Selected Topics on Hermite-Hadamard Inequalities and Applications

Sever S. Dragomir

Charles E.M. Pearce

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia.

E-mail address: sever@matilda.vu.edu.au

URL: http://rgmia.vu.edu.au/SSDragomirWeb.html

Department of Applied Mathematics, University of Adelaide, Adelaide 5000.

E-mail address: cpearce@maths.adelaide.edu.au

URL: http://www.maths.adelaide.edu.au/Applied/staff/cpearce.html

1991 *Mathematics Subject Classification.* Primary 26D15, 26D10; Secondary 26D99.

The first author would like to thank the University of Adelaide, Australia, for its support in completing a large part of this monograph during his visits in 1995, 1996 and 1997.

Abstract. The Hermite-Hadamard double inequality is the first fundamental result for convex functions defined on a interval of real numbers with a natural geometrical interpretation and a loose number of applications for particular inequalities. In this monograph we present the basic facts related to Hermite-Hadamard inequalities for convex functions and a large number of results for special means which can naturally be deduced. Hermite-Hadamard type inequalities for other concepts of convexities are also given. The properties of a number of functions and functionals or sequences of functions which can be associated in order to refine the $H. - H.$ result are pointed out. Recent references that are available online are mentioned as well.

Contents

1. Preface

As J.L.W.V. Jensen anticipated in 1906:

"Il me semble que la notion de fonction convexe est à peu prés aussi fondamentale que celles-ci: fonction positive, fonction croissante. Si je ne tromp pas en ceci, la notion devra trouver sa place dans les expositions élémentaires de la théorie des fonctions réelles"

the concept of convex functions has indeed found an important place in *Modern Mathematics* as can be seen in a large number of research articles and books devoted to the field these days.

In this context, the Hermite-Hadamard inequality, which, we can say, is the first fundamental result for convex functions with a natural geometrical interpretation and many applications, has attracted and continues to attract much interest in elementary mathematics.

Many mathematicians have devoted their efforts to generalise, refine, counterpart and extend it for different classes of functions such as: quasi-convex functions, Godunova-Levin class of functions, log-convex and $r-$convex functions, $p-$functions, etc or apply it for special means ($p-$logarithmic means, identric mean, Stolarsky means, etc).

The present work endeavours to present some of the fundamental results connected to the Hermite-Hadamard inequality in which the authors have been involved during the last ten years. It does not claim that it contains all the significant results about Hermite-Hadamard ($H.-H.$) inequalities and their companions, but at least it has those results that have natural applications for special means, which is a subsequent aim of this work.

In the Introduction, after considering some historical considerations, we present a number of fundamental facts as can be found in the book [**147**] which has devoted almost a whole chapter to this important inequality in the larger context of convex functions.

In Chapter 2, we consider some new generalisations related to the Hermite-Hadamard inequality. Hadamard's inferior and superior sums are introduced, refinements of the $H.-H.$ inequality for modulus are presented and natural generalisations for $n-$time differentiable functions and for isotonic linear and sublinear functionals are pointed out. A large number of applications for special means are obtained.

Chapter 3 is completely devoted to the functionals which can naturally be associated to the $H.-H.$ inequality. Their properties, such as: superadditivity, monotonicity and supermultiplicity, are studied. The monotonicity and convexity properties of other functions, defined in terms of simple or double integrals, are also considered.

Chapter 4 contains some sequences of mapping defined in terms of multiple integrals which refine the $H.-H.$ inequalities. Their convergence to $f\left(\frac{a+b}{2}\right)$ is investigated.

In Chapter 5, we present a number of $H.-H.$ type inequalities which can be obtained for functions that are: log-convex or $r-$convex, or belong to the class of Godunova-Levin. Similar results for quasi-convex, $p-$functions, multiplicatively convex, $s-$convex functions in the first and second sense and $m-$convex functions are also derived. Generalisation for convex-dominated and Lipschitzian functions and some applications are also presented.

The last chapter, Chapter 6, is devoted to some recent result on Hermite-Hadamard type inequalities for mappings of several variables, including functions defined on a disk in a plane, functions defined on a ball in a space and a result for convex domains in $\mathbb{R}^3$.

This book is intended for use in the fields of integral inequalities, approximation theory, special means theory, optimisation theory, information theory and numerical analysis.

The authors would like to thank Ms. M. R. Pinheiro for some valuable remarks which have been incorporated into this book.

The Authors,
Melbourne and Adelaide, July 2002.

CHAPTER 1

Introduction

1. Historical Considerations

We start with the following historical considerations (see [**112**] and [**147**, p. 137]).

On November 22, 1881, Hermite (1822-1901) sent a letter to the journal *Mathesis*. An extract from that letter was published in *Mathesis* **3** (1883, p. 82). It reads as follows:

"Sur deux limites d'une intégrale définie. *Soit $f(x)$ une fonction qui varie toujours dans le même sens de $x = a$, á $x = b$. On aura les relations*

$$(b-a) f\left(\frac{a+b}{2}\right) < \int_a^b f(x)\,dx < (b-a)\frac{f(a)+f(b)}{2} \tag{1.1}$$

ou bien

$$(b-a) f\left(\frac{a+b}{2}\right) > \int_a^b f(x)\,dx > (b-a)\frac{f(a)+f(b)}{2}$$

suivant que la courbe $y = f(x)$ tourne sa convexité ou sa concavité vers l'axe des abcisses.

En faisant dans ces formules $f(x) = 1/(1+x)$, $a = 0$, $b = x$ il vient

$$x - \frac{x^2}{2+x} < \log(1+x) < x - \frac{x^2}{2(1+x)}.\text{"}$$

It is interesting to note that this short note of Hermite is nowhere mentioned in mathematical literature, and that these important inequalities (of Hermite) are not widely known as Hermite's result. His note is recorded neither in the authoritative journal *Jahrbuch über die Fortschritte der Mathematik* nor in Hermite's collected papers, which were published "*sous les auspices de l'Académie des sciences de Paris par* Emile Picard (1905-1917), *membre de l'Institut.*" In the booklet on Hermite by Jordan and Mansion (1901), Mansion published a bibliography of Hermite's writings, but this note in *Mathesis* was not included [**112**]. Beckenbach, a leading expert on the history and theory of complex functions, wrote that the first inequality in (1.1) was proved by Hadamard in 1893 [**7**, p. 441] and apparently was not aware of Hermite's result.

It should be mentioned that Fejér (1880-1959), while studying trigonometric polynomials (1906), obtained inequalities which generalise that of Hermite, but again Hermite's work was not acknowledged. In its original form, Fejér's result reads [**72**] (see also [**147**, p. 138]):

Theorem 1. *Consider the integral $\int_a^b f(x)\,g(x)\,dx$, where f is a convex function in the interval (a,b) and g is a positive function in the same interval such* *that*

$$g(a+t)=g(b-t),\ \ 0\le t\le \frac{1}{2}(a+b),$$

i.e., $y=g(x)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b),0\right)$ and is normal to the $x-$axis. Under those conditions the following inequalities are valid:

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\le\int_a^b f(x)\,g(x)\,dx\le\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \tag{1.2}$$

Clearly, for $g(x)\equiv 1$ and $x\in(a,b)$, we obtain Hermite's inequalities. Therefore, Hermite's important result in (1.1), which provides a necessary and sufficient condition for a function f to be convex in (a,b), has not been credited to him in mathematical literature. In fact, the term "convex" also stems from a result obtained by Hermite in 1881 and published in 1883 as a short note in *Mathesis*, a journal of elementary mathematics. There are results of lesser importance which have received more attention in the area of inequalities, but unfortunately this fundamental work of Hermite has been frequently cited without the correct identification of its original author [**112**].

It is obvious that (1.1), is an interpolating inequality for

$$f\left(\frac{a+b}{2}\right)\le\frac{f(a)+f(b)}{2}. \tag{1.3}$$

More than twenty years after Hermite's work was published, J. L. W. V. Jensen (1905, 1906) defined convex functions (i.e., $J-$convex functions) using inequality (1.3) [**88**]. His remark, which we cite here, was shown to be justified: "*Il me semble que la notion de fonction convexe est à peu prés aussi fondamentale que celles-ci: fonction positive, fonction croissante. Si je ne tromp pas en ceci, la notion devra trouver sa place dans les expositions élémentaires de la théorie des fonctions réelles*" (Jensen, 1906).

Indeed, it is not easy to give a complete treatment of the literature when studying convex functions, but the importance of Hermite's result is obvious. Since the inequalities in (1.1) have been known as Hadamard's inequalities, in this work, following [**112**] and [**147**], we shall call them the Hermite-Hadamard inequalities, or $H.-H.$ inequalities, for simplicity.

REMARK 1. ([**147**, p. 140]) *Note that the first inequality is stronger than the second inequality in (1.1); i.e., the following inequality is valid for a convex function f:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\le\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.4}$$

Indeed, (1.4) can be written as

$$\frac{2}{b-a}\int_a^b f(x)\,dx\le\frac{1}{2}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right],$$

which is

$$\begin{aligned}&\frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx+\frac{2}{b-a}\int_{\frac{a+b}{2}}^b f(x)\,dx\\ \le\ &\frac{1}{2}\left[f(a)+f\left(\frac{a+b}{2}\right)\right]+\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+f(b)\right].\end{aligned}$$

This immediately follows by applying the second inequality in (1.1) twice (on the interval $[a,(a+b)/2]$ *and* $[(a+b)/2,b]$*). By letting* $a=-1$, $b=1$, *we obtain the result due to Bullen (1978). Further on, we shall call (1.4) as Bullen's inequality.*

2. Characterisations of Convexity via $H. - H.$ Inequalities

In the classic book of Hardy, Littlewood, and Pólya ([**84**, p. 98]) the following result in characterising the convex functions is given (see also [**147**, p. 139]):

THEOREM 2. *A necessary and sufficient condition that a continuous function* f *be convex in* (a,b) *is that*

$$f(x) \leq \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt \quad \text{for} \quad a \leq x-h < x+h \leq b. \tag{1.5}$$

It can be shown that this result is equivalent to the first inequality in (1.1) when f is continuous on $[a,b]$. However, it remains unclear by who and when the transition from the inequality (1.1) to the convexity criterion (1.5) was made [**147**, p. 139].

For $f \in C(I)$, $h>0$, and $x \in I_1(h) = \{t : t-h, t+h \in I\}$, the operator S_h defined by

$$S_h(f,x) = \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt \tag{1.6}$$

is often called a *Steklov function* , although it is an operator mapping $C(I)$ into $C(I_1)$. For a finite interval $I=[a,b]$, the maximum value of h can be $\frac{b-a}{2}$. In this case, I_1 contains a single point and S_h becomes a functional. The Hermite-Hadamard inequality (1.5) now has the form $f(x) \leq S_h(f,x)$ for $x \in I_1(h)$ and is equivalent to the convexity of the function. The iterated Steklov operators (with step $h>0$) S_h^n $(n \in \mathbb{N})$ are defined by (see for example [**147**, p. 140]):

$$S_h^0(f,x) = f(x), \quad S_h^n(f,h) = \frac{1}{2h}\int_{x-h}^{x+h} S_h^{n-1}(f,x)\,dt, \tag{1.7}$$

where $n \in \mathbb{N}$, $x \in I_n(h) = \{t : t-nh, t+nh \in I\}$. For convenience we write S_h instead of S_h^1, and (1.5) becomes $S_h^0(f,x) \leq S_h^1(f,x)$.

The following two theorems are generalisations of Theorem 2 (see for instance [**147**, p. 140]):

THEOREM 3. *A function* $f \in C(I)$ *is convex iff for every* $h>0$ *and* $x \in I_n(h)$ *the inequality*

$$f(x) \leq S_h^n(f,x) \tag{1.8}$$

holds for every fixed n.

THEOREM 4. *A function* $f \in C(I)$ *is convex iff for every* $h>0$ *and* $x \in I_n(h)$ *the inequality*

$$S_h^{n-1}(f,x) \leq S_h^n(f,x) \tag{1.9}$$

holds for every fixed n.

It is easy to see that Theorem 4 generalises the convexity criterion based on the inequality (1.5) (we obtain (1.5) by letting $h \to 0$).

In Roberts and Varberg ([**158**, p. 15]), the following result is given (see also [**147**, p. 147]).

THEOREM 5. *A function $f \in C[a,b]$ is convex iff for every $s < t$ in $[a,b]$ we have*

$$\frac{1}{t-s}\int_s^t f(x)\,dx \leq \frac{f(s)+f(t)}{2}. \tag{1.10}$$

More general results are given by Rado ([**157**, 1935]). In the following we state some characterisation results given in this paper. Let $f(x)$ be a positive and continuous function on (a,b) and let $u, v \in \mathbb{R}$. We define (see for example [**147**, p. 141])

$$I(f,x,h,u) = \begin{cases} \left(\frac{1}{2h}\int_{-h}^{h} f(x+t)^u\,dt\right)^{\frac{1}{u}} & \text{for } u \neq 0 \\ \exp\left(\frac{1}{2h}\int_{-h}^{h} \log f(x+t)\,dt\right) & \text{for } u = 0; \end{cases}$$

$$A(f,x,h,v) = \begin{cases} \left(\frac{1}{2}f(x-h)^v + f(x+h)^v\right)^{\frac{1}{v}} & \text{for } v \neq 0 \\ \left(f(x+h)\,f(x-h)\right)^{\frac{1}{2}} & \text{for } v = 0. \end{cases}$$

Further, let E denote the set of all pairs (u,v) such that

$$I(f,x,h,u) \leq A(f,x,h,v) \tag{1.11}$$

holds for all x and h satisfying $a < x-h < x+h < b$ and all positive continuous, and convex functions f on (a,b). Similarly, let $\bar{E}$ be the set of points on (u,v) such that (1.11) holds for all such x, h and all positive and continuous functions f on (a,b). The main result in Rado's paper [**157**] concerns the explicit determination of the sets E and $\bar{E}$, and the following theorems is proved (see also [**147**, p. 142]):

THEOREM 6.

(a) *(u,v) belongs to E iff one of the following conditions is satisfied:*
  (i) *$u \leq -2$ and $v \geq 0$;*
  (ii) *$-2 \leq u \leq -\frac{1}{2}$ and $v \geq \frac{u+2}{3}$;*
  (iii) *$-\frac{1}{2} \leq u \leq 1$ and $v \geq \frac{u\log 2}{\log(1+u)}$; and*
  (iv) *$1 \leq u$ and $v \geq \frac{u+2}{3}$.*
(b) *(u,v) belongs to $\bar{E}$ iff $3v - u - 2 \leq 0$.*

In Theorem 5, if we replace the word "convexity" by "concavity", the inequality "$I \leq A$" in (1.11) by "$I \geq A$" and $E, \bar{E}$ by E^* and $\bar{E}^*$, respectively, then the following theorems are true (see for example [**147**, p. 142]):

THEOREM 7.

(a) *(u,v) belongs to E^* iff one of the following conditions is satisfied:*
  (i) *$u \leq -2$ and $v \leq \frac{u+2}{3}$;*
  (ii) *$-2 \leq u \leq -1$ and $v \leq 0$;*
  (iii) *$-1 \leq u \leq -\frac{1}{2}$ and $v \geq \frac{u\log 2}{\log(1+u)}$;*
  (iv) *$-\frac{1}{2} \leq u \leq 1$ and $v \leq \frac{u+2}{3}$ and*
  (v) *$1 \leq u$ and $v \leq \frac{u\log 2}{\log(1+u)}$.*
(b) *(u,v) belongs to $\bar{E}^*$ iff $3v - u - 2 \geq 0$.*

As a simple consequence of Theorems 6-7, Rado (1935) proved the following result.

THEOREM 8. *Let f be a positive and continuous function on (a,b). Then*

(a) *the inequality in (1.11) is equivalent to the convexity of f iff (u, v) satisfies the conditions:*
  (i) $3v - u - 2 - 0$ *and*
  (ii) *either* $-2 \leq u \leq -\frac{1}{2}$ *or* $1 \leq u < \infty$*;*

(b) *the reverse inequality in (1.11) is equivalent to the convexity of f iff (u, v) satisfies the conditions*
  (i) $3v - u - 2 = 0$ *and*
  (ii) *either* $u \leq -2$ *or* $-\frac{1}{2} \leq u \leq 1$.

Another generalisation of the Hermite-Hadamard inequalities has been given by Vasić and Lacković (1974, 1976) [**181**] and Lupaş (1976) [**103**] (see also [**147**, p. 143]):

THEOREM 9. *Let p, q be given positive numbers and $a_1 \leq a < b \leq b_1$. Then the inequalities*

$$f\left(\frac{pa+qb}{p+q}\right) \leq \frac{1}{2y}\int_{A-y}^{A+y} f(t)\,dt \leq \frac{pf(a)+qf(b)}{p+q} \tag{1.12}$$

hold for $A = \frac{pa+qb}{p+q}$, $y > 0$, and all continuous convex functions $f : [a_1, b_1] \to \mathbb{R}$ iff

$$y \leq \frac{b-a}{p+q}\min\{p, q\}. \tag{1.13}$$

REMARK 2.

(a) *Observe that (1.12) may be regarded as a refinement of the definition inequality for convex functions.*

(b) *For $p = q = 1$ and $y = \frac{b-a}{2}$, (1.12) is the Hermite-Hadamard inequality. It is known that* [**147**, p. 144] *under that same conditions Hermite-Hadamard's inequality yields, the following refinement of (1.12):*

$$\begin{aligned} f\left(\frac{pa+qb}{p+q}\right) &\leq \frac{1}{2y}\int_{A-y}^{A+y} f(t)\,dt \leq \frac{1}{2}\{f(A-y)+f(A+y)\} \\ &\leq \frac{pf(a)+qf(b)}{p+q} \end{aligned} \tag{1.14}$$

holds.

3. Some Generalisations

Generalisations of Theorem 9 for positive linear functionals were given by Pečarić and Beesack in 1986 [**142**] (see also [**147**, p. 146]):

THEOREM 10. *Let f be a continuous convex function on an interval $I \supset [m, M]$, where $-\infty < m < M < \infty$. Suppose that $g : E \to \mathbb{R}$ satisfies $m \leq g(t) \leq M$ for all $t \in E$, $g \in L$, and $f(g) \in L$. Let $A : L \to \mathbb{R}$ be an isotonic linear functional with $A(1) = 1$, and let $p = p_g$, $q = q_g$ be nonnegative real numbers (with $p + q > 0$) for which*

$$A(g) = \frac{pm+qM}{p+q}. \tag{1.15}$$

Then

$$f\left(\frac{pm+qM}{p+q}\right) \leq A(f(g)) \leq \frac{pf(m)+qf(M)}{p+q}. \tag{1.16}$$

THEOREM 11. *Suppose that L satisfies conditions L_1-L_3 defined by*

(L_1) $f, g \in L$ imply $af + bg \in L$ for all $a, b \in \mathbb{R}$;
(L_2) $1 \in L$, that is, if $f(t) = 1$ for $t \in E$, then $f \in L$;
(L_3) $f \in L$, $E_1 \in \mathcal{A}$ then $fC_{E_1} \in L$;

on a nonempty set E and that f is a continuous convex function on an interval I, while $g, h \in L$ with $f(g)$, $f(h) \in L$. Let A, B be isotonic linear functionals on L for which $A(1) = B(1) = 1$. If $A(h) = B(g)$, $E_1 \in \mathcal{A}$ satisfies $A(C_{E_1}) > 0$ and $A(C_{E_2}) > 0$ where $E_2 = E \backslash E_1$, and if

$$\frac{A(hC_{E_1})}{A(C_{E_1})} \leq g(t) \leq \frac{A(hC_{E_2})}{A(C_{E_2})} \quad \text{for all } t \in E, \tag{1.17}$$

then

$$f(A(h)) \leq B((f)(g)) \leq A(f(h)). \tag{1.18}$$

Note that again the inequality (1.18) is a refinement of Jessen's inequality and is also a generalisation of an inequality obtained by Vasić, Lacović, and Maksimović in 1980 [**182**] (see also [**147**, p. 147]).

In 1982, Wang and Wang [**183**] proved the following generalisation of Theorem 9.

Theorem 12. *Let $f : [a, b] \to \mathbb{R}$ be a convex function, $x_i \in [a, b]$, and $p_i > 0$ $(i = 0, ..., n)$. Then the following inequalities are valid:*

$$\begin{aligned} f\left(\frac{\sum_{i=0}^n p_i x_i}{\sum_{i=0}^n p_i}\right) &\leq \prod_{j=1}^n (\beta_j - \alpha_j)^{-1} \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_n}^{\beta_n} f\big(x_0(1 - t_1) \\ &\quad + \sum_{j=1}^{n-1} x_j (1 - t_{j+1}) t_1 \ldots t_j + x_n t_1 t_2 \ldots t_n\big) \prod_{i=1}^n dt_i \\ &\leq \frac{\sum_{j=0}^n p_j f(x_j)}{\sum_{j=0}^n p_j}, \end{aligned} \tag{1.19}$$

where

$$\frac{(\alpha_i + \beta_i)}{2} = \frac{\sum_{k=1}^n p_k}{\sum_{k=i-1}^n p_k} \quad \text{for } i = 1, \ldots, n \tag{1.20}$$

and

$$0 \leq \alpha_i < \beta_i \leq 1 \quad \text{for} \quad i = 1, \ldots, n. \tag{1.21}$$

For other remarks related to the above results, see [**147**, p. 148].

Another generalisation of the first inequality in (1.1) was done by Neumann in 1986 [**121**].

Let $x(t) = \sum_{r=u}^v a_r t^r$ (for $0 \leq u \leq v$ and $a_r \in \mathbb{R}$) be an algebraic polynomial of degree not exceeding v and let $a = \min\{x(t) : c \leq t \leq d\}$, $b = \max\{x(t) : c \leq t \leq d\}$ (see also [**147**, p. 149]).

Theorem 13. *Let f be a convex function on (a, b). Then*

$$f\left(\sum_{r=u}^v a_r m^r\right) \leq \int_c^d M_n(t) f\left(\sum_{r=u}^v a_r t^r\right) dt, \tag{1.22}$$

where M_n is a $B-$spline and m_r the rth generalised symmetric mean of $t_0, \ldots, t_n$.

The following generalisation of the first inequality in (1.1) for convex functions of several variables was given by Neuman and Pečarić in 1989, [**122**] (see also [**147**, p. 149]).

THEOREM 14. *Let f be a convex function on $\mathbb{R}^k$ and let $vol\left([\mathbf{x}_0, \ldots, \mathbf{x}_k]\right) > 0$, $\mathbf{x}_i \in \mathbb{R}^k$, $i = 0, 1, \ldots, n$. Then*

$$f\left(m_{\ell_1}, \ldots, m_{\ell_k}\right) \leq \int_{\mathbb{R}^k} f\left(x_1^{\ell_1}, \ldots, x_k^{\ell_k}\right) M\left(\mathbf{x}|\mathbf{x}_0, \ldots, \mathbf{x}_k\right) d\mathbf{x}, \tag{1.23}$$

where $\ell_i = 1, 2, \ldots;\ i = 1, 2, \ldots, k$.

Another result of this type is embodied in the following theorem (see [**147**, p. 150]):

THEOREM 15. *Under the assumptions of Theorem 14, we have*

$$f\left(\frac{1}{n+1}\sum_{i=0}^{n}\mathbf{x}_i\right) \leq \int_{\mathbb{R}^k} f\left(\mathbf{x}\right) M\left(\mathbf{x}|\mathbf{x}_0, \ldots, \mathbf{x}_k\right) d\mathbf{x} \leq \frac{1}{n+1}\sum_{i=0}^{n} f(\mathbf{x_i}), \tag{1.24}$$

and the equality (1.24) holds iff $f \in \prod_1\left(\mathbb{R}^k\right)$, where $\prod_1\left(\mathbb{R}^k\right)$ is the set of all polynomials with degree of, at most 1.

As a special case, we obtain:

THEOREM 16. *Let $\sigma = [\mathbf{x}_0, \ldots, \mathbf{x}_k]$, where $k \geq 1$ and $vol_k\left(\sigma\right) > 0$. If $f : \sigma \to \mathbb{R}$ is a convex function, then*

$$f\left(\frac{1}{k+1}\sum_{j=0}^{k}\mathbf{x}_j\right) \leq \frac{1}{vol_k\left(\sigma\right)}\int_{\sigma} f\left(\mathbf{x}\right) d\mathbf{x} \leq \frac{1}{k+1}\sum_{j=0}^{k} f(\mathbf{x_j}), \tag{1.25}$$

and equalities hold iff $f \in \prod_1\left(\mathbb{R}^k\right)$.

Let $\mathbf{X} = \{\mathbf{x}_0, \ldots, \mathbf{x}_k\}$ $(n > k \geq 1)$, and assume that any subset that consists of $k+1$ points spans a proper simplex. Let $\mathbf{X}_j = \mathbf{X}\backslash\{\mathbf{x}_j : 0 \leq j \leq n\}$. Then a multivariate $B-$spline can be written as $M\left(\cdot|\mathbf{X}\right)$ (with knot set $\mathbf{X}$). Similarly, let $M\left(\cdot|\mathbf{X}_j\right)$ denote the multivariate $B-$spline with knot set $\mathbf{X}_j$. For real numbers $\lambda_0, \ldots. \lambda_n$ with $\sum_{j=0}^{n}\lambda_j = 1$, let $\mathbf{y} = \sum_{j=0}^{n}\lambda_j\mathbf{x}_j$ and let $\mathbf{z} = \frac{1}{n+1}\sum_{j=0}^{n}\mathbf{x}_j$. The following generalisation of Theorem 9 is a special case of a more general result of Neuman (1990) [**123**] (see [**147**, p. 151]).

THEOREM 17. *Let f be a convex function on $\mathbb{R}^k$. Then*

$$\int_{\mathbb{R}^k} f\left(\mathbf{x}\right) M\left(\mathbf{x}|\mathbf{X}\right) d\mathbf{x} \leq \sum_{j=0}^{n}\lambda_j \int_{\mathbb{R}^k} f\left(\mathbf{x}\right) M\left(\mathbf{x}|\mathbf{X}_j\right) d\mathbf{x} \tag{1.26}$$

holds iff $\mathbf{y} = \mathbf{z}$, and equality in (1.26) holds iff $f \in \prod_1\left(\mathbb{R}^k\right)$.

Note that Neuman's general inequality is also a generalisation of Fejér's inequality given in (1.2).

For other results related to the Hermite-Hadamard inequality and a comprehensive list of references up to 1992, see the book [**147**]. For recent results, see [**1**], [**2**] – [**3**], [**5**], [**73**] – [**77**], [**83**], [**100**], [**104**], [**108**], [**116**] , [**136**] – [**134**], [**141**] – [**145**], [**154**] and [**157**] – [**168**].

CHAPTER 2

Some Results Related to the $H.-H.$ Inequality

1. Generalisations of the $H.-H.$ Inequality

1.1. Integral Inequalities. The following generalization of the first inequality due to Hermite-Hadamard holds:

THEOREM 18. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on I and $a, b \in \mathring{I}$ with $a < b$. Then for all $t \in [a, b]$ and $\lambda \in \left[f'_{-}(t), f'_{+}(t)\right]$ one has the inequality:*

$$f(t) + \lambda\left(\frac{a+b}{2} - t\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{2.1}$$

PROOF. Let $t \in [a, b]$. Then it is known [**147**, Theorem 1.6] that for all $\lambda \in \left[f'_{-}(t), f'_{+}(t)\right]$ one has the inequality:

$$f(x) - f(t) \geq \lambda(x - t) \text{ for all } x \in [a, b].$$

Integrating this inequality on $[a, b]$ over x we have

$$\int_a^b f(x)\,dx - (b-a) f(t) \geq \lambda(b-a)\left(\frac{a+b}{2} - t\right)$$

and the inequality (2.1) is proved. ∎

REMARK 3. *For $t = \frac{a+b}{2}$ we get the first part of the $H.-H.$ inequality.*

COROLLARY 1. *Let f be as above and $0 \leq a < b$.*

(*a*) *If $f'_{+}\left(\sqrt{ab}\right) \geq 0$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geq f\left(\sqrt{ab}\right);$$

(*b*) *If $f'_{+}\left(\frac{2ab}{a+b}\right) \geq 0$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geq f\left(\frac{2ab}{a+b}\right);$$

(*c*) *If f is differentiable in a and b then*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geq \max\left\{f(a) + f'(a)\frac{b-a}{2}, f(b) + f'(b)\frac{a-b}{2}\right\}$$

and

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f'(b)+f'(a)}{2}(b-a);$$

(d) *If $x_i \in [a,b]$ are points of differentiability for f and $p_i \geq 0$ are such that $P_n := \sum_{i=1}^{n} p_i > 0$ and*

$$\frac{a+b}{2}\sum_{i=1}^{n} f'(x_i)\, p_i \geq \sum_{i=1}^{n} p_i f'(x_i)\, x_i,$$

then one has the inequality:

$$\frac{1}{b-a}\int_a^b f(x)\, dx \geq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i).$$

REMARK 4. *If we assume that f is differentiable on (a,b), we recapture some of the results from* [**55**] *and* [**30**].

The second part of the $H. - H.$ inequality can be extended as follows [**30**]:

THEOREM 19. *Let f and a,b be as above. Then for all $t \in [a,b]$ we have the inequality:*

$$\frac{1}{b-a}\int_a^b f(x)\, dx \leq \frac{f(t)}{2} + \frac{1}{2}\cdot\frac{bf(b) - af(a) - t(f(b) - f(a))}{b-a}. \tag{2.2}$$

PROOF. Taking into account that the class of differentiable convex functions on (a,b) is dense in uniform topology in the class of all convex functions defined on (a,b), we can assume, without loss of generality, that f is differentiable on (a,b). Thus we can write the inequality:

$$f(t) - f(x) \geq (t-x)\, f'(x) \text{ for all } t,x \in (a,b).$$

Integrating this inequality over x on $[a,b]$ we get:

$$(b-a)\, f(t) - \int_a^b f(x)\, dx \geq t\,(f(b) - f(a)) - \int_a^b x f'(x)\, dx. \tag{2.3}$$

As a simple computation shows us that

$$\int_a^b x f'(x)\, dx = bf(b) - af(a) - \int_a^b f(x)\, dx,$$

then (2.3) becomes

$$(b-a)\, f(t) - t\,(f(b) - f(a)) + bf(b) - af(a) \geq 2\int_a^b f(x)\, dx$$

which is equivalent to (2.2). ∎

COROLLARY 2. *With the above assumptions, and under the condition that $0 \leq a < b$, one has the inequality:*

$$\frac{1}{b-a}\int_a^b f(x)\, dx \leq \min\{H_f(a,b), G_f(a,b), A_f(a,b)\} \tag{2.4}$$

where:

$$H_f(a,b) := \frac{1}{2}\left[f\left(\frac{2ab}{a+b}\right) + \frac{bf(b) + af(a)}{b+a}\right],$$

$$G_f(a,b) := \frac{1}{2}\left[f\left(\sqrt{ab}\right) + \frac{\sqrt{b}f(b) + \sqrt{a}f(a)}{\sqrt{b}+\sqrt{a}}\right],$$

and

$$A_f(a,b) := \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(b)+f(a)}{2}\right].$$

REMARK 5. *The inequality* (2.4) *for* $A_f(a,b)$ *has been proved by P. S. Bullen in 1978,* [**147**, p. 140] *and the inequality* (2.4) *for* $G_f(a,b)$ *has been proved by J. Sándor in 1988,* [**167**].

The following generalization of the above theorem has been proved by S.S. Dragomir and E. Pearce in paper [**65**]:

THEOREM 20. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a convex function,* $a, b \in \mathring{I}$ *with* $a < b$ *and* $x_i \in [a,b], p_i \geq 0$ *with* $P_n > 0$. *Then we have the following refinement of the second part of the H. – H. inequality:*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)\,dx \\ \leq\ & \frac{1}{2}\left\{\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) + \frac{1}{b-a}\left[(b-x_p)f(b) + (x_p-a)f(a)\right]\right\} \\ \leq\ & \frac{1}{2}\left[f(a)+f(b)\right], \end{aligned} \tag{2.5}$$

where

$$x_p = \frac{1}{P_n}\sum_{i=1}^n p_i x_i.$$

PROOF. As above, it is sufficient to prove (2.5) for convex functions which are differentiable on (a,b), then

$$f(y) - f(x) \geq f'(x)(y-x) \text{ for all } x, y \in (a,b).$$

Thus, we have:

$$f(x_i) - f(x) \geq f'(x)(x_i - x) \text{ for all } i \in \{1,...,n\}.$$

Integrating on $[a,b]$ over x we have:

$$\begin{aligned} & f(x_i) - \frac{1}{b-a}\int_a^b f(x)\,dx \\ \geq\ & \frac{x_i}{b-a}(f(b)-f(a)) - \frac{1}{b-a}\int_a^b x f'(x)\,dx \\ =\ & \frac{x_i}{b-a}(f(b)-f(a)) - \frac{1}{b-a}(bf(b)-af(a)) + \frac{1}{b-a}\int_a^b f(x)\,dx. \end{aligned}$$

By multiplying with $p_i \geq 0$ and summing over i from 1 to n, we obtain:

$$\begin{aligned} & \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) - \frac{1}{b-a}\int_a^b f(x)\,dx \\ \geq\ & \frac{x_p}{b-a}(f(b)-f(a)) - \frac{1}{b-a}(bf(b)-af(a)) + \frac{1}{b-a}\int_a^b f(x)\,dx \end{aligned}$$

where x_p is as above, from where we get

$$\begin{aligned}
&\frac{2}{b-a}\int_a^b f(x)\,dx \\
\leq\;& \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i)+\frac{1}{b-a}\left[(b-x_p)f(b)+(x_p-a)f(a)\right]
\end{aligned}$$

and the first inequality in (2.5) is proved.
Let

$$\alpha=\frac{b-x_i}{b-a},\ \beta=\frac{x_i-a}{b-a},\ x_i\in[a,b],\ i\in\{1,...,n\}.$$

Then it is clear that $\alpha+\beta=1$ and by the convexity of f we have that

$$\frac{b-x_i}{b-a}f(a)+\frac{x_i-a}{b-a}f(b)\geq f(x_i)$$

for all $i\in\{1,...,n\}$. By multiplying with $p_i\geq 0$ and summing over i from 1 to n, we derive:

$$\frac{1}{b-a}\left[(b-x_p)f(a)+(x_p-a)f(b)\right]\geq\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i)$$

which is well known in the literature as the Lah-Ribarić inequality [**114**, p. 9].
Using the previous inequality, we have that:

$$\begin{aligned}
&\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i)+\frac{1}{b-a}\left[(b-x_p)f(b)+(x_p-a)f(a)\right] \\
\leq\;& \frac{1}{b-a}\left[(b-x_p)f(a)+(x_p-a)f(b)+(b-x_p)f(b)+(x_p-a)f(a)\right] \\
=\;& f(a)+f(b).
\end{aligned}$$

Thus, the inequality (2.5) is proved. ∎

Corollary 3. *With the above assumptions for f,a,b and if $t\in[a,b]$, then we have the inequality:*

$$\frac{f(t)}{2}+\frac{1}{2}\cdot\frac{bf(b)-af(a)-t(f(b)-f(a))}{b-a}\leq\frac{f(a)+f(b)}{2}.$$

Proof. The argument follows by the above theorem if we choose $x_i=t, i\in\{1,...,n\}$. We shall omit the details. ∎

Remark 6. *The inequality (2.5) is also a generalization of Bullen's result. We recapture his result when $x_i=\frac{a+b}{2}, i=1,...,n$.*

1.2. Applications for Special Means. Let us recall the following means for two positive numbers.

(1) *The Arithmetic mean*

$$A=A(a,b):=\frac{a+b}{2},\ a,b>0;$$

(2) *The Geometric mean*

$$G=G(a,b):=\sqrt{ab},\ a,b>0;$$

(3) *The Harmonic mean*

$$H = H(a,b) := \frac{2ab}{a+b}, \ a,b>0;$$

(4) *The Logarithmic mean*

$$L = L(a,b) := \begin{cases} a & \text{if} \quad a=b \\ \frac{b-a}{\ln b - \ln a} & \text{if} \quad a \neq b; \end{cases}, \ a,b>0,$$

(5) *The Identric mean*

$$I = I(a,b) := \begin{cases} a & \text{if} \quad a=b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad a \neq b \end{cases}, \ a,b>0;$$

(6) *The p-Logarithmic mean*

$$L_p = L_p(a,b) := \begin{cases} a & \text{if} \quad a=b \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} \quad a \neq b \end{cases}, \ a,b>0.$$

The following inequality is well known in the literature:

$$H \le G \le L \le I \le A. \tag{2.6}$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

We shall start with the following proposition:

PROPOSITION 1. *Let $p \in (-\infty,0) \cup [1,\infty) \setminus \{-1\}$ and $[a,b] \subset (0,\infty)$. Then one has the inequality:*

$$\frac{L_p^p - t^p}{pt^{p-1}} \ge A - t \tag{2.7}$$

for all $t \in [a,b]$.

PROOF. If we choose in Theorem 18, $f:[a,b] \to [0,\infty)$, $f(x) = x^p$ and p as specified above, we get

$$\frac{1}{b-a}\int_a^b x^p dx \ge t^p + pt^{p-1}\left(\frac{a+b}{2} - t\right)$$

for all $t \in [a,b]$.
As

$$\frac{1}{b-a}\int_a^b x^p dx = L_p^p(a,b) = L_p^p$$

we get the desired inequality (2.7). ∎

REMARK 7. *Using the above inequality we deduce the following particular results:*

$$\frac{L_p^p - I^p}{pI^{p-1}} \ge A - I \ge 0, \ \frac{L_p^p - L^p}{pL^{p-1}} \ge A - L \ge 0$$

and

$$\frac{L_p^p - G^p}{pG^{p-1}} \ge A - G \ge 0, \ \frac{L_p^p - H^p}{pH^{p-1}} \ge A - H \ge 0$$

and

$$\frac{L_p^p - a^p}{pa^{p-1}} \geq A - a \geq 0,\ 0 \leq \frac{L_p^p - b^p}{pb^{p-1}} \leq b - A$$

respectively.

The following proposition also holds

PROPOSITION 2. *Let $0 < a < b$. Then for all $t \in [a, b]$ we have the inequality:*

$$\frac{L - t}{L} \leq \frac{A - t}{t}. \tag{2.8}$$

PROOF. If we choose in Theorem 18, $f(x) = \frac{1}{x}, x \in [a, b]$ we have:

$$\frac{1}{b-a} \int_a^b \frac{dx}{x} \geq \frac{1}{t} - \frac{1}{t^2} \left(\frac{a+b}{2} - t \right),$$

which is equivalent to

$$\frac{1}{L} \geq \frac{1}{t} - \frac{1}{t^2} (A - t),$$

and the inequality (2.8) is proved. ∎

REMARK 8. *Using the above inequality we can state the following interesting inequalities:*

$$\frac{L_p - L}{L} \geq \frac{L_p - A}{L_p},\ \frac{A - G}{G} \geq \frac{L - G}{L}$$

and

$$\frac{A - H}{H} \geq \frac{L - H}{L},\ \frac{L - a}{L} \geq \frac{A - a}{a}$$

and

$$\frac{b - L}{L} \geq \frac{b - A}{b}.$$

Finally, we have the following additional proposition:

PROPOSITION 3. *Let $0 < a < b$. Then one has the inequality*

$$\ln I - \ln t \leq \frac{A - t}{t} \tag{2.9}$$

for all $t \in [a, b]$.

PROOF. If we choose in Theorem 18, $f(x) = -\ln x, x \in [a, b]$ we get

$$-\frac{1}{b-a} \int_a^b \ln x dx \geq -\ln t - \frac{1}{t} \left(\frac{a+b}{2} - t \right)$$

which is equivalent to

$$-\ln I \geq -\ln t - \frac{1}{t} (A - t)$$

which is equivalent to (2.9). ∎

REMARK 9. *Using the inequality* (2.9) *we get that*

$$\ln L_p - \ln I \geq \frac{L_p - A}{L_p} \geq 0,\ \ln b - \ln I \geq \frac{b - A}{b}$$

and

$$0 \leq \ln I - \ln L \leq \frac{A - L}{L},\ 0 \leq \ln I - \ln G \leq \frac{A - G}{G}$$

and

$$0 \leq \ln I - \ln H \leq \frac{A-H}{H},\ 0 \leq \ln I - \ln a \leq \frac{A-a}{a}$$

respectively.

Now, we shall give some natural applications of Theorem 19.

PROPOSITION 4. *Let $p \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$ and $[a,b] \subset [0,\infty)$. Then one has the inequality:*

$$L_p^p - t^p \leq p\left(L_p^p - tL_{p-1}^{p-1}\right) \tag{2.10}$$

for all $t \in [a,b]$.

PROOF. If we choose in Theorem 19, $f : [a,b] \to [0,\infty)$, $f(x) = x^p$ (which is convex) we get that:

$$\frac{1}{b-a}\int_a^b x^p dx \leq \frac{t^p}{2} + \frac{1}{2}\left[\frac{b^{p+1}-a^{p+1}}{b-a} - t \cdot \frac{b^p - a^p}{b-a}\right]$$

for all $t \in [a,b]$.

As

$$\frac{1}{b-a}\int_a^b x^p dx = L_p^p, \frac{b^{p+1}-a^{p+1}}{b-a} = (p+1)\, L_p^p$$

and

$$\frac{b^p - a^p}{b-a} = pL_{p-1}^{p-1},$$

we get from the above inequality that

$$\begin{aligned} L_p^p &\leq \frac{t^p}{2} + \frac{1}{2}\left[(p+1)\, L_p^p - tpL_{p-1}^{p-1}\right] \\ &= \frac{t^p}{2} + \frac{L_p^p}{2} + \frac{1}{2}p\left(L_p^p - tL_{p-1}^{p-1}\right), \end{aligned}$$

which is equivalent to (2.10). ∎

REMARK 10. *We have the following particular interesting inequality for $p \geq 1$:*

$$0 \leq L_p^p - A^p \leq p\left(L_p^p - AL_{p-1}^{p-1}\right),\ 0 \leq L_p^p - L^p \leq p\left(L_p^p - LL_{p-1}^{p-1}\right)$$

and

$$0 \leq L_p^p - I^p \leq p\left(L_p^p - IL_{p-1}^{p-1}\right),\ 0 \leq L_p^p - G^p \leq p\left(L_p^p - GL_{p-1}^{p-1}\right)$$

respectively.

The following proposition also holds.

PROPOSITION 5. *Let $0 < a < b$. Then for all $t \in [a,b]$ we have the following inequality:*

$$\frac{t-L}{L} \leq \frac{1}{2} \cdot \frac{t^2 - G^2}{G^2}. \tag{2.11}$$

PROOF. If we choose in Theorem 19, $f(x) = \frac{1}{x}$, we have that

$$\frac{1}{b-a}\int_a^b \frac{dx}{x} \leq \frac{1}{2t} - \frac{1}{2} \cdot \frac{t\left(\frac{1}{b} - \frac{1}{a}\right)}{b-a} = \frac{1}{2t} + \frac{t}{2ab}.$$

That is,

$$\frac{1}{L} \leq \frac{1}{2t} + \frac{t}{2ab} \text{ or } \frac{1}{L} - \frac{1}{t} \leq \frac{t}{2ab} - \frac{1}{2t},$$

which is equivalent to (2.11). ∎

REMARK 11. *The above inequality gives us the following particular interesting results:*

$$0 \leq \frac{L_p - L}{L} \leq \frac{1}{2} \cdot \frac{L_p^2 - G^2}{G^2} \quad (p \geq 1)$$

and

$$0 \leq \frac{A - L}{L} \leq \frac{1}{2} \cdot \frac{A^2 - G^2}{G^2},\ 0 \leq \frac{I - L}{L} \leq \frac{1}{2} \cdot \frac{I^2 - G^2}{G^2}$$

and

$$0 \leq \frac{1}{2} \cdot \frac{G^2 - H^2}{G^2} \leq \frac{L - H}{L}$$

respectively.

Finally, we have the following application of Theorem 19.

PROPOSITION 6. *Let $0 < a < b$. Then one has the inequality*

$$\frac{L - t}{L} \leq \ln I - \ln t, \tag{2.12}$$

for all $t \in [a, b]$.

PROOF. If we choose in Theorem 19, $f(x) = -\ln x, x \in [a, b]$ we get

$$\begin{aligned} -\frac{1}{b-a}\int_a^b \ln x dx &\leq -\frac{\ln t}{2} - \frac{1}{2}\frac{b\ln b - a\ln a - t(\ln b - \ln a)}{b-a} \\ &= -\frac{\ln t}{2} - \frac{1}{2}\ln\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} + \frac{1}{2}t\left(\frac{\ln b - \ln a}{b-a}\right) \\ &= -\frac{\ln t}{2} - \frac{1}{2}\ln\left[eI(a,b)\right] + \frac{t}{2L}, \end{aligned}$$

which gives us

$$-\ln I \leq -\frac{\ln t}{2} - \frac{1}{2}(1 + \ln I) + \frac{t}{2L}.$$

That is,

$$-2\ln I \leq -\ln t - 1 - \ln I + \frac{t}{L},$$

which is equivalent to

$$1 - \frac{t}{L} \leq \ln I - \ln t,$$

and the inequality is proved. ∎

REMARK 12. *From the above inequality we deduce the following particular inequalities:*

$$0 \leq \frac{L - G}{L} \leq \ln I - \ln G,\ 0 \leq \frac{L - H}{L} \leq \ln I - \ln H$$

and

$$\frac{A - L}{L} \geq \ln A - \ln I \geq 0.$$

In what follows we shall point out some natural applications of Theorem 20.

Proposition 7. *Let* $r \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$ *and* $[a, b] \subset [0, \infty)$. *If* $x_i \in [a, b]$, $p_i \geq 0$ *with* $P_n := \sum_{i=1}^{n} p_i > 0$ $\left(i = \overline{1, n}\right)$, *then we have the inequality:*

$$\begin{aligned} & L_r^r(a,b) - \left[M_n^{[r]}(x,p)\right]^r \\ \leq\ & r\left[\left[L_r(a,b)\right]^r - A_n(x,p)\left[L_{r-1}(a,b)\right]^{r-1}\right] \\ \leq\ & 2A\left(b^r, a^r\right) - \left[L_r(a,b)\right]^r - \left[M_n^{[r]}(x;p)\right]^r \end{aligned} \tag{2.13}$$

where

$$A_n(x,p) := \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$$

is the arithmetic mean and

$$M_n^{[r]}(x;p) := \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^r\right)^{\frac{1}{r}}$$

is the $r-$*power mean.*

Proof. Choosing in Theorem 20, $f(x) = x^r,\ x \in [a, b]$, we get that

$$\begin{aligned} \frac{1}{b-a}\int_a^b x^r dx &\leq \frac{1}{2}\left[\left(M_n^{[r]}(x;p)\right)^r + \frac{b^{r+1}-a^{r+1}}{b-a} - A_n(x,p)\cdot\frac{b^r-a^r}{b-a}\right] \\ &\leq A\left(b^r, a^r\right). \end{aligned}$$

As

$$\frac{b^{r+1}-a^{r+1}}{b-a} = (r+1)\left[L_r(a,b)\right]^r$$

and

$$\frac{b^r-a^r}{b-a} = r\left[L_{r-1}(a,b)\right]^{r-1},$$

we get:

$$\begin{aligned} L_r^r(a,b) &\leq \frac{1}{2}\left[\left[M_n^{[r]}(x,p)\right]^r + (r+1)\left[L_r(a,b)\right]^r - rA_n(x,p)\left[L_{r-1}(a,b)\right]^{r-1}\right] \\ &\leq A\left(b^r, a^r\right) \end{aligned}$$

from where results the desired inequality (2.13). ∎

Remark 13. *If in* (2.13) *we choose* $x_i = t,\ i = \overline{1, n}$, *then we get*

$$L_r^r - t^r \leq r\left[L_r^r - tL_{r-1}^{r-1}\right] \leq 2A\left(b^r, a^r\right) - L_r^r - t^r,\ t \in [a, b], \tag{2.14}$$

which counterparts the inequality (2.10) *(for* $r = p$*).*
The second inequality in (2.14) *gives us the particular inequalities*

$$0 \leq r\left[L_r^r - AL_{r-1}^{r-1}\right] \leq 2A\left(b^r, a^r\right) - L_r^r - A^r,$$

$$0 \leq r\left[L_r^r - LL_{r-1}^{r-1}\right] \leq 2A\left(b^r, a^r\right) - L_r^r - L^r,$$

$$0 \leq r\left[L_r^r - IL_{r-1}^{r-1}\right] \leq 2A\left(b^r, a^r\right) - L_r^r - I^r,$$

and

$$0 \leq r\left[L_r^r - GL_{r-1}^{r-1}\right] \leq 2A\left(b^r, a^r\right) - L_r^r - G^r.$$

The following proposition holds:

PROPOSITION 8. *Let $0<a<b,\ x_i\in[a,b],\ i=\overline{1,n},\ p_i>0,\ i=\overline{1,n}$. Then one has the inequality:*

$$\begin{aligned}\frac{H_n(p,x)-L(a,b)}{L(a,b)} &\le \frac{1}{2}\cdot\frac{A_n(p,x)H_n(p,x)-G^2(a,b)}{G^2(a,b)}\\ &\le \frac{A(a,b)H_n(p,x)-G^2(a,b)}{G^2(a,b)},\end{aligned}\tag{2.15}$$

where $H_n(p,x)$ is the harmonic mean. That is,

$$H_n(p,x)=\left[M_n^{[-1]}(x,p)\right]^{-1}=\frac{P_n}{\sum_{i=1}^n\frac{p_i}{x_i}}.$$

PROOF. If we choose in Theorem 20, $f(x)=\frac{1}{x},\ x\in[a,b]$, we get:

$$\frac{1}{b-a}\int_a^b\frac{dx}{x}\le\frac{1}{2}\cdot\left[\frac{1}{P_n}\sum_{i=1}^n\frac{p_i}{x_i}+\frac{\frac{b}{b}-\frac{a}{a}}{b-a}-A_n(x,p)\frac{\frac{1}{b}-\frac{1}{a}}{b-a}\right]\le\frac{\frac{1}{a}+\frac{1}{b}}{2}.$$

That is,

$$\frac{1}{L(a,b)}\le\frac{1}{2}\cdot\left[\frac{1}{H_n(p,x)}+\frac{A_n(p,x)}{G^2(a,b)}\right]\le\frac{A(a,b)}{G^2(a,b)},$$

from where we get

$$\frac{1}{L(a,b)}-\frac{1}{H_n(p,x)}\le\frac{1}{2}\cdot\left[\frac{A_n(x,p)}{G^2(a,b)}-\frac{1}{H_n(p,x)}\right]\le\frac{A(a,b)}{G^2(a,b)}-\frac{1}{H_n(p,x)}.$$

That is,

$$\begin{aligned}\frac{H_n(p,x)-L(a,b)}{L(a,b)H_n(p,x)} &\le \frac{1}{2}\cdot\left[\frac{A_n(x,p)H_n(p,x)-G^2(a,b)}{G^2(a,b)H_n(p,x)}\right]\\ &\le \frac{A(a,b)H_n(p,x)-G^2(a,b)}{G^2(a,b)H_n(p,x)}\end{aligned}$$

and the inequality (2.15) is obtained. ∎

REMARK 14. *If in (2.15) we choose $x_i=t,\ i=\overline{1,n}$, then we get*

$$\frac{t-L}{L}\le\frac{1}{2}\cdot\frac{t^2-G^2}{G^2}\le\frac{tA-G^2}{G^2},\ t\in[a,b],\tag{2.16}$$

which counterparts the inequality (2.11).
The second inequality (2.16) gives us the following particular results:

$$0\le\frac{1}{2}\cdot\frac{I^2-G^2}{G^2}\le\frac{IA-G^2}{G^2},\ 0\le\frac{1}{2}\cdot\frac{L^2-G^2}{G^2}\le\frac{LA-G^2}{G^2}.$$

Finally, we also have the following proposition:

PROPOSITION 9. *Let $0<a<b$ and $x_i\in[a,b],\ p_i\ge0\ \left(i=\overline{1,n}\right)$ with $P_n>0$. Then we have the inequality:*

$$\begin{aligned}\ln I(a,b)-\ln G_n(p,x) &\ge \frac{L(a,b)-A_n(x,p)}{L(a,b)}\\ &\ge \ln G^2(a,b)-\ln I(a,b)-\ln G_n(p,x),\end{aligned}\tag{2.17}$$

where $G_n(p,x)$ is the geometric mean. That is:

$$G_n(p,x):=\left(\prod_{i=1}^n x_i^{p_i}\right)^{\frac{1}{P_n}}.$$

PROOF. If we choose in Theorem 20, $f(x) = -\ln x,\ x \in [a,b]$ we obtain:

$$\begin{aligned} & \frac{1}{b-a}\int_a^b (-\ln x)\,dx \\ \leq\ & -\frac{1}{2}\cdot\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i \ln x_i + \frac{b\ln b - a\ln a}{b-a} - A_n(x,p)\,\frac{\ln b - \ln a}{b-a}\right] \\ \leq\ & -\frac{\ln b + \ln a}{2}. \end{aligned}$$

That is,

$$\frac{1}{b-a}\int_a^b \ln x dx \geq \frac{1}{2}\left[\ln G_n(p,x) + \ln\left[e\cdot I(a,b)\right] - \frac{A_n(x,p)}{L(a,b)}\right] \geq \ln G(a,b)$$

or

$$\ln I(a,b) \geq \frac{1}{2}\ln G_n(p,x) + \frac{1}{2} + \frac{1}{2}\ln I(a,b) - \frac{A_n(x,p)}{2L(a,b)} \geq \ln G(a,b).$$

That is,

$$\frac{1}{2}\ln I(a,b) \geq \frac{1}{2}\left[\ln G_n(p,x) + 1 - \frac{A_n(x,p)}{L(a,b)}\right] \geq \ln G(a,b) - \frac{1}{2}\ln I(a,b)$$

or, additionally,

$$\begin{aligned} \frac{1}{2}\left[\ln I(a,b) - \ln G_n(p,x)\right] &\geq \frac{1}{2}\left[\frac{L(a,b) - A_n(x,p)}{L(a,b)}\right] \\ &\geq \ln G(a,b) - \frac{1}{2}\ln I(a,b) - \frac{1}{2}\ln G_n(p,x) \end{aligned}$$

and the inequality (2.17) is obtained. ∎

REMARK 15. *If in* (2.17) *we put* $x_i = t, i = \overline{1,n}$, *then we get*

$$\ln I - \ln t \geq \frac{L-t}{L} \geq \ln G^2 - \ln I - \ln t,\ t \in [a,b], \tag{2.18}$$

which counterparts the inequality (2.12).
This last inequality also gives us the following particular inequalities

$$0 \leq \frac{A-L}{L} \leq \ln\left(\frac{IA}{G^2}\right) \text{ and } 0 \leq \frac{I-L}{L} \leq \ln\left(\frac{I^2}{G^2}\right).$$

Furthermore,

$$1 \leq \exp\left(\frac{A}{L} - 1\right) \leq \frac{IA}{G^2} \text{ and } 1 \leq \exp\left(\frac{I}{L} - 1\right) \leq \frac{I^2}{G^2}.$$

2. Hadamard's Inferior and Superior Sums

2.1. Some Inequalities. Let $[a,b]$ be a compact interval of real numbers, $d := \left\{x_i | i = \overline{0,n}\right\} \subset [a,b]$, a division of the interval $[a,b]$, given by

$$d : a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b \ \ (n \geq 1)$$

and f a bounded mapping on $[a,b]$. We consider the following sums [**42**]:

$$h_d(f) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i)$$

which is called *Hadamard's inferior sum*, and

$$H_d(f) := \sum_{i=0}^{n-1} \frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)$$

which is called *Hadamard's superior sum*. We also consider *Darboux's sums*

$$s_d(f) := \sum_{i=0}^{n-1} m_i(x_{i+1}-x_i), S_d(f) := \sum_{i=0}^{n-1} M_i(x_{i+1}-x_i)$$

where

$$m_i = \inf_{x\in[x_i,x_{i+1}]} f(x),\ M_i = \sup_{x\in[x_i,x_{i+1}]} f(x), i=0,...,n-1.$$

It is well-known that f is Riemann integrable on $[a,b]$ iff

$$\sup_d s_d(f) = \inf_d S_d(f) = I \in \mathbb{R}.$$

In this case,

$$I = \int_a^b f(x)\,dx.$$

The following theorem was proved by S.S. Dragomir in the paper [**42**].

