequences of nonnegative real numbers. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\frac{1}{p}\sum_{k=1}^{n} p_k |x_k|^p \sum_{k=1}^{n} q_k |y_k|^p + \frac{1}{q}\sum_{k=1}^{n} q_k |x_k|^q \sum_{k=1}^{n} p_k |y_k|^q \geq \sum_{k=1}^{n} p_k |x_k y_k| \sum_{k=1}^{n} q_k |x_k y_k|. \tag{1.46}$$

Proof. We shall follow the proof in [6].

Choosing $x = |x_j| |y_i|$, $y = |x_i| |y_j|$, $i, j \in \{1, \ldots, n\}$, we get from (1.45)

$$q |x_i|^p |y_j|^p + p |x_j|^q |y_i|^q \geq pq |x_i y_i| |x_j y_j| \tag{1.47}$$

for any $i, j \in \{1, \ldots, n\}$.

Multiplying with $p_i q_j \geq 0$ and summing over i and j from 1 to n, we deduce the desired result (1.46). ■

The following corollary is a natural consequence of the above theorem [6, Corollary 2.2] (see also [7, p. 105]).

Corollary 28 *If $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are as in Theorem 27 and $\bar{\mathbf{m}} = (m_1, \dots, m_n)$ is a sequence of nonnegative real numbers, then*

$$\frac{1}{p}\sum_{k=1}^{n} m_k |x_k|^p \sum_{k=1}^{n} m_k |y_k|^p + \frac{1}{q}\sum_{k=1}^{n} m_k |x_k|^q \sum_{k=1}^{n} m_k |y_k|^q \geq \left(\sum_{k=1}^{n} m_k |x_k y_k|\right)^2, \quad (1.48)$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 29 *If in (1.48) we assume that $m_k = 1$, $k \in \{1, \dots, n\}$, then we obtain [6, p. 7] (see also [7, p. 105])*

$$\frac{1}{p}\sum_{k=1}^{n} |x_k|^p \sum_{k=1}^{n} |y_k|^p + \frac{1}{q}\sum_{k=1}^{n} |x_k|^q \sum_{k=1}^{n} |y_k|^q \geq \left(\sum_{k=1}^{n} |x_k y_k|\right)^2, \quad (1.49)$$

which, in the particular case $p = q = 2$ will provide the (CBS) −inequality.

The second generalisation of the (CBS) −inequality via Young's inequality is incorporated in the following theorem [6, Theorem 2.4] (see also [7, Theorem 2]).

Theorem 30 *Let $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$, $\bar{\mathbf{p}}$, $\bar{\mathbf{q}}$ and p, q be as in Theorem 27. Then one has the inequality*

$$\frac{1}{p}\sum_{k=1}^{n} p_k |x_k|^p \sum_{k=1}^{n} q_k |y_k|^q + \frac{1}{q}\sum_{k=1}^{n} q_k |x_k|^q \sum_{k=1}^{n} p_k |y_k|^p \geq \sum_{k=1}^{n} p_k |x_k| |y_k|^{p-1} \sum_{k=1}^{n} q_k |x_k| |y_k|^{q-1}. \quad (1.50)$$

Proof. We shall follow the proof in [6].

Choosing in (1.45), $x = \frac{|x_j|}{|y_j|}$, $y = \frac{|x_i|}{|y_i|}$, we get

$$p\left(\frac{|x_j|}{|y_j|}\right)^q + q\left(\frac{|x_i|}{|y_i|}\right)^p \geq pq\frac{|x_i| |x_j|}{|y_i| |y_j|} \quad (1.51)$$

for any $y_i \neq 0$, $i, j \in \{1, \dots, n\}$.

It is easy to see that (1.51) is equivalent to

$$q\left|x_i\right|^p\left|y_j\right|^q+p\left|y_i\right|^p\left|x_j\right|^q \geq pq\left|x_i\right|\left|y_i\right|^{p-1}\left|x_j\right|\left|y_j\right|^{q-1} \tag{1.52}$$

for any $i, j \in \{1, \ldots, n\}$.

Multiplying (1.52) by $p_i q_j \geq 0$ ($i, j \in \{1, \ldots, n\}$) and summing over i and j from 1 to n, we deduce the desired inequlality (1.50). ■

The following corollary holds [6, Corollary 2.5] (see also [7, p. 106]).

Corollary 31 *Let $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$, $\bar{\mathbf{m}}$ and $\bar{\mathbf{p}}$, $\bar{\mathbf{q}}$ be as in Corollary 28. Then*

$$\frac{1}{p}\sum_{k=1}^{n} m_k\left|x_k\right|^p \sum_{k=1}^{n} m_k\left|y_k\right|^q+\frac{1}{q}\sum_{k=1}^{n} m_k\left|x_k\right|^q \sum_{k=1}^{n} m_k\left|y_k\right|^p \geq \sum_{k=1}^{n} m_k\left|x_k\right|\left|y_k\right|^{p-1} \sum_{k=1}^{n} m_k\left|x_k\right|\left|y_k\right|^{q-1}. \tag{1.53}$$

Remark 32 *If in (1.53) we assume that $m_k = 1$, $k \in \{1, \ldots, n\}$, then we obtain [6, p. 8] (see also [7, p. 106])*

$$\frac{1}{p}\sum_{k=1}^{n} \left|x_k\right|^p \sum_{k=1}^{n} \left|y_k\right|^q+\frac{1}{q}\sum_{k=1}^{n} \left|x_k\right|^q \sum_{k=1}^{n} \left|y_k\right|^p \geq \sum_{k=1}^{n} \left|x_k\right|\left|y_k\right|^{p-1} \sum_{k=1}^{n} \left|x_k\right|\left|y_k\right|^{q-1}, \tag{1.54}$$

which, in the particular case $p = q = 2$ will provide the (CBS) −inequality.

The third result is embodied in the following theorem [6, Theorem 2.7] (see also [7, Theorem 3]).

Theorem 33 *Let $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$, $\bar{\mathbf{p}}$, $\bar{\mathbf{q}}$ and p, q be as in Theorem 27. Then one has the inequality*

$$\frac{1}{p}\sum_{k=1}^{n} p_k\left|x_k\right|^p \sum_{k=1}^{n} q_k\left|y_k\right|^q+\frac{1}{q}\sum_{k=1}^{n} q_k\left|x_k\right|^p \sum_{k=1}^{n} p_k\left|y_k\right|^q \geq \sum_{k=1}^{n} p_k\left|x_k y_k\right| \sum_{k=1}^{n} p_k\left|x_k\right|^{p-1}\left|y_k\right|^{q-1}. \tag{1.55}$$

Proof. We shall follow the proof in [6].

If we choose $x = \frac{|y_i|}{|y_j|}$ and $y = \frac{|x_i|}{|x_j|}$ in (1.45) we get

$$p\left(\frac{|y_i|}{|y_j|}\right)^q + q\left(\frac{|x_i|}{|x_j|}\right)^p \geq pq\frac{|x_i|\,|y_i|}{|x_j|\,|y_j|},$$

for any $x_i, y_j \neq 0$, $i, j \in \{1, \dots, n\}$, giving

$$q\,|x_i|^p\,|y_j|^q + p\,|y_i|^q\,|x_j|^p \geq pq\,|x_i y_i|\,|x_j|^{p-1}\,|y_j|^{q-1} \tag{1.56}$$

for any $i, j \in \{1, \dots, n\}$.

Multiplying (1.56) by $p_i q_j \geq 0$ ($i, j \in \{1, \dots, n\}$) and summing over i and j from 1 to n, we deduce the desired inequality (1.55). ■

The following corollary is a natural consequence of the above theorem [7, p. 106].

Corollary 34 *Let $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$, $\bar{\mathbf{m}}$ and $\bar{\mathbf{p}}$, $\bar{\mathbf{q}}$ be as in Corollary 28. Then one has the inequality:*

$$\sum_{k=1}^{n} m_k\,|x_k|^p \sum_{k=1}^{n} m_k\,|y_k|^q \geq \sum_{k=1}^{n} m_k\,|x_k y_k| \sum_{k=1}^{n} m_k\,|x_k|^{p-1}\,|y_k|^{q-1}. \tag{1.57}$$

Remark 35 *If in (1.57) we assume that $m_k = 1$, $k = \{1, \dots, n\}$, then we obtain [6, p. 8] (see also [7, p. 10])*

$$\sum_{k=1}^{n} |x_k|^p \sum_{k=1}^{n} |y_k|^q \geq \sum_{k=1}^{n} |x_k y_k| \sum_{k=1}^{n} |x_k|^{p-1}\,|y_k|^{q-1}, \tag{1.58}$$

which, in the particular case $p = q = 2$ will provide the $(CBS)-$inequality.

The fourth generalisation of the $(CBS)-$inequality is embodied in the following theorem [6, Theorem 2.9] (see also [7, Theorem 4]).

Theorem 36 *Let $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$, $\bar{\mathbf{p}}$, $\bar{\mathbf{q}}$ and p, q be as in Theorem 27. Then one has the inequality*

$$\frac{1}{q}\sum_{k=1}^{n} p_k\,|x_k|^2 \sum_{k=1}^{n} q_k\,|y_k|^q + \frac{1}{p}\sum_{k=1}^{n} p_k\,|y_k|^2 \sum_{k=1}^{n} q_k\,|x_k|^p$$
$$\geq \sum_{k=1}^{n} q_k\,|x_k y_k| \sum_{k=1}^{n} p_k\,|x_k|^{\frac{2}{q}}\,|y_k|^{\frac{2}{p}}. \tag{1.59}$$

Proof. We shall follow the proof in [6].

Choosing in (1.45), $x = |x_i|^{\frac{2}{q}} |y_j|$, $y = |x_j| |y_i|^{\frac{2}{p}}$, we get

$$p|x_i|^2 |y_j|^q + q|x_j|^p |y_i|^2 \geq pq|x_i|^{\frac{2}{q}} |y_i|^{\frac{2}{p}} |x_j y_j| \tag{1.60}$$

for any $i, j \in \{1, \ldots, n\}$.

Multiply (1.60) by $p_i q_j \geq 0$ $(i, j \in \{1, \ldots, n\})$ and summing over i and j from 1 to n, we deduce the desired inequality (1.60). ■

The following corollary holds [6, Corollary 2.10] (see also [7, p. 107]).

Corollary 37 *Let* $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$, $\bar{\mathbf{m}}$ *and* p, q *be as in Corollary 28. Then one has the inequality:*

$$\frac{1}{q}\sum_{k=1}^{n} m_k |x_k|^2 \sum_{k=1}^{n} m_k |y_k|^q + \frac{1}{p}\sum_{k=1}^{n} m_k |y_k|^2 \sum_{k=1}^{n} m_k |x_k|^p \geq \sum_{k=1}^{n} m_k |x_k y_k| \sum_{k=1}^{n} m_k |x_k|^{\frac{2}{q}} |y_k|^{\frac{2}{p}}. \tag{1.61}$$

Remark 38 *If in (1.61) we take* $m_k = 1$, $k \in \{1, \ldots, n\}$, *then we get*

$$\frac{1}{q}\sum_{k=1}^{n} |x_k|^2 \sum_{k=1}^{n} |y_k|^q + \frac{1}{p}\sum_{k=1}^{n} |y_k|^2 \sum_{k=1}^{n} |x_k|^p \geq \sum_{k=1}^{n} |x_k y_k| \sum_{k=1}^{n} |x_k|^{\frac{2}{q}} |y_k|^{\frac{2}{p}}, \tag{1.62}$$

which, in the particular case $p = q = 2$ *will provide the* (CBS) *−inequality.*

The fifth result generalising the (CBS) −inequality is embodied in the following theorem [6, Theorem 2.12] (see also [7, Theorem 5]).

Theorem 39 *Let* $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$, $\bar{\mathbf{p}}$, $\bar{\mathbf{q}}$ *and* p, q *be as in Theorem 27. Then one has the inequality*

$$\frac{1}{p}\sum_{k=1}^{n} p_k |x_k|^2 \sum_{k=1}^{n} q_k |y_k|^q + \frac{1}{q}\sum_{k=1}^{n} p_k |y_k|^2 \sum_{k=1}^{n} q_k |x_k|^p \geq \sum_{k=1}^{n} p_k |x_k|^{\frac{2}{p}} |y_k|^{\frac{2}{q}} \sum_{k=1}^{n} q_k |x_k|^{p-1} |y_k|^{q-1}. \tag{1.63}$$

Proof. We will follow the proof in [6].

Choosing in (1.45), $x = \frac{|y_i|^{\frac{2}{q}}}{|y_j|}$, $y = \frac{|x_i|^{\frac{2}{p}}}{|x_j|}$, $y_i, x_j \neq 0$, $i, j \in \{1, \dots, n\}$, we may write

$$p\left(\frac{|y_i|^{\frac{2}{q}}}{|y_j|}\right)^q + q\left(\frac{|x_i|^{\frac{2}{p}}}{|x_j|}\right)^p \geq pq\frac{|y_i|^{\frac{2}{q}}|x_i|^{\frac{2}{p}}}{|x_j||y_j|},$$

from where results

$$p|y_i|^2|x_j|^p + q|x_i|^2|y_j|^q \geq pq|x_i|^{\frac{2}{p}}|y_i|^{\frac{2}{q}}|x_j|^{p-1}|y_j|^{q-1} \tag{1.64}$$

for any $i, j \in \{1, \dots, n\}$.

Multiplying (1.64) by $p_i q_j \geq 0$ $(i, j \in \{1, \dots, n\})$ and summing over i and j from 1 to n, we deduce the desired inequality (1.63). ■

The following corollary holds [6, Corollary 2.13] (see also [7, p. 108]).

Corollary 40 *Let $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$, $\bar{\mathbf{m}}$ and p, q be as in Corollary 28. Then one has the inequality:*

$$\frac{1}{p}\sum_{k=1}^{n} m_k|x_k|^2\sum_{k=1}^{n} m_k|y_k|^q + \frac{1}{q}\sum_{k=1}^{n} m_k|y_k|^2\sum_{k=1}^{n} m_k|x_k|^p \geq \sum_{k=1}^{n} m_k|x_k|^{\frac{2}{p}}|y_k|^{\frac{2}{q}}\sum_{k=1}^{n} m_k|x_k|^{p-1}|y_k|^{q-1}. \tag{1.65}$$

Remark 41 *If in (1.46) we choose $m_k = 1$, $k \in \{1, \dots, n\}$, then we get [6, p. 10] (see also [7, p. 108])*

$$\frac{1}{p}\sum_{k=1}^{n}|x_k|^2\sum_{k=1}^{n}|y_k|^q + \frac{1}{q}\sum_{k=1}^{n}|y_k|^2\sum_{k=1}^{n}|x_k|^p \geq \sum_{k=1}^{n}|x_k|^{\frac{2}{p}}|y_k|^{\frac{2}{q}}\sum_{k=1}^{n}|x_k|^{p-1}|y_k|^{q-1}, \tag{1.66}$$

which in the particular case $p = q = 2$ will provide the (CBS) −inequality.

Finally, the following result generalising the (CBS) −inequality holds [6, Theorem 2.15] (see also [7, Theorem 6]).

Theorem 42 *Let* $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$, $\bar{\mathbf{p}}$, $\bar{\mathbf{q}}$ *and* p, q *be as in Theorem 27. Then one has the inequality:*

$$\frac{1}{p}\sum_{k=1}^{n} p_k |x_k|^2 \sum_{k=1}^{n} q_k |y_k|^p + \frac{1}{q}\sum_{k=1}^{n} q_k |y_k|^2 \sum_{k=1}^{n} p_k |x_k|^q \geq \sum_{k=1}^{n} p_k |x_k|^{\frac{2}{p}} |y_k| \sum_{k=1}^{n} q_k |x_k|^{\frac{2}{q}} |y_k| . \quad (1.67)$$

Proof. We shall follow the proof in [6].

From (1.45) one has the inequality

$$q\left(|x_i|^{\frac{2}{p}} |y_j|\right)^p + p\left(|x_j|^{\frac{2}{q}} |y_i|\right)^q \geq pq |x_i|^{\frac{2}{p}} |y_i| |x_j|^{\frac{2}{q}} |y_j| \quad (1.68)$$

for any $i, j \in \{1, \ldots, n\}$.

Multiplying (1.68) by $p_i q_j \geq 0$ ($i, j \in \{1, \ldots, n\}$) and summing over i and j from 1 to n, we deduce the desired inequality (1.67). ∎

The following corollary also holds [6, Corollary 2.16] (see also [7, p. 108]).

Corollary 43 *With the assumptions in Corollary 28, one has the inequality*

$$\sum_{k=1}^{n} m_k |x_k|^2 \sum_{k=1}^{n} m_k \left(\frac{1}{p}|y_k|^p + \frac{1}{q}|y_k|^q\right) \geq \sum_{k=1}^{n} m_k |x_k|^{\frac{2}{p}} |y_k| \sum_{k=1}^{n} m_k |x_k|^{\frac{2}{q}} |y_k| . \quad (1.69)$$

Remark 44 *If in (1.69) we choose* $m_k = 1$ $(k \in \{1, \ldots, n\})$, *then we get*

$$\sum_{k=1}^{n} |x_k|^2 \sum_{k=1}^{n} \left(\frac{1}{p}|y_k|^p + \frac{1}{q}|y_k|^q\right) \geq \sum_{k=1}^{n} |x_k|^{\frac{2}{p}} |y_k| \sum_{k=1}^{n} |x_k|^{\frac{2}{q}} |y_k| , \quad (1.70)$$

which, in the particular case $p = q = 2$, *provides the* $(CBS)-$*inequality.*

1.8 A Generalisation Involving $J-$Convex Functions

For $a > 1$, we denote by $\exp_a$ the function

$$\exp_a : \mathbb{R} \to (0, \infty), \quad \exp_a(x) = a^x. \quad (1.71)$$

Definition 45 *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be $J-$convex on an interval I if*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \quad \text{for any } x, y \in I. \tag{1.72}$$

It is obvious that any convex function on I is a J convex function on I, but the converse does not generally hold.

The following lemma holds (see [6, Lemma 4.3]).

Lemma 46 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a $J-$convex function on I, $a > 1$ and $x, y \in \mathbb{R}\backslash\{0\}$ with $\log_a x^2$, $\log_a y^2 \in I$. Then $\log_a |xy| \in I$ and*

$$\{\exp_b [f(\log_a |xy|)]\}^2 \leq \exp_b \left[f\left(\log_a x^2\right)\right] \exp_b \left[f\left(\log_a y^2\right)\right] \tag{1.73}$$

for any $b > 1$.

Proof. I, being an interval, is a convex set in $\mathbb{R}$ and thus

$$\log_a |xy| = \frac{1}{2}\left[\log_a x^2 + \log_a y^2\right] \in I.$$

Since f is $J-$convex, one has

$$\begin{aligned} f(\log_a |xy|) &= f\left[\frac{1}{2}\left(\log_a x^2 + \log_a y^2\right)\right] \\ &\leq \frac{f(\log_a x^2) + f(\log_a y^2)}{2}. \end{aligned} \tag{1.74}$$

Taking the $\exp_b$ in both parts, we deduce

$$\begin{aligned} \exp_b [f(\log_a |xy|)] &\leq \exp_b \left[\frac{f(\log_a x^2) + f(\log_a y^2)}{2}\right] \\ &= \left\{\exp_b \left[f\left(\log_a x^2\right)\right] \exp_b \left[f\left(\log_a y^2\right)\right]\right\}^{\frac{1}{2}}, \end{aligned}$$

which is equivalent to (1.73). ■

The following generalisation of the $(CBS)-$inequality in terms of a $J-$convex function holds [6, Theorem 4.4].

Theorem 47 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a $J-$convex function on I, $a, b > 1$ and $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ sequences of nonzero real numbers. If $\log_a a_k^2$, $\log_a b_k^2 \in I$ for all $k \in \{1, \dots, n\}$, then one has the inequality:*

$$\left\{\sum_{k=1}^{n} \exp_b \left[f\left(\log_a |a_k b_k|\right)\right]\right\}^2 \leq \sum_{k=1}^{n} \exp_b \left[f\left(\log_a a_k^2\right)\right] \sum_{k=1}^{n} \exp_b \left[f\left(\log_a b_k^2\right)\right]. \quad (1.75)$$

Proof. Using Lemma 46 and the $(CBS)-$inequality one has

$$\begin{aligned}
&\sum_{k=1}^{n} \exp_b \left[f\left(\log_a |a_k b_k|\right)\right] \\
&\leq \sum_{k=1}^{n} \left[\exp_b \left[f\left(\log_a a_k^2\right)\right] \exp_b \left[f\left(\log_a b_k^2\right)\right]\right]^{\frac{1}{2}} \\
&\leq \left(\sum_{k=1}^{n} \left\{\left[\exp_b \left[f\left(\log_a a_k^2\right)\right]\right]^{\frac{1}{2}}\right\}^2 \sum_{k=1}^{n} \left\{\left[\exp_b \left[f\left(\log_a b_k^2\right)\right]\right]^{\frac{1}{2}}\right\}^2\right)^{\frac{1}{2}}
\end{aligned}$$

which is clearly equivalent to (1.75). ■

Remark 48 *If in (1.75) we choose $a = b > 1$ and $f(x) = x$, $x \in \mathbb{R}$, then we reapture the $(CBS)-$inequality.*

1.9 A Functional Generalisation

The following result was proved in [9, Theorem 2].

Theorem 49 *Let A be a subset of real numbers $\mathbb{R}$, $f : A \to \mathbb{R}$ and $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ sequences of real numbers with the properties that*

(i) $a_i b_i$, a_i^2, $b_i^2 \in A$ *for any* $i \in \{1, \dots, n\}$,

(ii) $f(a_i^2)$, $f(b_i^2) \geq 0$ *for any* $i \in \{1, \dots, n\}$,

(iii) $f^2(a_i b_i) \leq f(a_i^2) f(b_i^2)$ *for any* $i \in \{1, \dots, n\}$.

Then one has the inequality:

$$\left[\sum_{i=1}^{n} f(a_i b_i)\right]^2 \leq \sum_{i=1}^{n} f\left(a_i^2\right) \sum_{i=1}^{n} f\left(b_i^2\right). \tag{1.76}$$

Proof. We give here a simpler proof than that found in [9].

We have

$$\begin{aligned}
&\left|\sum_{i=1}^{n} f(a_i b_i)\right| \\
&\quad \leq \sum_{i=1}^{n} |f(a_i b_i)| \leq \sum_{i=1}^{n} \left[f\left(a_i^2\right)\right]^{\frac{1}{2}} \left[f\left(b_i^2\right)\right]^{\frac{1}{2}} \\
&\quad \leq \left[\sum_{i=1}^{n} \left(\left[f\left(a_i^2\right)\right]^{\frac{1}{2}}\right)^2 \sum_{i=1}^{n} \left(\left[f\left(b_i^2\right)\right]^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}} \text{ (by the } (CBS)\text{-inequality)} \\
&\quad = \left[\sum_{i=1}^{n} f\left(a_i^2\right) \sum_{i=1}^{n} f\left(b_i^2\right)\right]^{\frac{1}{2}}
\end{aligned}$$

and the inequality (1.76) is proved. ■

Remark 50 *It is obvious that for $A = \mathbb{R}$ and $f(x) = x$, we recapture the (CBS) −inequality.*

Assume that $\varphi : \mathbb{N} \to \mathbb{N}$ is Euler's indicator. In 1940, T. Popoviciu [10] proved the following inequality for φ

$$[\varphi(ab)]^2 \leq \varphi\left(a^2\right) \varphi\left(b^2\right) \quad \text{for any natural number } a, b \tag{1.77}$$

with equality iff a and b have the same prime factors.

A simple proof of this fact may be done by using the representation

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_k}\right),$$

where $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ [8, p. 109].

The following generalisation of Popoviciu's result holds [9, Theorem 1].

Theorem 51 *Let $a_i, b_i \in \mathbb{N}$ $(i = 1, \ldots, n)$. Then one has the inequality*

$$\left[\sum_{i=1}^{n} \varphi(a_i b_i)\right]^2 \le \sum_{i=1}^{n} \varphi\left(a_i^2\right) \sum_{i=1}^{n} \varphi\left(b_i^2\right). \tag{1.78}$$

Proof. Follows by Theorem 49 on taking into account that, by (1.77), $[\varphi(a_i b_i)]^2 \le \varphi(a_i^2)\varphi(b_i^2)$ for any $i \in \{1, \ldots, n\}$. ∎

Further, let us denote by $s(n)$ the sum of all relatively prime numbers with n and less than n. Then the following result also holds [9, Theorem 1].

Theorem 52 *Let $a_i, b_i \in \mathbb{N}$ $(i = 1, \ldots, n)$. Then one has the inequality*

$$\left[\sum_{i=1}^{n} s(a_i b_i)\right]^2 \le \sum_{i=1}^{n} s\left(a_i^2\right) \sum_{i=1}^{n} s\left(b_i^2\right). \tag{1.79}$$

Proof. It is known (see for example [8, p. 109]) that for any $n \in \mathbb{N}$ one has

$$s(n) = \frac{1}{2} n \varphi(n). \tag{1.80}$$

Thus

$$[s(a_i b_i)]^2 = \frac{1}{4} a_i^2 b_i^2 \varphi^2(a_i b_i) \le \frac{1}{4} a_i^2 b_i^2 \varphi\left(a_i^2\right)\varphi\left(b_i^2\right) = s\left(a_i^2\right) s\left(b_i^2\right) \tag{1.81}$$

for each $i \in \{1, \ldots, n\}$.

Using Theorem 49 we then deduce the desired inequality (1.79). ∎

The following corollaries of Theorem 49 are also natural to be considered [9, p. 126].

Corollary 53 *Let $a_i, b_i \in \mathbb{R}$ $(i = 1, \ldots, n)$ and $a > 1$. Denote $\exp_a x = a^x$, $x \in \mathbb{R}$. Then one has the inequality*

$$\left[\sum_{i=1}^{n} \exp_a(a_i b_i)\right]^2 \le \sum_{i=1}^{n} \exp_a\left(a_i^2\right) \sum_{i=1}^{n} \exp_a\left(b_i^2\right). \tag{1.82}$$

Corollary 54 *Let $a_i, b_i \in (-1, 1)$ $(i = 1, \ldots, n)$ and $m > 0$. Then one has the inequality:*

$$\left[\sum_{i=1}^{n} \frac{1}{(1 - a_i b_i)^m}\right]^2 \le \sum_{i=1}^{n} \frac{1}{\left(1 - a_i^2\right)^m} \sum_{i=1}^{n} \frac{1}{\left(1 - b_i^2\right)^m}. \tag{1.83}$$

1.10 A Generalisation for Power Series

The following result holds [11, Remark 2].

Theorem 55 *Let $F : (-r, r) \to \mathbb{R}$, $F(x) = \sum_{k=0}^{\infty} \alpha_k x^k$ with $\alpha_k \geq 0$, $k \in \mathbb{N}$. If $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ are sequences of real numbers such that*

$$a_i b_i,\ a_i^2,\ b_i^2 \in (-r, r) \quad \textit{for any} \quad i \in \{1, \ldots, n\}, \tag{1.84}$$

then one has the inequality:

$$\sum_{i=1}^{n} F\left(a_i^2\right) \sum_{i=1}^{n} F\left(b_i^2\right) \geq \left[\sum_{i=1}^{n} F\left(a_i b_i\right)\right]^2. \tag{1.85}$$

Proof. Firstly, let us observe that if $x, y \in \mathbb{R}$ such that $xy, x^2, y^2 \in (-r, r)$, then one has the inequality

$$[F(xy)]^2 \leq F\left(x^2\right) F\left(y^2\right). \tag{1.86}$$

Indeed, by the (CBS) −inequality, we have

$$\left[\sum_{k=0}^{n} \alpha_k x^k y^k\right]^2 \leq \sum_{k=0}^{n} \alpha_k x^{2k} \sum_{k=0}^{n} \alpha_k y^{2k}, \quad n \geq 0. \tag{1.87}$$

Taking the limit as $n \to \infty$ in (1.87), we deduce (1.86).

Using the (CBS) −inequality and (1.86) we have

$$\begin{aligned}
\left|\sum_{i=1}^{n} F\left(a_i b_i\right)\right| &\leq \sum_{i=1}^{n} \left|F\left(a_i b_i\right)\right| \leq \sum_{i=1}^{n} \left[F\left(a_i^2\right)\right]^{\frac{1}{2}} \left[F\left(b_i^2\right)\right]^{\frac{1}{2}} \\
&\leq \left\{\sum_{i=1}^{n} \left(\left[F\left(a_i^2\right)\right]^{\frac{1}{2}}\right)^2 \sum_{i=1}^{n} \left(\left[F\left(b_i^2\right)\right]^{\frac{1}{2}}\right)^2\right\}^{\frac{1}{2}} \\
&= \left[\sum_{i=1}^{n} F\left(a_i^2\right) \sum_{i=1}^{n} F\left(b_i^2\right)\right]^{\frac{1}{2}},
\end{aligned}$$

which is clearly equivalent to (1.85). ■

The following particular inequalities of (CBS) −type hold [11, p. 164].

1. If $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ are sequences of real numbers, then one has the inequality

$$\sum_{k=1}^{n} \exp\left(a_k^2\right) \sum_{k=1}^{n} \exp\left(b_k^2\right) \geq \left[\sum_{k=1}^{n} \exp\left(a_k b_k\right)\right]^2; \tag{1.88}$$

$$\sum_{k=1}^{n} \sinh\left(a_k^2\right) \sum_{k=1}^{n} \sinh\left(b_k^2\right) \geq \left[\sum_{k=1}^{n} \sinh\left(a_k b_k\right)\right]^2; \tag{1.89}$$

$$\sum_{k=1}^{n} \cosh\left(a_k^2\right) \sum_{k=1}^{n} \cosh\left(b_k^2\right) \geq \left[\sum_{k=1}^{n} \cosh\left(a_k b_k\right)\right]^2. \tag{1.90}$$

2. If $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ are such that $a_i, b_i \in (-1, 1)$, $i \in \{1, \ldots, n\}$, then one has the inequalities

$$\sum_{k=1}^{n} \tan\left(a_k^2\right) \sum_{k=1}^{n} \tan\left(b_k^2\right) \geq \left[\sum_{k=1}^{n} \tan\left(a_k b_k\right)\right]^2; \tag{1.91}$$

$$\sum_{k=1}^{n} \arcsin\left(a_k^2\right) \sum_{k=1}^{n} \arcsin\left(b_k^2\right) \geq \left[\sum_{k=1}^{n} \arcsin\left(a_k b_k\right)\right]^2; \tag{1.92}$$

$$\ln\left[\prod_{k=1}^{n}\left(\frac{1+a_k^2}{1-a_k^2}\right)\right] \ln\left[\prod_{k=1}^{n}\left(\frac{1+b_k^2}{1-b_k^2}\right)\right] \geq \left\{\ln\left[\prod_{k=1}^{n}\left(\frac{1+a_k b_k}{1-a_k b_k}\right)\right]\right\}^2; \tag{1.93}$$

$$\ln\left[\prod_{k=1}^{n}\left(\frac{1}{1-a_k^2}\right)\right] \ln\left[\prod_{k=1}^{n}\left(\frac{1}{1-b_k^2}\right)\right] \geq \left\{\ln\left[\prod_{k=1}^{n}\left(\frac{1}{1-a_k b_k}\right)\right]\right\}^2; \tag{1.94}$$

$$\sum_{k=1}^{n} \frac{1}{\left(1-a_k^2\right)^m} \sum_{k=1}^{n} \frac{1}{\left(1-b_k^2\right)^m} \geq \left[\sum_{k=1}^{n} \frac{1}{\left(1-a_k b_k\right)^m}\right]^2, \quad m > 0. \tag{1.95}$$

1.11 A Generalisation of Callebaut's Inequality

The following result holds (see also [11, Theorem 2] for a generalisation for positive linear functionals).

Theorem 56 *Let* $F:(-r,r)\to\mathbb{R}$, $F(x)=\sum_{k=0}^{\infty}\alpha_k x^k$ *with* $\alpha_k\geq 0$, $k\in\mathbb{N}$. *If* $\bar{\mathbf{a}}=(a_1,\dots,a_n)$, $\bar{\mathbf{b}}=(b_1,\dots,b_n)$ *are sequences of nonnegative real numbers such that*

$$a_i b_i,\ a_i^{\alpha}b_i^{2-\alpha},\ a_i^{2-\alpha}b_i^{\alpha}\in(0,r)\ \text{for any } i\in\{1,\dots,n\};\ \alpha\in[0,2], \tag{1.96}$$

then one has the inequality

$$\left[\sum_{i=1}^{n}F(a_i b_i)\right]^2\leq\sum_{i=1}^{n}F\left(a_i^{\alpha}b_i^{2-\alpha}\right)\sum_{i=1}^{n}F\left(a_i^{2-\alpha}b_i^{\alpha}\right). \tag{1.97}$$

Proof. Firstly, we note that for any $x,y>0$ such that xy, $x^{\alpha}y^{2-\alpha}$, $x^{2-\alpha}y^{\alpha}\in(0,r)$ one has

$$[F(xy)]^2\leq F\left(x^{\alpha}y^{2-\alpha}\right)F\left(x^{2-\alpha}y^{\alpha}\right). \tag{1.98}$$

Indeed, using Callebaut's inequality, i.e., we recall it [3]

$$\left(\sum_{i=1}^{m}\alpha_i x_i y_i\right)^2\leq\sum_{i=1}^{m}\alpha_i x_i^{\alpha}y_i^{2-\alpha}\sum_{i=1}^{m}\alpha_i x_i^{2-\alpha}y_i^{\alpha}, \tag{1.99}$$

we may write, for $m\geq 0$, that

$$\left(\sum_{i=0}^{m}\alpha_i x^i y^i\right)^2\leq\sum_{i=0}^{m}\alpha_i\left(x^{\alpha}y^{2-\alpha}\right)^i\sum_{i=0}^{m}\alpha_i\left(x^{2-\alpha}y^{\alpha}\right)^i. \tag{1.100}$$

Taking the limit as $m\to\infty$, we deduce (1.98).

Using the (CBS) −inequality and (1.98) we may write:

$$\begin{aligned}
&\left|\sum_{i=1}^{n} F(a_i b_i)\right| \\
&\leq \sum_{i=1}^{n} |F(a_i b_i)| \leq \sum_{i=1}^{n} \left[F\left(a_i^{\alpha} b_i^{2-\alpha}\right)\right]^{\frac{1}{2}} \left[F\left(a_i^{2-\alpha} b_i^{\alpha}\right)\right]^{\frac{1}{2}} \\
&\leq \left\{\sum_{i=1}^{n} \left(\left[F\left(a_i^{\alpha} b_i^{2-\alpha}\right)\right]^{\frac{1}{2}}\right)^2 \sum_{i=1}^{n} \left(\left[F\left(a_i^{2-\alpha} b_i^{\alpha}\right)\right]^{\frac{1}{2}}\right)^2\right\}^{\frac{1}{2}} \\
&= \left[\sum_{i=1}^{n} F\left(a_i^{\alpha} b_i^{2-\alpha}\right) \sum_{i=1}^{n} F\left(a_i^{2-\alpha} b_i^{\alpha}\right)\right]^{\frac{1}{2}}
\end{aligned}$$

which is clearly equivalent to (1.97). ■

The following particular inequalities also hold [11, pp. 165-166].

1. Let $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ be sequences of nonnegative real numbers. Then one has the inequalities

$$\left[\sum_{k=1}^{n} \exp(a_k b_k)\right]^2 \leq \sum_{k=1}^{n} \exp\left(a_k^{\alpha} b_k^{2-\alpha}\right) \sum_{k=1}^{n} \exp\left(a_k^{2-\alpha} b_k^{\alpha}\right); \tag{1.101}$$

$$\left[\sum_{k=1}^{n} \sinh(a_k b_k)\right]^2 \leq \sum_{k=1}^{n} \sinh\left(a_k^{\alpha} b_k^{2-\alpha}\right) \sum_{k=1}^{n} \sinh\left(a_k^{2-\alpha} b_k^{\alpha}\right); \tag{1.102}$$

$$\left[\sum_{k=1}^{n} \cosh(a_k b_k)\right]^2 \leq \sum_{k=1}^{n} \cosh\left(a_k^{\alpha} b_k^{2-\alpha}\right) \sum_{k=1}^{n} \cosh\left(a_k^{2-\alpha} b_k^{\alpha}\right). \tag{1.103}$$

2. Let $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ be such that $a_k, b_k \in (0,1)$ for any $k \in \{1, \ldots, n\}$. Then one has the inequalities:

$$\left[\sum_{k=1}^{n} \tan(a_k b_k)\right]^2 \leq \sum_{k=1}^{n} \tan\left(a_k^{\alpha} b_k^{2-\alpha}\right) \sum_{k=1}^{n} \tan\left(a_k^{2-\alpha} b_k^{\alpha}\right); \tag{1.104}$$

$$\left[\sum_{k=1}^{n}\arcsin\left(a_k b_k\right)\right]^2 \leq \sum_{k=1}^{n}\arcsin\left(a_k^{\alpha} b_k^{2-\alpha}\right)\sum_{k=1}^{n}\arcsin\left(a_k^{2-\alpha} b_k^{\alpha}\right); \quad (1.105)$$

$$\left\{\ln\left[\prod_{k=1}^{n}\left(\frac{1+a_k b_k}{1-a_k b_k}\right)\right]\right\}^2 \leq \ln\left[\prod_{k=1}^{n}\left(\frac{1+a_k^{\alpha} b_k^{2-\alpha}}{1-a_k^{\alpha} b_k^{2-\alpha}}\right)\right]\ln\left[\prod_{k=1}^{n}\left(\frac{1+a_k^{2-\alpha} b_k^{\alpha}}{1-a_k^{2-\alpha} b_k^{\alpha}}\right)\right]; \quad (1.106)$$

$$\left\{\ln\left[\prod_{k=1}^{n}\left(\frac{1}{1-a_k b_k}\right)\right]\right\}^2 \leq \ln\left[\prod_{k=1}^{n}\left(\frac{1}{1-a_k^{\alpha} b_k^{2-\alpha}}\right)\right]\ln\left[\prod_{k=1}^{n}\left(\frac{1}{1-a_k^{2-\alpha} b_k^{\alpha}}\right)\right]. \quad (1.107)$$

1.12 Wagner's Inequality for Real Numbers

The following generalisation of the (CBS) −inequality for sequences of real numbers is known in the literature as Wagner's inequality [14], or [13] (see also [3, p. 85]).

Theorem 57 *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ *and* $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be sequences of real numbers. If* $0 \leq x \leq 1$, *then one has the inequality*

$$\left(\sum_{k=1}^{n} a_k b_k + x\sum_{1\leq i\neq j\leq n} a_i b_j\right)^2 \leq \left[\sum_{k=1}^{n} a_k^2 + 2x\sum_{1\leq i<j\leq n} a_i a_j\right]\left[\sum_{k=1}^{n} b_k^2 + 2x\sum_{1\leq i<j\leq n} b_i b_j\right]. \quad (1.108)$$

Proof. We shall follow the proof in [12] (see also [3, p. 85]).

For any $x \in [0,1]$, consider the quadratic polynomial in y

$$\begin{aligned}
P(y) &:= (1-x)\sum_{k=1}^{n}(a_k y - b_k)^2 + x\left[\sum_{k=1}^{n}(a_k y - b_k)\right]^2 \\
&= (1-x)\left[y^2\sum_{k=1}^{n}a_k^2 - 2y\sum_{k=1}^{n}a_k b_k + \sum_{k=1}^{n}b_k^2\right] \\
&\quad + x\left[y^2\left(\sum_{k=1}^{n}a_k\right)^2 - 2y\left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}b_k\right) + \left(\sum_{k=1}^{n}b_k\right)^2\right] \\
&= \left[(1-x)\sum_{k=1}^{n}a_k^2 + x\left(\sum_{k=1}^{n}a_k\right)^2\right]y^2 \\
&\quad - 2y\left[(1-x)\sum_{k=1}^{n}a_k b_k + x\sum_{k=1}^{n}a_k\sum_{k=1}^{n}b_k\right] \\
&\quad + (1-x)\sum_{k=1}^{n}b_k^2 + x\left(\sum_{k=1}^{n}b_k\right)^2 \\
&= \left\{\sum_{k=1}^{n}a_k^2 + x\left[\left(\sum_{k=1}^{n}a_k\right)^2 - \sum_{k=1}^{n}a_k^2\right]\right\}y^2 \\
&\quad - 2y\left[\sum_{k=1}^{n}a_k b_k + x\left(\sum_{k=1}^{n}a_k\sum_{k=1}^{n}b_k - \sum_{k=1}^{n}a_k b_k\right)\right] \\
&\quad + \sum_{k=1}^{n}b_k^2 + x\left[\left(\sum_{k=1}^{n}b_k\right)^2 - \sum_{k=1}^{n}b_k^2\right].
\end{aligned}$$

Since, it is obvious that:

$$\begin{aligned}
\left(\sum_{k=1}^{n}a_k\right)^2 - \sum_{k=1}^{n}a_k^2 &= 2\sum_{1\le i<j\le n}a_i a_j, \\
\sum_{k=1}^{n}a_k\sum_{k=1}^{n}b_k - \sum_{k=1}^{n}a_k b_k &= \sum_{1\le i\neq j\le n}a_i b_j
\end{aligned}$$

and

$$\left(\sum_{k=1}^{n} b_k\right)^2 - \sum_{k=1}^{n} b_k^2 = 2\sum_{1\le i<j\le n} b_i b_j,$$

we get

$$P(y) = \left(\sum_{k=1}^{n} a_k^2 + 2x\sum_{1\le i<j\le n} a_i a_j\right) y^2 - 2y\left(\sum_{k=1}^{n} a_k b_k + x\sum_{1\le i\ne j\le n} a_i b_j\right) + \sum_{k=1}^{n} b_k^2 + 2x\sum_{1\le i<j\le n} b_i b_j.$$

Taking into consideration, by the definition of P, that $P(y) \ge 0$ for any $y \in \mathbb{R}$, it follows that the discriminant $\Delta \le 0$, i.e.,

$$0 \ge \frac{1}{4}\Delta = \left(\sum_{k=1}^{n} a_k b_k + x\sum_{1\le i\ne j\le n} a_i b_j\right)^2 - \left(\sum_{k=1}^{n} a_k^2 + 2x\sum_{1\le i<j\le n} a_i a_j\right)\left(\sum_{k=1}^{n} b_k^2 + 2x\sum_{1\le i<j\le n} b_i b_j\right)$$

and the inequality (1.108) is proved. ■

Remark 58 *If $x = 0$, then from (1.108) we recapture the (CBS) −inequality for real numbers.*

1.13 Wagner's inequality for Complex Numbers

The following inequality which provides a version for complex numbers of Wagner's result holds [15].

Theorem 59 *Let $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$ and $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ be sequences of complex numbers. Then for any $x \in [0,1]$ one has the inequality*

$$\left[\sum_{k=1}^{n} \operatorname{Re}\left(a_k \bar{b}_k\right) + x \sum_{\substack{1 \leq i,j \leq n \\ i \neq j}} \operatorname{Re}\left(a_i \bar{b}_j\right)\right]^2 \leq \left[\sum_{k=1}^{n} |a_k|^2 + 2x \sum_{1 \leq i < j \leq n} \operatorname{Re}\left(a_i \bar{a}_j\right)\right] \times \left[\sum_{k=1}^{n} |b_k|^2 + 2x \sum_{1 \leq i < j \leq n} \operatorname{Re}\left(b_i \bar{b}_j\right)\right]. \quad (1.109)$$

Proof. Start with the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(t) = (1-x) \sum_{k=1}^{n} |t a_k - b_k|^2 + x \left| \sum_{k=1}^{n} (t a_k - b_k) \right|^2. \quad (1.110)$$

We have

$$\begin{aligned} f(t) &= (1-x) \sum_{k=1}^{n} (t a_k - b_k)\left(t \bar{a}_k - \bar{b}_k\right) \quad (1.111) \\ &\quad + x \left(t \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k \right) \left(t \sum_{k=1}^{n} \bar{a}_k - \sum_{k=1}^{n} \bar{b}_k \right) \\ &= (1-x) \left[t^2 \sum_{k=1}^{n} |a_k|^2 - t \sum_{k=1}^{n} b_k \bar{a}_k - t \sum_{k=1}^{n} a_k \bar{b}_k + \sum_{k=1}^{n} |b_k|^2 \right] \\ &\quad + x \left[t^2 \sum_{k=1}^{n} |a_k|^2 - t \sum_{k=1}^{n} b_k \sum_{k=1}^{n} \bar{a}_k - t \sum_{k=1}^{n} a_k \sum_{k=1}^{n} \bar{b}_k + \sum_{k=1}^{n} |b_k|^2 \right] \\ &= \left[(1-x) \sum_{k=1}^{n} |a_k|^2 + x \left| \sum_{k=1}^{n} a_k \right|^2 \right] t^2 \\ &\quad + 2 \left[(1-x) \sum_{k=1}^{n} \operatorname{Re}\left(a_k \bar{b}_k\right) + x \operatorname{Re}\left[\sum_{k=1}^{n} a_k \sum_{k=1}^{n} \bar{b}_k \right] \right] t \\ &\quad + (1-x) \sum_{k=1}^{n} |b_k|^2 + x \left| \sum_{k=1}^{n} b_k \right|^2. \end{aligned}$$

Observe that

$$\begin{aligned} \left|\sum_{k=1}^{n} a_k\right|^2 &= \sum_{i,j=1}^{n} a_i \bar{a}_j = \sum_{i=1}^{n} |a_i|^2 + \sum_{1\le i\ne j\le n} a_i \bar{a}_j \\ &= \sum_{i=1}^{n} |a_i|^2 + \sum_{1\le i<j\le n} a_i \bar{a}_j + \sum_{1\le j<i\le n} a_i \bar{a}_j \\ &= \sum_{i=1}^{n} |a_i|^2 + 2 \sum_{1\le i<j\le n} \operatorname{Re}\left(a_i \bar{a}_j\right) \end{aligned} \tag{1.112}$$

and, similarly,

$$\left|\sum_{k=1}^{n} b_k\right|^2 = \sum_{i=1}^{n} |b_i|^2 + 2 \sum_{1\le i<j\le n} \operatorname{Re}\left(b_i \bar{b}_j\right). \tag{1.113}$$

Also

$$\sum_{k=1}^{n} a_k \sum_{k=1}^{n} \bar{b}_k = \sum_{i=1}^{n} a_i \bar{b}_i + \sum_{\substack{1\le i,j\le n \\ i\ne j}} a_i \bar{b}_j$$

and thus

$$\operatorname{Re}\left(\sum_{k=1}^{n} a_k \sum_{k=1}^{n} \bar{b}_k\right) = \sum_{i=1}^{n} \operatorname{Re}\left(a_i \bar{b}_i\right) + \sum_{\substack{1\le i,j\le n \\ i\ne j}} \operatorname{Re}\left(a_i \bar{b}_j\right). \tag{1.114}$$

Utilising (1.112) – (1.114), by (1.111), we deduce

$$\begin{aligned} f(t) \;=\; & \left[\sum_{k=1}^{n} |a_k|^2 + 2x \sum_{1\le i<j\le n} \operatorname{Re}\left(a_i \bar{a}_j\right)\right] t^2 \\ & +2\left[\sum_{k=1}^{n} \operatorname{Re}\left(a_k \bar{b}_k\right) + x \sum_{\substack{1\le i,j\le n \\ i\ne j}} \operatorname{Re}\left(a_i \bar{b}_j\right)\right] t \\ & +\sum_{k=1}^{n} |b_k|^2 + 2x \sum_{1\le i<j\le n} \operatorname{Re}\left(b_i \bar{b}_j\right). \end{aligned} \tag{1.115}$$

Since, by (1.110), $f(t) \geq 0$ for any $t \in \mathbb{R}$, it follows that the discriminant of the quadratic function given by (1.115) is negative, i.e.,

$$
\begin{aligned}
0 &\geq \frac{1}{4}\Delta \\
&= \left[\sum_{k=1}^{n} \operatorname{Re}\left(a_k \bar{b}_k\right) + x \sum_{\substack{1\leq i,j\leq n \\ i\neq j}} \operatorname{Re}\left(a_i \bar{b}_j\right)\right]^2 \\
&\quad - \left[\sum_{k=1}^{n} |a_k|^2 + 2x \sum_{1\leq i<j\leq n} \operatorname{Re}\left(a_i \bar{a}_j\right)\right]\left[\sum_{k=1}^{n} |b_k|^2 + 2x \sum_{1\leq i<j\leq n} \operatorname{Re}\left(b_i \bar{b}_j\right)\right]
\end{aligned}
$$

and the inequality (1.109) is proved. ■

Remark 60 *If $x = 0$, then we get the $(CBS)-$inequality*

$$
\left[\sum_{k=1}^{n} \operatorname{Re}\left(a_k \bar{b}_k\right)\right]^2 \leq \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 . \tag{1.116}
$$

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Chapter 2

Refinements of the (CBS) –Inequality

2.1 A Refinement in Terms of Moduli

The following result was proved in [1].

Theorem 61 *Let* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$ *and* $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ *be sequences of real numbers. Then one has the inequality*

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \left(\sum_{k=1}^{n} a_k b_k \right)^2 \geq \left| \sum_{k=1}^{n} a_k |a_k| \sum_{k=1}^{n} b_k |b_k| - \sum_{k=1}^{n} a_k |b_k| \sum_{k=1}^{n} |a_k| b_k \right| \geq 0. \quad (2.1)$$

Proof. We will follow the proof from [1].
For any $i, j \in \{1, \ldots, n\}$ the next elementary inequality is true:

$$|a_i b_j - a_j b_i| \geq ||a_i b_j| - |a_j b_i|| . \quad (2.2)$$

By multiplying this inequality with $|a_i b_j - a_j b_i| \geq 0$ we get

$$\begin{aligned} (a_i b_j - a_j b_i)^2 &\geq |(a_i b_j - a_j b_i)(|a_i| |b_j| - |a_j| |b_i|)| \quad (2.3) \\ &= |a_i |a_i| b_j |b_j| + b_i |b_i| a_j |a_j| - |a_i| b_i a_j |b_j| - a_i b_j |a_j| |b_i|| . \end{aligned}$$

Summing (2.3) over i and j from 1 to n, we deduce

$$\begin{aligned}
&\sum_{i,j=1}^{n}(a_ib_j-a_jb_i)^2\\
&\geq \sum_{i,j=1}^{n}\left|a_i|a_i|b_j|b_j|+b_i|b_i|a_j|a_j|-|a_i|b_ia_j|b_j|-a_ib_j|a_j||b_i|\right|\\
&\geq \left|\sum_{i,j=1}^{n}\left(a_i|a_i|b_j|b_j|+b_i|b_i|a_j|a_j|-|a_i|b_ia_j|b_j|-a_ib_j|a_j||b_i|\right)\right|,
\end{aligned}$$

giving the desired inequality (2.1). ■

The following corollary is a natural consequence of (2.1) [1, Corollary 4].