THEOREM 21. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then*

(*i*) *$h_d(f)$ increases monotonically over d. That is, for $d_1 \subseteq d_2$ one has $h_{d_1}(f) \le h_{d_2}(f)$;*
(*ii*) *$H_d(f)$ decreases monotonically over d;*
(*iii*) *We have the bounds*

$$\frac{1}{b-a}\inf_d h_d(f) = f\left(\frac{a+b}{2}\right),\ \sup_d h_d(f) = \int_a^b f(x)\,dx \tag{2.19}$$

and

$$\inf_d H_d(f) = \int_a^b f(x)\,dx,\ \frac{1}{b-a}\sup_d H_d(f) = \frac{f(a)+f(b)}{2}. \tag{2.20}$$

PROOF. The proof is as follows.

(*i*) Without loss of generality, we can assume that $d_1 \subseteq d_2$ with $d_1 = \{x_0,...,x_n\}$ and $d_2 = \{x_0,...,x_k,y,x_{k+1},...,x_n\}$ where $y\in[x_k,x_{k+1}]$ $(0\le k\le n-1)$. Then

$$\begin{aligned}&h_{d_2}(f)-h_{d_1}(f)\\ =\ & f\left(\frac{x_k+y}{2}\right)(y-x_k)+f\left(\frac{y+x_{k+1}}{2}\right)(x_{k+1}-y)\\ &-f\left(\frac{x_k+x_{k+1}}{2}\right)(x_{k+1}-x_k).\end{aligned}$$

Let us put

$$\alpha = \frac{y-x_k}{x_{k+1}-x_k},\ \beta = \frac{x_{k+1}-y}{x_{k+1}-x_k}$$

and

$$x = \frac{x_k+y}{2},\ z = \frac{y+x_{k+1}}{2}.$$

Then

$$\alpha+\beta = 1,\ \alpha x+\beta z = \frac{x_k+x_{k+1}}{2},$$

and, by the convexity of f we deduce that $\alpha f(x)+\beta f(z)\geq f(\alpha x+\beta z)$. That is, $h_{d_2}(f)\geq h_{d_1}(f)$.

(ii) For d_1, d_2 as above, we have

$$\begin{aligned}
& H_{d_2}(f)-H_{d_1}(f) \\
= & \frac{f(x_k)+f(y)}{2}(y-x_k)+\frac{f(y)+f(x_{k+1})}{2}(x_{k+1}-y) \\
& -\frac{f(x_k)+f(x_{k+1})}{2}(x_{k+1}-x_k) \\
= & \frac{f(y)(x_{k+1}-x_k)}{2}-\frac{f(x_k)(x_{k+1}-y)+f(x_{k+1})(y-x_k)}{2}.
\end{aligned}$$

Now, let α,β be as above and $v=x_k$, $u=x_{k+1}$. Then $\alpha u+\beta v=y$ and by the convexity of f we have $\alpha f(u)+\beta f(v)\geq f(y)$. That is, $H_{d_2}(f)\leq H_{d_1}(f)$ and the statement is proved.

(iii) Let $d=\{x_0,...,x_n\}$ with $a=x_0<x_1<...<x_n=b$. Put $p_i:=x_{i+1}-x_i$, $u_i=\frac{(x_i+x_{i+1})}{2}$, $i=0,...,n-1$. Then, by Jensen's discrete inequality we have

$$f\left(\frac{\sum_{i=0}^{n}p_iu_i}{\sum_{i=0}^{n}p_i}\right)\leq\frac{\sum_{i=0}^{n}p_if(u_i)}{\sum_{i=0}^{n}p_i}.$$

Since

$$\sum_{i=0}^{n}p_i=b-a,\sum_{i=0}^{n}p_iu_i=\frac{b^2-a^2}{2},$$

we can deduce the inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}h_d(f).$$

If $d=d_0=\{a,b\}$, we obtain

$$h_{d_0}(f)=(b-a)f\left(\frac{a+b}{2}\right),$$

which proves the first bound in (2.19).

By the first inequality in the Hermite-Hadamard result, we have

$$f\left(\frac{x_i+x_{i+1}}{2}\right)\leq\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(x)\,dx, i=0,...,n-1,$$

which gives, by addition,

$$\begin{aligned}
h_d(f) &= \sum_{i=0}^{n-1}f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i) \\
&\leq \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}f(x)\,dx=\int_a^b f(x)\,dx,
\end{aligned}$$

for all d a division of $[a,b]$.

Since

$$s_d(f)\leq h_d(f)\leq\int_a^b f(x)\,dx,$$

d is a division of $[a,b]$, and f is Riemann integrable on $[a,b]$, that is,

$$\sup_d s_d(f)=\int_a^b f(x)\,dx,$$

it follows that

$$\sup_d h_d(f)=\int_a^b f(x)\,dx,$$

which proves the second relation in (2.19).
To prove the relation (2.20), we observe, by the second inequality in the Hermite-Hadamard result, that

$$\begin{aligned}\int_a^b f(x)\,dx &= \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)\,dx\\ &\le \sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)=H_d(f),\end{aligned}$$

where d is a division of $[a,b]$.
Since

$$H_d(f)\le S_d(f)$$

for all d as above, and f is integrable on $[a,b]$, we conclude that

$$\inf_d H_d(f)=\int_a^b f(x)\,dx.$$

Finally, as for all d a division of $[a,b]$ we have $d\supseteqq d_0=\{a,b\}$, thus

$$\frac{1}{b-a}\sup_d H_d(f)=\frac{f(a)+f(b)}{2}$$

and the theorem is proved.

∎

The following corollary gives an improvement of the classical Hermite-Hadamard inequality [**42**]:

Corollary 4. *Let $f:[a,b]\to\mathbb{R}$ be a convex mapping on $[a,b]$. Then for all $a=x_0<x_1<...<x_n=b$ we have*

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\le \frac{1}{b-a}\sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)\\ &\le \frac{1}{b-a}\int_a^b f(x)\,dx\\ &\le \frac{1}{b-a}\sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)\\ &\le \frac{f(a)+f(b)}{2}.\end{aligned}\tag{2.21}$$

Define the sequences:

$$h_n(f):=\frac{1}{n}\sum_{i=0}^{n-1} f\left(a+\frac{2i+1}{2n}(b-a)\right)$$

and

$$H_n(f) := \frac{1}{2n} \sum_{i=0}^{n-1} \left[f\left(a + \frac{i}{n}(b-a)\right) + f\left(a + \frac{i+1}{n}(b-a)\right)\right]$$

for $n \geq 1$.

The following corollary also holds [**42**].

Corollary 5. *With the above assumptions, one has the inequality:*

$$f\left(\frac{a+b}{2}\right) \leq h_n(f) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq H_n(f) \leq \frac{f(a)+f(b)}{2}. \tag{2.22}$$

Moreover, one has the limits

$$\lim_{n\to\infty} h_n(f) = \lim_{n\to\infty} H_n(f) = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{2.23}$$

Proof. The inequality (2.22) follows by (2.21) for the division

$$d := \left\{ x_i = a + \frac{i}{n}(b-a) \,|\, i = \overline{0,n} \right\}.$$

The relation (2.23) is obvious by the integrability of f. We shall omit the details. ∎

Now, let us define the sequences:

$$t_n(f) := \frac{1}{2^n} \sum_{i=0}^{n-1} f\left(a + 3 \cdot \frac{2^i}{2^{n+1}}(b-a)\right) 2^i$$

and

$$T_n(f) := \frac{1}{2^{n+1}} \sum_{i=0}^{n-1} \left[f\left(a + \frac{2^i}{2^n}(b-a)\right) + f\left(a + \frac{2^{i+1}}{2^n}(b-a)\right)\right] 2^i$$

for $n \geq 1$.

Corollary 6. *Let $f : [a,b] \to \mathbb{R}$ be a convex mapping on $[a,b]$. Then we have:*

(*i*) *$t_n(f)$ is monotonic increasing;*
(*ii*) *$T_n(f)$ is monotonic decreasing;*
(*iii*) *The following bounds hold:*

$$\lim_{n\to\infty} t_n(f) = \sup_{n\geq 1} t_n(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

and

$$\lim_{n\to\infty} T_n(f) = \inf_{n\geq 1} T_n(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Proof. (*i*) and (*ii*) are obvious by (*i*) and (*ii*) of Theorem 21 for

$$d_n := \left\{ x_i = a + \frac{2^i}{2^n}(b-a) \,|\, i = \overline{0,n} \right\} \subseteq d_{n+1},\ n \in \mathbb{N}.$$

(*iii*) It follows from the bounds (2.19) and (2.20) and from the fact that f is Riemann integrable on $[a,b]$.

∎

REMARK 16. *The above result was proved in the paper* [**42**].

2.2. Applications for Special Means. Let $[a,b]$ be a compact interval of real numbers and $d \in \mathfrak{Div}\,[a,b]$. That is, $d := \left\{x_i | i = \overline{0,n}\right\} \subset [a,b]$ is a division of the interval $[a,b]$ given by $d : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$.

Define the sums

$$h_d^{[p]} := \sum_{i=0}^{n-1} A^p\left(x_i, x_{i+1}\right)\left(x_{i+1} - x_i\right)$$

and

$$H_d^{[p]} := \sum_{i=0}^{n-1} A\left(x_i^p, x_{i+1}^p\right)\left(x_{i+1} - x_i\right)$$

where $p \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$.

For every $d \in \mathfrak{Div}\,[a,b]$, we have the inequality:

$$\begin{aligned} A^p(a,b) &\leq \frac{1}{b-a}\sum_{i=0}^{n-1} A^p\left(x_i, x_{i+1}\right)\left(x_{i+1} - x_i\right) \\ &\leq L_p^p(a,b) \\ &\leq \frac{1}{b-a}\sum_{i=0}^{n-1} A\left(x_i^p, x_{i+1}^p\right)\left(x_{i+1} - x_i\right) \\ &\leq A\left(a^p, b^p\right) \end{aligned}$$

and the bounds

$$\frac{1}{b-a}\sup_d h_d^{[p]} = L_p^p(a,b)$$

and

$$\frac{1}{b-a}\inf_d H_d^{[p]} = L_p^p(a,b).$$

Now, let us define the sums; for $0 < a < b$:

$$h_d^{[-1]} = 2\sum_{i=0}^{n-1} \frac{x_{i+1} - x_i}{x_{i+1} + x_i},\ H_d^{[-1]} = \frac{1}{2}\sum_{i=0}^{n-1} \frac{x_{i+1}^2 - x_i^2}{x_i x_{i+1}}.$$

We have the inequality

$$\begin{aligned} A^{-1}(a,b) &\leq \frac{2}{b-a}\sum_{i=0}^{n-1} \frac{x_{i+1} - x_i}{x_{i+1} + x_i} \leq L^{-1}(a,b) \\ &\leq \frac{1}{2(b-a)}\sum_{i=0}^{n-1} \frac{x_{i+1}^2 - x_i^2}{x_i x_{i+1}} \leq H^{-1}(a,b) \end{aligned}$$

for all $d \in \mathfrak{Div}\,[a,b]$ and the bounds

$$\frac{1}{b-a}\sup_d h_d^{[-1]} = L^{-1}(a,b)$$

and

$$\frac{1}{b-a}\inf_d H_d^{[-1]} = L^{-1}(a,b).$$

Also, we can define the sequences

$$H_d^{[0]} := \prod_{i=0}^{n-1} \left[A\left(x_i, x_{i+1}\right)\right]^{\left(x_{i+1}-x_i\right)}, h_d^{[0]} := \prod_{i=0}^{n-1} \left[G\left(x_i, x_{i+1}\right)\right]^{\left(x_{i+1}-x_i\right)}$$

for a division d of the interval $[a, b] \subset (0, \infty)$.

Using the above results we have the inequality:

$$\begin{aligned} A(a, b) &\geq \prod_{i=0}^{n-1} \left[A\left(x_i, x_{i+1}\right)\right]^{\frac{x_{i+1}-x_i}{b-a}} \geq I(a, b) \\ &\geq \prod_{i=0}^{n-1} \left[G\left(x_i, x_{i+1}\right)\right]^{\frac{x_{i+1}-x_i}{b-a}} \geq G(a, b), \end{aligned}$$

which follows by the inequality (2.21) applied to the convex mapping $f : [a, b] \to \mathbb{R}$, $f(x) = -\ln x$.

By Theorem 21 we also deduce the bounds

$$\inf_d \left\{ \prod_{i=0}^{n-1} \left[A\left(x_i, x_{i+1}\right)\right]^{\frac{x_{i+1}-x_i}{b-a}} \right\} = I(a, b)$$

and

$$\sup_d \left\{ \prod_{i=0}^{n-1} \left[G\left(x_i, x_{i+1}\right)\right]^{\frac{x_{i+1}-x_i}{b-a}} \right\} = I(a, b).$$

3. A Refinement of the $H. - H.$ Inequality for Modulus

3.1. Some Inequalities for Modulus. We shall start with the following result containing a refinement of the second part of the $H. - H.$ inequality obtained by S. S. Dragomir [**38**]:

THEOREM 22. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval of real numbers I and let $a, b \in I$ with $a < b$. Then we have the following refinement of the right part of the $H. - H.$ inequality:*

$$\begin{aligned} &\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \qquad (2.24) \\ &\geq \begin{cases} \left| |f(a)| - \frac{1}{b-a} \int_a^b |f(x)|\, dx \right| & \text{if } \quad f(a) = f(b) \\ \left| \frac{1}{f(b)-f(a)} \int_{f(a)}^{f(b)} |x|\, dx - \frac{1}{b-a} \int_a^b |f(x)|\, dx \right| & \text{if } \quad f(a) \neq f(b) \end{cases}. \end{aligned}$$

PROOF. By the convexity of f on I and the continuity property of modulus we have:

$$\begin{aligned} & tf(a) + (1-t) f(b) - f(ta + (1-t) b) \\ = \; & |tf(a) + (1-t) f(b) - f(ta + (1-t) b)| \\ \geq \; & ||tf(a) + (1-t) f(b)| - |f(ta + (1-t) b)|| \geq 0 \end{aligned}$$

for all $a,b\in I$ and $t\in[0,1]$.
Integrating this inequality on $[0,1]$ over t we get the inequality:

$$\begin{aligned}&f(a)\int_0^1 tdt+f(b)\int_0^1(1-t)\,dt-\int_0^1 f(ta+(1-t)b)\,dt\\ \geq\ &\left|\int_0^1|tf(a)+(1-t)f(b)|\,dt-\int_0^1|f(ta+(1-t)b)|\,dt\right|.\end{aligned}$$

As it is easy to see that:

$$\int_0^1 tdt=\int_0^1(1-t)\,dt=\frac{1}{2},$$

$$\int_0^1 f(ta+(1-t)b)\,dt=\frac{1}{b-a}\int_a^b f(x)\,dx$$

and

$$\int_0^1|tf(a)+(1-t)f(b)|\,dt=\begin{cases}|f(a)| & \text{if}\quad f(b)=f(a)\\ \frac{1}{f(b)-f(a)}\int_{f(a)}^{f(b)}|x|\,dx & \text{if}\quad f(a)\neq f(b)\end{cases}$$

and

$$\int_0^1|f(ta+(1-t)b)|\,dt=\frac{1}{b-a}\int_a^b|f(x)|\,dx$$

respectively, then the inequality (2.24) is proved. ∎

The following corollary holds [**38**].

Corollary 7. *With the above assumptions, and the condition that $f(a+b-x)=f(x)$ for all $x\in[a,b]$, we have the inequality:*

$$f(a)-\frac{1}{b-a}\int_a^b f(x)\,dx\geq\left||f(a)|-\frac{1}{b-a}\int_a^b|f(x)|\,dx\right|\geq 0.$$

A refinement of the left hand side of the Hermite-Hadamard inequality is embodied in the following theorem by S. S. Dragomir [**38**].

Theorem 23. *With the assumptions of Theorem 22, we have the inequality:*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\\ \geq\ &\left|\frac{1}{b-a}\int_a^b\left|\frac{f(x)+f(a+b-x)}{2}\right|dx-\left|f\left(\frac{a+b}{2}\right)\right|\right|\geq 0.\end{aligned}\tag{2.25}$$

Proof. By the convexity of f we have that

$$\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\geq\left|\left|\frac{f(x)+f(y)}{2}\right|-\left|f\left(\frac{x+y}{2}\right)\right|\right|$$

for all $x,y\in I$.
Let us put $x=ta+(1-t)b,\ y=(1-t)a+tb$ with $t\in[0,1]$. Then we get

$$\begin{aligned}&\frac{f(ta+(1-t)b)+f((1-t)a+tb)}{2}-f\left(\frac{a+b}{2}\right)\\ \geq\ &\left|\left|\frac{f(ta+(1-t)b)+f((1-t)a+tb)}{2}\right|-\left|f\left(\frac{a+b}{2}\right)\right|\right|\end{aligned}$$

for all $t \in [0,1]$.
Integrating on $[0,1]$ we get that:

$$\frac{\int_0^1 f(ta+(1-t)b)\,dt + \int_0^1 f((1-t)a+tb)\,dt}{2} - f\left(\frac{a+b}{2}\right)$$
$$\geq \left|\int_0^1 \left|\frac{f(ta+(1-t)b)+f((1-t)a+tb)}{2}\right| dt - \left|f\left(\frac{a+b}{2}\right)\right|\right|.$$

However,

$$\int_0^1 f(ta+(1-t)b)\,dt = \int_0^1 f((1-t)a+tb)\,dt = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

and denoting $x := ta+(1-t)b,\ t\in[0,1]$, we also get that:

$$\int_0^1 \left|\frac{f(ta+(1-t)b)+f((1-t)a+tb)}{2}\right| dt$$
$$= \frac{1}{b-a}\int_a^b \left|\frac{f(x)+f(a+b-x)}{2}\right| dx.$$

Thus, the inequality (2.25) is proved. ∎

The following corollary also holds [**38**].

Corollary 8. *With the above assumptions and if the condition that $f(a+b-x) = f(x)$ is satisfied for all $x \in [a,b]$, then we have the inequality:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \geq \left|\frac{1}{b-a}\int_a^b |f(x)|\,dx - \left|f\left(\frac{a+b}{2}\right)\right|\right| \geq 0.$$

3.2. Applications for Special Means. It is well-known that the following inequality holds

$$(G-I-A) \qquad G(a,b) \leq I(a,b) \leq A(a,b)$$

where, we recall that

$$G(a,b) := \sqrt{ab}$$

is the *geometric mean,*

$$I(a,b) := \frac{1}{e}\cdot\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

is the *identric mean,* and

$$A(a,b) := \frac{a+b}{2}$$

is the *arithmetic mean* of the nonnegative real numbers $a<b$.

The following proposition holds [**38**].

Proposition 10. *If $a \in (0,1], b \in [1,\infty)$ with $a \neq b$; then one has the inequality:*

$$\frac{I(a,b)}{G(a,b)} \geq \exp\left[\left|\frac{(\ln b)^2+(\ln a)^2}{\ln\left(\frac{b}{a}\right)^2} - \ln\left[\left(b^b a^a e^{2-(a+b)}\right)^{\frac{1}{b-a}}\right]\right|\right] \geq 1 \tag{2.26}$$

which improves the first inequality $(G-I-A)$.

PROOF. Let us assume that $a \in (0,1], b \in [1,\infty)$ and $a \neq b$. Then we have for the convex mapping $f(x) = -\ln x, x > 0$:

$$\begin{aligned} A \quad : \quad &= \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = -\frac{\ln a + \ln b}{2} + \frac{1}{b-a}\int_a^b \ln x dx \\ &= \frac{1}{b-a}\left[b\ln b - a\ln a - (b-a)\right] - \ln G(a,b) = \ln\left[\frac{I(a,b)}{G(a,b)}\right]. \end{aligned}$$

Denote

$$B := \left|\frac{1}{f(b)-f(a)}\int_{f(a)}^{f(b)} |x|\,dx - \frac{1}{b-a}\int_a^b |\ln x|\,dx\right|.$$

We have

$$\int_{\ln b}^{\ln a} |x|\,dx = \frac{(\ln b)^2 + (\ln a)^2}{2}$$

and

$$\int_a^b |\ln x|\,dx = \ln\left[a^a b^b e^{-2-(a+b)}\right]$$

and thus

$$B = \left|\frac{(\ln b)^2+(\ln a)^2}{\ln\left(\frac{b}{a}\right)^2} - \ln\left[\left(b^b a^a e^{2-(a+b)}\right)^{\frac{1}{b-a}}\right]\right|.$$

Using the inequality (2.24) we can state that $A \geq B \geq 0$, and thus the proposition is proved. ∎

Finally, we have the following proposition [**38**].

PROPOSITION 11. *Let $a,b \in (0,\infty)$ with $a \neq b$. Then one has the inequality:*

$$\frac{A(a,b)}{I(a,b)} \geq \exp\left[\left|\frac{1}{b-a}\int_a^b \left|\ln\left(\sqrt{x(a+b-x)}\right)\right| dx - \left|\ln\left(\frac{a+b}{2}\right)\right|\right|\right] \geq 1, \tag{2.27}$$

which improves the second inequality $(G-I-A)$.

PROOF. Denote for $f(x) = -\ln x, x > 0$, that

$$\begin{aligned} C \quad : \quad &= \frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) = \ln\left(\frac{a+b}{2}\right) - \ln I(a,b) \\ &= \ln\left[\frac{A(a,b)}{I(a,b)}\right] \end{aligned}$$

and

$$\begin{aligned} D \quad : \quad &= \left|\frac{1}{b-a}\int_a^b \left|\frac{f(x)+f(a+b-x)}{2}\right| dx - \left|f\left(\frac{a+b}{2}\right)\right|\right| \\ &= \left|\frac{1}{b-a}\int_a^b \left|\ln\sqrt{x(a+b-x)}\right| dx - \left|\ln\left(\frac{a+b}{2}\right)\right|\right|. \end{aligned}$$

By the inequality (2.25) we have that $C \geq D \geq 0$ and the proposition is proved. ∎

4. Further Inequalities for Differentiable Convex Functions

4.1. Integral Inequalities. Let us assume that $I \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable mapping on $\mathring{\mathrm{I}}$, and let $a, b \in \mathring{\mathrm{I}}$ with $a < b$. If $f' \in L_1[a, b]$, then we have the equality

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b \left(x - \frac{a+b}{2}\right) f'(x)\,dx. \tag{2.28}$$

Indeed, by integrating by parts we obtain

$$\begin{aligned}\int_a^b \left(x - \frac{a+b}{2}\right) f'(x)\,dx &= \left(x - \frac{a+b}{2}\right) f(x)\Big|_a^b - \int_a^b f(x)\,dx \\ &= (b-a)\frac{f(a)+f(b)}{2} - \int_a^b f(x)\,dx,\end{aligned}$$

and the identity (2.28) is proved.

The following theorem holds [**34**]:

Theorem 24. *If the mapping f is differentiable on $\mathring{I}$ and the new mapping*

$$\varphi(x) := \left(x - \frac{a+b}{2}\right) f'(x)$$

is convex on $[a, b]$, then we have the inequality:

$$\frac{b-a}{8}\left(f'(a) - f'(b)\right) \geq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \geq 0. \tag{2.29}$$

Proof. Applying Hadamard's and Bullen's inequalities for the mapping φ, i.e.,

$$\frac{1}{2}\left[\varphi\left(\frac{a+b}{2}\right) + \frac{\varphi(a)+\varphi(b)}{2}\right] \geq \frac{1}{b-a}\int_a^b \varphi(x)\,dx \geq \varphi\left(\frac{a+b}{2}\right),$$

we get

$$\frac{1}{2}\left[\frac{\frac{b-a}{2}\cdot\left(f'(b) - f'(a)\right)}{2}\right] \geq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \geq 0,$$

and the inequality (2.29) is proved. ∎

Remark 17. *The above theorem contains a sufficient condition for the differentiable mapping f such that the second inequality in the H. – H. result remains true.*

In what follows, we need a lemma which is interesting in itself as it provides a refinement of the celebrated Chebychev's integral inequality (see also [**66**]).

Lemma 1. *Let $f, g : [a, b] \to \mathbb{R}$ be two integrable mappings which are synchronous, i.e., $(f(x) - f(y))(g(x) - g(y)) \geq 0$ for all $x, y \in [a, b]$. We then have:*

$$C(f, g) \geq \max\left\{\left|C(|f|, |g|)\right|, \left|C(|f|, g)\right|, \left|C(f, |g|)\right|\right\} \geq 0, \tag{2.30}$$

where

$$C(f, g) := (b-a)\int_a^b f(x)\,g(x)\,dx - \int_a^b f(x)\,dx \int_a^b g(x)\,dx.$$

Proof. As the mappings f,g are synchronous, we have that

$$0\leq\left(f\left(x\right)-f\left(y\right)\right)\left(g\left(x\right)-g\left(y\right)\right)=\left|\left(f\left(x\right)-f\left(y\right)\right)\left(g\left(x\right)-g\left(y\right)\right)\right|.$$

On the other hand, by the continuity property of modulus, we have

$$\left|\left(f\left(x\right)-f\left(y\right)\right)\left(g\left(x\right)-g\left(y\right)\right)\right|\geq\left|\left(\left|f\left(x\right)\right|-\left|f\left(y\right)\right|\right)\left(\left|g\left(x\right)\right|-\left|g\left(y\right)\right|\right)\right|$$

and

$$\left|\left(f\left(x\right)-f\left(y\right)\right)\left(g\left(x\right)-g\left(y\right)\right)\right|\geq\left|\left(\left|f\left(x\right)\right|-\left|f\left(y\right)\right|\right)\left(g\left(x\right)-g\left(y\right)\right)\right|$$

and

$$\left|\left(f\left(x\right)-f\left(y\right)\right)\left(g\left(x\right)-g\left(y\right)\right)\right|\geq\left|\left(f\left(x\right)-f\left(y\right)\right)\left(\left|g\left(x\right)\right|-\left|g\left(y\right)\right|\right)\right|$$

for all $x,y\in\left[a,b\right].$
Let us prove only the first inequality in (2.30).
Integrating on $\left[a,b\right]^{2}$ over $\left(x,y\right)$, we get

$$\begin{aligned}&\frac{1}{\left(b-a\right)^{2}}\int_{a}^{b}\int_{a}^{b}\left(f\left(x\right)-f\left(y\right)\right)\left(g\left(x\right)-g\left(y\right)\right)dxdy\\ \geq\ &\frac{1}{\left(b-a\right)^{2}}\int_{a}^{b}\int_{a}^{b}\left|\left(\left|f\left(x\right)\right|-\left|f\left(y\right)\right|\right)\left(\left|g\left(x\right)\right|-\left|g\left(y\right)\right|\right)\right|dxdy\\ \geq\ &\left|\frac{1}{\left(b-a\right)^{2}}\int_{a}^{b}\int_{a}^{b}\left(\left|f\left(x\right)\right|-\left|f\left(y\right)\right|\right)\left(\left|g\left(x\right)\right|-\left|g\left(y\right)\right|\right)dxdy\right|.\end{aligned}$$

As a simple calculation shows us that

$$C\left(f,g\right)=\frac{1}{2}\int_{a}^{b}\int_{a}^{b}\left(f\left(x\right)-f\left(y\right)\right)\left(g\left(x\right)-g\left(y\right)\right)dxdy$$

and

$$C\left(\left|f\right|,\left|g\right|\right)=\frac{1}{2}\int_{a}^{b}\int_{a}^{b}\left(\left|f\left(x\right)\right|-\left|f\left(y\right)\right|\right)\left(\left|g\left(x\right)\right|-\left|g\left(y\right)\right|\right)dxdy,$$

we deduce the first part of (2.30). ∎

The following refinement of the $H.-H.$ inequality holds [**34**].

Theorem 25. *Let $f:I\subseteq\mathbb{R}\rightarrow\mathbb{R}$ be a differentiable convex mapping on $\mathring{I}$ and $a,b\in\mathring{I}$ with $a<b$. Then, one has the inequality*

$$\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\geq\max\left\{\left|A\right|,\left|B\right|,\left|C\right|\right\}\leq0 \tag{2.31}$$

where

$$A:=\frac{1}{b-a}\int_{a}^{b}\left|x-\frac{a+b}{2}\right|\left|f'\left(x\right)\right|dx-\frac{1}{4}\int_{a}^{b}\left|f'\left(x\right)\right|dx;$$

$$B:=\frac{f\left(b\right)-f\left(a\right)}{4}+\frac{1}{b-a}\left[\int_{a}^{\frac{a+b}{2}}f\left(x\right)dx-\int_{\frac{a+b}{2}}^{b}f\left(x\right)dx\right]$$

and

$$C:=\frac{1}{b-a}\int_{a}^{b}\left(x-\frac{a+b}{2}\right)\left|f'\left(x\right)\right|dx.$$

PROOF. As f is convex on I, the mappings f' and $\left(x-\frac{a+b}{2}\right)$ are synchronous on $[a,b]$ and we can apply Lemma 1. Thus, we have:

$$\begin{aligned}(2.32)\qquad & (b-a)\int_a^b\left(x-\frac{a+b}{2}\right)f'(x)\,dx-\int_a^b\left(x-\frac{a+b}{2}\right)dx\int_a^b f'(x)\,dx \\ \geq\quad & \max\left\{\left|\bar{A}\right|,\left|\bar{B}\right|,\left|\bar{C}\right|\right\}\geq 0\end{aligned}$$

where

$$\bar{A}:=(b-a)\int_a^b\left|x-\frac{a+b}{2}\right|\left|f'(x)\right|dx-\int_a^b\left|x-\frac{a+b}{2}\right|dx\int_a^b\left|f'(x)\right|dx,$$

$$\bar{B}:=(b-a)\int_a^b\left|x-\frac{a+b}{2}\right|f'(x)\,dx-\int_a^b\left|x-\frac{a+b}{2}\right|dx\int_a^b f'(x)\,dx$$

and

$$\bar{C}:=(b-a)\int_a^b\left(x-\frac{a+b}{2}\right)\left|f'(x)\right|dx-\int_a^b\left(x-\frac{a+b}{2}\right)dx\int_a^b\left|f'(x)\right|dx.$$

However,

$$\int_a^b\left(x-\frac{a+b}{2}\right)dx=0\text{ and }\int_a^b\left|x-\frac{a+b}{2}\right|dx=\frac{(b-a)^2}{4}.$$

Thus,

$$\bar{A}:=(b-a)\int_a^b\left|x-\frac{a+b}{2}\right|\left|f'(x)\right|dx-\frac{(b-a)^2}{4}\int_a^b\left|f'(x)\right|dx,$$

$$\begin{aligned}\bar{B}\quad:\quad & =(b-a)\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)f'(x)\,dx+\int_{\frac{a+b}{2}}^b\left(x-\frac{a+b}{2}\right)f'(x)\,dx\right] \\ & -\frac{(b-a)^2}{4}\left(f(b)-f(a)\right) \\ = & \quad\frac{(b-a)^2}{4}\left(f(b)-f(a)\right)+(b-a)\left[\int_a^{\frac{a+b}{2}}f(x)\,dx-\int_{\frac{a+b}{2}}^b f(x)\,dx\right]\end{aligned}$$

and

$$\bar{C}:=(b-a)\int_a^b\left(x-\frac{a+b}{2}\right)\left|f'(x)\right|dx.$$

Using the inequality (2.32) we get

$$\frac{1}{b-a}\int_a^b\left(x-\frac{a+b}{2}\right)f'(x)\,dx\geq\max\left\{|A|,|B|,|C|\right\}\geq 0$$

where A,B,C are as given above. By the identity (2.28) we get (2.31).
Hence, the proof is completed. ∎

REMARK 18. *Taking into account that the class of differentiable convex functions defined on* $\mathring{I}$ *is dense in the class of all convex mappings defined on* $\mathring{I}$*, we can* *state that*

$$\begin{aligned} & \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx \\ \geq & \left|\frac{f(b)-f(a)}{4}+\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\,dx-\int_{\frac{a+b}{2}}^b f(x)\,dx\right]\right| \geq 0 \end{aligned}$$

for every $f: I \to \mathbb{R}$ *a convex function on* $\mathring{I}$.

The following theorem is interesting as well [**34**].

THEOREM 26. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a differentiable mapping on* $\mathring{I}$, $a,b\in\mathring{I}$, *with* $a<b$ *and* $p>1$. *If* $|f'|$ *is* $q-$*integrable on* $[a,b]$ *where* $q=\frac{p}{p-1}$, *then we have the inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{1}{2}\cdot\frac{(b-a)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\left(\int_a^b |f'(x)|^q\,dx\right)^{\frac{1}{q}}. \tag{2.33}$$

PROOF. Using Hölder's integral inequality for $p>1$ and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we can state that:

$$\begin{aligned} & \left|\frac{1}{b-a}\int_a^b\left(x-\frac{a+b}{2}\right)f'(x)\,dx\right| \\ \leq & \left(\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|^p dx\right)^{\frac{1}{p}}\times\left(\frac{1}{b-a}\int_a^b|f'(x)|^q\,dx\right)^{\frac{1}{q}}. \end{aligned}$$

However,

$$\begin{aligned} \int_a^b\left|x-\frac{a+b}{2}\right|^p dx &= 2\int_{\frac{a+b}{2}}^b\left(x-\frac{a+b}{2}\right)^p dx \\ &= \frac{(b-a)^{p+1}}{(p+1)\,2^p}. \end{aligned}$$

Thus,

$$\begin{aligned} & \left(\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|^p dx\right)^{\frac{1}{p}}\left(\frac{1}{b-a}\int_a^b|f'(x)|^q\,dx\right)^{\frac{1}{q}} \\ = & \left(\frac{(b-a)^p}{(p+1)\,2^p}\right)^{\frac{1}{p}}\left(\frac{1}{b-a}\int_a^b|f'(x)|^q\,dx\right)^{\frac{1}{q}} \\ = & \frac{1}{2}\frac{(b-a)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\left(\int_a^b|f'(x)|^q\,dx\right)^{\frac{1}{q}}, \end{aligned}$$

and the inequality (2.33) is obtained by the utilization of the identity (2.28). ∎

COROLLARY 9. *With the above assumptions, and provided that* f *is convex on* $\mathring{I}$, *we have the following reverse* $H.-H.$ *inequality:*

$$0\leq\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{1}{2}\frac{(b-a)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\left(\int_a^b|f'(x)|^q\,dx\right)^{\frac{1}{q}}. \tag{2.34}$$

The following result is well-known in the literature as Grüss' integral inequality:

LEMMA 2. *Let $f, g : [a, b] \to \mathbb{R}$ be two integrable functions such that $\varphi \leq f(x) \leq \phi$, $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a, b]$. Then we have the inequality:*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \frac{1}{b-a} \int_a^b g(x) \, dx \right| \leq \frac{1}{4} (\phi - \varphi)(\Gamma - \gamma).$$

For the proof of this classical result see the monograph [**114**, p. 296] where further details are given.

The following theorem holds [**34**]:

THEOREM 27. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$, $a, b \in \mathring{I}$, with $a < b$ and $m \leq f'(x) \leq M$ for all $x \in [a, b]$. If $f' \in L_1[a, b]$, then we have the inequality*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(M-m)(b-a)}{4}. \tag{2.35}$$

PROOF. Define the mapping $g(x) = x - \frac{a+b}{2}$, $x \in [a, b]$. Then,

$$-\left(\frac{b-a}{2} \right) \leq g(x) \leq \frac{b-a}{2}$$

for all $x \in [a, b]$. Therefore, by Grüss' inequality, we have that

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b \left(x - \frac{a+b}{2} \right) f'(x) \, dx \right. \\ & \left. - \frac{1}{b-a} \int_a^b \left(x - \frac{a+b}{2} \right) dx \cdot \frac{1}{b-a} \int_a^b f'(x) \, dx \right| \\ \leq & \frac{(M-m)(b-a)}{4}. \end{aligned}$$

Using the fact that

$$\int_a^b \left(x - \frac{a+b}{2} \right) dx = 0,$$

and the identity (2.28), we deduce the desired result (2.35). ∎

COROLLARY 10. *With the above assumptions, provided f is convex on $\mathring{I}$, we have*

$$0 \leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{(f'(b) - f'(a))(b-a)}{4}. \tag{2.36}$$

REMARK 19. *For a comprehensive list of results on the trapezoid inequality, see the expository work by Cerone and Dragomir [**15**]. Most of those results can be stated in the particular case of convex functions, but they will not be considered in this book.*

Now, we shall point out another identity which will allow us to establish some new inequalities connected with the first part of the celebrated Hermite-Hadamard integral inequality.

The following lemma is interesting [**34**].

LEMMA 3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$, $a, b \in I$, with $a < b$ and $f' \in L_1[a, b]$. Then, one has the identity:*

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b p(x) f'(x)\,dx, \tag{2.37}$$

where

$$p(x) = \begin{cases} x-a, x \in \left[a, \frac{a+b}{2}\right) \\ \\ x-b, x \in \left[\frac{a+b}{2}, b\right] \end{cases}.$$

PROOF. Using the formula for integration by parts, we have successively:

$$\int_a^{\frac{a+b}{2}} (x-a) f'(x)\,dx = \frac{b-a}{2} f\left(\frac{a+b}{2}\right) - \int_a^{\frac{a+b}{2}} f(x)\,dx$$

and

$$\int_{\frac{a+b}{2}}^b (x-b) f'(x)\,dx = \frac{b-a}{2} f\left(\frac{a+b}{2}\right) - \int_{\frac{a+b}{2}}^b f(x)\,dx.$$

Adding the above two identities, we deduce

$$\int_a^{\frac{a+b}{2}} (x-a) f'(x)\,dx + \int_{\frac{a+b}{2}}^b (x-b) f'(x)\,dx = (b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(x)\,dx.$$

As it is clear that

$$\int_a^b p(x) f'(x)\,dx = \int_a^{\frac{a+b}{2}} (x-a) f'(x)\,dx + \int_{\frac{a+b}{2}}^b (x-b) f'(x)\,dx,$$

the required identity is proved. ∎

REMARK 20. *It is obvious that we also have the representation:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) = \frac{1}{b-a}\int_a^b q(x) f'(x)\,dx \tag{2.38}$$

where

$$q(x) := \begin{cases} a-x, x \in \left[a, \frac{a+b}{2}\right) \\ \\ b-x, x \in \left[\frac{a+b}{2}, b\right] \end{cases}$$

which will be more appropriate, later, for our purposes.

The following theorem also holds [**34**]:

THEOREM 28. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$, $a, b \in \mathring{I}$, with $a < b$ and $p > 1$. If $|f'|$ is $q-$integrable on $[a, b]$ where $q = \frac{p}{p-1}$, then we have the inequality:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \frac{1}{2}\frac{(b-a)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\left(\int_a^b |f'(x)|^q\,dx\right)^{\frac{1}{q}}. \tag{2.39}$$

PROOF. Using Hölder's inequality we have that:

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b p(x) f'(x)\,dx\right| \\ \leq & \left(\frac{1}{b-a}\int_a^b |p(x)|^p\,dx\right)^{\frac{1}{p}} \times \left(\frac{1}{b-a}\int_a^b |f'(x)|^q\,dx\right)^{\frac{1}{q}}.\end{aligned}$$

However,

$$\begin{aligned}\int_a^b |p(x)|^p\,dx &= \int_a^{\frac{a+b}{2}} |x-a|^p\,dx + \int_{\frac{a+b}{2}}^b |x-b|^p\,dx \\ &= \frac{(b-a)^{p+1}}{2^p(p+1)}.\end{aligned}$$

Thus, the inequality (2.39) is proved. ∎

COROLLARY 11. *With the above assumptions and provided that f is convex on $\mathring{I}$, we have the reverse inequality*

$$0 \leq \frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\frac{(b-a)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\left(\int_a^b |f'(x)|^q\,dx\right)^{\frac{1}{q}}.$$

Finally, the following theorem also holds [**34**]:

THEOREM 29. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$, $a, b \in \mathring{I}$, with $a < b$ and $m \leq f'(x) \leq M$ for all $x \in [a, b]$. It $f' \in L_1[a, b]$, then we have the inequality:*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(M-m)(b-a)}{4}.$$

The proof is similar to the proof of Theorem 27 via Grüss' identity (2.36). We shall omit the details.

REMARK 21. *For a comprehensive list of results on the mid-point inequality, see the expository work by Cerone and Dragomir* [**16**]. *Most of those results can be stated in the particular case of convex functions, but they will not be considered in this book.*

4.2. Applications for Special Means. We shall start with the following proposition:

PROPOSITION 12. *Let $p > 1$ and $[a, b] \subset [0, \infty)$. Then we have the inequality:*

$$0 \leq A(a^p, b^p) - L_p^p(a, b) \leq \frac{p(b-a)}{2(p+1)^{\frac{1}{p}}}\left[L_p(a, b)\right]^{\frac{p}{q}}, \tag{2.40}$$

where $q := \frac{p}{p-1}$.

PROOF. By Theorem 26 applied to the convex mapping $f(x) = x^p$ we have:

$$\begin{aligned} \frac{a^p + b^p}{2} - \frac{1}{b-a}\int_a^b x^p dx &\leq \frac{1}{2}\frac{(b-a)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\left(\int_a^b \left|px^{p-1}\right|^q dx\right)^{\frac{1}{q}} \\ &= \frac{1}{2}\frac{(b-a)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}p\left(\int_a^b x^{(p-1)q}dx\right)^{\frac{1}{q}}. \end{aligned}$$

However,

$$\int_a^b x^{(p-1)q}dx = \frac{b^{pq-q+1} - a^{pq-q+1}}{pq-q+1} = \frac{b^{p+1} - a^{p+1}}{p+1} = L_p^p(a,b)(b-a),$$

and thus we have:

$$\begin{aligned} A(a^p, b^p) - L_p^p(a,b) &\leq \frac{p(b-a)^{\frac{1}{p}}}{2(p+1)^{\frac{1}{p}}}(b-a)^{\frac{1}{q}}\left[L_p(a,b)\right]^{\frac{p}{q}} \\ &= \frac{p(b-a)\left[L_p(a,b)\right]^{\frac{p}{q}}}{2(p+1)^{\frac{1}{p}}}, \end{aligned}$$

and the inequality (2.40) is proved. ∎

Another result which is connected with the logarithmic mean $L(a,b)$ is the following one:

PROPOSITION 13. *Let $p > 1$ and $0 < a < b$. Then one has the inequality:*

$$0 \leq H^{-1}(a,b) - L^{-1}(a,b) \leq \frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left[L_{\frac{2p}{1-p}}(a,b)\right]^{\frac{p-1}{p}}. \tag{2.41}$$

PROOF. By Theorem 26 applied for the convex mapping $f(x) := \frac{1}{x}$ we have:

$$0 \leq \frac{\frac{1}{a} + \frac{1}{b}}{2} - \frac{\ln b - \ln a}{b-a} \leq \frac{1}{2}\cdot\frac{(b-a)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\left(\int_a^b \frac{dx}{x^{2q}}\right)^{\frac{1}{q}}.$$

However,

$$\int_a^b x^{-2q}dx = (b-a)L_{-2q}^{-2q}(a,b),$$

and as

$$-2q = \frac{2p}{1-p},$$

we deduce that

$$\begin{aligned} 0 &\leq H^{-1}(a,b) - L^{-1}(a,b) \\ &\leq \frac{1}{2}\frac{(b-a)^{\frac{1}{p}}(b-a)^{\frac{1}{q}}}{(p+1)^{\frac{1}{p}}}\left[L_{-2q}^{-2q}(a,b)\right]^{\frac{1}{q}} \\ &= \frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left[L_{\frac{2p}{1-p}}(a,b)\right]^{\frac{p-1}{p}}, \end{aligned}$$

and the proposition is thus proved. ∎

The next proposition contains an inequality for the identric mean $I(a,b)$.

Proposition 14. *Let $p>1$ and $0<a<b$. Then one has the inequality:*

$$1\leq\frac{I(a,b)}{G(a,b)}\leq\exp\left[\frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left[L_{-q}^{-1}(a,b)\right]\right]. \tag{2.42}$$

Proof. If we apply Theorem 26 for the convex mapping $f(x)=-\ln x, x>0$, we have that

$$\begin{aligned}0 &\leq \frac{1}{b-a}\int_a^b \ln x dx-\frac{\ln a+\ln b}{2}\leq\frac{1}{2}\frac{(b-a)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\left(\int_a^b\frac{dx}{x^q}\right)^{\frac{1}{q}}\\ &= \frac{1}{2}\frac{(b-a)^{\frac{1}{p}}(b-a)^{\frac{1}{q}}}{(p+1)^{\frac{1}{p}}}\left[L_{-q}^{-q}(a,b)\right]^{\frac{1}{q}}\\ &= \frac{1}{2}\frac{(b-a)}{(p+1)^{\frac{1}{p}}}\left[L_{-q}^{-1}(a,b)\right],\end{aligned}$$

from where results the inequality (2.42). ∎

Further on, we shall point out some natural applications of Theorem 27.

The following proposition holds.

Proposition 15. *Let $p>1$ and $0\leq a<b$. Then one has the inequality:*

$$0\leq A(a^p,b^p)-L_p^p(a,b)\leq\frac{p(p-1)}{4}(b-a)^2L_{p-2}^{p-2}(a,b). \tag{2.43}$$

Proof. If we choose in Theorem 27, $f(x)=x^p, p>1$, we have that $pa^{p-1}\leq f'(x)\leq pb^{p-1}$ for all $x\in[a,b]$. Thus, by the inequality (2.35) we obtain:

$$\begin{aligned}0 &\leq A(a^p,b^p)-L_p^p(a,b)\leq\frac{p\left(b^{p-1}-a^{p-1}\right)(b-a)}{4}\\ &= \frac{p(p-1)}{4}(b-a)^2L_{p-2}^{p-2}(a,b),\end{aligned}$$

and the proposition is proved. ∎

For the logarithmic mean, we have the following result.

Proposition 16. *Let $0<a<b$. The we have the inequality*

$$0\leq H^{-1}(a,b)-L^{-1}(a,b)\leq\frac{\left(b^2-a^2\right)(b-a)}{4a^2b^2}. \tag{2.44}$$

Proof. Indeed, if we choose in Theorem 27, $f(x)=\frac{1}{x}$, then we have

$$-\frac{1}{a^2}\leq f'(x)=-\frac{1}{x^2}\leq-\frac{1}{b^2}$$

and thus

$$M-m=-\frac{1}{b^2}+\frac{1}{a^2}=\frac{b^2-a^2}{4a^2b^2}.$$

Using inequality (2.35) the proof is completed. ∎

For the identric mean we have:

Proposition 17. *If $0<a<b$, one has the inequality:*

$$1\leq\frac{I(a,b)}{G(a,b)}\leq\exp\left[\frac{(b-a)^2}{4ab}\right]. \tag{2.45}$$

PROOF. Follows by Theorem 27. ∎

Now, if we use Theorem 28 we can state the following inequalities:

$$0 \leq A\left(a^{p}, b^{p}\right) - L_{p}^{p}(a, b) \leq \frac{p(b-a) L_{p}^{\frac{p}{q}}(a, b)}{2(p+1)^{\frac{1}{p}}},$$

and

$$0 \leq H^{-1}(a, b) - L^{-1}(a, b) \leq \frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left[L_{\frac{(p+1)}{1-p}}(a, b)\right]^{\frac{p+1}{p}},$$

and

$$1 \leq \frac{I(a, b)}{G(a, b)} \leq \exp\left[\frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left[L_{-p}^{1-p}(a, b)\right]\right],$$

respectively, where $p>1$ and $q:=\frac{p}{p-1}$.
On the other hand, if we apply Theorem 29, we have the following inequalities:

$$0 \leq A\left(a^{p}, b^{p}\right) - L_{p}^{p}(a, b) \leq \frac{p(p-1)}{4}(b-a)^{2} L_{p-2}^{p-2}(a, b),$$

and

$$0 \leq H^{-1}(a, b) - L^{-1}(a, b) \leq \frac{\left(b^{2}-a^{2}\right)(b-a)}{4 a^{2} b^{2}},$$

and

$$1 \leq \frac{I(a, b)}{G(a, b)} \leq \exp\left[\frac{(b-a)^{2}}{4 a b}\right],$$

respectively, where $p>1$.

5. Further Inequalities for Twice Differentiable Convex Functions

5.1. Integral Inequalities for Twice Differentiable Convex Functions. We shall start with the following well known lemma which is interesting in itself.

LEMMA 4. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathring{I}$ with f'' integrable on $[a, b] \subset \mathring{I}$. Then we have the identity:*

$$\frac{1}{2}\int_{a}^{b}(x-a)(b-x) f''(x)\, dx = \frac{b-a}{2}(f(a)+f(b)) - \int_{a}^{b} f(x)\, dx. \tag{2.46}$$

PROOF. Indeed, by an integration by parts, we have that

$$\begin{aligned}
& \frac{1}{2}\int_{a}^{b}(x-a)(b-x) f''(x)\, dx \\
= & \left[\left.\frac{1}{2}(x-a)(b-x) f'(x)\right|_{a}^{b} - \int_{a}^{b}[-2x+(a+b)] f'(x)\, dx\right] \\
= & \frac{1}{2}\int_{a}^{b}[2x-(a+b)] f'(x)\, dx \\
= & \frac{1}{2}\left[\left.(2x-(a+b)) f(x)\right|_{a}^{b} - 2\int_{a}^{b} f(x)\, dx\right] \\
= & \frac{b-a}{2}(f(a)+f(b)) - \int_{a}^{b} f(x)\, dx,
\end{aligned}$$

and the identity (2.46) is proved. ∎

The following estimation result holds [**50**].

THEOREM 30. *With the above assumptions, given that* $k \leq f''(x) \leq K$ *on* $[a,b]$, *we have the inequality*

$$k \cdot \frac{(b-a)^2}{12} \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \leq K \cdot \frac{(b-a)^2}{12}. \tag{2.47}$$

PROOF. We have

$$k(x-a)(b-x) \leq (x-a)(b-x) f''(x) \leq K(x-a)(b-x)$$

for all $x \in [a,b]$. Thus,

$$\begin{aligned}\frac{k}{2} \cdot \int_a^b (x-a)(b-x)\,dx &\leq \frac{1}{2}\int_a^b (x-a)(b-x) f''(x)\,dx \\ &\leq \frac{K}{2} \cdot \int_a^b (x-a)(b-x)\,dx.\end{aligned}$$

However, by (2.46)

$$\frac{1}{2}\int_a^b (x-a)(b-x) f''(x)\,dx = \frac{(b-a)}{2}(f(a)+f(b)) - \int_a^b f(x)\,dx$$

and

$$\int_a^b (x-a)(b-x)\,dx = \frac{(b-a)^3}{6}.$$

Hence, the inequality (2.47) is proved. ∎

COROLLARY 12. *With the above assumption, given that* $\|f''\|_\infty := \sup_{x\in[a,b]} |f''(x)| \leq \infty$, *then we have the known inequality:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \leq \|f''\|_\infty \frac{(b-a)^2}{12}. \tag{2.48}$$

The following result also holds.

THEOREM 31. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be as above. If we assume that the new mapping* $\varphi : [a,b] \to \mathbb{R}$, $\varphi(x) = (x-a)(b-x) f''(x)$ *is convex on* $[a,b]$, *then we have the inequality:*

$$\begin{aligned}\frac{(b-a)^2}{16} f''\left(\frac{a+b}{2}\right) &\geq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\geq \frac{(b-a)^2}{8} f''\left(\frac{a+b}{2}\right).\end{aligned} \tag{2.49}$$

PROOF. Applying the first inequality of Hermite-Hadamard for the mapping φ we can state:

$$\frac{1}{b-a}\int_a^b \varphi(x)\,dx \geq \varphi\left(\frac{a+b}{2}\right) = \frac{(b-a)^2}{4} f''\left(\frac{a+b}{2}\right),$$

and, by Bullen's inequality:

$$\begin{aligned}\frac{1}{b-a}\int_a^b \varphi(x)\,dx &\leq \frac{1}{2}\left[\varphi\left(\frac{a+b}{2}\right) + \frac{\varphi(a)+\varphi(b)}{2}\right] \\ &= \frac{(b-a)^2}{8} f''\left(\frac{a+b}{2}\right),\end{aligned}$$

and the inequality (2.49) is proved. ∎

Another estimation result containing $q-$norms also holds [**50**].

THEOREM 32. *With the above assumption, assuming that $p>1$, $q:=\frac{p}{p-1}$ and $|f''|$ is $q-$Lebesque integrable on $[a,b]$, then we have the inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{1}{2}(b-a)^{\frac{p+1}{p}}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\left\|f''\right\|_q, \tag{2.50}$$

where B is Euler's Beta-function.

PROOF. By (2.46) we have that:

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| &= \frac{1}{2}\left|\frac{1}{b-a}\int_a^b (x-a)(b-x)f''(x)\,dx\right| \\ &\leq \frac{1}{2(b-a)}\left(\int_a^b (x-a)^p(b-x)^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b |f''(x)|^q\,dx\right)^{\frac{1}{q}}.\end{aligned} \tag{2.51}$$

Note that for the last inequality we have used Hölder's inequality.
Denote $x=(1-t)a+tb$. We have $dx=(b-a)\,dt$ and then

$$\begin{aligned}&\int_a^b (x-a)^p(b-x)^p\,dx \\ &= (b-a)\int_0^1 ((1-t)a+tb-a)^p(b-(1-t)a-tb)^p\,dt \\ &= (b-a)^{2p+1}\int_0^1 t^p(1-t)^p\,dt \\ &= (b-a)^{2p+1}B(p+1,p+1),\end{aligned}$$

where

$$B(p,q):=\int_0^1 t^{p-1}(1-t)^{q-1}\,dt,\ p,q>0$$

is Euler's Beta function.
Using the inequality (2.51) we deduce

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\leq \frac{(b-a)^{\frac{2p+1}{p}}}{2(b-a)}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\left\|f''\right\|_q \\ &= \frac{1}{2}(b-a)^{\frac{p+1}{p}}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\left\|f''\right\|_q,\end{aligned}$$

and the theorem is proved. ∎

REMARK 22. *If $p > 1, p \in \mathbb{N},$ as*

$$B(p+1,p+1) = \frac{p!}{(p+1)\dots(2p+1)} = \frac{[p!]^2}{(2p+1)!},$$

we deduce the estimation

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{2}(b-a)^{\frac{p+1}{p}}\left[\frac{[p!]^2}{(2p+1)!}\right]^{\frac{1}{p}}\|f''\|_q,$$

which gives for $p = 2$ that

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^{\frac{3}{2}}\|f''\|_2\sqrt{30}}{60}.$$

The following result also holds [**50**]:

THEOREM 33. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable mapping on $\mathring{I}$ with f'' being integrable and $\gamma \le f''(x) \le \Gamma$ on $[a,b] \subset \mathring{I}$. Then one has the inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \frac{b-a}{12}\left(f'(b)-f'(a)\right)\right| \le \frac{(b-a)^2}{32}(\Gamma-\gamma). \tag{2.52}$$

PROOF. By Grüss inequality, we have that

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b (x-a)(b-x) f''(x)\,dx\right.\\ &\left.-\frac{1}{b-a}\int_a^b (x-a)(b-x)\,dx \cdot \frac{1}{b-a}\int_a^b f''(x)\,dx\right|\\ \le\ & \frac{1}{4}(L-l)(\Gamma-\gamma),\end{aligned}$$

where

$$L = \sup_{x\in[a,b]}\{(x-a)(b-x)\} = \frac{(b-a)^2}{4}$$

and

$$l = \inf_{x\in[a,b]}\{(x-a)(b-x)\} = 0,$$

where Γ, γ are as above.
As a simple calculation shows us that

$$\int_a^b (x-a)(b-x)\,dx = \frac{(b-a)^3}{6},$$

and by Lemma 4 we have that

$$\begin{aligned} I \ :&= \frac{1}{2(b-a)}\int_a^b (x-a)(b-x) f''(x)\,dx\\ &= \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx,\end{aligned}$$

we obtain

$$\begin{aligned}&\left|I-\frac{1}{2(b-a)}\cdot\frac{(b-a)^3}{6}\cdot\frac{1}{b-a}\left(f'(b)-f'(a)\right)\right|\\ \le\ &\frac{1}{8}\cdot\frac{(b-a)^2}{4}\cdot(\Gamma-\gamma).\end{aligned}$$

That is,

$$\left|I-\frac{b-a}{12}\cdot\left(f'(b)-f'(a)\right)\right|\le\frac{(b-a)^2}{32}(\Gamma-\gamma),$$

and the theorem is thus proved. ∎

The following lemma is itself interesting [**40**]

Lemma 5. *Let $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $g'(x)\neq 0$ on (a,b) and*

$$l\le\frac{f'(x)}{g'(x)}\le L \text{ on } (a,b),$$

then one has the inequality:

$$\begin{aligned}(2.53)\qquad &l\left[(b-a)\int_a^b g^2(x)\,dx-\left(\int_a^b g(x)\,dx\right)^2\right]\\ \le\ &(b-a)\int_a^b f(x)\,g(x)\,dx-\int_a^b f(x)\,dx\int_a^b g(x)\,dx\\ \le\ &L\left[(b-a)\int_a^b g^2(x)\,dx-\left(\int_a^b g(x)\,dx\right)^2\right].\end{aligned}$$

Proof. First of all, we will show that for all $x,y\in[a,b]$ we have the inequality

$$(2.54)\qquad l\left(g(x)-g(y)\right)^2\le\left(f(x)-f(y)\right)\left(g(x)-g(y)\right)\le L\left(g(x)-g(y)\right)^2.$$

If $g(x)=g(y)$, then the above inequality becomes an identity.
If $g(x)\neq g(y)$, and (assume) $x<y$, then by Cauchy's theorem , there exists an $\xi\in(x,y)$ such that

$$\frac{f(x)-f(y)}{g(x)-g(y)}=\frac{f'(\xi)}{g'(\xi)}\in[l,L],$$

and thus

$$l\le\frac{f(x)-f(y)}{g(x)-g(y)}\le L.$$

If we multiply by $\left(g(x)-g(y)\right)^2>0$ we get (2.54).
Now, if we integrate (2.54) on $[a,b]^2$ we can state that

$$\begin{aligned}l\int_a^b\int_a^b\left(g(x)-g(y)\right)^2dxdy\ &\le\ \int_a^b\int_a^b\left(f(x)-f(y)\right)\left(g(x)-g(y)\right)dxdy\\ &\le\ L\int_a^b\int_a^b\left(g(x)-g(y)\right)^2dxdy.\end{aligned}$$

As a simple calculation shows us that

$$\begin{aligned}&\frac{1}{2}\int_a^b\int_a^b (g(x)-g(y))^2\,dxdy\\ =\;& (b-a)\int_a^b g^2(x)\,dx-\left(\int_a^b g(x)\,dx\right)^2\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\int_a^b\int_a^b (f(x)-f(y))(g(x)-g(y))\,dxdy\\ =\;& (b-a)\int_a^b f(x)g(x)\,dx-\int_a^b f(x)\,dx\int_a^b g(x)\,dx,\end{aligned}$$

the desired inequality (2.53) is obtained. ∎

Remark 23. *We shall show that the inequality* (2.47) *can also be proved by the use of the above lemma. Indeed, we can state that:*

$$\begin{aligned}& k\left[(b-a)\int_a^b\left(x-\frac{a+b}{2}\right)^2dx-\left[\int_a^b\left(x-\frac{a+b}{2}\right)dx\right]^2\right]\\ \leq\;& (b-a)\int_a^b\left(x-\frac{a+b}{2}\right)f'(x)\,dx-\int_a^b\left(x-\frac{a+b}{2}\right)dx\int_a^b f'(x)\,dx\\ \leq\;& K\left[(b-a)\int_a^b\left(x-\frac{a+b}{2}\right)^2dx-\left[\int_a^b\left(x-\frac{a+b}{2}\right)dx\right]^2\right].\end{aligned}$$

As a simple calculation shows us that

$$\int_a^b\left(x-\frac{a+b}{2}\right)dx=0$$

and

$$\int_a^b\left(x-\frac{a+b}{2}\right)^2dx=\frac{(b-a)^3}{12},$$

we get the inequality

$$k\cdot\frac{(b-a)^4}{12}\leq (b-a)\int_a^b\left(x-\frac{a+b}{2}\right)f'(x)\,dx\leq K\cdot\frac{(b-a)^4}{12}.$$

Now, if we use the identity

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{1}{b-a}\int_a^b\left(x-\frac{a+b}{2}\right)f'(x)\,dx$$

we get the desired result.

The following theorem also holds [**40**].

THEOREM 34. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on $\mathring{I}$ such that $k \leq f''(x) \leq K$ on $[a,b] \subset \mathring{I}$. Then one has the double inequality:*

$$
\begin{aligned}
(2.55) \qquad & k \cdot \frac{(b-a)^2}{48} \\
\leq\; & \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] - \frac{1}{b-a}\int_a^b f(x)\,dx \\
\leq\; & K \cdot \frac{(b-a)^2}{48}.
\end{aligned}
$$

PROOF. Recall the identity (proved in Lemma 3):

$$
\begin{aligned}
(2.56) \qquad & f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \\
=\; & \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (x-a) f'(x)\,dx + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (x-b) f'(x)\,dx.
\end{aligned}
$$

If we now apply Lemma 5 we obtain:

$$
\begin{aligned}
& k\left[\left(\frac{a+b}{2}-a\right)\int_a^{\frac{a+b}{2}} (x-a)^2\,dx - \left(\int_a^{\frac{a+b}{2}} (x-a)\,dx\right)^2\right] \\
\leq\; & \left(\frac{a+b}{2}-a\right)\int_a^{\frac{a+b}{2}} (x-a) f'(x)\,dx - \int_a^{\frac{a+b}{2}} (x-a)\,dx \int_a^{\frac{a+b}{2}} f'(x)\,dx \\
\leq\; & K\left[\left(\frac{a+b}{2}-a\right)\int_a^{\frac{a+b}{2}} (x-a)^2\,dx - \left(\int_a^{\frac{a+b}{2}} (x-a)\,dx\right)^2\right].
\end{aligned}
$$

As a simple computation gives us:

$$
\int_a^{\frac{a+b}{2}} (x-a)^2\,dx = \frac{(b-a)^3}{24}
$$

and

$$
\int_a^{\frac{a+b}{2}} (x-a)\,dx = \frac{(b-a)^2}{8}
$$

and

$$
I_1 := \frac{b-a}{2} \cdot \frac{(b-a)^3}{24} - \frac{(b-a)^4}{64} = \frac{(b-a)^4}{192},
$$

thus we get that

$$
\begin{aligned}
& k \cdot \frac{(b-a)^4}{192} \\
\leq\; & \frac{b-a}{2}\int_a^{\frac{a+b}{2}} (x-a) f'(x)\,dx - \frac{(b-a)^2}{8}\left[f\left(\frac{a+b}{2}\right) - f(a)\right] \\
\leq\; & K \cdot \frac{(b-a)^4}{192},
\end{aligned}
$$

from where we derive

$$
\begin{aligned}
(2.57)\qquad & k\cdot\frac{(b-a)^2}{96} \\
&\leq \frac{1}{b-a}\int_a^{\frac{a+b}{2}}(x-a)\,f'(x)\,dx-\frac{1}{4}\left[f\left(\frac{a+b}{2}\right)-f(a)\right] \\
&\leq K\cdot\frac{(b-a)^2}{96},
\end{aligned}
$$

Similarly, Lemma 5 gives us that

$$
\begin{aligned}
& k\left[\left(b-\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^b (x-b)^2\,dx-\left(\int_{\frac{a+b}{2}}^b (x-b)\,dx\right)^2\right] \\
\leq\ & \left(b-\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^b (x-b)\,f'(x)\,dx-\int_{\frac{a+b}{2}}^b (x-b)\,dx\int_{\frac{a+b}{2}}^b f'(x)\,dx \\
\leq\ & K\left[\left(b-\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^b (x-b)^2\,dx-\left(\int_{\frac{a+b}{2}}^b (x-b)\,dx\right)^2\right].
\end{aligned}
$$

As we have

$$\int_{\frac{a+b}{2}}^b (x-b)^2\,dx=\frac{(b-a)^3}{24}$$

and

$$\int_{\frac{a+b}{2}}^b (x-b)\,dx=-\frac{(b-a)^2}{8},$$

then, by the above inequality we can state that

$$
\begin{aligned}
& k\cdot\frac{(b-a)^4}{192} \\
\leq\ & \frac{b-a}{2}\int_{\frac{a+b}{2}}^b (x-b)\,f'(x)\,dx+\frac{(b-a)^2}{8}\left[f(b)-f\left(\frac{a+b}{2}\right)\right] \\
\leq\ & K\cdot\frac{(b-a)^2}{192},
\end{aligned}
$$

which is equivalent with

$$
\begin{aligned}
(2.58)\qquad & k\cdot\frac{(b-a)^2}{96} \\
&\leq \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (x-b)\,f'(x)\,dx+\frac{1}{4}\left[f(b)-f\left(\frac{a+b}{2}\right)\right] \\
&\leq K\cdot\frac{(b-a)^4}{96}.
\end{aligned}
$$

If we now add the inequalities (2.57) and (2.58), taking into account the identity (2.56), we obtain:

$$\begin{aligned} & k\cdot\frac{(b-a)^2}{48} \\ \leq & f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{4}\left[2f\left(\frac{a+b}{2}\right)-(f(a)+f(b))\right] \\ \leq & K\cdot\frac{(b-a)^2}{48}, \end{aligned}$$

which is equivalent with the desired inequality (2.55). ∎

The following two corollaries also hold.

COROLLARY 13. *With the above assumptions, with the condition that* $\|f''\|_\infty<\infty$, *we have the inequality*

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\|f''\|_\infty\cdot\frac{(b-a)^2}{48}.$$

COROLLARY 14. *If f is twice differentiable on $\mathring{I}$ and convex on this interval, then we have the following converse of Bullen's inequality:*

$$0\leq\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\leq\|f''\|_\infty\cdot\frac{(b-a)^2}{48},$$

where $[a,b]\subset\mathring{I}$.

The following result also holds [**40**]:

THEOREM 35. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be twice differentiable mapping on $\mathring{I}$ and $k\leq f''(x)\leq K$ for all $x\in[a,b]\subset\mathring{I}$. Then we have the inequality:*

$$\begin{aligned} B+K\cdot\frac{(b-a)^3}{12} &\leq (b-a)\cdot\frac{f(a)+f(b)}{2}-\int_a^b f(x)\,dx \\ &\leq A+k\cdot\frac{(b-a)^3}{12}, \end{aligned}\tag{2.59}$$

where

$$\begin{aligned} A &:= \frac{1}{2}\left\{(b-a)\left[f'(b)-k\left(\frac{b-a}{2}\right)\right]-(f(b)-f(a))\right\} \\ &\quad\times\frac{\left\{f(b)-f(a)-(b-a)\left[f'(a)-k\left(\frac{b-a}{2}\right)\right]\right\}}{f'(b)-f'(a)-k(b-a)} \end{aligned}$$

and

$$\begin{aligned} B &:= \frac{1}{2}\left\{(b-a)\left[K\left(\frac{b-a}{2}\right)-f'(b)\right]+(f(b)-f(a))\right\} \\ &\quad\times\frac{\left\{f(b)-f(a)-(b-a)\left[K\left(\frac{b-a}{2}\right)+f'(a)\right]\right\}}{K(b-a)-f'(b)+f'(a)}, \end{aligned}$$

provided that:

$$f'(b)-f'(a)\neq k(b-a)\ \text{ and }\ f'(b)-f'(a)\neq K(b-a).$$

PROOF. Let us denote

$$I := \frac{1}{2}\int_a^b (x-a)(b-x) f''(x)\,dx = \frac{b-a}{2}(f(a)+f(b)) - \int_a^b f(x)\,dx.$$

For the last inequality we used Lemma 4.
It is easy to see that

$$\begin{aligned}\frac{1}{2}\int_a^b (x-a)(b-x)(f''(x)-k)\,dx &= I - k\cdot\frac{1}{2}\int_a^b (x-a)(b-x)\,dx \\ &= I - k\cdot\frac{(b-a)^3}{12},\end{aligned}$$

and, similarly,

$$\frac{1}{2}\int_a^b (x-a)(b-x)(K-f''(x))\,dx = K\cdot\frac{(b-a)^3}{12} - I.$$

By the classical Chebychev integral inequality for asynchronous mappings we have that:

$$\begin{aligned}&\frac{1}{2}\int_a^b (x-a)(b-x)(f''(x)-k)\,dx \\ \leq\ &\frac{1}{2}\frac{\int_a^b (x-a)(f''(x)-k)\,dx \int_a^b (b-x)(f''(x)-k)\,dx}{\int_a^b (f''(x)-k)\,dx}\end{aligned} \tag{2.60}$$

and

$$\begin{aligned}&\frac{1}{2}\int_a^b (x-a)(b-x)(K-f''(x))\,dx \\ \leq\ &\frac{1}{2}\frac{\int_a^b (x-a)(K-f''(x))\,dx \int_a^b (b-x)(K-f''(x))\,dx}{\int_a^b (K-f''(x))\,dx}.\end{aligned} \tag{2.61}$$

By a simple calculation we get that:

$$\begin{aligned}&\int_a^b (x-a)(f''(x)-k)\,dx \\ =\ &(f'(x)-kx)(x-a)\big|_a^b - \int_a^b (f'(x)-k)\,dx \\ =\ &(f'(b)-kb)(b-a) - \left[f(b)-f(a)-k\cdot\frac{x^2}{2}\bigg|_a^b\right] \\ =\ &(f'(b)-kb)(b-a) - \left[f(b)-f(a)-k\left(\frac{b^2-a^2}{2}\right)\right] \\ =\ &(b-a)\left[f'(b)-kb+k\cdot\frac{a+b}{2}\right] - f(b)+f(a) \\ =\ &(b-a)\left[f'(b)-k\left(\frac{b-a}{2}\right)\right] - (f(b)-f(a))\end{aligned}$$

and

$$\begin{aligned}
& \int_a^b (b-x)\left(f''(x)-k\right)dx \\
= \;& \left(f'(x)-kx\right)(b-x)\Big|_a^b - \int_a^b \left(f'(x)-k\right)dx \\
= \;& -\left(f'(a)-ka\right)(b-a) + f(b) - f(a) - k\cdot\frac{b^2-a^2}{2} \\
= \;& (b-a)\left[-f'(a)+ka-k\cdot\frac{a+b}{2}\right] + f(b) - f(a) \\
= \;& (b-a)\left[-f'(a)-k\cdot\frac{b-a}{2}\right] + f(b) - f(a) \\
= \;& f(b) - f(a) - (b-a)\left[f'(a)+k\cdot\frac{b-a}{2}\right].
\end{aligned}$$

Using the inequality (2.60) we obtain:

$$\begin{aligned}
& I - \frac{k(b-a)^3}{12} \\
\le \;& \frac{1}{2}\left\{(b-a)\left[f'(b)-k\left(\frac{b-a}{2}\right)\right] - \left(f(b)-f(a)\right)\right\} \\
& \times\frac{\left\{f(b)-f(a)-(b-a)\left[f'(a)+k\frac{b-a}{2}\right]\right\}}{f'(b)-f'(a)-k(b-a)}.
\end{aligned}$$

That is,

$$I \le \frac{k(b-a)^3}{12} + A,$$

and the right inequality in (2.59) is proved.
Similarly, we have that:

$$\begin{aligned}
& \int_a^b (x-a)\left(K-f''(x)\right)dx \\
= \;& \left(Kb-f'(b)\right)(b-a) - K\cdot\frac{\left(b^2-a^2\right)}{2} + f(b) - f(a) \\
= \;& (b-a)\left[Kb-f'(b)+K\cdot\frac{a+b}{2}\right] + f(b) - f(a) \\
= \;& (b-a)\left[K\cdot\frac{b-a}{2}-f'(b)\right] + f(b) - f(a)
\end{aligned}$$

and

$$\begin{aligned}
& \int_a^b (b-x)\left(K-f''(x)\right)dx \\
= \;& -(b-a)\left(K\cdot a-f'(a)\right) + K\cdot\frac{\left(b^2-a^2\right)}{2} - f(b) + f(a) \\
= \;& (b-a)\left[K\cdot\left(\frac{b-a}{2}\right)+f'(a)\right] - f(b) + f(a).
\end{aligned}$$

Using the inequality (2.61), we get that

$$\begin{aligned} & K\cdot\frac{(b-a)^3}{12}-I \\ \leq\; & \frac{1}{2}\left\{(b-a)\left[K\cdot\left(\frac{b-a}{2}\right)-f'(b)\right]+f(b)-f(a)\right\} \\ & \times\frac{\left\{(b-a)\left[K\cdot\left(\frac{b-a}{2}\right)+f'(a)\right]-f(b)+f(a)\right\}}{K\cdot(b-a)-f'(b)+f'(a)}, \end{aligned}$$

from where we get

$$I\geq K\cdot\frac{(b-a)^3}{12}+B,$$

and the theorem is proved. ∎

The following corollary for convex mappings holds [**40**].

COROLLARY 15. *If f is a twice differentiable convex mapping on $\mathring{I}$, and $[a,b]\subset\mathring{I}$, then we have the inequality*

$$\begin{aligned} 0 \;\leq\; & \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx \\ \leq\; & \frac{1}{2}\frac{[(b-a)f'(b)-(f(b)-f(a))][f(b)-f(a)-(b-a)f'(a)]}{(b-a)(f'(b)-f'(a))}, \end{aligned}$$

provided that $f'(b)\neq f'(a)$.

5.2. Applications for Special Means. We shall start with the following proposition:

PROPOSITION 18. *Let $p>1$ and $0\leq a<b$. Then we have the inequality:*

$$\begin{aligned} 0 \;\leq\; & A(a^p,b^p)-L_p^p(a,b) \\ \leq\; & \frac{1}{2}(b-a)^2 p(p-1)[B(p+1,p+1)]^{\frac{1}{p}} L_{\frac{p(p-2)}{(p-1)}}^{p-2}(a,b), \end{aligned} \tag{2.62}$$

where B is Euler's Beta function.

PROOF. If we apply Theorem 32 for $f(x):=x^p$ on $[a,b]$ we get that:

$$\begin{aligned} 0 \;\leq\; & A(a^p,b^p)-L_p^p(a,b) \\ \leq\; & \frac{1}{2}(b-a)^{\frac{p+1}{p}}[B(p+1,p+1)]^{\frac{1}{p}}\left(\int_a^b\left[p(p-1)x^{p-2}\right]^q dx\right)^{\frac{1}{q}}, \end{aligned} \tag{2.63}$$

where B is Euler's Beta function.
As

$$\int_a^b x^{pq-2q}dx=\frac{b^{pq-2q+1}-a^{pq-2q+1}}{pq-2q+1}=\frac{b^{p-q+1}-a^{p-q+1}}{p-q+1},$$

and

$$\frac{b^{p-q+1}-a^{p-q+1}}{p-q+1}=L_{p-q}^{p-q}(a,b)(b-a),$$

we deduce, by (2.63), that

$$\begin{aligned} 0 &\leq A\left(a^{p}, b^{p}\right)-L_{p}^{p}(a, b) \\ &\leq \frac{1}{2}(b-a)^{\frac{p+1}{p}}[B(p+1, p+1)]^{\frac{1}{p}} p(p-1) L_{p-q}^{\frac{p-q}{q}}(a, b)(b-a)^{\frac{1}{q}} \\ &= \frac{1}{2}(b-a)^{2} p(p-1)[B(p+1, p+1)]^{\frac{1}{p}} L_{\frac{p(p-2)}{(p-1)}}^{p-2}(a, b), \end{aligned}$$

as

$$\frac{p-q}{q}=p-2$$

and

$$p-q=\frac{p(p-2)}{p-1};$$

and the proposition is proved. ∎

The following proposition also holds.

PROPOSITION 19. *Let $p>1$ and $0<a<b$. Then one has the inequality:*

$$0 \leq L(a, b)-H(a, b) \leq(b-a)^{2} \frac{L(a, b) H(a, b)[B(p+1, p+1)]^{\frac{1}{p}}}{L_{-\frac{3 p}{p-1}}^{3}(a, b)}. \tag{2.64}$$

PROOF. If we apply Theorem 32 for the function $f(x)=\frac{1}{x}$ on $[a, b]$, we obtain:

$$\begin{aligned} 0 &\leq H^{-1}(a, b)-L^{-1}(a, b) \\ &\leq \frac{1}{2}(b-a)^{\frac{p+1}{p}}[B(p+1, p+1)]^{\frac{1}{p}}\left(\int_{a}^{b} \frac{2^{q}}{x^{3 q}} d x\right)^{\frac{1}{q}}. \end{aligned} \tag{2.65}$$

However,

$$\int_{a}^{b} \frac{d x}{x^{3 q}}=\left.\frac{x^{-3 q+1}}{-3 q+1}\right|_{a} ^{b}=\frac{b^{-3 q+1}-a^{-3 q+1}}{-3 q+1}=L_{-3 q}^{-3 q}(a, b)(b-a).$$

Then, by (2.65) we get that

$$\begin{aligned} 0 &\leq H^{-1}(a, b)-L^{-1}(a, b) \\ &\leq \frac{1}{2}(b-a)^{\frac{p+1}{p}}(b-a)^{\frac{1}{q}}[B(p+1, p+1)]^{\frac{1}{p}} L_{-3 q}^{-3}(a, b) \\ &= (b-a)^{2} \frac{[B(p+1, p+1)]^{\frac{1}{p}}}{L_{-\frac{3 p}{p-1}}^{3}(a, b)}, \end{aligned}$$

and the inequality (2.64) is obtained. ∎

The following inequality for the geometric and identric mean also holds:

PROPOSITION 20. *Let $p>1$ and $0<a<b$. Then we have the inequality:*

$$1 \leq \frac{I(a, b)}{G(a, b)} \leq \exp \left[\frac{1}{2}(b-a)^{2} \frac{[B(p+1, p+1)]^{\frac{1}{p}}}{L_{-\frac{2 p}{p+1}}^{2}(a, b)}\right]. \tag{2.66}$$

Proof. If we apply Theorem 32 for the convex mapping $f(x) = -\ln x$ on $[a,b]$, we obtain:

$$\begin{aligned} 0 &\leq \ln I(a,b) - \ln G(a,b) \\ &\leq \frac{1}{2}(b-a)^{\frac{p+1}{p}}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\left(\int_a^b \frac{dx}{x^{2q}}\right)^{\frac{1}{q}} \\ &= \frac{1}{2}(b-a)^2 \frac{\left[B(p+1,p+1)\right]^{\frac{1}{p}}}{L^2_{-\frac{2p}{p+1}}(a,b)}, \end{aligned}$$

from where results the desired inequality (2.66). ∎

We shall now point out some applications of Theorem 33 for special means.

Proposition 21. *Let $p \geq 2$ and $0 \leq a < b$. Then we have the inequality:*

$$\left|A(a^p,b^p) - L_p^p(a,b) - \frac{p(p-1)}{12}(b-a)^2 L_{p-2}^{p-2}(a,b)\right| \leq \frac{p(p-1)(b-a)^3}{32} L_{p-3}^{p-3}(a,b). \tag{2.67}$$

Proof. If we choose in Theorem 33, $f(x) = x^p$, $p \geq 2$, then

$$k = p(p-1)a^{p-2} \leq f(x) \leq p(p-1)b^{p-2} = K, \ x \in [a,b].$$

Applying inequality (2.52) we get

$$\left|A(a^p,b^p) - L_p^p(a,b) - \frac{p(b-a)}{12}\left(b^{p-1} - a^{p-1}\right)\right| \leq \frac{p(p-1)}{32}(b-a)^2\left(b^{p-2} - a^{p-2}\right). \tag{2.68}$$

As

$$b^{p-1} - a^{p-1} = (p-1) L_{p-2}^{p-2}(a,b)(b-a)$$

and

$$b^{p-2} - a^{p-2} = (p-2) L_{p-3}^{p-3}(a,b)(b-a),$$

then, by (2.68) we can conclude that the desired inequality (2.67) holds true. ∎

The following proposition also holds.

Proposition 22. *Let $0 < a < b$. Then we have the inequality:*

$$\left|L(a,b) - H(a,b) - \frac{(b-a)^2}{6}\cdot\frac{A(a,b)L(a,b)H(a,b)}{G^4(a,b)}\right| \leq \frac{(b-a)^3}{16}\cdot\frac{L(a,b)H(a,b)\left[4A^2(a,b) - G^2(a,b)\right]}{G^6(a,b)}. \tag{2.69}$$

Proof. If we choose in Theorem 33, $f(x) = \frac{1}{x}$, then it is clear that $f''(x) \in \left[\frac{2}{b^3}, \frac{2}{a^3}\right]$. That is, we can choose in the inequality (2.52), $k = \frac{2}{b^3}$ and $K = \frac{2}{a^3}$, to obtain:

$$\left|L^{-1}(a,b) - H^{-1}(a,b) - \frac{b-a}{12}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\right| \leq \frac{(b-a)^3}{32}\left(\frac{2}{a^3} - \frac{2}{b^3}\right),$$

That is,

$$\left|L^{-1}(a,b)-H^{-1}(a,b)-\frac{(b-a)^2}{6}\cdot\frac{a+b}{2}\cdot\frac{1}{a^2b^2}\right| \tag{2.70}$$
$$\leq \frac{(b-a)^2}{16}\left(\frac{(b-a)\left(a^2+ab+b^2\right)}{a^3b^3}\right)$$
$$= \frac{(b-a)^3}{16}\frac{\left(a^2+ab+b^2\right)}{a^3b^3}.$$

However,

$$a^2+b^2+ab=(a+b)^2-ab=4A^2(a,b)-G^2(a,b),$$

and thus (2.70) becomes

$$\left|L^{-1}(a,b)-H^{-1}(a,b)-\frac{(b-a)^2}{6}\cdot\frac{A(a,b)}{G^4(a,b)}\right|$$
$$\leq \frac{(b-a)^3}{16}\frac{\left(4A^2(a,b)-G^2(a,b)\right)}{G^6(a,b)}.$$

which is equivalent with

$$\left|L(a,b)-H(a,b)-\frac{(b-a)^2}{6}\cdot\frac{A(a,b)L(a,b)H(a,b)}{G^4(a,b)}\right|$$
$$\leq \frac{(b-a)^3}{16}\cdot\frac{\left(4A^2(a,b)-G^2(a,b)\right)L(a,b)H(a,b)}{G^6(a,b)}.$$

∎

As a final application of Theorem 33 we have the following proposition:

PROPOSITION 23. *Let $0<a<b$. Then*

$$\left|\ln I(a,b)-\ln G(a,b)-\frac{(b-a)^2}{12G^2(a,b)}\right|\leq\frac{(b-a)^3}{12}A(a,b). \tag{2.71}$$

PROOF. If we choose in Theorem 33, $f(x)=-\ln x$, then $f''(x)=\frac{1}{x^2}\in\left[\frac{1}{b^2},\frac{1}{a^2}\right]$. That is, $k=\frac{1}{b^2}$, $K=\frac{1}{a^2}$, and by inequality (2.52) we obtain:

$$\left|\ln I(a,b)-\ln G(a,b)+\frac{b-a}{12}\left(\frac{1}{b}-\frac{1}{a}\right)\right|\leq\frac{(b-a)^2}{32}\left(\frac{1}{b^2}-\frac{1}{a^2}\right).$$

That is,

$$\left|\ln I(a,b)-\ln G(a,b)+\frac{(b-a)^2}{12ab}\right|\leq\frac{(b-a)^3}{16}\frac{b+a}{2},$$

which is equivalent with the desired inequality (2.71). ∎

Finally, we shall point out some natural applications of Theorem 34 for special means.

The following proposition holds:

PROPOSITION 24. *Let $p \geq 2$ and $a, b \in [0, \infty)$ with $a < b$. Then we have the inequality:*

$$\begin{aligned} \frac{p(p-1)a^{p-2}(b-a)^2}{48} &\leq \frac{1}{2}\left[A^p(a,b) + A(a^p, b^p)\right] - L_p^p(a,b) \\ &\leq \frac{p(p-1)b^{p-2}(b-a)^2}{48}. \end{aligned}$$

The argument follows by Theorem 34 applied for the mapping $f(x) = x^p$, $p \geq 2$ on $[a,b]$.

Another inequality for the harmonic and logarithmic means is embodied in the following proposition.

PROPOSITION 25. *If $0 < a < b$, then:*

$$\frac{(b-a)^2}{24b^2} \leq \frac{1}{2}\left[A^{-1}(a,b) + H^{-1}(a,b)\right] - L^{-1}(a,b) \leq \frac{(b-a)^2}{24a^2}.$$

The proof is obvious by Theorem 34 applied for the mapping $f(x) := \frac{1}{x}$, $x \in [a,b]$.

Finally, we have:

PROPOSITION 26. *If $0 < a < b$, then:*

$$\exp\left[\frac{(b-a)^2}{48b^2}\right] \leq \frac{I(a,b)}{G(A(a,b), G(a,b))} \leq \exp\left[\frac{(b-a)^2}{48a^2}\right].$$

The proof goes by Theorem 34 applied for the mapping $f(x) = -\ln x$, we shall omit the details.

REMARK 24. *Similar results can be obtained from Theorem 35, but we omit the details.*

6. A Best Possible $H. - H.$ Inequality in Fink's Sense

6.1. Introduction. The $H. - H.$ inequality is

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{2.72}$$

which holds for f convex. This inequality is a special case of a result of Fejer [**72**]

$$f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt \leq \int_a^b f(t)\,p(t)\,dt \leq \frac{f(a)+f(b)}{2}\int_a^b p(t)\,dt, \tag{2.73}$$

which holds when f is convex and p is a nonnegative function whose graph is symmetric with respect to the center $\frac{a+b}{2}$. As Fink noted in [**73**], one wonders what the symmetry has to do with this result and if such an inequality holds for other functions. In particular, one would like to have a result which cannot be generalised by being a 'best possible inequality', see [**76**], [**78**] and [**73**]. Here it would mean being able to prove the two statements.

(A) The inequality (2.73) holds for all functions $p \in M$ if and only if f is convex; and
(B) The inequality (2.73) holds for all convex f if and only if $p \in M$.

The problem is to find the correct class of functions or measures M. It turns out that the class M will not be a subset of the positive measures.

To answer these questions, we will consider the following (see [**73**]).

6.2. The Lower Bound. For convenience, we will take the interval to be $[-1,1]$ and concentrate on the left hand inequality of (2.73) first. To see that symmetry is not essential in Fejer's result, we first see how one might establish a result in this direction. To do this, we will replace $p(x)\,dx$ by a nonnegative regular Borel measure μ with the requirement that $\int_{-1}^{1} d\mu(x) > 0$.

Since f is convex, its graph lies above its tangent lines. Let y be an arbitrary number in $[-1,1]$ and write down this condition

$$f(x) \geq f(y) + f'(y)(x-y). \tag{2.74}$$

Let the moments of the measure be defined by $P_k = \int_{-1}^{1} x^k d\mu(x)$. If we integrate the inequality (2.74) we arrive at

$$\int_{-1}^{1} f(x)\,d\mu(x) \geq f(y)\,P_0 + f'(y)(P_1 - yP_0). \tag{2.75}$$

This inequality holds for any y in $[-1,1]$ so we may choose any y we please. But we get the best one by maximizing the right hand side of this inequality. Assuming that f has two derivatives, one gets the derivative of this quantity to be $f''(y)(P_1 - yP_0)$. Since $P_1 - P_0 \leq 0$ and $P_1 + P_0 > 0$, the maximum is at $y_0 = \frac{P_1}{P_0}$. So we arrive at

$$\int_{-1}^{1} f(x)\,d\mu(x) \geq P_0 f\left(\frac{P_1}{P_0}\right). \tag{2.76}$$

Of course, if μ is an even measure we have $P_1 = 0$ and Fejer's result. At this stage we are able to prove statement (A) (with (2.73) replaced by (2.76)) if we take M to be the nonnegative regular Borel measures. For the sufficiency is the above argument and the necessity is obtained by taking the measure

$$d\mu = \alpha\delta_x + (1-\alpha)\,\delta_y \tag{2.77}$$

for δ_z the unit mass at z and $0 \leq \alpha \leq 1$. Then (2.76) becomes the convexity of f. Of course the sufficiency in the statement (B) also is obtained from the above argument. It is the necessity that fails. That is, we cannot prove that if (2.76) holds for all convex f, then the measure must be nonnegative. This turns out to be false. If we allow μ to be a signed measure, the above proof fails since we may not integrate an inequality. However, here is what we can prove. Let (EP for end positive)

$$\int_{-1}^{t} (t-x)\,d\mu(x) \geq 0 \quad \text{and} \quad \int_{t}^{1} (x-t)\,d\mu(x) \geq 0 \quad \text{for} \quad t \in [-1,1]. \tag{EP}$$

This condition will be revisited in the section on upper bounds (see [**73**]).

THEOREM 36. *(*[**73**]*) Let f be continuous on $[-1,1]$ and μ a regular Borel measure such that $\mu[-1,1] > 0$. Then*

i) *the inequality (2.76) holds for all measures μ satisfying (EP) if and only if F is convex; and*
ii) *the inequality (2.76) holds for all convex f if and only if μ satisfies (EP).*

Equality holds in (2.76) for linear f.

PROOF. We first argue the sufficiency. For f convex, we can write

$$f(x) = a + b(x-y) + \int_{y}^{x} (x-t)\,d\sigma(t). \tag{2.78}$$

The nonnegative measure σ, a and b depend on the choice of y. The reader may take $d\sigma$ to be $f''(t)\,dt$ for first understanding. For the general case, a bounded convex function f has a derivative f' a.e. and f' is an increasing function. So f' can be written as $b+\int_y^x d\sigma$ where $d\sigma$ may contain point masses and b is the slope of some supporting line at y. Then

$$\int_{-1}^{1} f(x)\,d\mu(x) = (a-yb)\,P_0 + bP_1 + R \tag{2.79}$$

where $R=\int_{-1}^{1}\int_y^x (x-t)\,d\sigma(t)\,d\mu(x)$ which can be written as

$$\int_{-1}^{y}\left(\int_{-1}^{t}(t-x)\,d\mu(x)\right)d\sigma(t)+\int_{y}^{1}\left(\int_{t}^{1}(x-t)\,d\mu(x)\right)d\sigma(t).$$

It is now obvious that $\sigma\geq 0$ and $\mu\in$(EP) make $R\geq 0$, and again we may choose $y=y_0=\frac{P_1}{P_0}$ to get (2.76) (since $a=f(y_0)$). To prove the converse in i) we observe that the measure defined in (2.77) is in (EP). To prove that if (2.76) holds for all convex f then $\mu\in$(EP) we take for $f(x)$ the function $f(x)=(x-t)_+$ for $t\in[-1,1]$. Then

$$\int_{t}^{1}(x-t)\,d\mu(x)\geq\int_{-1}^{1}d\mu\left(\frac{P_1}{P_0}-t\right)_+. \tag{2.80}$$

Since the right hand side is nonnegative, we get the second condition in (EP). Note that for $t=-1$, (2.80) reads $P_1+P_0\geq 0$ so that $y_0=\frac{P_1}{P_0}\geq -1$. If $t\leq y_0$ then (2.80) becomes

$$\int_{t}^{1}(x-t)\,d\mu(x)\geq\int_{-1}^{1}x\,d\mu(x)-t\int_{-1}^{1}d\mu(x),$$

which becomes $\int_{-1}^{t}(t-x)\,d\mu(x)\geq 0$, the first of (EP) for $t\leq y_0$. If $t>y_0$ we have $tP_0>P_1$. The identity

$$\int_{-1}^{t}(t-x)\,d\mu(x)=(tP_0-P_1)+\int_{t}^{1}(x-t)\,d\mu(x)$$

gives the first term in (EP) as the sum of two positive terms. Note that at $t=1$ this condition gives $P_0-P_1\geq 0$ or $y_0\leq 1$. ∎

EXAMPLE 1. *([**73**])Let $d\mu_a(x)=\left(x^2-a\right)dx$, $0<a<\frac{1}{3}$. Then*

$$\int_{-1}^{t}(t-x)\,d\mu(x)=\int_{t}^{1}(x-t)\,d\mu(x)=\frac{1}{12}\left(t^2-1\right)^2\quad\text{and}\quad P_0=P_1=0.$$

Thus, for $0<a<\frac{1}{3}$ $d\mu$ is not a nonnegative measure and

$$\int_{-1}^{1}\left(x^2-a\right)f(x)\,dx\geq 2\left(\frac{1}{3}-a\right)f(0) \tag{2.81}$$

for any convex f. For $a=\frac{1}{3}$ we have (2.82).

$$\int_{-1}^{1}x^2 f(x)\,dx\geq\frac{1}{3}\int_{-1}^{1}f(x)\,dx \tag{2.82}$$

for any convex f.

REMARK 25. *If f is concave, all of the above inequalities are reversed.*

6.3. The n^{th} Order Case. One can obtain inequalities with $f'' \geq 0$ replaced by $f^{(n+1)} \geq 0$. For then, we look at the simple case when $\mu \geq 0$.

$$f(x) \geq \sum_{k=0}^{n} \frac{f^{(k)}(y)}{k!}(x-y)^k. \tag{2.83}$$

If $\mu \geq 0$ we have

$$\int_{-1}^{1} f(x)\, d\mu(x) \geq \sum_{k=0}^{n} \frac{f^{(k)}(y)}{k!} \int_{-1}^{1} (x-y)^k\, d\mu(x) \equiv g(y). \tag{2.84}$$

If follows that

$$g'(y) = \frac{f^{(n+1)}(y)}{n!} \int_{-1}^{1} (x-y)^n\, d\mu(x).$$

THEOREM 37. *([**73**]) If n is even, $f^{(n+1)} \geq 0$ on $[-1,1]$ and $\mu \geq 0$, then*

$$\int_{-1}^{1} f(x)\, d\mu(x) \geq \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!} \int_{-1}^{1} (x-1)^k\, d\mu(x) \tag{2.85}$$

with equality if f is a polynomial of degree n.

PROOF. If n is even, then $g' \geq 0$ and thus we get the best inequality by taking $g(1)$. ∎

THEOREM 38. *([**73**]) If n is odd, $f^{(n+1)} \geq 0$ on $[-1,1]$ and $\mu \geq 0$ then*

$$\int_{-1}^{1} f(x)\, d\mu(x) \geq \sum_{k=0}^{n-1} \frac{f^{(k)}(y_0)}{k!} \int_{-1}^{1} (x-y_0)^k\, d\mu(x), \tag{2.86}$$

where $\int_{-1}^{1} (x-y_0)^n\, d\mu(x) = 0$.

PROOF. If n is odd then g' has a factor $\int_{-1}^{1} (x-y)^n\, d\mu(x)$ which has opposite signs at ± 1 and this factor has a derivative which is less than zero, so it has a unique zero. This zero maximizes $g(y)$. ∎

This result is reversed if $f^{(n+1)} \leq 0$. Moreover, if μ is even, then $\int_{-1}^{1} x^n d\mu(x) = 0$ if n is odd, so that $\mu_0 = 0$.

REMARK 26. *([**73**]) If $n = 1$ then (2.86) becomes (2.76) since $y_0 = \frac{P_1}{P_0}$ in this case.*

EXAMPLE 2. *([**73**]) If n is odd, μ an even measure and $f^{(n+1)} \geq 0$ then*

$$\int_{-1}^{1} f(x)\, d\mu(x) \geq \sum_{k=0}^{\frac{n-1}{2}} \frac{f^{(2k)}(0)}{(2k)!} \int_{-1}^{1} x^{2k} d\mu(x).$$

Having disposed of the easy case we look at the replacement of nonnegative measures by signed measures. Here (2.83) is replaced by

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(y)}{k!}(x-y)^k + R^1, \tag{2.87}$$

where

$$R^1 = \int_{y}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt.$$

THEOREM 39. *([**73**]) Let $f \in C^{n+1}[-1,1]$ and μ a regular Borel measure. Then*

(i) *The inequality (2.84) holds for all* $y \in [-1,1]$ *and all* f *with* $f^{(n+1)} \geq 0$ *if and only if* μ *satisfies* $(EP)_n$.

(ii) *The inequality (2.84) holds for all* $y \in [-1,1]$ *and measures* μ *satisfying* $(EP)_n$ *if and only if* $f^{(n+1)} \geq 0$.

PROOF. Proceeding as in the proof of Theorem 36, we arrive at

$$R = \int_{-1}^{y} \frac{f^{(n+1)}(t)}{n!} \int_{-1}^{t} (-1)(x-t)^n \, d\mu(x)\, dt + \int_{y}^{1} \frac{f^{(n+1)}(t)}{n!} \int_{t}^{1} (x-t)^n \, d\mu(x)\, dt. \tag{2.88}$$

So the sufficiency of (2.84) is that

$$(EP_n) \qquad \int_{t}^{1} (x-t)^n \, d\mu(x) \geq 0 \quad \text{and} \quad \int_{-1}^{t} (x-t)^n \, d\mu(x) \geq 0; \quad t \in [-1,1].$$

The necessity is achieved by taking $f^{(n+1)}(t) = \delta t_0$, i.e., $f(t) = \frac{(t-t_0)_+^n}{n!}$. For then $R \geq 0$ gives $(\mathrm{EP})_n$.

To show that (2.84) holding implies that $f^{(n+1)} \geq 0$, one needs to assume the existence of $f^{(n+1)}$. The $(n+1)^{st}$ divided difference $g[x_1,\ldots,x_{n+2}]$ at distinct points of a function g is a linear combination of the values of g at these points. That is,

$$g[x_1,\ldots,x_{n+2}] = \sum_{1}^{n+2} \alpha_i x_i$$

the coefficients being determined by the $(n+2)$ points $x_1,\ldots,x_{n+2}$. Consequently, the measure $d\mu = \sum_1^{n+2} \alpha_i \delta x_i$ has the property that

$$g[x_1,\ldots,x_{n+2}] = \int_{-1}^{1} g(x)\, d\mu(x).$$

We take this measure in (2.84). Now

$$g[x_1,\ldots,x_{n+2}] = \frac{g^{(n+1)}(s)}{(n+1)!}$$

by a generalised mean value theorem. Consequently $\int_{-1}^{1} (x-y)^k \, d\mu(x) = 0$ for $k = 0,\ldots,n$ and (2.84) becomes

$$f[x_1,\ldots,x_{n+2}] = \int_{-1}^{1} f(x)\, d\mu(x) \geq 0.$$

Now, it is known that

$$\lim f[x_1,\ldots,x_{n+2}] = \frac{f^{(n+1)}(x)}{(n+1)!}$$

where the limit has all the $x_i \to x$. To complete the proof we must argue that this measure μ satisfies $(\mathrm{EP})_n$. Since we are assuming (2.84) for all $y \in [-1,1]$, we may take $y = 1$ so that $(\mathrm{EP})_n$ reduces to the simple condition $\int_{-1}^{1} (x-t)_t^n \, d\mu(x) \geq 0$, see (2.88).

Now $\int_{-1}^{1} (x-t)_t^n \, d\mu(x)$ is the $(n+1)^{st}$ divided difference of the function $(x-t)_t^n$ as a function of x. This is the classical $B-$spline $M(t, x_0, \ldots, x_{n+2})$ which is known to be nonnegative. See [**169**, page 2]. This completes the proof. ∎

6.4. Upper Bounds. One could begin a study of the upper bound by using (2.78) to compute

$$f(x)-f(1)\frac{1+x}{2}-f(-1)\frac{1-x}{2}=h(x)$$

and then $\int_{-1}^{1} h d\mu(x)$ as a linear combination of $f(y)$, $f'(y)$ as in formula (2.79) and an integral which is generally like R in (2.79). When one does this, the integral terms turns out to be independent of y, and the coefficients of $f(y)$ and $f'(y)$ are zero. If $R\leq 0$, one gets

$$\int_{-1}^{1} f(x)\,d\mu(x)-\frac{f(1)}{2}(P_0+P_1)-f(-1)\left(\frac{P_0-P_1}{2}\right)\leq 0$$

which replicates Fejer's upper bound if μ is nonnegative and even. The general condition on μ obtained in this way suggests a much easier proof and statement of the theorem.

Theorem 40. ([**73**]) *Let f be a twice differentiable convex function and let μ be a measure such that the solution to the boundary value problem $y''=d\mu$; $y(-1)=y(1)=0$, is less than or equal to zero on $[-1,1]$, then*

$$\int_{-1}^{1} f d\mu\leq P_0\frac{f(-1)+f(1)}{2}+P_1\frac{f(1)-f(-1)}{2}.$$

Remark 27. ([**73**]) *The meaning of the boundary value problem is this. Let $G(x,t)$ be the Green's function for the problem $Ly=y''$, $y(1)=y(-1)=0$ (note the change in sign in L) then $y(x)=\int_{-1}^{1}G(x,t)\,d\mu(t)$ is a C' function satisfying the boundary conditions and if $d\mu(t)=p(t)\,dt$, $y''=p$ a.e.. The boundary value problem is self adjoint so $G(x,y)=G(y,x)$.*

Proof of the theorem. Let $y(x)=\int_{-1}^{1}G(x,t)\,d\mu(t)$, then

$$\begin{aligned}\int_{-1}^{1} f''(x)\,y(x)\,dx &= \int_{-1}^{1} f''(x)\int_{-1}^{1}G(x,t)\,d\mu(t)\,dx\\ &=\int_{-1}^{1}\int_{-1}^{1}G(x,t)\,f''(x)\,dx d\mu(t)=\int_{-1}^{1}\left(\int_{-1}^{1}G(t,x)\,f''(x)\,dx\right)d\mu(t).\end{aligned}$$

Now, $\int_{-1}^{1}G(t,x)\,f''(x)\,dx$ is a function whose second derivative is f'' and whose values at ± 1 are zero since G is the Green's function. This function is

$$f(x)-f(-1)\frac{1-x}{2}-f(1)\frac{1+x}{2},$$

a.e.. Thus we have

$$\int_{-1}^{1} f(x)\,d\mu(x)-\frac{f(-1)}{2}(P_0-P_1)-\frac{f(1)}{2}(P_0+P_1)=\int_{-1}^{1} f''(x)\,y(x)\,dx.$$

Now $f''\geq 0$ and $y\leq 0$ by hypothesis. This completes the proof. ∎

Note that it can be verified that $G\leq 0$ so that if $\mu\geq 0$, then $y\leq 0$.

At this point it is instructive to look at the condition (EP). The functions in (EP) $\int_{-1}^{t}(t-x)\,d\mu(x)$ and $\int_{t}^{1}(x-t)\,d\mu(x)$ are solutions of initial value problems (respectively)

$$y''=d\mu;\ y(-1)=y'(-1)=0;\ y(1)=y'(1)=0.$$

For $(\mathrm{EP})_n$ the initial value problem is

$$y^{(n+1)} = \frac{(-1)^{n+1}}{n!} d\mu$$

and $y^{(k)}(-1) = 0$; $k = 0, \ldots, n$; $y^{(k)}(1) = 0$, $k = 0, \ldots, n$ respectively.

EXAMPLE 3. *([**73**]) Let $p(x) = x^2 - \frac{1}{6}$. Then y in Theorem 40 is*

$$y(x) = \frac{x^2}{12}\left(x^2 - 1\right) \leq 0.$$

Moreover, $P_0 = \frac{1}{3}$ and $P_1 = 0$ so we get

$$\int_{-1}^{1} f(x)\left(x^2 - \frac{1}{6}\right) dx \leq \frac{f(1) + f(-1)}{6}$$

for f convex. For a non-symmetric example, let $p(x) = x^2 - x$ so that $P_0 = -P_1 = \frac{2}{3}$. Then

$$y(x) = \frac{1}{12}\left(x^2 - 1\right)(x-1)^2 \leq 0$$

and we have

$$\int_{-1}^{1} f(x)\left(x^2 - x\right) dx \leq \frac{2}{3} f(1) \quad \text{for } f \text{ convex.}$$

We cannot expect the result in Theorem 40 to be best possible. Convexity from an inequality which is an upper bound on f seems impossible.

7. Generalised Weighted Mean Values of Convex Functions

7.1. Introduction. Let $f(x)$ be a positive integrable function on the interval $[a, b]$, then the power mean of $f(x)$ is defined as follows [**83**]

$$M_\alpha(f) = \begin{cases} \left(\dfrac{\int_a^b f^\alpha(x)dx}{b-a}\right)^{\frac{1}{\alpha}}, & \alpha \neq 0, \\ \exp\left(\dfrac{\int_a^b \ln f(x)dx}{b-a}\right), & \alpha = 0. \end{cases} \tag{2.89}$$

The generalized logarithmic mean (or Stolarsky's mean) on the interval $[a, b]$ is defined for $x > 0$, $y > 0$ by

$$S_\alpha(x, y) = \begin{cases} \left(\dfrac{x^\alpha - y^\alpha}{\alpha(x-y)}\right)^{\frac{1}{(\alpha-1)}}, & \alpha \neq 0, 1 \quad x - y \neq 0; \\ \dfrac{y-x}{\ln y - \ln x}, & \alpha = 0, \quad x - y \neq 0; \\ \dfrac{1}{e}\left(\dfrac{x^x}{y^y}\right)^{\frac{1}{(x-y)}}, & \alpha = 1, \quad x - y \neq 0; \\ x, & x - y = 0. \end{cases} \tag{2.90}$$

In [**114**, p. 12] and [**190**], Zhen-Hang Yang has given the following generalizations of the Hermite-Hadamard's inequality (2.72):

If $f(x) > 0$ has derivative of second order and $f''(x) > 0$, for $\lambda > 1$, we have

(i) $f^\lambda\Big(\dfrac{a+b}{2}\Big) < \dfrac{1}{b-a}\displaystyle\int_a^b f^\lambda(x)dx < \dfrac{f^\lambda(a)+f^\lambda(b)}{2}$;

(ii) $f\Big(\dfrac{a+b}{2}\Big) < M_\lambda(f) < N_\lambda(f(a), f(b))$, where

$$N_\lambda(x,y) = \begin{matrix} \dfrac{x^\lambda + y^\lambda}{2}, & \lambda \neq 0, \\ \sqrt{xy}, & \lambda = 0; \end{matrix}$$

(iii) For all real number α, $M_\alpha(f) < S_{\alpha+1}\big(f(a), f(b)\big)$;

(iv) For $\alpha \geq 1$, $f\Big(\dfrac{a+b}{2}\Big) < M_\alpha(f) < S_{\alpha+1}\big(f(a), f(b)\big)$.

(v) If $f''(x) < 0$ for $x \in (a,b)$, the above inequalities are all reversed.

In [**173**], two-parameter mean is defined as

$$M_{p,q}(f) = \begin{matrix} \left(\dfrac{\int_a^b f^p(x)dx}{\int_a^b f^q(x)dx}\right)^{1/(p-q)}, & p \neq q, \\ \exp\left(\dfrac{\int_a^b f^p(x)\ln f(x)dx}{\int_a^b f^p(x)dx}\right), & p = q. \end{matrix} \tag{2.91}$$

When $q = 0$, $M_{p,0}(f) = M_p(f)$; when $f(x) = x$, the two-parameter mean is reduced to the extended mean values $E(r,s;x,y)$ for positive x and y:

$$E(r,s;x,y) = \left[\frac{r}{s}\cdot\frac{y^s - x^s}{y^r - x^r}\right]^{1/(s-r)}, \qquad rs(r-s)(x-y) \neq 0; \tag{2.92}$$

$$E(r,0;x,y) = \left[\frac{1}{r}\cdot\frac{y^r - x^r}{\ln y - \ln x}\right]^{1/r}, \qquad r(x-y) \neq 0; \tag{2.93}$$

$$E(r,r;x,y) = e^{-1/r}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r - y^r)}, \qquad r(x-y) \neq 0; \tag{2.94}$$

$$E(0,0;x,y) = \sqrt{xy}, \qquad x \neq y;$$

$$E(r,s;x,x) = x, \qquad x = y.$$

In 1997, Ming-Bao Sun [**173**] generalized Hermite-Hadamard's inequality (2.72) and the results derived by Yang in [**114, 190**] to obtain that, if the positive function $f(x)$ has derivative of second order and $f''(x) > 0$, then, for all real numbers p and q,

$$M_{p,q}(f) < E\big(p+1, q+1; f(a), f(b)\big). \tag{2.95}$$

If $f''(x) < 0$, then inequality (2.95) is reversed.

Recently Feng Qi introduced in [**152, 153**] the generalized weighted mean values $M_{p,f}(r,s;x,y)$ of a positive function f defined on the interval between x and

y with two parameters $r, s \in \mathbb{R}$ and nonnegative weight $p \not\equiv 0$ by

$$
\begin{aligned}
&M_{p,f}(r,s;x,y) = \left(\frac{\int_x^y p(u) f^s(u)du}{\int_x^y p(u) f^r(u)du}\right)^{1/(s-r)}, && (r-s)(x-y) \neq 0; \\
&M_{p,f}(r,r;x,y) = \exp\left(\frac{\int_x^y p(u) f^r(u) \ln f(u)du}{\int_x^y p(u) f^r(u)du}\right), && x-y \neq 0; \\
&M_{p,f}(r,s;x,x) = f(x), && x = y.
\end{aligned}
\tag{2.96}
$$

It is well-known that the concepts of means and their inequalities not only are basic and important concepts in mathematics (for example, some definitions of norms are often special means) and have explicit geometric meanings [**155**], but also have applications in electrostatics [**149**], heat conduction and chemistry [**175**]. Moreover, some applications to medicine are given in [**21**].

In this section, using the Chebychev integral inequality, suitable properties of double integral and the Cauchy's mean value theorem in integral form, the following result is obtained (see [**83**]):

THEOREM 41. *Suppose $f(x)$ is a positive differentiable function and $p(x) \not\equiv 0$ an integrable nonnegative weight on the interval $[a,b]$, if $f'(x)$ and $f'(x)/p(x)$ are both increasing or both decreasing and integrable, then for all real numbers r and s, we have*

$$
M_{p,f}(r,s;a,b) < E\big(r+1, s+1; f(a), f(b)\big); \tag{$*$}
$$

if one of the functions $f'(x)$ or $f'(x)/p(x)$ is nondecreasing and the other nonincreasing, then the inequality ($*$) reverses.

7.2. Main Results. In order to verify Theorem 41, the following lemmas are necessary [**83**].

LEMMA 6. *Let $G, H : [a,b] \to \mathbb{R}$ be integrable functions, both increasing or both decreasing. Furthermore, let $Q : [a,b] \to [0,+\infty)$ be an integrable function. Then*

$$
\int_a^b Q(u)du \int_a^b Q(u)G(u)H(u)du \geq \int_a^b Q(u)H(u)du \int_a^b Q(u)G(u)du \tag{2.97}
$$

with equality if and only if one of the functions G or H reduces to a constant.

If one of the functions of G or H is nonincreasing and the other nondecreasing, then the inequality (2.97) reverses.

Inequality (2.97) is called the Chebychev integral inequality [**95, 114**], [**83**].

LEMMA 7 ([**156**]). *Suppose that $f(t)$ and $g(t) \geq 0$ are integrable on $[a,b]$ and the ratio $f(t)/g(t)$ has finitely many removable discontinuity points. Then there exists at least one point $\theta \in (a,b)$ such that*

$$
\frac{\int_a^b f(t)dt}{\int_a^b g(t)dt} = \lim_{t \to \theta} \frac{f(t)}{g(t)}. \tag{2.98}
$$

We call Lemma 7 the revised Cauchy's mean value theorem in integral form.

PROOF. Since $f(t)/g(t)$ has finitely many removable discontinuity points, without loss of generality, suppose it is continuous on $[a,b]$. Furthermore, using $g(t) \geq 0$, from the mean value theorem for integrals in standard textbook of mathematical analysis or calculus, there exists at least one point $\theta \in (a,b)$ satisfying

$$\int_a^b f(t)dt = \int_a^b \Big(\frac{f(t)}{g(t)}\Big) g(t)dt = \frac{f(\theta)}{g(\theta)} \int_a^b g(t)dt. \tag{2.99}$$

Lemma 7 follows. ∎

Proof of Theorem 41. It is sufficient to prove Theorem 41 only for $s > r$ and for $f'(x)$ and $\frac{f'(x)}{p(x)}$ both being increasing. The remaining cases can be done similarly.

Case 1. *When $s > r$ and $f(a) \neq f(b)$, inequality (∗) is equivalent to*

$$\int_a^b p(x)f^s(x)dx \left|\int_a^b f^r(x)f'(x)dx\right| < \int_a^b p(x)f^r(x)dx \left|\int_a^b f^s(x)f'(x)dx\right|. \tag{2.100}$$

Take $G(x) = f^{s-r}(x), H(x) = f'(x)/p(x)$ (being increasing) and $Q(x) = p(x)f^r(x) \geq 0$ in inequality (2.97). If $f'(x) > 0$, then $f^{s-r}(x)$ is increasing, inequality (2.100) holds. If $f'(x) < 0$, then $f^{s-r}(x)$ decreases, inequality (2.100) is still valid.

If $f'(x)$ does not keep the same sign on (a,b), then there exists an unique point $\theta \in (a,b)$ such that $f'(x) > 0$ on (θ, b) and $f'(x) < 0$ on (a,θ). Further, if $f(a) < f(b)$, then there exists an unique point $\xi \in (\theta, b)$ such that $f(\xi) = f(a)$. Therefore, inequality (2.100) is also equivalent to

$$\int_a^b p(x)f^s(x)dx \int_\xi^b f^r(x)f'(x)dx < \int_a^b p(x)f^r(x)dx \int_\xi^b f^s(x)f'(x)dx. \tag{2.101}$$

Using inequality (2.97) again produces

$$\int_\xi^b p(x)f^s(x)dx \int_\xi^b f^r(x)f'(x)dx < \int_\xi^b p(x)f^r(x)dx \int_\xi^b f^s(x)f'(x)dx. \tag{2.102}$$

For $x \in (a,\xi), y \in (\xi, b)$, we have $f'(y) > 0, f(x) < f(a) = f(\xi) < f(y)$ and $f^{s-r}(x) < f^{s-r}(y)$, therefore, suitable properties of double integral leads to

$$\begin{aligned} &\int_a^\xi p(x)f^s(x)dx \int_\xi^b f^r(x)f'(x)dx - \int_a^\xi p(x)f^r(x)dx \int_\xi^b f^s(x)f'(x)dx \\ &= \iint\limits_{[a,\xi]\times[\xi,b]} p(x)f^r(x)f^r(y)f'(y)\big[f^{s-r}(x) - f^{s-r}(y)\big]dxdy < 0. \end{aligned} \tag{2.103}$$

From this, we conclude that inequality (2.101) is valid, namely, inequality (2.100) holds.