Corollary 62 *Let* $\bar{\mathbf{a}}$ *be a sequence of real numbers. Then*

$$\frac{1}{n}\sum_{k=1}^{n}a_k^2-\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^2\geq\left|\frac{1}{n}\sum_{k=1}^{n}a_k|a_k|-\frac{1}{n}\sum_{k=1}^{n}a_k\cdot\frac{1}{n}\sum_{k=1}^{n}|a_k|\right|\geq 0. \quad (2.4)$$

There are some particular inequalities that may also be deduced from the above Theorem 61 (see [1, p. 80]).

1. Suppose that for $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ sequences of real numbers, one has $\operatorname{sgn}(a_k)=\operatorname{sgn}(b_k)=e_k\in\{-1,1\}$. Then one has the inequality

$$\begin{aligned}
&\sum_{k=1}^{n}a_k^2\sum_{k=1}^{n}b_k^2-\left(\sum_{k=1}^{n}a_kb_k\right)^2\\
&\qquad\geq\left|\sum_{k=1}^{n}e_ka_k^2\sum_{k=1}^{n}e_kb_k^2-\left(\sum_{k=1}^{n}e_ka_kb_k\right)^2\right|\geq 0. \quad (2.5)
\end{aligned}$$

2. If $\bar{\mathbf{a}}=(a_1,\ldots,a_{2n})$, then we have the inequality

$$2n\sum_{k=1}^{2n}a_k^2-\left[\sum_{k=1}^{2n}(-1)^k a_k\right]^2\geq\left|\sum_{k=1}^{2n}a_k\sum_{k=1}^{2n}(-1)^k|a_k|\right|\geq 0. \qquad (2.6)$$

3. If $\bar{\mathbf{a}} = (a_1, \ldots, a_{2n+1})$, then we have the inequality

$$(2n+1)\sum_{k=1}^{2n+1} a_k^2 - \left(\sum_{k=1}^{2n+1} (-1)^k a_k\right)^2 \geq \left|\sum_{k=1}^{2n+1} a_k \sum_{k=1}^{2n+1} (-1)^k |a_k|\right| \geq 0. \quad (2.7)$$

The following version for complex numbers is valid as well.

Theorem 63 *Let* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$ *and* $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ *be sequences of complex numbers. Then one has the inequality*

$$\sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \left|\sum_{i=1}^{n} a_i b_i\right|^2 \geq \left|\sum_{i=1}^{n} |a_i| \bar{a}_i \sum_{i=1}^{n} |b_i| b_i - \sum_{i=1}^{n} |a_i| b_i \sum_{i=1}^{n} |b_i| \bar{a}_i\right| \geq 0. \quad (2.8)$$

Proof. We have for any $i, j \in \{1, \ldots, n\}$ that

$$|\bar{a}_i b_j - \bar{a}_j b_i| \geq ||a_i| |b_j| - |a_j| |b_i|| .$$

Multiplying by $|\bar{a}_i b_j - \bar{a}_j b_i| \geq 0$, we get

$$|\bar{a}_i b_j - \bar{a}_j b_i|^2 \geq ||a_i| \bar{a}_i |b_j| b_j + |a_j| \bar{a}_j |b_i| b_i - |a_i| b_i |b_j| \bar{a}_j - |b_i| \bar{a}_i |a_j| b_j| .$$

Summing over i and j from 1 to n and using the Lagrange's identity for complex numbers:

$$\sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \left|\sum_{i=1}^{n} a_i b_i\right|^2 = \frac{1}{2} \sum_{i,j=1}^{n} |\bar{a}_i b_j - \bar{a}_j b_i|^2$$

we deduce the desired inequality (2.8). ■

Remark 64 *Similar particular inequalities may be stated, but we omit the details.*

2.2 A Refinement for a Sequence Whose Norm is One

The following result holds [1, Theorem 6].

Theorem 65 *Let* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ *be sequences of real numbers and* $\bar{\mathbf{e}} = (e_1, \ldots, e_n)$ *be such that* $\sum_{i=1}^{n} e_i^2 = 1$. *Then the following inequality holds*

$$\begin{aligned} & \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \\ & \geq \left[\left| \sum_{k=1}^{n} a_k b_k - \sum_{k=1}^{n} e_k a_k \sum_{k=1}^{n} e_k b_k \right| + \left| \sum_{k=1}^{n} e_k a_k \sum_{k=1}^{n} e_k b_k \right| \right]^2 \\ & \geq \left(\sum_{k=1}^{n} a_k b_k \right)^2 . \end{aligned} \tag{2.9}$$

Proof. We will follow the proof from [1].
From the (CBS) −inequality, one has

$$\begin{aligned} & \sum_{k=1}^{n} \left[a_k - \left(\sum_{i=1}^{n} e_i a_i \right) e_k \right]^2 \sum_{k=1}^{n} \left[b_k - \left(\sum_{i=1}^{n} e_i b_i \right) e_k \right]^2 \\ & \qquad \geq \left\{ \sum_{k=1}^{n} \left[a_k - \left(\sum_{i=1}^{n} e_i a_i \right) e_k \right] \left[b_k - \left(\sum_{i=1}^{n} e_i b_i \right) e_k \right] \right\}^2 . \end{aligned} \tag{2.10}$$

Since $\sum_{k=1}^{n} e_k^2 = 1$, a simple calculation shows that

$$\begin{aligned} \sum_{k=1}^{n} \left[a_k - \left(\sum_{i=1}^{n} e_i a_i \right) e_k \right]^2 &= \sum_{k=1}^{n} a_k^2 - \left(\sum_{k=1}^{n} e_k a_k \right)^2 , \\ \sum_{k=1}^{n} \left[b_k - \left(\sum_{i=1}^{n} e_i b_i \right) e_k \right]^2 &= \sum_{k=1}^{n} b_k^2 - \left(\sum_{k=1}^{n} e_k b_k \right)^2 , \end{aligned}$$

and

$$\sum_{k=1}^{n}\left[a_k-\left(\sum_{i=1}^{n}e_ia_i\right)e_k\right]\left[b_k-\left(\sum_{i=1}^{n}e_ib_i\right)e_k\right] = \sum_{k=1}^{n}a_kb_k-\sum_{k=1}^{n}e_ka_k\sum_{k=1}^{n}e_kb_k$$

and then the inequality (2.10) becomes

$$\left[\sum_{k=1}^{n}a_k^2-\left(\sum_{k=1}^{n}e_ka_k\right)^2\right]\left[\sum_{k=1}^{n}b_k^2-\left(\sum_{k=1}^{n}e_kb_k\right)^2\right] \geq \left(\sum_{k=1}^{n}a_kb_k-\sum_{k=1}^{n}e_ka_k\sum_{k=1}^{n}e_kb_k\right)^2 \geq 0. \quad (2.11)$$

Using the elementary inequality

$$\left(m^2-l^2\right)\left(p^2-q^2\right)\leq (mp-lq)^2, \quad m,l,p,q\in\mathbb{R}$$

for the choices

$$m=\left(\sum_{k=1}^{n}a_k^2\right)^{\frac{1}{2}}, \quad l=\left|\sum_{k=1}^{n}e_ka_k\right|, \quad p=\left(\sum_{k=1}^{n}b_k^2\right)^{\frac{1}{2}}$$
$$\text{and} \quad q=\left|\sum_{k=1}^{n}e_kb_k\right|$$

the above inequality (2.11) provides the following result

$$\left[\left(\sum_{k=1}^{n}a_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}b_k^2\right)^{\frac{1}{2}}-\left|\sum_{k=1}^{n}e_ka_k\sum_{k=1}^{n}e_kb_k\right|\right]^2 \geq \left|\sum_{k=1}^{n}a_kb_k-\sum_{k=1}^{n}e_ka_k\sum_{k=1}^{n}e_kb_k\right|^2. \quad (2.12)$$

Since

$$\left(\sum_{k=1}^{n}a_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}b_k^2\right)^{\frac{1}{2}}\geq\left|\sum_{k=1}^{n}e_ka_k\sum_{k=1}^{n}e_kb_k\right|$$

then, by taking the square root in (2.12) we deduce the first part of (2.9).

The second part is obvious, and the theorem is proved. ∎

The following corollary is a natural consequence of the above theorem [1, Corollary 7].

Corollary 66 *Let* $\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{e}}$ *be as in Theorem 65. If* $\sum_{k=1}^{n} a_k b_k = 0$, *then one has the inequality:*

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \geq 4 \left(\sum_{k=1}^{n} e_k a_k \right)^2 \left(\sum_{k=1}^{n} e_k b_k \right)^2 . \tag{2.13}$$

The following inequalities are interesting as well [1, p. 81].

1. For any $\bar{\mathbf{a}}, \bar{\mathbf{b}}$ one has the inequality

$$\begin{aligned} &\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \\ &\geq \left[\left| \sum_{k=1}^{n} a_k b_k - \frac{1}{n} \sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k \right| + \frac{1}{n} \left| \sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k \right| \right]^2 \\ &\geq \left(\sum_{k=1}^{n} a_k b_k \right)^2 . \end{aligned} \tag{2.14}$$

2. If $\sum_{k=1}^{n} a_k b_k = 0$, then

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \geq \frac{4}{n^2} \left(\sum_{k=1}^{n} a_k \right)^2 \left(\sum_{k=1}^{n} b_k \right)^2 . \tag{2.15}$$

In a similar manner, we may state and prove the following result for complex numbers.

Theorem 67 *Let* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ *be sequences of complex numbers and* $\bar{\mathbf{e}} = (e_1, \ldots, e_n)$ *a sequence of complex numbers satisfying the* *condition* $\sum_{i=1}^{n} e_i^2 = 1$. *Then the following refinement of the* (CBS) −*inequality holds*

$$\sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 \tag{2.16}$$
$$\geq \left[\left| \sum_{k=1}^{n} a_k \bar{b}_k - \sum_{k=1}^{n} a_k \bar{e}_k \cdot \sum_{k=1}^{n} e_k \bar{b}_k \right| + \left| \sum_{k=1}^{n} a_k \bar{e}_k \cdot \sum_{k=1}^{n} e_k \bar{b}_k \right| \right]^2$$
$$\geq \left| \sum_{k=1}^{n} a_k \bar{b}_k \right|^2 .$$

The proof is similar to the one in Theorem 65 on using the corresponding (CBS) −inequality for complex numbers.

Remark 68 *Similar particular inequalities may be stated, but we omit the details.*

2.3 A Second Refinement in Terms of Moduli

The following lemma holds.

Lemma 69 *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ *be a sequence of real numbers and* $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ *a sequence of positive real numbers with* $\sum_{i=1}^{n} p_i = 1$. *Then one has the inequality:*

$$\sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i \right)^2 \geq \left| \sum_{i=1}^{n} p_i |a_i| a_i - \sum_{i=1}^{n} p_i |a_i| \sum_{i=1}^{n} p_i a_i \right| . \tag{2.17}$$

Proof. By the properties of moduli we have

$$(a_i - a_j)^2 = |(a_i - a_j)(a_i - a_j)| \geq |(|a_i| - |a_j|)(a_i - a_j)|$$

for any $i, j \in \{1, \dots, n\}$. This is equivalent to

$$a_i^2 - 2a_i a_j + a_j^2 \geq ||a_i| a_i + |a_j| a_j - |a_i| a_j - |a_j| a_i| \tag{2.18}$$

for any $i, j \in \{1, \dots, n\}$.

If we multiply (2.18) by $p_i p_j \geq 0$ and sum over i and j from 1 to n we deduce

$$\begin{aligned}\sum_{j=1}^{n} p_j \sum_{i=1}^{n} p_i a_i^2 - 2\sum_{i=1}^{n} p_i a_i \sum_{j=1}^{n} p_j a_j + \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j a_j^2 \\ \geq \sum_{i,j=1}^{n} p_i p_j \left| |a_i| a_i + |a_j| a_j - |a_i| a_j - |a_j| a_i \right| \\ \geq \left| \sum_{i,j=1}^{n} p_i p_j \left(|a_i| a_i + |a_j| a_j - |a_i| a_j - |a_j| a_i \right) \right|,\end{aligned}$$

which is clearly equivalent to (2.17). ■

Using the above lemma, we may prove the following refinement of the (CBS) -inequality.

Theorem 70 *Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ and $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ be two sequences of real numbers. Then one has the inequality*

$$\begin{aligned}\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i \right)^2 \\ \geq \left| \sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} sgn\,(a_i)\, |b_i|\, b_i - \sum_{i=1}^{n} |a_i b_i| \sum_{i=1}^{n} a_i b_i \right| \geq 0. \quad (2.19)\end{aligned}$$

Proof. If we choose (for $a_i \neq 0$, $i \in \{1, \dots, n\}$) in (2.17), that

$$p_i := \frac{a_i^2}{\sum_{k=1}^{n} a_k^2}, \quad x_i = \frac{b_i}{a_i}, \quad i \in \{1, \dots, n\},$$

we get

$$\begin{aligned}\sum_{i=1}^{n} \frac{a_i^2}{\sum_{k=1}^{n} a_k^2} \cdot \left(\frac{b_i}{a_i} \right)^2 - \left(\sum_{i=1}^{n} \frac{a_i^2}{\sum_{k=1}^{n} a_k^2} \cdot \frac{b_i}{a_i} \right)^2 \\ \geq \left| \sum_{i=1}^{n} \frac{a_i^2}{\sum_{k=1}^{n} a_k^2} \left| \frac{b_i}{a_i} \right| \cdot \frac{b_i}{a_i} - \sum_{i=1}^{n} \frac{a_i^2}{\sum_{k=1}^{n} a_k^2} \left| \frac{b_i}{a_i} \right| \sum_{i=1}^{n} \frac{a_i^2}{\sum_{k=1}^{n} a_k^2} \cdot \frac{b_i}{a_i} \right|\end{aligned}$$

from where we get

$$\sum_{i=1}^{n} \frac{b_i^2}{\sum_{k=1}^{n} a_k^2} - \frac{\left(\sum_{i=1}^{n} a_i b_i\right)^2}{\left(\sum_{k=1}^{n} a_k^2\right)^2} \geq \left| \frac{\sum_{i=1}^{n} \frac{|a_i|}{a_i} |b_i| b_i}{\sum_{k=1}^{n} a_k^2} - \frac{\sum_{i=1}^{n} |a_i b_i| \sum_{i=1}^{n} a_i b_i}{\left(\sum_{k=1}^{n} a_k^2\right)^2} \right|$$

which is clearly equivalent to (2.19). ■

The case for complex numbers is as follows.

Lemma 71 *Let $\bar{\mathbf{z}} = (z_1, \dots, z_n)$ be a sequence of complex numbers and $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ a sequence of positive real numbers with $\sum_{i=1}^{n} p_i = 1$. Then one has the inequality:*

$$\sum_{i=1}^{n} p_i |z_i|^2 - \left| \sum_{i=1}^{n} p_i z_i \right|^2 \geq \left| \sum_{i=1}^{n} p_i |z_i| z_i - \sum_{i=1}^{n} p_i |z_i| \sum_{i=1}^{n} p_i z_i \right|. \tag{2.20}$$

Proof. By the properties of moduli for complex numbers we have

$$|z_i - z_j|^2 \geq |(|z_i| - |z_j|)(z_i - z_j)|$$

for any $i, j \in \{1, \dots, n\}$, which is clearly equivalent to

$$|z_i|^2 - 2\operatorname{Re}(z_i \bar{z}_j) + |z_j|^2 \geq ||z_i| z_i + |z_j| z_j - z_i |z_j| - |z_i| z_j|$$

for any $i, j \in \{1, \dots, n\}$.

If we multiply with $p_i p_j \geq 0$ and sum over i and j from 1 to n, we deduce the desired inequality (2.20). ■

Now, in a similar manner to the one in Theorem 70, we may state the following result for complex numbers.

Theorem 72 *Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ $(a_i \neq 0,\ i = 1, \dots, n)$ and $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ be two sequences of complex numbers. Then one has the inequality:*

$$\sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \left| \sum_{i=1}^{n} \bar{a}_i b_i \right|^2 \geq \left| \sum_{i=1}^{n} \frac{|a_i|}{a_i} |b_i| b_i - \sum_{i=1}^{n} |a_i| b_i \sum_{i=1}^{n} \bar{a}_i b_i \right| \geq 0. \tag{2.21}$$

2.4 A Refinement for a Sequence Less than the Weights

The following result was obtained in [1, Theorem 9] (see also [2, Theorem 3.10]).

Theorem 73 *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be sequences of real numbers and* $\bar{\mathbf{p}} = (p_1, \dots, p_n)$, $\bar{\mathbf{q}} = (q_1, \dots, q_n)$ *be sequences of nonnegative real numbers such that* $p_k \geq q_k$ *for any* $k \in \{1, \dots, n\}$. *Then we have the inequality*

$$\begin{aligned} &\sum_{k=1}^{n} p_k a_k^2 \sum_{k=1}^{n} p_k b_k^2 &(2.22)\\ &\geq \left[\left| \sum_{k=1}^{n} (p_k - q_k) a_k b_k \right| + \left(\sum_{k=1}^{n} q_k a_k^2 \sum_{k=1}^{n} q_k b_k^2 \right)^{\frac{1}{2}} \right]^2 \\ &\geq \left[\left| \sum_{k=1}^{n} (p_k - q_k) a_k b_k \right| + \left| \sum_{k=1}^{n} q_k a_k b_k \right| \right]^2 \\ &\geq \left(\sum_{k=1}^{n} p_k a_k b_k \right)^2 . \end{aligned}$$

Proof. We shall follow the proof in [1].

Since $p_k - q_k \geq 0$, then the (CBS) −inequality for the weights $r_k := p_k - q_k$ will give

$$\left(\sum_{k=1}^{n} p_k a_k^2 - \sum_{k=1}^{n} q_k a_k^2 \right) \left(\sum_{k=1}^{n} p_k b_k^2 - \sum_{k=1}^{n} q_k b_k^2 \right) \geq \left[\sum_{k=1}^{n} (p_k - q_k) a_k b_k \right]^2 . \quad (2.23)$$

Using the elementary inequality

$$(ac - bd)^2 \geq \left(a^2 - b^2\right) \left(c^2 - d^2\right), \quad a, b, c, d \in \mathbb{R}$$

for the choices

$$a=\left(\sum_{k=1}^{n}p_ka_k^2\right)^{\frac{1}{2}},\quad b=\left(\sum_{k=1}^{n}q_ka_k^2\right)^{\frac{1}{2}},\quad c=\left(\sum_{k=1}^{n}p_kb_k^2\right)^{\frac{1}{2}}$$

and

$$d=\left(\sum_{k=1}^{n}q_kb_k^2\right)^{\frac{1}{2}}$$

we deduce by (2.23) that

$$\left[\left(\sum_{k=1}^{n}p_ka_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}p_kb_k^2\right)^{\frac{1}{2}}-\left(\sum_{k=1}^{n}q_ka_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}q_kb_k^2\right)^{\frac{1}{2}}\right]^2 \geq\left[\sum_{k=1}^{n}(p_k-q_k)\,a_kb_k\right]^2. \quad (2.24)$$

Since, obviously,

$$\left(\sum_{k=1}^{n}p_ka_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}p_kb_k^2\right)^{\frac{1}{2}}\geq\left(\sum_{k=1}^{n}q_ka_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}q_kb_k^2\right)^{\frac{1}{2}}$$

then, by (2.24), on taking the square root, we would get

$$\left(\sum_{k=1}^{n}p_ka_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}p_kb_k^2\right)^{\frac{1}{2}} \geq\left(\sum_{k=1}^{n}q_ka_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}q_kb_k^2\right)^{\frac{1}{2}}+\left|\sum_{k=1}^{n}(p_k-q_k)\,a_kb_k\right|,$$

which provides the first inequality in (2.22).

The other inequalities are obvious and we omit the details. ■

The following corollary is a natural consequence of the above theorem [2, Corollary 3.11].

Corollary 74 *Let* $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ *be sequences of real numbers and* $\bar{\mathbf{s}} = (s_1, \dots, s_n)$ *be such that* $0 \le s_k \le 1$ *for any* $k \in \{1, \dots, n\}$. *Then one has the inequalities*

$$\begin{aligned}\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 &\geq \left[\left|\sum_{k=1}^{n} (1 - s_k)\, a_k b_k\right| + \left(\sum_{k=1}^{n} s_k a_k^2 \sum_{k=1}^{n} s_k b_k^2\right)^{\frac{1}{2}}\right]^2 \qquad (2.25)\\ &\geq \left[\left|\sum_{k=1}^{n} (1 - s_k)\, a_k b_k\right| + \left|\sum_{k=1}^{n} s_k a_k b_k\right|\right]^2 \\ &\geq \left(\sum_{k=1}^{n} a_k b_k\right)^2 .\end{aligned}$$

Remark 75 *Assume that* $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ *and* $\bar{\mathbf{s}}$ *are as in Corollary 74. The following inequalities hold (see [2, p. 15]).*

a) If $\sum_{k=1}^{n} a_k b_k = 0,$ *then*

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \geq 4 \left(\sum_{k=1}^{n} s_k a_k b_k\right)^2 . \qquad (2.26)$$

b) If $\sum_{k=1}^{n} s_k a_k b_k = 0,$ *then*

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \geq \left[\left|\sum_{k=1}^{n} a_k b_k\right| + \left(\sum_{k=1}^{n} \alpha_k a_k^2 \sum_{k=1}^{n} \alpha_k b_k^2\right)^{\frac{1}{2}}\right]^2 . \qquad (2.27)$$

In particular, we may obtain the following particular inequalities involving trigonometric functions (see [2, p. 15])

$$\begin{aligned}&\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \qquad (2.28)\\ &\geq \left[\left|\sum_{k=1}^{n} a_k b_k \cos^2 \alpha_k\right| + \left(\sum_{k=1}^{n} a_k^2 \sin^2 \alpha_k \sum_{k=1}^{n} b_k^2 \sin^2 \alpha_k\right)^{\frac{1}{2}}\right]^2\end{aligned}$$

$$\geq \left[\left| \sum_{k=1}^{n} a_k b_k \cos^2 \alpha_k \right| + \left| \sum_{k=1}^{n} a_k b_k \sin^2 \alpha_k \right| \right]^2$$
$$\geq \left(\sum_{k=1}^{n} a_k b_k \right)^2,$$

where $a_k, b_k, \alpha_k \in \mathbb{R},\ k = 1, \ldots, n.$

If one would assume that $\sum_{k=1}^{n} a_k b_k = 0$, then

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \geq 4 \left(\sum_{k=1}^{n} a_k b_k \sin^2 \alpha_k \right)^2. \tag{2.29}$$

If $\sum_{k=1}^{n} a_k b_k \sin^2 \alpha_k = 0$, then

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \geq \left[\left| \sum_{k=1}^{n} a_k b_k \right| + \left(\sum_{k=1}^{n} a_k^2 \sin^2 \alpha_k \sum_{k=1}^{n} b_k^2 \sin^2 \alpha_k \right)^{\frac{1}{2}} \right]^2. \tag{2.30}$$

2.5 A Conditional Inequality Providing a Refinement

The following lemma holds [2, Lemma 4.1].

Lemma 76 *Consider the sequences of real numbers* $\bar{\mathbf{x}} = (x_1, \ldots, x_n)$, $\bar{\mathbf{y}} = (y_1, \ldots, y_n)$ *and* $\bar{\mathbf{z}} = (z_1, \ldots, z_n)$. *If*

$$y_k^2 \leq |x_k z_k| \quad \text{for any } k \in \{1, \ldots, n\}, \tag{2.31}$$

then one has the inequality:

$$\left(\sum_{k=1}^{n} |y_k| \right)^2 \leq \sum_{k=1}^{n} |x_k| \sum_{k=1}^{n} |z_k|. \tag{2.32}$$

Proof. We will follow the proof in [2]. Using the condition (2.31) and the (CBS) −inequality, we have

$$\sum_{k=1}^{n}|y_k| \le \sum_{k=1}^{n}|x_k|^{\frac{1}{2}}|z_k|^{\frac{1}{2}} \le \left[\sum_{k=1}^{n}\left(|x_k|^{\frac{1}{2}}\right)^2\sum_{k=1}^{n}\left(|z_k|^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}}$$
$$= \left(\sum_{k=1}^{n}|x_k|\sum_{k=1}^{n}|z_k|\right)^{\frac{1}{2}}$$

which is clearly equivalent to (2.32). ■

The following result holds [2, Theorem 4.6].

Theorem 77 *Let* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ *and* $\bar{\mathbf{c}} = (c_1, \ldots, c_n)$ *be sequences of real numbers such that*

(i) $|b_k| + |c_k| \neq 0 \quad (k \in \{1, \ldots, n\})$

(ii) $|a_k| \le \frac{2|b_k c_k|}{|b_k|+|c_k|}$ *for any* $k \in \{1, \ldots, n\}$.

Then one has the inequality

$$\sum_{k=1}^{n}|a_k| \le \frac{2\sum_{k=1}^{n}|b_k|\sum_{k=1}^{n}|c_k|}{\sum_{k=1}^{n}(|b_k|+|c_k|)}. \tag{2.33}$$

Proof. We will follow the proof in [2]. By (ii) we observe that

$$|a_k| \le \frac{2|b_k c_k|}{|b_k|+|c_k|} \le \begin{cases} 2|b_k| \\ \\ 2|c_k| \end{cases} \text{for any } k \in \{1, \ldots, n\}$$

and thus

$$\begin{aligned} x_k &:= 2|b_k| - |a_k| \ge 0 \qquad \text{and} \\ z_k &:= 2|c_k| - |a_k| \ge 0 \qquad \text{for any } k \in \{1, \ldots, n\}. \end{aligned} \tag{2.34}$$

A simple calculation also shows that the relation (ii) is equivalent to

$$a_k^2 \le (2|b_k| - |a_k|)(2|c_k| - |a_k|) \quad \text{for any } k \in \{1, \ldots, n\}. \tag{2.35}$$

If we consider $y_k := a_k$ and take x_k, z_k $(k = 1, \dots, n)$ as defined by (2.34), then we get $y_k^2 \le x_k z_k$ (with x_k, $z_k \ge 0$) for any $k \in \{1, \dots, n\}$. Applying Lemma 76 we deduce

$$\left(\sum_{k=1}^{n} |a_k|\right)^2 \le \left(2\sum_{k=1}^{n} |b_k| - \sum_{k=1}^{n} |a_k|\right)\left(2\sum_{k=1}^{n} |c_k| - \sum_{k=1}^{n} |a_k|\right) \tag{2.36}$$

which is clearly equivalent to (2.33). ∎

The following corollary is a natural consequence of the above theorem [2, Corollary 4.7].

Corollary 78 *For any sequence* $\bar{\mathbf{x}}$ *and* $\bar{\mathbf{y}}$ *of real numbers, with* $|x_k| + |y_k| \neq 0$ $(k = 1, \dots, n)$, *one has:*

$$\sum_{k=1}^{n} \frac{|x_k y_k|}{|x_k| + |y_k|} \le \frac{2\sum_{k=1}^{n} |x_k| \sum_{k=1}^{n} |y_k|}{\sum_{k=1}^{n} (|x_k| + |y_k|)}. \tag{2.37}$$

For two positive real numbers, let us recall the following means

$$A(a,b) := \frac{a+b}{2} \qquad (\textit{the arithmetic mean})$$

$$G(a,b) := \sqrt{ab} \qquad (\textit{the geometric mean})$$

and

$$H(a,b) := \frac{2}{\frac{1}{a} + \frac{1}{b}} \qquad (\textit{the harmonic mean}).$$

We remark that if $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ are sequences of real numbers, then obviously

$$\sum_{i=1}^{n} A(a_i, b_i) = A\left(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i\right), \tag{2.38}$$

and, by the (CBS) −inequality,

$$\sum_{i=1}^{n} G(a_i, b_i) \le G\left(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i\right). \tag{2.39}$$

The following similar result for harmonic means also holds [2, p. 19].

Theorem 79 *For any two sequences of positive real numbers* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *we have the property:*

$$\sum_{i=1}^{n} H(a_i, b_i) \leq H\left(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i\right). \tag{2.40}$$

Proof. Follows by Corollary 78 on choosing $x_k = a_k$, $y_k = b_k$ and multiplying the inequality (2.37) with 2. ■

The following refinement of the (CBS) −inequality holds [2, Corollary 4.9]. This result is known in the literature as **Milne's inequality** [8].

Theorem 80 *For any two sequences of real numbers* $\bar{\mathbf{p}} = (p_1, \dots, p_n)$, $\bar{\mathbf{q}} = (q_1, \dots, q_n)$ *with* $|p_k| + |q_k| \neq 0$ $(k = 1, \dots, n)$, *one has the inequality:*

$$\left(\sum_{k=1}^{n} p_k q_k\right)^2 \leq \sum_{k=1}^{n} \left(p_k^2 + q_k^2\right) \sum_{k=1}^{n} \frac{p_k^2 q_k^2}{p_k^2 + q_k^2} \leq \sum_{k=1}^{n} p_k^2 \sum_{k=1}^{n} q_k^2. \tag{2.41}$$

Proof. We shall follow the proof in [2]. The first inequality is obvious by Lemma 76 on choosing $y_k = p_k q_k$, $x_k = p_k^2 + q_k^2$ and $z_k = \frac{p_k^2 q_k^2}{p_k^2 + q_k^2}$ $(k = 1, \dots, n)$.

The second inequality follows by Corollary 78 on choosing $x_k = p_k^2$ and $y = q_k^2$ $(k = 1, \dots, n)$. ■

Remark 81 *The following particular inequality is obvious by (2.41)*

$$\begin{aligned} \left(\sum_{i=1}^{n} \sin \alpha_k \cos \alpha_k\right)^2 &\leq n \sum_{i=1}^{n} \sin^2 \alpha_k \cos^2 \alpha_k \\ &\leq \sum_{i=1}^{n} \sin^2 \alpha_k \sum_{i=1}^{n} \cos^2 \alpha_k; \end{aligned} \tag{2.42}$$

for any $\alpha_k \in \mathbb{R}$, $k \in \{1, \dots, n\}$.

2.6 A Refinement for Non-Constant Sequences

The following result was proved in [3, Theorem 1].

Theorem 82 *Let* $\bar{\mathbf{a}} = (a_i)_{i \in \mathbb{N}}$, $\bar{\mathbf{b}} = (b_i)_{i \in \mathbb{N}}$, $\bar{\mathbf{p}} = (p_i)_{i \in \mathbb{N}}$ *be sequences of real numbers such that*

(i) $a_i \neq a_j$ *and* $b_i \neq b_j$ *for* $i \neq j$, $i, j \in \mathbb{N}$*;*

(ii) $p_i > 0$ *for all* $i \in \mathbb{N}$*.*

Then for any H *a finite part of* $\mathbb{N}$ *one has the inequality:*

$$\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i\right)^2 \geq \max\{A, B\} \geq 0, \tag{2.43}$$

where

$$A := \max_{\substack{J\subseteq H\\ J\neq\emptyset}} \frac{\left[\sum_{i\in H} p_i a_i b_i \sum_{j\in J} p_j a_j - \sum_{i\in H} p_i a_i^2 \sum_{j\in J} p_j b_j\right]^2}{P_J \sum_{i\in H} p_i a_i^2 - \left(\sum_{i\in J} p_i a_i\right)^2} \tag{2.44}$$

and

$$B := \max_{\substack{J\subseteq H\\ J\neq\emptyset}} \frac{\left[\sum_{i\in H} p_i a_i b_i \sum_{j\in J} p_j b_j - \sum_{i\in H} p_i b_i^2 \sum_{j\in J} p_j a_j\right]^2}{P_J \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in J} p_i b_i\right)^2} \tag{2.45}$$

and $P_J := \sum_{j\in J} p_j$*.*

Proof. We shall follow the proof in [3].

Let J be a part of H. Define the mapping $f_J : \mathbb{R} \to \mathbb{R}$ given by

$$f_J(t) = \sum_{i\in H} p_i a_i^2 \left[\sum_{i\in H\setminus J} p_i b_i^2 + \sum_{i\in J} p_i (b_i + t)^2\right] - \left[\sum_{i\in H\setminus J} p_i a_i b_i + \sum_{i\in J} p_i a_i (b_i + t)\right]^2.$$

Then by the (CBS) $-$inequality we have that $f_J(t) \geq 0$ for all $t \in \mathbb{R}$.

On the other hand we have

$$\begin{aligned} f_J(t) &= \sum_{i\in H} p_i a_i^2 \left[\sum_{i\in H} p_i b_i^2 + 2t \sum_{i\in H} p_i b_i + t^2 P_J \right] \\ &\quad - \left[\sum_{i\in H} p_i a_i b_i + t \sum_{i\in J} p_i a_i \right]^2 \\ &= t^2 \left[P_J \sum_{i\in H} p_i a_i^2 - \left(\sum_{i\in J} p_i a_i \right)^2 \right] \\ &\quad + 2t \left[\sum_{i\in H} p_i a_i^2 \sum_{i\in J} p_i b_i - \sum_{i\in H} p_i a_i b_i \sum_{i\in J} p_i a_i \right] \\ &\quad + \left[\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i \right)^2 \right] \end{aligned}$$

for all $t \in \mathbb{R}$.

Since

$$P_J \sum_{i\in H} p_i a_i^2 - \left(\sum_{i\in J} p_i a_i \right)^2 \geq P_J \sum_{i\in J} p_i a_i^2 - \left(\sum_{i\in J} p_i a_i \right)^2 > 0$$

as $a_i \neq a_j$ for all $i, j \in \{1, \dots, n\}$ with $i \neq j$, then, by the inequality $f_J(t) \geq 0$ for any $t \in \mathbb{R}$ we get that

$$\begin{aligned} 0 \geq \frac{1}{4}\Delta &= \left[\sum_{i\in H} p_i a_i b_i \sum_{j\in J} p_j a_j - \sum_{i\in H} p_i a_i^2 \sum_{j\in J} p_j b_j \right]^2 \\ &\quad - \left[P_J \sum_{i\in H} p_i a_i^2 - \left(\sum_{i\in J} p_i a_i \right)^2 \right] \left[\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i \right)^2 \right] \end{aligned}$$

from where results the inequality

$$\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i \right)^2 \geq A.$$

The second part of the proof goes likewise for the mapping $G_J : \mathbb{R} \to \mathbb{R}$ given by

$$g_J(t) = \left[\sum_{i\in H\setminus J} p_i a_i^2 + \sum_{i\in J} p_i (a_i + t)^2\right] \sum_{i\in H} p_i b_i^2 - \left[\sum_{i\in H\setminus J} p_i a_i b_i + \sum_{i\in J} p_i b_i (a_i + t)\right]^2$$

and we omit the details. ■

The following corollary also holds [3, Corollary 1].

Corollary 83 *With the assumptions of Theorem 82 and if*

$$C := \frac{\left[\sum_{i\in H} p_i a_i b_i \sum_{i\in H} p_i a_i - \sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i\right]^2}{P_H \sum_{i\in H} p_i a_i^2 - \left(\sum_{i\in H} p_i a_i\right)^2}, \tag{2.46}$$

$$D := \frac{\left[\sum_{i\in H} p_i a_i b_i \sum_{i\in H} p_i b_i - \sum_{i\in H} p_i b_i^2 \sum_{i\in H} p_i a_i\right]^2}{P_H \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i b_i\right)^2}, \tag{2.47}$$

then one has the inequality

$$\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i\right)^2 \geq \max\{C, D\} \geq 0. \tag{2.48}$$

The following corollary also holds [3, Corollary 2].

Corollary 84 *If a_i, $b_i \neq 0$ for $i \in \mathbb{N}$ and H is a finite part of $\mathbb{N}$, then one has the inequality*

$$\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i\right)^2 \geq \frac{1}{card(H) - 1} \max\left\{\frac{\sum_{j\in H} p_j c_j^2}{\sum_{i\in H} p_i a_i^2}, \frac{\sum_{j\in H} p_j d_j^2}{\sum_{i\in H} p_i b_i^2}\right\} \geq 0, \tag{2.49}$$

where

$$c_j := a_j \sum_{i\in H} p_i a_i b_i - b_j \sum_{i\in H} p_i a_i^2, \quad j \in H \tag{2.50}$$

and

$$d_j := a_j \sum_{i\in H} p_i b_i^2 - b_j \sum_{i\in H} p_i a_i b_i, \quad j \in H. \tag{2.51}$$

Proof. Choosing in Theorem 82, $J = \{j\}$, we get the inequality

$$\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i\right)^2 \geq \frac{p_j^2 c_j^2}{p_j \sum_{i\in H} p_i a_i^2 - p_j^2 a_j^2}, \quad j \in H$$

from where we obtain

$$\left(\sum_{i\in H} p_i a_i^2 - p_j a_j^2\right) \left[\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i\right)^2\right] \geq p_j c_j^2 \quad \text{for any} \quad j \in H.$$

Summing these inequalities over $j \in H$, we get

$$[card\,(H) - 1] \sum_{i\in H} p_i a_i^2 \left[\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i\right)^2\right] \geq \sum_{j\in H} p_j c_j^2$$

from where we get the first part of (2.49).

The second part goes likewise and we omit the details. ■

Remark 85 *The following particular inequalities provide refinement for the* (CBS) *−inequality [3, p. 60 – p. 61].*

1. *Assume that* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$ *are nonconstant se-*

quences of real numbers. Then

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \geq \max\left\{ \frac{\left[\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} a_i b_i\right]^2}{n\sum_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n} a_i\right)^2}, \frac{\left[\sum_{i=1}^{n} b_i \sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i^2\right]^2}{n\sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} b_i\right)^2} \right\}. \quad (2.52)$$

2. *Assume that* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *are sequences of real numbers with not all elements equal to zero, then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \geq \frac{1}{n-1} \max\left\{ \frac{\sum_{j=1}^{n}\left(a_j \sum_{i=1}^{n} a_i b_i - b_j \sum_{i=1}^{n} a_i^2\right)^2}{\sum_{i=1}^{n} a_i^2}, \frac{\sum_{j=1}^{n}\left(a_j \sum_{i=1}^{n} a_i^2 - b_j \sum_{i=1}^{n} a_i b_i\right)^2}{\sum_{i=1}^{n} b_i^2} \right\}. \quad (2.53)$$

2.7 De Bruijn's Inequality

The following refinement of the (CBS) −inequality was proved by N.G. de Bruijn in 1960, [4] (see also [5, p. 89]).

Theorem 86 *If $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$ is a sequence of real numbers and $\bar{\mathbf{z}} = (z_1, \ldots, z_n)$ is a sequence of complex numbers, then*

$$\left|\sum_{k=1}^{n} a_k z_k\right|^2 \leq \frac{1}{2} \sum_{k=1}^{n} a_k^2 \left[\sum_{k=1}^{n} |z_k|^2 + \left|\sum_{k=1}^{n} z_k^2\right|\right]. \tag{2.54}$$

Equality holds in (2.54) if and only if for $k \in \{1, \ldots, n\}$, $a_k = \operatorname{Re}(\lambda z_k)$, where λ is a complex number such that $\sum_{k=1}^{n} \lambda^2 z_k^2$ is a nonnegative real number.

Proof. We shall follow the proof in [5, p. 89 – p. 90].

By a simultaneous rotation of all the z_k's about the origin, we get

$$\sum_{k=1}^{n} a_k z_k \geq 0.$$

This rotation does not affect the moduli

$$\left|\sum_{k=1}^{n} a_k z_k\right|, \quad \left|\sum_{k=1}^{n} z_k^2\right| \quad \text{and} \quad |z_k| \quad \text{for} \quad k \in \{1, \ldots, n\}.$$

Hence, it is sufficient to prove inequality (2.54) for the case where $\sum_{k=1}^{n} a_k z_k \geq 0$.

If we put $z_k = x_k + i y_k$ $(k \in \{1, \ldots, n\})$, then, by the $(CBS)-$inequality for real numbers, we have

$$\left|\sum_{k=1}^{n} a_k z_k\right|^2 = \left(\sum_{k=1}^{n} a_k z_k\right)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} x_k^2. \tag{2.55}$$

Since

$$2x_k^2 = |z_k|^2 + \operatorname{Re} z_k^2 \quad \text{for any} \quad k \in \{1, \ldots, n\}$$

we obtain, by (2.55), that

$$\left|\sum_{k=1}^{n} a_k z_k\right|^2 \leq \frac{1}{2} \sum_{k=1}^{n} a_k^2 \left[\sum_{k=1}^{n} |z_k|^2 + \sum_{k=1}^{n} \operatorname{Re} z_k^2\right]. \tag{2.56}$$

As

$$\sum_{k=1}^{n} \operatorname{Re} z_k^2 = \operatorname{Re}\left(\sum_{k=1}^{n} z_k^2\right) \leq \left|\sum_{k=1}^{n} z_k^2\right|,$$

then by (2.56) we deduce the desired inequality (2.54). ■

2.8 McLaughlin's Inequality

The following refinement of the (CBS) −inequality for sequences of real numbers was obtained in 1966 by H.W. McLaughlin [7, p. 66].

Theorem 87 *If* $\bar{\mathbf{a}} = (a_1, \ldots, a_{2n})$, $\bar{\mathbf{b}} = (b_1, \ldots, b_{2n})$ *are sequences of real numbers, then*

$$\left(\sum_{i=1}^{2n} a_i b_i\right)^2 + \left[\sum_{i=1}^{n} (a_i b_{n+i} - a_{n+i} b_i)\right]^2 \le \sum_{i=1}^{2n} a_i^2 \sum_{i=1}^{2n} b_i^2 \tag{2.57}$$

with equality if and only if for any $i, j \in \{1, \ldots, n\}$

$$a_i b_j - a_j b_i - a_{n+i} b_{n+j} + a_{n+j} b_{n+i} = 0 \tag{2.58}$$

and

$$a_i b_{n+j} - a_j b_{n+i} + a_{n+i} b_j - a_{n+j} b_i = 0. \tag{2.59}$$

Proof. We shall follow the proof in [6] by M.O. Drîmbe.

The following identity may be obtained by direct computation

$$\begin{aligned}\sum_{i=1}^{2n} a_i^2 \sum_{i=1}^{2n} b_i^2 - \left(\sum_{i=1}^{2n} a_i b_i\right)^2 - \left[\sum_{i=1}^{n} (a_i b_{n+i} - a_{n+i} b_i)\right]^2 \\ = \sum_{1 \le i < j \le n} (a_i b_j - a_j b_i - a_{n+i} b_{n+j} + a_{n+j} b_{n+i})^2 \\ + \sum_{1 \le i < j \le n} (a_i b_{n+j} - a_j b_{n+i} + a_{n+i} b_j - a_{n+j} b_i)^2 .\end{aligned} \tag{2.60}$$

It is obvious that (2.57) is a simple consequence of the identity (2.60). The case of equality is also obvious. ■

Remark 88 *For other similar* (CBS) *−type inequalties see the survey paper [7]. An analogous inequality to (2.57) for sequences* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *having* $4n$ *terms each may be found in [7, p. 70].*

2.9 A Refinement due to Daykin-Eliezer-Carlitz

We will present now the version due to Mitrinović, Pečarić and Fink [5, p. 87] of Daykin-Eliezer-Carlitz's refinement of the discrete (CBS) −inequality [8].

Theorem 89 *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ *and* $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be two sequences of positive numbers. The ineuality*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} f(a_i b_i) \sum_{i=1}^{n} g(a_i b_i) \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \tag{2.61}$$

holds if and only if

$$f(a,b)\, g(a,b) = a^2 b^2, \tag{2.62}$$

$$f(ka,kb) = k^2 f(a,b), \tag{2.63}$$

$$\frac{bf(a,1)}{af(b,1)} + \frac{af(b,1)}{bf(a,1)} \leq \frac{a}{b} + \frac{b}{a} \tag{2.64}$$

for any $a, b, k > 0$.

Proof. We shall follow the proof in [5, p. 88 – p. 89].
Necessity. Indeed, for $n = 1$, the inequality (2.61) becomes

$$(ab)^2 \leq f(a,b)\, g(a,b) \leq a^2 b^2, \qquad a, b > 0$$

which gives the condition (2.62).

For $n = 2$ in (2.61), using (2.62), we get

$$2a_1b_1a_2b_2 \leq f(a_1,b_1)\, g(a_2,b_2) + f(a_2,b_2)\, g(a_1,b_1) \leq a_1^2 b_2^2 + a_2^2 b_1^2.$$

By eliminating g, we get

$$2 \leq \frac{f(a_1,b_1)}{f(a_2,b_2)} \cdot \frac{a_2b_2}{a_1b_1} + \frac{f(a_2,b_2)}{f(a_1,b_1)} \cdot \frac{a_1b_1}{a_2b_2} \leq \frac{a_1b_2}{a_2b_1} + \frac{a_2b_1}{a_1b_2}. \tag{2.65}$$

By substituting in (2.65) a, b for a_1, b_1 and ka, kb for a_2, b_2 $(k > 0)$, we get

$$2 \leq \frac{f(a,b)}{f(ka,kb)} k^2 + \frac{f(ka,kb)}{f(a,b)} k^{-2} \leq 2$$

and this is valid only if $k^2 f(a,b)(f(ka,kb)) = 1$, i.e., the condition (2.63) holds.

Using (2.65), for $a_1 = a$, $b_1 = b$, $a_2 = b$, $b_2 = 1$, we have

$$2 \le \frac{\frac{f(a,1)}{a}}{\frac{f(b,1)}{b}} + \frac{\frac{f(b,1)}{b}}{\frac{f(a,1)}{a}} \le \frac{a}{b} + \frac{b}{a}. \tag{2.66}$$

The first inequality in (2.66) is always satisfied while the second inequality is equivalent to (2.64).

Sufficiency. Suppose that (2.62) holds. Then inequality (2.61) can be written in the form

$$\begin{aligned} 2 \sum_{1 \le i < j \le n} a_i b_i a_j b_j &\le \sum_{1 \le i < j \le n} \left[f(a_i, b_i) g(a_j, b_j) + f(a_j, b_j) g(a_i, b_i) \right] \\ &\le \sum_{1 \le i < j \le n} \left(a_i^2 b_j^2 + a_j^2 b_i^2 \right). \end{aligned}$$

Therefore, it is enough to prove

$$\begin{aligned} 2a_i b_i a_j b_j &\le f(a_i, b_i) g(a_j, b_j) + f(a_j, b_j) g(a_i, b_i) \\ &\le a_i^2 b_j^2 + a_j^2 b_i^2. \end{aligned} \tag{2.67}$$

Suppose that (2.64) holds. Then (2.66) holds and putting $a = \frac{a_i}{b_i}$, $b = \frac{a_j}{b_j}$ in (2.66) and using (2.63), we get

$$2 \le \frac{f(a_i, b_i)}{f(a_j, b_j)} \cdot \frac{a_j b_j}{a_i b_i} + \frac{f(a_j, b_j)}{f(a_i, b_i)} \cdot \frac{a_i b_i}{a_j b_j} \le \frac{a_i b_j}{a_j b_i} + \frac{a_j b_i}{a_i b_j}.$$

Multiplying the last inequality by $a_i b_i a_j b_j$ and using (2.62), we obtain (2.67). ■

Remark 90 *In [8] (see [5, p. 89]) the condition (2.64) is given as*

$$f(b,1) \le f(a,1), \quad \frac{f(a,1)}{a^2} \le \frac{f(b,1)}{b^2} \quad \text{for } a \ge b > 0. \tag{2.68}$$

Remark 91 *O.E. Daykin, C.J. Eliezer and C. Carlitz [8] stated that examples for f, g satisfying (2.62) – (2.64) were obtained in the literature. The choice $f(x,y) = x^2 + y^2$, $g(x,y) = \frac{x^2 y^2}{x^2 + y^2}$ will give the* ***Milne's inequality***

$$\left(\sum_{i=1}^n a_i b_i \right)^2 \le \sum_{i=1}^n \left(a_i^2 + b_i^2 \right) \cdot \sum_{i=1}^n \frac{a_i^2 b_i^2}{a_i^2 + b_i^2} \le \sum_{i=1}^n a_i^2 \cdot \sum_{i=1}^n b_i^2. \tag{2.69}$$

For a different proof of this fact, see Section 2.5.

The choice $f(x,y) = x^{1+\alpha}y^{1-\alpha}$, $g(x,y) = x^{1-\alpha}y^{1+\alpha}$ $(\alpha \in [0,1])$ will give the ***Callebaut inequality***

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^{1+\alpha} b_i^{1-\alpha} \sum_{i=1}^{n} a_i^{1-\alpha} b_i^{1+\alpha} \leq \sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2. \tag{2.70}$$

2.10 A Refinement via Dunkl-Williams' Inequality

We will use the following version of Dunkl-Williams' inequality established in 1964 in inner product spaces [9].