If $f'(x)$ does not keep the same sign on (a,b) and $f(b) < f(a)$, from the same arguments as the case of $f(b) > f(a)$, inequality ((2.100)) follows.

Case 2. *When $s > r$ and $f(a) = f(b)$, since $f'(x)$ increases, we have $f(x) < f(a) = f(b), x \in (a,b)$. From the definition of $E(r,s;x,y)$, inequality (∗) is equivalent to*

$$M_{p,f}(r,s;a,b) < f(a) = f(b), \tag{2.104}$$

that is

$$\frac{\int_a^b p(x)f^s(x)dx}{\int_a^b p(x)f^r(x)dx} < f^{s-r}(a) = f^{s-r}(b). \tag{2.105}$$

This follows from Lemma 7.

The proof of Theorem 41 is completed. ∎

7.3. Applications. It is well-known that mean S_0 is called the logarithmic mean denoted by L, and S_1 the identric mean or the exponential mean by I.

The logarithmic mean $L(x,y)$ can be generalized to the one-parameter means:

$$\begin{aligned} &J_p(x,y) = \frac{p(y^{p+1}-x^{p+1})}{(p+1)(y^p-x^p)}, \quad x \neq y, \quad p \neq 0,-1; \\ &J_0(x,y) = L(x,y), \quad J_{-1}(x,y) = \frac{G^2}{L}; \\ &J_p(x,x) = x. \end{aligned} \tag{2.106}$$

Here, $J_{1/2}(x,y) = h(x,y)$ is called the Heron's mean and $J_2(x,y) = c(x,y)$ the centroidal mean. Moreover, $J_{-2}(x,y) = H(x,y)$, $J_1(x,y) = A(x,y)$, $J_{-1/2}(x,y) = G(x,y)$.

The extended Heron's means $h_n(x,y)$ is defined by

$$h_n(x,y) = \frac{1}{n+1} \cdot \frac{x^{1+1/n} - y^{1+1/n}}{x^{1/n} - y^{1/n}}. \tag{2.107}$$

Let f and p be defined and integrable functions on the closed interval $[a,b]$. The weighted mean $M^{[r]}(f;p;x,y)$ of order r of the function f on $[a,b]$ with the weight p is defined [**111**, pp. 75–76] by

$$M^{[r]}(f;p;x,y) = \begin{cases} \left(\dfrac{\int_x^y p(t) f^r(t) dt}{\int_x^y p(t) dt} \right)^{1/r}, & r \neq 0; \\ \exp\left(\dfrac{\int_x^y p(t) \ln f(t) dt}{\int_x^y p(t) dt} \right), & r = 0. \end{cases} \tag{2.108}$$

It is clear that $M^{[r]}(f;p;x,y) = M_{p,f}(r,0;x,y)$, $E(r,s;x,y) = M_{1,x}(r-1,s-1;x,y)$, $E(r,r+1;x,y) = J_r(x,y)$. From these definitions of mean values and some relationships between them, we can easily get the following inequalities [**83**]:

Corollary 16. *Let $f(x)$ be a positive differentiable function and $p(x) \not\equiv 0$ an integrable nonnegative weight on the interval $[a,b]$. If $f'(x)$ and $f'(x)/p(x)$ are integrable and both increasing or both decreasing, then for all real numbers r, s, we* *have*

$$M^{[r]}(f;p;a,b) < S_{r+1}\big(f(a),f(b)\big), M_{p,f}(0,-1;a,b) < L\big(f(a),f(b)\big), \tag{2.109}$$

$$M_{p,f}(0,0;a,b) < I\big(f(a),f(b)\big), M_{p,f}(0,1;a,b) < A\big(f(a),f(b)\big), \tag{2.110}$$

$$M_{p,f}(-1,-1;a,b) < G\big(f(a),f(b)\big), M_{p,f}(-3,-2;a,b) < H\big(f(a),f(b)\big), \tag{2.111}$$

$$M_{p,f}\Big(\frac{1}{2},-\frac{1}{2};a,b\Big) < h\big(f(a),f(b)\big),\ M_{p,f}\Big(\frac{1}{n},\frac{1}{n}-1;a,b\Big) < h_n\big(f(a),f(b)\big), \tag{2.112}$$

$$M_{p,f}(2,1;a,b) < c\big(f(a),f(b)\big), M_{p,f}(-1,-2;a,b) < \frac{G^2\big(f(a),f(b)\big)}{L\big(f(a),f(b)\big)}, \tag{2.113}$$

$$M_{p,f}(r,r+1;a,b) < J_r\big(f(a),f(b)\big). \tag{2.114}$$

If one of the functions $f'(x)$ or $f'(x)/p(x)$ is nondecreasing and the other nonincreasing, then all of the inequalities from (2.109) to (2.114) reverse.

Remark 28. *([**83**]) If we take $p(x) \equiv 1$ and special values of r and s in Theorem 41 or Corollary 16, we can derive the Hermite-Hadamard's inequality (2.72) and all of the related inequalities in* [**114, 173, 190**], *and the like.*

Remark 29. *([**83**]) The mean $M_{p,f}(0,1;a,b)$ is called the weighted arithmetic mean, $M_{p,f}(-1,-1;a,b)$ the weighted geometric mean, $M_{p,f}(-3,-2;a,b)$ the weighted harmonic mean of the function $f(x)$ on the interval $[a,b]$ with weight $p(x)$, respectively. So, we can seemingly call $M^{[r]}(f;p;a,b)$, $M_{p,f}(0,-1;a,b)$, $M_{p,f}(0,0;a,b)$, $M_{p,f}(\frac{1}{n},\frac{1}{n}-1;a,b)$ and $M_{p,f}(1,2;a,b)$ the weighted Stolarsky's (or generalized logarithmic) mean, the weighted logarithmic mean, the weighted exponential mean, the weighted Heron's mean and the weighted centroidal mean of the function $f(x)$ on the interval $[a,b]$ with weight $p(x)$, respectively.*

8. Generalisations for $n-$Time Differentiable Functions

8.1. A Generalisation for Positive Functionals. Let $f:[a,b]\to\mathbb{R}$ be a (continuous) convex function and $L: C[a,b]\to\mathbb{R}$ a positive, linear functional on $C[a,b]$ – the space of all continuous functions defined on $[a,b]$. Let us denote by $e_k(x) = x^k$, $x\in[a,b]$, $k\in\mathbb{N}$.

Theorem 42. *([**166**]) If the above conditions are satisfied, with $L(e_0) = 1$, then:*

$$f(L(e_1)) \le L(f) \le L(e_1)\left[\frac{f(b)-f(a)}{b-a}\right] + \frac{bf(a)-af(b)}{b-a}. \tag{2.115}$$

Proof. Since f is convex, it is well known that

$$f(x) - f(y) \ge f'_+(y)(x-y),\quad x,y\in[a,b].$$

By setting $y = L(e_1)$ and applying the positive linear functional L we get $L(f) \ge f(L(e_1))\cdot L(e_0) + f'_+(L(e_1))\cdot(L(e_1)-L(e_1)) = f(L(e_1))$ by $L(e_0) = 1$. This gives the left side of (2.115), where clearly, from $a\le e_1(x)\le b$ we have $aL(e_0)\le L(e_1)\le bL(e_0)$, i.e., $L(e_1)\in[a,b]$.

For the right side of (2.115) let us consider the inequality

$$f(x) \leq (x-a)\frac{f(b)}{b-a} + (b-x)\frac{f(a)}{b-a},$$

which means intuitively that the graph of f on $[a,b]$ is below the line segment joining $(a, f(a))$ and $(b, f(b))$. From $e_1(x) = x$, $e_0(x) = 1$, $x \in [a,b]$ by application of L, after simple calculations we get the desired result. ∎

Remark 30. [**166**] *1) For $L(f) = \frac{1}{b-a}\int_a^b f(t)\,dt$ we have $L(e_0) = 1$ and L is positive linear functional. For this L, relation (2.115) exactly gives inequality (2.72).*

2) Let $w_i \geq 0$ $(i = 1, \ldots, n)$ with $\sum_{i=1}^n w_i = 1$, and let $a_i \in [a,b]$, $i = 1, \ldots, n$. Let us define $L(f) = \sum_{i=1}^n w_i f(a_i)$. Then clearly L is positive linear functional, so by (2.115) we get:

$$f\left(\sum_{i=1}^n w_i a_i\right) \leq \sum_{i=1}^n w_i f(a_i) \leq \left(\sum_{i=1}^n w_i a_i\right)\left[\frac{f(b)-f(a)}{b-a}\right] + \frac{bf(a)-af(b)}{b-a} \tag{2.116}$$

for a convex function $f : [a,b] \to \mathbb{R}$. The left side of this relation is the well known Jensen inequality for n numbers.

8.2. On an Inequality of Sándor and Alzer. In this subsection, following [**166**], we shall present a unified method to prove certain generalisations of (2.72) discovered by J. Sándor [**167**] and H. Alzer [**2**]. First we state two lemmas.

Lemma 8. *([**166**]) For $x \in [a,b]$ one has*

$$\frac{(b-a)^n}{2^{n-1}} \leq (x-a)^n + (b-x)^n \leq (b-a)^n, \quad n \geq 1. \tag{2.117}$$

Proof. We consider the functions $h : [a,b] \to \mathbb{R}$ defined by $h(x) = (x-a)^n + (b-x)^n$. Here $h(a) = h(b) = (b-a)^n$ and $h\left(\frac{a+b}{2}\right) = \frac{(b-a)^n}{2^{n-1}}$. Obviously, $h'(x) = n[2x-(a+b)]\cdot q(x)$, with

$$q(x) = (x-a)^{n-2} + (x-a)^{n-3}(b-x) + \cdots + (b-x)^{n-2} > 0,$$

so $h'(x) \leq 0$ for $x \leq \frac{(a+b)}{2}$; and $h'(x) \geq 0$ for $x \geq \frac{(a+b)}{2}$. We get $h(x) \leq h(a)$ for $x \in \left[a, \frac{a+b}{2}\right]$ and $h(x) \leq h(b)$ for $x \in \left[\frac{a+b}{2}, b\right]$. In all cases $h(x) \leq (b-a)^n$ and $h(x) \geq \frac{(b-a)^n}{2^{n-1}}$. ∎

Lemma 9. *([**166**]) For $f \in C^n[a,b]$ and $t \in [a,b]$ one has*

$$\begin{aligned} &(-1)^n \int_a^b f(x)\,dx \\ &= \sum_{i=1}^n \left[\frac{(t-a)^i - (t-b)^i}{i!}\right] f^{(i-1)}(t)\cdot(-1)^{n-i+1} \\ &\quad + \frac{1}{n!}\left[\int_a^t (x-a)^n f^{(n)}(x)\,dx + \int_t^b (x-b)^n f^{(n)}(x)\,dx\right]. \end{aligned} \tag{2.118}$$

PROOF. Applying the generalised partial integration formula (also called the "Green-Lagrange identity") we can write:

$$\begin{aligned}
&\int_a^t (x-a)^n f^{(n)}(x)\,dx \\
= {} & (t-a)^n f^{(n-1)}(t) - n(t-a)^{n-1} f^{(n-2)}(t) + n(n-1)\cdot(t-a)^{n-2} f^{(n-3)}(t) \\
& - \cdots + (-1)^k n(n-1)\cdots(n-k+1)(t-a)^{n-k}\cdot f^{(n-k-1)}(t) \\
& + \cdots + (-1)^{n-1} n!\,(t-a) f(t) + (-1)^n \int_a^t n! f(x)\,dx
\end{aligned}$$

and likewise

$$\begin{aligned}
&\int_t^b (x-b)^n f^{(n)}(x)\,dx \\
= {} & -(t-b)^n f^{(n-1)}(t) - n(t-b)^{n-1} f^{(n-2)}(t) - n(n-1)\cdot(t-b)^{n-2} f^{(n-3)}(t) \\
& + \cdots + (-1)^{k+1} n(n-1)\cdots(n-k+1)(t-b)^{n-k}\cdot f^{(n-k-1)}(t) \\
& + \cdots + (-1)^{n-1} n!\,(t-b) f(t) + (-1)^n \int_t^b n! f(x)\,dx.
\end{aligned}$$

Adding these two relations and dividing with $n!$ we obtain (2.118). ∎

We now prove the following.

THEOREM 43. *([**166**]). Let $f \in C^{2k}[a,b]$ with $f^{(2k)}(x) \ge 0$ for $x \in (a,b)$ ($k \ge 1$, positive integer) and let $t \in [a,b]$ be arbitrary. Then:*

$$\begin{aligned}
&\int_a^b f(x)\,dx \\
\ge {} & \sum_{i=1}^{2k} \left[\frac{(t-a)^i - (t-b)^i}{i!}\right](-1)^{i-1} f^{(i-1)}(t) \\
& + \frac{1}{(2k)!}\left\{\frac{(b-a)^{2k}}{2^{2k-1}}\cdot\left[f^{(2k-1)}(t) - f^{(2k-1)}(a)\right] + S_{k,a,b}(t)\right\}
\end{aligned} \tag{2.119}$$

and

$$\begin{aligned}
&\int_a^b f(x)\,dx \\
\le {} & \sum_{i=1}^{2k} \left[\frac{(t-a)^i - (t-b)^i}{i!}\right](-1)^{i-1} f^{(i-1)}(t) \\
& + \frac{1}{(2k)!}\left\{(b-a)^{2k}\cdot\left[f^{(2k-1)}(t) - f^{(2k-1)}(a)\right] + S_{k,a,b}(t)\right\},
\end{aligned} \tag{2.120}$$

where

$$S_{k,a,b}(t) = \int_a^b (b-x)^{2k} f^{(2k)}(x)\,dx - 2\int_a^b (b-x)^{2k} f^{(2k)}(x)\,dx.$$

If $f^{(2k)}(x) > 0$, the inequalities are strict.

PROOF. We apply Lemma 9 for $n=2k$ and first the right side of (2.117), then the left side of (2.117). Since $\int_t^b=\int_a^b-\int_t^a$, the theorem follows by simple computations. From the proofs of (2.117) and (2.118), we can see that for $f^{(2k)}(x)>0$, $x\in(a,b)$, the inequalities in (2.119) and (2.120) are strict. ∎

THEOREM 44. *([**166**]) Under the same conditions,*

$$\begin{aligned}
(2.121)\qquad & \sum_{j=0}^{k-1}\frac{(b-a)^{2j+1}}{2^{2j}(2j+1)!}f^{(2j)}\left(\frac{a+b}{2}\right)\\
\leq\ & \int_a^b f(x)\,dx\leq\sum_{j=0}^{k-1}\frac{(b-a)^{2j+1}}{2^{2j}(2j+1)!}f^{(2j)}\left(\frac{a+b}{2}\right)\\
& +\frac{1}{(2k)!2^{2k}}(b-a)^{2k}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right].
\end{aligned}$$

PROOF. Let us apply Lemma 9 with $t=\frac{a+b}{2}$. Since

$$\left(\frac{b-a}{2}\right)^i-\left(\frac{a-b}{2}\right)^i=2\left(\frac{b-a}{2}\right)^i$$

for i odd; and equals zero for i even; with the notation $i=2j+1$ we can easily find the left side of (2.121). In order to prove the right-hand side inequality, we can remark that $(x-a)^{2k}\leq\left(\frac{b-a}{2}\right)^{2k}$, if $x\in\left[a,\frac{a+b}{2}\right]$, and $(b-x)^{2k}\leq\left(\frac{b-a}{2}\right)^{2k}$, for $x\in\left[\frac{a+b}{2},b\right]$. So, in all cases the second term is less than

$$\frac{1}{(2k)!}\cdot\frac{(b-a)^{2k}}{2^{2k}}\int_a^b f^{(2k)}(x)\,dx=\frac{1}{(2k)!}\cdot\frac{(b-a)^{2k}}{2^{2k}}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right].$$

∎

REMARK 31. *The left hand side of Theorem 44 is due to J. Sándor* [**167**].

THEOREM 45. *([**166**]) With the same conditions,*

$$\begin{aligned}
(2.122)\qquad & \frac{1}{2}\sum_{i=1}^{2k}\frac{(b-a)^i}{i!}\left[f^{(i-1)}(a)+(-1)^{i-1}f^{(i-1)}(b)\right]\\
& +\frac{(b-a)^{2k}}{2^{2k-2}(2k)!}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right]\\
\leq\ & \int_a^b f(x)\,dx\\
\leq\ & \frac{1}{2}\sum_{i=1}^{2k}\frac{(b-a)^i}{i!}\left[f^{(i-1)}(a)+(-1)^{i-1}f^{(i-1)}(b)\right].
\end{aligned}$$

If $f^{(2k)}(x)>0$, the inequalities are strict.

PROOF. Setting $t=a$ and $t=b$ in (2.119), after addition we get the left side inequality. By doing the same thing with (2.120) we get the right hand side of (2.122). ∎

REMARK 32. *The right side of (2.122) is due to H. Alzer* [**2**].

8.3. Some Related Inequalities. Finally, we will proved two related results (see [**166**]).

THEOREM 46. *If* $f \in C^n[a,b]$*, then:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.123}$$
$$\leq \sum_{j=1}^{\left[\frac{n-1}{2}\right]}\frac{(b-a)^{2j+1}}{2^{2j}}\left|f^{(2j)}\left(\frac{a+b}{2}\right)\right|+\frac{1}{2^n\cdot n!}\int_a^b\left|f^{(n)}(x)\right|dx.$$

PROOF. We apply Lemma 9 with $t=\frac{a+b}{2}$. The modulus inequality for sums and integrals applies at once the result if we observe that $\left(\frac{b-a}{2}\right)^i-\left(\frac{a-b}{2}\right)^i=0$, for i even, and $\left(\frac{b-a}{2}\right)^i-\left(\frac{a-b}{2}\right)^i=2\left(\frac{b-a}{2}\right)^i$ otherwise. Note that $i=2j+1\leq n\Leftrightarrow j\leq\left[\frac{n-1}{2}\right]$, where $[x]$ denotes the integer part of x. ∎

REMARK 33. *For* $n=1$ *we obtain the inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{1}{2}\int_a^b\left|f'(x)\right|dx \tag{2.124}$$

for $f\in C^1[a,b]$*. This improves the relation*

$$\left|f\left(\frac{a+b}{2}\right)\right|\leq\frac{1}{b-a}\int_a^b f(x)\,dx+\frac{1}{2}\int_a^b\left|f'(x)\right|dx \tag{2.125}$$

which is known as the "Gallagher-Sobolev inequality" ([**118**]*).*

THEOREM 47. *(*[**166**]*) If* $f\in C^n[a,b]$ *and* $\left|f^{(n)}(t)\right|\leq M$ *for all* $t\in[a,b]$*, then*

$$\left|\int_a^b f(x)\,dx+\sum_{i=1}^n\left[\frac{(t-a)^i-(t-b)^i}{i!}\right](-1)^i f^{(i-1)}(t)\right|\leq\frac{M(b-a)^{n+1}}{n!}. \tag{2.126}$$

PROOF. The result follows by an application of Lemma 9 and the remark that

$$\begin{aligned}\int_a^t(x-a)^n\,dx+\int_t^b(b-x)^n\,dx &\leq \int_a^b\left[(x-a)^n+(b-x)^n\right]dx\\ &\leq (b-a)^{n+1}\end{aligned}$$

by Lemma 8. ∎

COROLLARY 17. *(*[**166**]*) Under the same conditions,*

$$2\int_a^b f(x)\,dx+\sum_{i=1}^n\frac{(b-a)^i}{i!}\left[(-1)^i f^{(i-1)}(b)-f^{(i-1)}(a)\right] \tag{2.127}$$
$$\leq\frac{2M(b-a)^{n+1}}{n!}.$$

PROOF. Using (2.126) for $t=a$ and $t=b$ respectively, from the modulus inequality we get relation (2.127). ∎

9. The Euler Formulae and Convex functions

9.1. Introduction. Recently, Dragomir and Agarwal [**47**] considered the trapezoid formula for numerical integration for functions such that $\left|f'\right|^q$ is a convex function for some ($q \geq 1$). Their approach was based on estimating the difference between the two sides of the right–hand inequality in the $H.-H.$ inequality (2.72). Improvements of their results were obtained in [**135**]. In particular, the following tool was established.

THEOREM 48. *Suppose* $f : \mathring{I} \subseteq \mathbb{R} \to \mathbb{R}$ *is differentiable on* $\mathring{I}$ *and that* $\left|f'\right|^q$ *is convex on* $[a,b]$ *for some* $q \geq 1$, *where* $a, b \in \mathring{I}$ $(a < b)$. *Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{1/q}. \tag{2.128}$$

Some generalizations to higher–order convexity and applications of these results are given in [**19**].

In this section we consider further related results. The most natural nexus for these developments would appear to be the well–known Euler formula

$$\begin{aligned} f(x) &= \frac{1}{b-a}\int_a^b f(t)dt + \sum_{k=1}^{n-1}\frac{(b-a)^{k-1}}{k!} \\ &\quad \times B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] \\ &\quad - \frac{(b-a)^{n-1}}{n!}\int_a^b f^{(n)}(t)\left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n^*\left(\frac{x-a}{b-a}\right)\right]dt \end{aligned} \tag{2.129}$$

(see [**94**, p. 17]), which holds for every $x \in [a,b]$ and every function $f : [a,b] \to \mathbb{R}$ with $n \geq 2$ continuous derivatives. Here $B_k(\cdot)$ $k \geq 0$ is the kth Bernoulli polynomial and $B_k = B_k(0) = B_k(1)$ $(k \geq 1)$ the kth Bernoulli number. We denote by $B_k^*(\cdot)$ $(k \geq 0)$ the function of period one with $B_k^*(x) = B_k(x)$ for $0 \leq x \leq 1$.

For $x = b$ and $n = 2r$, (2.129) becomes

$$\begin{aligned} f(b) = \frac{1}{b-a}\int_a^b f(t)dt + [f(b) - f(a)]\, B_1(1) \\ + \sum_{k=2}^{2r-1}\frac{(b-a)^{k-1}B_k}{k!}\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] \\ - \frac{(b-a)^{2r-1}}{(2r)!}\int_a^b f^{(2r)}(t)\left[B_{2r}^*\left(\frac{b-t}{b-a}\right) - B_{2r}\right]dt. \end{aligned}$$

Since $B_1(1) = \frac{1}{2}$ and $B_{2j+1} = 0$ for $j \geq 1$, this may be rearranged, after a change of variable to $s = \frac{(t-a)}{(b-a)}$ in the final term, as

$$\begin{aligned} \frac{f(a)+f(b)}{2} &= \frac{1}{b-a}\int_a^b f(t)dt + \sum_{j=1}^{r-1}\frac{(b-a)^{2j-1}B_{2j}}{(2j)!}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right] \\ &\quad - \frac{(b-a)^{2r}}{(2r)!}\int_0^1 f^{(2r)}(a+s(b-a))\left[B_{2r}(1-s) - B_{2r}\right]ds. \end{aligned}$$

Here as subsequently an empty sum (in this case for $r=1$) is interpreted as zero.

Further, $B_{2r}(1-s)=B_{2r}(s)$, so we may write this as the Euler trapezoidal formula

$$\begin{aligned}\int_a^b f(x)dx &= \frac{b-a}{2}[f(a)+f(b)]-\sum_{k=1}^{r-1}\frac{(b-a)^{2k}B_{2k}}{(2k)!}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right]\\ &\quad +(b-a)^{2r+1}\int_0^1 P_{2r}(s)\,f^{(2r)}(a+s(b-a))\,ds,\end{aligned}$$

where $P_k(s):=[B_k(s)-B_k]/k!$ $(k\geq 1)$ (see [**8**, p. 274]).

This has many applications and was the starting point of the analysis in [**19**], where it was used to prove some integral inequalities germane to numerical integration. Analysis based on the trapezoidal formula devolves eventually on finding a method for handling the uncompromising–looking final term.

A natural quantity in the analysis in [**19**] is

$$\begin{aligned}I_r &:= (-1)^r\left\{\int_a^b f(x)dx-\frac{b-a}{2}[f(a)+f(b)]\right.\\ &\quad\left.+\sum_{k=1}^{r-1}\frac{(b-a)^{2k}B_{2k}}{(2k)!}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right]\right\}.\end{aligned}$$

The results of [**19**] include in particular the following.

Theorem 49. *Suppose $f:[a,b]\to\mathbb{R}$ is $(2r+2)$–convex. Then*

$$\begin{aligned}&(b-a)^{2r+1}\frac{|B_{2r}|}{(2r)!}f^{(2r)}\left(\frac{a+b}{2}\right)\\ &\leq I_r\leq (b-a)^{2r+1}\frac{|B_{2r}|}{(2r)!}\frac{f^{(2r)}(b)+f^{(2r)}(a)}{2}.\end{aligned}\tag{2.130}$$

If f is $(2r+2)$–concave, the inequality is reversed.

Theorem 50. *Suppose $f:[a,b]\to\mathbb{R}$ is $(2r)$–times differentiable. If $\left|f^{(2r)}\right|^q$ is convex for some $q\geq 1$, then*

$$|I_r|\leq (b-a)^{2r+1}\frac{|B_{2r}|}{(2r)!}\left[\frac{\left|f^{(2r)}(a)\right|^q+\left|f^{(2r)}(b)\right|^q}{2}\right]^{1/q}.$$

If $\left|f^{(2r)}\right|$ is concave, then

$$|I_r|\leq (b-a)^{2r+1}\frac{|B_{2r}|}{(2r)!}\left|f^{(2r)}\left(\frac{a+b}{2}\right)\right|.$$

The displayed inequalities are manifestly higher–order cousins of (2.128)

In the next section we take a different path from (2.129), one leading to the Euler midpoint formula instead of the Euler trapezoidal formula. In place of the function P_{2r} of the trapezoidal formula, it turns out that we shall have recourse to

$$p_{2r}(t)=B_{2r}^*\left(\frac{a+b-2t}{2(b-a)}\right)-B_{2r}\left(\frac{1}{2}\right).$$

We note that this does not change sign on the interval $[a,b]$ and that it is symmetric about $t=\frac{(a+b)}{2}$. Further

$$(-1)^{r-1}p_{2r}(t)\geq 0\quad\text{for}\quad t\in[a,b].$$

In Subsection 9.3 we explore briefly a third path, one that is associated with the Euler–Simpson formula.

The reader will have noted an asymmetry between the conditions applying in the convex and concave cases of Theorem 50. The reason is that if $|f^{(2r)}|^q$ is concave for some $q \geq 1$, then $|f^{(2r)}|$ must also be concave (see [**19**]). The omission of the index q in the concave case thus allows a weaker assumption to be made. This motif occurs also in the present section.

9.2. The Euler Midpoint Formula. Put $x = \frac{(a+b)}{2}$ and $n = 2r$ in (2.129). Since $B_{2j+1}\left(\frac{1}{2}\right) = 0$ for $j \geq 0$, we obtain the Euler midpoint formula

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &= \frac{1}{b-a}\int_a^b f(t)dt + \sum_{k=1}^{r-1}\frac{(b-a)^{k-1}}{(2k)!} \\
&\quad \times B_{2k}\left(\frac{1}{2}\right)\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right] \\
&\quad - \frac{(b-a)^{2r-1}}{(2r)!}\int_a^b f^{(2r)}(t)p_{2r}(t)dt.
\end{aligned} \tag{2.131}
$$

Note that $B_{2k}\left(\frac{1}{2}\right) = -\left(1-2^{1-2k}\right)B_{2k}$.

For the sequel we shall utilise

$$
\begin{aligned}
I_r^*(a,b) &:= (-1)^{r-1}\left\{\int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right)\right. \\
&\quad \left. - \sum_{k=1}^{r-1}\frac{B_{2k}(b-a)^{2k}}{(2k)!}\left(-1+\frac{1}{2^{2k-1}}\right)\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right]\right\},
\end{aligned}
$$

which serves the role assumed by I_r in [**19**]. Where a fixed interval is understood, we drop the argument from I_r^*.

THEOREM 51. *([**20**]) Suppose $f : [a,b] \to \mathbb{R}$ is $(2r+2)$–convex. Then*

$$
\begin{aligned}
&(b-a)^{2r+1}\frac{|B_{2r}|}{(2r)!}\left(1-\frac{1}{2^{2r-1}}\right)f^{(2r)}\left(\frac{a+b}{2}\right) \\
\leq\ & I_r^* \leq (b-a)^{2r+1}\frac{|B_{2r}|}{(2r)!}\left(1-\frac{1}{2^{2r-1}}\right)\frac{f^{(2r)}(b)+f^{(2r)}(a)}{2}.
\end{aligned} \tag{2.132}
$$

If f is $(2r+2)$–concave, the inequality is reversed.

PROOF. We have from (2.131) that

$$
\begin{aligned}
I_r^* &= (-1)^{r-1}\frac{(b-a)^{2r}}{(2r)!}\int_a^b f^{(2r)}(t)p_{2r}(t)dt \\
&= \frac{(b-a)^{2r}}{(2r)!}\int_a^b f^{(2r)}(t)\left|p_{2r}(t)\right|dt \\
&= \frac{(b-a)^{2r}}{(2r)!}\int_a^b f^{(2r)}\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b\right)\left|p_{2r}(t)\right|dt.
\end{aligned} \tag{2.133}
$$

Using the discrete Jensen inequality for the convex function $f^{(2r)}$, we have

$$
\begin{aligned}
(2.134)\qquad & \int_a^b f^{(2r)}\left(\frac{b-t}{b-a}\cdot a+\frac{t-a}{b-a}\cdot b\right)|p_{2r}(t)|\,dt \\
\leq\ & f^{(2r)}(a)\int_a^b \frac{b-t}{b-a}|p_{2r}(t)|\,dt+f^{(2r)}(b)\int_a^b \frac{t-a}{b-a}|p_{2r}(t)|\,dt. \\
=\ & f^{(2r)}(a)K_1+f^{(2r)}(b)K_2, \quad \text{say.}
\end{aligned}
$$

Since $p_{2r}(t)$ is symmetric about $t=\frac{(a+b)}{2}$ and has constant sign on $[a,b]$, we have $K_1=K_2$. On the other hand

$$
\begin{aligned}
K_1+K_2 &= \int_a^b |p_{2r}(t)|\,dt=(-1)^{r-1}\int_a^b p_{2r}(t)dt \\
&= (-1)^{r-1}\left(1-2^{1-2r}\right)(b-a)B_{2r} \\
&= \left(1-2^{1-2r}\right)|B_{2r}|(b-a),
\end{aligned}
$$

so that

$$
K_1=K_2=\frac{1}{2}\left(1-2^{1-2r}\right)|B_{2r}|(b-a). \tag{2.135}
$$

The second inequality in (2.132) follows at once from (2.133)–(2.135).

By Jensen's integral inequality

$$
\begin{aligned}
(2.136)\qquad & \int_a^b f^{(2k)}\left(\frac{b-t}{b-a}\cdot a+\frac{t-a}{b-a}\cdot b\right)|p_{2r}(t)|\,dt \\
\geq\ & \left(\int_a^b |p_{2r}(t)|\,dt\right)f^{(2k)}\left(\frac{\int_a^b\left(\frac{b-t}{b-a}\cdot a+\frac{t-a}{b-a}\cdot b\right)|p_{2r}(t)|\,dt}{\int_a^b |p_{2r}(t)|\,dt}\right) \\
=\ & \left(1-\frac{1}{2^{2r-1}}\right)|B_{2r}|(b-a)f^{(2k)}\left(\frac{a+b}{2}\right).
\end{aligned}
$$

The first inequality in (2.132) now derives from (2.133), (2.135) and (2.136). ∎

The proof of the following theorem is similar to that of the theorem above and to that of [**19**, Theorem 2].

THEOREM 52. *([**20**]) Suppose $f:[a,b]\to\mathbb{R}$ is $(2r)$-times differentiable.*

(a) *If $|f^{(2r)}|^q$ is convex for some $q\geq 1$, then*

$$
|I_r^*|\leq (b-a)^{2r+1}\frac{|B_{2r}|}{(2r)!}\left(1-\frac{1}{2^{2r-1}}\right)\left[\frac{\left|f^{(2r)}(a)\right|^q+\left|f^{(2r)}(b)\right|^q}{2}\right]^{1/q}.
$$

(b) *If $\left|f^{(2r)}\right|$ is concave, then*

$$
|I_r^*|\leq (b-a)^{2r+1}\frac{|B_{2r}|}{(2r)!}\left(1-\frac{1}{2^{2r-1}}\right)\left|f^{(2r)}\left(\frac{a+b}{2}\right)\right|.
$$

To obtain appropriate results for numerical integration from the Euler midpoint formula, we apply the results above to each interval of the subdivision

$$
[a,a+h],[a+h,a+2h],...,[a+(n-1)h,a+nh].
$$

Let us denote

$$T(f;h) := h\left[\frac{1}{2}f(a) + \sum_{k=1}^{n-1} f(a+kh) + \frac{1}{2}f(a+nh)\right],$$

$$M(f;h) := h\sum_{k=1}^{n} f\left(a+kh-\frac{h}{2}\right)$$

and

$$H_r := (-1)^{r-1}\left\{\int_a^{a+nh} f(x)dx - M(f;h)\right.$$
$$\left. -\sum_{k=1}^{r-1}\frac{B_{2k}h^{2k}}{(2k)!}\left(1-\frac{1}{2^{2k-1}}\right)\left[f^{(2k-1)}(a+nh)-f^{(2k-1)}(a)\right]\right\}.$$

THEOREM 53. *We have:*

(a) *([**20**])If $f:[a,a+nh]\to\mathbb{R}$ is $(2r+2)$–convex, then*

$$h^{2r}\frac{|B_{2r}|}{(2r)!}\left(1-\frac{1}{2^{2r-1}}\right)M\left(f^{(2r)};h\right)$$
$$\leq \quad H_r \leq h^{2r}\frac{|B_{2r}|}{(2r)!}\left(1-\frac{1}{2^{2r-1}}\right)T\left(f^{(2r)};h\right).$$

(b) *If f is $(2r+2)$–concave, the inequalities are reversed.*

PROOF. The result is immediate from Theorem 51, since

$$H_r = \sum_{m=1}^{n} I_r^*(a+(m-1)h, a+mh).$$

∎

THEOREM 54. *([**20**]) Suppose $f:[a,a+nh]\to\mathbb{R}$ is $(2r)$–times differentiable.*

(a) *If $\left|f^{(2r)}\right|^q$ is convex for some $q\geq 1$, then*

$$|H_r| \leq nh^{2r+1}\frac{|B_{2r}|}{(2r)!}\left(1-\frac{1}{2^{2r-1}}\right)\max\left\{\left|f^{(2r)}(a)\right|,\left|f^{(2r)}(a+nh)\right|\right\}.$$

(b) *If $\left|f^{(2r)}\right|$ is concave, then*

$$|H_r| \leq h^{2r}\frac{|B_{2r}|}{(2r)!}\left(1-\frac{1}{2^{2r-1}}\right)M\left(\left|f^{(2r)}\right|;h\right).$$

PROOF. The proof is as follows.

$$\begin{aligned}|H_r| &= \left|\sum_{m=1}^{n} I_r^*(a+(m-1)h, a+mh)\right| \\ &\leq \sum_{m=1}^{n}\left|I_r^*(a+(m-1)h, a+mh)\right| \\ &\leq \sum_{m=1}^{n} h^{2r+1}\frac{|B_{2r}|}{(2r)!}(1-2^{1-2r})\left[\frac{\left|f^{(2r)}(a+mh)\right|^q+\left|f^{(2r)}(a+(m-1)h)\right|^q}{2}\right]^{1/q}\end{aligned}$$

by Theorem 52 applied to each interval $[a+(m-1)h, a+mh]$. Hence

$$\begin{aligned}|H_r| &\le h^{2r+1}\frac{|B_{2r}|}{(2r)!}(1-2^{1-2r}) \\ &\quad \times \sum_{m=1}^{n} \max\left\{\left|f^{(2r)}(a+mh)\right|, \left|f^{(2r)}(a+(m-1)h)\right|\right\}.\end{aligned}$$

The result of (a) now follows from the convexity of $\left|f^{(2r)}\right|^q$.

The proof (b) is similar. ∎

9.3. The Euler–Simpson Formula. If f is defined on an arbitrary finite segment $[a,b]$ and has $2r$ continuous derivatives there, then the Euler–Simpson formula (see, for example, [**94**, p. 221]) states that

$$\begin{aligned}\int_a^b f(x)dx &= \frac{b-a}{6}\left[f(b)+4f\left(\frac{a+b}{2}\right)+f(a)\right] \\ &\quad + \sum_{k=1}^{r-1}\frac{(b-a)^{2k}B_{2k}}{3(2k)!}\left(1-2^{2-2k}\right)\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right] \\ &\quad + \frac{(b-a)^{2r+1}}{3(2r)!}\int_0^1 f^{(2r)}(a+u(b-a))F(u)du,\end{aligned}$$

where

$$F(u) = y_{2r}(u) + 2\left[y_{2r}^*\left(\frac{1}{2}-u\right) - y_{2r}\left(\frac{1}{2}\right)\right],$$

$y_{2r}(u) = B_{2r}(u) - B_{2r}$ and $y_{2r}^*(u) = B_{2r}^*(u) - B_{2r}$.

It was proved in [**94**, p. 224] that $F(1-u) = F(u)$ and that $(-1)^{r-1}F(u) \ge 0$. Also,

$$\begin{aligned}\int_0^1 |F(u)|du &= (-1)^{r-1}\int_0^1 F(u)du \\ &= (-1)^{r-1}\left\{\int_0^1 y_{2r}(u)du + 2\int_0^1 y_{2r}^*\left(\frac{1}{2}-u\right)du - 2y_{2r}\left(\frac{1}{2}\right)\right\} \\ &= (-1)^{r-1}\left\{3\int_0^1 y_{2r}(u)du - 2y_{2r}\left(\frac{1}{2}\right)\right\} \\ &= (-1)^{r-1}\left\{-3B_{2r} + 4\left(1-2^{-2r}\right)B_{2r}\right\} \\ &= (-1)^{r-1}\left(1-2^{2-2r}\right)B_{2r} = \left(1-2^{2-2r}\right)|B_{2r}|\end{aligned}$$

and

$$\int_0^1 u|F(u)|du = \int_0^1 (1-u)\,|F(u)|\,du = \frac{1}{2}\left(1-2^{2-2r}\right)|B_{2r}|.$$

We can parallel the development of the previous subsection with the following two theorems. Define

$$\begin{aligned}L_r \ :&= (-1)^{r-1}\left\{\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right. \\ &\quad \left. - \sum_{k=2}^{r-1}\frac{(b-a)^{2k}}{3(2k)!}\left(1-2^{2-2k}\right)B_{2k}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right]\right\}.\end{aligned}$$

THEOREM 55. *([**20**]) If $f:[a,b]\to\mathbb{R}$ is $(2r+2)$–convex, then*

$$\begin{aligned}&(b-a)^{2r+1}\frac{\left(1-2^{2-2r}\right)|B_{2r}|}{3(2r)!}f^{(2r)}\left(\frac{a+b}{2}\right)\\ \leq\quad & L_r\leq(b-a)^{2r+1}\frac{\left(1-2^{2-2r}\right)|B_{2r}|}{3(2r)!}\cdot\frac{f^{(2r)}(a)+f^{(2r)}(b)}{2}.\end{aligned}$$

If f is $(2r+2)$–concave, the inequalities are reversed.

THEOREM 56. *([**20**]) Suppose $f:[a,b]\to\mathbb{R}$ is $(2r)$–times differentiable.*

(a) *If $\left|f^{(2r)}\right|^q$ is convex for some $q\geq 1$, then*

$$|L_r|\leq(b-a)^{2r+1}\frac{\left(1-2^{2-2r}\right)|B_{2r}|}{3(2r)!}\left[\frac{\left|f^{(2r)}(a)\right|^q+\left|f^{(2r)}(b)\right|^q}{2}\right]^{1/q}.$$

(b) *If $\left|f^{(2r)}\right|$ is concave, then*

$$|L_r|\leq(b-a)^{2r+1}\frac{1-2^{2-2r}}{3(2r)!}|B_{2r}|\left|f^{(2r)}\left(\frac{a+b}{2}\right)\right|.$$

To obtain appropriate results for integration *via* the Simpson formula we apply the above results to each interval of the subdivision

$$[a,a+2h],[a+2h,a+4h],...,[a+2(n-1)h,a+2nh].$$

First we introduce

$$S(f;h):=\frac{h}{3}\left[f(a)+f(a+2nh)+2\sum_{i=1}^{n-1}f(a+2ih)+4\sum_{i=1}^{n-1}f(a+(2i-1)h)\right],$$

$$\begin{aligned}X_r\quad:\quad&=(-1)^{r-1}\left\{\int_a^{a+2nh}f(x)dx-S(f;h)-\sum_{k=2}^{r-1}\frac{(2h)^{2k}}{3(2k)!}\left(1-2^{2-2k}\right)\right.\\&\left.\times B_{2k}\left[f^{(2k-1)}(a+2nh)-f^{(2k-1)}(a)\right]\right\}.\end{aligned}$$

The following theorems apply.

THEOREM 57. *([**20**]) If $f:[a,a+2nh]\to\mathbb{R}$ is $(2r+2)$–convex, then*

$$\begin{aligned}&(2h)^{2r}|B_{2r}|\frac{\left(1-2^{2-2r}\right)}{3(2r)!}M\left(f^{(2r)};2h\right)\\ \leq\quad & X_r\leq(2h)^{2r}\frac{\left(1-2^{2-2r}\right)}{3(2r)!}|B_{2r}|T(f^{(2r)};2h).\end{aligned}$$

If f is $(2r+2)$–concave, the reverse inequalities hold.

THEOREM 58. *([**20**]) Suppose $f:[a,a+2nh]\to\mathbb{R}$ is $(2r)$–times differentiable.*

(a) *If* $\left|f^{(2r)}\right|^q$ *is convex for some* $q \geq 1$, *then*

$$\begin{aligned}
|X_r| &\leq (2h)^{2r+1}|B_{2r}|\frac{\left(1-2^{2-2r}\right)}{3(2r)!} \\
&\quad \times \sum_{m=1}^{n}\left[\frac{\left|f^{(2r)}(a+2mh)\right|^q+\left|f^{(2r)}(a+2(m-1)h)\right|^q}{2}\right]^{1/q} \\
&\leq n(2h)^{2r+1}|B_{2r}|\frac{\left(1-2^{2-2r}\right)}{3(2r)!}\max\left\{\left|f^{(2r)}(a)\right|,\left|f^{(2r)}(a+2nh)\right|\right\}.
\end{aligned}$$

(b) *If* $\left|f^{(2r)}\right|$ *is concave, then*

$$|X_r| \leq (2h)^{2r+1}|B_{2r}|\frac{\left(1-2^{2-2r}\right)}{3(2r)!}M\left(\left|f^{(2r)}\right|;2h\right).$$

The resultant formulæ in Theorems 55–58 when $r=2$ and the sums in L_r and X_r are empty are of special interest and we isolate them as corollaries.

Corollary 18. *([**20**]) If* $f:[a,b]\to\mathbb{R}$ *is* 6*–convex, then*

$$\begin{aligned}
&\frac{(b-a)^5}{2880}f^{(4)}\left(\frac{a+b}{2}\right) \\
&\leq \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\int_a^b f(x)dx \\
&\leq \frac{(b-a)^5}{2880}\frac{f^{(4)}(a)+f^{(4)}(b)}{2}.
\end{aligned}$$

If f *is* 6*–concave, the reversed inequalities apply.*

Corollary 19. *([**20**]) Suppose* $f:[a,b]\to\mathbb{R}$ *is* 4*–times differentiable.*

(a) *If* $\left|f^{(4)}\right|^q$ *is convex for some* $q\geq 1$, *then*

$$\begin{aligned}
&\left|\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
&\leq \frac{(b-a)^5}{2880}\left[\frac{\left|f^{(4)}(a)\right|^q+\left|f^{(4)}(b)\right|^q}{2}\right]^{1/q}.
\end{aligned}$$

(b) *If* $\left|f^{(4)}\right|$ *is concave, then*

$$\begin{aligned}
&\left|\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
&\leq \frac{(b-a)^5}{2880}\left|f^{(4)}\left(\frac{a+b}{2}\right)\right|.
\end{aligned}$$

Corollary 20. *([**20**]) If* $f:[a,a+2nh]\to\mathbb{R}$ *is* 6*–convex, then*

$$\frac{(2h)^4}{2880}M\left(f^{(4)};2h\right)\leq S(f,h)-\int_a^{a+2nh}f(x)dx\leq\frac{(2h)^4}{2880}T\left(f^{(4)};2h\right).$$

If f *is* 6*–concave, the inequalities are reversed.*

Corollary 21. *([**20**]) Suppose* $f:[a,a+2nh]\to\mathbb{R}$ *is* 4*–times differentiable.*

(a) *If* $\left|f^{(4)}\right|^q$ *is convex for some* $q \geq 1$, *then*

$$\begin{aligned}
&\left|\int_a^{a+2nh} f(x)dx - S(f;h)\right| \\
\leq\ & \frac{(2h)^5}{2880}\sum_{m=1}^{n}\left[\frac{\left|f^{(4)}(a+2nh)\right|^q + \left|f^{(4)}(a+2(m-1)h)\right|^q}{2}\right]^{1/q} \\
\leq\ & \frac{n(2h)^5}{2880}\max\left\{\left|f^{(4)}(a)\right|, \left|f^{(4)}(a+2nh)\right|\right\}.
\end{aligned}$$

(b) *If* $\left|f^{(4)}\right|$ *is concave, then*

$$\left|\int_a^{a+2nh} f(x)dx - S(f;h)\right| \leq \frac{n(2h)^5}{2880} M\left(\left|f^{(4)}\right|; 2h\right).$$

10. $H. - H.$ Inequality for Isotonic Linear Functionals

10.1. Inequalities for Isotonic Linear Functionals. In this section we shall give some generalizations of the Hermite-Hadamard inequality for isotonic linear functionals.

Let E be a non-empty set and let L be a linear class of real valued functions $g : E \to \mathbb{R}$ having the properties:

L1: $f, g \in L$ implies $(af + bg) \in L$ for all $a, b \in \mathbb{R}$;
L2: $\mathbf{1} \in L$, that is, if $f(t) = 1$ $(t \in E)$ then $f \in L$.

We also consider isotonic linear functionals $A : L \to \mathbb{R}$. That is, we suppose:

A1: $A(af + bg) = aA(f) + bA(g)$ for $f, g \in L$, $a, b \in \mathbb{R}$;
A2: $f \in L, f(t) \geq 0$ on E implies $A(f) \geq 0$.

We note that common examples of such isotonic linear functionals A are given by

$$A(g) = \int_E g d\mu \text{ or } A(g) = \sum_{k\in E} p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second $(p_k \geq 0, k \in E)$.

We shall use the following result which is well-known in the literature as *Jessen's Inequality* (see for example [**146**] or [**114**, p. 47] and [**23**]):

Theorem 59. *Let L satisfy properties L1 and L2 on a non-empty set E and suppose ϕ is a convex function on an interval $I \subseteq \mathbb{R}$. If A is any isotonic functional; with $A(\mathbf{1}) = 1$, then, for all $g \in L$ such that $\phi(g) \in L$, we have $A(g) \in I$ and*

$$\phi(A(g)) \leq A(\phi(g)). \tag{2.137}$$

The following lemma holds [**146**]:

Lemma 10. *Let X be a real linear space and C its convex subset. Then the following statements are equivalent for a mapping $f : X \to \mathbb{R}$:*

(*i*) *f is convex on C;*
(*ii*) *for all $x, y \in C$ the mapping $g_{x,y} : [0,1] \to \mathbb{R}$, $g_{x,y}(t) := f(tx + (1-t)y)$ is convex on $[0,1]$.*

PROOF. "$(i) \Rightarrow (ii)$". Suppose $x, y \in C$ and let $t_1, t_2 \in [0,1]$, $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$. Then

$$\begin{aligned} g_{x,y}(\lambda_1 t_1 + \lambda_2 t_2) &= f[(\lambda_1 t_1 + \lambda_2 t_2)x + (1 - \lambda_1 t_1 - \lambda_2 t_2)y] \\ &= f[(\lambda_1 t_1 + \lambda_2 t_2)x + [\lambda_1(1-t_1) + \lambda_2(1-t_2)]y] \\ &\leq \lambda_1 f(t_1 x + (1-t_1)y) + \lambda_2 f(t_2 x + (1-t_2)y). \end{aligned}$$

That is, $g_{x,y}$ is convex on $[0,1]$.
"$(ii) \Rightarrow (i)$". Now, let $x, y \in C$ and $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$. Then we have:

$$\begin{aligned} f(\lambda_1 x + \lambda_2 y) &= f(\lambda_1 x + (1-\lambda_1)y) = g_{x,y}(\lambda_1 \cdot 1 + \lambda_2 \cdot 0) \\ &\leq \lambda_1 g_{x,y}(1) + \lambda_2 g_{x,y}(0) = \lambda_1 f(x) + \lambda_2 f(y). \end{aligned}$$

That is, f is convex on C and the statement is proved. ∎

The following generalization of Hermite-Hadamard's inequality for isotonic linear functionals holds [**146**]:

THEOREM 60. *Let $f : C \subseteq X \to \mathbb{R}$ be a convex function on C, L and A satisfy conditions L1, L2 and A1, A2, and $h : E \to \mathbb{R}$, $0 \leq h(t) \leq 1$, $h \in L$ is such that $g_{x,y} \circ h \in L$ for x, y given in C. If $A(\mathbf{1}) = 1$, then we have the inequality*

$$\begin{aligned} f(A(h)x + (1 - A(h))y) &\leq A[f(hx + (\mathbf{1} - h)y)] \\ &\leq A(h)f(x) + (1 - A(h))f(y). \end{aligned} \tag{2.138}$$

PROOF. Consider the mapping $g_{x,y} : [0,1] \to \mathbb{R}$, $g_{x,y}(s) := f(sx + (1-s)y)$. Then, by Lemma 10, we have that $g_{x,y}$ is convex on $[0,1]$. For each $t \in E$ we have:

$$g_{x,y}(h(t) \cdot 1 + (1 - h(t)) \cdot 0) \leq h(t) g_{x,y}(1) + (1 - h(t)) g_{x,y}(0),$$

which implies that

$$A(g_{x,y}(h)) \leq A(h) g_{x,y}(1) + (1 - A(h)) g_{x,y}(0).$$

That is,

$$A[f(hx + (\mathbf{1} - h)y)] \leq A(h) f(x) + (1 - A(h)) f(y).$$

On the other hand, by Jessen's inequality, applied for $g_{x,y}$ we have:

$$g_{x,y}(A(h)) \leq A(g_{x,y}(h)),$$

which gives:

$$f(A(h)x + (1 - A(h))y) \leq A[f(hx + (\mathbf{1} - h)y)]$$

and the proof is completed. ∎

REMARK 34. *If $h : E \to [0,1]$ is such that $A(h) = \frac{1}{2}$, we get from the inequality* (2.138) *that*

$$f\left(\frac{x+y}{2}\right) \leq A[f(hx + (\mathbf{1} - h)y)] \leq \frac{f(x) + f(y)}{2}, \tag{2.139}$$

for all x, y in C.

Consequences

a) If $A=\int_0^1, E=[0,1], h(t)=t, t\in[0,1]$, $C=[x,y]\subset\mathbb{R}$, then we recapture from (2.138) the classical inequality of Hermite and Hadamard, because

$$\int_0^1 f(tx+(1-t)y)\,dt=\frac{1}{y-x}\int_x^y f(t)\,dt.$$

b) If $A=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}, E=\left[0,\frac{\pi}{2}\right], h(t)=\sin^2 t, C\subseteq\mathbb{R}$, then, from (2.139) we get

$$f\left(\frac{x+y}{2}\right)\le\frac{2}{\pi}\int_0^{\frac{\pi}{2}} f\left(x\sin^2 t+y\cos^2 t\right)dt\le\frac{f(x)+f(y)}{2},$$

$x,y\in C$, which is a new inequality of Hadamard's type. This is as $\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin^2 t dt=\frac{1}{2}$.

c) If $A=\int_0^1, E=[0,1], h(t)=t$ and X is a normed linear space, then (2.139) implies that for $f(x)=\|x\|^p$, $x\in X, p\ge 1$:

$$\left\|\frac{x+y}{2}\right\|^p\le\int_0^1\|tx+(1-t)y\|^p\,dt\le\frac{\|x\|^p+\|y\|^p}{2} \tag{2.140}$$

for all $x,y\in X$.

d) If $A=\frac{1}{n}\sum_{i=1}^{n}, E=\{1,...,n\}, \sum_{i=1}^{n}t_i=\frac{n}{2}, C\subseteq\mathbb{R}, n\ge 1$, then from (2.139) we also have

$$f\left(\frac{x+y}{2}\right)\le\frac{1}{n}\sum_{i=1}^{n}f(t_i x+(1-t_i)y)\le\frac{f(x)+f(y)}{2}$$

for all $x,y\in C$, which is a discrete variant of the Hermite-Hadamard inequality.

To give a symmetric generalization of the Hermite-Hadamard inequality, we present the following lemma which is interesting in itself [**23**].

Lemma 11. *Let X be a real linear space and C be its convex subset. If $f:C\to\mathbb{R}$ is convex on C, then for all x,y in C the mapping $g_{x,y}:[0,1]\to\mathbb{R}$ given by*

$$g_{x,y}(t):=\frac{1}{2}\left[f(tx+(1-t)y)+f((1-t)x+ty)\right]$$

is also convex on $[0,1]$. In addition, we have the inequality

$$f\left(\frac{x+y}{2}\right)\le g_{x,y}(t)\le\frac{f(x)+f(y)}{2} \tag{2.141}$$

for all $x,y\in C$ and $t\in[0,1]$.

PROOF. Suppose $x,y\in C$ and let $t_1,t_2\in[0,1]$, $\alpha,\beta\geq 0$ and $\alpha+\beta=1$. Then

$$\begin{aligned}
& g_{x,y}(\alpha t_1+\beta t_2) \\
= & \frac{1}{2}\left[f\left((\alpha t_1+\beta t_2)x+(1-\alpha t_1-\beta t_2)y\right)\right. \\
& \left.+f\left((1-\alpha t_1-\beta t_2)x+(\alpha t_1+\beta t_2)y\right)\right] \\
= & \frac{1}{2}\left(f\left[\alpha\left(t_1x+(1-t_1)y\right)+\beta\left(t_2x+(1-t_2)y\right)\right]\right. \\
& \left.+f\left[\alpha\left((1-t_1x)+t_1y\right)+\beta\left((1-t_2)x+t_2y\right)\right]\right) \\
\leq & \frac{1}{2}\left(\alpha f\left[t_1x+(1-t_1)y\right]+\beta f\left[t_2x+(1-t_2)y\right]\right. \\
& \left.+\alpha f\left[(1-t_1x)+t_1y\right]+\beta f\left[(1-t_2)x+t_2y\right]\right) \\
= & \alpha g_{x,y}(t_1)+\beta g_{x,y}(t_2),
\end{aligned}$$

which shows that $g_{x,y}$ is convex on $[0,1]$.
By the convexity of f we can state that

$$g_{x,y}(t)\geq f\left[\frac{1}{2}\left(tx+(1-t)y+(1-t)x+ty\right)\right]=f\left(\frac{x+y}{2}\right).$$

In addition,

$$g_{x,y}(t)\leq\frac{1}{2}\left[tf(x)+(1-t)f(y)+(1-t)f(x)+tf(y)\right]=\frac{f(x)+f(y)}{2}$$

for all t in $[0,1]$, which completes the proof. ∎

REMARK 35. *By the inequality* (2.141) *we deduce the bounds*

$$\sup_{t\in[0,1]}g_{x,y}(t)=\frac{f(x)+f(y)}{2}\quad\text{and}\quad\inf_{t\in[0,1]}g_{x,y}(t)=f\left(\frac{x+y}{2}\right)$$

for all x,y in C.

The following symmetric generalization of the Hermite-Hadamard inequality holds [**23**]:

THEOREM 61. *Let $f:C\subseteq X\to\mathbb{R}$ be a convex function on the convex set C, where L and A satisfy the conditions L1, L2 and A1,A2. Also, $h:E\to\mathbb{R}$, $0\leq h(t)\leq 1$ $(t\in E)$, and $h\in L$ is such that $f(hx+(1-h)y)$, $f((1-h)x+hy)$ belong to L for x,y fixed in C. If $A(\mathbf{1})=1$, then we have the inequality:*

$$\begin{aligned}
& f\left(\frac{x+y}{2}\right) \\
\leq & \frac{1}{2}\left[f\left(A(h)x+(1-A(h))y\right)+f\left((1-A(h))x+A(h)y\right)\right] \\
\leq & \frac{1}{2}\left(A\left[f\left(hx+(\mathbf{1}-h)y\right)\right]+A\left[f\left((\mathbf{1}-h)x+hy\right)\right]\right) \\
\leq & \frac{f(x)+f(y)}{2}
\end{aligned}\tag{2.142}$$

PROOF. Let us consider the mapping $g_{x,y}:[0,1]\to\mathbb{R}$ given in Lemma 11. Then, by the above Lemma we know that $g_{x,y}$ is convex on $[0,1]$.
Applying Jessen's inequality to the mapping $g_{x,y}$ we get:

$$g_{x,y}(A(h))\leq A(g_{x,y}(h)).$$

However,

$$g_{x,y}\left(A\left(h\right)\right)=\frac{1}{2}\left[f\left(A\left(h\right)x+\left(1-A\left(h\right)\right)y\right)+f\left(\left(1-A\left(h\right)\right)x+A\left(h\right)y\right)\right]$$

and

$$A\left(g_{x,y}\left(h\right)\right)=\frac{1}{2}\left(A\left[f\left(hx+\left(\mathbf{1}-h\right)y\right)\right]+A\left[f\left(\left(\mathbf{1}-h\right)x+hy\right)\right]\right)$$

and the second inequality in (2.142) is proved.
To prove the first inequality in (2.142) we observe, by (2.141), that

$$f\left(\frac{x+y}{2}\right)\leq g_{x,y}\left(A\left(h\right)\right)\ \text{ as } 0\leq A\left(h\right)\leq 1,$$

which is exactly the desired outcome.
Finally, by the convexity of f, we observe that

$$\frac{1}{2}\left[f\left(hx+\left(\mathbf{1}-h\right)y\right)+f\left(\left(\mathbf{1}-h\right)x+hy\right)\right]\leq\frac{f\left(x\right)+f\left(y\right)}{2}$$

on E.
By applying the functional A, since $A\left(\mathbf{1}\right)=1$, we obtain the last part of (2.142). ∎

REMARK 36. *The above theorem can also be proved by the use of Theorem 60 and by Lemma 11. We shall omit the details.*

Note that, if we choose $A=\int_0^1, E=\left[0,1\right], h\left(t\right)=t, C=\left[x,y\right]\subset\mathbb{R}$, we recapture, by (2.142), the Hermite-Hadamard inequality for integrals. This is because

$$\int_0^1 f\left(tx+\left(1-t\right)y\right)dt=\int_0^1 f\left(\left(1-t\right)x+ty\right)dt=\frac{1}{y-x}\int_x^y f\left(t\right)dt.$$

Consequences

a) Let $h:\left[0,1\right]\to\left[0,1\right]$ be a Riemann integrable function on $\left[0,1\right]$ and $p\geq 1$. Then, for all x,y vectors in the normed space $\left(X;\left\|\cdot\right\|\right)$ we have the inequality:

$$\begin{aligned}
&\left\|\frac{x+y}{2}\right\|^p\\
\leq\ &\frac{1}{2}\left[\left\|\left(1-\int_0^1 h\left(t\right)dt\right)x+\left(\int_0^1 h\left(t\right)dt\right)y\right\|^p\right.\\
&\left.+\left\|\left(\int_0^1 h\left(t\right)dt\right)x+\left(1-\int_0^1 h\left(t\right)dt\right)y\right\|^p\right]\\
\leq\ &\frac{1}{2}\left[\int_0^1\left\|\left(h\left(t\right)\right)x+\left(1-h\left(t\right)\right)y\right\|^p dt+\int_0^1\left\|\left(1-h\left(t\right)\right)x+\left(h\left(t\right)\right)y\right\|^p dt\right]\\
\leq\ &\frac{\left\|x\right\|^p+\left\|y\right\|^p}{2}.
\end{aligned}$$

If we choose $h\left(t\right)=t$, we get the inequality obtained at (2.140).

b) Let $f : C \subseteq X \to \mathbb{R}$ be a convex function on the convex set C of a linear space X, $t_i \in [0,1]$ $\left(i = \overline{1,n}\right)$. Then we have the inequality:

$$\begin{aligned} & f\left(\frac{x+y}{2}\right) \\ \leq\ & \frac{1}{2}\left[f\left(\frac{1}{n}\sum_{i=1}^{n} t_i x + \frac{1}{n}\sum_{i=1}^{n}(1-t_i)\,y\right) + f\left(\frac{1}{n}\sum_{i=1}^{n}(1-t_i)\,x + \frac{1}{n}\sum_{i=1}^{n} t_i y\right)\right] \\ \leq\ & \frac{1}{2n}\left[\sum_{i=1}^{n} f\left(t_i x + (1-t_i)\,y\right) + \sum_{i=1}^{n} f\left((1-t_i)\,x + t_i y\right)\right] \\ \leq\ & \frac{f(x)+f(y)}{2}. \end{aligned}$$

If we put in the above inequality $t_i = \sin^2\alpha_i, \alpha_i \in \mathbb{R}$ $\left(i = \overline{1,n}\right)$, then we have:

$$\begin{aligned} & f\left(\frac{x+y}{2}\right) \\ \leq\ & \frac{1}{2}\left(f\left[\left(\frac{1}{n}\sum_{i=1}^{n}\sin^2\alpha_i\right)x + \left(\frac{1}{n}\sum_{i=1}^{n}\cos^2\alpha_i\right)y\right]\right. \\ & \left. + f\left[\left(\frac{1}{n}\sum_{i=1}^{n}\cos^2\alpha_i\right)x + \left(\frac{1}{n}\sum_{i=1}^{n}\sin^2\alpha_i\right)y\right]\right) \\ \leq\ & \frac{1}{2n}\sum_{i=1}^{n}\left(f\left[\left(\sin^2\alpha_i\right)x + \left(\cos^2\alpha_i\right)y\right]\right. \\ & \left. + f\left[\left(\cos^2\alpha_i\right)x + \left(\sin^2\alpha_i\right)y\right]\right) \\ \leq\ & \frac{f(x)+f(y)}{2}. \end{aligned}$$

10.2. Applications for Special Means.

(1) For $x, y \geq 0$, let us consider the weighted means:

$$A_\alpha(x,y) := \alpha x + (1-\alpha)\,y$$

and

$$G_\alpha(x,y) := x^\alpha y^{1-\alpha}$$

where $\alpha \in [0,1]$.

If $h : [0,1] \to [0,1]$ is an integrable mapping on $[0,1]$, then, by Theorem 60 for $f(x) = -\ln x$, $x > 0$, we have the inequality:

$$A_{\int_0^1 h(t)dt}(x,y) \geq \exp\left[\int_0^1 \ln\left[A_{h(t)}(x,y)\right]dt\right] \geq G_{\int_0^1 h(t)dt}(x,y). \tag{2.143}$$

If $\int_0^1 h(t)\,dt = \frac{1}{2}$, we get

$$A(x,y) \geq \exp\left[\int_0^1 \ln\left[A_{h(t)}(x,y)\right]dt\right] \geq G(x,y) \tag{2.144}$$

which is a refinement of the classic $A. - G.$ inequality.

In particular, if in this inequality we choose $h(t) = t,\ t \in [0,1]$, we recapture the well-known result for the identric mean:

$$A(x,y) \geq I(x,y) \geq G(x,y).$$

Now, if we use Theorem 61, we can state the following weighted refinement of the classical $A. - G.$ inequality:

$$\begin{aligned} A(x,y) &\geq G\left(A_{\int_0^1 h(t)dt}(x,y), A_{\int_0^1 h(t)dt}(x,y)\right) \\ &\geq \exp\left[\int_0^1 \ln\left[G\left(A_{h(t)}(x,y), A_{h(t)}(y,x)\right)\right] dt\right] \\ &\geq G(x,y). \end{aligned} \tag{2.145}$$

If $\int_0^1 h(t)\,dt = \frac{1}{2}$, then, by (2.145) we get the following refinement of the $A. - G.$ inequality:

$$A(x,y) \geq \exp\left[\int_0^1 \ln\left[G\left(A_{h(t)}(x,y), A_{h(t)}(y,x)\right)\right] dt\right] \geq G(x,y). \tag{2.146}$$

If, in the above inequality we choose $h(t) = t, t \in [0,1]$, then we get the inequality

$$A(x,y) \geq \exp\left[\int_0^1 \ln\left[G\left(A_t(x,y), A_t(y,x)\right)\right] dt\right] \geq G(x,y). \tag{2.147}$$

(2) Some discrete refinements of $A. - G.$ means inequality can also be done. If $\bar{x} = (x_1, ..., x_n) \in \mathbb{R}_+^n$, we can denote by $G_n(\bar{x})$ the geometric mean of $\bar{x}$, i.e., $G_n(\bar{x}) := \left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}}$.
If $\bar{t} = (t_1, ..., t_n) \in [0,1]^n$, we can define the vector in $\mathbb{R}_+^n$ given by

$$\bar{A}_{\bar{t}}(x,y) := \left(A_{t_1}(x,y), ..., A_{t_n}(x,y)\right)$$

where $x, y \geq 0$.
Applying, now, Theorem 60 for the convex mapping $f(x) = -\ln x$ and the linear functional $A := \frac{1}{n}\sum_{i=1}^n t_i$, we get the inequality

$$A_{\tilde{t}}(x,y) \geq G_n\left(\bar{A}_{\bar{t}}(x,y)\right) \geq G_{\tilde{t}}(x,y) \tag{2.148}$$

where $\tilde{t} := \frac{1}{n}\sum_{i=1}^n t_i \in [0,1]$ and $x, y \geq 0$.
If we choose t_i so that $\tilde{t} = \frac{1}{2}$, we get

$$A(x,y) \geq G_n\left(\bar{A}_{\bar{t}}(x,y)\right) \geq G(x,y) \tag{2.149}$$

which is a discrete refinement of the classical $A. - G.$ inequality.
In addition, if we use Theorem 61, we can state that

$$\begin{aligned} A(x,y) &\geq G_n\left(A_{\bar{t}}(x,y), A_{\bar{t}}(y,x)\right) \\ &\geq G\left(G_n\left(\bar{A}_{\bar{t}}(x,y)\right), G_n\left(\bar{A}_{\bar{t}}(y,x)\right)\right) \geq G(x,y), \end{aligned} \tag{2.150}$$

which is another refinement of the $A. - G.$ inequality.

11. $H. - H.$ Inequality for Isotonic Sublinear Functionals

11.1. Inequalities for Isotonic Sublinear Functionals. Let L be a linear class of real-valued functions $g : E \to \mathbb{R}$ having the properties:

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f(t) = 1$ for all $t \in E$, then $f \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying the conditions:

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$;
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.
The mapping A is said to be *normalized* if
(A3) $A(\mathbf{1}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality and a functional Hermite-Hadamard inequality.
In this section we show that these ideas carry over to a sublinear setting [**64**].
Let E be a non-empty set and K a class of real-valued functions $g : E \to \mathbb{R}$ having the properties

(K1) $\mathbf{1} \in K$;
(K2) $f, g \in K$ imply $f + g \in K$;
(K3) $f \in K$ implies $\alpha \cdot \mathbf{1} + \beta \cdot f \in K$ for all $\alpha, \beta \in \mathbb{R}$.

We define the family of *isotonic sublinear functionals* $S : K \to \mathbb{R}$ by the properties [**64**]

(S1) $S(f + g) \leq S(f) + S(g)$ for all $f, g \in K$;
(S2) $S(\alpha f) = \alpha S(f)$ for all $\alpha \geq 0$ and $f \in K$;
(S3) If $f \geq g, f, g \in K$, then $S(f) \geq S(g)$.
An isotonic sublinear functional is said to be *normalized* if
(S4) $S(\mathbf{1}) = 1$
and *totally normalized* if, in addition,
(S5) $S(-\mathbf{1}) = -1$.
We note some immediate consequences from (K2) and (K3), $f - g$ belongs to K whenever $f, g \in K$, so that from (S1)

$$S(f) = S((f - g) + g) \leq S(f - g) + S(g)$$

and hence
(S6) $S(f - g) \geq S(f) - S(g)$ if $f, g \in K$.
Moreover, if S is a totally normalized isotonic sublinear functional, then we have
(S7) $S(\alpha \cdot \mathbf{1}) = \alpha$ for all $\alpha \in \mathbb{R}$
and
(S8) $S(f + \alpha \cdot \mathbf{1}) = S(f) + \alpha$ for all $\alpha \in \mathbb{R}$.

Equation (S7) is immediate from (S2) when $\alpha \geq 0$. When $\alpha < 0$ we have

$$S(\alpha \cdot \mathbf{1}) = S((-\alpha) \cdot (-\mathbf{1})) = (-\alpha) S(-\mathbf{1}) = (-\alpha)(-1) = \alpha.$$

Also, by (S6) and (S7), we have for $\alpha \in \mathbb{R}$

$$S(f - \alpha \cdot \mathbf{1}) \geq S(f) - S(\alpha \cdot \mathbf{1}) = S(f) - \alpha,$$

which by (S1) and (S7)

$$S(f-\alpha\cdot\mathbf{1})\leq S(f)+S(-\alpha\cdot\mathbf{1})=S(f)-\alpha,$$

so that

$$S(f-\alpha\cdot\mathbf{1})=S(f)-\alpha.$$

Since this holds for all $\alpha\in\mathbb{R}$, we have (S8).

It is clear that every normalized isotonic linear functional is a totally normalized isotonic sublinear functional.

In what follows, we shall present some simple examples of sublinear functionals that are not linear.

EXAMPLE 4. *Let $A_1,...,A_n:L\to\mathbb{R}$ be normalized isotonic linear functionals and $p_{i,j}\in\mathbb{R}$ $(i,j\in\{1,...,n\})$ such that*

$$p_{i,j}\geq 0 \text{ for all } i,j\in\{1,...,n\} \text{ and } \sum_{i=1}^{n}p_{i,j}=1 \text{ for all } j\in\{1,...,n\}.$$

Define the mapping $S:L\to\mathbb{R}$ by

$$S(f)=\max_{1\leq j\leq n}\left\{\sum_{i=1}^{n}p_{i,j}A_i(f)\right\}.$$

Then S is a totally normalized isotonic sublinear functional on L. As particular cases of this functional, we have the mappings

$$S_0(f):=\max_{1\leq j\leq n}\{A_i(f)\}$$

and

$$S_Q(f):=\max_{1\leq j\leq n}\left\{\frac{1}{Q_j}\sum_{i=1}^{j}q_iA_i(f)\right\}$$

where $q_i\geq 0$ for all $i\in\{1,...,n\}$ and $Q_j>0$ for $j=1,...,n$. If we choose $q_i=1$ for all $i\in\{1,...,n\}$, we also have that

$$S_1(f):=\max_{1\leq j\leq n}\left\{\frac{1}{j}\sum_{i=1}^{j}A_i(f)\right\}$$

is a totally normalized isotonic sublinear functional on L.

EXAMPLE 5. *If $A_1,...,A_n$ are as above and $A:L\to\mathbb{R}$ is also a normalized isotonic linear functional, then the mapping*

$$S_A(f):=\frac{1}{P_n}\sum_{i=1}^{n}p_i\max\{A(f),A_i(f)\}$$

where $p_i\geq 0$ $(1\leq i\leq n)$ with $P_n=\sum_{i=1}^{n}p_i>0$, is also a totally normalized isotonic sublinear functional.

The following provide concrete examples.

EXAMPLE 6. *Suppose $x=(x_1,...,x_n)$ and $y=(y_1,...,y_n)$ are points in $\mathbb{R}^n$. Then the mappings*

$$S(x):=\max_{1\leq j\leq n}\left\{\sum_{i=1}^{n}p_{i,j}x_i\right\},$$

where $p_i \geq 0$ *and* $\sum_{i=1}^{n} p_{i,j} = 1$ *for* $j \in \{1, ..., n\}$,

$$S_0(x) := \max_{1 \leq i \leq n} \{x_i\}$$

and

$$S_Q(x) := \max_{1 \leq j \leq n} \left\{ \frac{1}{Q_j} \sum_{i=1}^{j} q_i A_x \right\}$$

where $q_i \geq 0$ *and* $Q_j > 0$ *for all* $i, j \in \{1, ..., n\}$, *are totally normalized isotonic sublinear functionals on* $\mathbb{R}^n$.
Suppose $i_0 \in \{1, .., n\}$ *is fixed and* $p_i \geq 0$ *for all* $i \in \{1, ..., n\}$, *with* $P_n > 0$.
Then the mapping

$$S_{i_0}(x) := \frac{1}{P_n} \sum_{i=1}^{n} p_i \max\{x_{i_0}, x_i\}$$

is also totally normalized.

Example 7. *Denote by* $R[a, b]$ *the linear space of Riemann integrable functions on* $[a, b]$. *Suppose that* $p \in R[a, b]$ *with* $p(t) > 0$ *for all* $t \in [a, b]$. *Then the mappings*

$$S_p(f) := \sup_{x \in (a,b]} \left[\frac{\int_a^x p(t) f(t) \, dt}{\int_a^x p(t) \, dt} \right]$$

and

$$s_1(f) := \sup_{x \in (a,b]} \left[\frac{1}{x-a} \int_a^x f(t) \, dt \right]$$

are totally normalized isotonic sublinear functionals on $R[a, b]$.
If $c \in [a, b]$, *then*

$$S_{c,p}(f) := \frac{\int_a^b p(t) \max(f(c), f(t)) \, dt}{\int_a^b p(t) \, dt}$$

and

$$s_c(f) := \frac{1}{b-a} \int_a^b \max(f(c), f(t)) \, dt$$

are also totally normalized on $R[a, b]$.

We can give the following generalization of the well-known Jessen's inequality due to S. S. Dragomir, C. E. M. Pearce and J. E. Pečarić [**64**]:

Theorem 62. *Let* $\phi : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ *be a continuous convex function and* $f : E \to [\alpha, \beta]$ *such that* $f, \phi \circ f \in K$. *Then, if* S *is a totally normalized isotonic sublinear functional on* K, *we have* $S(f) \in [\alpha, \beta]$ *and:*

$$S(\phi \circ f) \geq \phi(S(f)). \tag{2.151}$$

Proof. By (S3) and (S7), $\alpha \cdot \mathbf{1} \leq f \leq \beta \cdot \mathbf{1}$ implies

$$\alpha = S(\alpha \cdot \mathbf{1}) \leq S(f) \leq S(\beta \cdot \mathbf{1}) = \beta$$

so that $S(f) \in [\alpha, \beta]$.
Set $l_1(x) = x$ for all $x \in [\alpha, \beta]$. For an arbitrary but fixed $q > 0$, we have by convexity of ϕ that there exist real numbers $u, v \in \mathbb{R}$ such that

(i) $p \leq \phi$ and

(ii) $p(S(f)) \geq \phi(S(f)) - q$
where

$$p(t) = u \cdot \mathbf{1} + v \cdot l_1(t).$$

If $\alpha < S(f) < \beta$ or if ϕ has a finite derivative in $[\alpha, \beta]$, we can replace (ii) by $p(S(f)) = \phi(S(f))$. Now (i) implies $p \circ f \leq \phi \circ f$. Hence, by (S3)

$$S(\phi \circ f) \geq S(p \circ f) = S(u \cdot \mathbf{1} + v \cdot f).$$

If $v \geq 0$, by (S8) and (S2) we have

$$S(u \cdot \mathbf{1} + v \cdot f) = u + vS(f) = p(S(f)),$$

while if $v < 0$, by (S6), (S7) and (S2) we have

$$\begin{aligned} S(u \cdot \mathbf{1} + v \cdot f) &= S(u \cdot \mathbf{1} - |v| f) \geq u - S(|v| f) \\ &= u - |v| S(f) = u + vS(f) = p(S(f)). \end{aligned}$$

Therefore, we have in either case

$$S(\phi \circ f) \geq \phi(S(f)) - q.$$

Since q is arbitrary, the proof is complete.

∎

Remark 37. *If $S = A$, a normalized isotonic linear functional on L, then* (2.151) *becomes the well-known Jessen's inequality.*

The following generalizations of Jessen's inequality for isotonic linear functionals also hold:

Corollary 22. *Let $A_1, ..., A_n : L \to \mathbb{R}$ be normalized isotonic linear functionals and $p_{i,j} \in \mathbb{R}$ be such that:*

$$p_{i,j} \geq 0 \text{ and } \sum_{i=1}^{n} p_{i,j} = 1 \text{ for all } i, j \in \{1, ..., n\}.$$

If $\phi : [\alpha, \beta] \to \mathbb{R}$ is convex and $f : E \to [\alpha, \beta]$ is such that $f, \phi \circ f \in L$ then:

$$\max_{1 \leq j \leq n} \left\{ \sum_{i=1}^{n} p_{i,j} A_i(\phi \circ f) \right\} \geq \phi\left(\max_{1 \leq j \leq n} \left\{ \sum_{i=1}^{n} p_{i,j} A_i(f) \right\} \right).$$

The proof follows by Theorem 62 applied for the following mapping

$$S(f) := \max_{1 \leq j \leq n} \left\{ \sum_{i=1}^{n} p_{i,j} A_i(f) \right\},$$

which is a totally normalized isotonic sublinear functional on L.

Remark 38. *If $A_1, ..., A_n, \phi$ and f are as above, then*

$$\max_{1 \leq j \leq n} \{A_i(\phi \circ f)\} \geq \phi\left(\max_{1 \leq j \leq n} \{A_i(f)\} \right)$$

and

$$\max_{1 \leq j \leq n} \left\{ \frac{1}{Q_j} \sum_{i=1}^{j} q_i A_i(\phi \circ f) \right\} \geq \phi\left(\max_{1 \leq j \leq n} \left\{ \frac{1}{Q_j} \sum_{i=1}^{j} q_i A_i(f) \right\} \right)$$

where $q_i \geq 0$ with $Q_j > 0$ for all $i, j \in \{1, ..., n\}$.

Corollary 23. *If $A_1,...,A_n,\phi$ and f are as shown, $p_i \geq 0, i \in \{1,...,n\}, P_n > 0$ and $A : L \to \mathbb{R}$ is also a normalized isotonic linear functional, then we have the inequality*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i \max\{A(\phi\circ f), A_i(\phi\circ f)\} \geq \phi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i \max\{A(f), A_i(f)\}\right).$$

The following *reverse of Jessen's inequality* for sublinear functionals was proved by S. S. Dragomir, C. E. M. Pearce and J. E. Pečarić in [**64**]:

Theorem 63. *Let $\phi : [\alpha,\beta] \subset \mathbb{R} \to \mathbb{R}$ be a convex function $(\alpha < \beta)$ and $f : E \to [\alpha,\beta]$ such that $\phi\circ f, f \in K$. Let $\lambda = sgn(\phi(\beta) - \phi(\alpha))$. Then, if S is a totally normalized isotonic sublinear functional on K we have*

$$S(\phi\circ f) \leq \frac{\beta\phi(\alpha) - \alpha\phi(\beta)}{\beta-\alpha} + \frac{|\phi(\beta)-\phi(\alpha)|}{\beta-\alpha} S(\lambda f). \tag{2.152}$$

Proof. Since ϕ is convex on $[\alpha,\beta]$ we have:

$$\phi(v) \leq \frac{w-v}{w-u}\phi(u) + \frac{v-u}{w-u}\phi(w),$$

where $u \leq v \leq w$ and $u < w$.
Set $u = \alpha, v = f(t), w = \beta$. Then

$$\phi(f(t)) \leq \frac{\beta - f(t)}{\beta-\alpha}\phi(\alpha) + \frac{f(t)-\alpha}{\beta-\alpha}\phi(\beta), t \in E,$$

or, alternatively,

$$\phi\circ f \leq \frac{\beta\phi(\alpha)-\alpha\phi(\beta)}{\beta-\alpha}\cdot\mathbf{1} + \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\cdot f.$$

Applying the functional S and using its properties we have

$$\begin{aligned} S(\phi\circ f) &\leq S\left(\frac{\beta\phi(\alpha)-\alpha\phi(\beta)}{\beta-\alpha}\cdot\mathbf{1} + \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\cdot f\right) \\ &= \frac{\beta\phi(\alpha)-\alpha\phi(\beta)}{\beta-\alpha} + S\left(\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\cdot f\right) \\ &= \frac{\beta\phi(\alpha)-\alpha\phi(\beta)}{\beta-\alpha} + \frac{|\phi(\beta)-\phi(\alpha)|}{\beta-\alpha}S(\lambda f). \end{aligned}$$

Hence, the theorem is proved. ∎

Remark 39. *If $S = A$, and A is a normalized isotonic linear functional, then, by* (2.152) *we deduce the inequality*

$$A(\phi(f)) \leq \frac{\{(\beta - A(f))\phi(\alpha) + (A(f)-\alpha)\phi(\beta)\}}{(\beta-\alpha)}.$$

Note that this last inequality is a generalization of the inequality

$$A(\phi) \leq \frac{\{(b - A(l_1))\phi(a) + (A(l_1)-a)\phi(b)\}}{(b-a)}$$

due to A. Lupaş. Here, $E = [a,b]$ $(-\infty < a < b < \infty)$, L satisfies (L1), (L2), $A : L \to \mathbb{R}$ satisfies (A1), (A2), $A(\mathbf{1}) = 1, \phi$ is convex on E and $\phi \in L, l_1 \in L$, where $l_1(x) = x, x \in [a,b]$.

By the use of Jessen's and Lupaş' inequalities for totally normalized sublinear functionals, we can state the following generalization of the classical Hermite-Hadamard's integral inequality due to S.S. Dragomir, C.E.M. Pearce and J.E. Pečarić [**64**].

THEOREM 64. *Let $\phi : [\alpha, \beta] \to \mathbb{R}$ be a convex function and $e : E \to [\alpha, \beta]$ a mapping such that $\phi \circ e$ and e belong to K and let $\lambda := sgn(\phi(\beta) - \phi(\alpha))$. If S is a totally normalized isotonic sublinear functional on K with*

$$S(\lambda e) = \lambda \cdot \frac{\alpha + \beta}{2} \text{ and } S(e) = \frac{\alpha + \beta}{2},$$

then we have the inequality

$$\phi\left(\frac{\alpha + \beta}{2}\right) \leq S(\phi \circ e) \leq \frac{\phi(\alpha) + \phi(\beta)}{2}. \tag{2.153}$$

PROOF. The first inequality in (2.153) follows by Jessen's inequality (2.151) applied to the mapping e.
By inequality (2.152), we have

$$\begin{aligned} S(\phi \circ e) &\leq \frac{\beta\phi(\alpha) - \alpha\phi(\beta)}{\beta - \alpha} + \frac{(\phi(\beta) - \phi(\alpha))(\beta + \alpha)}{2(\beta - \alpha)} \\ &= \frac{\phi(\alpha) + \phi(\beta)}{2}, \end{aligned}$$

and the statement is proved. ∎

REMARK 40. *If $S = A, \phi$ is as above and $e : E \to [\alpha, \beta]$ is such that $\phi \circ e, e \in L$ and $A(e) = \frac{\alpha+\beta}{2}$, then the Hermite-Hadamard inequality*

$$\phi\left(\frac{\alpha + \beta}{2}\right) \leq A(\phi \circ e) \leq \frac{\phi(\alpha) + \phi(\beta)}{2},$$

holds for normalized isotonic linear functionals (see also [**146**] *and* [**23**]*).*

REMARK 41. *If in the above theorem we assume that $\phi(\beta) \geq \phi(\alpha)$, then we can drop the assumption $S(\lambda e) = \lambda \cdot \frac{\alpha+\beta}{2}$.*

THEOREM 65. *Let ϕ, f and S be defined as in Theorem 63 with $\phi(\beta) \geq \phi(\alpha)$. Then*

$$S(\phi(f)) \leq \frac{\{(\beta - S(f))\phi(\alpha) + (S(f) - \alpha)\phi(\beta)\}}{\beta - \alpha}. \tag{2.154}$$

The proof is a simple consequence of Theorem 63.
Finally, we have the following result [**64**]:

THEOREM 66. *Let the hypothesis of Theorem 65 be fulfilled and let T be an interval which is such that $T \supset \phi([\alpha, \beta])$. If $F(u, v)$ is a real-valued function defined on $T \times T$ and increasing in u, then*

$$\begin{aligned} & F[S(\phi(f)), \phi(S(f))] \\ \leq & \max_{x \in [a,b]} F\left[\frac{\beta - x}{\beta - \alpha}\phi(\alpha) + \frac{x - \alpha}{\beta - \alpha}\phi(\beta), \phi(x)\right] \\ = & \max_{\theta \in [0,1]} F[\theta\phi(\alpha) + (1 - \theta)\phi(\beta), \phi(\theta\alpha + (1 - \theta)\beta)]. \end{aligned} \tag{2.155}$$

PROOF. By (2.154) and the increasing property of $F(\cdot, y)$ we have

$$\begin{aligned} F[S(\phi(f)), \phi(S(f))] &\leq F\left[\frac{\beta - S(f)}{\beta - \alpha}\phi(\alpha) + \frac{S(f) - \alpha}{\beta - \alpha}\phi(\beta, \phi(S(f)))\right] \\ &\leq \max_{x \in [a,b]} F\left[\frac{\beta - x}{\beta - \alpha}\phi(\alpha) + \frac{x - \alpha}{\beta - \alpha}\phi(\beta), \phi(x)\right]. \end{aligned}$$

Of course the equality in (2.155) follows immediately from the change of variable $\theta = \frac{\beta - x}{\beta - a}$, so that $x = \theta\alpha + (1 - \theta)\beta$ with $0 \leq \theta \leq 1$. ∎

11.2. Applications for Special Means.

(1) Suppose that $e \in K, p \geq 1, e^p \in K$ and S is as above. We can define the mean

$$L_p(S, e) := [S(e^p)]^{\frac{1}{p}}.$$

By the use of Theorem 64 we have the inequality

$$A(\alpha, \beta) \leq L_p(S, e) \leq [A(\alpha^p, \beta^p)]^{\frac{1}{p}},$$

provided that

$$S(e) = \frac{\alpha + \beta}{2}.$$

A particular case which generates in its turn the classical L_p-mean is where $S = A$, where A is a linear isotonic functional defined on K.

(2) Now, if $e \in K$ is such that $e^{-1} \in K$, we can define the mean as

$$L(S, e) := \left[S\left(e^{-1}\right)\right]^{-1}.$$

If we assume that $S(-e) = -\frac{\alpha+\beta}{2}$ and $S(e) = \frac{\alpha+\beta}{2}$, then, by Theorem 64 we have the inequality:

$$H(\alpha, \beta) \leq L(S, e) \leq A(\alpha, \beta).$$

A particular case which generalizes in its turn the classical logarithmic mean is where $S = A$, where A is as above.

(3) Finally, if we suppose that $e \in K$ is such that $\ln e \in K$, we can also define the mean

$$I(S, e) := \exp[-S(-\ln e)].$$

Now, if we assume that $S(-e) = -\frac{\alpha+\beta}{2}$ and $S(e) = \frac{\alpha+\beta}{2}$, then, by Theorem 64 we get the inequality:

$$G(\alpha, \beta) \leq I(S, e) \leq A(\alpha, \beta),$$

which generalizes the corresponding inequality for the identric mean.

CHAPTER 3

Some Functionals Associated with the $H.-H.$ Inequality

1. Two Difference Mappings

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the real interval I and $a, b \in \mathring{I}$ ($\mathring{I}$ is the interior of I) with $a < b$. We shall consider here the *difference* mappings L and P defined by [**48**]:

$$(L) \qquad L : [a, b] \to \mathbb{R}, \ L(t) := \frac{f(t) + f(a)}{2}(t - a) - \int_a^t f(s)\, ds$$

and

$$(P) \qquad P : [a, b] \to \mathbb{R}, \ P(t) := \int_a^t f(s)\, ds - (t - a) f\left(\frac{t + a}{2}\right).$$

We shall point out the main properties of these mappings, obtaining some refinements of the $H.-H.$ inequality.

The main properties of L are given in the following theorem [**48**]:

Theorem 67. *Let f, a, b be as above. Consider the mapping L defined in (L). Then: (i) L is non-negative, monotonically nondecreasing and convex on $[a, b]$; (ii) One has the following refinement of the Hermite-Hadamard inequality:*

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\, dx &\leq \frac{1}{b-a}\int_y^b f(s)\, ds + \left(\frac{y-a}{b-a}\right)\frac{f(a)+f(y)}{2} \\ &\leq \frac{f(a)+f(b)}{2}, \end{aligned} \tag{3.1}$$

for each $y \in [a, b]$.
(iii) We have the inequalities:

$$\begin{aligned} &\alpha \cdot \frac{f(t)+f(a)}{2}(t-a) + (1-\alpha)\cdot\frac{f(s)+f(a)}{2}(s-a) \\ &-\frac{f(\alpha t + (1-\alpha)s) + f(a)}{2}[\alpha t + (1-\alpha)s - a] \\ \geq\ &\alpha\int_a^t f(u)\, du + (1-\alpha)\int_a^s f(u)\, du - \int_a^{\alpha t + (1-\alpha)s} f(u)\, du, \end{aligned} \tag{3.2}$$

for every $t, s \in [a, b]$ and each $\alpha \in [0, 1]$.

Proof. (i) The fact that L is non-negative follows from the $H.-H.$ inequality.

In order to prove the monotonicity and the convexity of L, we shall show the following inequality

$$L(x) - L(y) \geq (x - y) L'_{+}(y) \tag{3.3}$$

holds, for all $x,y\in[a,b]$.

Let us suppose that $x>y$. Then we have:

$$L(x)-L(y)=\frac{f(x)+f(a)}{2}(x-a)-\frac{f(y)+f(a)}{2}(y-a)-\int_y^x f(s)\,ds. \tag{3.4}$$

By the $H.-H.$ inequality, we deduce

$$\begin{aligned}\frac{L(x)-L(y)}{x-y} &\geq \frac{(f(x)+f(a))(x-a)}{2(x-y)}\\ &\quad -\frac{(f(y)+f(a))(y-a)}{2(x-y)}-\frac{f(x)+f(y)}{2}.\end{aligned}$$

On the other hand, since f is convex, $f'_+(y)$ exists for all $y\in[a,b)$ and then a simple calculation yields

$$L'_+(y)=\frac{f'_+(y)(y-a)}{2}-\frac{f(y)-f(a)}{2},\ y\in[a,b). \tag{3.5}$$

Therefore, the relation (3.3) will be proved if we can demonstrate that

$$\begin{aligned}&A\\ :&=\frac{(f(x)+f(a))(x-a)}{x-y}-\frac{(f(y)+f(a))(y-a)}{x-y}-(f(x)+f(a))\\ \geq\ & f'_+(y)(y-a).\end{aligned} \tag{3.6}$$

A simple calculation shows that

$$A=\frac{(y-a)(f(x)-f(y))}{x-y},$$

and then the relation (3.6) is equivalent with

$$\frac{f(x)-f(y)}{x-y}\geq f'_+(y),$$

which holds by the convexity of f.

The fact that (3.3) holds also for $y>x$ goes likewise and we shall omit the details.

Consequently, the mapping L is convex on $[a,b]$.

Now, let $x>y$ ($x,\ y\in[a,b]$). Since L is convex on $[a,b]$ we have:

$$\frac{L(x)-L(y)}{x-y}\geq L'_+(y)=\frac{f'_+(y)(y-a)-(f(y)-f(a))}{2}\geq 0$$

as, by the convexity of f we have:

$$f(a)-f(y)\geq(a-y)f'_+(y)\ \text{ for all } y\in[a,b],$$

and so L is nondecreasing on $[a,b]$.

(ii) By the monotonicity property of L inequality we have:

$$0\leq L(y)\leq L(b)\ \text{ for all } y\in[a,b].$$

That is,

$$\frac{f(y)+f(a)}{2}(y-a)-\int_a^y f(s)\,ds\leq\frac{f(b)+f(a)}{2}(b-a)-\int_a^b f(s)\,ds,$$

which gives us

$$\int_a^b f(s)\,ds - \int_a^y f(s)\,ds \le \frac{f(b)+f(a)}{2}(b-a) - \frac{f(y)+f(a)}{2}(y-a).$$

Therefore,

$$\frac{1}{b-a}\int_y^b f(s)\,ds \le \frac{f(b)+f(a)}{2} - \frac{f(y)+f(a)}{2}\left(\frac{y-a}{b-a}\right),$$

i.e., the second inequality in (3.1).

By the $H.-H.$ inequality we have:

$$\begin{aligned} & \frac{1}{b-a}\int_y^b f(s)\,ds + \left(\frac{y-a}{b-a}\right)\frac{f(y)+f(a)}{2} \\ \ge\ & \frac{1}{b-a}\int_y^b f(s)\,ds + \frac{y-a}{b-a}\cdot\frac{1}{y-a}\int_a^y f(s)\,ds \\ =\ & \frac{1}{b-a}\left(\int_y^b f(s)\,ds + \int_a^y f(s)\,ds\right) = \frac{1}{b-a}\int_a^b f(s)\,ds \end{aligned}$$

for all $y \in (a,b]$, and the first inequality in (3.1) is also proved.

(iii) The inequality (3.2) follows by the convexity of L, i.e.,

$$L(\alpha t + (1-\alpha)s) \le \alpha L(t) + (1-\alpha)L(s)$$

for all $s,t \in [a,b]$ and $\alpha \in [0,1]$. We shall omit the details.

∎

Remark 42. *Since L is nondecreasing, then we have the bounds:*

$$\inf_{t\in[a,b]} L(t) = L(a) = 0$$

and

$$\sup_{t\in[a,b]} L(t) = L(b) = \frac{f(b)+f(a)}{2}(b-a) - \int_a^b f(s)\,ds \ge 0.$$

Remark 43. *If f is a monotonic nondecreasing function on $[a,b]$, then the mapping $\Phi(t) := \int_a^t f(u)\,du$ is convex on $[a,b]$. Consider the new mapping $\Psi : [a,b] \to \mathbb{R}$ given by*

$$\Psi(t) := \frac{f(t)+f(a)}{2}(t-a).$$

If f is assumed to be convex and nondecreasing, then Φ is also convex on $[a,b]$, and, by the inequality (3.2) *one has:*

$$\begin{aligned} & \alpha\Psi(t) + (1-\alpha)\Psi(s) - \Psi(\alpha t + (1-\alpha)s) \\ \ge\ & \alpha\Phi(t) + (1-\alpha)\Phi(s) - \Phi(\alpha t + (1-\alpha)s) \ge 0, \end{aligned} \tag{3.7}$$

for all $s,t \in [a,b]$ and $\alpha \in [0,1]$.

The main properties of P are given by the following theorem [**48**].

Theorem 68. *Let f be as in Theorem 67 and consider the mapping P defined in (P). Then*

(i) *P is non-negative and monotonically nondecreasing on $[a,b]$;*

(ii) *One has the inequality:*

$$0 \le P(t) \le L(t) \ \text{ for all } t \in [a,b]; \tag{3.8}$$

(*iii*) *We have the following refinement of the Hermite-Hadamard inequality:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\left[(b-a) f\left(\frac{a+b}{2}\right)-(y-a) f\left(\frac{a+y}{2}\right)\right] + \frac{1}{b-a} \int_a^y f(s)\, ds \leq \frac{1}{b-a} \int_a^b f(s)\, ds, \tag{3.9}$$

for all $y \in [a,b]$.

Proof. The proof is as follows.

(*i*) Clearly, by the $H. - H.$ inequality, P is non-negative on $[a,b]$.
Let $a \leq x < y \leq b$. Then we have:

$$P(y)-P(x)=\int_x^y f(s)\, ds-(y-a) f\left(\frac{y+a}{2}\right)+(x-a) f\left(\frac{x+a}{2}\right).$$

By the Hermite-Hadamard inequality one has

$$\int_x^y f(s)\, ds \geq (y-x) f\left(\frac{x+y}{2}\right).$$

Hence, by the above inequality, we can state

$$\begin{aligned} & P(y)-P(x) \\ \geq\ & (y-x) f\left(\frac{x+y}{2}\right)-(y-a) f\left(\frac{y+a}{2}\right)+(x-a) f\left(\frac{x+a}{2}\right). \end{aligned}$$

Now, using the convexity of f, we get

$$\begin{aligned} & \frac{y-x}{y-a} \cdot f\left(\frac{x+y}{2}\right)+\frac{x-a}{y-a} \cdot f\left(\frac{x+a}{2}\right) \\ \geq\ & f\left[\frac{(y-x)(x+y)}{2(y-a)}+\frac{(x-a)(x+a)}{2(y-a)}\right]=f\left(\frac{y+a}{2}\right), \end{aligned}$$

and then $P(y)-P(x) \geq 0$, which shows that P is nondecreasing on $[a,b]$.

(*ii*) By the Hermite-Hadamard inequality we have:

$$\frac{2}{t-a} \int_a^{\frac{a+t}{2}} f(s)\, ds \leq \frac{f\left(\frac{t+a}{2}\right)+f(a)}{2}$$

and

$$\frac{2}{t-a} \int_{\frac{a+t}{2}}^t f(s)\, ds \leq \frac{f\left(\frac{a+t}{2}\right)+f(t)}{2},$$

for all $a < t \leq b$. Summing these inequalities, we obtain

$$\frac{2}{t-a} \int_a^t f(s)\, ds \leq f\left(\frac{a+t}{2}\right)+\frac{f(a)+f(t)}{2},\ t \in [a,b],$$

which implies the inequality (3.8).

(*iii*) The first inequality in (3.9) is derived from the fact that

$$\int_a^y f(s)\, ds \geq (y-a) f\left(\frac{y+a}{2}\right) \text{ for all } y \in [a,b].$$

For the second inequality in (3.9), we use the fact that, by (i) :

$$0 \leq P(y) \leq P(b) \text{ for all } y \in [a,b].$$

That is:

$$\int_a^y f(s)\,ds - (y-a)\,f\left(\frac{y+a}{2}\right) \le \int_a^b f(s)\,ds - (b-a)\,f\left(\frac{a+b}{2}\right),$$

which is clearly equivalent with the second part of (3.9).

∎

REMARK 44. *With the above assumptions one has:*

$$\inf_{t\in[a,b]} P(t) = P(a) = 0$$

and

$$\sup_{t\in[a,b]} P(t) = P(b) = \int_a^b f(s)\,ds - (b-a)\,f\left(\frac{a+b}{2}\right) \ge 0.$$

REMARK 45. *The condition "f is convex on $[a,b]$" does not imply the convexity of P on $[a,b]$.*
Indeed, if $f(t)=\frac{1}{t}$, $t\in[1,6]$, then f is convex on $[1,6]$ and

$$P'(t) = \frac{(t-1)^2}{t(t+1)^2},\ P''(t) = \frac{8t^2-(t+1)^3}{t^2(t+1)^3}$$

and $P''(5)<0$, which shows that P is not convex on $[1,6]$.

The following proposition contains a sufficient condition for the convexity of P.

PROPOSITION 27. *Let f be twice differentiable on $\mathring{I}$ and suppose that f and f' are convex on $\mathring{I}$. Then P is also convex on $[a,b]$.*

PROOF. As

$$P'(t) = f(t) - f\left(\frac{t+a}{2}\right) - \left(\frac{t-a}{2}\right) f'\left(\frac{t+a}{2}\right)$$

and

$$P''(t) = f'(t) - f'\left(\frac{t+a}{2}\right) - \left(\frac{t-a}{4}\right) f''\left(\frac{t+a}{2}\right)$$

for all $t\in[a,b]$, then, from the convexity of f', we have that

$$f'(t) - f'\left(\frac{t+a}{2}\right) \ge \left(\frac{t-a}{2}\right) f''\left(\frac{t+a}{2}\right) \text{ for all } t\in[a,b],$$

which implies that

$$P''(t) \ge \left(\frac{t-a}{4}\right) f''\left(\frac{t+a}{2}\right) \ge 0 \text{ for all } t\in[a,b],$$

as f is convex.
Consequently, P is convex on $[a,b]$ and the statement is proved. ∎

1.1. Applications for Special Means. We define the mappings $\bar{L}_p : [a,b] \subset [0,\infty) \to \mathbb{R}$, given by

$$\bar{L}_p(t) := (t-a)\left[A(t^p,a^p) - L_p^p(t,a)\right],\ p \in (-\infty,0)\cup[1,\infty)\setminus\{-1\}$$

and

$$\begin{aligned} \bar{L}_{-1} &: [a,b]\subset[0,\infty)\to\mathbb{R}, \\ \bar{L}_{-1}(t) &:= \frac{(t-a)\left[L(t,a)-H(t,a)\right]}{L(t,a)\,H(t,a)} \end{aligned}$$

and

$$\begin{aligned} \bar{L}_{0} &: [a,b]\subset[0,\infty)\to\mathbb{R}, \\ \bar{L}_{0}(t) &:= \ln\left[\frac{I(t,a)}{G(t,a)}\right]^{(t-a)}, \end{aligned}$$

which come from (L) for the convex mappings $f(x) = x^p$, $f(x) = \frac{1}{x}$ and $f(x) = -\ln x$, respectively.

The following proposition holds:

Proposition 28. *Let $0 \le a < b$ and $p \in (-\infty,0)\cup[1,\infty)\setminus\{-1\}$. Then:*

(i) *$\bar{L}_p$ is non negative, monotonically nondecreasing and convex on $[a,b]$;*

(ii) *One has the following inequalities:*

$$L_p^p(a,b) \le \frac{b-y}{b-a}L^p(y,b) + \frac{y-a}{b-a}A(a^p,y^p) \le A(a^p,b^p)$$

for all $y \in [a,b]$.

The proof follows by Theorem 67, (i) and (ii), applied for the convex mapping $f : [a,b] \to [0,\infty)$, $f(x) = x^p$. We shall omit the details.

Using the same result we can state the next proposition.

Proposition 29. *Let $0 < a < b$. Then*

(i) *The mapping $\bar{L}_{-1}$ is nonnegative, monotonically nondecreasing and convex on $[a,b]$;*

(ii) *One has the inequality:*

$$L^{-1}(a,b) \le \frac{b-y}{b-a}L^{-1}(y,b) + \frac{y-a}{b-a}H^{-1}(a,y) \le H^{-1}(a,b)$$

for all $y \in [a,b]$.

The proof is obvious by Theorem 67 applied for $f : [a,b] \subset (0,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$.

Proposition 30. *Let $0 < a < b$. Then*

(i) *The mapping $\bar{L}_0$ is nonnegative, monotonically nondecreasing and convex on $[a,b]$;*

(ii) *One has the inequality:*

$$I(a,b) \ge [I(b,y)]^{\frac{b-y}{b-a}}\,[G(a,y)]^{\frac{y-a}{b-a}} \ge G(a,b)$$

The proof follows by Theorem 67 applied for the mapping $f : [a,b] \to \mathbb{R}$, $f(x) = -\ln x$.

Now we can also define the following mappings

$$\begin{aligned} P_p &: [a,b]\subset[0,\infty)\to\mathbb{R}, \\ P_p(t) &:= (t-a)\left[L_p^p(a,t) - A^p(a,t)\right], \end{aligned}$$

where $p \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$ and

$$\begin{aligned} P_{-1} &: [a,b] \subset [0,\infty) \to \mathbb{R}, \\ P_{-1}(t) &: = (t-a)\left[\frac{A(a,t) - L(a,t)}{A(a,t)\,L(a,t)}\right], \end{aligned}$$

and

$$\begin{aligned} P_0 &: [a,b] \subset [0,\infty) \to \mathbb{R}, \\ P_0(t) &: = \ln\left[\frac{A(a,t)}{I(a,t)}\right]^{(t-a)} \end{aligned}$$

respectively.

The following proposition holds.

Proposition 31. *Let* $0 \le a < b$ *and* $p \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$. *Then*

(*i*) P_p *is nonnegative and monotonically nondecreasing on* $[a,b]$;
(*ii*) *We have the inequality*

$$\begin{aligned} A^p(a,b) &\le \frac{1}{b-a}\left[(b-a)\,A^p(a,b) - (y-a)\,A^p(a,y)\right] + \frac{y-a}{b-a} L_p^p(a,y) \\ &\le L_p^p(a,b) \end{aligned}$$

for all $y \in [a,b]$
(*iii*) *If* $p \ge 2$, *then* P_p *is convex on* $[a,b]$.

The proof follows by Theorem 68 and Proposition 27 applied for the convex function $f : [a,b] \to \mathbb{R}$, $f(x) = x^p$.

Another result is embodied in the following proposition:

Proposition 32. *Let* $0 < a < b$. *Then:*

(*i*) *The mapping* P_{-1} *is nonnegative and monotonically nondecreasing on* $[a,b]$;
(*ii*) *We have the inequality*

$$A^{-1}(a,b) \le \frac{1}{b-a}\left[\frac{b-a}{A(a,b)} - \frac{y-a}{A(a,y)}\right] + \frac{y-a}{b-a} L^{-1}(a,y) \le L^{-1}(a,b)$$

for all $y \in [a,b]$.

The proof goes likewise for the function $f : [a,b] \to \mathbb{R}$, $f(x) = \frac{1}{x}$.
Finally, we have

Proposition 33. *Let* $0 < a < b$. *Then:*

(*i*) *The mapping* P_0 *is nonnegative and monotonically nondecreasing on* $[a,b]$;
(*ii*) *We have the inequality*

$$A(a,b) \ge \left\{\frac{[A(a,b)]^{b-a}}{[A(a,y)]^{y-a}}\right\}^{\frac{1}{b-a}} \times [I(a,b)]^{\frac{y-a}{b-a}} \ge I(a,b)$$

for all $y \in [a,b]$.

The proof follows by Theorem 68 applied for the convex mapping $f(x) = -\ln x$. We shall omit the details.

2. Properties of Superadditivity and Supermultiplicity

Suppose that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a function defined on the interval I, $a, b \in I$ with $a < b$. If $f \in L_1[a,b]$, then, we can consider the mappings

$$P(f;a,b) := \int_a^b f(x)\,dx - (b-a) f\left(\frac{a+b}{2}\right)$$

and

$$L(f;a,b) := \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)\,dx.$$

The following theorem contains some properties of superadditivity and monotonicity of the mappings P and L, [**44**]:

THEOREM 69. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on I. Then*

(*i*) *For all $a, b, c \in I$ with $a \le c \le b$ one has the inequalities:*

$$0 \le P(f;a,c) + P(f;c,b) \le P(f;a,b) \tag{3.10}$$

and

$$0 \le L(f;a,c) + L(f;c,b) \le L(f;a,b). \tag{3.11}$$

(*ii*) *For all $[c,d] \subseteq [a,b] \subseteq I$ we have:*

$$0 \le P(f;c,d) \le P(f;a,b) \tag{3.12}$$

and

$$0 \le L(f;c,d) \le L(f;a,b). \tag{3.13}$$

PROOF. (*i*) Let $a < b$ and $c \in [a,b]$. Put $\alpha := \frac{c-a}{b-a}$, $\beta = \frac{b-c}{b-a}$. Then $\alpha + \beta = 1$ $(\alpha, \beta \ge 0)$ and by the convexity of f written for $x = \frac{a+c}{2}$, $y = \frac{b+c}{2} \in I$, we have

$$\begin{aligned} & \frac{c-a}{b-a} f\left(\frac{a+c}{2}\right) + \frac{b-c}{b-a} f\left(\frac{b+c}{2}\right) \\ = \ & \alpha f(x) + \beta f(y) \\ \ge \ & f(\alpha x + \beta y) \\ = \ & f\left(\frac{c-a}{b-a}\cdot\frac{a+c}{2} + \frac{b-c}{b-a}\cdot\frac{b+c}{2}\right) \\ = \ & f\left(\frac{a+b}{2}\right). \end{aligned}$$

Now,

$$\begin{aligned} & P(f;a,b) - P(f;a,c) - P(f;c,b) \\ = \ & (c-a) f\left(\frac{a+c}{2}\right) + (b-c) f\left(\frac{b+c}{2}\right) - (b-a) f\left(\frac{a+b}{2}\right) \ge 0 \end{aligned}$$

and the statement (3.10) is proved.

Since f is convex on $[a,b]$, then for all $c \in [a,b]$ we have

$$\begin{vmatrix} a & f(a) & 1 \\ c & f(c) & 1 \\ b & f(b) & 1 \end{vmatrix} \ge 0$$

i.e.,

$$f(a)(b-c)+f(c)(a-b)+f(b)(c-a)\geq 0.$$

Therefore, we get

$$\begin{aligned} & L(f;a,b)-L(f;a,c)-L(f;c,b) \\ = & \frac{1}{2}\left[f(a)(b-c)+f(c)(a-b)+f(b)(c-a)\right]\geq 0, \end{aligned}$$

which shows that the inequality (3.11) holds.

(ii) Using the first part of Theorem 69, we have for $[c,d]\subseteq[a,b]$ that

$$P(f;a,b)\geq P(f;a,c)+P(f;c,b)\geq P(f;a,c)+P(f;c,d)+P(f;d,b)$$

which gives

$$P(f;a,b)-P(f;c,d)\geq P(f;a,c)+P(f;d,b)\geq 0$$

and the inequality (3.12) is proved.

The argument of (3.13) goes likewise and we shall omit the details.

∎

Now, suppose that $f:I\to(0,\infty)$ is logarithmically convex on I. We can define the following two mappings [**44**]:

$$\Phi(f;a,b):=\exp\left[\int_a^b\ln\left[\frac{f(x)}{f\left(\frac{a+b}{2}\right)}\right]dx\right]$$

and

$$\Psi(f;a,b):=\exp\left[\int_a^b\ln\left[\frac{\sqrt{f(a)f(b)}}{f(x)}\right]dx\right]$$

where $a,b\in I$ and $a<b$.

The following corollary is interesting:

Corollary 24. *Let $f:I\subseteq\mathbb{R}\to(0,\infty)$ be a logarithmically convex function. Then:*

(i) *For all $a,b,c\in I$ with $a\leq c\leq b$, one has the inequalities*

$$\Phi(f;a,b)\geq\Phi(f;a,c)\,\Phi(f;c,b)\geq 1$$

and

$$\Psi(f;a,b)\geq\Psi(f;a,c)\,\Psi(f;c,b)\geq 1.$$

(ii) *For all $a,b,c,d\in I$ with $[c,d]\subseteq[a,b]$, we have:*

$$\Phi(f;a,b)\geq\Phi(f;c,d)\geq 1 \text{ and } \Psi(f;a,b)\geq\Psi(f;c,d)\geq 1.$$

Proof. The argument goes by the above theorem on observing that:

$$P(\ln f;a,b):=\int_a^b\ln f(x)\,dx-(b-a)\ln f\left(\frac{a+b}{2}\right)=\int_a^b\ln\left[\frac{f(x)}{f\left(\frac{a+b}{2}\right)}\right]dx.$$

That is,

$$\Phi(f;a,b)=\exp\left[P(\ln f;a,b)\right]$$

and

$$\Psi(f;a,b)=\exp\left[L(\ln f;a,b)\right].$$

We shall omit the details. ∎

For an arbitrary function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ we can consider the following mapping

$$S(f;a,b) := (b-a)\left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right].$$

The following proposition holds [**44**]:

Proposition 34. *If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex on I, then*

(i) *For all $a \leq c \leq b, (a,b,c \in I)$ one has*

$$0 \leq S(f;a,c) + S(f,c,b) \leq S(f;a,b).$$

(ii) *For all $[c,d] \subseteq [a,b]$ we have:*

$$0 \leq S(f,c,d) \leq S(f;a,b).$$

The proof is obvious by Theorem 69 on observing that:

$$S(f;a,b) = P(f;a,b) + L(f;a,b).$$

Also, we can consider the functional;

$$R(f;a,b) := \left[\frac{\sqrt{f(a)f(b)}}{f\left(\frac{a+b}{2}\right)}\right]^{(b-a)}$$

for positive mappings $f : I \subseteq \mathbb{R} \to (0,\infty)$.

Taking into account that for this class of functions we have:

$$R(f;a,b) = \Phi(f;a,b)\,\Psi(f;a,b) \text{ with } a<b,$$

thus we can state the following corollary:

Corollary 25. *If $f : I \subseteq \mathbb{R} \to (0,\infty)$ is a logarithmically convex function on I, then*

(i) *For all $a,b,c \in I$ with $a \leq c \leq b$, one has*

$$R(f;a,b) \geq R(f;a,c) \cdot R(f;c,b) \geq 1;$$

(ii) *For all $[c,d] \subseteq [a,b] \subseteq I$ one has*

$$R(f;c,d) \leq R(f;a,b).$$

Now, if we assume that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping on I and $a,b \in I$ with $a<b$, we can also define the functionals as:

$$V(f;a,b) := \left[\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right]^{(b-a)}$$

and

$$W(f;a,b) := \left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right]^{(b-a)}.$$

For these functionals, we can state and prove the following theorem [**44**].