Lemma 92 *Let a, b be two non-null complex numbers. Then*

$$|a-b| \geq \frac{1}{2}(|a|+|b|)\left|\frac{a}{|a|} - \frac{b}{|b|}\right|. \tag{2.71}$$

Proof. We start with the identity (see also [5, pp. 515 – 516])

$$\begin{aligned}
\left|\frac{a}{|a|} - \frac{b}{|b|}\right|^2 &= \left(\frac{a}{|a|} - \frac{b}{|b|}\right)\left(\frac{\bar{a}}{|a|} - \frac{\bar{b}}{|b|}\right) \\
&= 2 - 2\operatorname{Re}\left(\frac{a}{|a|} \cdot \frac{\bar{b}}{|b|}\right) \\
&= \frac{1}{|a||b|}\left(2|a||b| - 2\operatorname{Re}\left(a \cdot \bar{b}\right)\right) \\
&= \frac{1}{|a||b|}\left[2|a||b| - \left(|a|^2 + |b|^2 - |a-b|^2\right)\right] \\
&= \frac{1}{|a||b|}\left[|a-b|^2 - (|a|-|b|)^2\right].
\end{aligned}$$

Hence

$$\begin{aligned}
&|a-b|^2 - \left[\frac{1}{2}(|a|+|b|)\right]^2 \left|\frac{a}{|a|} - \frac{b}{|b|}\right|^2 \\
&\qquad = \frac{(|a|-|b|)^2}{4|a||b|}\left[(|a|+|b|)^2 - |a-b|^2\right]
\end{aligned}$$

and (2.71) is proved. ■

Using the above result, we may prove the following refinement of the (CBS) −inequality for complex numbers.

Theorem 93 *If* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *are two sequences of nonzero complex numbers, then*

$$\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 - \left| \sum_{k=1}^{n} a_k b_k \right|^2 \geq \frac{1}{8} \sum_{i,j=1}^{n} \left| \bar{a}_i b_j - \bar{a}_j b_i + \frac{|b_i|}{|a_i|} \bar{a}_i \cdot \frac{|a_j|}{|b_j|} b_j - \frac{|a_i|}{|b_i|} b_i \cdot \frac{|b_j|}{|a_j|} \bar{a}_j \right|^2 \geq 0 \quad (2.72)$$

Proof. The inequality (2.71) is clearly equivalent to

$$|a-b|^2 \geq \frac{1}{4} \left| a - b + \frac{|b|}{|a|} \cdot a - \frac{|a|}{|b|} \cdot b \right|^2 \quad (2.73)$$

for any $a, b \in \mathbb{C}$, $a, b \neq 0$.

We know the Lagrange's identity for sequences of complex numbers (see Chapter I, Section 1.2)

$$\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 - \left| \sum_{k=1}^{n} a_k b_k \right|^2 = \frac{1}{2} \sum_{i,j=1}^{n} |\bar{a}_i b_j - \bar{a}_j b_i|^2 . \quad (2.74)$$

By (2.73), we have

$$|\bar{a}_i b_j - \bar{a}_j b_i|^2 \geq \frac{1}{4} \left| \bar{a}_i b_j - \bar{a}_j b_i + \frac{|a_j| \, |b_i|}{|a_i| \, |b_j|} \bar{a}_i b_j - \frac{|a_i| \, |b_j|}{|a_j| \, |b_i|} \bar{a}_j b_i \right|^2 .$$

Summing over i, j from 1 to n and using the (CBS) −inequality for double sums, we deduce (2.72). ■

2.11 Some Refinements due to Alzer and Zheng

In 1992, H. Alzer [10] presented the following refinement of the Cauchy-Schwartz inequality written in the form

$$\left(\sum_{k=1}^{n} x_k y_k \right)^2 \leq \sum_{k=1}^{n} y_k \sum_{k=1}^{n} x_k^2 y_k . \quad (2.75)$$

Theorem 94 *Let* x_k *and* y_k $(k = 1, \ldots, n)$ *be real numbers satisfying* $0 = x_0 < x_1 \leq \frac{x_2}{2} \leq \cdots \leq \frac{x_n}{n}$ *and* $0 < y_n \leq y_{n-1} \leq \cdots \leq y_1$. *Then*

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \sum_{k=1}^{n} y_k \sum_{k=1}^{n} \left[x_k^2 - \frac{1}{4} x_{k-1} x_k\right] y_k, \tag{2.76}$$

with equality holding if and only if $x_k = kx_1$ $(k = 1, \ldots, n)$ *and* $y_1 = \cdots = y_n$.

In 1998, Liu Zheng [11] pointed out an error in the proof given in [10], which can be corrected as shown in [11]. Moreover, Liu Zheng established the following result which sharpens (2.76).

Theorem 95 *Let* x_k *and* y_k $(k = 1, \ldots, n)$ *be real numbers satisfying* $0 < x_1 \leq \frac{x_2}{2} \leq \cdots \leq \frac{x_n}{n}$ *and* $0 < y_n \leq y_{n-1} \leq \cdots \leq y_1$. *Then*

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \sum_{k=1}^{n} y_k \sum_{k=1}^{n} \delta_k y_k, \tag{2.77}$$

with

$$\delta_1 = x_1^2 \quad \text{and} \quad \delta_k = \frac{7k+1}{8k} x_k^2 - \frac{k}{8(k-1)} x_{k-1}^2 \quad (k \geq 2). \tag{2.78}$$

Equality holds in (2.77) if and only if $x_k = kx_1$ $(k = 1, \ldots, n)$ *and* $y_1 = \cdots = y_n$.

In 1999, H. Alzer improved the above results as follows.

To present his results, we will follow [12].

In order to prove the main result, we need some technical lemmas.

Lemma 96 *Let* x_k $(k = 1, \ldots, n)$ *be real numbers such that*

$$0 < x_1 \leq \frac{x_2}{2} \leq \cdots \leq \frac{x_n}{n}.$$

Then

$$2 \sum_{k=1}^{n} x_k \leq (n+1) x_n, \tag{2.79}$$

with equality holding if and only if $x_k = kx_1$ $(k = 1, \ldots, n)$.

A proof of Lemma 96 is given in [10].

Lemma 97 *Let* x_k $(k = 1, \ldots, n)$ *be real numbers such that*

$$0 < x_1 \leq \frac{x_2}{2} \leq \cdots \leq \frac{x_n}{n}.$$

Then

$$\left(\sum_{k=1}^{n} x_k\right)^2 \leq n \sum_{k=1}^{n} \frac{3k+1}{4k} x_k^2, \tag{2.80}$$

with equality holding if and only if $x_k = kx_1$ $(k = 1, \ldots, n)$.

Proof. Let

$$S_n = S_n(x_1, \ldots, x_n) = n \sum_{k=1}^{n} \frac{3k+1}{4k} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2.$$

Then we have for $n \geq 2$:

$$\begin{aligned} S_n - S_{n-1} &= \sum_{k=1}^{n-1} \frac{3k+1}{4k} x_k^2 - 2x_n \sum_{k=1}^{n-1} x_k + \frac{3(n-1)}{4} x_n^2 \\ &= f(x_n), \quad \text{say.} \end{aligned} \tag{2.81}$$

We differentiate with respect to x_n and use (2.79) and $x_n \geq \frac{n}{n-1} x_{n-1}$. This yields

$$f'(x_n) = \frac{3(n-1)}{2} x_n - 2 \sum_{k=1}^{n-1} x_k \geq \frac{n}{2} x_{n-1} > 0$$

and

$$\begin{aligned} f(x_n) &\geq f\left(\frac{n}{n-1} x_{n-1}\right) \\ &= \sum_{k=1}^{n} \frac{3k+1}{4k} x_k^2 - \frac{2n}{n-1} x_{n-1} \sum_{k=1}^{n-1} x_k + \frac{3n^2}{4(n-1)} x_{n-1}^2 \\ &= T_{n-1}(x_1, \ldots, x_{n-1}), \quad \text{say.} \end{aligned} \tag{2.82}$$

We use induction on n to establish that $T_{n-1}(x_1,\ldots,x_{n-1}) \geq 0$ for $n \geq 2$. We have $T_1(x_1) = 0$. Let $n \geq 3$; applying (2.79) we obtain

$$\begin{aligned} \frac{\partial}{\partial x_{n-1}} T_{n-1}(x_1,\ldots,x_{n-1}) &= \frac{3n+2}{2} x_{n-1} - \frac{2n}{n-1} \sum_{k=1}^{n-1} x_k \\ &\geq \frac{(n-2)(n+1)}{2(n-1)} x_{n-1} > 0 \end{aligned}$$

and

$$T_{n-1}(x_1,\ldots,x_{n-1}) \geq T_{n-1}\left(x_1,\ldots,x_{n-2},\frac{n-1}{n-2}x_{n-2}\right). \tag{2.83}$$

Using the induction hypothesis $T_{n-2}(x_1,\ldots,x_{n-2}) \geq 0$ and (2.79), we get

$$T_{n-1}\left(x_1,\ldots,x_{n-2},\frac{n-1}{n-2}x_{n-2}\right) \geq \frac{x_{n-2}}{n-2}\left[(n-1)x_{n-2} - 2\sum_{k=1}^{n-2} x_k\right]. \tag{2.84}$$

From (2.83) and (2.84) we conclude $T_{n-1}(x_1,\ldots,x_{n-1}) \geq 0$ for $n \geq 2$, so that (2.81) and (2.82) imply

$$S_n \geq S_{n-1} \geq \cdots \geq S_2 \geq S_1 = 0. \tag{2.85}$$

This proves inequality (2.80). We discuss the cases of equality. A simple calculation reveals that $S_n(x_1, 2x_1, \ldots, nx_1) = 0$. We use induction on n to prove the implication

$$S_n(x_1,\ldots,x_n) = 0 \Longrightarrow x_k = kx_1 \text{ for } k = 1,\ldots,n. \tag{2.86}$$

If $n = 1$, then (2.86) is obviously true. Next, we assume that (2.86) holds with $n-1$ instead of n. Let $n \geq 2$ and $S_n(x_1,\ldots,x_n) = 0$. Then (2.85) leads to $S_{n-1}(x_1,\ldots,x_{n-1}) = 0$ which implies $x_k = kx_1$ for $k = 1,\ldots,n-1$. Thus, we have $S_n(x_1, 2x_1, \ldots, (n-1)x_1, x_n) = 0$ which is equivalent to $(x_n - nx_1)(3x_n - nx_1) = 0$. Since $3x_n > nx_1$, we get $x_n = nx_1$. ■

Lemma 98 *Let x_k $(k = 1,\ldots,n)$ be real numbers such that*

$$0 < x_1 \leq \frac{x_2}{2} \leq \cdots \leq \frac{x_n}{n}.$$

If the natural numbers n and q satisfy $n \geq q+1$, then

$$0 < \left(\sum_{k=1}^{q} x_k\right)^2 - 2qx_n \sum_{k=1}^{q} x_k + \frac{(3n+1)q^2}{4n} x_n^2. \tag{2.87}$$

Proof. We denote the expression on the right-hand side of (2.87) by $u(x_n)$. Then we differentiate with respect to x_n and apply (2.79), $x_n \geq \left(\frac{n}{q}\right) x_q$ and $n \geq q+1$. This yields

$$\begin{aligned}\frac{1}{q}u'(x_n) &= \frac{(3n+1)\,q}{2n}x_n - 2q\sum_{k=1}^{q} x_k \\ &\geq \frac{(3n+1)\,q}{2n}x_n - (q+1)\,x_q \\ &\geq \frac{3n-2q-1}{2}x_q \\ &> 0.\end{aligned}$$

Hence, we get

$$u(x_n) \geq u\left(\frac{n}{q}x_q\right) = \frac{(3n+1)\,n}{4}x_q^2 - 2nx_q\sum_{k=1}^{q} x_k + \left(\sum_{k=1}^{q} x_k\right)^2. \tag{2.88}$$

Let

$$v(t) = \frac{(3t+1)\,t}{4}x_q - 2t\sum_{k=1}^{q} x_k \quad \text{and} \quad t \geq q+1;$$

from (2.79) we conclude that

$$v'(t) = \frac{6t+1}{4}x_q - 2\sum_{k=1}^{q} x_k \geq \frac{2q+3}{4}x_q > 0.$$

This implies that the expression on the right-hand side of (2.88) is increasing on $[q+1,\infty)$ with respect to n. Since $n \geq q+1$, we get from (2.88):

$$\begin{aligned}u(x_n) &\geq \frac{(3q+4)\,(q+1)}{4}x_q^2 - 2\,(q+1)\,x_q\sum_{k=1}^{q} x_k + \left(\sum_{k=1}^{q} x_k\right)^2 \\ &= P_q(x_1,\ldots,x_q)\,, \quad \text{say.}\end{aligned} \tag{2.89}$$

We use induction on q to show that $P_q(x_1,\ldots,x_q) > 0$ for $q \geq 1$. We have $P_1(x_1) = \frac{1}{2}x_1^2$. If $P_{q-1}(x_1,\ldots,x_{q-1}) > 0$, then we obtain for $q \geq 2$:

$$\begin{aligned}P_q(x_1,\ldots,x_q) &> 2q\,(x_{q-1} - x_q)\sum_{k=1}^{q-1} x_k - \frac{(3q+1)\,q}{4}x_{q-1}^2 \\ &\quad + \frac{q\,(3q-1)}{4}x_q^2 = w(x_q)\,, \quad \text{say.}\end{aligned} \tag{2.90}$$

We differentiate with respect to x_q and use (2.79) and $x_q \geq \left(\frac{q}{q-1}\right) x_{q-1}$. Then we get

$$\begin{aligned} w'(x_q) &= q\left[\frac{3q-1}{2}x_q - 2\sum_{k=1}^{q-1} x_k\right] \\ &\geq \frac{q^2(q+1)}{2(q-1)}x_{q-1} \\ &> 0 \end{aligned}$$

and

$$\begin{aligned} w(x_q) &\geq w\left(\frac{q}{q-1}x_{q-1}\right) \\ &= \frac{q}{q-1}x_{q-1}\left[\frac{4q^2-q-1}{4(q-1)}x_{q-1} - 2\sum_{k=1}^{q-1} x_k\right] \\ &\geq \frac{(3q-1)q}{4(q-1)}x_{q-1}^2 \\ &> 0. \end{aligned} \tag{2.91}$$

From (2.89), (2.90) and (2.91), we obtain $u(x_n) > 0$. ■

We are now in a position to prove the following companion of inequalities (2.76) and (2.77) (see [12]).

Theorem 99 *The inequality*

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \sum_{k=1}^{n} y_k \sum_{k=1}^{n}\left(\alpha + \frac{\beta}{k}\right) y_k, \tag{2.92}$$

holds for all natural numbers n *and for all real numbers* x_k *and* y_k $(k = 1, \ldots, n)$ *with*

$$0 < x_1 \leq \frac{x_2}{2} \leq \cdots \leq \frac{x_n}{n} \quad \text{and} \quad 0 < y_n \leq y_{n-1} \leq \cdots \leq y_1, \tag{2.93}$$

if and only if

$$\alpha \geq \frac{3}{4} \quad \text{and} \quad \beta \geq 1 - \alpha.$$

Proof. First, we assume that (2.92) is valid for all $n \geq 1$ and for all real numbers x_k and y_k $(k = 1, \ldots, n)$ which satisfy (2.93). We set $x_k = k$ and $y_k = 1$ $(k = 1, \ldots, n)$. Then (2.92) leads to

$$0 \leq \left(\alpha - \frac{3}{4}\right) 2n + \alpha + 3\beta - \frac{3}{2} \qquad (n \geq 1). \tag{2.94}$$

This implies $\alpha \geq \frac{3}{4}$. And, (2.94) with $n = 1$ yields $\alpha + \beta \geq 1$.

Now, we suppose that $\alpha \geq \frac{3}{4}$ and $\beta \geq 1 - \alpha$. Then we obtain for $k \geq 1$:

$$\alpha + \frac{\beta}{k} \geq \alpha + \frac{1-\alpha}{k} \geq \frac{3}{4} + \frac{1}{4k},$$

so that is suffices to show that (2.92) holds with $\alpha = \frac{3}{4}$ and $\beta = \frac{1}{4}$. Let

$$F(y_1, \ldots, y_n) = \sum_{k=1}^{n} y_k \sum_{k=1}^{n} \frac{3k+1}{4k} x_k^2 y_k - \left(\sum_{k=1}^{n} x_k y_k\right)^2$$

and

$$F_q(y) = F(y, \ldots, y, y_{q+1}, \ldots, y_n) \qquad (1 \leq q \leq n-1).$$

We shall prove that F_q is strictly increasing on $[y_{q+1}, \infty)$. Since $y_{q+1} \leq y_q$, we obtain

$$F_q(y_q) \geq F_q(y_{q+1}) = F_{q+1}(y_{q+1}) \quad (1 \leq q \leq n-1), \tag{2.95}$$

and Lemma 97 imply

$$\begin{aligned} F(y_1, \ldots, y_n) &= F_1(y_1) \geq F_1(y_2) = F_2(y_2) \geq F_2(y_3) \\ &\geq \cdots \geq F_{n-1}(y_{n-1}) \geq F_{n-1}(y) \\ &= y_n^2 \left[n \sum_{k=1}^{n} \frac{3k+1}{4k} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2 \right] \geq 0. \end{aligned}$$

If $F(y_1, \ldots, y_n) = 0$, then we conclude from the strict monotonicity of F_q and from Lemma 97 that $y_1 = \cdots = y_n$ and $x_k = kx_1$ $(k = 1, \ldots, n)$.

It remains to show that F_q is strictly increasing on $[y_{q+1}, \infty)$. Let $y \geq y_{q+1}$; we differentiate F_q and apply Lemma 97. This yields

$$F_q'(y) = 2y\left[q\sum_{k=1}^{q}\frac{3k+1}{4k}x_k^2 - \left(\sum_{k=1}^{q}x_k\right)^2\right] + q\sum_{k=q+1}^{n}\frac{3k+1}{4k}x_k^2 y_k + \sum_{k=q+1}^{n}y_k\sum_{k=1}^{q}\frac{3k+1}{4k}x_k^2 - 2\sum_{k=q+1}^{n}x_k y_k\sum_{k=1}^{q}x_k$$

and

$$\frac{1}{2}F_q''(y) = q\sum_{k=1}^{q}\frac{3k+1}{4k}x_k^2 - \left(\sum_{k=1}^{q}x_k\right)^2 \geq 0.$$

Hence, we have

$$\begin{aligned} F_q'(y) &\geq F_q'(y_{q+1}) \\ &= \left(2qy_{q+1} + \sum_{k=q+1}^{n}y_k\right)\left[\sum_{k=1}^{q}\frac{3k+1}{4k}x_k^2 - \frac{1}{q}\left(\sum_{k=1}^{q}x_k\right)^2\right] \\ &+ \frac{1}{q}\sum_{k=q+1}^{n}y_k\left\{\frac{(3k+1)q^2}{4k}x_k^2 - 2qx_k\sum_{i=1}^{q}x_i + \left(\sum_{i=1}^{q}x_i\right)^2\right\}. \end{aligned} \tag{2.96}$$

From Lemma 97 and Lemma 98 we obtain $F_q'(y_{q+1}) > 0$, so that (2.96) implies $F_q'(y) > 0$ for $y \geq y_{q+1}$. This completes the proof of the theorem. ■

Remark 100 *The proof of the theorem reveals that the sign of equality holds in (2.92) (with $\alpha = \frac{3}{4}$ and $\beta = \frac{1}{4}$) if and only if $x_k = kx_1$ $(k = 1, \ldots, n)$ and $y_1 = \cdots = y_n$.*

Remark 101 *If δ_k is defined by (2.78), then we have for $k \geq 2$:*

$$\delta_k - \left(\frac{3}{4} + \frac{1}{4k}\right)x_k^2 = \frac{k(k-1)}{8}\left[\left(\frac{x_k}{k}\right)^2 - \left(\frac{x_{k-1}}{k-1}\right)^2\right],$$

which implies that inequality (2.92) (with $\alpha = \frac{3}{4}$ and $\beta = \frac{1}{4}$) sharpens (2.77).

Remark 102 *It is shown in [10] that if a sequence (x_k) satisfies $x_0 = 0$ and $2x_k \leq x_{k-1} + x_{k+1}$ $(k \geq 1)$, then $\left(\frac{x_k}{k}\right)$ is increasing. Hence, inequality (2.92) is valid for all sequences (x_k) which are positive and convex.*

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Chapter 3

Functional Properties

3.1 A Monotonicity Property

The following result was obtained in [1, Theorem].

Theorem 103 *Let* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ *be sequences of real numbers and* $\bar{\mathbf{p}} = (p_1, \ldots, p_n)$, $\bar{\mathbf{q}} = (q_1, \ldots, q_n)$ *be sequences of nonnegative real numbers such that* $p_k \geq q_k$ *for any* $k \in \{1, \ldots, n\}$. *Then one has the inequality*

$$\left(\sum_{i=1}^{n} p_i a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} p_i b_i^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} p_i a_i b_i\right| \geq \left(\sum_{i=1}^{n} q_i a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} q_i b_i^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} q_i a_i b_i\right| \geq 0. \quad (3.1)$$

Proof. We shall follow the proof in [1].

Since $p_k - q_k \geq 0$, then the (CBS) −inequality for the weights $r_k = p_k - q_k$ ($k \in \{1, \ldots, n\}$) will produce

$$\left(\sum_{k=1}^{n} p_k a_k^2 - \sum_{k=1}^{n} q_k a_k^2\right)\left(\sum_{k=1}^{n} p_k b_k^2 - \sum_{k=1}^{n} q_k b_k^2\right) \geq \left[\sum_{k=1}^{n} (p_k - q_k) a_k b_k\right]^2. \quad (3.2)$$

Using the elementary inequality

$$(ac - bd)^2 \geq \left(a^2 - b^2\right)\left(c^2 - d^2\right), \quad a, b, c, d \in \mathbb{R}$$

for the choices

$$a = \left(\sum_{k=1}^{n} p_k a_k^2\right)^{\frac{1}{2}}, \quad b = \left(\sum_{k=1}^{n} q_k a_k^2\right)^{\frac{1}{2}}, \quad c = \left(\sum_{k=1}^{n} p_k b_k^2\right)^{\frac{1}{2}} \quad \text{and}$$
$$d = \left(\sum_{k=1}^{n} q_k b_k^2\right)^{\frac{1}{2}}$$

we deduce by (3.2), that

$$\left(\sum_{k=1}^{n} p_k a_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} p_k b_k^2\right)^{\frac{1}{2}} - \left(\sum_{k=1}^{n} q_k a_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} q_k b_k^2\right)^{\frac{1}{2}}$$
$$\geq \left|\sum_{k=1}^{n} p_k a_k b_k - \sum_{k=1}^{n} q_k a_k b_k\right| \geq \left|\sum_{k=1}^{n} p_k a_k b_k\right| - \left|\sum_{k=1}^{n} q_k a_k b_k\right|$$

proving the desired inequality (3.1). ■

The following corollary holds [1, Corollary 1].

Corollary 104 *Let* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *be as in Theorem 103. Denote*

$$S_n(\mathbf{1}) := \{\bar{\mathbf{x}} = (x_1, \dots, x_n) \,|\, 0 \leq x_i \leq 1,\ i \in \{1, \dots, n\}\}.$$

Then

$$\left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} a_i b_i\right|$$
$$= \sup_{\bar{\mathbf{x}} \in S_n(1)} \left[\left(\sum_{i=1}^{n} x_i a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} x_i b_i^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} x_i a_i b_i\right|\right] \geq 0. \quad (3.3)$$

Remark 105 *The following inequality is a natural particular case that may* *be obtained from (3.1) [1, p. 79]*

$$\left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} a_i b_i\right| \geq \left[\sum_{i=1}^{n} a_i^2 \operatorname{trig}^2(\alpha_i)\right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} b_i^2 \operatorname{trig}^2(\alpha_i)\right]^{\frac{1}{2}} - \left|\sum_{i=1}^{n} a_i b_i \operatorname{trig}^2(\alpha_i)\right| \geq 0, \quad (3.4)$$

where $\operatorname{trig}(x) = \sin x$ *or* $\cos x$, $x \in \mathbb{R}$ *and* $\bar{\boldsymbol{\alpha}} = (\alpha_1, \ldots, \alpha_n)$ *is a sequence of real numbers.*

3.2 A Superadditivity Property in Terms of Weights

Let $\mathcal{P}_f(\mathbb{N})$ be the family of finite parts of the set of natural numbers $\mathbb{N}$, $S(\mathbb{K})$ the linear space of real or complex numbers, i.e.,

$$S(\mathbb{K}) := \left\{x | x = (x_i)_{i\in\mathbb{N}}, \;\; x_i \in \mathbb{K}, \; i \in \mathbb{N}\right\}$$

and $S_+(\mathbb{R})$ the family of nonnegative real sequences. Define the mapping

$$S(p, I, x, y) := \left(\sum_{i\in I} p_i |x_i|^2 \sum_{i\in I} p_i |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i\in I} p_i x_i \bar{y}_i\right|, \qquad (3.5)$$

where $p \in S_+(\mathbb{R})$, $I \in \mathcal{P}_f(\mathbb{N})$ and $x, y \in S(\mathbb{K})$.

The following superadditivity property in terms of weights holds [2, p. 16].

Theorem 106 *For any* $p, q \in S_+(\mathbb{R})$, $I \in \mathcal{P}_f(\mathbb{N})$ *and* $x, y \in S(\mathbb{K})$ *we have*

$$S(p + q, I, x, y) \geq S(p, I, x, y) + S(q, I, x, y) \geq 0. \qquad (3.6)$$

Proof. Using the (CBS) −inequality for real numbers

$$\left(a^2 + b^2\right)^{\frac{1}{2}} \left(c^2 + d^2\right)^{\frac{1}{2}} \geq ac + bd; \;\; a, b, c, d \geq 0, \qquad (3.7)$$

we have

$$
\begin{aligned}
& S(p,I,x,y) \\
&= \left(\sum_{i\in I} p_i |x_i|^2 + \sum_{i\in I} q_i |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i\in I} p_i |y_i|^2 + \sum_{i\in I} q_i |y_i|^2\right)^{\frac{1}{2}} \\
&- \left|\sum_{i\in I} p_i x_i \bar{y}_i + \sum_{i\in I} q_i x_i \bar{y}_i\right| \\
&\geq \left(\sum_{i\in I} p_i |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i\in I} p_i |y_i|^2\right)^{\frac{1}{2}} + \left(\sum_{i\in I} q_i |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i\in I} q_i |y_i|^2\right)^{\frac{1}{2}} \\
&- \left|\sum_{i\in I} p_i x_i \bar{y}_i\right| - \left|\sum_{i\in I} q_i x_i \bar{y}_i\right| \\
&= S(p,I,x,y) + S(q,I,x,y),
\end{aligned}
$$

and the inequality (3.6) is proved. ■

The following corollary concerning the monotonicity of $S(\cdot,I,x,y)$ also holds [2, p. 16].

Corollary 107 *For any $p,q \in S_+(\mathbb{R})$ with $p \geq q$ and $I \in \mathcal{P}_f(\mathbb{N})$, $x,y \in S(\mathbb{K})$ one has the inequality:*

$$S(p,I,x,y) \geq S(q,I,x,y) \geq 0. \tag{3.8}$$

Proof. Using Theorem 106, we have

$$S(p,I,x,y) = S((p-q)+q,I,x,y) \geq S(p-q,I,x,y) + S(q,I,x,y)$$

giving

$$S(p,I,x,y) - S(q,I,x,y) \geq S(p-q,I,x,y) \geq 0$$

and the inequality (3.8) is proved. ■

Remark 108 *The following inequalities follow by the above results [2, p. 17].*

1. *Let* $\alpha_i \in \mathbb{R}$ $(i \in \{1, \ldots, n\})$ *and* $x_i, y_i \in \mathbb{K}$ $(i \in \{1, \ldots, n\})$. *Then one has the inequality:*

$$\left(\sum_{i=1}^{n} |x_i|^2 \sum_{i=1}^{n} |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} x_i \bar{y}_i\right|$$
$$\geq \left(\sum_{i=1}^{n} |x_i|^2 \sin^2 \alpha_i \sum_{i=1}^{n} |y_i|^2 \sin^2 \alpha_i\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} x_i \bar{y}_i \sin^2 \alpha_i\right|$$
$$+ \left(\sum_{i=1}^{n} |x_i|^2 \cos^2 \alpha_i \sum_{i=1}^{n} |y_i|^2 \cos^2 \alpha_i\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} x_i \bar{y}_i \cos^2 \alpha_i\right| \geq 0. \quad (3.9)$$

2. *Denote* $S_n(\mathbf{1}) := \{p \in S_+(\mathbb{R}) \,|p_i \leq 1 \text{ for all } i \in \{1, \ldots, n\}\}$. *Then for all* $x, y \in S(\mathbb{K})$ *one has the bound (see also Corollary 104):*

$$0 \leq \left(\sum_{i=1}^{n} |x_i|^2 \sum_{i=1}^{n} |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} x_i \bar{y}_i\right| \quad (3.10)$$
$$= \sup_{p \in S_n(\mathbf{1})} \left\{\left(\sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} p_i x_i \bar{y}_i\right|\right\}.$$

3.3 The Superadditivity as an Index Set Mapping

We assume that we are under the hypothesis and notations in Section 3.2. Reconsider the functional $S(\cdot,\cdot,\cdot,\cdot) : S_+(\mathbb{R}) \times \mathcal{P}_f(\mathbb{N}) \times S(\mathbb{K}) \times S(\mathbb{K}) \to \mathbb{R}$,

$$S(p, I, x, y) := \left(\sum_{i \in I} p_i |x_i|^2 \sum_{i \in I} p_i |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i \in I} p_i x_i \bar{y}_i\right|. \quad (3.11)$$

The following superadditivity property as an index set mapping holds [2].

Theorem 109 *For any* $I, J \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ *with* $I \cap J = \emptyset$, *one has the inequality*

$$S(p, I \cup J, x, y) \geq S(p, I, x, y) + S(p, J, x, y) \geq 0. \quad (3.12)$$

Proof. Using the elementary inequality for real numbers

$$\left(a^2+b^2\right)^{\frac{1}{2}}\left(c^2+d^2\right)^{\frac{1}{2}} \geq ac+bd; \quad a,b,c,d \geq 0, \tag{3.13}$$

we have

$$\begin{aligned}
& S\left(p, I \cup J, x, y\right) \\
&= \left(\sum_{i\in I} p_i\left|x_i\right|^2 + \sum_{j\in J} p_j\left|x_j\right|^2\right)^{\frac{1}{2}}\left(\sum_{i\in I} p_i\left|y_i\right|^2 + \sum_{j\in J} p_j\left|y_j\right|^2\right)^{\frac{1}{2}} \\
&- \left|\sum_{i\in I} p_i x_i \bar{y}_i + \sum_{j\in J} p_j x_j \bar{y}_j\right| \\
&\geq \left(\sum_{i\in I} p_i\left|x_i\right|^2\right)^{\frac{1}{2}}\left(\sum_{i\in I} p_i\left|y_i\right|^2\right)^{\frac{1}{2}} + \left(\sum_{j\in J} p_j\left|x_j\right|^2\right)^{\frac{1}{2}}\left(\sum_{j\in J} p_j\left|y_j\right|^2\right)^{\frac{1}{2}} \\
&- \left|\sum_{i\in I} p_i x_i \bar{y}_i\right| - \left|\sum_{j\in J} p_j x_j \bar{y}_j\right| \\
&= S\left(p, I, x, y\right) + S\left(p, J, x, y\right)
\end{aligned}$$

and the inequality (3.12) is proved. ■

The following corollary concerning the monotonicity of $S\left(p, \cdot, x, y\right)$ as an index set mapping also holds [2, p. 16].

Corollary 110 *For any $I, J \in \mathcal{P}_f\left(\mathbb{N}\right)$ with $I \supseteq J \neq \emptyset$, one has*

$$S\left(p, I, x, y\right) \geq S\left(p, J, x, y\right) \geq 0. \tag{3.14}$$

Proof. Using Theorem 109, we may write

$$S\left(p, I, x, y\right) = S\left(p, \left(I \backslash J\right) \cup J, x, y\right) \geq S\left(p, I \backslash J, x, y\right) + S\left(p, J, x, y\right)$$

giving

$$S\left(p, I, x, y\right) - S\left(p, J, x, y\right) \geq S\left(p, I \backslash J, x, y\right) \geq 0$$

which proves the desired inequality (3.14). ■

Remark 111 *The following inequalities follow by the above results [2, p. 17].*

1. *Let* $p_i \geq 0$ $(i \in \{1, \ldots, 2n\})$ *and* $x_i, y_i \in \mathbb{K}$ $(i \in \{1, \ldots, 2n\})$. *Then we have the inequality*

$$\begin{aligned}
&\left(\sum_{i=1}^{2n} p_i |x_i|^2 \sum_{i=1}^{2n} p_i |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{2n} p_i x_i \bar{y}_i\right| \quad (3.15)\\
&\geq \left(\sum_{i=1}^{n} p_{2i} |x_{2i}|^2 \sum_{i=1}^{n} p_{2i} |y_{2i}|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} p_{2i} x_i \bar{y}_i\right| \\
&+ \left(\sum_{i=1}^{n} p_{2i-1} |x_{2i-1}|^2 \sum_{i=1}^{n} p_{2i-1} |y_{2i-1}|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} p_{2i-1} x_{2i-1} \bar{y}_{2i-1}\right| \\
&\geq 0.
\end{aligned}$$

2. *We have the bound*

$$\begin{aligned}
&\left(\sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} p_i x_i \bar{y}_i\right| \\
&= \sup_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} \left[\left(\sum_{i \in I} p_i |x_i|^2 \sum_{i \in I} p_i |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i \in I} p_i x_i \bar{y}_i\right|\right] \geq 0. \quad (3.16)
\end{aligned}$$

3. *Define the sequence*

$$S_n := \left(\sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} p_i x_i \bar{y}_i\right| \geq 0 \quad (3.17)$$

where $p = (p_i)_{i \in \mathbb{N}} \in S_+(\mathbb{R})$, $x = (x_i)_{i \in \mathbb{N}}$, $y = (y_i)_{i \in \mathbb{N}} \in S(\mathbb{K})$. *Then* S_n *is* ***monotontic nondecreasing*** *and we have the following* ***lower bound***

$$\begin{aligned}
S_n &\geq \max_{1 \leq i,j \leq n} \Big\{ \left(p_i |x_i|^2 + p_j |x_j|^2\right)^{\frac{1}{2}} \left(p_i |y_i|^2 + p_j |y_j|^2\right)^{\frac{1}{2}} \quad (3.18)\\
&\qquad - |p_i x_i \bar{y}_i + p_j x_j \bar{y}_j| \Big\} \\
&\geq 0.
\end{aligned}$$

3.4 Strong Superadditivity in Terms of Weights

With the notations in Section 3.2, define the mapping

$$\bar{S}(p,I,x,y) := \sum_{i\in I} p_i |x_i|^2 \sum_{i\in I} p_i |y_i|^2 - \left| \sum_{i\in I} p_i x_i \bar{y}_i \right|^2, \tag{3.19}$$

where $p \in S_+(\mathbb{R})$, $I \in \mathcal{P}_f(\mathbb{N})$ and $x, y \in S(\mathbb{K})$.

Denote also by $\|\cdot\|_{\ell,H}$ the weighted Euclidean norm

$$\|x\|_{\ell,H} := \left(\sum_{i\in H} \ell_i |x_i|^2 \right)^{\frac{1}{2}}, \quad \ell \in S_+(\mathbb{R}),\ H \in \mathcal{P}_f(\mathbb{N}). \tag{3.20}$$

The following strong superadditivity property in terms of weights holds [2, p. 18].

Theorem 112 *For any $p, q \in S_+(\mathbb{R})$, $I \in \mathcal{P}_f(\mathbb{N})$ and $x, y \in S(\mathbb{K})$ we have*

$$\bar{S}(p+q,I,x,y) - \bar{S}(p,I,x,y) - \bar{S}(q,I,x,y) \geq \left(\det \begin{bmatrix} \|x\|_{p,I} & \|y\|_{p,I} \\ \|x\|_{q,I} & \|y\|_{q,I} \end{bmatrix} \right)^2 \geq 0. \tag{3.21}$$

Proof. We have

$$\begin{aligned} &\bar{S}(p+q,I,x,y) \\ &= \left(\sum_{i\in I} p_i |x_i|^2 + \sum_{i\in I} q_i |x_i|^2 \right) \left(\sum_{i\in I} p_i |y_i|^2 + \sum_{i\in I} q_i |y_i|^2 \right) \\ &\quad - \left| \sum_{i\in I} p_i x_i \bar{y}_i + \sum_{i\in I} q_i x_i \bar{y}_i \right|^2 \\ &\geq \sum_{i\in I} p_i |x_i|^2 \sum_{i\in I} p_i |y_i|^2 + \sum_{i\in I} q_i |x_i|^2 \sum_{i\in I} q_i |y_i|^2 \\ &\quad - \left(\left| \sum_{i\in I} p_i x_i \bar{y}_i \right| + \left| \sum_{i\in I} q_i x_i \bar{y}_i \right| \right)^2 \end{aligned} \tag{3.22}$$

$$= \bar{S}(p,I,x,y) + \bar{S}(q,I,x,y) + \sum_{i\in I} p_i |x_i|^2 \sum_{i\in I} q_i |y_i|^2 + \sum_{i\in I} q_i |x_i|^2 \sum_{i\in I} p_i |y_i|^2 - 2\left|\sum_{i\in I} p_i x_i \bar{y}_i\right| \left|\sum_{i\in I} q_i x_i \bar{y}_i\right|.$$

By (CBS) −inequality, we have

$$\left|\sum_{i\in I} p_i x_i \bar{y}_i\right| \left|\sum_{i\in I} q_i x_i \bar{y}_i\right| \leq \left[\sum_{i\in I} p_i |x_i|^2 \sum_{i\in I} p_i |y_i|^2 \sum_{i\in I} q_i |x_i|^2 \sum_{i\in I} q_i |y_i|^2\right]^{\frac{1}{2}}$$

and thus

$$\sum_{i\in I} p_i |x_i|^2 \sum_{i\in I} q_i |y_i|^2 + \sum_{i\in I} q_i |x_i|^2 \sum_{i\in I} p_i |y_i|^2 - 2\left|\sum_{i\in I} p_i x_i \bar{y}_i\right| \left|\sum_{i\in I} q_i x_i \bar{y}_i\right| \geq \left[\left(\sum_{i\in I} p_i |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i\in I} q_i |y_i|^2\right)^{\frac{1}{2}} - \left(\sum_{i\in I} q_i |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i\in I} p_i |y_i|^2\right)^{\frac{1}{2}}\right]^2. \quad (3.23)$$

Utilising (3.22) and (3.23) we deduce the desired inequality (3.21). ■

The following corollary concerning a strong monotonicity result also holds [2, p. 18].

Corollary 113 *For any $p,q \in S_+(\mathbb{R})$ with $p \geq q$ one has the inequality:*

$$\bar{S}(p,I,x,y) - \bar{S}(q,I,x,y) \geq \left(\det\begin{bmatrix} \|x\|_{q,I} & \|y\|_{q,I} \\ \|x\|_{p-q,I} & \|y\|_{p-q,I} \end{bmatrix}\right)^2 \geq 0. \quad (3.24)$$

Remark 114 *The following refinement of the $(CBS)-$inequality is a natural consequence of (3.21) [2, p. 19]*

$$\begin{aligned}
&\sum_{i\in I}|x_i|^2\sum_{i\in I}|y_i|^2-\left|\sum_{i\in I}x_i\bar{y}_i\right|^2 \\
&\qquad\geq \sum_{i\in I}|x_i|^2\sin^2\alpha_i\sum_{i\in I}|y_i|^2\sin^2\alpha_i-\left|\sum_{i\in I}x_i\bar{y}_i\sin^2\alpha_i\right|^2 \\
&\qquad+\sum_{i\in I}|x_i|^2\cos^2\alpha_i\sum_{i\in I}|y_i|^2\cos^2\alpha_i-\left|\sum_{i\in I}x_i\bar{y}_i\cos^2\alpha_i\right|^2 \\
&\qquad+\left(\det\begin{bmatrix}\left(\sum_{i\in I}|x_i|^2\sin^2\alpha_i\right)^{\frac{1}{2}} & \left(\sum_{i\in I}|y_i|^2\sin^2\alpha_i\right)^{\frac{1}{2}} \\ \left(\sum_{i\in I}|x_i|^2\cos^2\alpha_i\right)^{\frac{1}{2}} & \left(\sum_{i\in I}|y_i|^2\cos^2\alpha_i\right)^{\frac{1}{2}}\end{bmatrix}\right)^2 \\
&\qquad\geq 0. \qquad (3.25)
\end{aligned}$$

where $\alpha_i \in \mathbb{R}$, $i \in I$.

3.5 Strong Superadditivity as an Index Set Mapping

We assume that we are under the hypothesis and notations in Section 3.2. Reconsider the functional $\bar{S}(\cdot,\cdot,\cdot,\cdot): S_+(\mathbb{R})\times\mathcal{P}_f(\mathbb{N})\times S(\mathbb{K})\times S(\mathbb{K})\to\mathbb{R}$,

$$\bar{S}(p,I,x,y):=\sum_{i\in I}p_i|x_i|^2\sum_{i\in I}p_i|y_i|^2-\left|\sum_{i\in I}p_ix_i\bar{y}_i\right|^2. \qquad (3.26)$$

The following strong supperadditivity property as an index set mapping holds [2, p. 18].

Theorem 115 *For any $p \in S_+(\mathbb{R})$, $I, J \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ with $I \cap J = \emptyset$ and $x, y \in S(\mathbb{K})$, we have*

$$\bar{S}(p, I \cup J, x, y) - \bar{S}(p, I, x, y) - \bar{S}(p, J, x, y) \geq \left(\det \begin{bmatrix} \|x\|_{p,I} & \|y\|_{p,I} \\ \|x\|_{p,J} & \|y\|_{p,J} \end{bmatrix}\right)^2 \geq 0. \quad (3.27)$$

Proof. We have

$$\begin{aligned}
&\bar{S}(p, I \cup J, x, y) \quad (3.28)\\
&= \left(\sum_{i\in I} p_i |x_i|^2 + \sum_{j\in J} p_j |x_j|^2\right)\left(\sum_{i\in I} p_i |y_i|^2 + \sum_{j\in J} p_j |y_j|^2\right) \\
&\quad - \left|\sum_{i\in I} p_i x_i \bar{y}_i + \sum_{j\in I} p_j x_j \bar{y}_j\right|^2 \\
&\geq \sum_{i\in I} p_i |x_i|^2 \sum_{i\in I} p_i |y_i|^2 + \sum_{j\in J} p_j |x_j|^2 \sum_{j\in J} p_j |y_j|^2 \\
&\quad + \sum_{i\in I} p_i |x_i|^2 \sum_{j\in J} p_j |y_j|^2 + \sum_{i\in I} p_i |y_i|^2 \sum_{j\in J} p_j |x_j|^2 \\
&\quad - \left(\left|\sum_{i\in I} p_i x_i \bar{y}_i\right| + \left|\sum_{j\in I} p_j x_j \bar{y}_j\right|\right)^2 \\
&= \bar{S}(p, I, x, y) + \bar{S}(p, J, x, y) + \sum_{i\in I} p_i |x_i|^2 \sum_{j\in J} p_j |y_j|^2 \\
&\quad + \sum_{i\in I} p_i |y_i|^2 \sum_{j\in J} p_j |x_j|^2 - 2\left|\sum_{i\in I} p_i x_i \bar{y}_i\right|\left|\sum_{j\in I} p_j x_j \bar{y}_j\right|.
\end{aligned}$$

By the (CBS) −inequality, we have

$$\left|\sum_{i\in I} p_i x_i \bar{y}_i\right|\left|\sum_{j\in I} p_j x_j \bar{y}_j\right| \leq \left[\sum_{i\in I} p_i |x_i|^2 \sum_{i\in I} p_i |y_i|^2 \sum_{j\in J} p_j |x_j|^2 \sum_{j\in J} p_j |y_j|^2\right]^{\frac{1}{2}}$$

and thus

$$\sum_{i\in I} p_i |x_i|^2 \sum_{j\in J} p_j |y_j|^2 + \sum_{i\in I} p_i |y_i|^2 \sum_{j\in J} p_j |x_j|^2$$
$$- 2\left|\sum_{i\in I} p_i x_i \bar{y}_i\right| \left|\sum_{j\in I} p_j x_j \bar{y}_j\right| \geq \left[\left(\sum_{i\in I} p_i |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{j\in J} p_j |y_j|^2\right)^{\frac{1}{2}}\right.$$
$$\left. - \left(\sum_{i\in I} p_i |y_i|^2\right)^{\frac{1}{2}} \left(\sum_{j\in J} p_j |x_j|^2\right)^{\frac{1}{2}}\right]^2. \quad (3.29)$$

If we use now (3.28) and (3.29), we may deduce the desired inequality (3.27). ■

The following corollary concerning strong monotonicity also holds [2, p. 18].

Corollary 116 *For any $I, J \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ with $I \supseteq J$ one has the inequality*

$$\bar{S}(p, I, x, y) - \bar{S}(p, J, x, y) \geq \left(\det \begin{bmatrix} \|x\|_{p,J} & \|y\|_{p,J} \\ \|x\|_{p,I\setminus J} & \|y\|_{p,I\setminus J} \end{bmatrix}\right)^2 \geq 0. \quad (3.30)$$

Remark 117 *The following refinement of the $(CBS)-$inequality is a natural consequence of (3.27) [2, p. 19].*

Suppose $p_i \geq 0$, $i \in \{1, \ldots, 2n\}$ and $x_i, y_i \in \mathbb{K}$, $i \in \{1, \ldots, 2n\}$. Then we have the inequality

$$\sum_{i=1}^{2n} p_i |x_i|^2 \sum_{i=1}^{2n} p_i |y_i|^2 - \left|\sum_{i=1}^{2n} p_i x_i \bar{y}_i\right|^2$$
$$\geq \sum_{i=1}^{n} p_{2i} |x_{2i}|^2 \sum_{i=1}^{n} p_{2i} |y_{2i}|^2 - \left|\sum_{i=1}^{n} p_{2i} x_{2i} \bar{y}_{2i}\right|^2$$
$$+ \sum_{i=1}^{n} p_{2i-1} |x_{2i-1}|^2 \sum_{i=1}^{n} p_{2i-1} |y_{2i-1}|^2 - \left|\sum_{i=1}^{n} p_{2i-1} x_{2i-1} \bar{y}_{2i-1}\right|^2$$

$$+\left(\det\begin{bmatrix}\left(\sum_{i=1}^{n}p_{2i}\left|x_{2i}\right|^{2}\right)^{\frac{1}{2}} & \left(\sum_{i=1}^{n}p_{2i}\left|y_{2i}\right|^{2}\right)^{\frac{1}{2}}\\ \left(\sum_{i=1}^{n}p_{2i-1}\left|x_{2i-1}\right|^{2}\right)^{\frac{1}{2}} & \left(\sum_{i=1}^{n}p_{2i-1}\left|y_{2i-1}\right|^{2}\right)^{\frac{1}{2}}\end{bmatrix}\right)^{2}\geq 0. \quad (3.31)$$

3.6 Another Superadditivity Property

Let $\mathcal{P}_f(\mathbb{N})$ be the family of finite parts of the set of natural numbers, $S(\mathbb{R})$ the linear space of real sequences and $S_+(\mathbb{R})$ the family of nonnegative real sequences.

Consider the mapping $C : S_+(\mathbb{R}) \times \mathcal{P}_f(\mathbb{N}) \times S(\mathbb{R}) \times S(\mathbb{R}) \to \mathbb{R}$

$$C(p, I, a, b) := \sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2 - \left(\sum_{i\in I} p_i a_i b_i\right)^2. \quad (3.32)$$

The following identity holds [3, p. 115].

Lemma 118 *For any $p, q \in S_+(\mathbb{R})$ one has*

$$\begin{aligned}&C(p+q, I, a, b)\\&\quad = C(p, I, a, b) + C(q, I, a, b) + \sum_{(i,j)\in I\times I} p_i q_j (a_i b_j - a_j b_i)^2. \quad (3.33)\end{aligned}$$

Proof. Using the well-known Lagrange's identity, we have

$$C(p, I, a, b) = \frac{1}{2} \sum_{(i,j)\in I\times I} p_i p_j (a_i b_j - a_j b_i)^2. \quad (3.34)$$

Thus

$$\begin{aligned}&C(p+q, I, a, b)\\&= \frac{1}{2} \sum_{(i,j)\in I\times I} (p_i + q_i)(p_j + q_j)(a_i b_j - a_j b_i)^2\end{aligned}$$

$$= \frac{1}{2} \sum_{(i,j)\in I\times I} p_i p_j (a_i b_j - a_j b_i)^2 + \frac{1}{2} \sum_{(i,j)\in I\times I} q_i q_j (a_i b_j - a_j b_i)^2$$
$$+ \frac{1}{2} \sum_{(i,j)\in I\times I} p_i q_j (a_i b_j - a_j b_i)^2 + \frac{1}{2} \sum_{(i,j)\in I\times I} p_j q_i (a_i b_j - a_j b_i)^2$$
$$= C(p, I, a, b) + C(q, I, a, b) + \sum_{(i,j)\in I\times I} p_i q_j (a_i b_j - a_j b_i)^2$$

since, by symmetry,

$$\sum_{(i,j)\in I\times I} p_i q_j (a_i b_j - a_j b_i)^2 = \sum_{(i,j)\in I\times I} p_j q_i (a_i b_j - a_j b_i)^2 .$$

■

Consider the following mapping:

$$D(p, I, a, b) := [C(p, I, a, b)]^{\frac{1}{2}} = \left[\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2 - \left(\sum_{i\in I} p_i a_i b_i \right)^2 \right]^{\frac{1}{2}} .$$

The following result has been obtained in [4, p. 88] as a particular case of a more general result holding in inner product spaces.

Theorem 119 *For any $p, q \in S_+(\mathbb{R})$, $I \in \mathcal{P}_f(\mathbb{N})$ and $a, b \in S(\mathbb{R})$, we have the superadditive property*

$$D(p + q, I, a, b) \geq D(p, I, a, b) + D(q, I, a, b) \geq 0. \tag{3.35}$$

Proof. We will give here an elementary proof following the one in [3, p. 116 – p. 117].

By Lemma 118, we obviously have

$$D^2(p + q, I, a, b)$$
$$= D^2(p, I, a, b) + D^2(q, I, a, b) + \sum_{(i,j)\in I\times I} p_i q_j (a_i b_j - a_j b_i)^2 . \tag{3.36}$$

We claim that

$$\sum_{(i,j)\in I\times I} p_i q_j (a_i b_j - a_j b_i)^2 \geq 2D(p, I, a, b) D(q, I, a, b) . \tag{3.37}$$

Taking the square in both sides of (3.37), we must prove that

$$\left[\sum_{i\in I} p_i a_i^2 \sum_{i\in I} q_i b_i^2 + \sum_{i\in I} q_i a_i^2 \sum_{i\in I} p_i b_i^2 \; -2 \sum_{i\in I} p_i a_i b_i \sum_{i\in I} q_i a_i b_i\right]^2 \geq 4\left[\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2 - \left(\sum_{i\in I} p_i a_i b_i\right)^2\right] \times \left[\sum_{i\in I} q_i a_i^2 \sum_{i\in I} q_i b_i^2 - \left(\sum_{i\in I} q_i a_i b_i\right)^2\right]. \tag{3.38}$$

Let us denote

$$a := \left(\sum_{i\in I} p_i a_i^2\right)^{\frac{1}{2}}, \quad x := \left(\sum_{i\in I} q_i a_i^2\right)^{\frac{1}{2}}, \quad b := \left(\sum_{i\in I} p_i b_i^2\right)^{\frac{1}{2}},$$
$$y := \left(\sum_{i\in I} q_i b_i^2\right)^{\frac{1}{2}}, \quad c := \sum_{i\in I} p_i a_i b_i, \quad z := \sum_{i\in I} q_i a_i b_i.$$

With these notations (3.38) may be written in the following form

$$\left(a^2 y^2 + b^2 x^2 - 2cz\right)^2 \geq 4\left(a^2 b^2 - c^2\right)\left(x^2 y^2 - z^2\right). \tag{3.39}$$

Using the elementary inequality

$$\left(m^2 - n^2\right)\left(p^2 - q^2\right) \leq (mp - nq)^2, \quad m, n, p, q \in \mathbb{R}$$

we may state that

$$4(abxy - cz)^2 \geq 4\left(a^2 b^2 - c^2\right)\left(x^2 y^2 - z^2\right) \geq 0. \tag{3.40}$$

Since, by the (CBS) −inequality, we observe that $abxy \geq |cz| \geq |cz|$, we can state that

$$a^2 y^2 + b^2 x^2 - 2cz \geq 2(abxy - cz) \geq 0$$

giving

$$\left(a^2 y^2 + b^2 x^2 - 2cz\right)^2 \geq 4(abxy - cz)^2. \tag{3.41}$$

Utilizing (3.40) and (3.41) we deduce the inequality (3.39), and (3.37) is proved.

Finally, by (3.36) and (3.37) we have

$$D^2(p+q,I,a,b) \geq [D(p,I,a,b)+D(q,I,a,b)]^2,$$

i.e., the superadditivity property (3.35). ■

Remark 120 *The following refinement of the $(CBS)-$ inequality holds [4, p. 89]*

$$\begin{aligned}
&\left[\sum_{i\in I} a_i^2 \sum_{i\in I} b_i^2 - \left(\sum_{i\in I} a_i b_i\right)^2\right]^{\frac{1}{2}} \\
&\qquad \geq \left[\sum_{i\in I} a_i^2 \sin^2\alpha_i \sum_{i\in I} b_i^2 \sin^2\alpha_i - \left(\sum_{i\in I} a_i b_i \sin^2\alpha_i\right)^2\right]^{\frac{1}{2}} \\
&\qquad + \left[\sum_{i\in I} a_i^2 \cos^2\alpha_i \sum_{i\in I} b_i^2 \cos^2\alpha_i - \left(\sum_{i\in I} a_i b_i \cos^2\alpha_i\right)^2\right]^{\frac{1}{2}} \\
&\qquad\qquad \geq 0 \quad (3.42)
\end{aligned}$$

for any $\alpha_i \in \mathbb{R}$, $i \in \{1,\dots,n\}$.

3.7 The Case of Index Set Mapping

Assume that we are under the hypothesis and notations in Section 3.6. Reconsider the functional $C : S_+(\mathbb{R}) \times \mathcal{P}_f(\mathbb{N}) \times S(\mathbb{R}) \times S(\mathbb{R}) \to \mathbb{R}$ given by

$$C(p,I,a,b) := \sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2 - \left(\sum_{i\in I} p_i a_i b_i\right)^2. \qquad (3.43)$$

The following identity holds.

Lemma 121 *For any $I, J \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ with $I \cap J \neq \emptyset$ one has the identity:*

$$\begin{aligned}
&C(p, I\cup J, a, b) \\
&\qquad = C(p,I,a,b) + C(p,J,a,b) + \sum_{(i,j)\in I\times J} p_i p_j (a_i b_j - a_j b_i)^2. \quad (3.44)
\end{aligned}$$

Proof. Using Lagrange's identity [5, p. 84], we may state

$$C(p,K,a,b)=\frac{1}{2}\sum_{(i,j)\in K\times K}p_ip_j(a_ib_j-a_jb_i)^2,\quad K\in\mathcal{P}_f(\mathbb{N})\setminus\{\emptyset\}. \tag{3.45}$$

Thus

$$\begin{aligned}
&C(p,I\cup J,a,b)\\
=&\ \frac{1}{2}\sum_{(i,j)\in(I\cup J)\times(I\cup J)}p_ip_j(a_ib_j-a_jb_i)^2\\
=&\ \frac{1}{2}\sum_{(i,j)\in I\times I}p_ip_j(a_ib_j-a_jb_i)^2+\frac{1}{2}\sum_{(i,j)\in I\times J}p_ip_j(a_ib_j-a_jb_i)^2\\
&+\frac{1}{2}\sum_{(i,j)\in J\times I}p_ip_j(a_ib_j-a_jb_i)^2+\frac{1}{2}\sum_{(i,j)\in J\times J}p_ip_j(a_ib_j-a_jb_i)^2\\
=&\ C(p,I,a,b)+C(p,J,a,b)+\sum_{(i,j)\in I\times J}p_ip_j(a_ib_j-a_jb_i)^2
\end{aligned}$$

since, by symmetry,

$$\sum_{(i,j)\in I\times J}p_ip_j(a_ib_j-a_jb_i)^2=\sum_{(i,j)\in J\times I}p_ip_j(a_ib_j-a_jb_i)^2.$$

■

Now, if we consider the mapping

$$D(p,I,a,b):=[C(p,I,a,b)]^{\frac{1}{2}}=\left[\sum_{i\in I}p_ia_i^2\sum_{i\in I}p_ib_i^2-\left(\sum_{i\in I}p_ia_ib_i\right)^2\right]^{\frac{1}{2}},$$

then the following superadditivity property as an index set mapping holds:

Theorem 122 *For any $I,J\in\mathcal{P}_f(\mathbb{N})\setminus\{\emptyset\}$ with $I\cap J\neq\emptyset$ one has*

$$D(p,I\cup J,a,b)\geq D(p,I,a,b)+D(p,J,a,b)\geq 0. \tag{3.46}$$

Proof. By Lemma 121, we have

$$\begin{aligned}
&D^2(p,I\cup J,a,b)\\
&\quad=D^2(p,I,a,b)+D^2(p,J,a,b)+\sum_{(i,j)\in I\times J}p_ip_j(a_ib_j-a_jb_i)^2
\end{aligned}\tag{3.47}$$

To prove (3.46) it is sufficient to show that

$$\sum_{(i,j)\in I\times J} p_i p_j \left(a_i b_j - a_j b_i\right)^2 \geq 2D\left(p, I, a, b\right) D\left(p, J, a, b\right). \tag{3.48}$$

Taking the square in (3.48), we must demonstrate that

$$\begin{aligned}\left[\sum_{i\in I} p_i a_i^2 \sum_{j\in J} p_j b_j^2 + \sum_{j\in J} p_j a_j^2 \sum_{i\in I} p_i b_i^2 - 2\sum_{i\in I} p_i a_i b_i \sum_{j\in J} p_j a_j b_j\right]^2 \\ \geq 4\left[\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2 - \left(\sum_{i\in I} p_i a_i b_i\right)^2\right] \\ \times \left[\sum_{j\in J} p_j a_j^2 \sum_{j\in J} p_j b_j^2 - \left(\sum_{j\in J} p_j a_j b_j\right)^2\right].\end{aligned}$$

If we denote

$$\begin{aligned} a &:= \left(\sum_{i\in I} p_i a_i^2\right)^{\frac{1}{2}}, \quad x := \left(\sum_{j\in J} p_j a_j^2\right)^{\frac{1}{2}}, \quad b := \left(\sum_{i\in I} p_i b_i^2\right)^{\frac{1}{2}}, \\ y &:= \left(\sum_{j\in J} p_j b_j^2\right)^{\frac{1}{2}}, \quad c := \sum_{i\in I} p_i a_i b_i, \quad z := \sum_{j\in J} p_j a_j b_j, \end{aligned}$$

then we need to prove

$$\left(a^2y^2 + b^2x^2 - 2cz\right)^2 \geq 4\left(a^2b^2 - c^2\right)\left(x^2y^2 - z^2\right), \tag{3.49}$$

which has been shown in Section 3.6.

This completes the proof. ■

Remark 123 *The following refinement of the* (CBS) *−inequality holds*

$$\left[\sum_{i=1}^{2n} p_i a_i^2 \sum_{i=1}^{2n} p_i b_i^2 - \left(\sum_{i=1}^{2n} p_i a_i b_i\right)^2\right]^{\frac{1}{2}}$$

$$\geq \left[\sum_{i=1}^{n} p_{2i} a_{2i}^2 \sum_{i=1}^{n} p_{2i} b_{2i}^2 - \left(\sum_{i=1}^{n} p_{2i} a_{2i} b_{2i} \right)^2 \right]^{\frac{1}{2}}$$
$$+ \left[\sum_{i=1}^{n} p_{2i-1} a_{2i-1}^2 \sum_{i=1}^{n} p_{2i-1} b_{2i-1}^2 - \left(\sum_{i=1}^{n} p_{2i-1} a_{2i-1} b_{2i-1} \right)^2 \right]^{\frac{1}{2}} \geq 0.$$

3.8 Supermultiplicity in Terms of Weights

Denote by $S_+(\mathbb{R})$ the set of nonnegative sequences. Assume that $A : S_+(\mathbb{R}) \to \mathbb{R}$ is *additive* on $S_+(\mathbb{R})$, i.e.,

$$A(p+q) = A(p) + A(q), \quad p, q \in S_+(\mathbb{R}) \tag{3.50}$$

and $L : S_+(\mathbb{R}) \to \mathbb{R}$ is *superadditive* on $S_+(\mathbb{R})$, i.e.,

$$L(p+q) \geq L(p) + L(q), \quad p, q \in S_+(\mathbb{R}). \tag{3.51}$$

Define the following associated functionals

$$F(p) := \frac{L(p)}{A(p)} \quad \text{and} \quad H(p) := [F(p)]^{A(p)}. \tag{3.52}$$

The following result holds [3, Theorem 2.1].

Lemma 124 *With the above assumptions, we have*

$$H(p+q) \geq H(p) H(q); \tag{3.53}$$

for any $p, q \in S_+(\mathbb{R})$, i.e., $H(\cdot)$ is ***supermultiplicative*** *on $S_+(\mathbb{R})$.*

Proof. We shall follow the proof in [3].

Using the well-known arithmetic mean-geometric mean inequality for real numbers

$$\frac{\alpha x + \beta y}{\alpha + \beta} \geq x^{\frac{\alpha}{\alpha+\beta}} y^{\frac{\beta}{\alpha+\beta}} \tag{3.54}$$

for any $x, y \geq 0$ and $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$, we have successively

$$\begin{aligned} F(p+q) &= \frac{L(p+q)}{A(p+q)} = \frac{L(p+q)}{A(p)+A(q)} \geq \frac{L(p)+L(q)}{A(p)+A(q)} \qquad (3.55) \\ &= \frac{A(p)\frac{L(p)}{A(p)} + A(q)\frac{L(q)}{A(q)}}{A(p)+A(q)} = \frac{A(p)F(p)+A(q)F(q)}{A(p)+A(q)} \\ &\geq [F(p)]^{\frac{A(p)}{A(p)+A(q)}} \cdot [F(q)]^{\frac{A(q)}{A(p)+A(q)}} \end{aligned}$$

for all $p, q \in S_+(\mathbb{R})$. However, $A(p) + A(q) = A(p+q)$, and thus (3.55) implies the desired inequality (3.53). ■

We are now able to point out the following inequality related to the $(CBS)-$ inequality.

The first result is incorporated in the following theorem [3, p. 115].

Theorem 125 *For any $p, q \in S_+(\mathbb{R})$, and $a, b \in S(\mathbb{R})$, one has the inequality*

$$\begin{aligned} &\left\{ \frac{1}{P_I + Q_I} \left[\sum_{i\in I} (p_i + q_i) a_i^2 \sum_{i\in I} (p_i + q_i) b_i^2 \right.\right. \\ &\qquad \left.\left. - \left(\sum_{i\in I} (p_i + q_i) a_i b_i \right)^2 \right] \right\}^{P_I + Q_I} \\ &\geq \left\{ \frac{1}{P_I} \left[\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2 - \left(\sum_{i\in I} p_i a_i b_i \right)^2 \right] \right\}^{P_I} \\ &\quad \times \left\{ \frac{1}{Q_I} \left[\sum_{i\in I} q_i a_i^2 \sum_{i\in I} q_i b_i^2 - \left(\sum_{i\in I} q_i a_i b_i \right)^2 \right] \right\}^{Q_I} \\ &> 0, \qquad (3.56) \end{aligned}$$

where $P_I := \sum_{i\in I} p_i > 0$, $Q_I := \sum_{i\in I} q_i > 0$.

Proof. Consider the functionals

$$A(p) := \sum_{i\in I} p_i = P_I;$$

$$C(p) := \sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2 - \left(\sum_{i\in I} p_i a_i b_i \right)^2.$$

Then $A(\cdot)$ is additive and $C(\cdot)$ is superadditive (see for example Lemma 118) on $S_+(\mathbb{R})$.

Applying Lemma 124 we deduce the desired inequality (3.56). ■

The following refinement of the (CBS) −inequality holds.

Corollary 126 *For any $a, b, \alpha \in S(\mathbb{R})$, one has the inequality*

$$\begin{aligned}
&\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \\
&\quad \geq \frac{1}{\left(\frac{1}{n}\sum_{i=1}^{n}\sin^2\alpha_i\right)^{\frac{1}{n}\sum_{i=1}^{n}\sin^2\alpha_i}\left(\frac{1}{n}\sum_{i=1}^{n}\cos{}^2\alpha_i\right)^{\frac{1}{n}\sum_{i=1}^{n}\cos{}^2\alpha_i}} \\
&\quad \times \left[\sum_{i=1}^{n} a_i^2 \sin^2\alpha_i \sum_{i=1}^{n} b_i^2 \sin^2\alpha_i - \left(\sum_{i=1}^{n} a_i b_i \sin^2\alpha_i\right)^2\right]^{\frac{1}{n}\sum_{i=1}^{n}\sin^2\alpha_i} \\
&\quad \times \left[\sum_{i=1}^{n} a_i^2 \cos^2\alpha_i \sum_{i=1}^{n} b_i^2 \cos^2\alpha_i - \left(\sum_{i=1}^{n} a_i b_i \cos^2\alpha_i\right)^2\right]^{\frac{1}{n}\sum_{i=1}^{n}\cos^2\alpha_i} \\
&\quad \geq 0. \qquad (3.57)
\end{aligned}$$

The following result holds [3, p. 116].

Theorem 127 *For any $p, q \in S_+(\mathbb{R})$, and $a, b \in S(\mathbb{R})$, one has the inequality*

$$\begin{aligned}
&\left\{\left[\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)\,a_i^2\right]^{\frac{1}{2}}\left[\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)\,b_i^2\right]^{\frac{1}{2}}\right. \\
&\qquad\qquad \left. - \left|\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)\,a_i b_i\right|\right\}^{P_I+Q_I} \\
&\quad \geq \left[\left(\frac{1}{P_I}\sum_{i\in I}p_i a_i^2\right)^{\frac{1}{2}}\left(\frac{1}{P_I}\sum_{i\in I}p_i b_i^2\right)^{\frac{1}{2}} - \left|\frac{1}{P_I}\sum_{i\in I}p_i a_i b_i\right|\right]^{P_I} \\
&\quad \times \left[\left(\frac{1}{Q_I}\sum_{i\in I}q_i a_i^2\right)^{\frac{1}{2}}\left(\frac{1}{Q_I}\sum_{i\in I}q_i b_i^2\right)^{\frac{1}{2}} - \left|\frac{1}{Q_I}\sum_{i\in I}q_i a_i b_i\right|\right]^{Q_I} \\
&\quad \geq 0. \qquad (3.58)
\end{aligned}$$

Proof. Follows by Lemma 124 on taking into account that the functional

$$B(p) := \left(\sum_{i\in I} p_i a_i^2\right)^{\frac{1}{2}} \left(\sum_{i\in I} p_i b_i^2\right)^{\frac{1}{2}} - \left|\sum_{i\in I} p_i a_i b_i\right|$$

is superadditive on $S_+(\mathbb{R})$ (see Section 3.2). ■

The following refinement of the (CBS) −inequality holds.

Corollary 128 *For any $a, b, \alpha \in S(\mathbb{R})$, one has the inequality*

$$\begin{aligned}
&\left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} a_i b_i\right| \\
&\quad \geq \frac{1}{\left(\frac{1}{n}\sum_{i=1}^{n} \sin^2 \alpha_i\right)^{\frac{1}{n}\sum_{i=1}^{n} \sin^2 \alpha_i}} \cdot \frac{1}{\left(\frac{1}{n}\sum_{i=1}^{n} \cos^2 \alpha_i\right)^{\frac{1}{n}\sum_{i=1}^{n} \cos^2 \alpha_i}} \\
&\quad \times \left[\left(\sum_{i=1}^{n} a_i^2 \sin^2 \alpha_i\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2 \sin^2 \alpha_i\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} a_i b_i \sin^2 \alpha_i\right|\right]^{\frac{1}{n}\sum_{i=1}^{n} \sin^2 \alpha_i} \\
&\quad \times \left[\left(\sum_{i=1}^{n} a_i^2 \cos^2 \alpha_i\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2 \cos^2 \alpha_i\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} a_i b_i \cos^2 \alpha_i\right|\right]^{\frac{1}{n}\sum_{i=1}^{n} \cos^2 \alpha_i} \\
&\quad \geq 0. \qquad (3.59)
\end{aligned}$$

Finally, we may also state [3, p. 117].

Theorem 129 *For any $p, q \in S_+(\mathbb{R})$, and $a, b \in S(\mathbb{R})$, one has the inequality*

$$\begin{aligned}
&\left[\frac{1}{P_I + Q_I}\sum_{i\in I} (p_i + q_i) a_i^2 \cdot \frac{1}{P_I + Q_I}\sum_{i\in I} (p_i + q_i) b_i^2 \right. \\
&\qquad \left. - \left(\frac{1}{P_I + Q_I}\sum_{i\in I} (p_i + q_i) a_i b_i\right)^2\right]^{\frac{P_I + Q_I}{2}} \\
&\quad \geq \left[\frac{1}{P_I}\sum_{i\in I} p_i a_i^2 \cdot \frac{1}{P_I}\sum_{i\in I} p_i b_i^2 - \left(\frac{1}{P_I}\sum_{i\in I} p_i a_i b_i\right)^2\right]^{\frac{P_I}{2}}
\end{aligned}$$

$$\times\left[\frac{1}{Q_I}\sum_{i\in I}q_ia_i^2\cdot\frac{1}{Q_I}\sum_{i\in I}q_ib_i^2-\left(\frac{1}{Q_I}\sum_{i\in I}q_ia_ib_i\right)^2\right]^{\frac{Q_I}{2}}. \quad (3.60)$$

Proof. Follows by Lemma 124 on taking into account that the functional

$$D(p):=\left[\sum_{i=1}^n p_ia_i^2\sum_{i=1}^n p_ib_i^2-\left(\sum_{i=1}^n p_ia_ib_i\right)^2\right]^{\frac{1}{2}}$$

is superadditive on $S_+(\mathbb{R})$ (see Section 3.6). ■

The following corollary also holds.

Corollary 130 *For any $a,b,\alpha\in S(\mathbb{R})$, one has the inequality*

$$\begin{aligned}&\left[\sum_{i=1}^n a_i^2\sum_{i=1}^n b_i^2-\left(\sum_{i=1}^n a_ib_i\right)^2\right]^{\frac{1}{2}}\\ &\geq\frac{1}{\left(\frac{1}{n}\sum_{i=1}^n\sin^2\alpha_i\right)^{\frac{1}{n}\sum_{i=1}^n\sin^2\alpha_i}}\cdot\frac{1}{\left(\frac{1}{n}\sum_{i=1}^n\cos^2\alpha_i\right)^{\frac{1}{n}\sum_{i=1}^n\cos^2\alpha_i}}\\ &\times\left[\sum_{i=1}^n a_i^2\sin^2\alpha_i\sum_{i=1}^n b_i^2\sin^2\alpha_i-\left(\sum_{i=1}^n a_ib_i\sin^2\alpha_i\right)^2\right]^{\frac{1}{2n}\sum_{i=1}^n\sin^2\alpha_i}\\ &\times\left[\sum_{i=1}^n a_i^2\cos^2\alpha_i\sum_{i=1}^n b_i^2\cos^2\alpha_i-\left(\sum_{i=1}^n a_ib_i\cos^2\alpha_i\right)^2\right]^{\frac{1}{2n}\sum_{i=1}^n\cos^2\alpha_i}\\ &\geq 0. \quad (3.61)\end{aligned}$$

3.9 Supermultiplicity as an Index Set Mapping

Denote by $\mathcal{P}_f(\mathbb{N})$ the set of all finite parts of the natural number set $\mathbb{N}$ and assume that $B:\mathcal{P}_f(\mathbb{N})\to\mathbb{R}$ is *set-additive* on $\mathcal{P}_f(\mathbb{N})$, i.e.,

$$B(I\cup J)=B(I)+B(J)\quad\text{for any}\quad I,J\in\mathcal{P}_f(\mathbb{N}),\quad I\cap J\neq\emptyset, \quad (3.62)$$

and $G:\mathcal{P}_f(\mathbb{N})\to\mathbb{R}$ is *set-superadditive* on $\mathcal{P}_f(\mathbb{N})$, i.e.,

$$G(I\cup J)\geq G(I)+G(J)\quad\text{for any}\quad I,J\in\mathcal{P}_f(\mathbb{N}),\quad I\cap J\neq\emptyset. \tag{3.63}$$

We may define the following associated functionals

$$M(I):=\frac{G(I)}{B(I)}\quad\text{and}\quad N(I):=[M(I)]^{A(I)}. \tag{3.64}$$

With these notations we may prove the following lemma that is interesting in itself as well.

Lemma 131 *Under the above assumptions one has*

$$N(I\cup J)\geq N(I)\,N(J) \tag{3.65}$$

for any $I,J\in\mathcal{P}_f(\mathbb{N})\setminus\{\emptyset\}$ *with* $I\cap J\neq\emptyset$, *i.e.,* $N(\cdot)$ *is set-supermultiplicative on* $\mathcal{P}_f(\mathbb{N})$.

Proof. Using the arithmetic mean – geometric mean inequality

$$\frac{\alpha x+\beta y}{\alpha+\beta}\geq x^{\frac{\alpha}{\alpha+\beta}}y^{\frac{\beta}{\alpha+\beta}} \tag{3.66}$$

for any $x,y\geq 0$ and $\alpha,\beta\geq 0$ with $\alpha+\beta>0$, we have successively for $I,J\in\mathcal{P}_f(\mathbb{N})\setminus\{\emptyset\}$ with $I\cap J\neq\emptyset$ that

$$\begin{aligned}M(I\cup J)&=\frac{G(I\cup J)}{B(I\cup J)}=\frac{G(I\cup J)}{B(I)+B(J)}\geq\frac{G(I)+G(J)}{B(I)+B(J)}\\&=\frac{B(I)\frac{G(I)}{B(I)}+B(J)\frac{G(J)}{B(J)}}{B(I)+B(J)}=\frac{B(I)M(I)+B(J)M(J)}{B(I)+B(J)}\\&\geq(M(I))^{\frac{B(I)}{B(I)+B(J)}}\cdot(M(J))^{\frac{B(J)}{B(I)+B(J)}}.\end{aligned} \tag{3.67}$$

Since $B(I)+B(J)=B(I\cup J)$, we deduce by (3.67) the desired inequality (3.65). ■

Now, we are able to point out some set-superadditivity properties for some functionals associates to the (CBS) −inequality.

The first result is embodied in the following theorem.

Theorem 132 *If $a, b \in S(\mathbb{R})$, $p \in S_+(\mathbb{R})$ and $I, J \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ so that $I \cap J \neq \emptyset$, then one has the inequality*

$$\left\{ \frac{1}{P_{I\cup J}} \left[\sum_{k\in I\cup J} p_k a_k^2 \sum_{k\in I\cup J} p_k b_k^2 - \left(\sum_{k\in I\cup J} p_k a_k b_k \right)^2 \right] \right\}^{P_{I\cup J}} \geq \left\{ \frac{1}{P_I} \left[\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2 - \left(\sum_{i\in I} p_i a_i b_i \right)^2 \right] \right\}^{P_I} \times \left\{ \frac{1}{P_J} \left[\sum_{j\in J} p_j a_j^2 \sum_{j\in J} p_j b_j^2 - \left(\sum_{j\in J} p_j a_j b_j \right)^2 \right] \right\}^{P_J}, \quad (3.68)$$

when $P_J := \sum_{j\in J} p_j$.

Proof. Consider the functionals

$$B(I) := \sum_{i\in I} p_i;$$

$$G(I) := \sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2 - \left(\sum_{i\in I} p_i a_i b_i \right)^2.$$

The functional $B(\cdot)$ is obviously *set-additive* and (see Section 3.7) the functional $G(\cdot)$ is *set-superadditive*. Applying Lemma 131 we then deduce the desired inequality (3.68). ■

The following corollary is a natural application.

Corollary 133 *If $a, b \in S(\mathbb{R})$ and $p \in S_+(\mathbb{R})$, then for any $n \geq 1$ one has the inequality*

$$\left\{ \frac{1}{P_{2n}} \left[\sum_{i=1}^{2n} p_i a_i^2 \sum_{i=1}^{2n} p_i b_i^2 - \left(\sum_{i=1}^{2n} p_i a_i b_i \right)^2 \right] \right\}^{P_{2n}} \geq \left\{ \frac{1}{\sum_{i=1}^{n} p_{2i}} \left[\sum_{i=1}^{n} p_{2i} a_{2i}^2 \sum_{i=1}^{n} p_{2i} b_{2i}^2 - \left(\sum_{i=1}^{n} p_{2i} a_{2i} b_{2i} \right)^2 \right] \right\}^{\sum_{i=1}^{n} p_{2i}}$$

$$\times\left\{\frac{1}{\sum_{i=1}^{n}p_{2i-1}}\left[\sum_{i=1}^{n}p_{2i-1}a_{2i-1}^{2}\sum_{i=1}^{n}p_{2i-1}b_{2i-1}^{2}\right.\right.$$
$$\left.\left.-\left(\sum_{i=1}^{n}p_{2i-1}a_{2i-1}b_{2i-1}\right)^{2}\right]\right\}^{\sum_{i=1}^{n}p_{2i-1}}. \quad (3.69)$$

The following result also holds.

Theorem 134 *If $a,b\in S(\mathbb{R})$, $p\in S_{+}(\mathbb{R})$ and $I,J\in\mathcal{P}_{f}(\mathbb{N})\setminus\{\emptyset\}$ so that $I\cap J\neq\emptyset$, then one has the inequality*

$$\left\{\left[\frac{1}{P_{I\cup J}}\sum_{k\in I\cup J}p_{k}a_{k}^{2}\right]^{\frac{1}{2}}\left[\frac{1}{P_{I\cup J}}\sum_{k\in I\cup J}p_{k}b_{k}^{2}\right]^{\frac{1}{2}}\right.$$
$$\left.-\left|\frac{1}{P_{I\cup J}}\sum_{k\in I\cup J}p_{k}a_{k}b_{k}\right|\right\}^{P_{I\cup J}}$$
$$\geq\left\{\left(\frac{1}{P_{I}}\sum_{i\in I}p_{i}a_{i}^{2}\right)^{\frac{1}{2}}\left(\frac{1}{P_{I}}\sum_{i\in I}p_{i}b_{i}^{2}\right)^{\frac{1}{2}}-\left|\frac{1}{P_{I}}\sum_{i\in I}p_{i}a_{i}b_{i}\right|\right\}^{P_{I}}$$
$$\times\left\{\left(\frac{1}{P_{J}}\sum_{j\in J}p_{j}a_{j}^{2}\right)^{\frac{1}{2}}\left(\frac{1}{P_{J}}\sum_{j\in J}p_{j}b_{j}^{2}\right)^{\frac{1}{2}}-\left|\frac{1}{P_{J}}\sum_{j\in J}p_{j}a_{j}b_{j}\right|\right\}^{P_{J}}. \quad (3.70)$$

Proof. Follows by Lemma 131 on taking into account that the functional

$$G(I):=\left(\sum_{i\in I}p_{i}a_{i}^{2}\right)^{\frac{1}{2}}\left(\sum_{i\in I}p_{i}b_{i}^{2}\right)^{\frac{1}{2}}-\left|\sum_{i\in I}p_{i}a_{i}b_{i}\right|$$

is *set-superadditive* on $\mathcal{P}_{f}(\mathbb{N})$. ■

The following corollary is a natural application.

Corollary 135 *If $a,b\in S(\mathbb{R})$ and $p\in S_{+}(\mathbb{R})$, then for any $n\geq 1$ one has* *the inequality*

$$\left[\left(\frac{1}{P_{2n}}\sum_{i=1}^{2n}p_ia_i^2\right)^{\frac{1}{2}}\left(\frac{1}{P_{2n}}\sum_{i=1}^{2n}p_ib_i^2\right)^{\frac{1}{2}}-\left|\frac{1}{P_{2n}}\sum_{i=1}^{2n}p_ia_ib_i\right|\right]^{P_{2n}}$$
$$\geq\left[\left(\frac{1}{\sum_{i=1}^n p_{2i}}\sum_{i=1}^{n}p_{2i}a_{2i}^2\right)^{\frac{1}{2}}\left(\frac{1}{\sum_{i=1}^n p_{2i}}\sum_{i=1}^{n}p_{2i}b_{2i}^2\right)^{\frac{1}{2}}\right.$$
$$\left.-\left|\frac{1}{\sum_{i=1}^n p_{2i}}\sum_{i=1}^{n}p_{2i}a_{2i}b_{2i}\right|\right]^{\sum_{i=1}^n p_{2i}}$$
$$\times\left[\left(\frac{1}{\sum_{i=1}^n p_{2i-1}}\sum_{i=1}^{n}p_{2i-1}a_{2i-1}^2\right)^{\frac{1}{2}}\left(\frac{1}{\sum_{i=1}^n p_{2i-1}}\sum_{i=1}^{n}p_{2i-1}b_{2i-1}^2\right)^{\frac{1}{2}}\right.$$
$$\left.-\left|\frac{1}{\sum_{i=1}^n p_{2i-1}}\sum_{i=1}^{n}p_{2i-1}a_{2i-1}b_{2i-1}\right|\right]^{\sum_{i=1}^n p_{2i-1}}. \quad (3.71)$$

Finally, we may also state:

Theorem 136 *If $a,b\in S(\mathbb{R})$, $p\in S_+(\mathbb{R})$ and $I,J\in\mathcal{P}_f(\mathbb{N})\setminus\{\emptyset\}$ so that $I\cap J\neq\emptyset$, then one has the inequality*

$$\left[\frac{1}{P_{I\cup J}}\sum_{k\in I\cup J}p_ka_k^2\cdot\frac{1}{P_{I\cup J}}\sum_{k\in I\cup J}p_kb_k^2-\left(\frac{1}{P_{I\cup J}}\sum_{k\in I\cup J}p_ka_kb_k\right)^2\right]^{\frac{P_{I\cup J}}{2}}$$
$$\geq\left[\frac{1}{P_I}\sum_{i\in I}p_ia_i^2\cdot\frac{1}{P_I}\sum_{i\in I}p_ib_i^2-\left(\frac{1}{P_I}\sum_{i\in I}p_ia_ib_i\right)^2\right]^{\frac{P_I}{2}}$$
$$\times\left[\frac{1}{P_J}\sum_{j\in J}p_ja_j^2\cdot\frac{1}{P_J}\sum_{j\in J}p_jb_j^2-\left(\frac{1}{P_J}\sum_{j\in J}p_ja_jb_j\right)^2\right]^{\frac{P_J}{2}}. \quad (3.72)$$

Proof. Follows by Lemma 131 on taking into account that the functional

$$Q(I):=\left[\sum_{i\in I}p_ia_i^2\sum_{i\in I}p_ib_i^2-\left(\sum_{i\in I}p_ia_ib_i\right)^2\right]^{\frac{1}{2}}$$

is *set-superadditive* on $\mathcal{P}_f(\mathbb{N})$ (see Section 3.7). ■

The following corollary holds as well.

Corollary 137 *If $a, b \in S(\mathbb{R})$ and $p \in S_+(\mathbb{R})$, then for any $n \geq 1$ one has the inequality*

$$\left[\frac{1}{P_{2n}}\sum_{i=1}^{2n} p_i a_i^2 \cdot \frac{1}{P_{2n}}\sum_{i=1}^{2n} p_i b_i^2 - \left(\frac{1}{P_{2n}}\sum_{i=1}^{2n} p_i a_i b_i\right)^2\right]^{\frac{P_{2n}}{2}}$$
$$\geq \left[\left(\frac{1}{\sum_{i=1}^n p_{2i}}\sum_{i=1}^n p_{2i} a_{2i}^2\right)\left(\frac{1}{\sum_{i=1}^n p_{2i}}\sum_{i=1}^n p_{2i} b_{2i}^2\right) - \left(\frac{1}{\sum_{i=1}^n p_{2i}}\sum_{i=1}^n p_{2i} a_{2i} b_{2i}\right)^2\right]^{\frac{1}{2}\sum_{i=1}^n p_{2i}}$$
$$\times \left[\left(\frac{1}{\sum_{i=1}^n p_{2i-1}}\sum_{i=1}^n p_{2i-1} a_{2i-1}^2\right)\left(\frac{1}{\sum_{i=1}^n p_{2i-1}}\sum_{i=1}^n p_{2i-1} b_{2i-1}^2\right) - \left(\frac{1}{\sum_{i=1}^n p_{2i-1}}\sum_{i=1}^n p_{2i-1} a_{2i-1} b_{2i-1}\right)^2\right]^{\frac{1}{2}\sum_{i=1}^n p_{2i-1}}. \quad (3.73)$$

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Chapter 4

Counterpart Inequalities

4.1 The Cassels' Inequality

The following result was proved by J.W.S. Cassels in 1951 (see Appendix 1 of [2] or Appendix of [3]):

Theorem 138 *Let* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ *be sequences of positive real numbers and* $\bar{\mathbf{w}} = (w_1, \ldots, w_n)$ *a sequence of nonnegative real numbers. Suppose that*

$$m = \min_{i=\overline{1,n}} \left\{ \frac{a_i}{b_i} \right\} \quad \text{and} \quad M = \max_{i=\overline{1,n}} \left\{ \frac{a_i}{b_i} \right\}. \tag{4.1}$$

Then one has the inequality

$$\frac{\sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2}{\left(\sum_{i=1}^{n} w_i a_i b_i\right)^2} \leq \frac{(m+M)^2}{4mM}. \tag{4.2}$$

The equality holds in (4.2) when $w_1 = \frac{1}{a_1 b_1}$, $w_n = \frac{1}{a_n b_n}$, $w_2 = \cdots = w_{n-1} = 0$, $m = \frac{a_n}{b_1}$ *and* $M = \frac{a_1}{b_n}$.

Proof. 1. The original proof by Cassels (1951) is of interest. We shall follow the paper [5] in sketching this proof.

We begin with the assertion that

$$\frac{(1+kw)\left(1+k^{-1}w\right)}{(1+w)^2} \leq \frac{(1+k)\left(1+k^{-1}\right)}{4}, \quad k > 0, \; w \geq 0 \tag{4.3}$$

which, being an equivalent form of (4.2) for $n = 2$, shows that it holds for $n = 2$.

To prove that the maximum of (4.2) is obtained when we have more than two w_i's being nonzero, Cassels then notes that if for example, $w_1, w_2, w_3 \neq 0$ lead to an extremum M of $\frac{XY}{2^2}$, then we would have the linear equations

$$a_n^2 X + b_n^2 Y - 2Ma_n b_n Z = 0, \quad k = 1, 2, 3.$$

Nontrivial solutions exist if and only if the three vectors $[a_n^2, b_n^2, a_n b_n]$ are linearly dependent. But this will be so only if, for some $i \neq j \quad (i, j = 1, 2, 3)$ $a_i = \gamma a_j$, $b_i = \gamma b_j$. And if that were true, we could, for example, drop the a_i, b_i terms and so deal with the same problem with one less variable. If only one $w_i \neq 0$, then $M = 1$, the lower bound. So we need only examine all pairs $w_i \neq 0$, $w_j \neq 0$. The result (4.2) then quickly follows.

2. We will now use the *barycentric method* of Frucht [1] and Watson [4]. We will follow the paper [5].

We substitute $w_i = \frac{u_i}{b_i^2}$ in the left hand side of (4.2), which may then be expressed as the ratio

$$\frac{N}{D^2}$$

where

$$N = \sum_{i=1}^{n} \left(\frac{a_i}{b_i}\right)^2 u_i \text{ and } D = \sum_{i=1}^{n} \left(\frac{a_i}{b_i}\right) u_i,$$

assuming without loss of generality, that $\sum_{i=1}^{n} a_i = 1$. But the point with coordinates (D, N) must lie within the convex closure of the n points $\left(\frac{a_i}{b_i}, \frac{a_i^2}{b_i^2}\right)$. The value of $\frac{N}{D^2}$ at points on the parabola is one unit. If $m = \min\limits_{i=\overline{1,n}} \left\{\frac{a_i}{b_i}\right\}$ and $M = \max\limits_{i=\overline{1,n}} \left\{\frac{a_i}{b_i}\right\}$, then the minimum must lie on the chord joining the point (m, m^2) and (M, M^2). Some easy calculus then leads to (4.2). ■

The following "unweighted" Cassels' inequality holds.

Corollary 139 *If* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *satisfy the assumptions in Theorem 138, one has the inequality*

$$\frac{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}{\left(\sum_{i=1}^{n} a_i b_i\right)^2} \leq \frac{(m + M)^2}{4mM}. \tag{4.4}$$

The following two additive versions of Cassels inequality hold.

Corollary 140 *With the assumptions of Theorem 138, one has*

$$0 \le \left(\sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2\right)^{\frac{1}{2}} - \sum_{i=1}^{n} w_i a_i b_i \qquad (4.5)$$
$$\le \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \sum_{i=1}^{n} w_i a_i b_i.$$

and

$$0 \le \sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2 - \left(\sum_{i=1}^{n} w_i a_i b_i\right)^2 \qquad (4.6)$$
$$\le \frac{(M-m)^2}{4mM} \left(\sum_{i=1}^{n} w_i a_i b_i\right)^2.$$

Proof. Taking the square root in (4.2) we get

$$1 \le \frac{\left(\sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2\right)^{\frac{1}{2}}}{\sum_{i=1}^{n} w_i a_i b_i} \le \frac{M+m}{2\sqrt{mM}}.$$

Subtracting 1 on both sides, a simple calculation will lead to (4.5).

The second inequality follows by (4.2) on subtracting 1 and appropriate computation. ■

The following additive version of unweighted Cassels inequality also holds.

Corollary 141 *With the assumption of Theorem 138 for* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *one has the inequalities*

$$0 \le \left(\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}} - \sum_{i=1}^{n} a_i b_i \le \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \sum_{i=1}^{n} a_i b_i \qquad (4.7)$$

and

$$0 \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \frac{(M-m)^2}{4mM} \left(\sum_{i=1}^{n} a_i b_i\right)^2. \qquad (4.8)$$

4.2 The Pólya-Szegö Inequality

The following inequality was proved in 1925 by Pólya and Szegö [6, pp. 57, 213 – 214], [7, pp. 71– 72, 253 – 255].

Theorem 142 *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ *and* $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be two sequences of positive real numbers. If*

$$0 < a \le a_i \le A < \infty, \quad 0 < b \le b_i \le B < \infty \quad \text{for each } i \in \{1, \dots, n\}, \tag{4.9}$$

then one has the inequality

$$\frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2}{\left(\sum_{i=1}^n a_i b_i\right)^2} \le \frac{(ab + AB)^2}{4abAB}. \tag{4.10}$$

The equality holds in (4.10) if and only if

$$p = n \cdot \frac{A}{a} \bigg/ \left(\frac{A}{a} + \frac{B}{b}\right) \quad \text{and} \quad q = n \cdot \frac{B}{b} \bigg/ \left(\frac{A}{a} + \frac{B}{b}\right)$$

are integers and if p *of the numbers* $a_1, \dots, a_n$ *are equal to* a *and* q *of these numbers are equal to* A, *and if the corresponding numbers* b_i *are equal to* B *and* b *respectively.*

Proof. Following [5], we shall present here the original proof of Pólya and Szegö.

We may, without loss of generality, suppose that $a_1 \ge \cdots \ge a_n$, then to maximise the left-hand side of (4.10) we must have that the critical b_i's be reversely ordered (for if $b_k > b_m$ with $k < m$, then we can interchange b_k and b_m such that $b_k^2 + b_m^2 = b_m^2 + b_k^2$ and $a_k b_k + a_m b_m \ge a_k b_m + a_m b_k$), i.e., that $b_1 \le \cdots \le b_n$.

Pólya and Szegö then continue by defining nonnegative numbers u_i and v_i for $i = 1, \dots, n-1$ and $n > 2$ such that

$$a_i^2 = u_i a_1^2 + v_i a_n^2 \quad \text{and} \quad b_i^2 = u_i b_1^2 + v_i b_n^2. \tag{4.11}$$

Since $a_i b_i > u_i a_1 b_1 + v_i a_n b_n$ the left hand side of (4.10),

$$\frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2}{\left(\sum_{i=1}^n a_i b_i\right)^2} \le \frac{\left(U a_1^2 + V a_n^2\right)\left(U b_1^2 + V b_n^2\right)}{\left(U a_1 b_1 + V a_n b_n\right)^2},$$

where $U = \sum_{i=1}^{n} u_i$ and $V = \sum_{i=1}^{n} v_i$.

This reduces the problem to that with $n = 2$, which is solvable by elementary methods, leading to

$$\frac{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}{\left(\sum_{i=1}^{n} a_i b_i\right)^2} \leq \frac{\left(a_1 b_1 + a_n b_n\right)^2}{4 a_1 a_n b_1 b_n}, \tag{4.12}$$

where, since the a_i's and b_i's here are reversely ordered,

$$a_1 = \max_{i=\overline{1,n}} \{a_i\}, \quad a_n = \min_{i=\overline{1,n}} \{a_i\}, \quad b_1 = \min_{i=\overline{1,n}} \{b_i\}, \quad b_n = \max_{i=\overline{1,n}} \{b_i\}. \tag{4.13}$$

If we now assume, as in (4.9), that

$$0 < a \leq a_i \leq A, \;\; 0 < b \leq b_i \leq B, \quad i = (1, \ldots, n),$$

then

$$\frac{\left(a_1 b_1 + a_n b_n\right)^2}{4 a_1 a_n b_1 b_n} \leq \frac{(ab + AB)^2}{4abAB}$$

(because $\frac{(k+1)^2}{4k} \leq \frac{(\alpha+1)^2}{4\alpha}$ for $k \leq \alpha$), and the inequality (4.10) is proved. ■

Remark 143 *The inequality (4.10) may also be obtained from the "unweighted" Cassels' inequality*

$$\frac{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}{\left(\sum_{i=1}^{n} a_i b_i\right)^2} \leq \frac{(m + M)^2}{4mM}, \tag{4.14}$$

where $0 < m \leq \frac{a_i}{b_i} \leq M$ *for each* $i \in \{1, \ldots, n\}$.

The following additive versions of the Pólya-Szegö inequality also hold.

Corollary 144 *With the assumptions in Theorem 142, one has the inequality*

$$0 \leq \left(\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}} - \sum_{i=1}^{n} a_i b_i \leq \frac{\left(\sqrt{AB} - \sqrt{ab}\right)^2}{2\sqrt{abAB}} \sum_{i=1}^{n} a_i b_i \tag{4.15}$$

and

$$0 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \frac{(AB - ab)^2}{4abAB} \left(\sum_{i=1}^{n} a_i b_i\right)^2. \tag{4.16}$$

4.3 The Greub-Rheinboldt Inequality

The following weighted version of the Pólya-Szegö inequality was obtained by Greub and Rheinboldt in 1959, [6].

Theorem 145 *Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ and $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ be two sequences of positive real numbers and $\bar{\mathbf{w}} = (w_1, \dots, w_n)$ a sequence of nonnegative real numbers. Suppose that*

$$0 < a \le a_i \le A < \infty, \;\; 0 < b \le b_i \le B < \infty \quad (i = 1, \dots, n). \tag{4.17}$$

Then one has the inequality

$$\frac{\sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2}{\left(\sum_{i=1}^{n} w_i a_i b_i\right)^2} \le \frac{(ab + AB)^2}{4abAB}. \tag{4.18}$$

Equality holds in (4.18) when $w_i = \frac{1}{a_1 b_1}$, $w_n = \frac{1}{a_n b_n}$, $w_2 = \cdots = w_{n-1} = 0$, $m = \frac{a_n}{b_1}$, $M = \frac{a_1}{b_n}$ with $a_1 = A$, $a_n = a$, $b_1 = b$ and $b_n = b$.

Remark 146 *This inequality follows by Cassels' result which states that*

$$\frac{\sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2}{\left(\sum_{i=1}^{n} w_i a_i b_i\right)^2} \le \frac{(m + M)^2}{4mM}, \tag{4.19}$$

provided $0 < m \le \frac{a_i}{b_i} \le M < \infty$ for each $i \in \{1, \dots, n\}$.

The following additive versions of Greub-Rheinboldt also hold.

Corollary 147 *With the assumptions in Theorem 145, one has the inequalities*

$$\begin{aligned} 0 &\le \left(\sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2\right)^{\frac{1}{2}} - \sum_{i=1}^{n} w_i a_i b_i \\ &\le \frac{\left(\sqrt{AB} - \sqrt{ab}\right)^2}{2\sqrt{abAB}} \sum_{i=1}^{n} w_i a_i b_i \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} 0 &\le \sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2 - \left(\sum_{i=1}^{n} w_i a_i b_i\right)^2 \\ &\le \frac{(AB - ab)^2}{4abAB} \left(\sum_{i=1}^{n} w_i a_i b_i\right)^2. \end{aligned} \tag{4.21}$$

4.4 A Cassels' Type Inequality for Complex Numbers

The following counterpart inequality for the (CBS) −inequality holds [9].

Theorem 148 *Let $a, A \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ such that* $\operatorname{Re}(\bar{a}A) > 0$.

If $\bar{\mathbf{x}} = (x_1, \dots, x_n)$, $\bar{\mathbf{y}} = (y_1, \dots, y_n)$ are sequences of complex numbers and $\bar{\mathbf{w}} = (w_1, \dots, w_n)$ is a sequence of nonnegative real numbers with the property that

$$\sum_{i=1}^{n} w_i \operatorname{Re}\left[(Ay_i - x_i)(\bar{x}_i - \bar{a}\bar{y}_i)\right] \geq 0, \tag{4.22}$$

then one has the inequality

$$\begin{aligned} \left[\sum_{i=1}^{n} w_i |x_i|^2 \sum_{i=1}^{n} w_i |y_i|^2\right]^{\frac{1}{2}} &\leq \frac{1}{2} \cdot \frac{\sum_{i=1}^{n} w_i \operatorname{Re}\left[A\bar{x}_i y_i + \bar{a} x_i \bar{y}_i\right]}{\left[\operatorname{Re}(\bar{a}A)\right]^{\frac{1}{2}}} \\ &\leq \frac{1}{2} \cdot \frac{|A| + |a|}{\left[\operatorname{Re}(\bar{a}A)\right]^{\frac{1}{2}}} \left|\sum_{i=1}^{n} w_i x_i \bar{y}_i\right|. \end{aligned} \tag{4.23}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller one.

Proof. We have, obviously, that

$$\begin{aligned} \Gamma &:= \sum_{i=1}^{n} w_i \operatorname{Re}\left[(Ay_i - x_i)(\bar{x}_i - \bar{a}\bar{y}_i)\right] \\ &= \sum_{i=1}^{n} w_i \operatorname{Re}\left[A\bar{x}_i y_i + \bar{a} x_i \bar{y}_i\right] - \sum_{i=1}^{n} w_i |x_i|^2 - \operatorname{Re}(\bar{a}A) \sum_{i=1}^{n} w_i |y_i|^2 \end{aligned}$$

and then, by (4.22), one has

$$\sum_{i=1}^{n} w_i |x_i|^2 + \operatorname{Re}(\bar{a}A) \sum_{i=1}^{n} w_i |y_i|^2 \leq \sum_{i=1}^{n} w_i \operatorname{Re}\left[A\bar{x}_i y_i + \bar{a} x_i \bar{y}_i\right]$$

giving

$$\frac{1}{[\mathrm{Re}(\bar{a}A)]^{\frac{1}{2}}}\sum_{i=1}^{n} w_i |x_i|^2 + [\mathrm{Re}(\bar{a}A)]^{\frac{1}{2}}\sum_{i=1}^{n} w_i |y_i|^2 \leq \frac{\sum_{i=1}^{n} w_i \,\mathrm{Re}\,[A\bar{x}_i y_i + \bar{a} x_i \bar{y}_i]}{[\mathrm{Re}(\bar{a}A)]^{\frac{1}{2}}}. \tag{4.24}$$

On the other hand, by the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha} q^2 \geq 2pq$$

holding for any $p, q \geq 0$ and $\alpha > 0$, we deduce

$$2\left(\sum_{i=1}^{n} w_i |x_i|^2 \sum_{i=1}^{n} w_i |y_i|^2\right)^{\frac{1}{2}} \leq \frac{1}{[\mathrm{Re}(\bar{a}A)]^{\frac{1}{2}}}\sum_{i=1}^{n} w_i |x_i|^2 + [\mathrm{Re}(\bar{a}A)]^{\frac{1}{2}}\sum_{i=1}^{n} w_i |y_i|^2. \tag{4.25}$$

Utilising (4.24) and (4.25), we deduce the first part of (4.23).