Theorem 70. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on I. Then for all $a,b,c \in I$ with $a<c<b$ we have:*

$$V(f;a,c) \cdot V(f;c,b) \leq V(f;a,b) \tag{3.14}$$

and

$$W(f;a,c) \cdot W(f;c,b) \leq W(f;a,b). \tag{3.15}$$

PROOF. We observe that for all $a<b$ we have:

$$V(f;a,b)=\left[\frac{P(f;a,b)}{(b-a)}\right]^{(b-a)}$$

and

$$W(f;a,b)=\left[\frac{L(f;a,b)}{(b-a)}\right]^{(b-a)}.$$

To prove the inequality (3.14), we use the inequality (3.10) as follows:

$$V(f;a,b)=\left[\frac{P(f;a,b)}{(b-a)}\right]^{(b-a)}\geq\left[\frac{P(f;a,c)+P(f;c,b)}{b-a}\right]^{(b-a)}. \tag{3.16}$$

As

$$P(f;a,b)=(b-a)\left[V(f;a,b)\right]^{\frac{1}{(b-a)}},$$

thus, by (3.16) we obtain

$$\begin{aligned} & V(f;a,b) \\ \geq\ & \left[\frac{(c-a)\left[V(f;a,c)\right]^{\frac{1}{(c-a)}}+(b-c)\left[V(f;c,b)\right]^{\frac{1}{(b-c)}}}{b-a}\right]^{(b-a)} \\ =\ & \left[\frac{(c-a)\left[V(f;a,c)\right]^{\frac{1}{(c-a)}}+(b-c)\left[V(f;c,b)\right]^{\frac{1}{(b-c)}}}{(c-a)+(b-c)}\right]^{(b-a)}. \end{aligned} \tag{3.17}$$

Using the well-known arithmetic mean-geometric mean inequality:

$$\frac{px+qy}{p+q}\geq x^{\frac{p}{p+q}}\cdot y^{\frac{q}{p+q}}$$

with $p=c-a>0,\ q=b-c>0$ and

$$x=\left[V(f;a,c)\right]^{\frac{1}{(c-a)}},\ y=\left[V(f;c,b)\right]^{\frac{1}{(b-c)}},$$

we deduce:

$$\begin{aligned} & \frac{(c-a)\left[V(f;a,c)\right]^{\frac{1}{(c-a)}}+(b-c)\left[V(f;c,b)\right]^{\frac{1}{(b-c)}}}{(c-a)+(b-c)} \\ \geq\ & \left[\left[V(f;a,c)\right]^{\frac{1}{(c-a)}}\right]^{\frac{c-a}{b-a}}\left[\left[V(f;c,b)\right]^{\frac{1}{(b-c)}}\right]^{\frac{b-c}{b-a}} \\ =\ & \left[V(f;a,c)\,V(f;c,b)\right]^{\frac{1}{b-a}}. \end{aligned} \tag{3.18}$$

Now, using (3.17) and (3.18) we get (3.14).
The proof of the inequality (3.15) goes likewise via (3.11) and we shall omit the details. ∎

In what follows, let us suppose that $f:I\subseteq\mathbb{R}\to(0,\infty)$ is logarithmically convex on I. We can define the mappings:

$$\tau(f;a,b):=(b-a)\ln\left[\frac{1}{b-a}\int_a^b\ln\left[\frac{f(x)}{f\left(\frac{a+b}{2}\right)}\right]dx\right]$$

and

$$\sigma(f;a,b):=(b-a)\ln\left[\frac{1}{b-a}\int_a^b\ln\left[\frac{\sqrt{f(a)f(b)}}{f(x)}\right]dx\right],$$

where $a,b\in I$ with $a<b$.

The following corollary holds:

COROLLARY 26. *Let f be as above. Then for all $a < c < b$ one has the inequalities:*

$$\tau(f;a,b) \geq \tau(f;a,c) + \tau(f;c,b) \tag{3.19}$$

and

$$\sigma(f;a,b) \geq \sigma(f;a,c) + \sigma(f;c,b). \tag{3.20}$$

The proof follows by the above theorem, taking into account that

$$\tau(f;a,b) = \ln[V(\ln f;a,b)]$$

and

$$\sigma(f;a,b) = \ln[W(\ln f;a,b)].$$

Also, for a convex function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ we can consider the functional

$$Z(f;a,b) := \left\{ \frac{1}{b-a} \left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right]\right\}^{b-a}.$$

The following proposition holds:

PROPOSITION 35. *With the above assumptions, we have for $a, b, c \in I$, with $a < c < b$ that:*

$$Z(f;a,c)\,Z(f;c,b) \leq Z(f;a,b).$$

It follows by Proposition 34 and we omit the details.

In addition, if we consider the functional

$$\theta(f;a,b) := (b-a)\ln\left[\ln\left[\frac{\sqrt{f(a)f(b)}}{f\left(\frac{a+b}{2}\right)}\right]^{b-a}\right],$$

then we have the corollary:

COROLLARY 27. *With the above assumptions, we have:*

$$\theta(f;a,b) \geq \theta(f;a,c) + \theta(f;c,b),$$

for all $a, b, c \in I$, where $a < c < b$.

2.1. Applications for Special Means. We shall start with the following proposition:

PROPOSITION 36. *Let $p \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$. Then*

(*i*) *For all $0 < a \leq c \leq b$ one has the inequalities:*

$$\begin{aligned} & (b-a)\left[L_p^p(a,b) - A^p(a,b)\right] \\ \geq\ & (c-a)\left[L_p^p(a,c) - A^p(a,c)\right] + (b-c)\left[L_p^p(c,b) - A^p(c,b)\right] \\ \geq\ & 0 \end{aligned}$$

and

$$\begin{aligned} & (b-a)\left[A(a^p,b^p) - L_p^p(a,b)\right] \\ \geq\ & (c-a)\left[A(a^p,c^p) - L_p^p(a,c)\right] + (b-c)\left[A(c^p,b^p) - L_p^p(c,b)\right]. \end{aligned}$$

(*ii*) *For all* $0 < a \leq c \leq d \leq b$ *one has the inequalities:*

$$(b-a)\left[L_p^p(a,b) - A^p(a,b)\right] \geq (d-c)\left[L_p^p(c,d) - A^p(c,d)\right] \geq 0$$

and

$$(b-a)\left[A(a^p,b^p) - L_p^p(a,b)\right] \geq (d-c)\left[A(c^p,d^p) - L_p^p(c,d)\right] \geq 0.$$

Proof. Consider the convex mapping $f:(0,\infty)\to\mathbb{R}$, $f(x) = x^p$, $p \in (-\infty,0)\cup[1,\infty)\setminus\{-1\}$. A simple calculation shows us that

$$P(f;a,b) = (b-a)\left[L_p^p(a,b) - A^p(a,b)\right]$$

and

$$L(f;a,b) = (b-a)\left[A(a^p,b^p) - L_p^p(a,b)\right].$$

Now, using Theorem 69, we can easily derive the above inequalities. ∎

Our next result which contains some new inequalities for logarithmic means, $L(a,b)$, is embodied in the following proposition:

Proposition 37. *We have:*

(*i*) *For all* $0 < a \leq c \leq b$ *one has the inequalities:*

$$\begin{aligned} & (b-a)\frac{A(a,b)-L(a,b)}{A(a,b)L(a,b)} \\ \geq\ & (c-a)\frac{A(a,c)-L(a,c)}{A(a,c)L(a,c)} + (b-c)\frac{A(b,c)-L(b,c)}{A(b,c)L(b,c)} \\ \geq\ & 0 \end{aligned}$$

and

$$\begin{aligned} & (b-a)\frac{L(a,b)-H(a,b)}{L(a,b)H(a,b)} \\ \geq\ & (c-a)\frac{L(a,c)-H(a,c)}{L(a,c)H(a,c)} + (b-c)\frac{L(b,c)-H(b,c)}{L(b,c)H(b,c)} \\ \geq\ & 0. \end{aligned}$$

(*ii*) *For all* $0 < a \leq c \leq d \leq b$ *one has the inequalities:*

$$(b-a)\frac{A(a,b)-L(a,b)}{A(a,b)L(a,b)} \geq (d-c)\frac{A(c,d)-L(c,d)}{A(c,d)L(c,d)} \geq 0$$

and

$$(b-a)\frac{L(a,b)-H(a,b)}{L(a,b)H(a,b)} \geq (d-c)\frac{L(c,d)-H(c,d)}{L(c,d)H(c,d)} \geq 0.$$

Proof. Consider the convex mapping

$$f:(0,\infty)\to\mathbb{R}, f(x) = \frac{1}{x}.$$

A simple calculation shows us that

$$P(f;a,b) = (b-a)\frac{A(a,b)-L(a,b)}{A(a,b)L(a,b)}$$

and

$$L(f;a,b) = (b-a)\frac{L(a,b)-H(a,b)}{L(a,b)H(a,b)}.$$

Now, using Theorem 69, we can easily derive the above inequalities. ∎

The following proposition contains some inequalities for identric means.

Proposition 38. *We have:*

(*i*) *For all* $0<a\leq c\leq b$ *one has the inequalities:*

$$\left[\frac{A(a,b)}{I(a,b)}\right]^{(b-a)}\geq\left[\frac{A(a,c)}{I(a,c)}\right]^{(c-a)}\left[\frac{A(b,c)}{I(b,c)}\right]^{(b-c)}$$

and

$$\left[\frac{I(a,b)}{G(a,b)}\right]^{(b-a)}\geq\left[\frac{I(a,c)}{G(a,c)}\right]^{(c-a)}\left[\frac{I(c,b)}{G(c,b)}\right]^{(b-c)}.$$

(*ii*) *If* $0<a\leq c\leq d\leq b$*, then:*

$$\left[\frac{A(a,b)}{I(a,b)}\right]^{(b-a)}\geq\left[\frac{A(c,d)}{I(c,d)}\right]^{(d-c)}$$

and

$$\left[\frac{I(a,b)}{G(a,b)}\right]^{(b-a)}\geq\left[\frac{I(c,d)}{G(c,d)}\right]^{(d-c)}.$$

Proof. Consider the convex function $f:(0,\infty)\to\mathbb{R}$, $f(x)=-\ln x$. A simple calculation shows us that

$$P(f;a,b)=\ln\left[\frac{A(a,b)}{I(a,b)}\right]^{(b-a)}$$

and

$$L(f;a,b)=\ln\left[\frac{I(a,b)}{G(a,b)}\right]^{(b-a)}.$$

Now, using Theorem 69, we can easily derive the desired inequality stated above. ∎

In what follows, we shall use Theorem 70 to point out some other inequalities for the special means considered above.

Proposition 39. *Let* $p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$.
For all $0<a\leq c\leq b$ *one has the inequalities:*

$$\begin{aligned}&\left[L_p^p(a,b)-A^p(a,b)\right]^{(b-a)}\\ \geq\ &\left[L_p^p(a,c)-A^p(a,c)\right]^{(c-a)}\left[L_p^p(c,b)-A^p(c,b)\right]^{(b-c)}\\ \geq\ &0\end{aligned}$$

and

$$\begin{aligned}&\left[A(a^p,b^p)-L_p^p(a,b)\right]^{(b-a)}\\ \geq\ &\left[A(a^p,c^p)-L_p^p(a,c)\right]^{(c-a)}\left[A(c^p,b^p)-L_p^p(c,b)\right]^{(b-c)}\\ \geq\ &0.\end{aligned}$$

Proof. Consider the convex mapping $f:(0,\infty)\to\mathbb{R}$, $f(x)=x^p$, $p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$. A simple calculation shows us that

$$V(f;a,b)=\left[L_p^p(a,b)-A^p(a,b)\right]^{(b-a)}$$

and

$$W(f;a,b)=\left[A(a^p,b^p)-L_p^p(a,b)\right]^{(b-a)}.$$

Now, using Theorem 70, we deduce the above inequalities. ∎

The following proposition holds:

PROPOSITION 40. *For all $0 < a \leq c \leq b$ one has the inequalities:*

$$\begin{aligned} & \left[\frac{A(a,b) - L(a,b)}{A(a,b)\, L(a,b)}\right]^{(b-a)} \\ \geq & \left[\frac{A(a,c) - L(a,c)}{A(a,c)\, L(a,c)}\right]^{(c-a)} \left[\frac{A(c,b) - L(c,b)}{A(c,b)\, L(c,b)}\right]^{(b-c)} \\ \geq & \ 0 \end{aligned}$$

and

$$\begin{aligned} & \left[\frac{L(a,b) - H(a,b)}{L(a,b)\, H(a,b)}\right]^{(b-a)} \\ \geq & \left[\frac{L(a,c) - H(a,c)}{L(a,c)\, H(a,c)}\right]^{(c-a)} \left[\frac{L(c,b) - H(c,b)}{L(c,b)\, H(c,b)}\right]^{(b-c)} \\ \geq & \ 0. \end{aligned}$$

PROOF. Consider the convex mapping $f : (0, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$. A simple calculation shows us that

$$V(f; a, b) = \left[\frac{A(a,b) - L(a,b)}{A(a,b)\, L(a,b)}\right]^{(b-a)}$$

and

$$W(f; a, b) = \left[\frac{L(a,b) - H(a,b)}{L(a,b)\, H(a,b)}\right]^{(b-a)}.$$

Now, using Theorem 70, we deduce the above inequalities. ∎

Finally, we can state that:

PROPOSITION 41. *For all $0 < a \leq c \leq b$ one has the inequalities:*

$$\left(\ln\left[\frac{A(a,b)}{I(a,b)}\right]\right)^{(b-a)} \geq \left(\ln\left[\frac{A(a,c)}{I(a,c)}\right]\right)^{(c-a)} \cdot \left(\ln\left[\frac{A(c,b)}{I(c,b)}\right]\right)^{(b-c)}$$

and

$$\left(\ln\left[\frac{I(a,b)}{G(a,b)}\right]\right)^{(b-a)} \geq \left(\ln\left[\frac{I(a,c)}{G(a,c)}\right]\right)^{(c-a)} \cdot \left(\ln\left[\frac{I(c,b)}{G(c,b)}\right]\right)^{(b-c)}.$$

The proof is obvious by Theorem 70 applied for the convex function $f(x) = -\ln x$, and we shall omit the details.

3. Properties of Some Mappings Defined By Integrals

3.1. Fundamental Properties. Now for a given convex mapping $f : [a, b] \to \mathbb{R}$, let $H : [0, 1] \to \mathbb{R}$ be defined by

$$H(t) := \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx.$$

The following theorem holds (see also [**22**], [**45**], [**30**] and [**58**]):

THEOREM 71. *With the above assumptions, we have:*

(*i*) *H is convex on $[0, 1]$;*

(*ii*) *One has the bounds:*

$$\inf_{t\in[0,1]} H(t) = H(0) = f\left(\frac{a+b}{2}\right)$$

and

$$\sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx;$$

(*iii*) *H increases monotonically on* $[0,1]$;
(*iv*) *The following inequalities*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{2}{b-a}\int_{\frac{(3a+b)}{4}}^{\frac{(a+3b)}{4}} f(x)\,dx \\ &\leq \int_0^1 H(t)\,dt \\ &\leq \frac{1}{2}\left(f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)\,dx\right) \end{aligned} \tag{3.21}$$

hold.

Proof. (*i*) It is obvious by the convexity of f (see also [**22**]).
(*ii*) We shall prove the following inequalities:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq H(t) \\ &\leq t\cdot\frac{1}{b-a}\int_a^b f(x)\,dx+(1-t)\cdot f\left(\frac{a+b}{2}\right) \\ &\leq \frac{1}{b-a}\int_a^b f(x)\,dx \end{aligned} \tag{3.22}$$

for all $t\in[0,1]$.
By Jensen's integral inequality [**114**, p. 45] we have that

$$\begin{aligned} H(t) &\geq f\left(\frac{1}{b-a}\int_a^b \left[tx+(1-t)\frac{a+b}{2}\right]dx\right) \\ &= f\left(\frac{a+b}{2}\right). \end{aligned}$$

Now, using the convexity of f, we obtain

$$\begin{aligned} H(t) &\leq \frac{1}{b-a}\int_a^b \left[tx+(1-t)\cdot f\left(\frac{a+b}{2}\right)\right]dx \\ &= t\cdot\frac{1}{b-a}\int_a^b f(x)\,dx+(1-t)\cdot f\left(\frac{a+b}{2}\right) \end{aligned}$$

and the second inequality in (3.22) is also proved.
The last inequality is obvious as the mapping

$$g(t) := t\cdot\frac{1}{b-a}\int_a^b f(x)\,dx+(1-t)\cdot f\left(\frac{a+b}{2}\right)$$

is monotonically increasing on $[0,1]$.

(*iii*) We shall give here a simpler proof following the paper [**22**] (see also [**45**]). As H is convex on $(0,1)$, we have, for t_1, $t_2 \in (0,1]$ with $t_2 > t_1$, that

$$\frac{H(t_2) - H(t_1)}{t_2 - t_1} \geq \frac{H(t_1) - H(0)}{t_1 - 0} = \frac{H(t_1) - f\left(\frac{a+b}{2}\right)}{t_1} \geq 0.$$

Consequently, $H(t_2) - H(t_1) \geq 0$ for $1 \geq t_2 \geq t_1 \geq 0$, and the statement is proved.

(*iv*) As H is convex on $[0,1]$, the Hermite-Hadamard inequalities yield that

$$\begin{aligned} & \frac{1}{b-a}\int_a^b f\left(\frac{2x+a+b}{4}\right)dx \\ = \; & H\left(\frac{1}{2}\right) \leq \int_0^1 H(t)\,dt \leq \frac{H(0)+H(1)}{2} \\ = \; & \frac{1}{2}\left(f\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b f(x)\,dx\right), \end{aligned}$$

and the inequality (3.21) is proved.

∎

Now, we shall introduce another mapping which is connected with H and the $H.-H.$ result.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Define the mapping $G : [0,1] \to \mathbb{R}$, given by

$$G(t) := \frac{1}{2}\left[f\left(ta + (1-t)\frac{a+b}{2}\right) + f\left((1-t)\frac{a+b}{2} + tb\right)\right].$$

The following theorem contains some properties of this mapping [**58**]:

THEOREM 72. *Let f and G be as above. Then*

(*i*) *G is convex and monotonically increasing on $[0,1]$;*

(*ii*) *We have the bounds:*

$$\inf_{t\in[0,1]} G(t) = G(0) = f\left(\frac{a+b}{2}\right)$$

and

$$\sup_{t\in[0,1]} G(t) = G(1) = \frac{f(a)+f(b)}{2};$$

(*iii*) *One has the inequality*

$$H(t) \leq G(t) \;\; \textit{for all } t \in [0,1];$$

(*iv*) *One has the inequalities*

$$\begin{aligned} \frac{2}{b-a}\int_{\frac{(3a+b)}{4}}^{\frac{(a+3b)}{4}} f(x)\,dx &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \\ &\leq \int_0^1 G(t)\,dt \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]. \end{aligned} \tag{3.23}$$

PROOF. (*i*) The convexity is obvious and we shall omit the details.
Now, since G is convex on $[0,1]$, we have for $1 \geq t_2 \geq t_1 > 0$ that

$$\frac{G(t_2)-G(t_1)}{t_2-t_1} \geq \frac{G(t_1)-G(0)}{t_1} = \frac{G(t_1)-f\left(\frac{a+b}{2}\right)}{t_1} \geq 0,$$

which shows us the monotonicity of G.

(*ii*) As f is convex on $[a,b]$, we have

$$G(t) \geq f\left[\frac{1}{2}\left(ta+(1-t)\frac{a+b}{2}+(1-t)\frac{a+b}{2}+tb\right)\right] = f\left(\frac{a+b}{2}\right)$$

which implies that the first bound in (*ii*) holds.
On the other hand, we also have:

$$\begin{aligned} G(t) &\leq \frac{1}{2}\left[tf(a)+(1-t)f\left(\frac{a+b}{2}\right)+(1-t)f\left(\frac{a+b}{2}\right)+tf(b)\right] \\ &= t\cdot\frac{f(a)+f(b)}{2}+(1-t)\cdot f\left(\frac{a+b}{2}\right) \end{aligned}$$

for all $t \in [0,1]$, which implies that

$$G(t) \leq G(1) = \frac{f(a)+f(b)}{2}, \ t \in [0,1]$$

and the second bound in (*ii*) is hence proved.

(*iii*) Let us consider the mapping $g:[a,b]\to\mathbb{R}$, $g(x):=f\left(tx+(1-t)\frac{a+b}{2}\right)$. Clearly, g is convex on $[a,b]$, and by Hadamard's inequality, one has

$$H(t) = \frac{1}{b-a}\int_a^b g(x)\,dx \leq \frac{g(a)+g(b)}{2} = G(t)$$

for all $t \in [0,1]$.

(*iv*) Since f is convex on $\left[\frac{(3a+b)}{4},\frac{(a+3b)}{4}\right]$, the $H.-H.$ inequality shows the first part of (3.23). The same inequality applied for the convex mapping G yields the second part of the required inequality and we shall omit the details.

∎

Now, we shall consider another mapping associated with the Hermite-Hadamard inequality given by $L:[0,1]\to\mathbb{R}$,

$$L(t) := \frac{1}{2(b-a)}\int_a^b [f(ta+(1-t)x)+f((1-t)x+tb)]\,dx$$

where $f: I\subseteq\mathbb{R}\to\mathbb{R}$ and $a,b\in I$ with $a<b$.

The following theorem also holds [**58**]:

THEOREM 73. *With the above assumptions one has:*

(*i*) *L is convex on* $[0,1]$;
(*ii*) *We have the inequalities:*

$$G(t) \leq L(t) \leq \frac{1-t}{b-a}\cdot\int_a^b f(x)\,dx + t\cdot\frac{f(a)+f(b)}{2} \leq \frac{f(a)+f(b)}{2} \tag{3.24}$$

for all $t \in [0,1]$ *and the bound:*

$$\sup_{t\in[0,1]} L(t) = \frac{f(a)+f(b)}{2}; \tag{3.25}$$

(*iii*) *One has the inequalities:*

$$H(1-t) \leq L(t) \text{ and } \frac{H(t)+H(1-t)}{2} \leq L(t)$$

for all $t \in [0,1]$.

PROOF. (*i*) Follows by the convexity of f.

(*ii*) By Jensen's integral inequality one has

$$\begin{aligned} & L(t) \\ \geq\ & \frac{1}{2}\left[f\left(\frac{1}{b-a}\int_a^b [(1-t)x+ta]\,dx\right)+f\left(\frac{1}{b-a}\int_a^b [(1-t)x+tb]\,dx\right)\right] \\ =\ & \frac{1}{2}\left[f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}\right)\right]=G(t). \end{aligned}$$

By the convexity of f one has:

$$\begin{aligned} L(t) &\leq \frac{1}{2(b-a)}\int_a^b [(1-t)f(x)+tf(a)+(1-t)f(x)+tf(b)]\,dx \\ &= \frac{1-t}{b-a}\cdot\int_a^b f(x)\,dx+t\cdot\frac{f(a)+f(b)}{2} \end{aligned}$$

for all $t \in [0,1]$.

The last part of (3.24) is obvious.

The bound (3.25) follows from (3.24).

(*iii*) By the convexity of f one has:

$$\begin{aligned} L(t) &\geq \frac{1}{b-a}\int_a^b f\left(\frac{ta+(1-t)x+(1-t)x+tb}{2}\right)dx \\ &= \frac{1}{b-a}\int_a^b f\left((1-t)x+t\cdot\frac{a+b}{2}\right)dx \\ &= H(1-t). \end{aligned}$$

For the second part one has:

$$L(t) \geq H(1-t) \text{ and } L(t) \geq G(t) \geq H(t),\ t \in [0,1]$$

and the theorem is proved.

∎

Now, we shall introduce a new mapping defined by a double integral in connection with the Hermite-Hadamard inequalities:

$$F:[0,1]\to\mathbb{R}, F(t) := \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy$$

The following theorem holds [**45**] (see also [**30**]):

THEOREM 74. *Let* $f:[a,b]\to\mathbb{R}$ *be as above. Then*

(*i*) $F\left(\tau+\frac{1}{2}\right)=F\left(\frac{1}{2}-\tau\right)$ *for all* $\tau\in\left[0,\frac{1}{2}\right]$ *and* $F(t)=F(1-t)$ *for all* $t\in[0,1]$;

(*ii*) *F is convex on* $[0,1]$;

(*iii*) *We have the bounds:*

$$\sup_{t\in[0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

and

$$\inf_{t\in[0,1]} F(t) = F\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy;$$

(*iv*) *The following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \le F\left(\frac{1}{2}\right)$$

(*v*) *F decreases monotonically on* $\left[0,\frac{1}{2}\right]$ *and increases monotonically on* $\left[\frac{1}{2},1\right]$;

(*vi*) *We have the inequality:*

$$H(t) \le F(t) \text{ for all } t\in[0,1].$$

Proof. (*i*) and (*ii*) are obvious by the definition of F and by the convexity of f.

(*iii*) For all x,y in $[a,b]$ and t in $[0,1]$ we have:

$$f(tx+(1-t)y) \le tf(x) + (1-t)f(y).$$

Integrating this inequality in $[a,b]\times[a,b]$ we get:

$$\begin{aligned}\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy &\le \int_a^b\int_a^b [tf(x)+(1-t)f(y)]\,dxdy \\ &= (b-a)\int_a^b f(x)\,dx,\end{aligned}$$

which shows that $F(t)\le F(0)=F(1)$ for all $t\in[0,1]$.

Since f is convex on $[a,b]$ for all $t\in[0,1]$ and $x,y\in[a,b]$ we have:

$$\frac{1}{2}\left[f(tx+(1-t)y)+f(ty+(1-t)x)\right] \ge f\left(\frac{x+y}{2}\right).$$

Integrating this inequality in $[a,b]\times[a,b]$, we deduce

$$\begin{aligned}&\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy \\ \le\ & \frac{1}{2}\int_a^b\int_a^b \left[f(tx+(1-t)y)+f(ty+(1-t)x)\right]dxdy \\ =\ & \int_a^b\int_a^b f(tx+(1-t)y)\,dxdy\end{aligned}$$

which implies that $F\left(\frac{1}{2}\right)\le F(t)$ for all $t\in[0,1]$, and the statement is thus proved.

(*iv*) Using Jensen's inequality for double integrals, we have

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy \ge f\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b\left(\frac{x+y}{2}\right)dxdy\right).$$

As a simple computation shows that

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b\left(\frac{x+y}{2}\right)dxdy=\frac{a+b}{2},$$

the proof of the statement is thus completed.

(v) Since the function F is convex on $[0,1]$, we have for $1\geq t_2>t_1>\frac{1}{2}$ that

$$\begin{aligned}\frac{F(t_2)-F(t_1)}{t_2-t_1} &\geq \frac{F(t_1)-F\left(\frac{1}{2}\right)}{t_1-\frac{1}{2}}\\ &= \frac{F(t_1)-\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy}{t_1-\frac{1}{2}}\geq 0,\end{aligned}$$

which shows that F increases monotonically on $\left(\frac{1}{2},1\right]$.

The fact that F decreases monotonically on $\left[0,\frac{1}{2}\right)$ follows from the above conclusion using statement (i).

(vi) A simple computation shows that

$$H(t)=\frac{1}{b-a}\int_a^b f\left(\int_a^b (tx+(1-t)y)\,dy\right)dx.$$

Using Jensen's integral inequality, we derive

$$H(t)\leq\frac{1}{b-a}\int_a^b\left(\frac{1}{b-a}\int_a^b f(tx+(1-t)y)\,dy\right)dx=F(t)$$

for all $t\in[0,1]$, and the proof of the theorem is thus completed.

∎

In what follows, we shall point out some reverse inequalities for the mappings H, G, L and F considered above.
We shall start with the following result (see [**140**]).

Theorem 75. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex mapping on I and $a,b\in\mathring{I}$ with $a<b$. Then we have the inequality:*

$$\begin{aligned}0 &\leq \frac{1}{b-a}\int_a^b f(x)\,dx-H(t)\\ &\leq (1-t)\left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right]\end{aligned}\tag{3.26}$$

for all $t\in[0,1]$.

Proof. Taking into account that the class of differentiable convex mappings on (a,b) is dense in uniform topology in the class of all convex functions defined on (a,b), we can assume, without loss of generality, that f is differentiable on (a,b). Thus, we can write the inequality:

$$f(x)-f(y)\geq(x-y)f'(y)\ \text{ for all } x,y\in(a,b).$$

This implies that:

$$f\left(tx+(1-t)\cdot\frac{a+b}{2}\right)-f(x)\geq(1-t)\left(\frac{a+b}{2}-x\right)f'(x)$$

for all $x\in(a,b)$ and $t\in[0,1]$. Integrating this inequality over x on $[a,b]$, we obtain

$$H(t)-\frac{1}{b-a}\int_a^b f(x)\,dx\geq\frac{1-t}{b-a}\int_a^b\left(\frac{a+b}{2}-x\right)f'(x)\,dx$$

for all $t\in[0,1]$.
As a simple computation shows that

$$\int_a^b\left(\frac{a+b}{2}-x\right)f'(x)\,dx=\int_a^b f(x)\,dx-(b-a)\frac{f(a)+f(b)}{2},$$

the above inequality gives us the desired result (3.26). ∎

Corollary 28. *With the above assumptions, one has*

$$\begin{aligned}0&\leq\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{2}{b-a}\int_{\frac{(3a+b)}{4}}^{\frac{(a+3b)}{4}}f(x)\,dx\\&\leq\frac{1}{2}\left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right].\end{aligned}$$

Remark 46. *If in* (3.26) *we choose* $t=0$, *we obtain*

$$0\leq\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\leq\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx,$$

which is the well-known Bullen result [**147**, p. 140].

Another theorem of this type in which the mapping G defined above is involved, is the following one [**25**]:

Theorem 76. *Let* $f:I\subseteq\mathbb{R}\to\mathbb{R}$ *be a convex mapping on* I *and* $a,b\in\mathring{I}$ *with* $a<b$. *Then we have the inequality:*

$$0\leq H(t)-f\left(\frac{a+b}{2}\right)\leq G(t)-H(t)\tag{3.27}$$

for all $t\in[0,1]$.

Proof. As above, it is sufficient to prove the above inequality for differentiable convex functions.
By the convexity of f, we have that

$$f\left(\frac{a+b}{2}\right)-f\left(tx+(1-t)\frac{a+b}{2}\right)\geq t\left(\frac{a+b}{2}-x\right)f'\left(tx+(1-t)\frac{a+b}{2}\right)$$

for all x in (a,b) and $t\in[0,1]$.
Integrating this inequality over x on $[a,b]$ one gets

$$f\left(\frac{a+b}{2}\right)-H(t)\geq\frac{t}{b-a}\int_a^b\left(\frac{a+b}{2}-x\right)f'\left(tx+(1-t)\frac{a+b}{2}\right)dx.$$

As a simple calculation (an integration by parts) yields that

$$\frac{t}{b-a}\int_a^b\left(\frac{a+b}{2}-x\right)f'\left(tx+(1-t)\frac{a+b}{2}\right)dx=H(t)-G(t),\ t\in[0,1],$$

then, the above inequality gives us the desired result (3.27). ∎

Remark 47. *If in the above inequality we choose* $t=1$, *we also recapture Bullen's result* [**147**, p. 140].

Now, we shall investigate the case of the mapping F defined by the use of double integrals ([**140**])

Theorem 77. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping on I and $a, b \in \mathring{I}$ with $a < b$. Then we have the inequality:*

$$\begin{aligned} 0 &\leq \frac{1}{b-a}\int_a^b f(x)\,dx - F(t) \\ &\leq \min\{t, 1-t\}\left(\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right) \end{aligned} \tag{3.28}$$

for all $t \in [0,1]$.

Proof. As above, it is sufficient to prove the above inequality for differentiable convex functions.
Thus, for all $x, y \in (a,b)$ and $f \in [0,1]$ we have:

$$f(tx + (1-t)y) - f(y) \geq t(x-y) f'(y).$$

Integrating this inequality on $[a,b]^2$ over x and y, we obtain

$$\begin{aligned} &\int_a^b \int_a^b f(tx + (1-t)y)\,dxdy - (b-a)\int_a^b f(x)\,dx \\ \geq\ & t\int_a^b \int_a^b (x-y) f'(y)\,dxdy \end{aligned}$$

for all $t \in [0,1]$.
As a simple computation shows that

$$\int_a^b \int_a^b (x-y) f'(y)\,dxdy = (b-a)\int_a^b f(x)\,dx - (b-a)^2 \cdot \frac{f(a)+f(b)}{2},$$

the above inequality gives us that

$$\frac{1}{b-a}\int_a^b f(x)\,dx - F(t) \leq t\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right]$$

for all $t \in [0,1]$.
As $F(t) = F(1-t)$ for all $t \in [0,1]$, if we replace in the above inequality t with $1-t$ we get the desired result (3.28). ∎

Corollary 29. *With the above assumptions, one has:*

$$\begin{aligned} 0 &\leq \frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{(b-a)^2}\int_a^b \int_a^b f\left(\frac{x+y}{2}\right)dxdy \\ &\leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right]. \end{aligned}$$

Now, let us define the mapping $J : [0,1] \to \mathbb{R}$, $J(t) := L(1-t)$, i.e.,

$$J(t) = \frac{1}{2(b-a)}\int_a^b [f(tx + (1-t)a) + f(tx + (1-t)b)]\,dx,$$

where $t \in [0,1]$.
We have the following result [**25**]:

THEOREM 78. *Let f and $a,b\in\mathring{I}$ be as above. Then we have the inequality:*

$$0\leq F(t)-H(t)\leq J(t)-F(t) \tag{3.29}$$

for all $t\in[0,1]$.

PROOF. As above, it is sufficient to prove the above inequality for differentiable convex functions.
By the convexity of f on $[a,b]$ we have that

$$\begin{aligned} & f\left(tx+(1-t)\frac{a+b}{2}\right)-f(tx+(1-t)y) \\ \geq\ & (1-t)f'(tx+(1-t)y)\left(\frac{a+b}{2}-y\right) \end{aligned}$$

for all $x,y\in(a,b)$ and $t\in[0,1]$.
If we integrate over x and y on $[a,b]^2$, we get that:

$$\begin{aligned} & \int_a^b\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dxdy-\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy \\ \geq\ & (1-t)\int_a^b\int_a^b f'(tx+(1-t)y)\left(\frac{a+b}{2}-y\right)dxdy \end{aligned}$$

which gives us that:

$$\begin{aligned} 0 &\leq F(t)-H(t) \\ &\leq \frac{1-t}{(b-a)^2}\int_a^b\int_a^b f'(tx+(1-t)y)\left(y-\frac{a+b}{2}\right)dxdy=:A(t) \end{aligned}$$

for all $t\in[0,1]$.
Define

$$I_1(t):=\frac{1-t}{(b-a)^2}\int_a^b\int_a^b f'(tx+(1-t)y)\,ydxdy$$

and

$$I_2(t):=\frac{1-t}{(b-a)^2}\int_a^b\int_a^b f'(tx+(1-t)y)\,dxdy.$$

Note that, for $t=1$, the inequality (3.29) is obvious. Assume that $t\in[0,1)$. Integrating by parts, we get that:

$$\begin{aligned} & \int_a^b f'(tx+(1-t)y)\,ydy \\ =\ & \frac{f((1-t)b+tx)b-f((1-t)a+tx)a}{1-t}-\frac{1}{1-t}\int_a^b f((1-t)y+tx)\,dy. \end{aligned}$$

Thus, we deduce that

$$I_1(t)=\frac{b\int_a^b f(tx+(1-t)b)\,dx-a\int_a^b f(tx+(1-t)a)\,dx}{(b-a)^2}-F(t).$$

We also have

$$\int_a^b f'(tx+(1-t)y)\,dy=\frac{f(tx+(1-t)b)-f(tx+(1-t)a)}{1-t},$$

and thus

$$I_2(t)=\frac{\int_a^b f(tx+(1-t)b)\,dx-\int_a^b f(tx+(1-t)a)\,dx}{(b-a)^2}.$$

Now, we get that

$$\begin{aligned}A(t) &= \frac{b\int_a^b f(tx+(1-t)b)\,dx-a\int_a^b f(tx+(1-t)a)\,dx}{(b-a)^2}-F(t)\\ &\quad -\frac{a+b}{2}\cdot\frac{\int_a^b f(tx+(1-t)b)\,dx-\int_a^b f(tx+(1-t)a)\,dx}{(b-a)^2}\\ &= \frac{\frac{b-a}{2}\int_a^b f(tx+(1-t)b)\,dx+\frac{b-a}{2}\int_a^b f(tx+(1-t)a)\,dx}{(b-a)^2}-F(t)\\ &= J(t)-F(t)\end{aligned}$$

and the theorem is proved. ∎

Corollary 30. *With the above assumptions, we have:*

$$0\le F(t)-\frac{H(t)+H(1-t)}{2}\le\frac{L(t)+L(1-t)}{2}-F(t)$$

for all $t\in[0,1]$.

Finally, the following theorem holds [**25**].

Theorem 79. *Let* $f:I\subseteq\mathbb{R}\to\mathbb{R}$ *be a convex function on* I *and* $a,b\in\mathring{I}$ *with* $a<b$. *Then one has the inequality*

$$\begin{aligned}0 &\le F(t)-F\left(\frac{1}{2}\right)\\ &\le \frac{1}{2t(1-t)}\left[(1-2t)^2F(t)-\frac{1-2t}{b-a}\cdot\int_{(1-t)a+tb}^{ta+(1-t)b}f(x)\,dx\right]\end{aligned}\tag{3.30}$$

for all $t\in(0,1)$.

Proof. As above, we can prove the inequality (3.30) only for the case where f is a differentiable convex function.
By the convexity of f we have that:

$$\begin{aligned}&f\left(\frac{x+y}{2}\right)-f(tx+(1-t)y)\\ &\ge \left[\frac{x+y}{2}-(tx+(1-t)y)\right]f'(tx+(1-t)y)\\ &= \frac{1-2t}{2}(x-y)f'(tx+(1-t)y)\end{aligned}\tag{3.31}$$

for all $x,y\in(a,b)$ and $t\in[0,1]$.
If we integrate the inequality (3.31) over x,y on $[a,b]^2$ we can deduce

$$F\left(\frac{1}{2}\right)-F(t)\ge\frac{1-2t}{2}\cdot\frac{1}{(b-a)^2}\int_a^b\int_a^b(x-y)f'(tx+(1-t)y)\,dxdy.$$

Denote

$$\begin{aligned} I(t) &:= \frac{1}{(b-a)^2}\int_a^b\int_a^b (x-y)\,f'(tx+(1-t)y)\,dxdy, \\ I_1(t) &:= \frac{1}{(b-a)^2}\int_a^b\int_a^b x f'(tx+(1-t)y)\,dxdy \end{aligned}$$

and

$$I_2(t) := \frac{1}{(b-a)^2}\int_a^b\int_a^b y f'(tx+(1-t)y)\,dxdy.$$

Then we have $I(t) = I_1(t) - I_2(t)$ for all $t\in[0,1]$.
An integration by parts gives us that

$$\int_a^b x f'(tx+(1-t)y)\,dx = \left.\frac{f(tx+(1-t)y)}{t}\right|_a^b - \frac{1}{t}\int_a^b f(tx+(1-t)y)\,dx$$

then

$$\begin{aligned} & I_1(t) \\ = {} & \frac{1}{(b-a)^2}\int_a^b\left[\frac{f(tb+(1-t)y)\,b - f(ta+(1-t)y)\,a}{t}\right. \\ & \left.-\frac{1}{t}\int_a^b f(tx+(1-t)y)\,dx\right]dy \\ = {} & \frac{1}{(b-a)^2}\cdot\frac{1}{t}\left[b\int_a^b f(tb+(1-t)y)\,dy - a\int_a^b f(ta+(1-t)y)\,dy\right] - \frac{1}{t}F(t). \end{aligned}$$

Also, by an integration by parts, we have:

$$\int_a^b y f'(tx+(1-t)y)\,dy = \left.\frac{f(tx+(1-t)y)\,y}{1-t}\right|_a^b - \frac{1}{1-t}\int_a^b f(tx+(1-t)y)\,dy$$

then we obtain:

$$\begin{aligned} I_2(t) &= \frac{1}{(b-a)^2}\int_a^b\left[\frac{f(tx+(1-t)b)\,b - f(tx+(1-t)a)\,a}{1-t}\right. \\ &\quad \left.-\frac{1}{1-t}\int_a^b f(tx+(1-t)y)\,dy\right]dx \\ &= \frac{1}{(b-a)^2}\cdot\frac{1}{1-t}\left[b\int_a^b f(tx+(1-t)b)\,dx - a\int_a^b f(tx+(1-t)a)\,dx\right] \\ &\quad -\frac{1}{1-t}F(t). \end{aligned}$$

Thus, we have

$$\begin{aligned}
& I(t) \\
= \; & \frac{1}{t(b-a)^2} \cdot \left[b \int_a^b f(tb + (1-t)y)\,dy - a \int_a^b f(ta + (1-t)y)\,dy \right] - \frac{1}{t} F(t) \\
& - \frac{1}{(1-t)(b-a)^2} \cdot \left[b \int_a^b f(tx + (1-t)b)\,dx - a \int_a^b f(tx + (1-t)a)\,dx \right] \\
& + \frac{1}{1-t} F(t) \\
= \; & \frac{2t-1}{t(1-t)} \cdot F(t) + \frac{1}{t(1-t)(b-a)^2} \cdot V(t),
\end{aligned}$$

where

$$\begin{aligned}
& V(t) \\
= \; & (1-t)\,b \int_a^b f(tb + (1-t)y)\,dy - (1-t)\,a \int_a^b f(ta + (1-t)y)\,dy \\
& - tb \int_a^b f(tx + (1-t)b)\,dx + ta \int_a^b f(tx + (1-t)a)\,dx \\
= \; & b \int_{(1-t)a+tb}^{b} f(u)\,du - a \int_a^{(1-t)b+ta} f(u)\,du \\
& - b \int_{ta+(1-t)b}^{b} f(u)\,du + a \int_a^{tb+(1-t)a} f(u)\,du \\
= \; & b \int_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,du - a \int_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,du \\
= \; & (b-a) \int_{(1-t)a+tb}^{ta+(1-t)b} f(x)\,dx.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
& F\left(\frac{1}{2}\right) - F(t) \\
\geq \; & \frac{1-2t}{2} \cdot I(t) \\
= \; & \frac{1-2t}{2} \left[\frac{2t-1}{t(1-t)} \cdot F(t) + \frac{1}{(b-a)\,t(1-t)} \int_{(1-t)a+tb}^{ta+(1-t)b} f(x)\,dx \right] \\
= \; & \frac{1-2t}{2(b-a)\,t(1-t)} \int_{(1-t)a+tb}^{ta+(1-t)b} f(x)\,dx - \frac{(2t-1)^2}{2t(1-t)} F(t)
\end{aligned}$$

for all $t \in (0,1)$, which is equivalent with the desired inequality (3.30). ∎

3.2. Applications for Special Means. 1. Let us consider the convex mapping $f : (0,\infty) \to \mathbb{R},\ f(x) = x^p,\ p \in (-\infty, 0) \cup [1,\infty) \setminus \{-1\}$ and $0 < a < b$. Define the mapping

$$H_p(t) := \frac{1}{b-a} \int_a^b (tx + (1-t)\,A(a,b))^p\,dx,\ t \in [0,1].$$

It is obvious that $H_p(0) = A^p(a,b)$, $H_p(1) = L_p^p(a,b)$ and for $t \in (0,1)$

$$\begin{aligned} H_p(t) &= \frac{1}{[tb+(1-t)A(a,b)]-[ta+(1-t)A(a,b)]} \int_{ta+(1-t)A(a,b)}^{tb+(1-t)A(a,b)} y^p dy \\ &= L_p^p(ta+(1-t)A(a,b), tb+(1-t)A(a,b)). \end{aligned}$$

The following proposition holds, via Theorem 71, applied for the convex function $f(x) = x^p$.

Proposition 42. *With the above assumptions, we have*

(i) H_p *is convex on* $[0,1]$;

(ii) *One has the bounds:*

$$\begin{aligned} \inf_{t\in[0,1]} H_p(t) &= A^p(a,b), \\ \sup_{t\in[0,1]} H_p(t) &= L_p^p(a,b); \end{aligned}$$

(iii) H_p *increases monotonically on* $[0,1]$;

(iv) *The following inequalities hold*

$$\begin{aligned} A^p(a,b) &\le L_p^p(A(a,A(a,b)), A(b,A(a,b))) \\ &\le \int_0^1 H_p(t)\,dt \le A\left(A^p(a,b), L_p^p(a,b)\right). \end{aligned}$$

Now, define the mapping $G_p : [0,1] \to \mathbb{R}$,

$$G_p(t) = A\left((ta+(1-t)A(a,b))^p, (tb+(1-t)A(a,b))^p\right).$$

Using Theorem 72 we can state the following proposition:

Proposition 43. *With the above assumptions, we have:*

(i) G_p *is convex and monotonically increasing on* $[0,1]$.

(ii) *We have the bounds:*

$$\begin{aligned} \inf_{t\in[0,1]} G_p(t) &= A^p(a,b), \\ \sup_{t\in[0,1]} G_p(t) &= A(a^p, b^p); \end{aligned}$$

(iii) *One has the inequality*

$$H_p(t) \le G_p(t) \text{ for all } t \in [0,1];$$

(iv) *We have the inequalities*

$$\begin{aligned} & L_p^p(A(a,A(a,b)), A(b,A(a,b))) \\ \le\ & A(A^p(a,A(a,b)), A^p(b,A(a,b))) \\ \le\ & \int_0^1 G_p(t)\,dt \le A(A^p(a,b), A(a^p,b^p)). \end{aligned}$$

Now, we shall consider another mapping $L_p : [0,1] \to \mathbb{R}$

$$L_p(t) = \frac{1}{2(b-a)} \int_a^b \left[(ta+(1-t)x)^p + ((1-t)x+tb)^p\right] dx.$$

It is obvious that:

$$L_p(0) = \frac{1}{2}\left[\frac{1}{b-a}\int_a^b x^p dx + \frac{1}{b-a}\int_a^b x^p dx\right] = L_p^p(a,b)$$

and

$$L_p(1) = A(a^p, b^p).$$

Also, it is clear (by a change of variable) that

$$\begin{aligned} & \frac{1}{b-a}\int_a^b (ta + (1-t)x)^p \, dx \\ = & \frac{1}{ta + (1-t)b - a}\int_a^{ta+(1-t)b} y^p dy \\ = & L_p^p(a, ta + (1-t)b), \ t \in [0,1) \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{b-a}\int_a^b ((1-t)x + tb)^p \, dx \\ = & \frac{1}{b - [(1-t)a + tb]}\int_{(1-t)a+tb}^{b} y^p dy \\ = & L_p^p((1-t)a + tb, b), \ t \in [0,1). \end{aligned}$$

Consequently, for $t \in [0,1)$, we have that:

$$L_p(t) = A\left(L_p^p(a, ta + (1-t)b), L_p^p((1-t)a + tb, b)\right).$$

If we now use Theorem 73, we can state the following proposition containing the properties of L_p.

PROPOSITION 44. *With the above assumptions, we have that:*

(i) *L_p is convex on $[0,1]$;*

(ii) *We have the inequalities:*

$$G_p(t) \le L_p(t) \le (1-t)L_p^p(a,b) + tA(a^p, b^p) \le A(a^p, b^p)$$

for all $t \in [0,1]$, and we have the bound:

$$\sup_{t\in[0,1]} L_p(t) = A(a^p, b^p);$$

(iii) *One has the inequalities:*

$$\begin{aligned} H_p(1-t) &\le L_p(t) \ \text{ and} \\ A(H_p(t), H_p(1-t)) &\le L_p(t) \end{aligned}$$

for all $t \in [0,1]$.

Finally, using Theorem 75 and Theorem 76, we can state the following proposition.

PROPOSITION 45. *With the above assumptions we have the inequalities:*

$$\begin{aligned} 0 &\le L_p^p(a,b) - H_p(t) \\ &\le (1-t)\left[A(a^p, b^p) - L_p^p(a,b)\right] \end{aligned}$$

and

$$0 \leq H_p(t) - A^p(a,b) \leq G_p(t) - H_p(t)$$

for all $t \in [0,1]$.

2. Let us consider the convex mapping $f : (0,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$. Define the mapping

$$H_{-1}(t) := \frac{1}{b-a} \int_a^b \frac{dx}{tx + (1-t) A(a,b)}, \; t \in [0,1].$$

It is obvious that

$$H_{-1}(0) = A^{-1}(a,b), \; H_{-1}(1) = L^{-1}(a,b) = \frac{\ln b - \ln a}{b-a},$$

and for $t \in (0,1)$

$$\begin{aligned} H_{-1}(t) &= \frac{1}{[tb + (1-t) A(a,b)] - [ta + (1-t) A(a,b)]} \int_{ta+(1-t)A(a,b)}^{tb+(1-t)A(a,b)} \frac{dy}{y} \\ &= L^{-1}(ta + (1-t) A(a,b), tb + (1-t) A(a,b)). \end{aligned}$$

If we apply Theorem 71 for the above convex map $f(x) = \frac{1}{x}$, we obtain:

Proposition 46. *With the above assumption, we have:*

(*i*) H_{-1} *is convex on* $[0,1]$;

(*ii*) *One has the bounds:*

$$\begin{aligned} \inf_{t\in[0,1]} H_{-1}(t) &= A^{-1}(a,b), \\ \sup_{t\in[0,1]} H_{-1}(t) &= L^{-1}(a,b); \end{aligned}$$

(*iii*) H_{-1} *increases monotonically on* $[0,1]$;

(*iv*) *The following inequalities hold:*

$$\begin{aligned} A^{-1}(a,b) &\leq L^{-1}(A(a, A(a,b)), A(b, A(a,b))) \\ &\leq \int_0^1 H_{-1}(t)\, dt \leq A\left(A^{-1}(a,b), L^{-1}(a,b)\right). \end{aligned}$$

Now, we can define the mapping $G_{-1} : [0,1] \to \mathbb{R}$,

$$\begin{aligned} G_{-1}(t) &= \frac{1}{2}\left[\frac{1}{ta + (1-t)\frac{a+b}{2}} + \frac{1}{(1-t)\frac{a+b}{2} + tb}\right] \\ &= \frac{A(a,b)}{G^2(ta + (1-t) A(a,b), (1-t) A(a,b) + tb)}. \end{aligned}$$

Using Theorem 72, we can state the following proposition:

Proposition 47. *With the above assumptions, we have that:*

(*i*) G_{-1} *is convex on* $[0,1]$ *and monotonically increasing on* $[0,1]$;

(*ii*) *We have the bounds*

$$\begin{aligned} \inf_{t\in[0,1]} G_{-1}(t) &= A^{-1}(a,b), \\ \sup_{t\in[0,1]} G_{-1}(t) &= \frac{A(a,b)}{G^2(a,b)}; \end{aligned}$$

(*iii*) *One has the inequality*

$$H_{-1}(t) \leq G_{-1}(t) \text{ for all } t \in [0,1]$$

(*iv*) *We have the inequalities:*

$$\begin{aligned} & L^{-1}(A(a, A(a,b)), A(b, A(a,b))) \\ \leq & \frac{A(a,b)}{G^2(A(a, A(a,b)), A(b, A(a,b)))} \\ \leq & \int_0^1 G_{-1}(t)\,dt \leq A\left(A^{-1}(a,b), \frac{A(a,b)}{G^2(a,b)}\right). \end{aligned}$$

Now, we shall consider another mapping $L_{-1} : [0,1] \to \mathbb{R}$ given by

$$L_{-1}(t) = \frac{1}{2(b-a)}\left[\int_a^b \frac{dx}{ta+(1-t)x} + \int_a^b \frac{dx}{(1-t)x+tb}\right].$$

It is obvious that

$$\begin{aligned} & \frac{1}{b-a}\int_a^b \frac{dx}{ta+(1-t)x} \\ = & \frac{1}{ta+(1-t)b-a}\int_a^{ta+(1-t)b} \frac{dy}{y} \\ = & L_{-1}(a, ta+(1-t)b), \ t \in [0,1) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{b-a}\int_a^b \frac{dx}{(1-t)x+tb} &= \frac{1}{b-[ta+(1-t)b]}\int_{(1-t)a+tb}^b \frac{dy}{y} \\ &= L_{-1}((1-t)a+tb, b), \ t \in [0,1). \end{aligned}$$

Thus,

$$L_{-1}(t) = A(L_{-1}(a, ta+(1-t)b), L_{-1}((1-t)a+tb, b))$$

for all $t \in [0,1]$.

For $t = 1$, we have

$$L_{-1}(1) = \frac{A(a,b)}{G^2(a,b)}.$$

If we now use Theorem 73, we can state the following proposition containing the properties of the mapping L_{-1} defined above.

Proposition 48. *With the above assumptions, we have:*

(*i*) L_{-1} *is convex on* $[0,1]$;

(*ii*) *We have the inequalities:*

$$G_{-1}(t) \leq L_{-1}(t) \leq (1-t)\cdot L^{-1}(a,b) + t\cdot\frac{A(a,b)}{G^2(a,b)} \leq \frac{A(a,b)}{G^2(a,b)}$$

for all $t \in [0,1]$ *and the bound:*

$$\sup_{t\in[0,1]} L_{-1}(t) = \frac{A(a,b)}{G^2(a,b)};$$

(*iii*) *One has the inequalities*

$$H_{-1}(t) \le L_{-1}(t) \text{ and } \frac{H_{-1}(t)+H_{-1}(1-t)}{2} \le L_{-1}(t)$$

for all $t \in [0,1]$.

Finally, using Theorem 75 and Theorem 76, we can state the following proposition.

PROPOSITION 49. *With the above assumptions, we have the inequalities:*

$$0 \le L^{-1}(a,b) - H_{-1}(t) \le (1-t)\left[\frac{A(a,b)}{G^2(a,b)} - L^{-1}(a,b)\right]$$

and

$$0 \le H_{-1}(t) - A^{-1}(a,b) \le G_{-1}(t) - H_{-1}(t)$$

for all $t \in [0,1]$.

Let us consider now, the convex mapping $f(x) = \ln x,\ x > 0$. Define the mapping

$$H_0(t) := -\frac{1}{b-a}\int_a^b \ln(tx + (1-t)A(a,b))\,dx,\ t \in [0,1].$$

It is obvious that

$$H_0(0) = -\ln A(a,b) = \ln A^{-1}(a,b)$$

and

$$H_0(1) = -\frac{1}{b-a}\int_a^b \ln x dx = \ln I^{-1}(a,b)$$

and, for all $t \in (0,1)$,

$$\begin{aligned} H_0(t) &= -\frac{1}{[tb+(1-t)A(a,b)] - [ta+(1-t)A(a,b)]}\int_{ta+(1-t)A(a,b)}^{tb+(1-t)A(a,b)} \ln y dy \\ &= \ln I^{-1}(ta+(1-t)A(a,b), tb+(1-t)A(a,b)). \end{aligned}$$

Using Theorem 71 for the convex map $f(x) = -\ln x$, we can state the following proposition:

PROPOSITION 50. *With the above assumptions, we have:*

(*i*) H_0 *is convex on* $[0,1]$;

(*ii*) *One has the bounds*

$$\begin{aligned} \inf_{t\in[0,1]} H_0(t) &= \ln A^{-1}(a,b), \\ \sup_{t\in[0,1]} H_0(t) &= \ln I^{-1}(a,b); \end{aligned}$$

(*iii*) H_0 *increases monotonically on* $[0,1]$;

(*iv*) *The following inequalities hold:*

$$\begin{aligned} \ln A^{-1}(a,b) &\le \ln I^{-1}(A(a,A(a,b)), A(b,A(a,b))) \\ &\le \int_0^1 H_0(t)\,dt \le A\left(\ln A^{-1}(a,b), \ln I^{-1}(a,b)\right) \\ &= \ln\left[G^{-1}(A(a,b), I(a,b))\right]. \end{aligned}$$

Also, we can define the following mapping:

$$\begin{aligned} G_0 &: [0,1] \to \mathbb{R}, \\ G_0(t) &: = \ln\left[G^{-1}(ta + (1-t)A(a,b), tb + (1-t)A(a,b))\right]. \end{aligned}$$

Using Theorem 72, we can state the following proposition:

Proposition 51. *With the above assumptions, we have:*

(*i*) *G_0 is convex and monotonically decreasing on* $[0,1]$;

(*ii*) *We have the bounds*

$$\begin{aligned} \inf_{t\in[0,1]} G_0(t) &= \ln A^{-1}(a,b), \\ \sup_{t\in[0,1]} G_0(t) &= \ln G^{-1}(a,b); \end{aligned}$$

(*iii*) *One has the inequality*

$$H_0(t) \le G_0(t) \text{ for all } t \in [0,1];$$

(*iv*) *One has the inequalities*

$$\begin{aligned} & \ln\left[I^{-1}(A(a,A(a,b)), A(b,A(a,b)))\right] \\ \le\ & \ln\left[G^{-1}(A(a,A(a,b)), A(b,A(a,b)))\right] \\ \le\ & \int_0^1 G_0(t)\,dt \\ \le\ & \ln\left[G^{-1}(A(a,b), G(a,b))\right]. \end{aligned}$$

Now, we shall consider another mapping $L_0 : [0,1] \to \mathbb{R}$ given by:

$$L_0(t) := -\frac{1}{2(b-a)}\left[\int_a^b \ln(ta + (1-t)x)\,dx + \int_a^b \ln(tb + (1-t)x)\,dx\right].$$

It is obvious that:

$$-\frac{1}{b-a}\int_a^b \ln(ta + (1-t)x)\,dx = \ln\left[I^{-1}(a, ta + (1-t)b)\right]$$

and

$$-\frac{1}{b-a}\int_a^b \ln(tb + (1-t)x)\,dx = \ln\left[I^{-1}(b, (1-t)a + tb)\right]$$

for all $t \in [0,1)$, which gives us:

$$L_0(t) = \ln\left[G^{-1}(I(a, ta + (1-t)b), I(b, (1-t)a + tb))\right]$$

for all $t \in [0,1)$. For $t = 1$, we have:

$$L_0(1) = \ln\left[G^{-1}(a,b)\right].$$

If we now use Theorem 73, we can state the following proposition containing the properties of the mapping L_0 defined above.

Proposition 52. *With the above assumptions, we have:*

(*i*) *L_0 is convex on* $[0,1]$;

(ii) *We have the inequalities:*

$$G_0(t) \leq L_0(t) \leq (1-t)\cdot \ln\left[I^{-1}(a,b)\right] + t\cdot \ln\left[G^{-1}(a,b)\right] \leq \ln\left[G^{-1}(a,b)\right]$$

for all $t \in (0,1]$ *and the bound:*

$$\sup_{t\in[0,1]} L_0(t) = \ln\left[G^{-1}(a,b)\right];$$

(iii) *One has the inequalities*

$$H_0(1-t) \leq L_0(t) \ \ \textit{and} \ \ \frac{H_0(t)+H_0(1-t)}{2} \leq L_0(t)$$

for all $t \in [0,1]$.

Finally, using Theorem 75 and Theorem 76, we can state the following proposition.

Proposition 53. *With the above assumptions, we have the inequalities:*

$$0 \leq \ln\left[I^{-1}(a,b)\right] - H_0(t) \leq (1-t)\left[\ln\left[G^{-1}(a,b)\right] - \ln\left[I^{-1}(a,b)\right]\right]$$

and

$$0 \leq H_0(t) - \ln\left[A^{-1}(a,b)\right] \leq G_0(t) - H_0(t)$$

for all $t \in [0,1]$.

4. It is also natural to consider the following mapping which is connected with the *identric mean* $I(a,b) := \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{b-a}}$, $h_0 : [0,1] \to \mathbb{R}$:

$$h_0(t) := I(ta+(1-t)A(a,b), (1-t)A(a,b)+tb).$$

Taking into account that

$$h_0(t) = \exp\left[-H_0(t)\right] \text{ for all } t \in [0,1],$$

we can state the following proposition:

Proposition 54. *With the above assumptions, we have:*

(i) h_0 *is log-concave on* $[0,1]$;

(ii) *One has the bounds:*

$$\begin{aligned}\inf_{t\in[0,1]} h_0(t) &= I(a,b),\\ \sup_{t\in[0,1]} h_0(t) &= A(a,b);\end{aligned}$$

(iii) h_0 *decreases monotonically on* $[0,1]$;

(iv) *The following inequalities hold*

$$\begin{aligned} A(a,b) &\geq I(A(a,A(a,b)), A(b,A(a,b)))\\ &\geq \exp\left[\int_0^1 \ln h_0(t)\,dt\right]\\ &\geq G(A(a,b), I(a,b)).\end{aligned}$$

The proof is obvious by Proposition 50. We shall omit the details.

We can also consider the mapping: $g_0 : [0,1] \to \mathbb{R}$ given by

$$g_0(t) := -G^{-1}(ta+(1-t)A(a,b), (1-t)A(a,b)+tb),$$

which is closely connected with the geometric mean $G(a,b)$.

It is clear that

$$g_0(t) := \exp\left[-G_0(t)\right] \text{ for all } t \in [0,1]$$

and by Proposition 51, we can state the following.

Proposition 55. *With the above assumptions, we have:*

(*i*) g_0 *is log-concave and monotonically increasing on* $[0,1]$;

(*ii*) *We have the bounds*

$$\begin{aligned} \inf_{t\in[0,1]} g_0(t) &= G(a,b), \\ \sup_{t\in[0,1]} g_0(t) &= A(a,b); \end{aligned}$$

(*iii*) *One has the inequality*

$$g_0(t) \le h_0(t) \ \text{ for all } t \in [0,1];$$

(*iv*) *One has the inequalities:*

$$\begin{aligned} I(A(a,A(a,b)),A(b,A(a,b))) &\ge G(A(a,A(a,b)),A(b,A(a,b))) \\ &\ge \exp\int_0^1 \ln g_0(t)\,dt \\ &\ge G(A(a,b),G(a,b)). \end{aligned}$$

Now, if we consider the new mapping $l_0 : [0,1] \to \mathbb{R}$ given by:

$$l_0(t) := G(I(a,ta+(1-t)b),I(b,(1-t)a+tb))$$

then we have that

$$l_0(t) = \exp[-L_0(t)] \ \text{ for all } t \in [0,1]$$

and by Proposition 52 we have that:

Proposition 56. *With the above assumptions, we have:*

(*i*) l_0 *is log-concave on* $[0,1]$;

(*ii*) *We have the inequalities:*

$$g_0(t) \ge l_0(t) \ge [I(a,b)]^{1-t}[G(a,b)]^t \ge G(a,b)$$

for all $t \in [0,1]$, *and the bound:*

$$\inf_{t\in[0,1]} l_0(t) = G(a,b);$$

(*iii*) *One has the inequalities:*

$$h_0(1-t) \ge l_0(t) \ \text{ and } \ \frac{h_0(t)+h_0(1-t)}{2} \ge l_0(t)$$

for all $t \in [0,1]$.

Finally, using Proposition 53, we can state that:

Proposition 57. *With the above assumptions, we have that:*

$$1 \le \frac{h_0(t)}{I(a,b)} \le \left[\frac{I(a,b)}{G(a,b)}\right]^{(1-t)}$$

and

$$1 \le \frac{A(a,b)}{h_0(t)} \le \frac{h_0(t)}{g_0(t)}$$

for all $t \in [0,1]$.

4. Some Results due to B.G. Pachpatte

4.1. Introduction. Let $f,g:[a,b]\to\mathbb{R}$ be convex mappings. For x,y two elements in $[a,b]$, we shall define the mappings $F(x,y)$, $G(x,y):[0,1]\to\mathbb{R}$ given by (see [**130**])

$$F(x,y)(t)=\frac{1}{2}\left[f(tx+(1-t)y)+f((1-t)x+y)\right], \tag{3.32}$$

$$G(x,y)(t)=\frac{1}{2}\left[g(tx+(1-t)y)+g((1-t)x+y)\right]. \tag{3.33}$$

Recently in [**56**] Dragomir and Ionescu established some interesting properties of such mappings. In particular in [**56**], it is shown that $F(x,y)$, $G(x,y)$ are convex on $[0,1]$. In another paper [**146**], Pečarić and Dragomir proved that the following statements are equivalent for mappings $f,g:[a,b]\to\mathbb{R}$:

(i) f,g are convex on $[a,b]$;
(ii) for all $x,y\in[a,b]$ the mappings f_0, $g_0:[0,1]\to\mathbb{R}$ defined by $f_0(t)=f(tx+(1-t)y)$ or $f((1-t)x+ty)$, $g_0(t)=g(tx+(1-t)y)$ or $g((1-t)x+ty)$ are convex on $[0,1]$.

From these properties, it is easy to observe that if f_0 and g_0 are convex on $[0,1]$, then they are integrable on $[0,1]$ and hence f_0g_0 is also integrable on $[0,1]$. Similarly, if f and g are convex on $[a,b]$, they are integrable on $[a,b]$ and hence fg is also integrable on $[a,b]$. Consequently, it is easy to see that if f and g are convex on $[a,b]$, then $F=F(x,y)$ and $G=G(x,y)$ are convex and hence Fg Gf, Ff, Gg are also integrable on $[a,b]$. We shall use these facts in our discussion without further mention.

The object of this section is to establish some new integral inequalities involving the functions F and G as defined in (3.32) and (3.33).

4.2. The Results. The first main result is given in the following theorem [**130**].

THEOREM 80. *Let f and g be real-valued and convex functions on $[a,b]$ and the mappings $F(x,y)$ and $G(x,y)$ be defined by (3.32) and (3.33). Then for all t in $[0,1]$ we have*

$$\begin{aligned}&\frac{1}{(b-a)^2}\int_a^b(b-y)f(y)g(y)\,dy\\ \le\ &\frac{2}{5}\cdot\frac{1}{(b-a)^2}\int_a^b\left(\int_a^y\left[F(x,y)(t)g(x)+G(x,y)(t)f(x)\right]dx\right)dy\\ &+\frac{1}{10}f(a)g(a),\end{aligned} \tag{3.34}$$

$$\begin{aligned}&\frac{1}{(b-a)^2}\int_a^b(y-a)f(y)g(y)\,dy\\ \le\ &\frac{2}{5}\cdot\frac{1}{(b-a)^2}\int_a^b\left(\int_y^b\left[F(x,y)(t)g(x)+G(x,y)(t)f(x)\right]dx\right)dy\\ &+\frac{1}{10}f(b)g(b),\end{aligned} \tag{3.35}$$

$$
\begin{aligned}
(3.36)\qquad & \frac{1}{(b-a)^2}\int_a^b f(y)\,g(y)\,dy \\
\leq\ & \frac{2}{5}\cdot\frac{1}{(b-a)}\int_a^b\int_a^b \left[F(x,y)(t)\,g(x)+G(x,y)(t)\,f(x)\right]dxdy \\
& +\frac{1}{10}\left[f(a)\,g(a)+f(b)\,g(b)\right].
\end{aligned}
$$

Proof. The assumptions that f and g are convex imply that we may assume that f, g are differentiable and that we have the following estimates

$$f(tx+(1-t)\,y) \geq f(x)+(1-t)(y-x)\,f'(x), \tag{3.37}$$

$$f((1-t)\,x+y) \geq f(x)+t\,(y-x)\,f'(x), \tag{3.38}$$

$$g(tx+(1-t)\,y) \geq g(x)+(1-t)(y-x)\,g'(x), \tag{3.39}$$

$$g((1-t)\,x+y) \geq g(x)+t\,(y-x)\,g'(x), \tag{3.40}$$

for $x,y\in[a,b]$ and $t\in[0,1]$. From (3.37), (3.38) (3.32) and (3.39), (3.40), (3.33) it is easy to see that

$$F(x,y)(t) \geq f(x)+\frac{1}{2}(y-x)\,f'(x), \tag{3.41}$$

$$G(x,y)(t) \geq g(x)+\frac{1}{2}(y-x)\,g'(x), \tag{3.42}$$

for $x,y\in[a,b]$ and $t\in[0,1]$. Multiplying (3.41) by $g(x)$ and (3.42) by $f(x)$ and then adding, we obtain

$$
\begin{aligned}
(3.43)\qquad & F(x,y)(t)\,g(x)+G(x,y)(t)\,f(x) \\
\geq\ & 2f(x)\,g(x)+\frac{1}{2}(y-x)\frac{d}{dx}\left(f(x)\,g(x)\right).
\end{aligned}
$$

Integrating the inequality (3.43) over x from a to y we have

$$
\begin{aligned}
(3.44)\qquad & \int_a^y \left[F(x,y)(t)\,g(x)+G(x,y)(t)\,f(x)\right]dx \\
\geq\ & \frac{5}{2}\int_a^y f(x)\,g(x)\,dx-\frac{1}{2}(y-a)\,f(a)\,g(a).
\end{aligned}
$$

Further, integrating both sides of (3.44) with respect to y from a to b we get

$$
\begin{aligned}
(3.45)\qquad & \int_a^b\int_a^y \left[F(x,y)(t)\,g(x)+G(x,y)(t)\,f(x)\right]dxdy \\
\geq\ & \frac{5}{2}\int_a^b (b-y)\,f(y)\,g(y)\,dy-\frac{1}{4}(b-a)^2 f(a)\,g(a).
\end{aligned}
$$

Multiplying both sides of (3.45) by $\frac{2}{5}\cdot\frac{1}{(b-a)^2}$ and rewriting we get the required inequality in (3.34).

Similarly, by first integrating (3.43) over x from y to b and after that integrating the resulting inequality over y from a to b, we get the required inequality in (3.35). The inequality (3.36) is obtained by adding the inequalities (3.34) and (3.35). The proof is complete. ∎

The next result deals with the slight variants of the inequalities given in Theorem 80 [**130**].

THEOREM 81. *Let f and g be real-valued, nonnegative and convex functions on $[a,b]$ and the mappings $F(x,y)$ and $G(x,y)$ be defined by (3.32) and (3.33). Then for all t in $[0,1]$ we have*

$$\begin{aligned}(3.46)\qquad &\frac{1}{(b-a)^2}\int_a^b (b-y)\left[f^2(y)+g^2(y)\right]dy\\ \leq\ &\frac{4}{5}\cdot\frac{1}{(b-a)^2}\int_a^b\left(\int_a^y \left[F(x,y)(t)f(x)+G(x,y)(t)g(x)\right]dx\right)dy\\ &+\frac{1}{10}\left[f^2(a)+g^2(a)\right],\end{aligned}$$

$$\begin{aligned}(3.47)\qquad &\frac{1}{(b-a)^2}\int_a^b (y-a)\left[f^2(y)+g^2(y)\right]dy\\ \leq\ &\frac{4}{5}\cdot\frac{1}{(b-a)^2}\int_a^b\left(\int_y^b \left[F(x,y)(t)f(x)+G(x,y)(t)g(x)\right]dx\right)dy\\ &+\frac{1}{10}\left[f^2(b)+g^2(b)\right],\end{aligned}$$

$$\begin{aligned}(3.48)\qquad &\frac{1}{(b-a)}\int_a^b \left[f^2(y)+g^2(y)\right]dy\\ \leq\ &\frac{4}{5}\cdot\frac{1}{(b-a)^2}\int_a^b\left(\int_a^b \left[F(x,y)(t)f(x)+G(x,y)(t)g(x)\right]dx\right)dy\\ &+\frac{1}{10}\left[f^2(a)+g^2(a)+f^2(b)+g^2(b)\right].\end{aligned}$$

PROOF. As in the proof of Theorem 80, from the assumptions we have the estimates (3.41) and (3.42). Multiplying (3.41) by $f(x)$ and (3.42) by $g(x)$ and then adding, we obtain

$$\begin{aligned}(3.49)\qquad &F(x,y)(t)f(x)+G(x,y)(t)g(x)\\ \geq\ &f^2(x)+g^2(x)+\frac{1}{2}(y-x)\left[f(x)f'(x)+g(x)g'(x)\right].\end{aligned}$$

Integrating (3.49) over x from a to y, we have

$$\begin{aligned}(3.50)\qquad &\int_a^y \left[F(x,y)(t)f(x)+G(x,y)(t)g(x)\right]dx\\ \geq\ &\frac{5}{4}\int_a^y\left[f^2(x)+g^2(x)\right]dx-\frac{1}{4}(y-a)\left[f^2(a)+g^2(a)\right].\end{aligned}$$

Further, integrating both sides of (3.50) with respect to y from a to b we have

$$\begin{aligned}(3.51)\qquad &\int_a^b\left(\int_a^y \left[F(x,y)(t)f(x)+G(x,y)(t)g(x)\right]dx\right)dy\\ \geq\ &\frac{5}{4}\int_a^b(b-y)\left[f^2(y)+g^2(y)\right]dy-\frac{1}{8}(b-a)^2\left[f^2(a)+g^2(a)\right].\end{aligned}$$

Multiplying both sides of (3.51) by $\frac{4}{5}\cdot\frac{1}{(b-a)^2}$ and rewriting, we get the required inequality in (3.46).

The remainder of the proof follows by the same arguments as mentioned in the proof of Theorem 80 with suitable modifications and hence the proof is complete. ∎

4.3. Further Inequalities. In this section we shall give some inequalities that are analogous to those given in Theorem 80 involving only one convex function. We believe that these inequalities are interesting in their own right [**130**].

THEOREM 82. *Let f be a real-valued nonnegative convex function on $[a,b]$. Then*

$$\begin{aligned} &\frac{1}{(b-a)^2}\int_a^b (b-y)\, f(y)\, dy \\ &\leq \frac{2}{3}\cdot\frac{1}{(b-a)^2}\int_a^b \left[\int_a^y \left(\int_0^1 f(tx+(1-t)y)\, dt\right) dx\right] dy + \frac{1}{6} f(a), \end{aligned} \tag{3.52}$$

$$\begin{aligned} &\frac{1}{(b-a)^2}\int_a^b (y-a)\, f(y)\, dy \\ &\leq \frac{2}{3}\cdot\frac{1}{(b-a)^2}\int_a^b \left[\int_y^b \left(\int_0^1 f(tx+(1-t)y)\, dt\right) dx\right] dy + \frac{1}{6} f(b), \end{aligned} \tag{3.53}$$

$$\begin{aligned} &\frac{1}{(b-a)}\int_a^b f(y)\, dy \\ &\leq \frac{2}{3}\cdot\frac{1}{(b-a)^2}\int_a^b \left[\int_y^b \left(\int_0^1 f(tx+(1-t)y)\, dt\right) dx\right] dy \\ &\quad +\frac{1}{6}\left[f(a)+f(b)\right]. \end{aligned} \tag{3.54}$$

PROOF. To prove the inequality (3.52), as in the proof of Theorem 80 from the assumptions we have the estimate (3.37). Integrating both sides of (3.37) over t from 0 to 1 we have

$$\int_0^1 f(tx+(1-t)y)\, dt \geq f(x) + \frac{1}{2}(y-x)\, f'(x). \tag{3.55}$$

Now first integrating both sides of (3.55) over x from a to y and after that integrating the resulting inequality over y from a to b we get the required inequality in (3.52).

Similarly, by first integrating both sides of (3.55) over x from y to b and then integrating the resulting inequality over y from a to b we get the inequality in (3.53). By adding the inequalities (3.52) and (3.53) we get the inequality (3.54). The proof of Theorem 82 is thus completed. ∎

5. Fejér's Generalization of the $H. - H.$ Inequality

In 1906, Fejér [**72**] obtained the following result which is a generalization of that of Hermite and Hadamard (see for example [**147**, p. 138]) or [**61**].

THEOREM 83. *Let $g : [a,b] \to [0,\infty)$ be a density function on $[a,b]$. In other words, g is non-negative and integrable with $\int_a^b g(u)\, du = 1$. If $f : [a,b] \to \mathbb{R}$*

is a convex function and g is a symmetric density function on $[a,b]$, that is, $g(a+b-u)=g(u)$ for all $u\in[a,b]$, then we have:

$$f\left(\frac{a+b}{2}\right)\leq\int_a^b f(u)\,g(u)\,du\leq\frac{f(a)+f(b)}{2}. \tag{3.56}$$

In addition, these bounds are sharp.

PROOF. By the convexity of f, we have:

$$\begin{aligned}\int_a^b f(u)\,g(u)\,du &= \int_a^b f\left(\frac{b-u}{b-a}\cdot a+\frac{u-a}{b-a}\cdot b\right)g(u)\,du\\ &\leq \int_a^b\left[\frac{b-u}{b-a}\cdot f(a)+\frac{u-a}{b-a}\cdot f(b)\right]g(u)\,du.\end{aligned}$$

The symmetry of g gives that $\int_a^b ug(u)\,du=\frac{a+b}{2}$. Hence, the preceding calculation may be continued to yield

$$\begin{aligned}\int_a^b f(u)\,g(u)\,du &\leq \frac{bf(a)-af(b)}{b-a}+\frac{f(b)-f(a)}{b-a}\cdot\frac{a+b}{2}\\ &= \frac{f(a)+f(b)}{2}.\end{aligned}$$

On the other hand,

$$\frac{1}{2}\left[f\left(\frac{b-u}{b-a}\cdot a+\frac{u-a}{b-a}\cdot b\right)+f\left(\frac{b-u}{b-a}\cdot b+\frac{u-a}{b-a}\cdot a\right)\right]\geq f\left(\frac{a+b}{2}\right)$$

for each $u\in[a,b]$. Multiplying by $g(u)$ and then integrating over $[a,b]$, we obtain

$$\begin{aligned}&\frac{1}{2}\left[\int_a^b f\left(\frac{b-u}{b-a}\cdot a+\frac{u-a}{b-a}\cdot b\right)g(u)\,du+\int_a^b f\left(\frac{b-u}{b-a}\cdot b+\frac{u-a}{b-a}\cdot a\right)g(u)\,du\right]\\ \geq\ & f\left(\frac{a+b}{2}\right).\end{aligned}$$

Set $v=a+b-u$ in the second integral.
This then becomes

$$\begin{aligned}&\int_a^b f\left(\frac{v-a}{b-a}\cdot b+\frac{b-v}{b-a}\cdot a\right)g(a+b-v)\,dv\\ =\ &\int_a^b f\left(\frac{v-a}{b-a}\cdot b+\frac{b-v}{b-a}\cdot a\right)g(v)\,dv\end{aligned}$$

by the symmetry of g. Hence, it has the same value as the first integral, both being equal to $\int_a^b f(u)\,g(u)\,du$, and we have

$$\int_a^b f(u)\,g(u)\,du\geq f\left(\frac{a+b}{2}\right).$$

Finally, take $f(u)\equiv u$ on $[a,b]$. Then the upper and lower bounds in (3.56) both simplify down to $\frac{a+b}{2}$. Hence the bounds are sharp and the theorem is proved. ∎

A simple example of the situation envisaged in Theorem 83 is provided by expressions of the form

$$\frac{1}{\pi}\int_a^b\frac{f(u)\,du}{\sqrt{(u-a)(b-a)}}$$

which occur in various problems in elementary mechanic and in probability theory in connection with the arc-sin law. The function

$$\Pi^{-1}\left[(u-a)(b-u)\right]^{-\frac{1}{2}}$$

is non-negative and symmetric on $[a,b]$ and the substitution

$$u = a\sin^2 t + b\cos^2 t$$

shows that

$$\int_a^b g(u)\,du = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} dt = 1.$$

Hence, g is a symmetric density function on $[a,b]$.

*

First, we introduce the mapping $H_g : [0,1] \to \mathbb{R}$, defined by

$$H_g(t) := \int_a^b f\left(tu + (1-t)\frac{a+b}{2}\right) g(u)\,du.$$

This mapping reduces to $H(t)$ in the classical case $g(u) = \frac{1}{b-a}$. The basic properties of H_g are embodied in the following theorem [**61**].

Theorem 84. *If $f : [a,b] \to \mathbb{R}$ is convex and $g : [a,b] \to [0,\infty)$, a symmetric density function, then:*

(i) H_g is convex on $[0,1]$;
(ii) One has the inequalities:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq H_g(t) \leq t\int_a^b f(u)g(u)\,du + (1-t) f\left(\frac{a+b}{2}\right) \\ &\leq \int_a^b f(u)g(u)\,du; \end{aligned}$$

(iii) We have the bounds

$$\begin{aligned} \sup_{t\in[0,1]} H_g(t) &= H_g(1) = \int_a^b f(u)g(u)\,du, \\ \inf_{t\in[0,1]} H_g(t) &= H_g(0) = f\left(\frac{a+b}{2}\right); \end{aligned}$$

(iv) The mapping increases monotonically on $[0,1]$.

Proof. (i) The convexity of H_g follows directly from that of f.
(ii) By Jensen's integral inequality,

$$\begin{aligned} H_g(t) &\geq f\left(\int_a^b \left[tu + (1-t)\frac{a+b}{2}\right] g(u)\,du\right) \\ &= f\left(t\cdot\frac{a+b}{2} + (1-t)\cdot\frac{a+b}{2}\right) \\ &= f\left(\frac{a+b}{2}\right) \end{aligned}$$

for any $t\in[0,1]$. Also, from the convexity of f and by a previous result we have

$$\begin{aligned} H_g(t) &\leq \int_a^b \left[tf(u)+(1-t)f\left(\frac{a+b}{2}\right)\right]g(u)\,du \\ &= t\int_a^b f(u)\,g(u)\,du+(1-t)f\left(\frac{a+b}{2}\right) \\ &\leq \int_a^b f(u)\,g(u)\,du. \end{aligned}$$

(*iii*) This is immediate from (*ii*).

(*iv*) Suppose $0<t_1<t_2\leq 1$. The convexity of H_g gives

$$H_g(t_2)-H_g(t_1)\geq \frac{t_2-t_1}{t_1}\left[H_g(t_1)-H_g(0)\right]\geq 0$$

by the first part of (*iii*), whence the desired result.

∎

The second companion mapping $F_g:[0,1]\to\mathbb{R}$ given by

$$F_g(t):=\int_a^b\int_a^b f(tx+(1-t)y)\,g(x)\,g(y)\,dxdy.$$

Clearly, it reduces to F in the classical case when $g(u)=\frac{1}{b-a}$.

The basic properties of F_g are encapsulated in the following theorem [**61**].

THEOREM 85. *If $f:[a,b]\to\mathbb{R}$ is convex and $g:[a,b]\to[0,\infty)$, a symmetric density function, then*

(*i*) *F_g is symmetric about $t=\frac{1}{2}$;*

(*ii*) *F_g is convex on $[0,1]$;*

(*iii*) *We have the bounds*

$$\begin{aligned} \sup_{t\in[0,1]} F_g(t) &= F_g(0)=F_g(1)=\int_a^b f(u)\,g(u)\,du, \\ \inf_{t\in[0,1]} F_g(t) &= F_g\left(\frac{1}{2}\right)=\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)g(x)\,g(y)\,dxdy; \end{aligned}$$

(*iv*) *One has the inequality;*

$$f\left(\frac{a+b}{2}\right)\leq F_g\left(\frac{1}{2}\right);$$

(*v*) *F_g decreases monotonically on $\left[0,\frac{1}{2}\right]$ and increases monotonically on $\left[\frac{1}{2},1\right]$;*

(*vi*) *We have the inequality:*

$$F_g(t)\geq\max\{H_g(t),H_g(1-t)\}$$

for all $t\in[0,1]$.

PROOF. (*i*) The fact that $F_g(t)=F_g(1-t)$ follows from an interchange of dummies in the definition of F_g.

(*ii*) It is obvious by the convexity of f.

(*iii*) For $t\in[0,1]$, we have, by the convexity of f that

$$\begin{aligned}
F_g(t) &= \int_a^b\int_a^b f(tx+(1-t)y)\,g(x)\,g(y)\,dxdy \\
&\leq \int_a^b\int_a^b [tf(x)+(1-t)f(y)]\,g(x)\,g(y)\,dxdy \\
&= t\int_a^b f(x)\,g(x)\,dx+(1-t)\int_a^b f(y)\,g(y)\,dy \\
&= \int_a^b f(u)\,g(u)\,du = F_g(0)=F_g(1)
\end{aligned}$$

which gives the first relation in (iii).
By the convexity of f, we have

$$\frac{1}{2}\left[f(tx+(1-t)y)+f(ty+(1-t)x)\right]\geq f\left(\frac{x+y}{2}\right).$$

Multiplication by $g(x)\,g(y)$ and integration over $[a,b]\times[a,b]$ gives

$$F_g(t)\geq F_g\left(\frac{1}{2}\right)$$

for all $t\in[0,1]$.

(*iv*) By Jensen's Theorem,

$$\begin{aligned}
&\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)g(x)\,g(y)\,dxdy \\
\geq\ & f\left(\int_a^b\int_a^b \frac{x+y}{2}g(x)\,g(y)\,dxdy\right) \\
=\ & f\left(\int_a^b\left(\frac{x}{2}\right)g(x)\,dx+\int_a^b\left(\frac{y}{2}\right)g(y)\,dy\right) \\
=\ & f\left(\frac{a+b}{2}\right),
\end{aligned}$$

so that $F_g\left(\frac{1}{2}\right)\geq f\left(\frac{a+b}{2}\right)$.

(*v*) The property of monotonicity on the interval $\left[\frac{1}{2},1\right]$ is established as for the corresponding property in Theorem 74 (see also [**45**]). The monotonic property on $\left[0,\frac{1}{2}\right]$ then follows by (i).

(*vi*) We have

$$\begin{aligned}
H_g(t) &= \int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)g(x)\,dx \\
&= \int_a^b f\left(\int_a^b [(tx+(1-t)y)]\,g(y)\,dy\right)g(x)\,dx \\
&\leq \int_a^b\int_a^b f(tx+(1-t)y)\,g(x)\,g(y)\,dxdy = F_g(t)
\end{aligned}$$

by Jensen's inequality.
As $F_g(1-t)=F_g(t)$, the statement is proved.

∎

It is also natural to consider the possibility of further generality in the companion mappings. Therefore, suppose $\Phi : [0,1] \to \mathbb{R}$, $\Phi([0,1]) = [0,1]$ and define

$$H_{g,\Phi}(t) := \int_a^b f\left(\Phi(t)u + (1-\Phi(t))\frac{a+b}{2}\right) g(u)\,du.$$

We use the fact that $H_{g,\Phi}(t) = H_g(\Phi(t))$ to derive the following result [**61**].

THEOREM 86. *Suppose that f, g are as above and Φ is monotonic nondecreasing on $[0,1]$. Then*

(*i*) *$H_{g,\Phi}$ is convex if Φ is convex;*

(*ii*) *One has the inequality*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq H_{g,\Phi}(t) \\ &\leq \Phi(t)\int_a^b f(u)g(u)\,du + (1-\Phi(t)) f\left(\frac{a+b}{2}\right) \\ &\leq \int_a^b f(u)g(u)\,du; \end{aligned}$$

(*iii*) *We have the bounds*

$$\begin{aligned} \sup_{t\in[0,1]} H_{g,\Phi}(t) &= H_{g,\Phi}(1) = \int_a^b f(u)g(u)\,du, \\ \inf_{t\in[0,1]} H_{g,\Phi}(t) &= H_{g,\Phi}(0) = f\left(\frac{a+b}{2}\right); \end{aligned}$$

(*iv*) *The mapping $H_{g,\Phi}$ increases monotonically on $[0,1]$.*

PROOF. (*i*) Take $t_1, t_2 \in [0,1]$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$. We may argue for the monotonicity and convexity of $H_{g,\Phi}$ that

$$\begin{aligned} H_{g,\Phi}(\alpha t_1 + \beta t_2) &= H_g(\Phi(\alpha t_1 + \beta t_2)) \\ &\leq H_g(\alpha\Phi(t_1) + \beta\Phi(t_2)) \\ &\leq \alpha H_g(\Phi(t_1)) + \beta H_g(\Phi(t_2)) \\ &= \alpha H_{g,\Phi}(t_1) + \beta H_{g,\Phi}(t_2), \end{aligned}$$

establishing that $H_{g,\Phi}$ is convex.

(*ii*) and (*iii*) The argument of Theorem 84 carries over.

(*iv*) The result follows from $H_{g,\Phi}(t) = H_g(\Phi(t))$ and the monotonic character of H_g and Φ.

∎

Theorem 85 may be adapted to a version involving a function Φ. Define

$$F_{g,\Phi}(t) := \int_a^b \int_a^b f(\Phi(t)x + (1-\Phi(t))y)\, g(x)\, g(y)\,dxdy.$$

We have [**61**]:

THEOREM 87. *Let f, g be as above and $\Phi : [0,1] \to [0,1]$ be a monotonic nondecreasing function. Suppose that t_0 satisfy $\Phi(t_0) = \frac{1}{2}$. Then*

(*i*) *$F_{g,\Phi}$ is symmetric about $t = \frac{1}{2}$ if $\Phi(t) + \Phi(1-t) = 1$;*

(*ii*) *$F_{g,\Phi}$ is convex on any subinterval of $[0,t_0]$ on which Φ is concave and on any subinterval of $[t_0,1]$ on which Φ is convex;*

(*iii*) *We have the bounds*

$$\sup_{t\in[0,1]} F_{g,\Phi}(t) = F_{g,\Phi}(0) = F_{g,\Phi}(1) = \int_a^b f(u)\,g(u)\,du,$$
$$\inf_{t\in[0,1]} F_{g,\Phi}(t) = F_{g,\Phi}(t_0) = \int_a^b\int_a^b f\left(\frac{x+y}{2}\right) g(x)\,g(y)\,dxdy;$$

(*iv*) *One has the inequality:*

$$f\left(\frac{a+b}{2}\right) \le F_{g,\Phi}(t) \ \textit{ for all } t\in[0,1]$$

(*v*) $F_{g,\Phi}$ *decreases monotonically on* $[0,t_0]$ *and increases monotonically on* $[t_0,1]$;

(*vi*) *One has the inequality:*

$$H_{g,\Phi}(t) \le F_{g,\Phi}(t) \ \textit{ for all } t\in[0,1].$$

Proof.

(*i*) This is immediate from the definitions.

(*ii*) The given conditions provide the chain of inequalities used in the proof of Theorem 86 (*i*).

(*iii*) The proof follows that of the corresponding statement in Theorem 85.

(*iv*) We have

$$\begin{aligned}
F_{g,\Phi}(t) &= \int_a^b\int_a^b f(\Phi(t)x+(1-\Phi(t))y)\,g(x)\,g(y)\,dxdy \\
&\ge f\left(\int_a^b\int_a^b [\Phi(t)x+(1-\Phi(t))y]\,g(x)\,g(y)\,dxdy\right) \\
&= f\left(\int_a^b \Phi(t)\,xg(x)\,dx + \int_a^b (1-\Phi(t))\,yg(y)\,dy\right) \\
&= f\left(\int_a^b ug(u)\,du\right) = f\left(\frac{a+b}{2}\right).
\end{aligned}$$

(*v*) The result follows from $F_{g,\Phi}(t) = F_g(\Phi(t))$ and the monotonicity properties of F_g and Φ.

(*vi*) The proof mimics that of Theorem 85.

∎

If the map Φ considered above is absolutely continuous, then it may be represented canonically in the form

$$\Phi(t) = \int_0^t g'(u)\,du \ \ (0\le t\le 1) \tag{3.57}$$

where g' is a non-negative Bessel function on $[0,1]$ (cf. Loéve [**99**]). Almost sure strict positivity of g' corresponds to strict monotonicity of Φ. Furthermore, a necessary and sufficient condition for Φ to be (strictly) convex is that g' be (strictly) increasing (cf. Mitrinović, Pečarić and Fink [**114**]).

Suppose that g' is strictly positive. Then the mapping $\Phi : [0,1] \to [0,1]$ provided by (3.57) is invertible. We denote by $\tilde{\Phi} : [0,1] \to [0,1]$ the inverse map, which is also nondecreasing and satisfies $\tilde{\Phi}(0)=0$, $\tilde{\Phi}(1)=1$. In a natural way, we write $\tilde{t}_0$ for a solution to $\tilde{\Phi}(t)=\frac{1}{2}$.

The above results and notation set the foundations for the construction of functions Φ satisfying the conditions of Theorem 87 (i) and (ii). Our concluding theorem summarizes the basic results [**61**].

THEOREM 88. *Suppose that g' is symmetric and strictly positive on $[0,1]$. Then,*

- (i) *Φ and $\tilde{\Phi}$ satisfy the condition of Theorem 87 (i);*
- (ii) $t_0=\frac{1}{2}=\tilde{t}_0$;
- (iii) *Φ is convex (concave) on $\left[0,\frac{1}{2}\right]$ iff it is concave (convex) on $\left[\frac{1}{2},1\right]$. The same condition applies to $\tilde{\Phi}$.*
- (iv) *Φ is convex (concave) on $\left[0,\frac{1}{2}\right]$ iff $\tilde{\Phi}$ is concave (convex) on $\left[0,\frac{1}{2}\right]$. A similar result applies on $\left[\frac{1}{2},1\right]$;*
- (v) *If g' is increasing (decreasing) on $\left[0,\frac{1}{2}\right]$, then Φ is convex (concave) on $\left[0,\frac{1}{2}\right]$.*

PROOF. (i) By the symmetry of g', the equality

$$1-\Phi(t)=\Phi(1-t) \tag{3.58}$$

follows from considering the area under the graph of $\Phi(t)$ against t. Now, suppose $t=\int_0^x g'(u)\,du$, such that $x=\tilde{\Phi}(t)$. Then

$$1-t=\int_x^1 g'(u)\,du=\int_0^{1-x} g'(u)\,du,$$

so that

$$\tilde{\Phi}(1-t)=1-x=1-\tilde{\Phi}(t)$$

and $\tilde{\Phi}$ also satisfies (3.58).

(ii) By (i) with $t=\frac{1}{2}$ we have $2\Phi\left(\frac{1}{2}\right)=2\tilde{\Phi}\left(\frac{1}{2}\right)$.

(iii) Suppose Φ is convex on $\left[0,\frac{1}{2}\right]$ and $t_1,t_2\in\left[\frac{1}{2},1\right]$. Then, for $\lambda\in[0,1]$, we have, by (3.58) that:

$$\begin{aligned}\Phi(\lambda t_1+(1-\lambda)t_2) &= 1-\Phi(\lambda(1-t_1)+(1-\lambda)(1-t_2))\\ &\geq 1-[\lambda\Phi(1-t_1)+(1-\lambda)\Phi(1-t_2)]\end{aligned}$$

since $1-t_1,1-t_2\in\left[0,\frac{1}{2}\right]$. Therefore

$$\begin{aligned}\Phi(\lambda t_1+(1-\lambda)t_2) &\geq \lambda[1-\Phi(1-t_1)]+(1-\lambda)[\Phi(1-t_2)]\\ &= \lambda\Phi(t_1)+(1-\lambda)\Phi(t_2)\end{aligned}$$

by (3.58) and Φ is concave on $\left[\frac{1}{2},1\right]$. The other cases follow similarly.

(iv) The graph of $y=\tilde{\Phi}(x)$ is obtained from that of $y=\Phi(x)$ by reflection about the line $y=x$. The results follow.

(v) The first alternative is a recapitulation of the results leading to the enunciation of the theorem. For the second, suppose g' is decreasing on $\left[0,\frac{1}{2}\right]$. Then it is increasing on $\left[\frac{1}{2},1\right]$ and so Φ is convex on $\left[\frac{1}{2},1\right]$. Hence, by (ii) Φ is convex on $\left[0,\frac{1}{2}\right]$.

∎

A simple concrete instance of a function Φ satisfying the conditions of Theorem 87 to a full extent is provided by the example of the introduction. Put $a=0,\ b=1$ in that example and set

$$g'(u) := \Pi^{-1}\left[u(1-u)\right]^{-\frac{1}{2}}.$$

This is symmetric on $[0,1]$ and decreasing on $\left[0,\frac{1}{2}\right]$ and so

$$\Phi(t) = \frac{2}{\pi}\arcsin\sqrt{t}$$

is concave on $\left[0,\frac{1}{2}\right]$ and convex on $\left[\frac{1}{2},1\right]$.

5.1. Applications for the Beta Function. Let us consider the *Beta function* of Euler, that is,

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,dt,\ p,q>-1.$$

We have for $r\geq 1$, that

$$B(p+r,p) := \int_0^1 t^{p-1}(1-t)^{p-1}t^r dt. \tag{3.59}$$

Define $g:(0,1)\to\mathbb{R}$ given by

$$g(t) := \frac{1}{B(p,p)}t^{p-1}(1-t)^{p-1}.$$

It is clear that

$$g(t) = g(1-t) \text{ for all } t\in(0,1)$$

and

$$\int_0^1 g(t)\,dt = 1.$$

Define $f:[0,1]\to\mathbb{R}$ given by $f(t):=t^r,\ r\geq 1$. Then f is convex and we can apply Theorem 83. Thus, we obtain

$$\frac{1}{2^r} \leq \int_0^1 f(t)\,g(t)\,dt = \frac{1}{B(p,p)}\int_0^1 t^{p-1}(1-t)^{p-1}t^r dt \leq \frac{1}{2}$$

and by (3.59) we deduce:

$$\frac{1}{2^r} \leq \frac{B(p+r,p)}{B(p,p)} \leq \frac{1}{2} \text{ for all } r\geq 1,\ p>-1. \tag{3.60}$$

We can also introduce the following mapping:

$$H_B^{[r]}(t,p) := \frac{1}{B(p,p)}\int_0^1 u^{p-1}(1-u)^{p-1}\left[tu+\frac{1}{2}(1-t)\right]^r du$$

for all $t\in[0,1]$ and $p>-1,\ r\geq 1$.

Using Theorem 84, we can state the following properties of this mapping:

(i) $H_B^{[r]}(\cdot,p)$ is convex on $[0,1]$ for all $p>-1,\ r\geq 1$;

(ii) One has the inequalities:

$$\frac{1}{2^r} \leq H_B^{[r]}(t,p) \leq t\cdot\frac{B(p+r,p)}{B(p,p)} + (1-t)\cdot\frac{1}{2^r} \leq \frac{B(p+r,p)}{B(p,p)}$$

for all $t\in[0,1],\ p>-1,\ r\geq 1$;

(iii) We have the bounds:

$$\begin{aligned}\sup_{t\in[0,1]} H_B^{[r]}(t,p) &= \frac{B(p+r,p)}{B(p,p)},\ p>-1,\ r\geq 1\\ \inf_{t\in[0,1]} H_B^{[r]}(t,p) &= \frac{1}{2^r},\ p>-1,\ r\geq 1;\end{aligned}$$

(iv) The mapping $H_B^{[r]}(\cdot,p)$ increases monotonically on $[0,1]$ for $p>-1,\ r\geq 1$.

We can also introduce the mapping

$$F_B^{[r]}(t,p):=\frac{1}{B^2(p,p)}\int_0^1\int_0^1 x^{p-1}y^{p-1}(1-x)^{p-1}(1-y)^{p-1}[tx+(1-t)y]^r\,dxdy$$

for all $t\in[0,1]$, $p>-1$, and $r\geq 1$.

Using Theorem 85, we can state the following properties of this mapping:

(i) $F_\beta^{[r]}(\cdot,p)$ is symmetric about $t=\frac{1}{2}$; $p>-1,\ r\geq 1$;
(ii) $F_\beta^{[r]}(\cdot,p)$ is convex on $[0,1]$, $p>-1,\ r\geq 1$;
(iii) We have the bounds:

$$\sup_{t\in[0,1]} F_B^{[r]}(t,p)=\frac{B(p+r,p)}{B(p,p)},$$

$$\begin{aligned}&\inf_{t\in[0,1]} F_B^{[r]}(t,p)\\ &= \frac{1}{2^rB^2(p,p)}\int_0^1\int_0^1 x^{p-1}y^{p-1}(1-x)^{p-1}(1-y)^{p-1}(x+y)^r\,dxdy\\ &\geq \frac{1}{2^r}\text{ for } p>-1,\ r\geq 1;\end{aligned}$$

(iv) The mapping $F_\beta^{[r]}(\cdot,p)$ decreases monotonically on $\left[0,\frac{1}{2}\right]$ and increases monotonically on $\left[\frac{1}{2},1\right]$;
(v) We have the inequality:

$$F_B^{[r]}(t,p)\geq\max\left\{H_B^{[r]}(t,p),H_B^{[r]}(1-t,p)\right\}$$

for all $t\in[0,1]$, $p>-1$, and $r\geq 1$.

6. Further Results Refining the $H.-H.$ Inequality

Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex mapping on the interval I and $a,b\in\mathring{I}$ with $a<b$. Reconsider the mappings: $H,G:[0,1]\to\mathbb{R}$ given by

$$\begin{aligned}H_f(t) &:= \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx,\\ G_f(t) &= \frac{1}{2}\left[f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left((1-t)\frac{a+b}{2}+tb\right)\right].\end{aligned}$$

The following theorem holds [**29**]:

THEOREM 89. *Let f, a and b be as above. Then we have the inequalities:*

$$G_f(t) - H_f(t) \tag{3.61}$$
$$\geq \begin{cases} \left| \left| f\left(ta + (1-t)\frac{a+b}{2}\right)\right| - H_{|f|}(t) \right| \\ \text{if } f\left(ta + (1-t)\frac{a+b}{2}\right) = f\left(tb + (1-t)\frac{a+b}{2}\right) \\ \left| \frac{1}{\left|f\left(tb+(1-t)\frac{a+b}{2}\right) - f\left(ta+(1-t)\frac{a+b}{2}\right)\right|} \int\limits_{f\left(ta+(1-t)\frac{a+b}{2}\right)}^{f\left(tb+(1-t)\frac{a+b}{2}\right)} |x|\, dx - H_{|f|}(t) \right| \\ \text{otherwise} \end{cases};$$

and

$$H_f(t) - f\left(\frac{a+b}{2}\right) \tag{3.62}$$
$$\geq \left| \frac{1}{b-a} \int_a^b \left| \frac{f\left(tx + (1-t)\frac{a+b}{2}\right) + f\left((1+t)\frac{a+b}{2} - tx\right)}{2} \right| dx - \left| f\left(\frac{a+b}{2}\right) \right| \right|$$

for all $t \in [0,1]$.

PROOF. By the change of variable $y = tx + (1-t)\frac{a+b}{2}$, $x \in [a,b]$, $t \in (0,1]$, we have the equality

$$H_f(t) = \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx = \frac{1}{q-p} \int_p^q f(y)\, dy$$

where $p = ta + (1-t)\frac{a+b}{2}$ and $q = tb + (1-t)\frac{a+b}{2}$.
Now, using the inequality (2.24) from Chapter II, we get

$$\frac{f(p) + f(q)}{2} - \frac{1}{q-p} \int_p^q f(y)\, dy$$
$$\geq \begin{cases} \left| |f(p)| - \frac{1}{q-p} \int_p^q |f(y)|\, dy \right| & \text{if } \quad f(p) = f(q) \\ \left| \frac{1}{|f(q)-f(p)|} \int_{f(p)}^{f(q)} |x|\, dx - \frac{1}{q-p} \int_p^q |f(x)|\, dx \right| & \text{if } \quad f(p) \neq f(q). \end{cases}$$

However, it is clear that

$$\frac{f(p) + f(q)}{2} = G_f(t)$$

and

$$\frac{1}{q-p} \int_p^q |f(x)|\, dx = H_{|f|}(t)$$

Thus, the inequality (3.61) is proved.
Now, if we use the inequality (2.25) we can also state:

$$\frac{1}{q-p} \int_p^q f(y)\, dy - f\left(\frac{p+q}{2}\right)$$
$$\geq \left| \frac{1}{q-p} \int_p^q \left| \frac{f(y) + f(p+q-y)}{2} \right| dy - \left| f\left(\frac{p+q}{2}\right) \right| \right|.$$

However, it can be easily seen that

$$
\begin{aligned}
& \frac{1}{q-p}\int_p^q\left|\frac{f(y)+f(p+q-y)}{2}\right|dy \\
= & \frac{1}{b-a}\int_a^b\left|\frac{f\left(tx+(1-t)\frac{a+b}{2}\right)+f\left((1+t)\frac{a+b}{2}-tx\right)}{2}\right|dx
\end{aligned}
$$

for all $t\in[0,1]$. As the inequality (3.62) also holds for $t=0$, the proof is completed. ∎

We shall now give an improvement of the celebrated result of Bullen [**147**, p. 140] (see [**29**]):

Theorem 90. *With the above assumptions, we have the inequalities:*

$$
\begin{aligned}
& \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx \\
\geq & \begin{cases}
\left|\left|f\left(\frac{a+b}{2}\right)\right|-\int_0^1|G_f(t)|\,dt\right| \\
\text{if } f\left(\frac{a+b}{2}\right)=\frac{f(a)+f(b)}{2}; \\
\left|\frac{1}{\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)}\int_{f\left(\frac{a+b}{2}\right)}^{\frac{f(a)+f(b)}{2}}|x|\,dx-\int_0^1|G_f(t)|\,dt\right| \\
\text{if } f\left(\frac{a+b}{2}\right)<\frac{f(a)+f(b)}{2}
\end{cases}
\end{aligned}
\tag{3.63}
$$

and

$$
\begin{aligned}
& \frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\
\geq & \frac{1}{2}\left|\int_0^1|G_f(t)+G_f(1-t)|\,dt-\left|f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right|\right|.
\end{aligned}
\tag{3.64}
$$

Proof. Firstly, let us observe that

$$
\begin{aligned}
& \int_0^1 G_f(t)\,dt \\
= & \frac{1}{2}\left[\int_0^1 f\left(ta+(1-t)\frac{a+b}{2}\right)dt+\int_0^1 f\left((1-t)\frac{a+b}{2}+tb\right)dt\right] \\
= & \frac{1}{2}\left[\frac{2}{b-a}\int_a^{\frac{a+b}{2}}f(x)\,dx+\frac{2}{b-a}\int_{\frac{a+b}{2}}^b f(x)\,dx\right] \\
= & \frac{1}{b-a}\int_a^b f(x)\,dx.
\end{aligned}
$$

Now, if we use the inequalities (2.24) and (2.25) for the mapping G_f, we get

$$\begin{aligned}
&\frac{G_f(0)+G_f(1)}{2}-\int_0^1 G_f(t)\,dt \\
\geq\ &\begin{cases} \left|G_f(0)-\int_0^1 |G_f(t)|\,dt\right| & \text{if } \quad G_f(0)=G_f(1) \\ \left|\frac{1}{G_f(1)-G_f(0)}\int_{G_f(0)}^{G_f(1)} |x|\,dx-\int_0^1 |G_f(t)|\,dt\right| & \text{if } \quad G_f(0)\neq G_f(1) \end{cases}
\end{aligned}$$

and

$$\int_0^1 G_f(t)\,dt-G_f\left(\frac{1}{2}\right)\geq\left|\int_0^1\left|\frac{G_f(t)+G_f(1-t)}{2}\right|dt-\left|G_f\left(\frac{1}{2}\right)\right|\right|,$$

which are equivalent with the desired inequalities (3.63) and (3.64).
We shall omit the details. ∎

Now, under the same assumptions for the mapping f and $a,b\in\mathring{I}$, we can also reconsider the mapping $L:[0,1]\to\mathbb{R}$

$$L_f(t):=\frac{1}{2(b-a)}\int_a^b [f(ta+(1-t)x)+f((1-t)x+tb)]\,dx.$$

In Section 3, we proved among others (see Theorem 73) the following inequality

$$L_f(t)\geq H_f(1-t) \text{ for all } t\in[0,1].$$

We can improve this result here as follows [**29**]:

THEOREM 91. *With the above assumptions, one has the inequality*

$$\begin{aligned}
&L(t)-H(1-t) \\
\geq\ &\left|\frac{1}{(1-t)(b-a)}\int_a^{ta+(1-t)b}\left|\frac{f(u)+f(u+t(b-a))}{2}\right|du-H_{|f|}(1-t)\right| \\
\geq\ &0,
\end{aligned} \tag{3.65}$$

where $H_{|f|}$ is the function H written for $|f|$, for all $t\in[0,1)$.

PROOF. By the convexity of f and by the modulus properties, we have respectively:

$$\begin{aligned}
&\frac{f(ta+(1-t)x)+f((1-t)x+tb)}{2}-f\left((1-t)x+t\frac{a+b}{2}\right) \\
=\ &\left|\frac{f(ta+(1-t)x)+f((1-t)x+tb)}{2}-f\left((1-t)x+t\frac{a+b}{2}\right)\right| \\
\geq\ &\left|\left|\frac{f(ta+(1-t)x)+f((1-t)x+tb)}{2}\right|-\left|f\left((1-t)x+t\frac{a+b}{2}\right)\right|\right| \\
\geq\ &0
\end{aligned}$$

for all $x\in[a,b]$ and $t\in[0,1]$.
If we integrate the above inequality on $[a,b]$ over x we get that:

$$\begin{aligned}
&L_f(t)-H_f(1-t) \\
\geq\ &\left|\frac{1}{b-a}\int_a^b\left|\frac{f(ta+(1-t)x)+f((1-t)x+tb)}{2}\right|dx-H_{|f|}(1-t)\right| \\
\geq\ &0.
\end{aligned}$$

Denoting $u := ta + (1-t)x,\ x \in [a,b],\ t \in (0,1]$, we have that:

$$\begin{aligned} & \frac{1}{b-a}\int_a^b \left|\frac{f(ta+(1-t)x)+f((1-t)x+tb)}{2}\right| dx \\ = & \frac{1}{(1-t)(b-a)}\int_a^{ta+(1-t)b} \left|\frac{f(u)+f(u+t(b-a))}{2}\right| du \end{aligned}$$

and the inequality (3.65) is proved. ∎

Now, consider the mapping F given by a double integral

$$F_f(t) := \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy,$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on I, $a, b \in \mathring{I}$ with $a < b$.
The following theorem holds [**29**]:

THEOREM 92. *With the above assumptions, we have the inequalities:*

$$\begin{aligned} & \frac{1}{b-a}\int_a^b f(x)\,dx - F(t) \\ \geq & \left|\frac{1}{(b-a)^2}\int_a^b\int_a^b |tf(x)+(1-t)f(y)|\,dxdy - F_{|f|}(t)\right| \\ \geq & \ 0 \end{aligned} \tag{3.66}$$

and

$$\begin{aligned} & F(t) - \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right) dxdy \\ \geq & \left|\frac{1}{(b-a)^2}\int_a^b\int_a^b \left|\frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2}\right| dxdy \right. \\ & \left. -\frac{1}{(b-a)^2}\int_a^b\int_a^b \left|f\left(\frac{x+y}{2}\right)\right| dxdy\right| \\ \geq & \ 0 \end{aligned} \tag{3.67}$$

for all $t \in [0,1]$.

PROOF. As f is convex on I we have that

$$\begin{aligned} & tf(x)+(1-t)f(y)-f(tx+(1-t)y) \\ = & |tf(x)+(1-t)f(y)-f(tx+(1-t)y)| \\ \geq & ||tf(x)+(1-t)f(y)|-|f(tx+(1-t)y)|| \end{aligned}$$

for all $t \in (0,1]$ and $x \in [a,b]$.
Integrating the above inequality over x, y on $[a,b]^2$, we get that

$$\begin{aligned} & \frac{1}{(b-a)^2}\int_a^b\int_a^b [tf(x)+(1-t)f(y)]\,dxdy - F_f(t) \\ \geq & \left|\frac{1}{(b-a)^2}\int_a^b\int_a^b |tf(x)+(1-t)f(y)|\,dxdy - F_{|f|}(t)\right| \end{aligned}$$

for all $t \in [0,1]$.
As we have that

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b [tf(x)+(1-t)f(y)]\,dxdy=\frac{1}{b-a}\int_a^b f(x)\,dx,$$

the inequality (3.66) is obtained.
To prove the second inequality, we observe, by the convexity of f, that

$$\begin{aligned}&\frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2}-f\left(\frac{x+y}{2}\right)\\ =\;&\left|\frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2}-f\left(\frac{x+y}{2}\right)\right|\\ \geq\;&\left|\left|\frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2}\right|-\left|f\left(\frac{x+y}{2}\right)\right|\right|\end{aligned}$$

for all $t \in [0,1]$ and $x, y \in [a,b]$.
Integrating this inequality on $[a,b]^2$ over x, y and taking into account that

$$\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy=\int_a^b\int_a^b f((1-t)x+ty)\,dxdy,$$

we obtain the desired inequality (3.67). ∎

We know that $F_f(t) \geq H_f(t)$ for all $t \in [0,1]$ (see Theorem 74). This inequality can be improved as follows [**29**]:

Theorem 93. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping on the interval I and $a, b \in \mathring{I}$ with $a < b$. Then one has the inequality:*

$$\begin{aligned}&F_f(t)-H_f(t)\\ \geq\;&\left|\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|\frac{f(tx+(1-t)y)+f(tx+(1-t)(a+b-y))}{2}\right|dxdy\right.\\ &\left.-H_{|f|}(t)\right|\end{aligned}\tag{3.68}$$

for all $t \in [0,1]$.