The second part is obvious by the fact that for $z \in \mathbb{C}$, $|\mathrm{Re}(z)| \leq |z|$.

Now, assume that the first inequality in (4.23) holds with a constant $c > 0$, i.e.,

$$\sum_{i=1}^{n} w_i |x_i|^2 \sum_{i=1}^{n} w_i |y_i|^2 \leq c \cdot \frac{\sum_{i=1}^{n} w_i \,\mathrm{Re}\,[A\bar{x}_i y_i + \bar{a} x_i \bar{y}_i]}{[\mathrm{Re}(\bar{a}A)]^{\frac{1}{2}}}, \tag{4.26}$$

where $a, A, \bar{\mathbf{x}}, \bar{\mathbf{y}}$ satisfy (4.22).

If we choose $a = A = 1$, $y = x \neq 0$, then obviously (4.23) holds and from (4.26) we may get

$$\sum_{i=1}^{n} w_i |x_i|^2 \leq 2c \sum_{i=1}^{n} w_i |x_i|^2,$$

giving $c \geq \frac{1}{2}$.

The theorem is completely proved. ■

The following corollary is a natural consequence of the above theorem.

Corollary 149 *Let $m, M > 0$ and $\mathbf{\bar{x}}, \mathbf{\bar{y}}$, $\mathbf{\bar{w}}$ be as in Theorem 148 and with the property that*

$$\sum_{i=1}^{n} w_i \operatorname{Re}\left[\left(My_i - x_i\right)\left(\bar{x}_i - m\bar{y}_i\right)\right] \geq 0, \tag{4.27}$$

then one has the inequality

$$\begin{aligned} \left[\sum_{i=1}^{n} w_i \left|x_i\right|^2 \sum_{i=1}^{n} w_i \left|y_i\right|^2\right]^{\frac{1}{2}} &\leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} \sum_{i=1}^{n} w_i \operatorname{Re}\left(x_i \bar{y}_i\right) \\ &\leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} \left|\sum_{i=1}^{n} w_i x_i \bar{y}_i\right|. \end{aligned} \tag{4.28}$$

The following corollary also holds.

Corollary 150 *With the assumptions in Corollary 149, then one has the following inequality:*

$$\begin{aligned} 0 &\leq \left[\sum_{i=1}^{n} w_i \left|x_i\right|^2 \sum_{i=1}^{n} w_i \left|y_i\right|^2\right]^{\frac{1}{2}} - \left|\sum_{i=1}^{n} w_i x_i \bar{y}_i\right| \\ &\leq \left[\sum_{i=1}^{n} w_i \left|x_i\right|^2 \sum_{i=1}^{n} w_i \left|y_i\right|^2\right]^{\frac{1}{2}} - \sum_{i=1}^{n} w_i \operatorname{Re}\left(x_i \bar{y}_i\right) \\ &\leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \sum_{i=1}^{n} w_i \operatorname{Re}\left(x_i \bar{y}_i\right) \\ &\leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \left|\sum_{i=1}^{n} w_i x_i \bar{y}_i\right| \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} w_i \left|x_i\right|^2 \sum_{i=1}^{n} w_i \left|y_i\right|^2 - \left|\sum_{i=1}^{n} w_i x_i \bar{y}_i\right|^2 \\ &\leq \sum_{i=1}^{n} w_i \left|x_i\right|^2 \sum_{i=1}^{n} w_i \left|y_i\right|^2 - \left[\sum_{i=1}^{n} w_i \operatorname{Re}\left(x_i \bar{y}_i\right)\right]^2 \end{aligned} \tag{4.30}$$

$$\leq \frac{(M-m)^2}{4mM}\left[\sum_{i=1}^{n} w_i \operatorname{Re}(x_i \bar{y}_i)\right]^2$$
$$\leq \frac{(M-m)^2}{4mM}\left|\sum_{i=1}^{n} w_i x_i \bar{y}_i\right|^2 .$$

4.5 A Counterpart Inequality for Real Numbers

The following result holds [10, Proposition 5.1].

Theorem 151 *Let $a, A \in \mathbb{R}$ and $\bar{\mathbf{x}} = (x_1, \ldots, x_n)$, $\bar{\mathbf{y}} = (y_1, \ldots, y_n)$ be two sequences with the property that:*

$$a y_i \leq x_i \leq A y_i \quad \text{for each} \quad i \in \{1, \ldots, n\} . \tag{4.31}$$

Then for any $\bar{\mathbf{w}} = (w_1, \ldots, w_n)$ a sequence of positive real numbers, one has the inequality

$$0 \leq \sum_{i=1}^{n} w_i x_i^2 \sum_{i=1}^{n} w_i y_i^2 - \left(\sum_{i=1}^{n} w_i x_i y_i\right)^2 \tag{4.32}$$
$$\leq \frac{1}{4}(A-a)^2 \left(\sum_{i=1}^{n} w_i y_i^2\right)^2 .$$

The constant $\frac{1}{4}$ is sharp in (4.32).

Proof. Let us define

$$I_1 := \left(A \sum_{i=1}^{n} w_i y_i^2 - \sum_{i=1}^{n} w_i x_i y_i\right)\left(\sum_{i=1}^{n} w_i x_i y_i - a \sum_{i=1}^{n} w_i y_i^2\right)$$

and

$$I_2 := \left(\sum_{i=1}^{n} w_i y_i^2\right) \sum_{i=1}^{n} (A y_i - x_i)(x_i - a y_i) w_i .$$

Then

$$I_1 = (a+A) \sum_{i=1}^{n} w_i y_i^2 \sum_{i=1}^{n} w_i x_i y_i - \left(\sum_{i=1}^{n} w_i x_i y_i\right)^2 - aA \left(\sum_{i=1}^{n} w_i y_i^2\right)^2$$

and

$$I_2 = (a + A) \sum_{i=1}^{n} w_i y_i^2 \sum_{i=1}^{n} w_i x_i y_i - \sum_{i=1}^{n} w_i x_i^2 \sum_{i=1}^{n} w_i y_i^2 - aA \left(\sum_{i=1}^{n} w_i y_i^2 \right)^2$$

giving

$$I_1 - I_2 = \sum_{i=1}^{n} w_i x_i^2 \sum_{i=1}^{n} w_i y_i^2 - \left(\sum_{i=1}^{n} w_i y_i^2 \right)^2 . \tag{4.33}$$

If (4.31) holds, then $(Ay_i - x_i)(x_i - ay_i) \geq 0$ for each $i \in \{1, \dots, n\}$ and thus $I_2 \geq 0$ giving

$$\sum_{i=1}^{n} w_i x_i^2 \sum_{i=1}^{n} w_i y_i^2 - \left(\sum_{i=1}^{n} w_i y_i^2 \right)^2 \leq \left[\left(A \sum_{i=1}^{n} w_i y_i^2 - \sum_{i=1}^{n} w_i x_i y_i \right) \left(\sum_{i=1}^{n} w_i x_i y_i - a \sum_{i=1}^{n} w_i y_i^2 \right) \right] . \tag{4.34}$$

If we use the elementary inequality for real numbers $u, v \in \mathbb{R}$

$$uv \leq \frac{1}{4} (u + v)^2 , \tag{4.35}$$

then we have for

$$u := A \sum_{i=1}^{n} w_i y_i^2 - \sum_{i=1}^{n} w_i x_i y_i, \quad v := \sum_{i=1}^{n} w_i x_i y_i - a \sum_{i=1}^{n} w_i y_i^2$$

that

$$\left(A \sum_{i=1}^{n} w_i y_i^2 - \sum_{i=1}^{n} w_i x_i y_i \right) \left(\sum_{i=1}^{n} w_i x_i y_i - a \sum_{i=1}^{n} w_i y_i^2 \right) \leq \frac{1}{4} (A - a)^2 \left(\sum_{i=1}^{n} w_i y_i^2 \right)^2$$

and the inequality (4.32) is proved.

Now, assume that (4.32) holds with a constant $c > 0$, i.e.,

$$\sum_{i=1}^{n} w_i x_i^2 \sum_{i=1}^{n} w_i y_i^2 - \left(\sum_{i=1}^{n} w_i x_i y_i \right)^2 \leq c (A - a)^2 \left(\sum_{i=1}^{n} w_i y_i^2 \right)^2 , \tag{4.36}$$

where $a, A, \bar{\mathbf{x}}, \bar{\mathbf{y}}$ satisfy (4.31).

We choose $n = 2$, $w_1 = w_2 = 1$ and let $a, A, y_1, y_2, x, \alpha \in \mathbb{R}$ such that

$$\begin{aligned} ay_1 &< x_1 = Ay_1, \\ ay_2 &= x_2 < Ay_2. \end{aligned}$$

With these choices, we get from (4.36) that

$$\left(a^2y_1^2 + a^2y_2^2\right)\left(y_1^2 + y_2^2\right) - \left(A^2y_1^2 + a^2y_2^2\right)^2 \le c\,(A-a)^2\left(y_1^2 + y_2^2\right)^2,$$

which is equivalent to

$$(A-a)^2\,y_1^2y_2^2 \le c\,(A-a)^2\left(y_1^2 + y_2^2\right)^2.$$

Since we may choose $a \neq A$, we deduce

$$y_1^2y_2^2 \le c\left(y_1^2 + y_2^2\right)^2,$$

giving, for $y_1 = y_2 = 1$, $c \ge \frac{1}{4}$. ■

The following corollary is obvious.

Corollary 152 *With the above assumptions for a, A, $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$, we have the inequality*

$$0 \le \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} x_i y_i\right)^2 \le \frac{1}{4}\,(A-a)^2\left(\sum_{i=1}^{n} y_i^2\right)^2. \tag{4.37}$$

Remark 153 *Condition (4.31) may be replaced by the weaker condition*

$$\sum_{i=1}^{n} w_i\,(Ay_i - x_i)\,(x_i - ay_i) \ge 0 \tag{4.38}$$

and the conclusion in Theorem 151 will still be valid, i.e., the inequality (4.32) holds.

For (4.37) to be true it suffices that

$$\sum_{i=1}^{n} (Ay_i - x_i)\,(x_i - ay_i) \ge 0 \tag{4.39}$$

holds true.

4.6 A Counterpart Inequality for Complex Numbers

The following result holds [10, Proposition 5.1].

Theorem 154 *Let* $a, A \in \mathbb{C}$ *and* $\bar{\mathbf{x}} = (x_1, \dots, x_n)$, $\bar{\mathbf{y}} = (y_1, \dots, y_n) \in \mathbb{C}^n$, $\bar{\mathbf{w}} = (w_1, \dots, w_n) \in \mathbb{R}_+^n$. *If*

$$\sum_{i=1}^{n} w_i \operatorname{Re}\left[(Ay_i - x_i)(\bar{x}_i - \bar{a}\bar{y}_i)\right] \geq 0, \tag{4.40}$$

then one has the inequality

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} w_i |x_i|^2 \sum_{i=1}^{n} w_i |y_i|^2 - \left|\sum_{i=1}^{n} w_i x_i \bar{y}_i\right|^2 \\ &\leq \frac{1}{4} |A - a|^2 \left(\sum_{i=1}^{n} w_i |y_i|^2\right)^2 . \end{aligned} \tag{4.41}$$

The constant $\frac{1}{4}$ *is sharp in (4.41).*

Proof. Consider

$$A_1 := \operatorname{Re}\left[\left(A \sum_{i=1}^{n} w_i |y_i|^2 - \sum_{i=1}^{n} w_i x_i \bar{y}_i\right)\left(\sum_{i=1}^{n} w_i \bar{x}_i y_i - \bar{a} \sum_{i=1}^{n} w_i |y_i|^2\right)\right]$$

and

$$A_2 := \sum_{i=1}^{n} w_i |y_i|^2 - \operatorname{Re}\left[\sum_{i=1}^{n} w_i (Ay_i - x_i)(\bar{x}_i - \bar{a}\bar{y}_i)\right].$$

Then

$$\begin{aligned} A_1 &= \sum_{i=1}^{n} w_i |y_i|^2 - \operatorname{Re}\left[A \sum_{i=1}^{n} w_i \bar{x}_i y_i + \bar{a} \sum_{i=1}^{n} w_i x_i \bar{y}_i\right] \\ &\quad - \left|\sum_{i=1}^{n} w_i x_i \bar{y}_i\right|^2 - \operatorname{Re}(\bar{a}A)\left(\sum_{i=1}^{n} w_i |y_i|^2\right)^2 \end{aligned}$$

and

$$\begin{aligned} A_2 &= \sum_{i=1}^{n} w_i |y_i|^2 - \operatorname{Re}\left[A \sum_{i=1}^{n} w_i \bar{x}_i y_i + \bar{a} \sum_{i=1}^{n} w_i x_i \bar{y}_i\right] \\ &\quad - \sum_{i=1}^{n} w_i |x_i|^2 \sum_{i=1}^{n} w_i |y_i|^2 - \operatorname{Re}(\bar{a}A)\left(\sum_{i=1}^{n} w_i |y_i|^2\right)^2 \end{aligned}$$

giving

$$A_1 - A_2 = \sum_{i=1}^{n} w_i |x_i|^2 \sum_{i=1}^{n} w_i |y_i|^2 - \left|\sum_{i=1}^{n} w_i x_i \bar{y}_i\right|^2 . \tag{4.42}$$

If (4.40) holds, then $A_2 \geq 0$ and thus

$$\begin{aligned} &\sum_{i=1}^{n} w_i |x_i|^2 \sum_{i=1}^{n} w_i |y_i|^2 - \left|\sum_{i=1}^{n} w_i x_i \bar{y}_i\right|^2 \\ &\qquad \leq \operatorname{Re}\left[\left(A \sum_{i=1}^{n} w_i |y_i|^2 - \sum_{i=1}^{n} w_i x_i \bar{y}_i\right) \right. \\ &\qquad\qquad \left. \times \left(\sum_{i=1}^{n} w_i \bar{x}_i y_i - \bar{a} \sum_{i=1}^{n} w_i |y_i|^2\right)\right]. \end{aligned} \tag{4.43}$$

If we use the elementary inequality for complex numbers $z, t \in \mathbb{C}$

$$\operatorname{Re}[z\bar{t}] \leq \frac{1}{4} |z - t|^2 , \tag{4.44}$$

then we have for $z := A \sum_{i=1}^{n} w_i |y_i|^2 - \sum_{i=1}^{n} w_i x_i \bar{y}_i$, $t := \sum_{i=1}^{n} w_i x_i \bar{y}_i - a \sum_{i=1}^{n} w_i |y_i|^2$ that

$$\begin{aligned} &\operatorname{Re}\left[\left(A \sum_{i=1}^{n} w_i |y_i|^2 - \sum_{i=1}^{n} w_i x_i \bar{y}_i\right) \left(\sum_{i=1}^{n} w_i \bar{x}_i y_i - \bar{a} \sum_{i=1}^{n} w_i |y_i|^2\right)\right] \\ &\qquad \leq \frac{1}{4} |A - a|^2 \left(\sum_{i=1}^{n} w_i |y_i|^2\right)^2 \end{aligned} \tag{4.45}$$

and the inequality (4.41) is proved.

Now, assume that (4.41) holds with a constant $c > 0$, i.e.,

$$\sum_{i=1}^{n} w_i |x_i|^2 \sum_{i=1}^{n} w_i |y_i|^2 - \left| \sum_{i=1}^{n} w_i x_i \bar{y}_i \right|^2 \leq c |A-a|^2 \left(\sum_{i=1}^{n} w_i |y_i|^2 \right)^2, \quad (4.46)$$

where $\bar{\mathbf{x}}, \bar{\mathbf{y}}, a, A$ satisfy (4.40).

Consider $\bar{\mathbf{y}} \in \mathbb{C}^n$, $\sum_{i=1}^{n} |y_i|^2 w_i = 1$, $a \neq A$, $\bar{\mathbf{m}} \in \mathbb{C}^n$, $\sum_{i=1}^{n} w_i |m_i|^2 = 1$ with $\sum_{i=1}^{n} w_i y_i m_i = 0$. Define

$$x_i := \frac{A+a}{2} y_i + \frac{A+a}{2} m_i, \quad i \in \{1, \dots, n\}.$$

Then

$$\sum_{i=1}^{n} w_i (A y_i - x_i)(\bar{x}_i - \bar{a} y_i) = \left| \frac{A-a}{2} \right|^2 \sum_{i=1}^{n} w_i (y_i - m_i)(\bar{y}_i - \bar{m}_i) = 0$$

and thus the condition (4.40) is fulfilled.

From (4.46) we deduce

$$\sum_{i=1}^{n} \left| \frac{A+a}{2} y_i + \frac{A-a}{2} m_i \right|^2 w_i - \left| \sum_{i=1}^{n} \left(\frac{A+a}{2} y_i + \frac{A-a}{2} m_i \right) \bar{y}_i w_i \right|^2 \leq c |A-a|^2$$

and since

$$\sum_{i=1}^{n} w_i \left| \left(\frac{A+a}{2} y_i + \frac{A-a}{2} m_i \right) \right|^2 = \left| \frac{A+a}{2} \right|^2 - \left| \frac{A-a}{2} \right|^2$$

and

$$\left| \sum_{i=1}^{n} \left(\frac{A+a}{2} y_i + \frac{A-a}{2} m_i \right) \bar{y}_i w_i \right|^2 = \left| \frac{A+a}{2} \right|^2$$

then by (4.46) we get

$$\frac{|A-a|^2}{4} \leq c |A-a|^2$$

giving $c \geq \frac{1}{4}$ and the theorem is completely proved. ■

The following corollary holds.

Corollary 155 *Let $a, A \in \mathbb{C}$ and $\bar{\mathbf{x}} = (x_1, \dots, x_n)$, $\bar{\mathbf{y}} = (y_1, \dots, y_n) \in \mathbb{C}^n$ be with the property that*

$$\sum_{i=1}^{n} \operatorname{Re}\left[(Ay_i - x_i)(\bar{x}_i - \bar{a}\bar{y}_i)\right] \geq 0, \tag{4.47}$$

then one has the inequality

$$0 \leq \sum_{i=1}^{n} |x_i|^2 \sum_{i=1}^{n} |y_i|^2 - \left|\sum_{i=1}^{n} x_i \bar{y}_i\right|^2 \leq \frac{1}{4} |A - a|^2 \left(\sum_{i=1}^{n} |y_i|^2\right)^2. \tag{4.48}$$

The constant $\frac{1}{4}$ is best in (4.48).

Remark 156 *A sufficient condition for both (4.40) and (4.47) to hold is*

$$\operatorname{Re}\left[(Ay_i - x_i)(\bar{x}_i - \bar{a}\bar{y}_i)\right] \geq 0 \tag{4.49}$$

for any $i \in \{1, \dots, n\}$.

4.7 Shisha-Mond Type Inequalities

As some particular case for bounds on differences of means, O. Shisha and B. Mond obtained in 1967 (see [23]) the following counterpart of $(CBS)-$inequality:

Theorem 157 *Assume that $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ and $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ are such that there exists $a, A, b, B > 0$ with the property that:*

$$a \leq a_j \leq A \quad \text{and} \quad b \leq b_j \leq B \quad \text{for any } j \in \{1, \dots, n\} \tag{4.50}$$

then we have the inequality

$$\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 - \left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sqrt{\frac{A}{b}} - \sqrt{\frac{a}{B}}\right)^2 \sum_{j=1}^{n} a_j b_j \sum_{j=1}^{n} b_j^2. \tag{4.51}$$

The equality holds in (4.51) if and only if there exists a subsequence $(k_1, \dots, k_p)$ of $(1, 2, \dots, n)$ such that

$$\frac{n}{p} = 1 + \left(\frac{A}{a}\right)^{\frac{1}{2}} \left(\frac{B}{b}\right)^{\frac{3}{2}},$$

$a_{k_\mu} = A$, $b_{k_\mu} = b$ $(\mu = 1, \dots, p)$ and $a_k = a$, $b_k = B$ for every k distinct from all k_μ.

Using another result stated for weighted means in [23], we may prove the following counterpart of the (CBS) −inequality.

Theorem 158 *Assume that* $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ *are positive sequences and there exists* $\gamma, \Gamma > 0$ *with the property that*

$$0 < \gamma \leq \frac{a_i}{b_i} \leq \Gamma < \infty \quad \text{for any} \quad i \in \{1, \ldots, n\}. \tag{4.52}$$

Then we have the inequality

$$0 \leq \left(\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \right)^{\frac{1}{2}} - \sum_{i=1}^{n} a_i b_i \leq \frac{(\Gamma - \gamma)^2}{4(\gamma + \Gamma)} \sum_{j=1}^{n} b_j^2. \tag{4.53}$$

The equality holds in (4.53) if and only if there exists a subsequence $(k_1, \ldots, k_p)$ *of* $(1, 2, \ldots, n)$ *such that*

$$\sum_{m=1}^{p} b_{k_m}^2 = \frac{\Gamma + 3\gamma}{4(\gamma + \Gamma)} \sum_{j=1}^{n} b_j^2, \quad \frac{a_{k_m}}{b_{k_m}} = \Gamma \quad (m = 1, \ldots, p) \quad \text{and} \quad \frac{a_k}{b_k} = \gamma$$

for every k *distinct from all* k_m.

Proof. In [23, p. 301], Shisha and Mond have proved the following weighted inequality

$$0 \leq \left(\sum_{j=1}^{n} q_j x_j^2 \right)^{\frac{1}{2}} - \sum_{j=1}^{n} q_j x_j \leq \frac{(C - c)^2}{4(c + C)}, \tag{4.54}$$

provided $q_j \geq 0$ $(j = 1, \ldots, n)$ with $\sum_{j=1}^{n} q_j = 1$ and $0 < c \leq x_j < C < \infty$ for any $j \in \{1, \ldots, n\}$.

Equality holds in (4.54) if and only if there exists a subsequence $(k_1, \ldots, k_p)$ of $(1, 2, \ldots, n)$ such that

$$\sum_{m=1}^{p} q_{k_m} = \frac{C + 3c}{4(c + C)}, \tag{4.55}$$

$x_{k_m} = C$ $(m = 1, 2, \ldots, p)$ and $x_k = c$ for every k distinct from all k_m.

If in (4.54) we choose

$$x_j = \frac{a_j}{b_j}, \ \ q_j = \frac{b_j^2}{\sum_{k=1}^n b_k^2}, \ \ j \in \{1, \dots, n\};$$

then we get

$$\left(\frac{\sum_{j=1}^n a_j^2}{\sum_{k=1}^n b_k^2} \right)^{\frac{1}{2}} - \frac{\sum_{j=1}^n a_j b_j}{\sum_{k=1}^n b_k^2} \le \frac{(\Gamma - \gamma)^2}{4(\gamma + \Gamma)},$$

giving the desired inequality (4.53).

The case of equality follows by the similar case in (4.54) and we omit the details. ■

4.8 Zagier Type Inequalities

The following result was obtained by D. Zagier in 1995, [24].

Lemma 159 *Let $f, g : [0, \infty) \to \mathbb{R}$ be monotone decreasing nonnegative functions on $[0, \infty)$. Then*

$$\int_0^\infty f(x) g(x) dx \ge \frac{\int_0^\infty f(x) F(x) dx \int_0^\infty g(x) G(x) dx}{\max \left\{ \int_0^\infty F(x) dx, \int_0^\infty G(x) dx \right\}}, \tag{4.56}$$

for any integrable functions $F, G : [0, \infty) \to [0, 1]$.

Proof. We will follow the proof in [24].

For all $x \ge 0$ we have

$$\begin{aligned} \int_0^\infty f(t) F(t) dt &= f(x) \int_0^\infty F(t) dt + \int_0^\infty [f(t) - f(x)] F(t) dt \\ &\le f(x) \int_0^\infty F(t) dt + \int_0^x [f(t) - f(x)] dt \end{aligned}$$

and hence, since $\int_0^x G(t) dt$ is bounded from above by both x and $\int_0^\infty G(t) dt$,

$$\begin{aligned} &\int_0^\infty f(t) F(t) dt \cdot \int_0^x G(t) dt \\ &\le x f(x) \int_0^\infty F(t) dt + \int_0^\infty G(t) dt \cdot \int_0^x [f(t) - f(x)] dt \\ &\le \max \left\{ \int_0^\infty F(t) dt, \int_0^\infty G(t) dt \right\} \cdot \int_0^x f(t) dt. \end{aligned}$$

Now, multiply by $-dg(x)$ and integrate by parts from 0 to ∞. The left hand side gives $\int_{-\infty}^{\infty} f(t) F(t) dt \cdot \int_{-\infty}^{\infty} g(t) G(t) dt$, the right hand side gives

$$\max\left\{\int_0^{\infty} F(t) dt, \int_0^{\infty} G(t) dt\right\} \cdot \int_0^{\infty} f(t) g(t) dt,$$

and the inequality remains true because the measure $-dg(x)$ is nonnegative. ■

The following particular case is a counterpart of the $(CBS)-$integral inequality obtained by D. Zagier in 1977, [25].

Corollary 160 *If $f, g : [0,\infty) \to [0,\infty)$ are decreasing function on $[0,\infty)$, then*

$$\max\left[f(0)\int_0^{\infty} g(t) dt, g(0)\int_0^{\infty} f(t) dt\right] \cdot \int_0^{\infty} f(t) g(t) dt \geq \int_0^{\infty} f^2(t) dt \int_0^{\infty} g^2(t) dt. \quad (4.57)$$

Remark 161 *The following weighted version of (4.56) may be proved in a similar way, as noted by D. Zagier in [25]*

$$\int_0^{\infty} w(t) f(t) g(t) dt \geq \frac{\int_0^{\infty} w(t) f(t) F(t) dt \int_0^{\infty} w(t) f(t) G(t) dt}{\max\left\{\int_0^{\infty} w(t) F(t) dt, \int_0^{\infty} w(t) G(t) dt\right\}}. \quad (4.58)$$

provided $w(t) > 0$ on $[0,\infty)$, $f, g : [0,\infty) \to [0,\infty)$ are monotonic decreasing and $F, G : [0,\infty) \to [0,1]$ are integrable on $[0,\infty)$.

We may state and prove the following discrete inequality.

Theorem 162 *Consider the sequences of real numbers $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$, $\bar{\mathbf{p}} = (p_1, \dots, p_n)$, $\bar{\mathbf{q}} = (q_1, \dots, q_n)$ and $\bar{\mathbf{w}} = (w_1, \dots, w_n)$.*

If

(i) $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ are decreasing and nonnegative;

(ii) $p_i, q_i \in [0,1]$ and $w_i \geq 0$ for any $i \in \{1, \dots, n\}$,

then we have the inequality

$$\sum_{i=1}^{n} w_i a_i b_i \geq \frac{\sum_{i=1}^{n} w_i p_i a_i \sum_{i=1}^{n} w_i q_i b_i}{\max\left\{\sum_{i=1}^{n} w_i p_i, \sum_{i=1}^{n} w_i q_i\right\}}. \quad (4.59)$$

Proof. Consider the functions $f, g, F, G, W : [0, \infty) \to \mathbb{R}$ given by

$$f(t) = \begin{cases} a_1, & t \in [0,1) \\ a_2, & t \in [1,2) \\ \vdots & \\ a_n, & t \in [n-1,n) \\ 0 & t \in [n,\infty) \end{cases}, \qquad g(t) = \begin{cases} b_1, & t \in [0,1) \\ b_2, & t \in [1,2) \\ \vdots & \\ b_n, & t \in [n-1,n) \\ 0 & t \in [n,\infty) \end{cases},$$

$$F(t) = \begin{cases} p_1, & t \in [0,1) \\ p_2, & t \in [1,2) \\ \vdots & \\ p_n, & t \in [n-1,n) \\ 0 & t \in [n,\infty) \end{cases}, \qquad G(t) = \begin{cases} q_1, & t \in [0,1) \\ q_2, & t \in [1,2) \\ \vdots & \\ q_n, & t \in [n-1,n) \\ 0 & t \in [n,\infty) \end{cases},$$

and

$$W(t) = \begin{cases} w_1, & t \in [0,1) \\ w_2, & t \in [1,2) \\ \vdots & \\ w_n, & t \in [n-1,n) \\ 0 & t \in [n,\infty) \end{cases}.$$

We observe that, the above functions satisfy the hypothesis of Remark 161 and since, for example,

$$\begin{aligned} \int_0^\infty w(t) f(t) g(t)\, dt &= \sum_{i=1}^n \int_{i-1}^i w(t) f(t) g(t)\, dt + \int_n^\infty w(t) f(t) g(t)\, dt \\ &= \sum_{k=1}^n w_k a_k b_k, \end{aligned}$$

then by (4.58) we deduce the desired inequality (4.59). ∎

Remark 163 *A similar inequality for sequences under some monotonicity assumptions for* $\bar{\mathbf{p}}$ *and* $\bar{\mathbf{q}}$ *was obtained in 1995 by J. Pečarić in [26].*

The following counterpart of the (CBS) −discrete inequality holds.

Theorem 164 *Assume that* $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ *are decreasing nonnegative sequences with* $a_1, b_1 \neq 0$ *and* $\bar{\mathbf{w}}$ *a nonnegative sequence. Then*

$$\sum_{i=1}^n w_i a_i^2 \sum_{i=1}^n w_i b_i^2 \leq \max \left\{ b_1 \sum_{i=1}^n w_i a_i, a_1 \sum_{i=1}^n w_i b_i \right\} \sum_{i=1}^n w_i a_i b_i. \qquad (4.60)$$

The proof follows by Theorem 162 on choosing $p_i = \frac{a_i}{a_1} \in [0,1]$, $q_i = \frac{b_i}{b_1} \in [0,1]$, $i \in \{1,\ldots,n\}$. We omit the details.

Remark 165 *When $w_i = 1$, we recapture Alzer's result from 1992, [27].*

4.9 A Counterpart in Terms of the $\sup$ −Norm

The following result has been proved in [11].

Lemma 166 *Let $\bar{\boldsymbol{\alpha}} = (\alpha_1,\ldots,\alpha_n)$ and $\bar{\mathbf{x}} = (x_1,\ldots,x_n)$ be sequences of complex numbers and $\bar{\mathbf{p}} = (p_1,\ldots,p_n)$ a sequence of nonnegative real numbers such that $\sum_{i=1}^n p_i = 1$. Then one has the inequality*

$$\left|\sum_{i=1}^n p_i\alpha_i x_i - \sum_{i=1}^n p_i\alpha_i \sum_{i=1}^n p_i x_i\right| \leq \max_{i=\overline{1,n-1}} |\Delta\alpha_i| \max_{i=\overline{1,n-1}} |\Delta x_i| \left[\sum_{i=1}^n i^2 p_i - \left(\sum_{i=1}^n i p_i\right)^2\right], \quad (4.61)$$

where $\Delta\alpha_i$ is the forward difference, i.e., $\Delta\alpha_i := \alpha_{i+1} - \alpha_i$.

Inequality (4.61) is sharp in the sense that the constant $C = 1$ in the right membership cannot be replaced be a smaller one.

Proof. We shall follow the proof in [11]. We start with the following identity

$$\sum_{i=1}^n p_i\alpha_i x_i - \sum_{i=1}^n p_i\alpha_i \sum_{i=1}^n p_i x_i = \frac{1}{2}\sum_{i,j=1}^n p_i p_j (\alpha_i - \alpha_j)(x_i - x_j) = \sum_{1\leq i<j\leq n} p_i p_j (\alpha_i - \alpha_j)(x_i - x_j).$$

As $i < j$, we can write that

$$\alpha_j - \alpha_i = \sum_{k=i}^{j-1} \Delta\alpha_k$$

and

$$x_j - x_i = \sum_{k=i}^{j-1} \Delta x_k.$$

Using the generalised triangle inequality, we have successively

$$\begin{aligned}
&\left| \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i \right| \\
&= \left| \sum_{1 \le i < j \le n} p_i p_j \sum_{k=i}^{j-1} \Delta \alpha_k \sum_{k=i}^{j-1} \Delta x_k \right| \\
&\le \sum_{1 \le i < j \le n} p_i p_j \left| \sum_{k=i}^{j-1} \Delta \alpha_k \right| \left| \sum_{k=i}^{j-1} \Delta x_k \right| \\
&\le \sum_{1 \le i < j \le n} p_i p_j \sum_{k=i}^{j-1} |\Delta \alpha_k| \sum_{k=i}^{j-1} |\Delta x_k| \\
&:= A.
\end{aligned}$$

Note that

$$|\Delta \alpha_k| \le \max_{1 \le s \le n-1} |\Delta \alpha_s|$$

and

$$|\Delta x_k| \le \max_{1 \le s \le n-1} |\Delta x_s|$$

for all $k = i, \ldots, j-1$ and then by summation

$$\sum_{k=i}^{j-1} |\Delta \alpha_k| \le (j-i) \max_{1 \le s \le n-1} |\Delta \alpha_s|$$

and

$$\sum_{k=i}^{j-1} |\Delta x_k| \le (j-i) \max_{1 \le s \le n-1} |\Delta x_s|.$$

Taking into account the above estimations, we can write

$$A \le \left[\sum_{1 \le i < j \le n} p_i p_j (j-i)^2 \right] \max_{1 \le s \le n-1} |\Delta \alpha_s| \max_{1 \le s \le n-1} |\Delta x_s|.$$

As a simple calculation shows that

$$\sum_{1\le i<j\le n} p_i p_j (j-i)^2 = \sum_{i=1}^n i^2 p_i - \left(\sum_{i=1}^n i p_i\right)^2,$$

inequality (4.61) is proved.

To prove the sharpness of the constant, let us assume that (4.61) holds with a constant $C > 0$, i.e.,

$$\left|\sum_{i=1}^n p_i\alpha_i x_i - \sum_{i=1}^n p_i\alpha_i \sum_{i=1}^n p_i x_i\right| \le C \max_{i=\overline{1,n-1}} |\Delta\alpha_i| \max_{i=\overline{1,n-1}} |\Delta x_i| \left[\sum_{i=1}^n i^2 p_i - \left(\sum_{i=1}^n i p_i\right)^2\right]. \quad (4.62)$$

Now, choose the sequences $\alpha_k = \alpha + k\beta \ \ (\beta \neq 0)$ and $x_k = x + ky \ \ (y \neq 0)$, $k \in \{1, \dots, n\}$ to get

$$\left|\sum_{i=1}^n p_i\alpha_i x_i - \sum_{i=1}^n p_i\alpha_i \sum_{i=1}^n p_i x_i\right| = \frac{1}{2}\left|\sum_{i,j=1}^n p_i p_j (i-j)\beta y\right| = |\beta||y| \left[\sum_{i=1}^n i^2 p_i - \left(\sum_{i=1}^n i p_i\right)^2\right]$$

and

$$\max_{i=\overline{1,n-1}} |\Delta\alpha_i| \max_{i=\overline{1,n-1}} |\Delta x_i| \left[\sum_{i=1}^n i^2 p_i - \left(\sum_{i=1}^n i p_i\right)^2\right] = |\beta||y| \left[\sum_{i=1}^n i^2 p_i - \left(\sum_{i=1}^n i p_i\right)^2\right]$$

and then, by (4.62), we get $C \ge 1$. ■

The following counterpart of the (CBS) −inequality holds [12].

Theorem 167 *Let* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$ *and* $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ *be two sequences of real numbers with* $a_i \neq 0$, $(i = 1, \ldots, n)$. *Then one has the inequality*

$$0 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i \right)^2$$
$$\leq \max_{k=\overline{1,n-1}} \left\{ \Delta \left(\frac{b_k}{a_k} \right) \right\}^2 \left[\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} i^2 a_i^2 - \left(\sum_{i=1}^{n} i a_i^2 \right)^2 \right].$$

The constant $C = 1$ *is sharp in the sense that it cannot be replaced by a smaller constant.*

Proof. Follows by Lemma 166 on choosing

$$p_i = \frac{a_i^2}{\sum_{k=1}^{n} a_k^2}, \quad \alpha_i = \frac{b_i}{a_i}, \quad x_i = \frac{b_i}{a_i}, \quad i \in \{1, \ldots, n\}$$

and performing some elementary calculations.

We omit the details. ∎

4.10 A Counterpart in Terms of the $1-$Norm

The following result has been obtained in [13].

Lemma 168 *Let* $\bar{\boldsymbol{\alpha}} = (\alpha_1, \ldots, \alpha_n)$ *and* $\bar{\mathbf{x}} = (x_1, \ldots, x_n)$ *be sequences of complex numbers and* $\bar{\mathbf{p}} = (p_1, \ldots, p_n)$ *a sequence of nonnegative real numbers such that* $\sum_{i=1}^{n} p_i = 1$. *Then one has the inequality*

$$\left| \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i \right|$$
$$\leq \frac{1}{2} \sum_{i=1}^{n} p_i (1 - p_i) \sum_{i=1}^{n-1} |\Delta \alpha_i| \sum_{i=1}^{n-1} |\Delta x_i|, \quad (4.63)$$

where $\Delta \alpha_i := \alpha_{i+1} - \alpha_i$ *is the forward difference.*

The constant $\frac{1}{2}$ *is sharp in the sense that it cannot be replaced by a smaller constant.*

Proof. We shall follow the proof in [13].

As in the proof of Lemma 166 in Section 4.9, we have

$$\left|\sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i x_i\right| \leq \sum_{1\leq i<j\leq n} p_i p_j \sum_{k=i}^{j-1} |\Delta\alpha_k| \sum_{l=i}^{j-1} |\Delta x_l| := A. \quad (4.64)$$

It is obvious that for all $1 \leq i < j \leq n-1$, we have that

$$\sum_{k=i}^{j-1} |\Delta\alpha_k| \leq \sum_{k=1}^{n-1} |\Delta\alpha_k|$$

and

$$\sum_{l=i}^{j-1} |\Delta x_l| \leq \sum_{l=1}^{n-1} |\Delta x_l|.$$

Utilising these and the definition of A, we conclude that

$$A \leq \sum_{k=1}^{n-1} |\Delta\alpha_k| \sum_{l=1}^{n-1} |\Delta x_l| \sum_{1\leq i<j\leq n} p_i p_j. \quad (4.65)$$

Now, let us observe that

$$\begin{aligned} \sum_{1\leq i<j\leq n} p_i p_j &= \frac{1}{2}\left[\sum_{i,j=1}^{n} p_i p_j - \sum_{i=j} p_i p_j\right] \quad (4.66) \\ &= \frac{1}{2}\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j - \sum_{i=1}^{n} p_i^2\right] \\ &= \frac{1}{2}\sum_{i=1}^{n} p_i (1-p_i). \end{aligned}$$

Making use of (4.64) – (4.66), we deduce the desired inequality (4.63).

To prove the sharpness of the constant $\frac{1}{2}$, let us assume that (4.63) holds with a constant $C > 0$. That is

$$\left|\sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i x_i\right| \leq C\sum_{i=1}^{n} p_i (1-p_i) \sum_{i=1}^{n-1} |\Delta\alpha_i| \sum_{i=1}^{n-1} |\Delta x_i| \quad (4.67)$$

for all α_i, x_i, p_i $(i = 1, \ldots, n)$ as above and $n \geq 1$.

Choose in (4.63) $n = 2$ and compute

$$\begin{aligned} \sum_{i=1}^{2} p_i \alpha_i x_i - \sum_{i=1}^{2} p_i \alpha_i \sum_{i=1}^{2} p_i x_i &= \frac{1}{2} \sum_{i,j=1}^{2} p_i p_j (\alpha_i - \alpha_j)(x_i - x_j) \\ &= \sum_{1 \leq i < j \leq 2} p_i p_j (\alpha_i - \alpha_j)(x_i - x_j) \\ &= p_1 p_2 (\alpha_1 - \alpha_2)(x_1 - x_2). \end{aligned}$$

Also

$$\sum_{i=1}^{2} p_i (1 - p_i) \sum_{i=1}^{2} |\Delta \alpha_i| \sum_{i=1}^{2} |\Delta x_i| = (p_1 p_2 + p_1 p_2) |\alpha_1 - \alpha_2| |x_1 - x_2|.$$

Substituting in (4.67), we obtain

$$p_1 p_2 |\alpha_1 - \alpha_2| |x_1 - x_2| \leq 2C p_1 p_2 |\alpha_1 - \alpha_2| |x_1 - x_2|.$$

If we assume that $p_1, p_2 > 0$, $\alpha_1 \neq \alpha_2$, $x_1 \neq x_2$, then we obtain $C \geq \frac{1}{2}$, which proves the sharpness of the constant $\frac{1}{2}$. ■

We are now able to state the following counterpart of the (CBS) −inequality [12].

Theorem 169 *Let* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$ *and* $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ *be two sequences of real numbers with* $a_i \neq 0 \ \ (i = 1, \ldots, n)$. *Then one has the inequality*

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i \right)^2 \qquad (4.68) \\ &\leq \left[\sum_{k=1}^{n-1} \left| \Delta \left(\frac{b_k}{a_k} \right) \right| \right]^2 \sum_{1 \leq i < j \leq n} a_i^2 a_j^2. \end{aligned}$$

The constant $C = 1$ is sharp in (4.68), in the sense that it cannot be replaced by a smaller constant.

Proof. We choose

$$p_i = \frac{a_i^2}{\sum_{k=1}^{n} a_k^2}, \quad \alpha_i = x_i = \frac{b_i}{a_i}, \ \ i \in \{1, \ldots, n\}$$

in (4.63) to get

$$
\begin{aligned}
0 &\leq \frac{\sum_{i=1}^{n} b_i^2}{\sum_{k=1}^{n} a_k^2} - \frac{\left(\sum_{i=1}^{n} a_i b_i\right)^2}{\left(\sum_{k=1}^{n} a_k^2\right)^2} \\
&\leq \frac{1}{2} \cdot \frac{\sum_{i=1}^{n} a_i^2 \left(1 - \frac{a_i^2}{\sum_{k=1}^{n} a_k^2}\right)}{\sum_{k=1}^{n} a_k^2} \left(\sum_{j=1}^{n-1} \left|\Delta\left(\frac{b_j}{a_j}\right)\right|\right)^2 \\
&= \frac{1}{2} \cdot \frac{\sum_{i=1}^{n} a_i^2 \left(\sum_{k=1}^{n} a_k^2 - a_i^2\right)}{\left(\sum_{k=1}^{n} a_k^2\right)^2} \left(\sum_{j=1}^{n-1} \left|\Delta\left(\frac{b_j}{a_j}\right)\right|\right)^2
\end{aligned}
$$

which is clearly equivalent to

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \\
&\leq \frac{1}{2} \left[\left(\sum_{k=1}^{n} a_k^2\right)^2 - \sum_{i=1}^{n} a_i^4\right] \left(\sum_{j=1}^{n} \left|\Delta\left(\frac{b_j}{a_j}\right)\right|\right)^2 .
\end{aligned}
$$

Since

$$
\frac{1}{2} \left[\left(\sum_{k=1}^{n} a_k^2\right)^2 - \sum_{i=1}^{n} a_i^4\right] = \sum_{1 \leq i < j \leq n} a_i^2 a_j^2
$$

the inequality (4.68) is thus proved. ■

4.11 A Counterpart in Terms of the $p-$Norm

The following result has been obtained in [14].

Lemma 170 *Let $\bar{\boldsymbol{\alpha}} = (\alpha_1, \ldots, \alpha_n)$ and $\bar{\mathbf{x}} = (x_1, \ldots, x_n)$ be sequences of complex numbers and $\bar{\mathbf{p}} = (p_1, \ldots, p_n)$ a sequence of nonnegative real numbers such that $\sum_{i=1}^{n} p_i = 1$. Then one has the inequality*

$$
\begin{aligned}
&\left|\sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i\right| \\
&\qquad \leq \sum_{1 \leq j < i \leq n} (i - j)\, p_i p_j \left(\sum_{k=1}^{n-1} |\Delta \alpha_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} |\Delta x_k|^q\right)^{\frac{1}{q}}, \quad (4.69)
\end{aligned}
$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The constant $C = 1$ *in the right hand side of (4.69) is sharp in the sense that it cannot be replaced by a smaller constant.*

Proof. We shall follow the proof in [14].

As in the proof of Lemma 166 in Section 4.9, we have

$$\left|\sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i x_i\right| \le \sum_{1\le j<i\le n} p_i p_j \sum_{k=j}^{i-1} |\Delta\alpha_k| \sum_{l=j}^{i-1} |\Delta x_l| := A. \quad (4.70)$$

Using Hölder's discrete inequality, we can state that

$$\sum_{k=j}^{i-1} |\Delta\alpha_k| \le (i-j)^{\frac{1}{q}} \left(\sum_{k=j}^{i-1} |\Delta\alpha_k|^p\right)^{\frac{1}{p}}$$

and

$$\sum_{l=j}^{i-1} |\Delta x_l| \le (i-j)^{\frac{1}{p}} \left(\sum_{l=j}^{i-1} |\Delta x_l|^q\right)^{\frac{1}{q}},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and then we get

$$A \le \sum_{1\le j<i\le n} p_i p_j (i-j) \left(\sum_{k=j}^{i-1} |\Delta\alpha_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=j}^{i-1} |\Delta x_k|^q\right)^{\frac{1}{q}}. \quad (4.71)$$

Since

$$\sum_{k=j}^{i-1} |\Delta\alpha_k|^p \le \sum_{k=1}^{n-1} |\Delta\alpha_k|^p$$

and

$$\sum_{k=j}^{i-1} |\Delta x_k|^q \le \sum_{k=1}^{n-1} |\Delta x_k|^q.$$

for all $1 \le j < i \le n$, then by (4.70) and (4.71) we deduce the desired inequality (4.69).

To prove the sharpness of the constant, let us assume that (4.69) holds with a constant $C > 0$. That is,

$$\left|\sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i x_i\right| \leq C \sum_{1\leq j<i\leq n} (i-j)\, p_i p_j \left(\sum_{k=1}^{n-1} |\Delta\alpha_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} |\Delta x_k|^q\right)^{\frac{1}{q}}. \quad (4.72)$$

Note that, for $n = 2$, we have

$$\left|\sum_{i=1}^{2} p_i\alpha_i x_i - \sum_{i=1}^{2} p_i\alpha_i \sum_{i=1}^{2} p_i x_i\right| = p_1 p_2 |\alpha_1 - \alpha_2| |x_1 - x_2|$$

and

$$\sum_{1\leq j<i\leq 2} (i-j)\, p_i p_j \left(\sum_{k=1}^{1} |\Delta\alpha_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{1} |\Delta x_k|^q\right)^{\frac{1}{q}} = p_1 p_2 |\alpha_1 - \alpha_2| |x_1 - x_2|.$$

Therefore, from (4.72), we obtain

$$p_1 p_2 |\alpha_1 - \alpha_2| |x_1 - x_2| \leq C p_1 p_2 |\alpha_1 - \alpha_2| |x_1 - x_2|$$

for all $\alpha_1 \neq \alpha_2$, $x_1 \neq x_2$, $p_1 p_2 > 0$, giving $C \geq 1$. ■

We are able now to state the following counterpart of the (CBS) −inequality.

Theorem 171 *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be two sequences of real numbers with* $a_i \neq 0$, $(i = 1, \dots, n)$. *Then one has the inequality*

$$0 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{k=1}^{n-1} \left|\Delta\left(\frac{b_k}{a_k}\right)\right|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \left|\Delta\left(\frac{b_k}{a_k}\right)\right|^q\right)^{\frac{1}{q}} \sum_{1\leq j<i\leq n} (i-j)\, a_i^2 a_j^2,$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The constant $C = 1$ *is sharp in the above sense.*

Proof. Follows by Lemma 170 for

$$p_i = \frac{a_i^2}{\sum_{k=1}^n a_k^2}, \quad \alpha_i = x_i = \frac{b_i}{a_i}, \quad i \in \{1, \dots, n\}.$$

■

The following corollary is a natural consequence of Theorem 171 for $p = q = 2$.

Corollary 172 *With the assumptions of Theorem 171 for* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$, *we have*

$$\begin{aligned} 0 &\le \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2 \\ &\le \sum_{k=1}^{n-1} \left|\Delta\left(\frac{b_k}{a_k}\right)\right|^2 \sum_{1\le j<i\le n} (i-j)\, a_i^2 a_j^2. \end{aligned}$$

4.12 A Counterpart Via an Andrica-Badea Result

The following result is due to Andrica and Badea [15, p. 16].

Lemma 173 *Let* $\bar{\mathbf{x}} = (x_1, \dots, x_n) \in I^n = [m, M]^n$ *be a sequence of real numbers and let* S *be the subset of* $\{1, \dots, n\}$ *that minimises the expression*

$$\left|\sum_{i\in S} p_i - \frac{1}{2} P_n\right|, \tag{4.73}$$

where $P_n := \sum_{i=1}^n p_i > 0$, $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ *is a sequence of nonnegative real numbers. Then*

$$\begin{aligned} &\max_{\bar{\mathbf{x}}\in I^n} \left[\frac{1}{P_n}\sum_{i=1}^n p_i x_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)^2\right] \\ &\qquad = \sum_{i\in S} p_i \left(P_n - \sum_{i\in S} p_i\right) \frac{(M-m)^2}{P_n^2}. \end{aligned} \tag{4.74}$$

Proof. We shall follow the proof in [15, p. 161].

Define

$$D_n(\bar{\mathbf{x}}, \bar{\mathbf{p}}) := \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)^2 = \frac{1}{P_n}\sum_{1\le i<j\le n} p_i p_j (x_i - x_j)^2.$$

Keeping in mind the convexity of the quadratic function, we have

$$\begin{aligned}
& D_n(\alpha\bar{\mathbf{x}} + (1-\alpha)\bar{\mathbf{y}}, \bar{\mathbf{p}}) \\
&= \frac{1}{P_n^2}\sum_{1\le i<j\le n} p_i p_j [\alpha x_i + (1-\alpha) y_i - \alpha x_j - (1-\alpha) y_j]^2 \\
&= \frac{1}{P_n^2}\sum_{1\le i<j\le n} p_i p_j [\alpha (x_i - x_j) + (1-\alpha)(y_i - y_j)]^2 \\
&\le \frac{1}{P_n^2}\sum_{1\le i<j\le n} p_i p_j \left[\alpha (x_i - x_j)^2 + (1-\alpha)(y_i - y_j)^2\right] \\
&= \alpha D_n(\bar{\mathbf{x}}, \bar{\mathbf{p}}) + (1-\alpha) D_n(\bar{\mathbf{y}}, \bar{\mathbf{p}}),
\end{aligned}$$

hence $D_n(\cdot, \bar{\mathbf{p}})$ is a convex function on I^n.

Using a well known theorem (see for instance [16, p. 124]), we get that the maximum of $D_n(\cdot, \bar{\mathbf{p}})$ is attained on the boundary of I^n.

Let $(S, \bar{S})$ be the partition of $\{1, \dots, n\}$ such that the maximum of $D_n(\cdot, \bar{\mathbf{p}})$ is obtained for $\bar{\mathbf{x}}_0 = (x_1^0, \dots, x_n^0)$, where $x_i^0 = m$ if $i \in \bar{S}$ and $x_i^0 = M$ if $i \in S$. In this case we have

$$D_n(\bar{\mathbf{x}}_0, \bar{\mathbf{p}}) = \frac{1}{P_n^2}\sum_{1\le i<j\le n} p_i p_j (x_i - x_j)^2 = \frac{(M-m)^2}{P_n^2}\sum_{i\in S} p_i\left(P_n - \sum_{i\in S} p_i\right). \tag{4.75}$$

The expression

$$\sum_{i\in S} p_i\left(P_n - \sum_{i\in S} p_i\right)$$

is a maximum when the set S minimises the expression

$$\left|\sum_{i\in S} p_i - \frac{1}{2}P_n\right|.$$

From (4.75) it follows that $D_n(\bar{\mathbf{x}}, \bar{\mathbf{p}})$ is also a maximum and the proof of the above lemma is complete. ■

The following counterpart of the (CBS) −inequality holds.

Theorem 174 *Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ and $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ be two sequences of real numbers with $a_i \neq 0$ $(i = 1, \dots, n)$ and*

$$-\infty < m \le \frac{b_i}{a_i} \le M < \infty \text{ for each } i \in \{1, \dots, n\}. \tag{4.76}$$

Let S be the subset of $\{1, \dots, n\}$ that minimizes the expression

$$\left|\sum_{i\in S} a_i^2 - \frac{1}{2}\sum_{i=1}^{n} a_i^2\right|, \tag{4.77}$$

and denote $\bar{S} := \{1, \dots, n\} \setminus S$. Then we have the inequality

$$\begin{aligned} 0 &\le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \qquad (4.78) \\ &\le (M-m)^2 \sum_{i\in S} a_i^2 \sum_{i\in \bar{S}} a_i^2 \\ &\le \frac{1}{4}(M-m)^2 \left(\sum_{i=1}^{n} a_i^2\right)^2. \end{aligned}$$

Proof. The proof of the second inequality in (4.78) follows by Lemma 173 on choosing $p_i = a_i^2$, $x_i = \frac{b_i}{a_i}$, $i \in \{1, \dots, n\}$.

The third inequality is obvious as

$$\begin{aligned} \sum_{i\in S} a_i^2 \sum_{i\in \bar{S}} a_i^2 &= \sum_{i\in S} a_i^2 \left(\sum_{j=1}^{n} a_j^2 - \sum_{i\in S} a_i^2\right) \\ &\le \frac{1}{4}\left(\sum_{i\in S} a_i^2 + \sum_{j=1}^{n} a_j^2 - \sum_{i\in S} a_i^2\right)^2 \\ &= \frac{1}{4}\left(\sum_{j=1}^{n} a_j^2\right)^2. \end{aligned}$$

■

4.13 A Refinement of Cassels' Inequality

In 1914, P. Schweitzer [18] proved the following result.

Theorem 175 *If* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$ *is a sequence of real numbers such that* $0 < m \le a_i \le M < \infty$ $(i \in \{1, \ldots, n\})$, *then*

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i}\right) \le \frac{(M+m)^2}{4mM}. \tag{4.79}$$

In 1972, A. Lupaş [17] proved the following refinement of Schweitzer's result which gives the best bound for n odd as well.

Theorem 176 *With the assumptions in Theorem 175, one has*

$$\sum_{i=1}^{n} a_i \sum_{i=1}^{n} \frac{1}{a_i} \le \frac{\left(\left[\frac{n}{2}\right]M + \left[\frac{n+1}{2}\right]m\right)\left(\left[\frac{n+1}{2}\right]M + \left[\frac{n}{2}\right]m\right)}{Mm}, \tag{4.80}$$

where $[\cdot]$ *is the integer part.*

In 1988, Andrica and Badea [15] established a weighted version of Schweitzer and Lupaş inequalities via the use of the following weighted version of the Grüss inequality [15, Theorem 2].

Theorem 177 *If* $m_1 \le a_i \le M_1$, $m_2 \le b_i \le M_2$ $(i \in \{1, \ldots, n\})$ *and* S *is the subset of* $\{1, \ldots, n\}$ *which minimises the expression*

$$\left|\sum_{i \in S} p_i - \frac{1}{2}P_n\right|, \tag{4.81}$$

where $P_n := \sum_{i=1}^{n} p_i > 0$, *then*

$$\begin{aligned} &\left|P_n \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \cdot \sum_{i=1}^{n} p_i b_i\right| \\ &\le (M_1 - m_1)(M_2 - m_2)\sum_{i \in S} p_i \left(P_n - \sum_{i \in S} p_i\right). \\ &\le \frac{1}{4}P_n^2 (M_1 - m_1)(M_2 - m_2). \end{aligned} \tag{4.82}$$

Proof. Using the result in Lemma 173, Section 4.12, we have

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right)^2 \leq \frac{(M_1 - m_1)^2}{P_n^2}\sum_{i\in S} p_i \left(P_n - \sum_{i\in S} p_i\right) \quad (4.83)$$

and

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i b_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i b_i\right)^2 \leq \frac{(M_2 - m_2)^2}{P_n^2}\sum_{i\in S} p_i \left(P_n - \sum_{i\in S} p_i\right) \quad (4.84)$$

and since

$$\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i b_i\right)^2 \leq \left[\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right)^2\right] \times \left[\frac{1}{P_n}\sum_{i=1}^{n} p_i b_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i b_i\right)^2\right], \quad (4.85)$$

the first part of (4.82) holds true.

The second part follows by the elementary inequality

$$ab \leq \frac{1}{4}(a+b)^2, \quad a, b \in \mathbb{R}$$

for the choices $a := \sum_{i\in S} p_i$, $b := P_n - \sum_{i\in S} p_i$. ■

We are now able to state and prove the result of Andrica and Badea [15, Theorem 4], which is related to Schweitzer's inequality.

Theorem 178 *If $0 < m \leq a_i \leq M < \infty$, $i \in \{1, \ldots, n\}$ and S is a subset of $\{1, \ldots, n\}$ that minimises the expression*

$$\left|\sum_{i\in S} p_i - \frac{P_n}{2}\right|,$$

then we have the inequality

$$\left(\sum_{i=1}^{n} p_i a_i\right)\left(\sum_{i=1}^{n} p_i \frac{1}{a_i}\right) \leq P_n^2 + \frac{(M-m)^2}{Mm} \sum_{i\in S} p_i \left(P_n - \sum_{i\in S} p_i\right) \tag{4.86}$$
$$\leq \frac{(M+m)^2}{4Mm} P_n^2 .$$

Proof. We shall follow the proof in [15]. We obtain from Theorem 176 with $b_i = \frac{1}{a_i}$, $m_1 = m$, $M_1 = m$, $m_2 = \frac{1}{M}$, $M_2 = \frac{1}{m}$, the following

$$\left| P_n^2 - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i \frac{1}{a_i} \right| \leq (M-m)\left(\frac{1}{m} - \frac{1}{M}\right) \sum_{i\in S} p_i \left(P_n - \sum_{i\in S} p_i\right),$$

that leads, in a simple manner, to (4.86). ■

We may now prove the following counterpart for the weighted $(CBS)-$inequality that improves the additive version of Cassels' inequality.

Theorem 179 *Let* $\bar{\mathbf{a}} = (a_1,\ldots,a_n)$, $\bar{\mathbf{b}} = (b_1,\ldots,b_n)$ *be two sequences of positive real numbers with the property that*

$$0 < m \leq \frac{b_i}{a_i} \leq M < \infty \quad \text{for each } i \in \{1,\ldots,n\} \tag{4.87}$$

and $\bar{\mathbf{p}} = (p_1,\ldots,p_n)$ *a sequence of nonnegative real numbers such that* $P_n := \sum_{i=1}^{n} p_i > 0$. *If* S *is a subset of* $\{1,\ldots,n\}$ *that minimises the expression*

$$\left| \sum_{i\in S} p_i a_i b_i - \frac{1}{2}\sum_{i=1}^{n} p_i a_i b_i \right| \tag{4.88}$$

then one has the inequality

$$\sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2 - \left(\sum_{i=1}^{n} p_i a_i b_i\right)^2 \tag{4.89}$$
$$\leq \frac{(M-m)^2}{Mm} \sum_{i\in S} p_i a_i b_i \left(\sum_{i=1}^{n} p_i a_i b_i - \sum_{i\in S} p_i a_i b_i\right)$$
$$\leq \frac{(M-m)^2}{4Mm} \left(\sum_{i=1}^{n} p_i a_i b_i\right)^2 .$$

Proof. Applying Theorem 178 for $a_i = x_i$, $p_i = q_i x_i$ we may deduce the inequality

$$\sum_{i=1}^{n} q_i x_i^2 \sum_{i=1}^{n} q_i - \left(\sum_{i=1}^{n} q_i x_i\right)^2 \leq \frac{(M-m)^2}{Mm} \sum_{i\in S} q_i x_i \left(\sum_{i=1}^{n} q_i x_i - \sum_{i\in S} q_i x_i\right), \quad (4.90)$$

provided $q_i \geq 0$, $\sum_{i=1}^{n} q_i > 0$, $0 < m \leq x_i \leq M < \infty$, for $i \in \{1, \dots, n\}$ and S is a subset of $\{1, \dots, n\}$ that minimises the expression

$$\left|\sum_{i\in S} q_i x_i - \frac{\sum_{i=1}^{n} q_i x_i}{2}\right|. \quad (4.91)$$

Now, if in (4.90) we choose $q_i = p_i a_i^2$, $x_i = \frac{b_i}{a_i} \in [m, M]$ for $i \in \{1, \dots, n\}$, we deduce the desired result (4.89). ■

The following corollary provides a refinement of Cassels' inequality.

Corollary 180 *With the assumptions of Theorem 179, we have the inequality*

$$\begin{aligned} 1 &\leq \frac{\sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2}{\left(\sum_{i=1}^{n} p_i a_i b_i\right)^2} \quad (4.92)\\ &\leq 1 + \frac{(M-m)^2}{Mm} \cdot \frac{\sum_{i\in S} p_i a_i b_i}{\sum_{i=1}^{n} p_i a_i b_i} \left(1 - \frac{\sum_{i\in S} p_i a_i b_i}{\sum_{i=1}^{n} p_i a_i b_i}\right) \\ &\leq \frac{(M+m)^2}{4Mm}. \end{aligned}$$

The case of the "unweighted" Cassels' inequality is embodied in the following corollary as well.

Corollary 181 *Assume that* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *satisfy (4.88). If* S *is a subset of* $\{1, \dots, n\}$ *that minimises the expression*

$$\left|\sum_{i\in S} a_i b_i - \frac{1}{2}\sum_{i=1}^{n} a_i b_i\right| \quad (4.93)$$

then one has the inequality

$$\begin{aligned} 1 &\leq \frac{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}{\left(\sum_{i=1}^{n} a_i b_i\right)^2} \\ &\leq 1 + \frac{(M-m)^2}{Mm} \cdot \frac{\sum_{i\in S} a_i b_i}{\sum_{i=1}^{n} a_i b_i} \left(1 - \frac{\sum_{i\in S} a_i b_i}{\sum_{i=1}^{n} a_i b_i}\right) \\ &\leq \frac{(M+m)^2}{4Mm}. \end{aligned} \tag{4.94}$$

In particular, we may obtain the following refinement of the Pólya-Szegö's inequality.

Corollary 182 *Assume that*

$$0 < a \leq a_i \leq A < \infty, \ \ 0 < b \leq b_i \leq B < \infty \ \ \textit{for} \ \ i \in \{1, \dots, n\}. \tag{4.95}$$

If S is a subset of $\{1, \dots, n\}$ that minimises the expression (4.93), then one has the inequality

$$\begin{aligned} 1 &\leq \frac{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}{\left(\sum_{i=1}^{n} a_i b_i\right)^2} \\ &\leq 1 + \frac{(AB-ab)^2}{abAB} \cdot \frac{\sum_{i\in S} a_i b_i}{\sum_{i=1}^{n} a_i b_i} \left(1 - \frac{\sum_{i\in S} a_i b_i}{\sum_{i=1}^{n} a_i b_i}\right) \\ &\leq \frac{(AB+ab)^2}{4abAB}. \end{aligned} \tag{4.96}$$

4.14 Two Counterparts Via Diaz-Metcalf Results

In [19], J.B. Diaz and F.T. Metcalf proved the following inequality for sequences of complex numbers.

Lemma 183 *Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ and $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ be sequences of complex numbers such that $a_k \neq 0$, $k \in \{1, \dots, n\}$ and*

$$m \leq \operatorname{Re}\left(\frac{b_k}{a_k}\right) + \operatorname{Im}\left(\frac{b_k}{a_k}\right) \leq M, \ \ m \leq \operatorname{Re}\left(\frac{b_k}{a_k}\right) - \operatorname{Im}\left(\frac{b_k}{a_k}\right) \leq M, \tag{4.97}$$

where $m, M \in \mathbb{R}$ and $k \in \{1, \dots, n\}$. Then one has the inequality

$$\sum_{k=1}^{n} |b_k|^2 + mM \sum_{k=1}^{n} |a_k|^2 \leq (m+M) \operatorname{Re}\left[\sum_{k=1}^{n} a_k \bar{b}_k\right] \leq |M+m| \left|\sum_{k=1}^{n} a_k \bar{b}_k\right|. \tag{4.98}$$

Using the above result we may state and prove the following counterpart inequality.

Theorem 184 *If $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ are as in (4.97) and $m, M > 0$, then one has the inequality*

$$\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 \leq \frac{(M+m)^2}{4mM} \left(\operatorname{Re} \sum_{k=1}^{n} a_k \bar{b}_k\right)^2 \leq \frac{(M+m)^2}{4mM} \left|\sum_{k=1}^{n} a_k \bar{b}_k\right|^2. \tag{4.99}$$

Proof. Using the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha} q^2 \geq 2pq, \quad \alpha > 0, \ p, q \geq 0$$

we have

$$\sqrt{mM} \sum_{k=1}^{n} |a_k|^2 + \frac{1}{\sqrt{mM}} \sum_{k=1}^{n} |b_k|^2 \geq 2 \left(\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2\right)^{\frac{1}{2}}. \tag{4.100}$$

On the other hand, by (4.98), we have

$$\frac{1}{\sqrt{mM}} \sum_{k=1}^{n} |b_k|^2 + \sqrt{mM} \sum_{k=1}^{n} |a_k|^2 \leq \frac{(M+m)}{\sqrt{mM}} \operatorname{Re}\left[\sum_{k=1}^{n} a_k \bar{b}_k\right] \leq \frac{M+m}{\sqrt{mM}} \left|\sum_{k=1}^{n} a_k \bar{b}_k\right|. \tag{4.101}$$

Combining (4.100) and (4.101), we deduce the desired result (4.99). ∎

The following corollary is a natural consequence of the above lemma.

Corollary 185 *If* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *and* m, M *satisfy the hypothesis of Theorem 184, then*

$$\begin{aligned}
0 &\leq \left(\sum_{i=1}^{n}|a_i|^2\sum_{i=1}^{n}|b_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n}a_i\bar{b}_i\right| \qquad (4.102)\\
&\leq \left(\sum_{i=1}^{n}|a_i|^2\sum_{i=1}^{n}|b_i|^2\right)^{\frac{1}{2}} - \left|\mathrm{Re}\left(\sum_{k=1}^{n}a_k\bar{b}_k\right)\right| \\
&\leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}}\left|\mathrm{Re}\left(\sum_{k=1}^{n}a_k\bar{b}_k\right)\right| \\
&\leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}}\left|\sum_{k=1}^{n}a_k\bar{b}_k\right|
\end{aligned}$$

and

$$0 \leq \sum_{i=1}^{n}|a_i|^2\sum_{i=1}^{n}|b_i|^2 - \left|\sum_{i=1}^{n}a_i\bar{b}_i\right|^2 \qquad (4.103)$$

$$\leq \sum_{i=1}^{n}|a_i|^2\sum_{i=1}^{n}|b_i|^2 - \left|\mathrm{Re}\left(\sum_{i=1}^{n}a_i\bar{b}_i\right)\right|^2 \qquad (4.104)$$

$$\leq \frac{(M-m)^2}{4mM}\left|\mathrm{Re}\left(\sum_{i=1}^{n}a_i\bar{b}_i\right)\right|^2 \qquad (4.105)$$

$$\leq \frac{(M-m)^2}{4mM}\left|\sum_{i=1}^{n}a_i\bar{b}_i\right|^2. \qquad (4.106)$$

Another result obtained by Diaz and Metcalf in [19] is the following one.

Lemma 186 *Let* $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, m *and* M *be complex numbers such that*

$$\mathrm{Re}\,(m) + \mathrm{Im}\,(m) \leq \mathrm{Re}\left(\frac{b_k}{a_k}\right) + \mathrm{Im}\left(\frac{b_k}{a_k}\right) \leq \mathrm{Re}\,(M) + \mathrm{Im}\,(M)\,; \qquad (4.107)$$

$$\mathrm{Re}\,(m) - \mathrm{Im}\,(m) \leq \mathrm{Re}\left(\frac{b_k}{a_k}\right) - \mathrm{Im}\left(\frac{b_k}{a_k}\right) \leq \mathrm{Re}\,(M) - \mathrm{Im}\,(M)\,;$$

for each $k \in \{1, \ldots, n\}$. *Then*

$$\sum_{k=1}^{n} |b_k|^2 + \operatorname{Re}\left(m\bar{M}\right) \sum_{k=1}^{n} |a_k|^2 \le \operatorname{Re}\left[(M+m)\sum_{k=1}^{n} a_k \bar{b}_k\right] \le |M+m| \left|\sum_{k=1}^{n} a_k \bar{b}_k\right|. \tag{4.108}$$

The following counterpart result for the (CBS) −inequality may be stated as well.

Theorem 187 *With the assumptions in Lemma 186, and if* $\operatorname{Re}\left(m\bar{M}\right) > 0$, *then we have the inequality:*

$$\left[\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2\right]^{\frac{1}{2}} \le \frac{\operatorname{Re}\left[(M+m)\sum_{k=1}^{n} a_k \bar{b}_k\right]}{2\left[\operatorname{Re}\left(m\bar{M}\right)\right]^{\frac{1}{2}}} \le \frac{|M+m| \left|\sum_{k=1}^{n} a_k \bar{b}_k\right|}{2\left[\operatorname{Re}\left(m\bar{M}\right)\right]^{\frac{1}{2}}}. \tag{4.109}$$

The proof is similar to the one in Theorem 184 and we omit the details.

Remark 188 *Similar additive versions may be stated. They are left as an exercise to the interested reader.*

4.15 Some Counterparts Via the Čebyšev Functional

For $\bar{\mathbf{x}} = (x_1, \ldots, x_n)$, $\bar{\mathbf{y}} = (y_1, \ldots, y_n)$ two sequences of real numbers and $\bar{\mathbf{p}} = (p_1, \ldots, p_n)$ a sequence of nonnegative real numbers with $\sum_{i=1}^{n} p_i = 1$, define the *Čebyšev functional*

$$T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}, \bar{\mathbf{y}}) := \sum_{i=1}^{n} p_i x_i y_i - \sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i y_i. \tag{4.110}$$

For $\bar{\mathbf{x}}$ and $\bar{\mathbf{p}}$ as above consider the norms:

$$\|\bar{\mathbf{x}}\|_{\infty} := \max_{i=\overline{1,n}} |x_i|$$

$$\|\bar{\mathbf{x}}\|_{\bar{\mathbf{p}},\alpha} := \left(\sum_{i=1}^{n} p_i |x_i|^{\alpha} \right)^{\frac{1}{\alpha}}, \quad \alpha \in [1, \infty).$$

The following result holds [20].

Theorem 189 *Let $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$, $\bar{\mathbf{p}}$ be as above and $\bar{\mathbf{c}} = (c, \ldots, c)$ a constant sequence with $c \in \mathbb{R}$. Then one has the inequalities*

$$\begin{aligned} 0 &\le |T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}, \bar{\mathbf{y}})| \qquad (4.111) \\ &\le \begin{cases} \|\bar{y} - \bar{y}_{\mu,p}\|_{\bar{\mathbf{p}},1} \cdot \inf_{c\in\mathbb{R}} \|\bar{x} - \bar{c}\|_{\infty}; \\ \\ \|\bar{y} - \bar{y}_{\mu,p}\|_{\bar{\mathbf{p}},\beta} \cdot \inf_{c\in\mathbb{R}} \|\bar{x} - \bar{c}\|_{\bar{\mathbf{p}},\alpha}, \ \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \\ \|\bar{y} - \bar{y}_{\mu,p}\|_{\infty} \cdot \inf_{c\in\mathbb{R}} \|\bar{x} - \bar{c}\|_{\bar{\mathbf{p}},1}; \end{cases} \\ &\le \begin{cases} \|\bar{y} - \bar{y}_{\mu,p}\|_{\bar{\mathbf{p}},1} \min\left\{ \|\bar{x}\|_{\infty}, \|\bar{x} - \bar{x}_{\mu,p}\|_{\infty} \right\}; \\ \\ \|\bar{y} - \bar{y}_{\mu,p}\|_{\bar{\mathbf{p}},\beta} \min\left\{ \|\bar{x}\|_{\bar{\mathbf{p}},\alpha}, \|\bar{x} - \bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},\alpha} \right\}, \\ \qquad\qquad \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \|\bar{y} - \bar{y}_{\mu,p}\|_{\infty} \cdot \min\left\{ \|\bar{x}\|_{\bar{\mathbf{p}},1}, \|\bar{x} - \bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},1} \right\}; \end{cases} \end{aligned}$$

where

$$x_{\mu,p} := \sum_{i=1}^{n} p_i x_i, \quad y_{\mu,p} := \sum_{i=1}^{n} p_i y_i$$

and $\bar{\mathbf{x}}_{\mu}$, $\bar{\mathbf{y}}_{\mu}$ are the sequences with all components equal to $x_{\mu,p}$, $y_{\mu,p}$.

Proof. Firstly, let us observe that for any $c \in \mathbb{R}$, one has Sonin's identity

$$\begin{aligned} T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}, \bar{\mathbf{y}}) &= T_n(\bar{\mathbf{p}}; \bar{x} - \bar{c}, \bar{y} - \bar{y}_{\mu,p}) \qquad (4.112) \\ &= \sum_{i=1}^{n} p_i (x_i - c) \left(y_i - \sum_{j=1}^{n} p_j y_j \right). \end{aligned}$$

Taking the modulus and using Hölder's inequality, we have

$$|T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}, \bar{\mathbf{y}})| \le \sum_{i=1}^{n} p_i |x_i - c| |y_i - y_{\mu,p}| \qquad (4.113)$$

$$\leq \begin{cases} \max\limits_{i=\overline{1,n}} |x_i - c| \sum_{i=1}^n p_i (y_i - y_{\mu,p}) \\ \\ \left(\sum_{i=1}^n p_i |x_i - c|^\alpha\right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^n p_i |y_i - y_{\mu,p}|^\beta\right)^{\frac{1}{\beta}}, \\ \hfill \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum_{i=1}^n p_i |x_i - c| \max\limits_{i=\overline{1,n}} |y_i - y_{\mu,p}| \end{cases}$$

$$= \begin{cases} \|\bar{x} - \bar{c}\|_\infty \|\bar{y} - \bar{y}_{\mu,p}\|_{\bar{\mathbf{p}},1}; \\ \\ \|\bar{x} - \bar{c}\|_{\bar{\mathbf{p}},\alpha} \|\bar{y} - \bar{y}_{\mu,p}\|_{\bar{\mathbf{p}},\beta}, \quad \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \\ \|\bar{x} - \bar{c}\|_{\bar{\mathbf{p}},1} \|\bar{y} - \bar{y}_{\mu,p}\|_\infty. \end{cases}$$

Taking the inf over $c \in \mathbb{R}$ in (4.113), we deduce the second inequality in (4.111).

Since

$$\inf_{c\in\mathbb{R}} \|\bar{x} - \bar{c}\|_{\bar{\mathbf{p}},\alpha} \leq \begin{cases} \|\bar{x}\|_{\bar{\mathbf{p}},\alpha}, \\ \\ \|\bar{x} - \bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},\alpha} \end{cases} \text{for any } \alpha \in [1, \infty]$$

the last part of (4.110) is also proved. ■

For $\bar{\mathbf{p}}$ and $\bar{\mathbf{x}}$ as above, define

$$T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}) := \sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i\right)^2.$$

The following corollary holds [20].

Corollary 190 *With the above assumptions we have*

$$0 \leq |T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}})| \tag{4.114}$$

$$\leq \begin{cases} \|\bar{x} - \bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},1} \cdot \inf\limits_{c\in\mathbb{R}} \|\bar{x} - \bar{c}\|_\infty; \\ \\ \|\bar{x} - \bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},\beta} \cdot \inf\limits_{c\in\mathbb{R}} \|\bar{x} - \bar{c}\|_{\bar{\mathbf{p}},\alpha}, \ \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \\ \|\bar{x} - \bar{x}_{\mu,p}\|_\infty \cdot \inf\limits_{c\in\mathbb{R}} \|\bar{x} - \bar{c}\|_{\bar{\mathbf{p}},1}; \end{cases}$$

$$\leq \begin{cases} \|\bar{x}-\bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},1}\min\left\{\|\bar{x}\|_{\infty},\|\bar{x}-\bar{x}_{\mu,p}\|_{\infty}\right\}; \\ \\ \|\bar{x}-\bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},\beta}\min\left\{\|\bar{x}\|_{\bar{\mathbf{p}},\alpha},\|\bar{x}-\bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},\alpha}\right\}, \\ \qquad\qquad\qquad\qquad \alpha>1, \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \|\bar{x}-\bar{x}_{\mu,p}\|_{\infty}\cdot\min\left\{\|\bar{x}\|_{\bar{\mathbf{p}},1},\|\bar{x}-\bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},1}\right\}. \end{cases}$$

Remark 191 *If $p_i := \frac{1}{n}$, $i = 1, \dots, n$, then from Theorem 189 and Corollary 190 we recapture the results in [22].*

The following counterpart of the (CBS) −inequality holds [20].

Theorem 192 *Let $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ be two sequences of real numbers with $a_i \neq 0$, $i \in \{1, \dots, n\}$. Then one has the inequality*

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \qquad (4.115) \\ &\leq \inf_{c\in\mathbb{R}}\left[\max_{i=\overline{1,n}}\left|\frac{b_i}{a_i}-c\right|\right]\sum_{i=1}^{n}\left[|a_i|\left|\sum_{k=1}^{n}a_k\begin{vmatrix} a_k & a_i \\ b_k & b_i\end{vmatrix}\right|\right] \\ &\leq \sum_{i=1}^{n}\left[|a_i|\left|\sum_{k=1}^{n}a_k\begin{vmatrix} a_k & a_i \\ b_k & b_i\end{vmatrix}\right|\right]\times\begin{cases}\max_{i=\overline{1,n}}\left|\frac{b_i}{a_i}\right| \\ \\ \max_{i=\overline{1,n}}\left|\frac{b_i}{a_i}-\frac{\sum_{k=1}^{n}a_k b_k}{\sum_{k=1}^{n}a_k^2}\right|.\end{cases} \end{aligned}$$

Proof. By Corollary 190, we may state that

$$\begin{aligned} 0 &\leq T_n(\bar{\mathbf{p}};\bar{\mathbf{x}}) \leq \|\bar{x}-\bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},1}\cdot\inf_{c\in\mathbb{R}}\|\bar{x}-\bar{c}\|_{\infty} \qquad (4.116) \\ &\leq \|\bar{x}-\bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},1}\times\begin{cases}\|\bar{x}\|_{\infty}, \\ \\ \|\bar{x}-\bar{x}_{\mu,p}\|_{\infty}.\end{cases} \end{aligned}$$

For the choices

$$p_i = \frac{a_i^2}{\sum_{k=1}^{n} a_k^2}, \quad x_i = \frac{b_i}{a_i}, \quad i = 1, \dots, n;$$

we get

$$T_n(\bar{\mathbf{p}};\bar{\mathbf{x}}) = \frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2}{\left(\sum_{k=1}^n a_k^2\right)^2},$$

$$\begin{aligned}
\|\bar{x}-\bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},1} &= \sum_{i=1}^n p_i \left| x_i - \sum_{j=1}^n p_j x_j \right| \\
&= \frac{1}{\sum_{k=1}^n a_k^2} \sum_{i=1}^n a_i^2 \left| \frac{b_i}{a_i} - \frac{1}{\sum_{k=1}^n a_k^2} \sum_{j=1}^n a_j b_j \right| \\
&= \frac{1}{\left(\sum_{k=1}^n a_k^2\right)^2} \sum_{i=1}^n \left| a_i b_i \sum_{k=1}^n a_k^2 - a_i^2 \sum_{j=1}^n a_j b_j \right| \\
&= \frac{1}{\left(\sum_{k=1}^n a_k^2\right)^2} \sum_{i=1}^n |a_i| \left| \sum_{k=1}^n a_k \left(a_k b_i - a_i b_k\right) \right| \\
&= \frac{1}{\left(\sum_{k=1}^n a_k^2\right)^2} \sum_{i=1}^n |a_i| \left| \sum_{k=1}^n a_k \begin{vmatrix} a_k & a_i \\ b_k & b_i \end{vmatrix} \right|,
\end{aligned}$$

$$\|\bar{x}-\bar{c}\|_\infty = \max_{i=\overline{1,n}} \left| \frac{b_i}{a_i} - c \right|, \quad \|\bar{x}\|_\infty = \max_{i=\overline{1,n}} \left| \frac{b_i}{a_i} \right|$$

and

$$\|\bar{x}-\bar{x}_{\mu,p}\|_\infty = \max_{i=\overline{1,n}} \left| \frac{b_i}{a_i} - \frac{\sum_{j=1}^n a_j b_j}{\sum_{k=1}^n a_k^2} \right|.$$

Utilising the inequality (4.116) we deduce the desired result (4.115). ■

The following result also holds [20].

Theorem 193 *With the assumption in Theorem 192 and if $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, then we have the inequality:*

$$\begin{aligned}
0 &\le \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2 \qquad (4.117) \\
&\le \left(\sum_{i=1}^n \left[|a_i|^{2-\beta} \left| \sum_{k=1}^n a_k \begin{vmatrix} a_k & a_i \\ b_k & b_i \end{vmatrix} \right|^\beta \right] \right)^{\frac{1}{\beta}} \inf_{c\in\mathbb{R}} \left(\sum_{i=1}^n |a_i|^{2-\alpha} |b_i - c a_i|^\alpha \right)^{\frac{1}{\alpha}}
\end{aligned}$$

$$\leq \left(\sum_{i=1}^{n} \left[|a_i|^{2-\beta} \left| \sum_{k=1}^{n} a_k \left| \begin{array}{cc} a_k & a_i \\ b_k & b_i \end{array} \right| \right|^{\beta} \right] \right)^{\frac{1}{\beta}}$$

$$\times \begin{cases} \left(\sum_{i=1}^{n} |a_i|^{2-\alpha} |b_i|^{\alpha} \right)^{\frac{1}{\alpha}} \\ \\ \frac{1}{\sum_{k=1}^{n} a_k^2} \left(\sum_{i=1}^{n} |a_i|^{2-\alpha} \left| \sum_{k=1}^{n} a_k \left| \begin{array}{cc} a_k & a_i \\ b_k & b_i \end{array} \right| \right|^{\alpha} \right)^{\frac{1}{\alpha}} . \end{cases}$$

Proof. By Corollary 190, we may state that

$$0 \leq T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}) \leq \|\bar{x} - \bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},\beta} \cdot \inf_{c \in \mathbb{R}} \|\bar{x} - \bar{c}\|_{\bar{\mathbf{p}},\alpha} \tag{4.118}$$

$$\leq \|\bar{x} - \bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},\beta} \times \begin{cases} \|\bar{x}\|_{\bar{\mathbf{p}},\alpha} , \\ \\ \|\bar{x} - \bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},\alpha} , \end{cases}$$

for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

For the choices

$$p_i = \frac{a_i^2}{\sum_{k=1}^{n} a_k^2}, \quad x_i = \frac{b_i}{a_i}, \quad i = 1, \dots, n;$$

we get

$$\|\bar{x} - \bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},\beta} = \left(\sum_{i=1}^{n} p_i \left| x_i - \sum_{j=1}^{n} p_j x_j \right|^{\beta} \right)^{\frac{1}{\beta}}$$

$$= \left(\sum_{i=1}^{n} \frac{a_i^2}{\sum_{k=1}^{n} a_k^2} \left| \frac{b_i \sum_{k=1}^{n} a_k^2 - a_i \sum_{j=1}^{n} a_j b_j}{a_i \sum_{k=1}^{n} a_k^2} \right|^{\beta} \right)^{\frac{1}{\beta}}$$

$$= \frac{1}{\left(\sum_{k=1}^{n} a_k^2\right)^{1+\frac{1}{\beta}}} \left(\sum_{i=1}^{n} |a_i|^{2-\beta} \left| \sum_{k=1}^{n} a_k \left| \begin{array}{cc} a_k & a_i \\ b_k & b_i \end{array} \right| \right|^{\beta} \right)^{\frac{1}{\beta}},$$

$$\|\bar{x} - \bar{c}\|_{\bar{\mathbf{p}},\alpha} = \left(\sum_{i=1}^{n} p_i |x_i - c|^{\alpha} \right)^{\frac{1}{\alpha}} = \frac{1}{\left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{\alpha}}} \left(\sum_{i=1}^{n} |a_i|^{2-\alpha} |b_i - c a_i|^{\alpha} \right)^{\frac{1}{\alpha}},$$

$$\|\bar{x}\|_{\bar{\mathbf{p}},\alpha} = \frac{1}{\left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{\alpha}}} \left(\sum_{i=1}^{n} |a_i|^{2-\alpha} |b_i|^{\alpha}\right)^{\frac{1}{\alpha}}$$

and

$$\|\bar{x} - \bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},\alpha} = \frac{1}{\left(\sum_{k=1}^{n} a_k^2\right)^{1+\frac{1}{\alpha}}} \left(\sum_{i=1}^{n} |a_i|^{2-\alpha} \left|\sum_{k=1}^{n} a_k \begin{vmatrix} a_k & a_i \\ b_k & b_i \end{vmatrix}\right|^{\alpha}\right)^{\frac{1}{\alpha}}.$$

Utilising the inequality (4.118), we deduce the desired result (4.117). ■

Finally, the following result also holds [20].

Theorem 194 *With the assumptions in Theorem 192 we have the following counterpart of the (CBS) −inequality:*

$$\begin{aligned} 0 &\le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 &(4.119)\\ &\le \max_{i=\overline{1,n}} \left|\frac{b_i}{a_i}\sum_{k=1}^{n} a_k^2 - \sum_{j=1}^{n} a_j b_j\right| \inf_{c\in\mathbb{R}} \left[\sum_{i=1}^{n} |a_i|\,|b_i - c a_i|\right] \\ &\le \max_{i=\overline{1,n}} \left|\frac{b_i}{a_i}\sum_{k=1}^{n} a_k^2 - \sum_{j=1}^{n} a_j b_j\right| \\ &\times \begin{cases} \sum_{i=1}^{n} |a_i b_i| \\ \\ \frac{1}{\sum_{k=1}^{n} a_k^2} \sum_{i=1}^{n} |a_i| \left|\sum_{k=1}^{n} a_k \begin{vmatrix} a_k & a_i \\ b_k & b_i \end{vmatrix}\right|. \end{cases} \end{aligned}$$

Proof. By Corollary 190, we may state that

$$\begin{aligned} 0 &\le T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}) \le \|\bar{x} - \bar{x}_{\mu,p}\|_{\infty} \cdot \inf_{c\in\mathbb{R}} \|\bar{x} - \bar{c}\|_{\bar{\mathbf{p}},1} &(4.120)\\ &\le \|\bar{x} - \bar{x}_{\mu,p}\|_{\infty} \begin{cases} \|\bar{x}\|_{\bar{\mathbf{p}},1}, \\ \\ \|\bar{x} - \bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},1}. \end{cases} \end{aligned}$$

For the choices

$$p_i = \frac{a_i^2}{\sum_{k=1}^{n} a_k^2}, \quad x_i = \frac{b_i}{a_i}, \quad i = 1, \dots, n;$$

we get

$$\begin{aligned}\|\bar{x}-\bar{x}_{\mu,p}\|_{\infty} &= \max_{i=\overline{1,n}}\left|x_i-\sum_{j=1}^{n}p_jx_j\right| = \max_{i=\overline{1,n}}\left|\frac{b_i}{a_i}-\frac{\sum_{j=1}^{n}a_jb_j}{\sum_{k=1}^{n}a_k^2}\right| \\ &= \frac{1}{\sum_{k=1}^{n}a_k^2}\max_{i=\overline{1,n}}\left|\frac{b_i}{a_i}\sum_{k=1}^{n}a_k^2-\sum_{j=1}^{n}a_jb_j\right|,\end{aligned}$$

$$\begin{aligned}\|\bar{x}-\bar{c}\|_{\bar{\mathbf{p}},1} &= \sum_{i=1}^{n}p_i\left|x_i-c\right| = \sum_{i=1}^{n}\frac{a_i^2}{\sum_{k=1}^{n}a_k^2}\left|\frac{b_i}{a_i}-c\right| \\ &= \frac{1}{\sum_{k=1}^{n}a_k^2}\sum_{i=1}^{n}\left|a_i\right|\left|b_i-ca_i\right|,\end{aligned}$$

$$\|\bar{x}\|_{\bar{\mathbf{p}},1} = \sum_{i=1}^{n}p_i\left|x_i\right| = \sum_{i=1}^{n}\frac{a_i^2}{\sum_{k=1}^{n}a_k^2}\left|\frac{b_i}{a_i}\right| = \frac{1}{\sum_{k=1}^{n}a_k^2}\sum_{i=1}^{n}\left|a_ib_i\right|$$

and

$$\|\bar{x}-\bar{x}_{\mu,p}\|_{\bar{\mathbf{p}},1} = \frac{1}{\left(\sum_{k=1}^{n}a_k^2\right)^2}\sum_{i=1}^{n}\left|a_i\right|\left|\sum_{k=1}^{n}a_k\left|\begin{array}{cc}a_k & a_i\\ b_k & b_i\end{array}\right|\right|.$$

Utilising the inequality (4.120) we deduce (4.119). ■

4.16 Another Counterpart via a Grüss Type Result

The following Grüss type inequality has been obtained in [21].

Lemma 195 *Let $\bar{\mathbf{a}}=(a_1,\dots,a_n)$, $\bar{\mathbf{b}}=(b_1,\dots,b_n)$ be two sequences of real numbers and assume that there are $\gamma,\Gamma\in\mathbb{R}$ such that*

$$-\infty<\gamma\le a_i\le\Gamma<\infty \text{ for each } i\in\{1,\dots,n\}. \tag{4.121}$$

Then for any $\bar{\mathbf{p}}=(p_1,\dots,p_n)$ a nonnegative sequence with the property that $\sum_{i=1}^{n}p_i=1$, one has the inequality

$$\left|\sum_{i=1}^{n}p_ia_ib_i-\sum_{i=1}^{n}p_ia_i\sum_{i=1}^{n}p_ib_i\right|\le\frac{1}{2}\left(\Gamma-\gamma\right)\sum_{i=1}^{n}p_i\left|b_i-\sum_{k=1}^{n}p_kb_k\right|. \tag{4.122}$$

The constant $\frac{1}{2}$ *is sharp in the sense that it cannot be replaced by a smaller constant.*

Proof. We will give here a simpler direct proof based on Sonin's identity. A simple calculation shows that:

$$\sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i = \sum_{i=1}^{n} p_i \left(a_i - \frac{\gamma + \Gamma}{2} \right) \left(b_i - \sum_{k=1}^{n} p_k b_k \right). \quad (4.123)$$

By (4.121) we have

$$\left| a_i - \frac{\gamma + \Gamma}{2} \right| \leq \frac{\Gamma - \gamma}{2} \text{ for all } i \in \{1, \ldots, n\}$$

and thus, by (4.123), on taking the modulus, we get

$$\begin{aligned} \left| \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \right| &\leq \sum_{i=1}^{n} p_i \left| a_i - \frac{\gamma + \Gamma}{2} \right| \left| b_i - \sum_{k=1}^{n} p_k b_k \right| \\ &\leq \frac{1}{2} (\Gamma - \gamma) \sum_{i=1}^{n} p_i \left| b_i - \sum_{k=1}^{n} p_k b_k \right|. \end{aligned}$$

To prove the sharpness of the constant $\frac{1}{2}$, let us assume that (4.122) holds with a constant $c > 0$, i.e.,

$$\left| \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \right| \leq c (\Gamma - \gamma) \sum_{i=1}^{n} p_i \left| b_i - \sum_{k=1}^{n} p_k b_k \right|. \quad (4.124)$$

provided a_i satisfies (4.121).

If we choose $n = 2$ in (4.124) and take into account that

$$\sum_{i=1}^{2} p_i a_i b_i - \sum_{i=1}^{2} p_i a_i \sum_{i=1}^{2} p_i b_i = p_1 p_2 (a_1 - a_2)(b_1 - b_2)$$

provided $p_1 + p_2 = 1$, $p_1, p_2 \in [0, 1]$, and since

$$\begin{aligned}\sum_{i=1}^{2} p_i \left| b_i - \sum_{k=1}^{2} p_k b_k \right| &= p_1 \left| (p_1 + p_2) b_1 - p_1 b_1 - p_2 b_2 \right| \\ &\quad + p_2 \left| (p_1 + p_2) b_2 - p_1 b_1 - p_2 b_2 \right| \\ &= 2 p_1 p_2 \left| b_1 - b_2 \right|\end{aligned}$$

we deduce by (4.124)

$$p_1 p_2 \left| a_1 - a_2 \right| \left| b_1 - b_2 \right| \leq 2c (\Gamma - \gamma) \left| b_1 - b_2 \right| p_1 p_2. \tag{4.125}$$

If we assume that $p_1, p_2 \neq 0$, $b_1 \neq b_2$ and $a_1 = \Gamma$, $a_2 = \gamma$, then by (4.125) we deduce $c \geq \frac{1}{2}$, which proves the sharpness of the constant $\frac{1}{2}$. ■

The following corollary is a natural consequence of the above lemma.

Corollary 196 *Assume that* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ *satisfies the assumption (4.121) and* $\bar{\mathbf{p}}$ *is a probability sequence. Then*

$$0 \leq \sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i \right)^2 \leq \frac{1}{2} (\Gamma - \gamma) \sum_{i=1}^{n} p_i \left| a_i - \sum_{k=1}^{n} p_k a_k \right|. \tag{4.126}$$

The constant $\frac{1}{2}$ *is best possible in the sense mentioned above.*

The following counterpart of the (CBS) −inequality may be stated.

Theorem 197 *Assume that* $\bar{\mathbf{x}} = (x_1, \dots, x_n)$ *and* $\bar{\mathbf{y}} = (y_1, \dots, y_n)$ *are sequences of real numbers with* $y_i \neq 0$ $(i = 1, \dots, n)$. *If there exists the real numbers* m, M *such that*

$$m \leq \frac{x_i}{y_i} \leq M \text{ for each } i \in \{1, \dots, n\}, \tag{4.127}$$

then we have the inequality

$$\begin{aligned}0 &\leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} x_i y_i \right)^2 \qquad (4.128) \\ &\leq \frac{1}{2} (M - m) \sum_{i=1}^{n} |y_i| \left| \sum_{k=1}^{n} y_k \cdot \begin{vmatrix} x_i & y_i \\ x_k & y_k \end{vmatrix} \right|.\end{aligned}$$

Proof. If we choose $p_i = \frac{y_i^2}{\sum_{k=1}^n y_k^2}$, $a_i = \frac{x_i}{y_i}$ for $i = 1, \ldots, n$ and $\gamma = m$, $\Gamma = M$ in (4.126), we deduce

$$\begin{aligned}
&\frac{\sum_{i=1}^n x_i^2}{\sum_{k=1}^n y_k^2} - \left(\frac{1}{\sum_{k=1}^n y_k^2}\sum_{i=1}^n x_i y_i\right)^2 \\
&\qquad \le \frac{1}{2}(M-m)\frac{1}{\sum_{k=1}^n y_k^2}\sum_{i=1}^n y_i^2 \left|\frac{x_i}{y_i} - \frac{1}{\sum_{k=1}^n y_k^2}\sum_{k=1}^n x_k y_k\right| \\
&\qquad = \frac{1}{2}(M-m)\frac{1}{\left(\sum_{k=1}^n y_k^2\right)^2}\sum_{i=1}^n |y_i| \left|x_i \sum_{k=1}^n y_k^2 - y_i \sum_{k=1}^n x_k y_k\right| \\
&\qquad = \frac{1}{2}(M-m)\frac{1}{\left(\sum_{k=1}^n y_k^2\right)^2}\sum_{i=1}^n |y_i| \left|\sum_{k=1}^n y_k \cdot \begin{vmatrix} x_i & y_i \\ x_k & y_k \end{vmatrix}\right|.
\end{aligned}$$

giving the desired inequality (4.128). ■

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Chapter 5

Related Inequalities

5.1 Ostrowski's Inequality for Real Sequences

In 1951, A.M. Ostrowski [2, p. 289] gave the following result related to the (CBS) −inequality for real sequences (see also [1, p. 92]).

Theorem 198 *Let* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$ *and* $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ *be two non-proportional sequences of real numbers. Let* $\bar{\mathbf{x}} = (x_1, \ldots, x_n)$ *be a sequence of real numbers such that*

$$\sum_{i=1}^{n} a_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} b_i x_i = 1. \tag{5.1}$$

Then

$$\sum_{i=1}^{n} x_i^2 \geq \frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2} \tag{5.2}$$

with equality if and only if

$$x_k = \frac{b_k \sum_{i=1}^{n} a_i^2 - a_k \sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2} \tag{5.3}$$

for any $k \in \{1, \ldots, n\}$.

Proof. We shall follow the proof in [1, p. 93 – p. 94].
Let

$$A = \sum_{i=1}^{n} a_i^2, \quad B = \sum_{i=1}^{n} b_i^2, \quad C = \sum_{i=1}^{n} a_i b_i \tag{5.4}$$

and

$$y_i = \frac{Ab_i - Ca_i}{AB - C^2} \text{ for any } i \in \{1, \ldots, n\}. \tag{5.5}$$

It is easy to see that the sequence $\bar{\mathbf{y}} = (y_1, \ldots, y_n)$ as defined by (5.5) satisfies (5.1).

Any sequence $\bar{\mathbf{x}} = (x_1, \ldots, x_n)$ that satisfies (5.1) fulfills the equality

$$\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} x_i \cdot \frac{(Ab_i - Ca_i)}{AB - C^2} = \frac{A}{AB - C^2};$$

so, in particular

$$\sum_{i=1}^{n} y_i^2 = \frac{A}{AB - C^2}.$$

Any sequence $\bar{\mathbf{x}} = (x_1, \ldots, x_n)$ that satisfies (5.1) therefore satisfies

$$\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} (x_i - y_i)^2 \geq 0, \tag{5.6}$$

and thus

$$\sum_{i=1}^{n} x_i^2 \geq \sum_{i=1}^{n} y_i^2 = \frac{A}{AB - C^2}$$

and the inequality (5.2) is proved.

From (5.6) it follows that equality holds in (5.1) iff $x_i = y_i$ for each $i \in \{1, \ldots, n\}$, and the theorem is completely proved. ■

5.2 Ostrowski's Inequality for Complex Sequences

The following result that points out a natural generalisation of Ostrowski's inequality for complex numbers holds [3].

Theorem 199 *Let* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ *and* $\bar{\mathbf{x}} = (x_1, \ldots, x_n)$ *be sequences of complex numbers. If* $\bar{\mathbf{a}}$ *and* $\bar{\bar{\mathbf{b}}}$, *where* $\bar{\bar{\mathbf{b}}} = (\bar{b}_1, \ldots, \bar{b}_n)$, *are* *not proportional and*

$$\sum_{i=1}^{n} x_i \bar{a}_i = 0; \tag{5.7}$$

$$\left| \sum_{i=1}^{n} x_i \bar{b}_i \right| = 1, \tag{5.8}$$

then one has the inequality

$$\sum_{i=1}^{n} |x_i|^2 \geq \frac{\sum_{i=1}^{n} |a_i|^2}{\sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \left| \sum_{i=1}^{n} a_i \bar{b}_i \right|^2} \tag{5.9}$$

with equality iff

$$x_i = \mu \left[b_i - \frac{\sum_{k=1}^{n} b_k \bar{a}_k}{\sum_{k=1}^{n} |a_k|^2} \cdot a_i \right], \quad i \in \{1, \ldots, n\} \tag{5.10}$$

and $\mu \in \mathbb{C}$ *with*

$$|\mu| = \frac{\sum_{k=1}^{n} |a_k|^2}{\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 - \left| \sum_{k=1}^{n} a_k \bar{b}_k \right|^2}. \tag{5.11}$$

Proof. Recall the (CBS) −inequality for complex sequences

$$\sum_{k=1}^{n} |u_k|^2 \sum_{k=1}^{n} |v_k|^2 \geq \left| \sum_{k=1}^{n} u_k \bar{v}_k \right|^2 \tag{5.12}$$

with equality iff there is a complex number $\alpha \in \mathbb{C}$ such that

$$u_k = \alpha v_k, \quad k = 1, \ldots, n. \tag{5.13}$$

If we apply (5.12) for

$$u_k = z_k - \frac{\sum_{i=1}^{n} z_i \bar{c}_i}{\sum_{i=1}^{n} |c_i|^2} \cdot c_k,$$
$$v_k = d_k - \frac{\sum_{i=1}^{n} d_i \bar{c}_i}{\sum_{i=1}^{n} |c_i|^2} \cdot c_k, \quad \text{where } \bar{\mathbf{c}} \neq 0 \text{ and } \bar{\mathbf{c}}, \bar{\mathbf{d}}, \bar{\mathbf{z}} \in \mathbb{C}^n,$$

we have

$$\begin{aligned}\sum_{k=1}^{n}\left|z_k-\frac{\sum_{i=1}^{n}z_i\bar{c}_i}{\sum_{i=1}^{n}|c_i|^2}\cdot c_k\right|^2\sum_{k=1}^{n}\left|d_k-\frac{\sum_{i=1}^{n}d_i\bar{c}_i}{\sum_{i=1}^{n}|c_i|^2}\cdot c_k\right|^2 \\ \geq\left|\sum_{k=1}^{n}\left(z_k-\frac{\sum_{i=1}^{n}z_i\bar{c}_i}{\sum_{i=1}^{n}|c_i|^2}\cdot c_k\right)\left(\overline{d_k-\frac{\sum_{i=1}^{n}d_i\bar{c}_i}{\sum_{i=1}^{n}|c_i|^2}\cdot c_k}\right)\right|^2 \quad (5.14)\end{aligned}$$

with equality iff there is a $\beta\in\mathbb{C}$ such that

$$z_k=\frac{\sum_{i=1}^{n}z_i\bar{c}_i}{\sum_{i=1}^{n}|c_i|^2}\cdot c_k+\beta\left(d_k-\frac{\sum_{i=1}^{n}d_i\bar{c}_i}{\sum_{i=1}^{n}|c_i|^2}\cdot c_k\right). \quad (5.15)$$

Since a simple calculation shows that

$$\sum_{k=1}^{n}\left|z_k-\frac{\sum_{i=1}^{n}z_i\bar{c}_i}{\sum_{i=1}^{n}|c_i|^2}\cdot c_k\right|^2=\frac{\sum_{k=1}^{n}|z_k|^2\sum_{k=1}^{n}|c_k|^2-\left|\sum_{k=1}^{n}z_k\bar{c}_k\right|^2}{\left(\sum_{k=1}^{n}|c_k|^2\right)^2},$$

$$\sum_{k=1}^{n}\left|d_k-\frac{\sum_{i=1}^{n}d_i\bar{c}_i}{\sum_{i=1}^{n}|c_i|^2}\cdot c_k\right|^2=\frac{\sum_{k=1}^{n}|d_k|^2\sum_{k=1}^{n}|c_k|^2-\left|\sum_{k=1}^{n}d_k\bar{c}_k\right|^2}{\left(\sum_{k=1}^{n}|c_k|^2\right)^2}$$

and

$$\begin{aligned}\sum_{k=1}^{n}\left(z_k-\frac{\sum_{i=1}^{n}z_i\bar{c}_i}{\sum_{i=1}^{n}|c_i|^2}\cdot c_k\right)\left(\overline{d_k-\frac{\sum_{i=1}^{n}d_i\bar{c}_i}{\sum_{i=1}^{n}|c_i|^2}\cdot c_k}\right) \\ =\frac{\sum_{k=1}^{n}z_k\bar{d}_k\cdot\sum_{k=1}^{n}|c_k|^2-\sum_{k=1}^{n}z_k\bar{c}_k\cdot\sum_{k=1}^{n}c_k\bar{d}_k}{\left(\sum_{i=1}^{n}|c_i|^2\right)^2}\end{aligned}$$

then by (5.12) we deduce

$$\begin{aligned}\left[\sum_{k=1}^{n}|z_k|^2\sum_{k=1}^{n}|c_k|^2-\left|\sum_{k=1}^{n}z_k\bar{c}_k\right|^2\right]\left[\sum_{k=1}^{n}|d_k|^2\sum_{k=1}^{n}|c_k|^2-\left|\sum_{k=1}^{n}d_k\bar{c}_k\right|^2\right] \\ \geq\left|\sum_{k=1}^{n}z_k\bar{d}_k\cdot\sum_{k=1}^{n}|c_k|^2-\sum_{k=1}^{n}z_k\bar{c}_k\sum_{k=1}^{n}c_k\bar{d}_k\right|^2 \quad (5.16)\end{aligned}$$

with equality iff there is a $\beta\in\mathbb{C}$ such that (5.15) holds.

If $\bar{\mathbf{a}}, \bar{\mathbf{x}}, \bar{\mathbf{b}}$ satisfy (5.7) and (5.8), then by (5.16) and (5.15) for the choices $\bar{\mathbf{z}} = \bar{\mathbf{x}}$, $\bar{\mathbf{c}} = \bar{\mathbf{a}}$ and $\bar{\mathbf{d}} = \bar{\mathbf{b}}$, we deduce (5.9) with equality iff there is a $\mu \in \mathbb{C}$ such that

$$x_k = \mu \left(b_k - \frac{\overline{\sum_{i=1}^n a_i \bar{b}_i}}{\sum_{i=1}^n |a_i|^2} \cdot a_k \right),$$

and, by (5.8),

$$\left| \mu \sum_{k=1}^n \left(b_k - \frac{\sum_{i=1}^n a_i \bar{b}_i}{\sum_{i=1}^n |a_i|^2} \cdot a_k \right) \cdot \bar{b}_k \right| = 1. \tag{5.17}$$

Since (5.17) is clearly equivalent to (5.15), the theorem is completely proved. ■

5.3 Another Ostrowski's Inequality

In his book from 1951, [2, p. 130], A.M. Ostrowski proved the following inequality as well (see also [1, p. 94]).

Theorem 200 *Let $\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{x}}$ be sequences of real numbers so that $\bar{\mathbf{a}} \neq 0$ and*

$$\sum_{k=1}^n x_k^2 = 1 \tag{5.18}$$

$$\sum_{k=1}^n a_k x_k = 0. \tag{5.19}$$

Then

$$\frac{\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_n^2 - \left(\sum_{k=1}^n a_k b_k\right)^2}{\sum_{k=1}^n a_k^2} \geq \left(\sum_{k=1}^n b_k x_k \right)^2. \tag{5.20}$$

If $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ are non-proportional, then equality holds in (5.20) iff

$$x_k = q \cdot \frac{b_k \sum_{i=1}^n a_i^2 - a_k \sum_{i=1}^n a_i b_i}{\left(\sum_{k=1}^n a_k^2\right)^{\frac{1}{2}} \left[\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2 \right]^{\frac{1}{2}}},$$
$$k \in \{1, \ldots, n\}, \; q \in \{-1, 1\}. \tag{5.21}$$

We may extend this result for sequences of complex numbers as follows [4].

Theorem 201 *Let* $\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{x}}$ *be sequences of complex numbers so that* $\bar{\mathbf{a}} \neq 0$, $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ *are not proportional, and*

$$\sum_{k=1}^{n} |x_k|^2 = 1 \tag{5.22}$$

$$\sum_{k=1}^{n} x_k \bar{a}_k = 0. \tag{5.23}$$

Then

$$\frac{\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 - \left|\sum_{k=1}^{n} a_k \bar{b}_k\right|^2}{\sum_{k=1}^{n} |a_k|^2} \geq \left|\sum_{k=1}^{n} x_k \bar{b}_k\right|^2. \tag{5.24}$$

The equality holds in (5.24) iff

$$x_k = \beta \left(b_k - \frac{\sum_{i=1}^{n} b_i \bar{a}_i}{\sum_{i=1}^{n} |a_i|^2} \cdot a_k \right), \quad k \in \{1, \ldots, n\}; \tag{5.25}$$

where $\beta \in \mathbb{C}$ *is such that*

$$|\beta| = \frac{\left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}}}{\left(\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 - \left|\sum_{k=1}^{n} a_k \bar{b}_k\right|^2\right)^{\frac{1}{2}}}. \tag{5.26}$$

Proof. In Section 5.2, we proved the following inequality:

$$\begin{aligned} &\left[\sum_{k=1}^{n} |z_k|^2 \sum_{k=1}^{n} |c_k|^2 - \left|\sum_{k=1}^{n} z_k \bar{c}_k\right|^2\right] \\ &\qquad \times \left[\sum_{k=1}^{n} |d_k|^2 \sum_{k=1}^{n} |c_k|^2 - \left|\sum_{k=1}^{n} d_k \bar{c}_k\right|^2\right] \\ &\qquad \geq \left|\sum_{k=1}^{n} z_k \bar{d}_k \cdot \sum_{k=1}^{n} |c_k|^2 - \sum_{k=1}^{n} z_k \bar{c}_k \sum_{k=1}^{n} c_k \bar{d}_k\right|^2 \end{aligned} \tag{5.27}$$

for any $\bar{\mathbf{z}}, \bar{\mathbf{c}}, \bar{\mathbf{d}}$ sequences of complex numbers, with equality iff there is a $\beta \in \mathbb{C}$ such that

$$z_k = \frac{\sum_{i=1}^{n} z_i \bar{c}_i}{\sum_{i=1}^{n} |c_i|^2} \cdot c_k + \beta \left(d_k - \frac{\sum_{i=1}^{n} d_i \bar{c}_i}{\sum_{i=1}^{n} |c_i|^2} \cdot c_k \right) \tag{5.28}$$

for each $k \in \{1, \ldots, n\}$.

If in (5.27) we choose $\bar{\mathbf{z}} = \bar{\mathbf{x}}$, $\bar{\mathbf{c}} = \bar{\mathbf{a}}$ and $\bar{\mathbf{d}} = \bar{\mathbf{b}}$ and take into consideration that (5.22) and (5.23) hold, then we get

$$\sum_{k=1}^{n} |x_k|^2 \sum_{k=1}^{n} |a_k|^2 \left[\sum_{k=1}^{n} |b_k|^2 \sum_{k=1}^{n} |a_k|^2 - \left| \sum_{k=1}^{n} b_k \bar{a}_k \right|^2 \right] \geq \left| \sum_{k=1}^{n} x_k \bar{b}_k \right|^2 \left(\sum_{k=1}^{n} |a_k|^2 \right)^2$$

which is clearly equivalent to (5.24).

By (5.28) the equality holds in (5.24) iff

$$x_k = \beta \left(b_k - \frac{\sum_{i=1}^{n} b_i \bar{a}_i}{\sum_{i=1}^{n} |a_i|^2} \cdot a_k \right), \quad k \in \{1, \ldots, n\}.$$

Since $\bar{\mathbf{x}}$ should satisfy (5.22), we get

$$\begin{aligned} 1 = \sum_{k=1}^{n} |x_k|^2 &= |\beta|^2 \sum_{k=1}^{n} \left| b_k - \frac{\sum_{i=1}^{n} b_i \bar{a}_i}{\sum_{i=1}^{n} |a_i|^2} \cdot a_k \right|^2 \\ &= |\beta|^2 \left[\sum_{k=1}^{n} |b_k|^2 - \frac{\left|\sum_{k=1}^{n} b_k \bar{a}_k\right|^2}{\sum_{k=1}^{n} |a_k|^2} \right] \end{aligned}$$

from where we deduce that β satisfies (5.26). ■

5.4 Fan and Todd Inequalities

In 1955, K. Fan and J. Todd [5] proved the following inequality (see also [1, p. 94]).

Theorem 202 *Let $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$ and $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ be two sequences of real numbers such that $a_i b_j \neq a_j b_i$ for $i \neq j$. Then*

$$\frac{\sum_{i=1}^{n} a_i^2}{\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2} \leq \binom{n}{2} \sum_{i=1}^{n} \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{a_j}{a_j b_i - a_i b_j} \right)^2. \quad (5.29)$$

Proof. We shall follow the proof in [1, p. 94 – p. 95].
Define

$$x_i := \binom{n}{2}^{-1} \sum_{j\neq i} \frac{a_j}{a_j b_i - a_i b_j} \qquad (1 \leq i \leq n).$$

The terms in the sum on the right-hand side

$$\sum_{i=1}^{n} x_i a_i = \binom{n}{2}^{-1} \sum_{i=1}^{n} \left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{a_i a_j}{a_j b_i - a_i b_j} \right)$$

can be grouped in pairs of the form

$$\binom{n}{2}^{-1} \left(\frac{a_i a_j}{a_j b_i - a_i b_j} + \frac{a_j a_i}{a_i b_j - a_j b_i} \right) \quad (i \neq j)$$

and the sum of each such pair vanishes.

Hence, we deduce

$$\sum_{i=1}^{n} a_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} b_i x_i = 1.$$

Applying Ostrowski's inequality (see Section 5.1) we deduce the desired result (5.29). ■

A weighted version of the result is also due to K. Fan and J. Todd [5] (see also [1, p. 95]). We may state the result as follows.

Theorem 203 *Let p_{ij} $(i, j \in \{1, \ldots, n\},\ i \neq j)$ be real numbers such that*

$$p_{ij} = p_{ji}, \quad \text{for any } \ i, j \in \{1, \ldots, n\} \quad \text{with } \ i \neq j. \tag{5.30}$$

Denote $P := \sum_{1 \leq i < j \leq n} p_{ij}$ and assume that $P \neq 0$. Then for any two sequences of real numbers $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$ and $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ satisfying $a_i b_j \neq a_j b_i$ $(i \neq j)$, we have

$$\frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2} \leq \frac{1}{P^2} \sum_{i=1}^{n} \left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{p_{ij} a_j}{a_j b_i - a_i b_j} \right)^2. \tag{5.31}$$

5.5 Some Results for Asynchronous Sequences

If $S(\mathbb{R})$ is the linear space of real sequences, $S_+(\mathbb{R})$ is the subset of nonnegative sequences and $\mathcal{P}_f(\mathbb{N})$ denotes the set of finite parts of $\mathbb{N}$, then for the functional $T : \mathcal{P}_f(\mathbb{N}) \times S_+(\mathbb{R}) \times S^2(\mathbb{R}) \to \mathbb{R}$,

$$T(I,p,a,b) := \left(\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2\right)^{\frac{1}{2}} - \left|\sum_{i\in I} p_i a_i b_i\right| \tag{5.32}$$

we may state the following result [6, Theorem 3].

Theorem 204 *If $|\bar{\mathbf{a}}| = (|a_i|)_{i\in\mathbb{N}}$ and $|\bar{\mathbf{b}}| = (|b_i|)_{i\in\mathbb{N}}$ are asynchronous, i.e., $(|a_i| - |a_j|)(|b_i| - |b_j|) \le 0$ for all $i, j \in \mathbb{N}$, then*

$$T(I,p,a,b) \ge \frac{\sum_{i\in I} p_i |a_i| \sum_{i\in I} p_i |b_i|}{\sum_{i\in I} p_i} - \sum_{i\in I} p_i |a_i b_i| \ge 0. \tag{5.33}$$

Proof. We shall follow the proof in [6].
Consider the inequalities

$$\left(\sum_{i\in I} p_i \sum_{i\in I} p_i a_i^2\right)^{\frac{1}{2}} \ge \sum_{i\in I} p_i |a_i|$$

and

$$\left(\sum_{i\in I} p_i \sum_{i\in I} p_i b_i^2\right)^{\frac{1}{2}} \ge \sum_{i\in I} p_i |b_i|$$

that by multiplication give

$$\left(\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2\right)^{\frac{1}{2}} \ge \frac{\sum_{i\in I} p_i |a_i| \sum_{i\in I} p_i |b_i|}{\sum_{i\in I} p_i}.$$

Now, by the definition of T and by Čebyšev's inequality for asynchronous sequences, we have

$$\begin{aligned} T(I,p,a,b) &\ge \frac{\sum_{i\in I} p_i |a_i| \sum_{i\in I} p_i |b_i|}{\sum_{i\in I} p_i} - \left|\sum_{i\in I} p_i a_i b_i\right| \\ &\ge \frac{\sum_{i\in I} p_i |a_i| \sum_{i\in I} p_i |b_i|}{\sum_{i\in I} p_i} - \sum_{i\in I} p_i |a_i| |b_i| \\ &\ge 0 \end{aligned}$$

and the theorem is proved. ■

The following result also holds [6, Theorem 4].

Theorem 205 *If* $|\bar{\mathbf{a}}|$ *and* $|\bar{\mathbf{b}}|$ *are synchronous, i.e.,* $(|a_i| - |a_j|)(|b_i| - |b_j|) \geq 0$ *for all* $i, j \in \mathbb{N}$, *then one has the inequality*

$$0 \leq T(I, p, a, b) \leq T(I, p, ab, \mathbf{1}), \tag{5.34}$$

where $\mathbf{1} = (e_i)_{i\in\mathbb{N}}$, $e_i = 1,\ i \in \mathbb{N}$.

Proof. We have, by Čebyšev's inequality for the synchronous sequences $\bar{\mathbf{a}}^2 = (a_i^2)_{i\in\mathbb{N}}$ and $\bar{\mathbf{b}}^2 = (b_i^2)_{i\in\mathbb{N}}$, that

$$\begin{aligned} T(I, p, a, b) &= \left(\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2\right)^{\frac{1}{2}} - \left|\sum_{i\in I} p_i a_i b_i\right| \\ &\leq \left(\sum_{i\in I} p_i a_i^2 b_i^2 \sum_{i\in I} p_i\right)^{\frac{1}{2}} - \left|\sum_{i\in I} p_i a_i b_i\right| \\ &= T(I, p, ab, \mathbf{1}) \end{aligned}$$

and the theorem is proved. ■

5.6 An Inequality via $A - G - H$ Mean Inequality

The following result holds [6, Theorem 5].

Theorem 206 *Let* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *be sequences of positive real numbers. Define*

$$\Delta_i = \begin{vmatrix} a_i^2 & b_i^2 \\ \sum_{i\in I} a_i^2 & \sum_{i\in I} b_i^2 \end{vmatrix} \tag{5.35}$$

where $i \in I$ *and* I *is a finite part of* $\mathbb{N}$. *Then one has the inequality*

$$\begin{aligned} \frac{\left(\sum_{i\in I} a_i b_i\right)^2}{\sum_{i\in I} a_i^2 \sum_{i\in I} b_i^2} &\geq \left[\prod_{i\in I} \left(\frac{a_i}{b_i}\right)^{\Delta_i}\right]^{\frac{1}{\sum_{i\in I} a_i^2 \sum_{i\in I} b_i^2}} \\ &\geq \frac{\sum_{i\in I} a_i^2 \sum_{i\in I} b_i^2}{\sum_{i\in I} \frac{a_i^3}{b_i} \sum_{i\in I} \frac{b_i^3}{a_i}}. \end{aligned} \tag{5.36}$$

The equality holds in all the inequalities from (5.36) iff there exists a positive number $k>0$ such that $a_i = kb_i$ for all $i \in I$.

Proof. We shall follow the proof in [6].

We will use the $AGH-$inequality

$$\frac{1}{P_I}\sum_{i\in I} p_i x_i \geq \left(\prod_{i\in I} x_i^{p_i}\right)^{\frac{1}{P_I}} \geq \frac{P_I}{\sum_{i\in I}\frac{p_i}{x_i}}, \tag{5.37}$$

where $p_i > 0$, $x_i \geq 0$ for all $i \in I$, where $P_I := \sum_{i\in I} p_i > 0$.

We remark that the equality holds in (5.37) iff $x_i = x_j$ for each $i, j \in I$.

Choosing $p_i = a_i^2$ and $x_i = \frac{b_i}{a_i}$ $(i \in I)$ in (5.37), then we get

$$\frac{\sum_{i\in I} a_i b_i}{\sum_{i\in I} a_i^2} \geq \prod_{i\in I}\left(\frac{b_i}{a_i}\right)^{\frac{a_i^2}{\sum_{i\in I} a_i^2}} \geq \frac{\sum_{i\in I} a_i^2}{\sum_{i\in I}\frac{a_i^3}{b_i}} \tag{5.38}$$

and by $p_i = b_i^2$ and $x_i = \frac{a_i}{b_i}$, we also have

$$\frac{\sum_{i\in I} a_i b_i}{\sum_{i\in I} b_i^2} \geq \prod_{i\in I}\left(\frac{a_i}{b_i}\right)^{\frac{b_i^2}{\sum_{i\in I} b_i^2}} \geq \frac{\sum_{i\in I} b_i^2}{\sum_{i\in I}\frac{b_i^3}{a_i}}. \tag{5.39}$$

If we multiply (5.38) with (5.39) we easily deduce the desired inequality (5.36).

The case of equality follows by the same case in the arithmetic mean – geometric mean – harmonic mean inequality. We omit the details. ■

The following corollary holds [6, Corollary 5.1].

Corollary 207 *With $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ as above, one has the inequality*

$$\left[\frac{\sum_{i\in I}\frac{a_i^3}{b_i}\sum_{i\in I}\frac{b_i^3}{a_i}}{\left(\sum_{i\in I} a_i b_i\right)^2}\right]^{\frac{1}{2}} \geq \frac{\sum_{i\in I} a_i^2 \sum_{i\in I} b_i^2}{\left(\sum_{i\in I} a_i b_i\right)^2}. \tag{5.40}$$

The equality holds in (5.40) iff there is a $k>0$ such that $a_i = kb_i$, $i \in \{1,\ldots,n\}$.

5.7 A Related Result via Jensen's Inequality for Power Functions

The following result also holds [6, Theorem 6].

Theorem 208 *Let* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *be sequences of positive real numbers and* $p \geq 1$. *If* I *is a finite part of* $\mathbb{N}$, *then one has the inequality*

$$\frac{\left(\sum_{i\in I} a_i b_i\right)^2}{\sum_{i\in I} a_i^2 \sum_{i\in I} b_i^2} \leq \left[\frac{\sum_{i\in I} a_i^{2-p} b_i^p \sum_{i\in I} a_i^p b_i^{2-p}}{\sum_{i\in I} a_i^2 \sum_{i\in I} b_i^2}\right]^{\frac{1}{p}}. \tag{5.41}$$

The equality holds in (5.41) if and only if there exists a $k > 0$ *such that* $a_i = kb_i$ *for all* $i \in I$.

If $p \in (0,1)$, *the inequality in (5.41) reverses.*

Proof. We shall follow the proof in [6].

By Jensen's inequality for the convex mapping $f : \mathbb{R}_+ \to \mathbb{R}$,

$$f(x) = x^p, \;\; p \geq 1$$

one has

$$\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right)^p \leq \frac{\sum_{i\in I} p_i x_i^p}{P_I}, \tag{5.42}$$

where $P_I := \sum_{i\in I} p_i$, $p_i > 0$, $x_i \geq 0$, $i \in I$. The equality holds in (5.42) iff $x_i = x_j$ for all $i, j \in I$.

Now, choosing in (5.42) $p_i = a_i^2$, $x_i = \frac{b_i}{a_i}$, we get

$$\frac{\sum_{i\in I} a_i b_i}{\sum_{i\in I} a_i^2} \leq \left(\frac{\sum_{i\in I} a_i^{2-p} b_i^p}{\sum_{i\in I} a_i^2}\right)^{\frac{1}{p}} \tag{5.43}$$

and for $p_i = b_i^2$, $x_i = \frac{a_i}{b_i}$, the inequality (5.42) also gives

$$\frac{\sum_{i\in I} a_i b_i}{\sum_{i\in I} b_i^2} \leq \left(\frac{\sum_{i\in I} a_i^p b_i^{2-p}}{\sum_{i\in I} b_i^2}\right)^{\frac{1}{p}}. \tag{5.44}$$

By multiplying the inequalities (5.43) and (5.44), we deduce the desired result from (5.42).

The case of equality follows by the fact that in (5.42) the equality holds iff $(x_i)_{i\in I}$ is constant.

If $p \in (0,1)$, then a reverse inequality holds in (5.42) giving the corresponding result in (5.41). ■

Remark 209 *If $p = 2$, then (5.41) becomes the $(CBS)-$inequality.*

5.8 Inequalities Derived from the Double Sums Case

Let $A = (a_{ij})_{i,j=\overline{1,n}}$ and $B = (b_{ij})_{i,j=\overline{1,n}}$ be two matrices of real numbers. The following inequality is known as the $(CBS)-$inequality for double sums

$$\left(\sum_{i,j=1}^{n} a_{ij}b_{ij}\right)^2 \leq \sum_{i,j=1}^{n} a_{ij}^2 \sum_{i,j=1}^{n} b_{ij}^2 \tag{5.45}$$

with equality iff there is a real number r such that $a_{ij} = rb_{ij}$ for any $i,j \in \{1,\ldots,n\}$.

The following inequality holds [7, Theorem 5.2].

Theorem 210 *Let $\bar{\mathbf{a}} = (a_1,\ldots,a_n)$ and $\bar{\mathbf{b}} = (b_1,\ldots,b_n)$ be sequences of real numbers. Then*

$$\left|\left(\sum_{k=1}^{n} a_k\right)^2 + \left(\sum_{k=1}^{n} b_k\right)^2 - 2n\sum_{k=1}^{n} a_k b_k\right| \leq n\sum_{k=1}^{n}\left(a_k^2 + b_k^2\right) - 2\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k. \tag{5.46}$$

Proof. We shall follow the proof from [7].

Applying (5.45) for $a_{ij} = a_i - b_j$, $b_{ij} = b_i - a_j$ and taking into account that

$$\sum_{i,j=1}^{n} (a_i - b_j)(b_i - a_j) = 2n\sum_{k=1}^{n} a_k b_k - \left(\sum_{k=1}^{n} a_k\right)^2 - \left(\sum_{k=1}^{n} b_k\right)^2$$

$$\sum_{i,j=1}^{n} (a_i - b_j)^2 = n\sum_{k=1}^{n}\left(a_k^2 + b_k^2\right) - 2\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k$$

and

$$\sum_{i,j=1}^{n}(b_i-a_j)^2=n\sum_{k=1}^{n}\left(a_k^2+b_k^2\right)-2\sum_{k=1}^{n}a_k\sum_{k=1}^{n}b_k,$$

we may deduce the desired inequality (5.46). ■

The following result also holds [7, Theorem 5.3].

Theorem 211 *Let* $\bar{\mathbf{a}}=(a_1,\dots,a_n)$, $\bar{\mathbf{b}}=(b_1,\dots,b_n)$, $\bar{\mathbf{c}}=(c_1,\dots,c_n)$ *and* $\bar{\mathbf{d}}=(d_1,\dots,d_n)$ *be sequences of real numbers. Then one has the inequality:*

$$\left[\det\begin{bmatrix}\sum_{i=1}^n a_ic_i & \sum_{i=1}^n a_id_i\\ \sum_{i=1}^n b_ic_i & \sum_{i=1}^n b_id_i\end{bmatrix}\right]^2 \le \det\begin{bmatrix}\sum_{i=1}^n a_i^2 & \sum_{i=1}^n a_ib_i\\ \sum_{i=1}^n a_ib_i & \sum_{i=1}^n b_i^2\end{bmatrix} \times\det\begin{bmatrix}\sum_{i=1}^n c_i^2 & \sum_{i=1}^n c_id_i\\ \sum_{i=1}^n c_id_i & \sum_{i=1}^n d_i^2\end{bmatrix}. \quad (5.47)$$

Proof. We shall follow the proof in [7].

Applying (5.45) for $a_{ij}=a_ib_j-a_jb_i$, $b_{ij}=c_id_j-c_jd_i$ and using Cauchy-Binet's identity [1, p. 85]

$$\frac{1}{2}\sum_{i,j=1}^{n}(a_ib_j-a_jb_i)(c_id_j-c_jd_i) =\sum_{i=1}^{n}a_ic_i\sum_{i=1}^{n}b_id_i-\left(\sum_{i=1}^{n}a_id_i\right)\left(\sum_{i=1}^{n}b_ic_i\right) \quad (5.48)$$

and Lagrange's identity [1, p. 84]

$$\frac{1}{2}\sum_{i,j=1}^{n}(a_ib_j-a_jb_i)^2=\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2-\left(\sum_{i=1}^{n}a_ib_i\right)^2, \quad (5.49)$$

we deduce the desired result (5.47). ■

5.9 A Functional Generalisation for Double Sums

The following result holds [7, Theorem 5.5].

Theorem 212 *Let A be a subset of real numbers $\mathbb{R}$, $f : A \to \mathbb{R}$ and $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ sequences of real numbers with the property that*

(i) $a_k b_i,\ a_i^2, b_k^2 \in A$ *for any* $i, k \in \{1, \ldots, n\}$;

(ii) $f\left(a_k^2\right),\ f\left(b_k^2\right) \geq 0$ *for any* $k \in \{1, \ldots, n\}$;

(iii) $f^2\left(a_k b_i\right) \leq f\left(a_k^2\right) f\left(b_i^2\right)$ *for any* $i, k \in \{1, \ldots, n\}$.

Then one has the inequality

$$\left[\sum_{k,i=1}^{n} f\left(a_k b_i\right)\right]^2 \leq n^2 \sum_{k=1}^{n} f\left(a_k^2\right) \sum_{k=1}^{n} f\left(b_k^2\right). \tag{5.50}$$

Proof. We will follow the proof in [7].

Using the assumption (iii) and the (CBS) −inequality for double sums, we have

$$\begin{aligned}
\left|\sum_{k,i=1}^{n} f\left(a_k b_i\right)\right| &\leq \sum_{k,i=1}^{n} \left|f\left(a_k b_i\right)\right| \leq \sum_{k,i=1}^{n} \left[f\left(a_k^2\right) f\left(b_i^2\right)\right]^{\frac{1}{2}} \qquad (5.51)\\
&\leq \left\{\left(\sum_{k,i=1}^{n} \left[f\left(a_k^2\right)\right]^{\frac{1}{2}}\right)^2 \left(\sum_{k,i=1}^{n} \left[f\left(b_i^2\right)\right]^{\frac{1}{2}}\right)^2\right\}^{\frac{1}{2}}\\
&= \left[\sum_{k,i=1}^{n} f\left(a_k^2\right) \sum_{k,i=1}^{n} f\left(b_i^2\right)\right]^{\frac{1}{2}}\\
&= n\left[\sum_{k=1}^{n} f\left(a_k^2\right)\right]^{\frac{1}{2}} \left[\sum_{k=1}^{n} f\left(b_k^2\right)\right]^{\frac{1}{2}}
\end{aligned}$$

which is clearly equivalent to (5.50). ■

The following corollary is a natural consequence of the above theorem [7, Corollary 5.6].

Corollary 213 *Let A, f and $\bar{\mathbf{a}}$ be as above. If*

(i) $a_k a_i \in A$ *for any* $i, k \in \{1, \dots, n\}$;

(ii) $f(a_k^2) \geq 0$ *for any* $k \in \{1, \dots, n\}$;

(iii) $f^2(a_k a_i) \leq f(a_k^2) f(a_i^2)$ *for any* $i, k \in \{1, \dots, n\}$,

Then one has the inequality

$$\left| \sum_{k,i=1}^{n} f(a_k a_i) \right| \leq n \sum_{k=1}^{n} f\left(a_k^2\right). \tag{5.52}$$

The following particular inequalities also hold [7, p. 23].

1. If $\varphi : \mathbb{N} \to \mathbb{N}$ is Euler's indicator and $s(n)$ denotes the sum of all relatively prime numbers including and less than n, then for any $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ sequences of natural numbers, one has the inequalities

$$\left[\sum_{k,i=1}^{n} \varphi(a_k b_i) \right]^2 \leq n^2 \sum_{k=1}^{n} \varphi\left(a_k^2\right) \sum_{k=1}^{n} \varphi\left(b_k^2\right); \tag{5.53}$$

$$\sum_{k,i=1}^{n} \varphi(a_k a_i) \leq n \sum_{k=1}^{n} \varphi\left(a_k^2\right); \tag{5.54}$$

$$\left[\sum_{k,i=1}^{n} s(a_k b_i) \right]^2 \leq n^2 \sum_{k=1}^{n} s\left(a_k^2\right) \sum_{k=1}^{n} s\left(b_k^2\right); \tag{5.55}$$

$$\sum_{k,i=1}^{n} s(a_k a_i) \leq n \sum_{k=1}^{n} s\left(a_k^2\right). \tag{5.56}$$

2. If $a > 1$ and $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ are sequences of real numbers, then

$$\left[\sum_{k,i=1}^{n} \exp_a(a_k b_i) \right]^2 \leq n^2 \sum_{k=1}^{n} \exp_a\left(a_k^2\right) \sum_{k=1}^{n} \exp_a\left(b_k^2\right); \tag{5.57}$$

$$\sum_{k,i=1}^{n} \exp_a(a_k a_i) \leq n \sum_{k=1}^{n} \exp_a\left(a_k^2\right); \tag{5.58}$$

3. If $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ are sequences of real numbers such that $a_k, b_k \in (-1,1)$ $(k \in \{1,\dots,n\})$, then one has the inequalities:

$$\left[\sum_{k,i=1}^{n} \frac{1}{(1-a_k b_i)^m}\right]^2 \le n^2 \sum_{k=1}^{n} \frac{1}{(1-a_k^2)^m} \sum_{k=1}^{n} \frac{1}{(1-b_k^2)^m}, \tag{5.59}$$

$$\sum_{k,i=1}^{n} \frac{1}{(1-a_k a_i)^m} \le n \sum_{k=1}^{n} \frac{1}{(1-a_k^2)^m}, \tag{5.60}$$

where $m > 0$.

5.10 A $(CBS)-$Type Result for Lipschitzian Functions

The following result was obtained in [8, Theorem].

Theorem 214 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a Lipschitzian function with the constant M, i.e., it satisfies the condition*

$$|f(x) - f(y)| \le M|x-y| \text{ for any } x, y \in I. \tag{5.61}$$

If $\bar{\mathbf{a}} = (a_1,\dots,a_n)$, $\bar{\mathbf{b}} = (b_1,\dots,b_n)$ are sequences of real numbers with $a_i b_j \in I$ for any $i, j \in \{1,\dots,n\}$, then

$$\begin{aligned} 0 &\le \left|\sum_{i,j=1}^{n} f(a_i b_j)|f(a_i b_j)| - \sum_{i,j=1}^{n} |f(a_j b_i)| f(a_i b_j)\right| \qquad (5.62)\\ &\le \sum_{i,j=1}^{n} f^2(a_j b_i) - \sum_{i,j=1}^{n} f(a_i b_j) f(a_j b_i) \\ &\le M^2 \left[\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2\right]. \end{aligned}$$

Proof. We shall follow the proof in [8].

Since f is Lipschitzian with the constant M, we have

$$0 \le ||f(a_i b_j)| - |f(a_j b_i)|| \le |f(a_i b_j) - f(a_j b_i)| \le M|a_i b_j - a_j b_i| \tag{5.63}$$

for any $i, j \in \{1, \ldots, n\}$, giving

$$0 \leq |(|f(a_i b_j)| - |f(a_j b_i)|)(f(a_i b_j) - f(a_j b_i))| \tag{5.64}$$
$$\leq (f(a_i b_j) - f(a_j b_i))^2 \leq M^2 (a_i b_j - a_j b_i)^2$$

for any $i, j \in \{1, \ldots, n\}$.

The inequality (5.64) is obviously equivalent to

$$\Big| |f(a_i b_j)| f(a_i b_j) + |f(a_j b_i)| f(a_j b_i) \tag{5.65}$$
$$- |f(a_i b_j)| f(a_j b_i) - |f(a_j b_i)| f(a_i b_j) \Big|$$
$$\leq f^2(a_i b_j) - 2f(a_i b_j) f(a_j b_i) + f^2(a_j b_i)$$
$$\leq M^2 \left(a_i^2 b_j^2 - 2a_i b_i a_j b_j + a_j^2 b_i^2\right)$$

for any $i, j \in \{1, \ldots, n\}$. Summing over i and j from 1 to n in (5.65) and taking into account that:

$$\sum_{i,j=1}^{n} |f(a_i b_j)| f(a_i b_j) = \sum_{i,j=1}^{n} |f(a_j b_i)| f(a_i b_j),$$
$$\sum_{i,j=1}^{n} |f(a_i b_j)| f(a_j b_i) = \sum_{i,j=1}^{n} |f(a_j b_i)| f(a_i b_j),$$
$$\sum_{i,j=1}^{n} f^2(a_i b_j) = \sum_{i,j=1}^{n} f^2(a_j b_i),$$

we deduce the desired inequality. ■

The following particular inequalities hold [8, p. 27 – p. 28].

1. Let $\bar{\mathbf{x}} = (x_1, \ldots, x_n)$, $\bar{\mathbf{y}} = (y_1, \ldots, y_n)$ be sequences of real numbers such that $0 \leq |x_i| \leq M_1$, $0 \leq |y_i| \leq M_2$, $i \in \{1, \ldots, n\}$. Then for any $r \geq 1$ one has

$$0 \leq \sum_{i=1}^{n} x_i^{2r} \sum_{i=1}^{n} y_i^{2r} - \left(\sum_{i=1}^{n} |x_i y_i|^r\right)^2 \tag{5.66}$$
$$\leq r^2 (M_1 M_2)^{2(r-1)} \left[\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} |x_i y_i|\right)^2\right].$$

2. If $0 < m_1 \leq |x_i|$, $0 < m_2 \leq |y_i|$, $i \in \{1, \ldots, n\}$ and $r \in (0,1)$, then

$$0 \leq \sum_{i=1}^{n} x_i^{2r} \sum_{i=1}^{n} y_i^{2r} - \left(\sum_{i=1}^{n} |x_i y_i|^r \right)^2 \tag{5.67}$$
$$\leq \frac{r^2}{(m_1 m_2)^{2(r-1)}} \left[\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} |x_i y_i| \right)^2 \right].$$

3. If $0 \leq |x_i| \leq M_1$, $0 \leq |y_i| \leq M_2$, $i \in \{1, \ldots, n\}$, then for any natural number k one has

$$0 \leq \left| \sum_{i=1}^{n} x_i^{2k+1} |x_i|^{2k+1} \sum_{i=1}^{n} y_i^{2k+1} |y_i|^{2k+1} \right. \tag{5.68}$$
$$\left. - \sum_{i=1}^{n} x_i^{2k+1} |y_i|^{2k+1} \sum_{i=1}^{n} y_i^{2k+1} |x_i|^{2k+1} \right|$$
$$\leq \sum_{i=1}^{n} x_i^{2(2k+1)} \sum_{i=1}^{n} y_i^{2(2k+1)} - \left(\sum_{i=1}^{n} x_i^{2k+1} y_i^{2k+1} \right)^2$$
$$\leq (2k+1)^2 (M_1 M_2)^{4k} \left[\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} |x_i y_i| \right)^2 \right].$$

4. If $0 < m_1 \leq x_i$, $0 < m_2 \leq y_i$, for any $i \in \{1, \ldots, n\}$, then one has the inequality

$$0 \leq n \sum_{i=1}^{n} \left[\ln \left(\frac{x_i}{y_i} \right) \right]^2 - \left[\sum_{i=1}^{n} \ln \left(\frac{x_i}{y_i} \right) \right]^2 \tag{5.69}$$
$$\leq \frac{1}{(m_1 m_2)^2} \left[\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} x_i y_i \right)^2 \right].$$

5.11 An Inequality via Jensen's Discrete Inequality

The following result holds [9].

Theorem 215 *Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ be two sequences of real numbers with $a_i \neq 0$, $i \in \{1, \dots, n\}$. If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex (concave) function on I and $\frac{b_i}{a_i} \in I$ for each $i \in \{1, \dots, n\}$, and $\bar{\mathbf{w}} = (w_1, \dots, w_n)$ is a sequence of nonnegative real numbers, then*

$$f\left(\frac{\sum_{i=1}^n w_i a_i b_i}{\sum_{i=1}^n w_i a_i^2}\right) \leq (\geq) \frac{\sum_{i=1}^n w_i a_i^2 f\left(\frac{b_i}{a_i}\right)}{\sum_{i=1}^n w_i a_i^2}. \tag{5.70}$$

Proof. We shall use Jensen's discrete inequality for convex (concave) functions

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \leq (\geq) \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i), \tag{5.71}$$

where $p_i \geq 0$ with $P_n := \sum_{i=1}^n p_i > 0$ and $x_i \in I$ for each $i \in \{1, \dots, n\}$.

If in (5.71) we choose $x_i = \frac{b_i}{a_i}$ and $p_i = w_i a_i^2$, then by (5.71) we deduce the desired result (5.70). ■

The following corollary holds [9].

Corollary 216 *Let $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ be sequences of positive real numbers and assume that $\bar{\mathbf{w}}$ is as above. If $p \in (-\infty, 0) \cup [1, \infty)$ $(p \in (0, 1))$, then one has the inequality*

$$\left(\sum_{i=1}^n w_i a_i b_i\right)^p \leq (\geq) \left(\sum_{i=1}^n w_i a_i^2\right)^{p-1} \sum_{i=1}^n w_i a_i^{2-p} b_i^p. \tag{5.72}$$

Proof. Follows by Theorem 215 applied for convex (concave) function $f : [0, \infty) \to \mathbb{R}$, $f(x) = x^p$, $p \in (-\infty, 0) \cup [1, \infty)$ $(p \in (0, 1))$. ■

Remark 217 *If $p = 2$, then by (5.72) we deduce the (CBS)-inequality.*

5.12 An Inequality via Lah-Ribarić Inequality

The following counterpart of Jensen's discrete inequality was obtained in 1973 by Lah and Ribarić [10].

Lemma 218 *Let $f : I \to \mathbb{R}$ be a convex function, $x_i \in [m, M] \subseteq I$ for each $i \in \{1, \ldots, n\}$ and $\bar{\mathbf{p}} = (p_1, \ldots, p_n)$ be a positive $n-$tuple. Then*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \le \frac{M - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i}{M - m} f(m) + \frac{\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i - m}{M - m} f(M). \quad (5.73)$$

Proof. We observe for each $i \in \{1, \ldots, n\}$, that

$$x_i = \frac{(M - x_i)m + (x_i - m)M}{M - m}. \quad (5.74)$$

If in the definition of convexity, i.e., $\alpha, \beta \ge 0$, $\alpha + \beta > 0$

$$f\left(\frac{\alpha a + \beta b}{\alpha + \beta}\right) \le \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta} \quad (5.75)$$

we choose $\alpha = M - x_i$, $\beta = x_i - m$, $a = m$ and $b = M$, we deduce, by (5.75), that

$$f(x_i) = f\left(\frac{(M - x_i)m + (x_i - m)M}{M - m}\right) \le \frac{(M - x_i) f(m) + (x_i - m) f(M)}{M - m} \quad (5.76)$$

for each $i \in \{1, \ldots, n\}$.

If we multiply (5.76) by $p_i > 0$ and sum over i from 1 to n, we deduce (5.73). ■

The following result holds.

Theorem 219 *Let $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ be two sequences of real numbers with $a_i \neq 0$, $i \in \{1, \ldots, n\}$. If $I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex (concave) function on I and $\frac{b_i}{a_i} \in [m, M] \subseteq I$ for each $i \in \{1, \ldots, n\}$ and $\bar{\mathbf{w}} = (w_1, \ldots, w_n)$ is a sequence of nonnegative real numbers, then*

$$\frac{\sum_{i=1}^{n} w_i a_i^2 f\left(\frac{b_i}{a_i}\right)}{\sum_{i=1}^{n} w_i a_i^2} \le (\ge) \frac{M - \frac{\sum_{i=1}^{n} w_i a_i b_i}{\sum_{i=1}^{n} w_i a_i^2}}{M - m} f(m) + \frac{\frac{\sum_{i=1}^{n} w_i a_i b_i}{\sum_{i=1}^{n} w_i a_i^2} - m}{M - m} f(M). \quad (5.77)$$

Proof. Follows by Lemma 218 for the choices $p_i = w_i a_i^2$, $x_i = \frac{b_i}{a_i}$, $i \in \{1, \ldots, n\}$. ■

The following corollary holds.

Corollary 220 *Let* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ *be two sequences of positive real numbers and such that*

$$0 < m \leq \frac{b_i}{a_i} \leq M < \infty \text{ for each } i \in \{1, \ldots, n\}. \tag{5.78}$$

If $\bar{\mathbf{w}} = (w_1, \ldots, w_n)$ *is a sequence of nonnegative real numbers and* $p \in (-\infty, 0) \cup [1, \infty)$ $(p \in (0,1))$, *then one has the inequality*

$$\sum_{i=1}^{n} w_i a_i^{2-p} b_i^p + \frac{Mm\left(M^{p-1} - m^{p-1}\right)}{M - m} \sum_{i=1}^{n} w_i a_i^2 \leq (\geq) \frac{M^p - m^p}{M - m} \sum_{i=1}^{n} w_i a_i b_i. \tag{5.79}$$

Proof. If we write the inequality (5.77) for the convex (concave) function $f(x) = x^p$, $p \in (-\infty, 0) \cup [1, \infty)$ $(p \in (0,1))$, we get

$$\frac{\sum_{i=1}^{n} w_i a_i^{2-p} b_i^p}{\sum_{i=1}^{n} w_i a_i^2} \leq (\geq) \frac{M - \frac{\sum_{i=1}^{n} w_i a_i b_i}{\sum_{i=1}^{n} w_i a_i^2}}{M - m} m^p + \frac{\frac{\sum_{i=1}^{n} w_i a_i b_i}{\sum_{i=1}^{n} w_i a_i^2} - m}{M - m} M^p$$

which, after elementary calculations, is equivalent to (5.79). ■

Remark 221 *For* $p = 2$, *we get*

$$\sum_{i=1}^{n} w_i b_i^2 + Mm \sum_{i=1}^{n} w_i a_i^2 \leq (M + m) \sum_{i=1}^{n} w_i a_i b_i \tag{5.80}$$

which is the well known Diaz-Metcalf inequality [11].

5.13 An Inequality via Dragomir-Ionescu Inequality

The following counterpart of Jensen's inequality was proved in 1994 by S.S. Dragomir and N.M. Ionescu [12].

Lemma 222 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on $\mathring{I}$, $x_i \in \mathring{I}$ $(i \in \{1, \ldots, n\})$ and $p_i \geq 0$ $(i \in \{1, \ldots, n\})$ such that $P_n := \sum_{i=1}^{n} p_i > 0$. Then one has the inequality*

$$0 \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\right) \tag{5.81}$$
$$\leq \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i f'(x_i) - \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i f'(x_i).$$

Proof. Since f is differentiable convex on $\mathring{I}$, one has

$$f(x) - f(y) \geq (x - y) f'(y), \tag{5.82}$$

for any $x, y \in \mathring{I}$.

If we choose $x = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i$ and $y = y_k$, $k \in \{1, \ldots, n\}$, we get

$$f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\right) - f(y_k) \geq \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i - y_k\right) f'(y_k). \tag{5.83}$$

Multiplying (5.83) by $p_k \geq 0$ and summing over k from 1 to n, we deduce the desired result (5.81). ■

The following result holds [9].

Theorem 223 *Let $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ be two sequences of real numbers with $a_i \neq 0$, $i \in \{1, \ldots, n\}$. If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable convex (concave) function on $\mathring{I}$ and $\frac{b_i}{a_i} \in \mathring{I}$ for each $i \in \{1, \ldots, n\}$, and $\bar{\mathbf{w}} = (w_1, \ldots, w_n)$ is a sequence of nonnegative real numbers, then*

$$0 \leq \sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i a_i^2 f\left(\frac{b_i}{a_i}\right) \tag{5.84}$$
$$- \left(\sum_{i=1}^{n} w_i a_i^2\right)^2 \cdot f\left(\frac{\sum_{i=1}^{n} w_i a_i b_i}{\sum_{i=1}^{n} w_i a_i^2}\right)$$
$$\leq \sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i a_i b_i f'\left(\frac{b_i}{a_i}\right)$$
$$- \sum_{i=1}^{n} w_i a_i b_i \sum_{i=1}^{n} w_i a_i^2 f'\left(\frac{b_i}{a_i}\right).$$

Proof. Follows from Lemma 222 on choosing $p_i = w_i a_i^2$, $x_i = \frac{b_i}{a_i}$, $i \in \{1, \ldots, n\}$. ■

The following corollary holds [9].

Corollary 224 *Let $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ be two sequences of positive real numbers with $a_i \neq 0$, $i \in \{1, \ldots, n\}$. If $p \in [1, \infty)$ then one has the inequality*

$$\begin{aligned} 0 &\le \sum_{i=1}^n w_i a_i^2 \sum_{i=1}^n w_i a_i^{2-p} b_i^p - \left(\sum_{i=1}^n w_i a_i^2\right)^{2-p} \left(\sum_{i=1}^n w_i a_i b_i\right)^p \\ &\le p\left[\sum_{i=1}^n w_i a_i^2 \sum_{i=1}^n w_i a_i^{2-p} b_i - \sum_{i=1}^n w_i a_i b_i \sum_{i=1}^n w_i a_i^{3-p} b_i^{p-1}\right]. \end{aligned} \tag{5.85}$$

If $p \in (0,1)$, then

$$\begin{aligned} 0 &\le \left(\sum_{i=1}^n w_i a_i^2\right)^{2-p} \left(\sum_{i=1}^n w_i a_i b_i\right) - \sum_{i=1}^n w_i a_i^2 \sum_{i=1}^n w_i a_i^{2-p} b_i^p \\ &\le p\left[\sum_{i=1}^n w_i a_i b_i \sum_{i=1}^n w_i a_i^{3-p} b_i^p - \sum_{i=1}^n w_i a_i^2 \sum_{i=1}^n w_i a_i^{2-p} b_i\right]. \end{aligned} \tag{5.86}$$

5.14 An Inequality via a Refinement of Jensen's Inequality

We will use the following lemma which contains a refinement of Jensen's inequality obtained in [13].

Lemma 225 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I and $x_i \in I$, $p_i \geq 0$ with $P_n := \sum_{i=1}^n p_i > 0$. Then the following inequality holds:*

$$\begin{aligned} f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) &\le \frac{1}{P_n^{k+1}} \sum_{i_1, \ldots, i_{k+1}=1}^n p_{i_1} \cdots p_{i_k} f\left(\frac{x_{i_1} + \cdots + x_{i_{k+1}}}{k+1}\right) \\ &\le \frac{1}{P_n^k} \sum_{i_1, \ldots, i_k=1}^n p_{i_1} \cdots p_{i_k} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right) \end{aligned} \tag{5.87}$$

$$\leq \cdots \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i f\left(x_i\right),$$

where $k \geq 1,\ k \in \mathbb{N}$.

Proof. We shall follow the proof in [13].

The first inequality follows by Jensen's inequality for multiple sums

$$f\left(\frac{\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}}\left(\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right)}{\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}}}\right) = \frac{\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}} f\left(\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right)}{\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}}}$$

since

$$\frac{\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}}\left(\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right)}{\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}}} = P_n^k \sum_{i=1}^{n} p_i x_i$$

and

$$\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}} = P_n^{k+1}.$$

Now, applying Jensen's inequality for

$$y_1 = \frac{x_{i_1}+x_{i_2}\cdots+x_{i_{k-1}}+x_{i_k}}{k},\quad y_2 = \frac{x_{i_2}+x_{i_3}\cdots+x_{i_k}+x_{i_{k+1}}}{k}, \quad \ldots,\quad y_{k+1} = \frac{x_{i_{k+1}}+x_{i_1}+x_{i_2}+\cdots+x_{i_{k-1}}}{k}$$

we have

$$f\left(\frac{y_1+y_2+\cdots+y_k+y_{k+1}}{k+1}\right) \leq \frac{f\left(y_1\right)+f\left(y_2\right)+\cdots+f\left(y_k\right)+f\left(y_{k+1}\right)}{k+1},$$

which is equivalent to

$$f\left(\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right) \leq \frac{f\left(\frac{x_{i_1}+x_{i_2}\cdots+x_{i_{k-1}}+x_{i_k}}{k}\right)+\cdots+f\left(\frac{x_{i_{k+1}}+x_{i_1}+x_{i_2}+\cdots+x_{i_{k-1}}}{k}\right)}{k+1}. \quad (5.88)$$

Multiplying (5.88) with the nonnegative real numbers $p_{i_1}, \ldots, p_{i_{k+1}}$ and summing over $i_1, \ldots, i_{k+1}$ from 1 to n we deduce

$$\begin{aligned}
&\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}} f\left(\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right) \qquad (5.89)\\
&\leq \frac{1}{k+1}\left[\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)\right.\\
&\qquad \left. +\cdots+\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}} f\left(\frac{x_{i_{k+1}}+x_{i_1}+x_{i_2}+\cdots+x_{i_{k-1}}}{k}\right)\right]\\
&= P_n \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\cdots p_{i_k} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)
\end{aligned}$$

which proves the second part of (5.87). ■

The following result holds.

Theorem 226 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I, $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ real numbers such that $a_i \neq 0$, $i \in \{1, \ldots, n\}$ and $\frac{b_i}{a_i} \in I$, $i \in \{1, \ldots, n\}$. If $\bar{\mathbf{w}} = (w_1, \ldots, w_n)$ are positive real numbers, then*

$$\begin{aligned}
&f\left(\frac{\sum_{i=1}^{n} w_i a_i b_i}{\sum_{i=1}^{n} w_i a_i^2}\right) \qquad (5.90)\\
&\leq \frac{1}{\left(\sum_{i=1}^{n} w_i a_i^2\right)^{k+1}} \sum_{i_1,\ldots,i_{k+1}=1}^{n} w_{i_1}\cdots w_{i_{k+1}} a_{i_1}^2 \cdots a_{i_{k+1}}^2\\
&\qquad \times f\left(\frac{\frac{b_{i_1}}{a_{i_1}}+\cdots+\frac{b_{i_{k+1}}}{a_{i_{k+1}}}}{k+1}\right)\\
&\leq \frac{1}{\left(\sum_{i=1}^{n} w_i a_i^2\right)^{k}} \sum_{i_1,\ldots,i_k=1}^{n} w_{i_1}\cdots w_{i_k} a_{i_1}^2 \cdots a_{i_k}^2\\
&\qquad \times f\left(\frac{\frac{b_{i_1}}{a_{i_1}}+\cdots+\frac{b_{i_k}}{a_{i_k}}}{k}\right)
\end{aligned}$$

$$\leq \cdots \leq \frac{1}{\sum_{i=1}^{n} w_i a_i^2} \sum_{i=1}^{n} w_i a_i^2 f\left(\frac{b_i}{a_i}\right).$$

The proof is obvious by Lemma 225 applied for $p_i = w_i a_i^2$, $x_i = \frac{b_i}{a_i}$, $i \in \{1, \ldots, n\}$.

The following corollary holds.

Corollary 227 *Let* $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ *and* $\bar{\mathbf{w}}$ *be sequences of positive real numbers. If* $p \in (-\infty, 0) \cup [1, \infty)$ $(p \in (0,1))$, *then one has the inequalities*

$$\begin{aligned}
&\left(\sum_{i=1}^{n} w_i a_i b_i\right)^p \\
&\underset{(\geq)}{\leq} \frac{\left(\sum_{i=1}^{n} w_i a_i^2\right)^{p-k-1}}{(k+1)^p} \sum_{i_1, \ldots, i_{k+1}=1}^{n} w_{i_1} \cdots w_{i_{k+1}} a_{i_1}^2 \cdots a_{i_{k+1}}^2 \left(\frac{b_{i_1}}{a_{i_1}} + \cdots + \frac{b_{i_{k+1}}}{a_{i_{k+1}}}\right)^p \\
&\underset{(\geq)}{\leq} \frac{\left(\sum_{i=1}^{n} w_i a_i^2\right)^{p-k}}{k^p} \sum_{i_1, \ldots, i_k=1}^{n} w_{i_1} \cdots w_{i_k} a_{i_1}^2 \cdots a_{i_k}^2 \left(\frac{b_{i_1}}{a_{i_1}} + \cdots + \frac{b_{i_k}}{a_{i_k}}\right)^p \\
&\underset{(\geq)}{\leq} \cdots \underset{(\geq)}{\leq} \left(\sum_{i=1}^{n} w_i a_i^2\right)^{p-1} \sum_{i=1}^{n} w_i a_i^{2-p} b_i^p.
\end{aligned}$$

Remark 228 *If* $p = 2$, *then we deduce the following refinement of the* $(CBS)-$ *inequality*

$$\begin{aligned}
\left(\sum_{i=1}^{n} w_i a_i b_i\right)^2 &\leq \frac{\left(\sum_{i=1}^{n} w_i a_i^2\right)^{1-k}}{(k+1)^2} \sum_{i_1, \ldots, i_{k+1}=1}^{n} w_{i_1} \cdots w_{i_{k+1}} \left(\sum_{\ell=1}^{k+1} b_{i_\ell} \prod_{\substack{j=1 \\ j \neq \ell}}^{k+1} a_{ij}\right)^2 \\
&\leq \frac{\left(\sum_{i=1}^{n} w_i a_i^2\right)^{2-k}}{k^2} \sum_{i_1, \ldots, i_k=1}^{n} w_{i_1} \cdots w_{i_k} \left(\sum_{\ell=1}^{k} b_{i_\ell} \prod_{\substack{j=1 \\ j \neq \ell}}^{k} a_{ij}\right)^2 \\
&\leq \cdots \leq \frac{1}{4} \sum_{i,j=1}^{n} w_i w_j \left(b_i a_j + a_i b_j\right)^2 \\
&\leq \sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2.
\end{aligned}$$

5.15 Another Refinement via Jensen's Inequality

The following refinement of Jensen's inequality holds (see [15]).

Lemma 229 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable convex function on (a,b) and $x_i \in (a,b)$, $p_i \geq 0$ with $P_n := \sum_{i=1}^{n} p_i > 0$. Then one has the inequality*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \geq \left|\frac{1}{P_n}\sum_{i=1}^{n} p_i \left| f(x_i) - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)\right| - \left| f'\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)\right| \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i \left| x_i - \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right|\right| \geq 0. \quad (5.91)$$

Proof. Since f is differentiable convex on (a,b), then for each $x, y \in (a,b)$, one has the inequality

$$f(x) - f(y) \geq (x-y) f'(y). \quad (5.92)$$

Using the properties of the modulus, we have

$$\begin{aligned} f(x) - f(y) - (x-y) f'(y) &= |f(x) - f(y) - (x-y) f'(y)| \\ &\geq ||f(x) - f(y)| - |x-y| \, |f'(y)|| \end{aligned} \quad (5.93)$$

for each $x, y \in (a,b)$.

If we choose $y = \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j$ and $x = x_i$, $i \in \{1, \ldots, n\}$, then we have

$$f(x_i) - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) - \left(x_i - \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) f'\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) \geq \left|\left| f(x_i) - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)\right| - \left| x_i - \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right| \left| f'\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)\right|\right| \quad (5.94)$$

for any $i \in \{1, \ldots, n\}$.

If we multiply (5.94) by $p_i \geq 0$, sum over i from 1 to n, and divide by $P_n > 0$, we deduce

$$\begin{aligned}
&\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) \\
&\qquad - \frac{1}{P_n}\sum_{i=1}^{n} p_i \left(x_i - \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) f'\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) \\
&\geq \frac{1}{P_n}\sum_{i=1}^{n} p_i \left|\left|f(x_i) - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)\right| \right. \\
&\qquad \left. - \left|x_i - \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right|\left|f'\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)\right|\right| \\
&\geq \left|\frac{1}{P_n}\sum_{i=1}^{n} p_i \left|f(x_i) - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)\right| \right. \\
&\left. - \frac{1}{P_n}\sum_{i=1}^{n} p_i \left|x_i - \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right| \cdot \left|f'\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)\right|\right|.
\end{aligned}$$

Since

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i \left(x_i - \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) = 0,$$

the inequality (5.91) is proved. ■

In particular, we have the following result for unweighted means.

Corollary 230 *With the above assumptions for f and x_i, one has the inequality*

$$\begin{aligned}
&\frac{f(x_1) + \cdots + f(x_n)}{n} - f\left(\frac{x_1 + \cdots + x_n}{n}\right) \\
&\qquad\qquad \geq \left|\frac{1}{n}\sum_{i=1}^{n} \left|x_i - f\left(\frac{x_1 + \cdots + x_n}{n}\right)\right| \right. \\
&\qquad\qquad \left. - \left|f'\left(\frac{x_1 + \cdots + x_n}{n}\right)\right| \cdot \frac{1}{n}\sum_{i=1}^{n}\left|x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right|\right| \geq 0. \quad (5.95)
\end{aligned}$$

The following refinement of the (CBS) −inequality holds.

Theorem 231 *If $a_i, b_i \in \mathbb{R}$, $i \in \{1, \ldots, n\}$, then one has the inequality;*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \geq \frac{1}{\sum_{i=1}^{n} b_i^2} \left|\sum_{i=1}^{n} \left\| \begin{array}{cc} \dfrac{a_i^2}{\left(\sum_{j=1}^{n} a_j b_j\right)^2} & \dfrac{b_i^2}{\left(\sum_{j=1}^{n} b_j^2\right)^2} \end{array} \right\| -2\left|\sum_{k=1}^{n} a_k b_k\right| \cdot \sum_{i=1}^{n} |b_i| \left|\sum_{j=1}^{n} b_j \left| \begin{array}{cc} a_i & b_i \\ a_j & b_j \end{array} \right| \right|\right| \geq 0. \quad (5.96)$$

Proof. If we apply Lemma 229 for $f(x) = x^2$, we get

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)^2 \geq \left|\frac{1}{P_n}\sum_{i=1}^{n} p_i \left| x_i^2 - \left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)^2\right| - 2\left|\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k\right| \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i \left| x_i - \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right|\right| \geq 0. \quad (5.97)$$

If in (5.97), we choose $p_i = b_i^2$, $x_i = \frac{a_i}{b_i}$, $i \in \{1, \ldots, n\}$, we get

$$\frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} b_i^2} - \frac{\left(\sum_{i=1}^{n} a_i b_i\right)^2}{\left(\sum_{i=1}^{n} b_i^2\right)^2} \geq \left|\frac{1}{\sum_{i=1}^{n} b_i^2}\sum_{i=1}^{n} b_i^2 \cdot \left|\frac{a_i^2}{b_i^2} - \left(\frac{\sum_{j=1}^{n} a_j b_j}{\sum_{j=1}^{n} b_j^2}\right)^2\right| -2\left|\frac{\sum_{k=1}^{n} a_k b_k}{\sum_{i=1}^{n} b_i^2}\right| \cdot \frac{\sum_{i=1}^{n} b_i^2 \left|\frac{a_i}{b_i} - \sum_{j=1}^{n} a_j b_j \Big/ \sum_{j=1}^{n} b_j^2\right|}{\sum_{i=1}^{n} b_i^2}\right|, \quad (5.98)$$

which is clearly equivalent to (5.96). ■

5.16 An Inequality via Slater's Result

Suppose that I is an interval of real numbers with interior $\overset{\circ}{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\overset{\circ}{I}$ and has finite left and right derivatives at each point of $\overset{\circ}{I}$. Moreover, if $x, y \in \overset{\circ}{I}$ and $x < y$, then $D^- f(x) \le D^+ f(x) \le D^- f(y) \le D^+ f(y)$ which shows that both $D^- f$ and $D^+ f$ are nondecreasing functions on $\overset{\circ}{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the *subdifferential* of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\overset{\circ}{I}\right) \subset \mathbb{R}$ and

$$f(x) \ge f(a) + (x - a)\varphi(a) \quad \text{for any} \quad x, a \in I. \tag{5.99}$$

It is also well known that if f is convex on I, then ∂f is nonempty, $D^+ f, D^- f \in \partial f$ and if $\varphi \in \partial f$, then

$$D^- f(x) \le \varphi(x) \le D^+ f(x) \tag{5.100}$$

for every $x \in \overset{\circ}{I}$. In particular, φ is a nondecreasing function.

If f is differentiable convex on $\overset{\circ}{I}$, then $\partial f = \{f'\}$.

The following inequality is well known in the literature as Slater's inequality [16].

Lemma 232 *Let $f : I \to \mathbb{R}$ be a nondecreasing (nonincreasing) convex function, $x_i \in I$, $p_i \ge 0$ with $P_n := \sum_{i=1}^n p_i > 0$ and $\sum_{i=1}^n p_i \varphi(x_i) \ne 0$ where $\varphi \in \partial f$. Then one has the inequality:*

$$\frac{1}{P_n} \sum_{i=1}^n p_i f(x_i) \le f\left(\frac{\sum_{i=1}^n p_i x_i \varphi(x_i)}{\sum_{i=1}^n p_i \varphi(x_i)} \right). \tag{5.101}$$

Proof. Firstly, observe that since, for example, f is nondecreasing, then $\varphi(x) \ge 0$ for any $x \in I$ and thus

$$\frac{\sum_{i=1}^n p_i x_i \varphi(x_i)}{\sum_{i=1}^n p_i \varphi(x_i)} \in I, \tag{5.102}$$

since it is a convex combination of x_i with the positive weights

$$\frac{x_i \varphi(x_i)}{\sum_{i=1}^n p_i \varphi(x_i)}, \quad i = 1, \dots, n.$$

A similar argument applies if f is nonincreasing.

Now, if we use the inequality (5.99), we deduce

$$f(x) - f(x_i) \geq (x - x_i)\varphi(x_i) \quad \text{for any} \quad x, x_i \in I,\ i = 1, \dots, n. \tag{5.103}$$

Multiplying (5.103) by $p_i \geq 0$ and summing over i from 1 to n, we deduce

$$f(x) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \geq x \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \varphi(x_i) \tag{5.104}$$

for any $x \in I$.

If in (5.104) we choose

$$x = \frac{\sum_{i=1}^{n} p_i x_i \varphi(x_i)}{\sum_{i=1}^{n} p_i \varphi(x_i)},$$

which, we have proved that it belongs to I, we deduce the desired inequality (5.101). ■

If one would like to drop the assumption of monotonicity for the function f, then one can state and prove in a similar way the following result.

Lemma 233 *Let $f : I \to \mathbb{R}$ be a convex function, $x_i \in I$, $p_i \geq 0$ with $P_n > 0$ and $\sum_{i=1}^{n} p_i \varphi(x_i) \neq 0$, where $\varphi \in \partial f$. If*

$$\frac{\sum_{i=1}^{n} p_i x_i \varphi(x_i)}{\sum_{i=1}^{n} p_i \varphi(x_i)} \in I, \tag{5.105}$$

then the inequality (5.101) holds.

The following result in connection to the (CBS) −inequality holds.

Theorem 234 *Assume that $f : \mathbb{R}_+ \to \mathbb{R}$ is a convex function on $\mathbb{R}_+ := [0, \infty)$, $a_i, b_i \geq 0$ with $a_i \neq 0$, $i \in \{1, \dots, n\}$ and $\sum_{i=1}^{n} a_i^2 \varphi\left(\frac{b_i}{a_i}\right) \neq 0$ where $\varphi \in \partial f$.*

(*i*) *If f is monotonic nondecreasing (nonincreasing) in $[0, \infty)$ then*

$$\sum_{i=1}^{n} a_i^2 \varphi\left(\frac{b_i}{a_i}\right) \leq \sum_{i=1}^{n} a_i^2 \cdot f\left(\frac{\sum_{i=1}^{n} a_i b_i \varphi\left(\frac{b_i}{a_i}\right)}{\sum_{i=1}^{n} a_i^2 \varphi\left(\frac{b_i}{a_i}\right)}\right). \tag{5.106}$$

(ii) If

$$\frac{\sum_{i=1}^{n} a_i b_i \varphi\left(\frac{b_i}{a_i}\right)}{\sum_{i=1}^{n} a_i^2 \varphi\left(\frac{b_i}{a_i}\right)} \geq 0, \tag{5.107}$$

then (5.106) also holds.

Remark 235 *Consider the function $f : [0, \infty) \to \mathbb{R}$, $f(x) = x^p$, $p \geq 1$. Then f is convex and monotonic nondecreasing and $\varphi(x) = px^{p-1}$. Applying (5.106), we may deduce the following inequality:*

$$p\left(\sum_{i=1}^{n} a_i^{3-p} b_i^{p-1}\right)^{p+1} \leq \sum_{i=1}^{n} a_i^2 \left(\sum_{i=1}^{n} a_i^{2-p} b_i^p\right)^p \tag{5.108}$$

for $p \geq 1$, $a_i, b_i \geq 0$, $i = 1, \ldots, n$.

5.17 An Inequality via an Andrica-Raşa Result

The following Jensen type inequality has been obtained in [17] by Andrica and Raşa.

Lemma 236 *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable function and assume that*

$$m = \inf_{t \in (a,b)} f''(t) > -\infty \quad \text{and} \quad M = \sup_{t \in (a,b)} f''(t) < \infty.$$

If $x_i \in [a, b]$ and $p_i \geq 0$ $(i = 1, \ldots, n)$ with $\sum_{i=1}^{n} p_i = 1$, then one has the inequalities:

$$\frac{1}{2} m \left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right] \leq \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{2} M \left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right]. \tag{5.109}$$

Proof. Consider the auxiliary function $f_m(t) := f(t) - \frac{1}{2}mt^2$. This function is twice differentiable and $f''_m(t) \geq 0$, $t \in (a,b)$, showing that f_m is convex.

Applying Jensen's inequality for f_m, i.e.,

$$\sum_{i=1}^{n} p_i f_m(x_i) \geq f_m\left(\sum_{i=1}^{n} p_i x_i\right),$$

we deduce, by a simple calculation, the first inequality in (5.109).

The second inequality follows in a similar way for the auxiliary function $f_M(t) = \frac{1}{2}Mt^2 - f(t)$. We omit the details. ■

The above result may be naturally used to obtain the following inequality related to the (CBS) −inequality.

Theorem 237 *Let* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ *be two sequences of real numbers with the property that there exists* $\gamma, \Gamma \in \mathbb{R}$ *such that*

$$-\infty \leq \gamma \leq \frac{a_i}{b_i} \leq \Gamma \leq \infty, \quad \text{for each } i \in \{1, \ldots, n\}, \tag{5.110}$$

and $b_i \neq 0$, $i = 1, \ldots, n$. *If* $f : (\gamma, \Gamma) \to \mathbb{R}$ *is twice differentiable and*

$$m = \inf_{t \in (\gamma,\Gamma)} f''(t) \quad \text{and} \quad M = \sup_{t \in (\gamma,\Gamma)} f''(t),$$

then we have the inequality

$$\begin{aligned}
&\frac{1}{2}m\left[\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2\right] \qquad (5.111)\\
&\leq \sum_{i=1}^{n} b_i^2 \sum_{i=1}^{n} b_i^2 f\left(\frac{a_i}{b_i}\right) - \left(\sum_{i=1}^{n} b_i^2\right)^2 f\left(\frac{\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} b_i^2}\right)\\
&\leq \frac{1}{2}M\left[\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2\right].
\end{aligned}$$

Proof. We may apply Lemma 236 for the choices $p_i = \frac{b_i^2}{\sum_{k=1}^n b_k^2}$ and $x_i = \frac{a_i}{b_i}$ to get

$$\begin{aligned}
&\frac{1}{2}m\left[\frac{\sum_{i=1}^n a_i^2}{\sum_{k=1}^n b_k^2} - \left(\frac{\sum_{i=1}^n a_ib_i}{\sum_{k=1}^n b_k^2}\right)^2\right] \\
&\leq \frac{\sum_{i=1}^n b_i^2 f\left(\frac{a_i}{b_i}\right)}{\sum_{k=1}^n b_k^2} - f\left(\frac{\sum_{i=1}^n a_ib_i}{\sum_{k=1}^n b_k^2}\right) \\
&\leq \frac{1}{2}M\left[\frac{\sum_{i=1}^n a_i^2}{\sum_{k=1}^n b_k^2} - \left(\frac{\sum_{i=1}^n a_ib_i}{\sum_{k=1}^n b_k^2}\right)^2\right]
\end{aligned}$$

giving the desired result (5.111). ■

The following corollary is a natural consequence of the above theorem.

Corollary 238 *Assume that* $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ *are sequences of nonnegative real numbers and*

$$0 < \varphi \leq \frac{a_i}{b_i} \leq \Phi < \infty \quad \textit{for each} \quad i \in \{1, \ldots, n\}. \tag{5.112}$$

Then for any $p \in [1, \infty)$ *one has the inequalities*

$$\begin{aligned}
&\frac{1}{2}p(p-1)\varphi^{p-2}\left[\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_ib_i\right)^2\right] \\
&\leq \sum_{i=1}^n b_i^2 \left(\sum_{i=1}^n a_i^p b_i^{2-p}\right)^p - \left(\sum_{i=1}^n b_i^2\right)^{2-p}\left(\sum_{i=1}^n a_ib_i\right)^p \\
&\leq \frac{1}{2}p(p-1)\Phi^{p-2}\left[\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_ib_i\right)^2\right]
\end{aligned} \tag{5.113}$$

if $p \in [2, \infty)$ *and*

$$\begin{aligned}
&\frac{1}{2}p(p-1)\Phi^{p-2}\left[\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_ib_i\right)^2\right] \\
&\leq \sum_{i=1}^n b_i^2 \left(\sum_{i=1}^n a_i^p b_i^{2-p}\right)^p - \left(\sum_{i=1}^n b_i^2\right)^{2-p}\left(\sum_{i=1}^n a_ib_i\right)^p
\end{aligned} \tag{5.114}$$

$$\leq \frac{1}{2}p(p-1)\varphi^{p-2}\left[\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2-\left(\sum_{i=1}^{n}a_ib_i\right)^2\right]$$

if $p \in [1,2]$.

5.18 An Inequality via Jensen's Result for Double Sums

The following result for convex functions via Jensen's inequality also holds [18].

Lemma 239 *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a convex (concave) function and* $\bar{\mathbf{x}}=(x_1,\ldots,x_n)$, $\bar{\mathbf{p}}=(p_1,\ldots,p_n)$ *real sequences with the property that* $p_i\geq 0$ $(i=1,\ldots,n)$ *and* $\sum_{i=1}^{n}p_i=1$. *Then one has the inequality:*

$$f\left[\frac{\sum_{i=1}^{n}p_ix_i^2-\left(\sum_{i=1}^{n}p_ix_i\right)^2}{\sum_{i=1}^{n}i^2p_i-\left(\sum_{i=1}^{n}ip_i\right)^2}\right]$$
$$\leq(\geq)\frac{\sum_{1\leq i<j\leq n}p_ip_j\left[\sum_{k,l=i}^{j-1}f(\Delta x_k\cdot\Delta x_l)\right]}{\sum_{i=1}^{n}i^2p_i-\left(\sum_{i=1}^{n}ip_i\right)^2},\quad(5.115)$$

where $\Delta x_k := x_{k+1}-x_k$ $(k=1,\ldots,n-1)$ *is the forward difference.*

Proof. We have, by Jensen's inequality for multiple sums that

$$f\left[\frac{\sum_{i=1}^{n}p_ix_i^2-\left(\sum_{i=1}^{n}p_ix_i\right)^2}{\sum_{i=1}^{n}i^2p_i-\left(\sum_{i=1}^{n}ip_i\right)^2}\right]=f\left[\frac{\sum_{1\leq i<j\leq n}p_ip_j(x_i-x_j)^2}{\sum_{1\leq i<j\leq n}p_ip_j(j-i)^2}\right]\quad(5.116)$$
$$=f\left[\frac{\sum_{1\leq i<j\leq n}p_ip_j(j-i)^2\frac{(x_j-x_i)^2}{(j-i)^2}}{\sum_{1\leq i<j\leq n}p_ip_j(j-i)^2}\right]$$
$$\leq\frac{\sum_{1\leq i<j\leq n}p_ip_j(j-i)^2f\left(\frac{(x_j-x_i)^2}{(j-i)^2}\right)}{\sum_{1\leq i<j\leq n}p_ip_j(j-i)^2}=:I.$$

On the other hand, for $j>i$, one has

$$x_j-x_i=\sum_{k=i}^{j-1}(x_{k+1}-x_k)=\sum_{k=i}^{j-1}\Delta x_k\quad(5.117)$$

and thus

$$(x_j - x_i)^2 = \left(\sum_{k=i}^{j-1} \Delta x_k\right)^2 = \sum_{k,l=i}^{j-1} \Delta x_k \cdot \Delta x_l.$$

Applying once more the Jensen inequality for multiple sums, we deduce

$$f\left[\frac{(x_j - x_i)^2}{(j-i)^2}\right] = f\left[\frac{\sum_{k,l=i}^{j-1} \Delta x_k \cdot \Delta x_l}{(j-i)^2}\right] \leq (\geq) \frac{\sum_{k,l=i}^{j-1} f(\Delta x_k \cdot \Delta x_l)}{(j-i)^2} \quad (5.118)$$

and thus, by (5.118), we deduce

$$\begin{aligned} I &\leq (\geq) \frac{\sum_{1\leq i<j\leq n} p_i p_j (j-i)^2 \frac{\sum_{k,l=i}^{j-1} f(\Delta x_k \cdot \Delta x_l)}{(j-i)^2}}{\sum_{1\leq i<j\leq n} p_i p_j (j-i)^2} \quad (5.119) \\ &= \frac{\sum_{1\leq i<j\leq n} p_i p_j \left[\sum_{k,l=i}^{j-1} f(\Delta x_k \cdot \Delta x_l)\right]}{\sum_{i=1}^n i^2 p_i - \left(\sum_{i=1}^n i p_i\right)^2}, \end{aligned}$$

and then, by (5.116) and (5.119), we deduce the desired inequality (5.115). ■

The following inequality connected with the (CBS)−inequality may be stated.

Theorem 240 *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex (concave) function and $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$, $\bar{\mathbf{w}} = (w_1, \dots, w_n)$ sequences of real numbers such that $b_i \neq 0$, $w_i \geq 0$ $(i = 1, \dots, n)$ and not all w_i are zero. Then one has the inequality*

$$\begin{aligned} & f\left[\frac{\sum_{i=1}^n w_i a_i^2 \sum_{i=1}^n w_i b_i^2 - \left(\sum_{i=1}^n w_i a_i b_i\right)^2}{\sum_{i=1}^n w_i b_i^2 \sum_{i=1}^n i^2 w_i b_i^2 - \left(\sum_{i=1}^n i w_i b_i\right)^2}\right] \\ & \quad \leq (\geq) \frac{\sum_{1\leq i<j\leq n} w_i w_j b_i^2 b_j^2 \left[\sum_{k,l=i}^{j-1} f\left(\Delta\left(\frac{a_k}{b_k}\right) \cdot \Delta\left(\frac{a_l}{b_l}\right)\right)\right]}{\sum_{i=1}^n w_i b_i^2 \sum_{i=1}^n i^2 w_i b_i^2 - \left(\sum_{i=1}^n i w_i b_i\right)^2}. \quad (5.120) \end{aligned}$$

Proof. Follows by Lemma 239 on choosing $p_i = w_i b_i^2$ and $x_i = \frac{a_i}{b_i}$, $i = 1, \dots, n$. We omit the details. ■

5.19 Some Inequalities for the Čebyšev Functional

For two sequences of real numbers $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ and $\bar{\mathbf{p}} = (p_1, \ldots, p_n)$ with $p_i \geq 0$ $(i \in \{1, \ldots, n\})$ and $\sum_{i=1}^{n} p_i = 1$, consider the Čebyšev functional

$$T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) := \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i. \tag{5.121}$$

By Korkine's identity [1, p. 242] one has the representation

$$\begin{aligned} T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) &= \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \left(a_i - a_j\right)\left(b_i - b_j\right) \\ &= \sum_{1 \leq i < j \leq n} p_i p_j \left(a_j - a_i\right)\left(b_j - b_i\right). \end{aligned} \tag{5.122}$$

Using the (CBS) −inequality for double sums one may state the following result

$$\left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right)\right]^2 \leq T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}\right) T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right), \tag{5.123}$$

where, obviously

$$\begin{aligned} T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}\right) &= \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \left(a_i - a_j\right)^2 \\ &= \sum_{1 \leq i < j \leq n} p_i p_j \left(a_j - a_i\right)^2. \end{aligned} \tag{5.124}$$

The following result holds [14].

Lemma 241 *Assume that* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ *are real numbers such that for each* $i, j \in \{1, \ldots, n\}$, $i < j$, *one has*

$$m\left(b_j - b_i\right) \leq a_j - a_i \leq M\left(b_j - b_i\right), \tag{5.125}$$

where m, M *are given real numbers.*

If $\bar{\mathbf{p}} = (p_1, \ldots, p_n)$ *is a nonnegative sequence with* $\sum_{i=1}^{n} p_i = 1$, *then one has the inequality*

$$(m + M)\, T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) \geq T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}\right) + mMT\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right). \tag{5.126}$$

Proof. If we use the condition (5.125), we get

$$[M(b_i - b_j) - (a_i - a_j)][(a_i - a_j) - m(b_i - b_j)] \geq 0 \tag{5.127}$$

for $i, j \in \{1, \ldots, n\}$, $i < j$.

If we multiply in (5.127), then, obviously, for any $i, j \in \{1, \ldots, n\}$, $i < j$ we have

$$(a_j - a_i)^2 + mM(b_j - b_i)^2 \leq (m + M)(a_j - a_i)(b_j - b_i). \tag{5.128}$$

Multiplying (5.128) by $p_i p_j \geq 0$, $i, j \in \{1, \ldots, n\}$, $i < j$, summing over i and j, $i < j$ from 1 to n and using the identities (5.122) and (5.124), we deduce the required inequality (5.125). ■

The following result holds [14].

Theorem 242 *If* $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, $\bar{\mathbf{p}}$ *are as in Lemma 241 and* $M \geq m > 0$, *then one has the inequality providing a counterpart for (5.123)*

$$\left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right)\right]^2 \geq \frac{4mM}{(m + M)^2} T(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}) T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right). \tag{5.129}$$

Proof. We use the following elementary inequality

$$\alpha x^2 + \frac{1}{\alpha} y^2 \geq 2xy, \quad x, y \geq 0, \ \alpha > 0 \tag{5.130}$$

to get, for the choices

$$\alpha = \sqrt{mM} > 0, \quad x = \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right)\right]^{\frac{1}{2}} \geq 0, \quad y = [T(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}})]^{\frac{1}{2}} \geq 0$$

the following inequality:

$$\sqrt{mM} T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right) + \frac{1}{\sqrt{mM}} T(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}) \geq 2\left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right)\right]^{\frac{1}{2}} [T(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}})]^{\frac{1}{2}}. \tag{5.131}$$

Using (5.130) and (5.131), we deduce

$$\frac{(m + M)}{2\sqrt{mM}} T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) \geq \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right)\right]^{\frac{1}{2}} [T(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}})]^{\frac{1}{2}}$$

which is clearly equivalent to (5.129). ■

The following corollary also holds [14].

Corollary 243 *With the assumptions of Theorem 242, we have:*

$$0 \le \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right)\right]^{\frac{1}{2}} \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}\right)\right]^{\frac{1}{2}} - T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) \le \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) \tag{5.132}$$

and

$$0 \le T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}\right) T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right) - \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right)\right]^2 \le \frac{(M-m)^2}{4mM} \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right)\right]^2. \tag{5.133}$$

The following result is useful in practical applications [14].

Theorem 244 *Let $f, g : [\alpha, \beta] \to \mathbb{R}$ be continuous on $[\alpha, \beta]$ and differentiable on (α, β) with $g'(x) \neq 0$ for $x \in (\alpha, \beta)$. Assume*

$$-\infty < \gamma = \inf_{x \in (\alpha,\beta)} \frac{f'(x)}{g'(x)}, \quad \sup_{x \in (\alpha,\beta)} \frac{f'(x)}{g'(x)} = \Gamma < \infty. \tag{5.134}$$

If $\bar{\mathbf{x}}$ is a real sequence with $x_i \in [\alpha, \beta]$ and $x_i \neq x_j$ for $i \neq j$ and if we denote by $\mathbf{f}(\bar{\mathbf{x}}) := (f(x_1), \ldots, f(x_n))$, then we have the inequality:

$$(\gamma + \Gamma) T\left(\bar{\mathbf{p}}, \mathbf{f}(\bar{\mathbf{x}}), \mathbf{g}(\bar{\mathbf{x}})\right) \ge T\left(\bar{\mathbf{p}}, \mathbf{f}(\bar{\mathbf{x}}), \mathbf{f}(\bar{\mathbf{x}})\right) + \gamma\Gamma T\left(\bar{\mathbf{p}}, \mathbf{g}(\bar{\mathbf{x}}), \mathbf{g}(\bar{\mathbf{x}})\right) \tag{5.135}$$

for any $\bar{p}$ with $p_i \ge 0$ $(i \in \{1, \ldots, n\})$, $\sum_{i=1}^n p_i = 1$.

Proof. Applying the Cauchy Mean-Value Theorem, there exists $\xi_{ij} \in (\alpha, \beta)$ $(i < j)$ such that

$$\frac{f(x_j) - f(x_i)}{g(x_j) - g(x_i)} = \frac{f'\left(\xi_{ij}\right)}{g'\left(\xi_{ij}\right)} \in [\gamma, \Gamma] \tag{5.136}$$

for $i, j \in \{1, \ldots, n\}$, $i < j$. Then

$$\left[\Gamma - \frac{f(x_j) - f(x_i)}{g(x_j) - g(x_i)}\right] \left[\frac{f(x_j) - f(x_i)}{g(x_j) - g(x_i)} - \gamma\right] \ge 0, \quad 1 \le i < j \le n, \tag{5.137}$$

which, by a similar argument to the one in Lemma 241 will give the desired result (5.135). ■

The following corollary is natural [14].

Corollary 245 *With the assumptions in Theorem 244 and if* $\Gamma \geq \gamma > 0$, *then one has the inequalities:*

$$\left[T\left(\bar{\mathbf{p}}, \mathbf{f}\left(\bar{\mathbf{x}}\right), \mathbf{g}\left(\bar{\mathbf{x}}\right)\right)\right]^2 \geq \frac{4\gamma\Gamma}{\left(\gamma+\Gamma\right)^2} T\left(\bar{\mathbf{p}}, \mathbf{f}\left(\bar{\mathbf{x}}\right), \mathbf{f}\left(\bar{\mathbf{x}}\right)\right) T\left(\bar{\mathbf{p}}, \mathbf{g}\left(\bar{\mathbf{x}}\right), \mathbf{g}\left(\bar{\mathbf{x}}\right)\right), \quad (5.138)$$

$$\begin{aligned} 0 &\leq \left[T\left(\bar{\mathbf{p}}, \mathbf{f}\left(\bar{\mathbf{x}}\right), \mathbf{f}\left(\bar{\mathbf{x}}\right)\right)\right]^{\frac{1}{2}} \left[T\left(\bar{\mathbf{p}}, \mathbf{g}\left(\bar{\mathbf{x}}\right), \mathbf{g}\left(\bar{\mathbf{x}}\right)\right)\right]^{\frac{1}{2}} - T\left(\bar{\mathbf{p}}, \mathbf{f}\left(\bar{\mathbf{x}}\right), \mathbf{g}\left(\bar{\mathbf{x}}\right)\right) \quad (5.139) \\ &\leq \frac{\left(\sqrt{\Gamma}-\sqrt{\gamma}\right)^2}{2\sqrt{\gamma\Gamma}} T\left(\bar{\mathbf{p}}, \mathbf{f}\left(\bar{\mathbf{x}}\right), \mathbf{g}\left(\bar{\mathbf{x}}\right)\right) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq T\left(\bar{\mathbf{p}}, \mathbf{f}\left(\bar{\mathbf{x}}\right), \mathbf{f}\left(\bar{\mathbf{x}}\right)\right) T\left(\bar{\mathbf{p}}, \mathbf{g}\left(\bar{\mathbf{x}}\right), \mathbf{g}\left(\bar{\mathbf{x}}\right)\right) - T^2\left(\bar{\mathbf{p}}, \mathbf{f}\left(\bar{\mathbf{x}}\right), \mathbf{g}\left(\bar{\mathbf{x}}\right)\right) \quad (5.140) \\ &\leq \frac{\left(\Gamma-\gamma\right)^2}{4\gamma\Gamma} T^2\left(\bar{\mathbf{p}}, \mathbf{f}\left(\bar{\mathbf{x}}\right), \mathbf{g}\left(\bar{\mathbf{x}}\right)\right). \end{aligned}$$

5.20 Other Inequalities for the Čebyšev Functional

For two sequences of real numbers $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ and $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ with $p_i \geq 0 \ \ (i \in \{1, \dots, n\})$ and $\sum_{i=1}^n p_i = 1$, consider the Čebyšev functional

$$T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) = \sum_{i=1}^n p_i a_i b_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i. \quad (5.141)$$

By Sonin's identity [1, p. 246] one has the representation

$$T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) = \sum_{i=1}^n p_i \left(a_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}\right)\right)\left(b_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right)\right), \quad (5.142)$$

where

$$A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}\right) := \sum_{j=1}^n p_j a_j, \quad A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right) := \sum_{j=1}^n p_j b_j.$$

Using the $(CBS)-$inequality for weighted sums, we may state the following result

$$\left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right)\right]^2 \leq T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}\right) T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right), \tag{5.143}$$

where, obviously

$$T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}\right) = \sum_{i=1}^{n} p_i \left(a_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}\right)\right)^2. \tag{5.144}$$

The following result holds [14].

Lemma 246 *Assume that* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ *are real numbers,* $\bar{\mathbf{p}} = (p_1, \ldots, p_n)$ *are nonnegative numbers with* $\sum_{i=1}^{n} p_i = 1$ *and* $b_i \neq A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right)$ *for each* $i \in \{1, \ldots, n\}$. *If*

$$-\infty < l \leq \frac{a_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}\right)}{b_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right)} \leq L < \infty \quad \text{for all } i \in \{1, \ldots, n\}, \tag{5.145}$$

where l, L *are given real numbers, then one has the inequality*

$$(l + L)\, T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) \geq T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}\right) + LlT\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right). \tag{5.146}$$

Proof. Using (5.145) we have

$$\left(L - \frac{a_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}\right)}{b_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right)}\right)\left(\frac{a_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}\right)}{b_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right)} - l\right) \geq 0 \tag{5.147}$$

for each $i \in \{1, \ldots, n\}$.

If we multiply (5.147) by $\left(b_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right)\right)^2 \geq 0$, we get

$$\begin{aligned}\left(a_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}\right)\right)^2 &+ Ll\left(b_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right)\right)^2 \\ &\leq (L + l)\left(a_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}\right)\right)\left(b_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right)\right)\end{aligned} \tag{5.148}$$

for each $i \in \{1, \ldots, n\}$.

Finally, if we multiply (5.148) by $p_i \geq 0$, sum over i from 1 to n and use the identity (5.142) and (5.144), we obtain (5.146). ■

Using Lemma 246 and a similar argument to that in the previous section, we may state the following theorem [14].

Theorem 247 *With the assumption of Lemma 246 and if $L \geq l > 0$, then one has the inequality*

$$\left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right)\right]^2 \geq \frac{4lL}{(L+l)^2} T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}\right) T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right). \tag{5.149}$$

The following corollary is natural [14].

Corollary 248 *With the assumptions in Theorem 247 one has*

$$\begin{aligned} 0 &\leq \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}\right)\right]^{\frac{1}{2}} \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right)\right]^{\frac{1}{2}} - T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) \\ &\leq \frac{\left(\sqrt{L} - \sqrt{l}\right)^2}{2\sqrt{lL}} T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right), \end{aligned} \tag{5.150}$$

and

$$\begin{aligned} 0 &\leq T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}\right) T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right) - \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right)\right]^2 \\ &\leq \frac{(L-l)^2}{4lL} \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right)\right]^2. \end{aligned} \tag{5.151}$$

5.21 Bounds for the Čebyšev Functional

The following result holds.

Theorem 249 *Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ (with $b_i \neq b_j$ for $i \neq j$) be two sequences of real numbers with the property that there exists the real constants m, M such that for any $1 \leq i < j \leq n$ one has*

$$m \leq \frac{a_j - a_i}{b_j - b_i} \leq M. \tag{5.152}$$

Then we have the inequality

$$mT\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right) \leq T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) \leq MT\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right), \tag{5.153}$$

for any nonnegative sequence $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ with $\sum_{i=1}^{n} p_i = 1$.

Proof. From (5.152), by multiplying with $(b_j - b_i)^2 > 0$, one has

$$m (b_j - b_i)^2 \leq (a_j - a_i)(b_j - b_i) \leq M (b_j - b_i)^2$$

for any $1 \leq i < j \leq n$, giving by multiplication with $p_i p_j \geq 0$, that

$$\begin{aligned} m \sum_{1 \leq i < j \leq n} p_i p_j (b_i - b_j)^2 &\leq \sum_{1 \leq i < j \leq n} p_i p_j (a_j - a_i)(b_j - b_i) \\ &\leq M \sum_{1 \leq i < j \leq n} p_i p_j (b_i - b_j)^2 . \end{aligned}$$

Using Korkine's identity (see for example Section 5.19), we deduce the desired result (5.153). ■

The following corollary is natural.

Corollary 250 *Assume that the sequence* $\bar{\mathbf{b}}$ *in Theorem 249 is strictly increasing and there exists* m, M *such that*

$$m \leq \frac{\Delta a_k}{\Delta b_k} \leq M, \quad k = 1, \ldots, n-1; \tag{5.154}$$

where $\Delta a_k := a_{k+1} - a_k$ *is the forward difference, then (5.153) holds true.*

Proof. Follows from Theorem 249 on taking into account that for $j > i$ and from (5.154) one has

$$m \sum_{k=i}^{j-1} \Delta b_k \leq \sum_{k=i}^{j-1} \Delta a_k \leq M \sum_{k=i}^{j-1} \Delta b_k,$$

giving $m (b_j - b_i) \leq a_j - a_i \leq M (b_j - b_i)$. ■

Another possibility is to use functions that generate similar inequalities.

Theorem 251 *Let* $f, g : [\alpha, \beta] \to \mathbb{R}$ *be continuous on* $[\alpha, \beta]$ *and differentiable on* (α, β) *with* $g'(x) \neq 0$ *for* $x \in (\alpha, \beta)$. *Assume that*

$$-\infty < m = \inf_{x \in (\alpha,\beta)} \frac{f'(x)}{g'(x)}, \quad \sup_{x \in (\alpha,\beta)} \frac{f'(x)}{g'(x)} = M < \infty.$$

If $\bar{\mathbf{x}} = (x_1, \ldots, x_n)$ *is a real sequence with* $x_i \in [\alpha, \beta]$ *and* $x_i \neq x_j$ *for* $i \neq j$ *and if we denote* $\mathbf{f}(\bar{\mathbf{x}}) := (f(x_1), \ldots, f(x_n))$, *then we have the inequality*

$$mT(\bar{\mathbf{p}}, \mathbf{g}(\bar{\mathbf{x}}), \mathbf{g}(\bar{\mathbf{x}})) \leq T(\bar{\mathbf{p}}, \mathbf{f}(\bar{\mathbf{x}}), \mathbf{g}(\bar{\mathbf{x}})) \leq MT(\bar{\mathbf{p}}, \mathbf{g}(\bar{\mathbf{x}}), \mathbf{g}(\bar{\mathbf{x}})). \tag{5.155}$$

Proof. Applying the Cauchy Mean-Value Theorem, for any $j > i$ there exists $\xi_{ij} \in (\alpha, \beta)$ such that

$$\frac{f(x_j) - f(x_i)}{g(x_j) - g(x_i)} = \frac{f'(\xi_{ij})}{g'(\xi_{ij})} \in [m, M].$$

Then, by Theorem 249 applied for $a_i = f(x_i)$, $b_i = g(x_i)$, we deduce the desired inequality (5.155). ∎

The following inequality related to the $(CBS)-$inequality holds.

Theorem 252 *Let $\bar{\mathbf{a}}$, $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$ be sequences of real numbers such that $x_i \neq 0$ and $\frac{y_i}{x_i} \neq \frac{y_j}{x_j}$ for $i \neq j$, $(i, j \in \{1, \dots, n\})$. If there exist the real numbers γ, Γ such that*

$$\gamma \leq \frac{a_j - a_i}{\frac{y_j}{x_j} - \frac{y_i}{x_i}} \leq \Gamma \quad \text{for} \quad 1 \leq i < j \leq n, \tag{5.156}$$

then we have the inequality

$$\gamma \left[\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} x_i y_i \right)^2 \right] \tag{5.157}$$
$$\leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} a_i x_i y_i - \sum_{i=1}^{n} a_i x_i^2 \sum_{i=1}^{n} x_i y_i$$
$$\leq \Gamma \left[\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} x_i y_i \right)^2 \right].$$

Proof. Follows by Theorem 249 on choosing $p_i = \frac{x_i^2}{\sum_{k=1}^{n} x_k^2}$, $b_i = \frac{y_i}{x_i}$, $m = \gamma$ and $M = \Gamma$. We omit the details. ∎

The following different approach may be considered as well.

Theorem 253 *Assume that $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ are sequences of real numbers, $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ is a sequence of nonnegative real numbers with $\sum_{i=1}^{n} p_i = 1$ and $b_i \neq A_n(\bar{\mathbf{p}}, \bar{\mathbf{b}}) := \sum_{i=1}^{n} p_i b_i$. If*

$$-\infty < l \leq \frac{a_i - A_n(\bar{\mathbf{p}}, \bar{\mathbf{a}})}{b_i - A_n(\bar{\mathbf{p}}, \bar{\mathbf{b}})} \leq L < \infty \quad \text{for each} \quad i \in \{1, \dots, n\}, \tag{5.158}$$

where l, L are given real numbers, then one has the inequality

$$lT(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}) \leq T(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}) \leq LT(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}). \tag{5.159}$$

Proof. From (5.158), by multiplying with $\left(b_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right)\right)^2 > 0$, we deduce

$$\begin{aligned} l\left(b_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right)\right)^2 &\le \left(a_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}\right)\right)\left(b_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right)\right) \\ &\le L\left(b_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right)\right)^2, \end{aligned}$$

for any $i \in \{1, \dots, n\}$.

By multiplying with $p_i \ge 0$, and summing over i from 1 to n, we deduce

$$\begin{aligned} l\sum_{i=1}^n p_i\left(b_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right)\right)^2 &\le \sum_{i=1}^n p_i\left(a_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}\right)\right)\left(b_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right)\right) \\ &\le L\sum_{i=1}^n p_i\left(b_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right)\right)^2. \end{aligned}$$

Using Sonin's identity (see for example Section 5.20), we deduce the desired result (5.159). ■

The following result in connection with the (CBS) −inequality may be stated.

Theorem 254 *Let* $\bar{\mathbf{a}}, \bar{\mathbf{x}}, \bar{\mathbf{b}}$ *be sequences of real numbers such that* $x_i \ne 0$ *and* $\frac{y_i}{x_i} \ne \frac{1}{\sum_{i=1}^n x_i^2} A_n\left(\overline{\mathbf{x}^2}, \overline{\frac{\mathbf{y}}{\mathbf{x}}}\right)$ *for* $i \in \{1, \dots, n\}$. *If there exists the real numbers* ϕ, Φ *such that*

$$\phi \le \frac{a_i - \frac{1}{\sum_{i=1}^n x_i^2} A_n\left(\overline{\mathbf{x}^2}, \bar{\mathbf{a}}\right)}{\frac{y_i}{x_i} - \frac{1}{\sum_{i=1}^n x_i^2} A_n\left(\overline{\mathbf{x}^2}, \overline{\frac{\mathbf{y}}{\mathbf{x}}}\right)} \le \Phi, \tag{5.160}$$

where $\overline{\mathbf{x}^2} = \left(x_1^2, \dots, x_n^2\right)$ *and* $\overline{\frac{\mathbf{y}}{\mathbf{x}}} = \left(\frac{y_1}{x_1}, \dots, \frac{y_n}{x_n}\right)$, *then one has the inequality (5.157).*

Proof. Follows by Theorem 253 on choosing $p_i = \frac{x_i^2}{\sum_{i=1}^n x_i^2}$, $b_i = \frac{y_i}{x_i}$, $l = \phi$ and $L = \Phi$. We omit the details. ■

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