Proof. Using the equality pointed out above (see the proof of Theorem 89):

$$\frac{1}{b-a}\int_a^b f(tx+(1-t)y)\,dy=\frac{1}{p-q}\int_q^p f(u)\,du$$

where

$$u := tx+(1-t)y,\ y\in[a,b],\ t\in[0,1),\ p=tx+(1-t)b,\ q=tx+(1-t)a,$$

we have, by (2.25) that:

$$\begin{aligned}&\frac{1}{p-q}\int_q^p f(u)\,du-f\left(\frac{p+q}{2}\right)\\ \geq\;&\left|\frac{1}{p-q}\int_q^p\left|\frac{f(u)+f(p+q-u)}{2}\right|du-\left|f\left(\frac{p+q}{2}\right)\right|\right|\end{aligned}$$

which is equivalent with

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f\left(tx+(1-t)y\right)dy - f\left(tx+(1-t)\frac{a+b}{2}\right)\\
\geq\ &\left|\frac{1}{b-a}\int_a^b \left|\frac{f\left(tx+(1-t)y\right)+f\left(tx+(1-t)(a+b-y)\right)}{2}\right| dy\right.\\
&\left.-\left|f\left(tx+(1-t)\frac{a+b}{2}\right)\right|\right|
\end{aligned}$$

for all $x\in[a,b]$ and $t\in[0,1]$.
If we integrate this inequality over $x\in[a,b]$, we derive the desired inequality (3.68). ∎

Note that in the inequality (3.29) we proved the inequality

$$L_f(1-t)\geq F_f(t) \text{ for all } t\in[0,1].$$

This inequality can be improved as follows [**29**]:

Theorem 94. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex mapping on the interval I, $a,b\in I$ and $a<b$. Then one has the inequality:*

$$\begin{aligned}
(3.69)\quad & L_f(1-t)-F_f(t)\\
\geq\ & \left|\frac{1}{b-a}\int_a^b\int_0^1 \left|sf\left(tx+(1-t)a\right)+(1-s)f\left(tx+(1-t)b\right)\right| dsdx - F_{|f|}(t)\right|\\
\geq\ & 0
\end{aligned}$$

for all $t\in[0,1]$.

Proof. As in the proof of Theorem 22, we have

$$\begin{aligned}
&\frac{f\left(tx+(1-t)a\right)+f\left(tx+(1-t)b\right)}{2}-\frac{1}{b-a}\int_a^b f\left(tx+(1-t)y\right)dy\\
\geq\ &\left|\int_0^1\left|sf\left(tx+(1-t)a\right)+(1-s)f\left(tx+(1-t)b\right)\right| ds\right.\\
&\left.-\frac{1}{b-a}\int_a^b\left|f\left(tx+(1-t)y\right)\right| dy\right|
\end{aligned}$$

for all $x\in[a,b]$ and $t\in[0,1]$.
Now, if we integrate this inequality on $[a,b]$ over x, we can easily deduce the inequality (3.69), and the theorem is proved. ∎

6.1. Applications for Special Means. Now, let us consider the mapping $h_0:[0,1]\to\mathbb{R}$ given by

$$h_0(t):=I\left(ta+(1-t)A(a,b),(1-t)A(a,b)+tb\right)$$

where $a,b>0$ with $a\leq b$, and the mapping $g_0:[0,1]\to\mathbb{R}$ given by

$$g_0(t):=G\left(ta+(1-t)A(a,b),(1-t)A(a,b)+tb\right)$$

where a,b are as above.

It is clear, by the definitions in Section 6, that

$$G_{-\ln}(t)=-\ln g_0(t) \text{ and } H_{-\ln}(t)=-\ln h_0(t)$$

for all $t\in[0,1]$.

By the inequality (3.61), we have that:

$$
\begin{aligned}
& \ln h_0(t)-\ln g_0(t) \\
= {} & G_{-\ln}(t)-H_{-\ln}(t) \\
\geq {} & \begin{cases}
\left|\left|\ln(ta+(1-t)A(a,b))\right|-\frac{1}{b-a}\int_a^b\left|\ln(tx+(1-t)A(a,b))\right|dx\right|, \\
\text{if } a=b \text{ or } t=0; \\
\left|\frac{1}{\ln\left[\frac{tb+(1-t)A(a,b)}{ta+(1-t)A(a,b)}\right]}\int_{\ln(ta+(1-t)A(a,b))}^{\ln(tb+(1-t)A(a,b))}|x|\,dx\right. \\
\left.-\frac{1}{b-a}\int_a^b\left|\ln(tx+(1-t)A(a,b))\right|dx\right|, \\
\text{if } a\neq b \text{ and } t\neq 0,
\end{cases}
\end{aligned}
$$

from where we get the inequality:

$$
\frac{h_0(t)}{g_0(t)}\geq\exp[\lambda(t)]\geq 1 \text{ for all } t\in[0,1]. \tag{3.70}
$$

where:

$$
\lambda(t):=\begin{cases}
\left|\left|\ln(ta+(1-t)A(a,b))\right|-\frac{1}{b-a}\int_a^b\left|\ln(tx+(1-t)A(a,b))\right|dx\right|, \\
\text{if } a=b \text{ or } t=0; \\
\left|\frac{1}{\ln\left[\frac{tb+(1-t)A(a,b)}{ta+(1-t)A(a,b)}\right]}\int_{\ln(ta+(1-t)A(a,b))}^{\ln(tb+(1-t)A(a,b))}|x|\,dx\right. \\
\left.-\frac{1}{b-a}\int_a^b\left|\ln(tx+(1-t)A(a,b))\right|dx\right|, \\
\text{if } a\neq b \text{ and } t\neq 0.
\end{cases}
$$

By the use of the inequality (3.62) we also have:

$$
\begin{aligned}
& \ln A(a,b)-\ln h_0(t) \\
\geq {} & \left|\frac{1}{b-a}\int_a^b\left|\ln\sqrt{(tx+(1-t)A(a,b)),((1+t)A(a,b)-tx)}\right|dx\right. \\
& \left.-\left|\ln(A(a,b))\right|\right| \\
\geq {} & 0
\end{aligned}
$$

which gives us the inequality:

$$
\begin{aligned}
(3.71)\qquad & \frac{A(a,b)}{h_0(t)} \\
\geq\ & \exp\left[\left|\frac{1}{b-a}\int_a^b \left|\ln\sqrt{(tx+(1-t)A(a,b)),((1+t)A(a,b)-tx)}\right| dx \right.\right. \\
& \left.\left. - \left|\ln(A(a,b))\right|\right|\right] \\
\geq\ & 1
\end{aligned}
$$

for all $t\in[0,1]$.

Now, if we use the inequality (3.63) for the convex mapping $-\ln$, we get:

$$
\ln I(a,b) - \ln\left[G(A(a,b),G(a,b))\right]
$$

$$
\leq \begin{cases} \left| \left|\ln(A(a,b))\right| - \int_0^1 \left|\ln g_0(t)\right| dt \right| & \text{if} \quad a=b, \\ \left| \frac{1}{\ln\left[\frac{A(a,b)}{G(a,b)}\right]} \int_{\ln G(a,b)}^{\ln A(a,b)} |x|\,dx - \int_0^1 \left|\ln g_0(t)\right| dt \right| & \text{if} \quad a\neq b, \end{cases}
$$

from where we deduce:

$$
\frac{I(a,b)}{G(A(a,b),G(a,b))} \geq \exp\left[\gamma_{a,b}\right] \geq 1 \tag{3.72}
$$

where

$$
\gamma_{a,b} := \begin{cases} \left| \left|\ln(A(a,b))\right| - \int_0^1 \left|\ln g_0(t)\right| dt \right| & \text{if} \quad a=b, \\ \left| \frac{1}{\ln\left[\frac{A(a,b)}{G(a,b)}\right]} \int_{\ln G(a,b)}^{\ln A(a,b)} |x|\,dx - \int_0^1 \left|\ln g_0(t)\right| dt \right| & \text{if} \quad a\neq b. \end{cases}
$$

Finally, if we use the inequality (3.64) for the convex mapping -ln, we get that:

$$
\begin{aligned}
& \ln G(A(a,A(a,b)),A(A(a,b),b)) - \ln I(a,b) \\
\geq\ & \frac{1}{2}\left|\int_0^1 \left|\ln\left[g_0(t)\,g_0(1-t)\right]\right| dt - \left|\ln\left[A(a,A(a,b))\,A(b,A(a,b))\right]\right|\right| \\
=\ & \left|\int_0^1 \left|\ln G(g_0(t),g_0(1-t))\right| dt - \left|\ln\left[G(A(a,A(a,b)),A(A(a,b),b))\right]\right|\right|
\end{aligned}
$$

which is equivalent with

$$
\frac{G(A(a,A(a,b)),A(A(a,b),b))}{I(a,b)} \geq \exp\left[m_{a,b}\right] \geq 1 \tag{3.73}
$$

where

$$
m_{a,b} := \left|\int_0^1 \left|\ln G(g_0(t),g_0(1-t))\right| dt - \left|\ln\left[G(A(a,A(a,b)),A(A(a,b),b))\right]\right|\right|.
$$

7. Another Generalisation of Féjer's Result

7.1. Introduction. In [**72**], Féjer proved that if $g : [a,b] \to \mathbb{R}$ is nonnegative integrable and symmetric to $x = \frac{a+b}{2}$, and if f is convex on $[a,b]$, then

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \le \int_a^b f(x)\,g(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \tag{3.74}$$

In 1991, Brenner and Alzer, [**11**] obtained the following result generalising Féjer's result as well as the result of Vasić, Lacković and Lupaş (see Introduction, Theorem 9).

Theorem 95. *If $g : [a,b] \to [0,\infty)$ is integrable and symmetric to $x = A = \frac{pa+qb}{p+q}$ with positive numbers p and q, then*

$$\begin{aligned} f\left(\frac{pa+qb}{p+q}\right)\int_{A-y}^{A+y} g(t)\,dt &\le \int_{A-y}^{A+y} f(t)\,g(t)\,dt \\ &\le \frac{pf(a)+qf(b)}{p+q}\int_{A-y}^{A+y} g(t)\,dt, \end{aligned} \tag{3.75}$$

where $0 \le y \le \frac{b-a}{p+q}\min(p,q)$, and f is convex on $[a,b]$.

In [**45**], Dragomir and in [**188**], Yang and Hong found convex monotonically real functions H and $\bar{F}$ defined on $[0,1]$ by

$$H(t) = \frac{1}{b-a}\int_a^b f\left[tx + (1-t)\frac{a+b}{2}\right]dx, \tag{3.76}$$

and

$$\begin{aligned} \bar{F}(t) &= \frac{1}{2(b-a)}\int_a^b \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right)\right. \\ &\quad \left. + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right)\right]dx, \end{aligned} \tag{3.77}$$

respectively such that

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= H(0) \le H(t) \le H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx \\ &= \bar{F}(0) \le \bar{F}(t) \le \bar{F}(1) = \frac{f(a)+f(b)}{2}. \end{aligned}$$

7.2. Some Results Related to the Brenner-Alzer Inequality. Following [**189**], we can state

Theorem 96. *Let $f : [a,b] \to \mathbb{R}$ be a convex function, $0 < \alpha < 1$, $0 < \beta < 1$, $A = \alpha a + (1-\alpha)b$, $u_0 = (b-a)\min\left\{\frac{\alpha}{1-\beta}, \frac{1-\alpha}{\beta}\right\}$ and let h be defined by*

$$h(t) = (1-\beta)f(A-\beta t) + \beta f(A + (1-\beta)t), \quad t \in [0,u_0].$$

Then h is convex, increasing on $[0,u_0]$ and for all $t \in [0,u_0]$,

$$f[\alpha a + (1-\alpha)b] \le h(t) \le \alpha f(a) + (1-\alpha)f(b). \tag{3.78}$$

Proof. We note that if f is convex and g is linear, then the composition $f \circ g$ is convex. Also, we note that a positive constant multiplied by a convex function and a sum of two convex functions are convex, hence h is convex on $[0,u_0]$. Next, if $t\in[0,u_0]$, it follows from the convexity of f that

$$\begin{aligned} h(t) &= (1-\beta) f(A-\beta t)+\beta f(A+(1-\beta)t) \\ &\geq f[(1-\beta)(A-\beta t)+\beta(A+(1-\beta)t)] \\ &= f(A)=f[\alpha a+(1-\alpha)b]. \end{aligned}$$

Also, we observe that

$$\begin{aligned} 0 &< \alpha \leq \frac{\alpha(b-a)+\beta t}{b-a}\leq 1, \quad 0\leq \frac{(1-\alpha)(b-a)-\beta t}{b-a}\leq 1-\alpha<1, \\ 0 &\leq \frac{\alpha(b-a)-(1-\beta)t}{b-a}\leq \alpha<1, \end{aligned}$$

and

$$0<1-\alpha<\frac{(1-\alpha)(b-a)+(1-\beta)t}{b-a}\leq 1,$$

so that

$$\begin{aligned} h(t) &= (1-\beta) f[\alpha a+(1-\alpha)b-\beta t] \\ &\quad +\beta f[\alpha a+(1-\alpha)b+(1-\beta)t] \\ &= (1-\beta) f\left[\frac{\alpha(b-a)+\beta t}{b-a}a+\frac{(1-\alpha)(b-a)-\beta t}{b-a}b\right] \\ &\quad +\beta f\left[\frac{\alpha(b-a)-(1-\beta)t}{b-a}a+\frac{(1-\alpha)(b-a)+(1-\beta)t}{b-a}b\right] \\ &\leq (1-\beta)\left[\frac{\alpha(b-a)+\beta t}{b-a}f(a)+\frac{(1-\alpha)(b-a)-\beta t}{b-a}f(b)\right] \\ &\quad +\beta\left[\frac{\alpha(b-a)-(1-\beta)t}{b-a}f(a)+\frac{(1-\alpha)(b-a)+(1-\beta)t}{b-a}f(b)\right] \\ &= \alpha f(a)+(1-\alpha) f(b), \end{aligned}$$

hence (3.78) holds. Finally, for t_1, t_2 such that $0<t_1<t_2\leq u_0$, since h is convex, it follows from (3.78) that

$$\frac{h(t_2)-h(t_1)}{t_2-t_1}\geq\frac{h(t_1)-h(0)}{t_1-0}=\frac{h(t_1)-f[\alpha a+(1-\alpha)b]}{t_1}\geq 0,$$

hence $h(t_2)\geq h(t_1)$. This shows that h is increasing on $[0,u_0]$, and the proof is completed. ∎

The following result also holds [**189**, Theorem 2]:

THEOREM 97. *Let f,α,β,A and u_0 be defined as in Theorem 96 and let $g:[a,b]\to\mathbb{R}$ be nonnegative and integrable and*

$$g(A-\beta u)=g(A+(1-\beta)u), \quad u\in[0,u_0]. \tag{3.79}$$

Then

$$\begin{aligned}
(3.80)\qquad & f\left[\alpha a+(1-\alpha)b\right]\int_{A-\beta u}^{A+(1-\beta)u} g(t)\,dt \\
&\le \frac{1-\beta}{\beta}\int_{A-\beta u}^{A} f(t)\,g(t)\,dt+\frac{\beta}{1-\beta}\int_{A}^{A+(1-\beta)u} f(t)\,g(t)\,dt \\
&\le \left[\alpha f(a)+(1-\alpha)f(b)\right]\int_{A-\beta u}^{A+(1-\beta)u} g(t)\,dt.
\end{aligned}$$

PROOF. For every $u\in[0,u_0]$, we have the identity

$$\begin{aligned}
(3.81)\quad \int_{A-\beta u}^{A+(1-\beta)u} g(t)\,dt &= \int_{A-\beta u}^{A} g(t)\,dt+\int_{A}^{A+(1-\beta)u} g(t)\,dt \\
&= \beta\int_0^u g(A-\beta t)\,dt+(1-\beta)\int_0^u g(A-\beta t)\,dt \\
&= \int_0^u g(A-\beta t)\,dt.
\end{aligned}$$

Since g is nonnegative, multiplying (3.78) by $g(A-\beta t)$, integrating the resulting inequalities over $[0,u]$, and using (3.79) we have

$$\begin{aligned}
& f\left[\alpha a+(1-\alpha)b\right]\int_0^u g(A-\beta t)\,dt \\
\le\ & (1-\beta)\int_0^u f(A-\beta t)\,g(A-\beta t)\,dt \\
& +\beta\int_0^u f(A+(1-\beta)t)\,g(A+(1-\beta)t)\,dt \\
=\ & \frac{1-\beta}{\beta}\int_{A-\beta u}^{A} f(t)\,g(t)\,dt+\frac{\beta}{1-\beta}\int_{A}^{A+(1-\beta)u} f(t)\,g(t)\,dt \\
\le\ & \left[\alpha f(a)+(1-\alpha)f(b)\right]\int_0^u g(A-\beta t)\,dt.
\end{aligned}$$

Thus, the inequalities (3.80) follow by using the identity (3.81). ∎

REMARK 48. *If we choose $\alpha=\frac{p}{p+q}$, $\beta=\frac{1}{2}$, and $u=2y$ in Theorem 97, then the inequalities (3.80) reduce to the inequalities (3.75).*

REMARK 49. *If we choose $\alpha=\beta=\frac{1}{2}$, and $u=u_0=b-a$ in Theorem 97, then the inequalities (3.80) reduce to the inequalities (3.74).*

REMARK 50. *If we choose $\alpha=\beta=\frac{1}{2}$, and $u=u_0=b-a$, and $g(x)\equiv 1$ in Theorem 97, then the inequalities (3.80) reduce to the H. – H. inequality.*

It is natural to consider the following mapping $\bar{H}$ (see [**189**, Theorem 3]).

THEOREM 98. *Let f, A and u_0 be defined as in Theorem 96, $0<\alpha<1$, $0<\beta<1$, $\alpha+\beta\le 1$, and let $\bar{H}$ be defined by*

$$\begin{aligned}
(3.82)\qquad \bar{H}(t) &= \frac{1-\beta}{\alpha(b-a)}\int_0^{\frac{\alpha(b-a)}{1-\beta}} \left[(1-\beta)f(A-\beta tx)\right. \\
&\quad \left.+\,\beta f(A+(1-\beta)tx)\right]dx,\quad 0\le t\le 1.
\end{aligned}$$

Then, $\bar{H}$ is convex monotonically increasing on $[0,1]$, and

$$
\begin{aligned}
(3.83) \qquad & f[\alpha a+(1-\alpha)b] \\
&= \bar{H}(0)\leq \bar{H}(t)\leq \bar{H}(1) \\
&= \frac{1-\beta}{\alpha(b-a)}\int_0^{\frac{\alpha(b-a)}{1-\beta}}[(1-\beta)f(A-\beta x)+\beta f(A+(1-\beta)x)]\,dx \\
&\leq \alpha f(a)+(1-\alpha)f(b).
\end{aligned}
$$

Proof. The fact that $\bar{H}$ is convex follows immediately from the convexity of f. Next, the condition $\alpha+\beta\leq 1$ implies that $u_0=\frac{\alpha(b-a)}{1-\beta}$. It follows from Theorem 96 that

$$h(t)=(1-\beta)f(A-\beta t)+\beta f(A+(1-\beta)t)$$

is increasing on $[0,u_0]$ and hence $H(t)$ is increasing on $[0,1]$.

Finally, the last inequality in (3.83) follows from (3.78), and the proof is completed. ∎

Similarly, we have the following theorem (see [**189**]).

Theorem 99. *Let f, A, u_0, α, β be defined as in Theorem 98. If*

$$
\begin{aligned}
(3.84) \qquad & \bar{G}(t) \\
&:= \frac{1-\beta}{\alpha(b-a)}\int_0^{\frac{\alpha(b-a)}{1-\beta}}\left[(1-\beta)f\left(A-\beta\left(\frac{\alpha(b-a)}{1-\beta}-x(1-t)\right)\right)\right. \\
&\quad \left.+\beta f\left(A+(1-\beta)\left(\frac{\alpha(b-a)}{1-\beta}-x(1-t)\right)\right)\right]dx,\quad 0\leq t\leq 1.
\end{aligned}
$$

then $\bar{G}$ is convex and monotonically increasing on $[0,1]$, and

$$
\begin{aligned}
(3.85) \qquad & \frac{(1-\beta)^2}{\alpha\beta(b-a)}\int_{b-\frac{\alpha(b-a)}{1-\beta}}^{A}f(x)\,dx+\frac{\beta}{\alpha(b-a)}\int_A^b f(x)\,dx \\
&= \bar{G}(0)\leq \bar{G}(t)\leq \bar{G}(1)=(1-\beta)f\left(b-\frac{\alpha(b-a)}{1-\beta}\right)+\beta f(b) \\
&\leq \alpha f(a)+(1-\alpha)f(b),\quad 0\leq t\leq 1.
\end{aligned}
$$

Remark 51. *The identity (3.76) is a special case of (3.82) taking $\alpha=\beta=\frac{1}{2}$.*

Remark 52. *The identity (3.77) is a special case of (3.84) taking $\alpha=\beta=\frac{1}{2}$.*

We can also state the following results [**189**].

Theorem 100. *Let f, A, u_0, α, β be defined as in Theorem 98 and let g be defined as in Theorem 97. Let $\bar{P}$ be a function defined on $[0,1]$ by*

$$
\begin{aligned}
(3.86) \qquad \bar{P}(t) &= \int_0^u[(1-\beta)f(A-\beta tx)g(A-\beta tx) \\
&\quad +\beta f(A+(1-\beta)tx)g(A+(1-\beta)tx)]\,dx,\quad 0\leq t\leq 1.
\end{aligned}
$$

for some $u \in [0, u_0]$. *Then* $\bar{P}$ *is convex and monotonically increasing on* $[0,1]$ *and*

$$\begin{aligned} (3.87) \qquad & f[\alpha a + (1-\alpha) b] \int_{A-\beta u}^{A+(1-\beta)u} g(x)\,dx \\ &= \bar{P}(0) \le \bar{P}(t) \le \bar{P}(1) \\ &= \frac{1-\beta}{\beta} \int_{A-\beta u}^{A} f(x)\,g(x)\,dx + \frac{\beta}{1-\beta} \int_{A}^{A+(1-\beta)u} f(x)\,g(x)\,dx. \end{aligned}$$

Proof. Since f is convex and g is nonnegative, we see that $\bar{P}$ is convex on $[0,1]$. Next, for each $x \in [0,u]$, where $u \in [0,u_0]$, it follows from Theorem 96 that

$$h(tx) = (1-\beta) f(A - \beta t x) + \beta f(A + (1-\beta) t x)$$

is increasing for $t \in [0,1]$. Using the identity (3.79) we see that $\bar{P}(t)$ is increasing on $[0,1]$. Therefore the inequalities (3.87) follow immediately. ∎

Theorem 101. *Let* f*,* A*,* u_0*,* α*,* β *be defined as in Theorem 100 and let* $\bar{Q}$ *be defined on* $[0,1]$ *by*

$$\begin{aligned} (3.88) \qquad \bar{Q}(t) &= \int_0^u \big[(1-\beta) f(A - \beta u + \beta x(1-t))\, g(A - \beta(u-x)) \\ &\quad + \beta f(A + (1-\beta) u - (1-\beta)(1-t) x) \\ &\quad \times g(A + (1-\beta)(u-x)) \big] dx \end{aligned}$$

for some $u \in [0, u_0]$. *Then* $\bar{Q}$ *is monotonically increasing and convex on* $[0,1]$*, and*

$$\begin{aligned} (3.89) \qquad & \frac{1-\beta}{\beta} \int_{A-\beta u}^{A} f(x)\,g(x)\,dx + \frac{\beta}{1-\beta} \int_{A}^{A+(1-\beta)u} f(x)\,g(x)\,dx \\ &= \bar{Q}(0) \le \bar{Q}(t) \le \bar{Q}(1) \\ &= [(1-\beta) f(A-\beta u) + \beta f(A + (1-\beta) u)] \int_{A-\beta u}^{A+(1-\beta)u} g(x)\,dx \\ &\le [\alpha f(a) + (1-\alpha) f(b)] \int_{A-\beta u}^{A+(1-\beta)u} g(x)\,dx. \end{aligned}$$

Proof. The fact that $\bar{Q}$ is convex follows immediately from the convexity of f. Next, for each $x \in [0,u]$, where $u \in [0,u_0]$, it follows from Theorem 96 that

$$h(t) = (1-\beta) f(A - \beta t) + \beta f(A + (1-\beta) t)$$

and

$$k(t) = u - (1-t) x$$

are increasing on $[0,u_0]$ and $[0,1]$ respectively. Hence

$$\begin{aligned} h(k(t)) &= (1-\beta) f(A - \beta u + \beta x(1-t)) \\ &\quad + \beta f(A + (1-\beta) u - (1-\beta)(1-t) x) \end{aligned}$$

is increasing on $[0,1]$. Since g is nonnegative and satisfies (3.79), it follows that $\bar{Q}(t)$ is monotonically increasing on $[0,1]$. Finally, the last inequalities in (3.89) follow from (3.88) and (3.78). ∎

Remark 53. *Choose $\alpha=\beta=\frac{1}{2}$, $u=u_0=b-a$ in Theorems 100 and 101. Then the inequalities (3.87) and (3.89) reduce to*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &= \bar{P}(0)\le\bar{P}(t)\le\bar{P}(1)=\int_a^b f(x)\,g(x)\,dx \\ &= \bar{Q}(0)\le\bar{Q}(t)\le\bar{Q}(1) \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx, \end{aligned}$$

where

$$\bar{P}(t)=\int_a^b f\left[tx+(1-t)\frac{a+b}{2}\right]g(x)\,dx$$

and

$$\begin{aligned} \bar{Q}(t) &= \frac{1}{2}\int_a^b\left[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)g\left(\frac{x+a}{2}\right)\right. \\ &\quad \left.+f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)g\left(\frac{x+b}{2}\right)\right]dx, \end{aligned}$$

which is a refinement of (3.74).

Remark 54. *Choose $\alpha=\beta=\frac{1}{2}$, $u=u_0=b-a$ and $g(x)\equiv 1$ in Theorems 100 and 101. Then*

$$\bar{P}(t)=(b-a)\,\bar{H}(t),\quad \bar{Q}(t)=(b-a)\,\bar{F}(t),$$

where $\bar{H}(t)$ and $\bar{F}(t)$ are defined in (3.76) and (3.77) respectively. Hence (3.86) and (3.88) generalise (3.76) and (3.77) respectively.

Remark 55. *Choose α,β such that $0<\alpha<1$, $0<\beta<1$, $\alpha+\beta\le 1$, and choose $u=u_0=\frac{\alpha(b-a)}{1-\beta}$, $A=\alpha a+(1-\alpha)b$, $g(x)\equiv 1$ in Theorems 100 and 101. Then $\bar{P}(t)=\frac{\alpha(b-a)}{1-\beta}\bar{H}(t)$ and $\bar{Q}(t)=\frac{\alpha(b-a)}{1-\beta}\bar{G}(t)$ where $\bar{H}(t)$ and $\bar{G}(t)$ are defined in Theorems 98 and 99 respectively. Hence Theorem 100 generalises Theorem 98 and Theorem 101 generalises Theorem 99.*

CHAPTER 4

Sequences of Mappings Associated with the $H.-H.$ Inequality

1. Some Sequences Defined by Multiple Integrals

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I and $a, b \in I$ with $a < b$. Define the sequence:

$$A_n(f) := \frac{1}{(b-a)^n} \int_a^b \dots \int_a^b f\left(\frac{x_1 + \dots + x_n}{n}\right) dx_1 \dots dx_n$$

for $n \geq 1$.

This sequence has the following properties [**68**]:

Theorem 102. *With the above assumptions, we have:*

(i) *The sequence $A_n(f)$ is monotonic nonincreasing; i.e.,*

$$A_{n+1}(f) \leq A_n(f) \text{ for all } n \geq 1; \tag{4.1}$$

(ii) *One has the bounds:*

$$f\left(\frac{a+b}{2}\right) \leq A_n(f) \leq \frac{1}{(b-a)} \int_a^b f(x)\, dx; \tag{4.2}$$

for all $n \geq 1$.

Proof. (i) Define the elements:

$$\begin{aligned} y_1 &:= \frac{x_1 + \dots + x_n}{n}, \\ y_2 &:= \frac{x_2 + \dots + x_{n+1}}{n}, \\ &\dots\dots\dots\dots\dots\dots \\ y_{n+1} &:= \frac{x_{n+1} + \dots + x_{n-1}}{n}. \end{aligned}$$

It is easy to see that:

$$\frac{y_1 + \dots + y_{n+1}}{n+1} = \frac{x_1 + \dots + x_{n+1}}{n+1},$$

and thus, by Jensen's discrete inequality, we have that:

$$\frac{f(y_1) + \dots + f(y_{n+1})}{n+1} \geq f\left(\frac{y_1 + \dots + y_{n+1}}{n+1}\right).$$

That is,

$$\begin{aligned}
& \frac{1}{n+1}\left[f\left(\frac{x_1+\ldots+x_n}{n}\right)+f\left(\frac{x_2+\ldots+x_{n+1}}{n}\right)+\ldots\right. \\
& \left.+f\left(\frac{x_{n+1}+\ldots+x_{n-1}}{n}\right)\right] \\
\geq\ & f\left(\frac{x_1+\ldots+x_{n+1}}{n+1}\right)
\end{aligned}$$

for all $x_i \in [a,b]$, $i=\overline{1,n+1}$.
If we integrate over $x_1,\ldots,x_{n+1}$ in $[a,b]^{n+1}$, we get:

$$\begin{aligned}
& \frac{1}{n+1}\left[\int_a^b \ldots \int_a^b f\left(\frac{x_1+\ldots+x_n}{n}\right)dx_1\ldots dx_{n+1}+\ldots\right. \\
& \left.+\int_a^b \ldots \int_a^b f\left(\frac{x_{n+1}+\ldots+x_{n-1}}{n}\right)dx_1\ldots dx_{n+1}\right] \\
\geq\ & \int_a^b \ldots \int_a^b f\left(\frac{x_1+\ldots+x_{n+1}}{n+1}\right)dx_1\ldots dx_{n+1}
\end{aligned}$$

and, as

$$\begin{aligned}
& \int_a^b \ldots \int_a^b f\left(\frac{x_1+\ldots+x_n}{n}\right)dx_1\ldots dx_{n+1} \\
=\ & \ldots = \int_a^b \ldots \int_a^b f\left(\frac{x_{n+1}+\ldots+x_{n-1}}{n}\right)dx_1\ldots dx_{n+1} \\
=\ & (b-a)\int_a^b \ldots \int_a^b f\left(\frac{x_1+\ldots+x_n}{n}\right)dx_1\ldots dx_n,
\end{aligned}$$

we obtain the inequality (4.1).
By Jensen's integral inequality for multiple integrals we also have

$$\begin{aligned}
A_n(f) &= \frac{1}{(b-a)^n}\int_a^b \ldots \int_a^b f\left(\frac{x_1+\ldots+x_n}{n}\right)dx_1\ldots dx_n \\
&\geq f\left(\frac{1}{(b-a)^n}\int_a^b \ldots \int_a^b \left[\frac{x_1+\ldots+x_n}{n}\right]dx_1\ldots dx_n\right).
\end{aligned}$$

As a simple calculation shows us that:

$$\begin{aligned}
& \frac{1}{(b-a)^n}\int_a^b \ldots \int_a^b \left[\frac{x_1+\ldots+x_n}{n}\right]dx_1\ldots dx_n \\
=\ & \frac{(b-a)^{n-1}}{(b-a)^n}\int_a^b x\,dx = \frac{a+b}{2},
\end{aligned}$$

the first inequality in (4.2) is proved.

(ii) The second inequality in (4.2) follows by (i) and the theorem is thus proved.

∎

As above, it is natural to consider the following sequences associated with the convex mapping $f : I \subseteq \mathbb{R} \to \mathbb{R}$ [**6**]:

$$\begin{aligned} B_n^{(1)}(f) &:= \frac{1}{(b-a)^{n-1}} \int_a^b \cdots \int_a^b f\left(\frac{x_1 + \dots + x_{n-1} + \frac{a+b}{2}}{n}\right) dx_1 \dots dx_{n-1} \\ B_n^{(2)}(f) &:= \frac{1}{(b-a)^{n-2}} \int_a^b \cdots \int_a^b f\left(\frac{x_1 + \dots + x_{n-2} + \frac{2(a+b)}{2}}{n}\right) dx_1 \dots dx_{n-2} \\ & \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ B_n^{(n-1)}(f) &:= \frac{1}{b-a} \int_a^b f\left(\frac{x_1 + \frac{(n-1)(a+b)}{2}}{n}\right) dx_1 \end{aligned}$$

which are defined for $n \geq 2$.

The following theorem contains some properties of these sequences [**6**]:

THEOREM 103. *With the above assumptions, one has:*

$$B_n^{(i+1)}(f) \leq B_n^{(i)}(f) \text{ for } 1 \leq i \leq n-2,\ n \geq 3, \tag{4.3}$$

i.e., $\left\{B_n^{(i)}\right\}_{i=1,\dots,n-1}$ *is monotonic nonincreasing and*

$$f\left(\frac{a+b}{2}\right) \leq B_n^{(i)}(f) \leq A_n(f) \text{ for } n \geq 2,\ i = \overline{1, n-1}. \tag{4.4}$$

PROOF. By Jensen's integral inequality, we have

$$\begin{aligned} B_n^{(i)}(f) &:= \frac{1}{(b-a)^{n-i}} \int_a^b \cdots \int_a^b f\left(\frac{x_1 + \dots + x_{n-i} + i\frac{(a+b)}{2}}{n}\right) dx_1 \dots dx_{n-i} \\ &\geq \frac{1}{(b-a)^{n-i-1}} \int_a^b \cdots \int_a^b f\left[\frac{1}{b-a}\right. \\ &\quad \times \left.\int_a^b \left(\frac{x_1 + \dots + x_{n-i-1} + x_{n-i} + i\frac{(a+b)}{2}}{n}\right) dx_{n-i}\right] dx_1 \dots dx_{n-i-1} \\ &= \frac{1}{(b-a)^{n-i-1}} \\ &\quad \times \int_a^b \cdots \int_a^b f\left(\frac{x_1 + \dots + x_{n-i-1} + (i+1)\frac{(a+b)}{2}}{n}\right) dx_1 \dots dx_{n-i-1} \\ &= B_n^{(i+1)}(f) \end{aligned}$$

for all $1 \leq i \leq n-2$, and the inequality (4.3) is proved.
For the first part of (4.4), it is sufficient to prove that $B_n^{(n-1)}(f) \geq f\left(\frac{a+b}{2}\right)$.

By Jensen's integral inequality, we have:

$$\begin{aligned}
B_n^{(n-1)}(f) &= \frac{1}{b-a}\int_a^b f\left(\frac{x_1+\frac{(n-1)(a+b)}{2}}{n}\right)dx_1 \\
&\geq f\left[\frac{1}{b-a}\left(\int_a^b \frac{x_1+\frac{(n-1)(a+b)}{2}}{n}dx_1\right)\right] \\
&= f\left(\frac{a+b}{2}\right).
\end{aligned}$$

Finally, we also have:

$$\begin{aligned}
A_n(f) &= \frac{1}{(b-a)^n}\int_a^b \dots \int_a^b f\left(\frac{x_1+\dots+x_n}{n}\right)dx_1\dots dx_n \\
&\geq \frac{1}{(b-a)^{n-1}}\int_a^b \dots \int_a^b f\left(\frac{1}{b-a}\int_a^b\left[\frac{x_1+\dots+x_n}{n}\right]dx_n\right)dx_1\dots dx_{n-1} \\
&= \frac{1}{(b-a)^{n-1}}\int_a^b \dots \int_a^b f\left(\frac{x_1+\dots+x_{n-1}+\frac{(a+b)}{2}}{n}\right)dx_1\dots dx_{n-1} \\
&= B_n^{(1)}(f),
\end{aligned}$$

which, along with the monotonicity of $\left\{B_n^{(i)}(f)\right\}_{i=1,\dots,n-2}$, proves the second part of (4.4).
The proof is thus completed. ∎

Now, let us assume that $(q_i)_{i\geq 1}$ are positive real numbers and define the sequence [**49**]:

$$A_n(f,q) := \frac{1}{(b-a)^n}\int_a^b \dots \int_a^b f\left(\frac{q_1x_1+q_2x_2+\dots+q_nx_n}{Q_n}\right)dx_1\dots dx_n,$$

where $Q_n := \sum_{i=1}^n q_i$.

We have the following result proved by S. S. Dragomir and C. Buşe in [**49**].

Theorem 104. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then one has the inequalities:*

$$A_n(f) \leq A_n(f,q) \leq \frac{1}{b-a}\int_a^b f(x)\,dx,\ n\geq 1 \tag{4.5}$$

for all $q = (q_i)_{i\geq 1}$ with $q_i > 0$ $(i \geq 1)$.

Proof. Let us consider the elements:

$$\begin{aligned}
y_1 &:= \frac{1}{Q_n}(q_1x_1+q_2x_2+\dots+q_nx_n) \\
y_2 &:= \frac{1}{Q_n}(q_nx_1+q_1x_2+\dots+q_{n-1}x_n) \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
y_n &:= \frac{1}{Q_n}(q_2x_1+q_3x_2+\dots+q_nx_{n-1}+q_1x_n).
\end{aligned}$$

A simple computation shows that:

$$\frac{y_1+y_2+...+y_{n-1}+y_n}{n}=\frac{x_1+x_2+...+x_{n-1}+x_n}{n},$$

and then Jensen's inequality,

$$\frac{1}{n}\left(f\left(y_1\right)+f\left(y_2\right)+...+f\left(y_{n-1}\right)+f\left(y_n\right)\right)\geq f\left(\frac{y_1+...+y_n}{n}\right)$$

yields that:

$$\begin{aligned}&\frac{1}{n}\left(f\left[\frac{1}{Q_n}\left(q_1x_1+...+q_nx_n\right)\right]+...+f\left[\frac{1}{Q_n}\left(q_2x_1+...+q_1x_n\right)\right]\right)\\&\geq f\left(\frac{x_1+x_2+...+x_{n-1}+x_n}{n}\right)\end{aligned}$$

for all $x_i\in[a,b]$ and $q_i\geq 0$ with $Q_n>0$.
Integrating this inequality on the $n-$dimensional interval $[a,b]^n$, we deduce:

$$\begin{aligned}&\frac{1}{n}\left[\frac{1}{(b-a)^n}\int_a^b...\int_a^b f\left(\frac{1}{Q_n}\left(q_1x_1+q_2x_2+...+q_nx_n\right)\right)dx_1...dx_n+...\right.\\&\left.+\frac{1}{(b-a)^n}\int_a^b...\int_a^b f\left(\frac{1}{Q_n}\left(q_2x_1+...+q_1x_n\right)\right)dx_1...dx_n\right]\\\geq\;&\frac{1}{(b-a)^n}\int_a^b...\int_a^b f\left(\frac{x_1+x_2+...+x_{n-1}+x_n}{n}\right)dx_1...dx_n.\end{aligned}$$

Since it can easily be shown that

$$\begin{aligned}&\frac{1}{(b-a)^n}\int_a^b...\int_a^b f\left(\frac{1}{Q_n}\left(q_1x_1+q_2x_2+...+q_nx_n\right)\right)dx_1...dx_n\\=\;&...=\frac{1}{(b-a)^n}\int_a^b...\int_a^b f\left(\frac{1}{Q_n}\left(q_2x_1+...+q_1x_n\right)\right)dx_1...dx_n,\end{aligned}$$

the first inequality in (4.5) is proved.
By Jensen's inequality, we also have

$$f\left(\frac{q_1x_1+q_2x_2+...+q_nx_n}{Q_n}\right)\leq\frac{q_1f\left(x_1\right)+q_2f\left(x_2\right)+...+q_nf\left(x_n\right)}{Q_n}.$$

Integrating this inequality over $[a,b]^n$, we obtain

$$\begin{aligned}A_n(f,q)&=\frac{1}{(b-a)^n}\int_a^b...\int_a^b f\left(\frac{q_1x_1+q_2x_2+...+q_nx_n}{Q_n}\right)dx_1...dx_n\\&\leq\frac{1}{(b-a)^n}\int_a^b...\int_a^b\left[\frac{q_1f\left(x_1\right)+q_2f\left(x_2\right)+...+q_nf\left(x_n\right)}{Q_n}\right]dx_1...dx_n\\&=\frac{1}{b-a}\int_a^b f(x)\,dx,\end{aligned}$$

and the proof is thus completed. ∎

It is also natural to consider the following sequences associated with the convex function f, to the sequence of positive weights $q=(q_i)_{i\geq 1}$ and to the permutation of the indices $(1,2,...,n)$, σ :

$$\begin{aligned}
B_n^{(1)}(f,q,\sigma) \quad &: \quad = \frac{1}{(b-a)^{n-1}}\int_a^b ... \int_a^b f\left[\frac{q_{\sigma(1)}x_1+...+q_{\sigma(n-1)}x_{n-1}}{Q_n}\right. \\
&\qquad \left. +\frac{\frac{(a+b)}{2}\cdot q_{\sigma(n)}}{Q_n}\right] dx_1...dx_{n-1} \\
B_n^{(2)}(f,q,\sigma) \quad &: \quad = \frac{1}{(b-a)^{n-2}}\int_a^b ... \int_a^b f\left[\frac{q_{\sigma(1)}x_1+...+q_{\sigma(n-2)}x_{n-2}}{Q_n}\right. \\
&\qquad \left. +\frac{\frac{(a+b)}{2}\cdot \left(q_{\sigma(n-1)}+q_{\sigma(n)}\right)}{Q_n}\right] dx_1...dx_{n-2} \\
&\qquad .... \\
B_n^{(n-1)}(f,q,\sigma) \quad &: \quad = \frac{1}{b-a}\int_a^b f\left[\frac{q_{\sigma(1)}x_1+\frac{(a+b)}{2}\cdot\left(q_{\sigma(2)}+...+q_{\sigma(n)}\right)}{Q_n}\right] dx_1
\end{aligned}$$

which are defined for $n\geq 2$.

The following theorem contains some properties of the above sequences [**49**].

THEOREM 105. *Let f, q_i $\left(i=\overline{1,n}\right)$ be as above and σ a permutation of the indices $(1,2,...,n)$. Then one has the inequalities:*

$$f\left(\frac{a+b}{2}\right)\leq B_n^{(n-1)}(f,q,\sigma)\leq ...\leq B_n^{(1)}(f,q,\sigma)\leq A_n(f,q). \tag{4.6}$$

PROOF. By Jensen's integral inequality, one has

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f\left(\frac{q_{\sigma(1)}x_1+\frac{(a+b)}{2}\cdot\left(q_{\sigma(2)}+...+q_{\sigma(n)}\right)}{Q_n}\right) dx_1 \\
\geq \quad & f\left(\frac{1}{b-a}\int_a^b \left[\frac{q_{\sigma(1)}x_1+\frac{(a+b)}{2}\cdot\left(q_{\sigma(2)}+...+q_{\sigma(n)}\right)}{Q_n}\right] dx_1\right) \\
= \quad & f\left(\frac{a+b}{2}\right),
\end{aligned}$$

which proves the first inequality in (4.6).

Now, by Jensen's inequality, we also have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f\left(\frac{q_{\sigma(1)}x_1+...+q_{\sigma(k)}x_k+\frac{(a+b)}{2}\cdot\left(q_{\sigma(k+1)}+...+q_{\sigma(n)}\right)}{Q_n}\right) dx_k \\
\geq \quad & f\left(\frac{q_{\sigma(1)}x_1+...+q_{\sigma(k-1)}x_{k-1}+\frac{(a+b)}{2}\cdot\left(q_{\sigma(k)}+...+q_{\sigma(n)}\right)}{Q_n}\right)
\end{aligned}$$

for all $x_1,...,x_{k-1}\in[a,b]$, which gives, by integration on $[a,b]^{k-1}$, that

$$B_n^{(n-k+1)}(f,q,\sigma)\leq B_n^{(n-k)}(f,q,\sigma),\ 2\leq k\leq n-1,$$

and the second part of (4.6) is thus proved.
For the last part of (4.6), we observe that:

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f\left(\frac{q_{\sigma(1)}x_1+...+q_{\sigma(n)}x_n}{Q_n}\right)dx_n \\
\geq\;& f\left(\frac{q_{\sigma(1)}x_1+...+q_{\sigma(n-1)}x_{n-1}+\frac{(a+b)}{2}\cdot q_{\sigma(n)}}{Q_n}\right)
\end{aligned}$$

for all $x_1,...,x_{n-1}\in[a,b]$, which gives, by integration on $[a,b]^{n-1}$, that

$$\begin{aligned}
&\frac{1}{(b-a)^n}\int_a^b...\int_a^b f\left(\frac{q_{\sigma(1)}x_1+...+q_{\sigma(n)}x_n}{Q_n}\right)dx_1...dx_n \\
\geq\;& \frac{1}{(b-a)^{n-1}} \\
&\times\int_a^b...\int_a^b f\left(\frac{q_{\sigma(1)}x_1+...+q_{\sigma(n-1)}x_{n-1}+\left[\frac{(a+b)}{2}\right]q_{\sigma(n)}}{Q_n}\right)dx_1...dx_{n-1}.
\end{aligned}$$

Since

$$\frac{1}{(b-a)^n}\int_a^b...\int_a^b f\left(\frac{q_{\sigma(1)}x_1+...+q_{\sigma(n)}x_n}{Q_n}\right)dx_1...dx_n=A_n(f,q)$$

for every σ a permutation of the indices $(1,...,n)$. The proof of the theorem is thus completed. ∎

2. Convergence Results

We shall start with the following theorem which contains a result of convergence for the sequence $A_n(f)$ defined by:

$$A_n(f):=\frac{1}{(b-a)^n}\int_a^b...\int_a^b f\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n,\ n\geq 1,$$

where $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is a convex mapping on the interval of real numbers I and $a,b\in I$ with $a<b$ (see [**6**]).

Theorem 106. *With the above assumptions for f, I and a,b, we have:*

$$\lim_{n\to\infty}A_n(f)=\inf\{A_n(f)\,|n\geq 1\}=f\left(\frac{a+b}{2}\right).$$

Firstly, we shall state the next lemma which is also interesting in itself [**6**]:

Lemma 12. *Let $g:\mathbb{R}\to\mathbb{R}$ be a bounded and Lebesgue measurable function on $\mathbb{R}$. If $a,b\in\mathbb{R}$, $a<b$ and g is continuous at $\frac{a+b}{2}$, then*

$$\lim_{n\to\infty}\frac{1}{(b-a)^n}\int_a^b...\int_a^b g\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n=g\left(\frac{a+b}{2}\right).$$

Proof. We shall give a probabilistic argument following [**6**].
Let (X_n), $X_n:(\Omega,\mathcal{F},p)\to\mathbb{R}$ be a sequence of independent random variables which are uniformly distributed on the interval $[a,b]$. By the use of the "*strong low of large numbers*", we have

$$\frac{X_1+X_2+...+X_n}{n}\longrightarrow^{\text{a.e.}} M(X_1)=\frac{a+b}{2}.$$

The mapping g being Lebesgue measurable on $\mathbb{R}$ and continuous at $\frac{a+b}{2}$, we obtain:

$$g\left(\frac{X_1+...+X_n}{n}\right) \longrightarrow^{\text{a.e.}} g\left(\frac{a+b}{2}\right).$$

Using the dominated convergence theorem of Lebesgue, we obtain:

$$\int_\Omega g\left(\frac{X_1+...+X_n}{n}\right)dp \to \int_\Omega g\left(\frac{a+b}{2}\right)dp = g\left(\frac{a+b}{2}\right).$$

However, $X_1,...,X_n$ are independent and the repartition $p\circ(X_1,...,X_n)^{-1}$ of the vector $(X_1,...,X_n)$ has the density $p(x_1,...,x_n) = p(x_1)...p_n(x_n)$, where $p_i(x_i)$ is the density of the random variable X_i.
Since X_i $(i\in\mathbb{N},\ i\geq 1)$ are uniformly distributed, $p_i(x) = (b-a)^{-1}$ if $x\in[a,b]$ and $p_i(x)=0$ if $x\in\mathbb{R}\backslash[a,b]$, $i\geq 1$. Then

$$\begin{aligned} & \int_\Omega g\left(\frac{X_1+...+X_n}{n}\right)dp \\ = & \int_{\mathbb{R}^n} g\left(\frac{a+b}{2}\right)d\left(p\circ(X_1,...,X_n)^{-1}\right)(x_1,...,x_n) \\ = & \frac{1}{(b-a)^n}\int_a^b ...\int_a^b g\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n. \end{aligned}$$

and the desired result is obtained. ∎

Proof. (Theorem 106) If $f: I\subseteq\mathbb{R}\to\mathbb{R}$ is convex on I, then the map $g: \mathbb{R}\to\mathbb{R}$, $g(x)=f(x)$ if $x\in I$ and $g(x)=0$ if $x\in\mathbb{R}\backslash I$, satisfies the conditions in the above lemma and then

$$\lim_{n\to\infty} A_n(f) = f\left(\frac{a+b}{2}\right).$$

The fact that $\lim_{n\to\infty} A_n(f) = \inf\{A_n(f)\,|n\geq 1\}$ follows by the monotonicity of $A_n(f)$ (see Theorem 102). ∎

The following corollary holds [**6**]:

Corollary 31. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex mapping defined on the interval I and $a,b\in I$ with $a<b$. Then we have the limits:*

$$\lim_{n\to\infty} B_n^{(n-1)}(f) = ... = \lim_{n\to\infty} B_n^{(1)}(f) = f\left(\frac{a+b}{2}\right).$$

Proof. The argument follows by the above theorem and from the inequality:

$$f\left(\frac{a+b}{2}\right) \leq B_n^{(n-1)}(f) \leq ... \leq B_n^{(1)}(f) \leq A_n(f) \text{ for } n\geq 2,$$

which were proved in Theorem 103. ∎

Now, let us assume that $(q_i)_{i\geq 1}$ are positive real numbers and reconsider the sequence

$$A_n(f,q) := \frac{1}{(b-a)^n}\int_a^b ...\int_a^b f\left(\frac{q_1x_1+q_2x_2+...+q_nx_n}{Q_n}\right)dx_1...dx_n.$$

The following theorem contains a sufficient condition for the convergence of $A_n(f,q)$ as follows [**13**]:

Theorem 107. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I and $a, b \in \mathring{I}$ with $a < b$ and $q_i(n) > 0$ for all $i, n \geq 1$. If*

$$\lim_{n\to\infty} \frac{q_1^2(n) + ... + q_n^2(n)}{Q_n^2} = 0,$$

where $Q_n = \sum_{i=1}^n q_i(n)$, then we have the limit:

$$\begin{aligned} &\lim_{n\to\infty} A_n(f,q) \\ = &\lim_{n\to\infty} \frac{1}{(b-a)^n} \int_a^b ... \int_a^b f\left(\frac{q_1(n)x_1 + q_2(n)x_2 + ... + q_n(n)x_n}{Q_n}\right) dx_1...dx_n \\ = & f\left(\frac{a+b}{2}\right). \end{aligned}$$

Firstly, we shall prove the following lemma which is itself of importance [**13**]:

Lemma 13. *Let $f : [a,b] \to \mathbb{R}$ be a Lebesgue integrable function on $[a,b]$ and continuous at the point $\frac{a+b}{2}$. If $q_i(n) > 0$ $\left(i = \overline{1,n}\right)$, $n \geq 1$ are such that*

$$\lim_{n\to\infty} \frac{q_1^2(n) + ... + q_n^2(n)}{Q_n^2} = 0$$

then

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{(b-a)^n} \int_{[a,b]^n} f\left(\frac{q_1(n)x_1 + ... + q_n(n)x_n}{Q_n}\right) dx_1...dx_n \\ = & f\left(\frac{a+b}{2}\right), \end{aligned}$$

where the above integral is considered in the Lebesgue sense.

Proof. Now, it is clear that the above integral exists and choosing $x_i = a + t_i(b-a)$, $i = \overline{1,n}$, we get:

$$\begin{aligned} &\frac{1}{(b-a)^n} \int_{[a,b]^n} f\left(\frac{q_1(n)x_1 + ... + q_n(n)x_n}{Q_n}\right) dx_1...dx_n \\ = &\int_{[0,1]^n} f\left(a + (b-a)\cdot \frac{q_1(n)t_1 + ... + q_n(n)t_n}{Q_n}\right) dt_1...dt_n \end{aligned}$$

Since f is continuous at $x_0 = \frac{a+b}{2}$, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\left|x - \frac{a+b}{2}\right| < \delta$ implies that $\left|f(x) - f\left(\frac{a+b}{2}\right)\right| < \frac{\varepsilon}{2}$.

Consider the mappings $\varphi_n : [0,1]^n \to [a,b]$ given by

$$\varphi_n(t) = a + \frac{b-a}{Q_n}\left[q_1(n)t_1 + ... + q_n(n)t_n\right],$$

where $t = (t_1, ..., t_n)$, and define the sets:

$$A_\delta(\varphi_n) := \left\{ t \in [0,1]^n : \left| \varphi_n(t) - \int_{[0,1]^n} \varphi_n(s)\, ds \right| \geq \delta \right\}.$$

$A_\delta(\varphi_n)$ are Lebesgue measurable in $[0,1]^n$.

If $A_\delta(\varphi_n) \neq \phi$, then

$$\left| \varphi_n(t) - \int_{[0,1]^n} \varphi_n(s)\, ds \right|^2 \geq \delta^2 \text{ for all } t \in A_\delta(\varphi_n)$$

and thus

$$\begin{aligned}
&\int_{[0,1]^n}\left(\varphi_n(t)-\int_{[0,1]^n}\varphi_n(s)\,ds\right)^2 dt\\
\geq\ &\int_{A_\delta(\varphi_n)}\left(\varphi_n(t)-\int_{[0,1]^n}\varphi_n(s)\,ds\right)^2 dt\\
\geq\ &\delta^2 mes\left(A_\delta(\varphi_n)\right).
\end{aligned}$$

That is,

$$\int_{[0,1]^n}\varphi_n^2(t)\,dt-\left(\int_{[0,1]^n}\varphi_n(s)\,ds\right)^2 dt\geq\delta^2 mes\left(A_\delta(\varphi_n)\right). \tag{4.7}$$

However, a simple calculation shows that

$$\begin{aligned}
&\int_{[0,1]^n}\varphi_n^2(t)\,dt\\
=\ &\int_{[0,1]^n}\left[a^2+2a\cdot\frac{b-a}{Q_n}\left(q_1(n)\,t_1+...+q_n(n)\,t_n\right)\right.\\
&\left.+\frac{(b-a)^2}{Q_n^2}\left(q_1(n)\,t_1+...+q_n(n)\,t_n\right)^2\right]dt\\
=\ &a^2+a(b-a)+\frac{(b-a)^2}{Q_n^2}\left[\frac{1}{3}\sum_{i=1}^n q_i^2(n)+\frac{1}{2}\sum_{1\leq i\leq j\leq n}q_i(n)\,q_j(n)\right]
\end{aligned}$$

and

$$\left[\int_{[0,1]^n}\varphi_n(t)\,dt\right]^2=a^2+a(b-a)+\frac{(b-a)^2}{4}.$$

By the inequality (4.7), we obtain:

$$\begin{aligned}
&\int_{[0,1]^n}\varphi_n^2(t)\,dt-\left[\int_{[0,1]^n}\varphi_n(t)\,dt\right]^2\\
=\ &\frac{(b-a)^2}{12}\cdot\frac{q_1^2(n)+...+q_n^2(n)}{Q_n^2}\\
\geq\ &\delta^2 mes\left(A_\delta(\varphi_n)\right).
\end{aligned}$$

Now, we have successively:

$$\begin{aligned} & \left| \int_{[0,1]^n} f\left(\varphi_n\left(t\right)\right) dt - f\left(\frac{a+b}{2}\right) \right| \\ \leq & \int_{[0,1]^n} \left| f\left(\varphi_n\left(t\right)\right) - f\left(\frac{a+b}{2}\right) \right| dt \\ = & \int_{[0,1]^n \setminus A_\delta(\varphi_n)} \left| f\left(\varphi_n\left(t\right)\right) - f\left(\frac{a+b}{2}\right) \right| dt \\ & + \int_{A_\delta(\varphi_n)} \left| f\left(\varphi_n\left(t\right)\right) - f\left(\frac{a+b}{2}\right) \right| dt \\ \leq & \frac{\varepsilon}{2} + 2Mmes\left(A_\delta\left(\varphi_n\right)\right) \\ \leq & \frac{\varepsilon}{2} + \frac{2M}{\delta^2} \cdot \frac{\left(b-a\right)^2}{12} \cdot \frac{q_1^2\left(n\right) + ... + q_n^2\left(n\right)}{Q_n^2}, \end{aligned}$$

where $M = \sup\left\{\left|f\left(x\right)\right| : x \in [a,b]\right\}.$

As

$$\frac{q_1^2\left(n\right) + ... + q_n^2\left(n\right)}{Q_n^2} \to 0,$$

hence, there exists an $n_\varepsilon \in \mathbb{N}$ such that

$$\frac{2M}{\delta^2} \cdot \frac{\left(b-a\right)^2}{12} \cdot \frac{q_1^2\left(n\right) + ... + q_n^2\left(n\right)}{Q_n^2} < \frac{\varepsilon}{2}$$

for all $n \geq n_\varepsilon$. Thus, we have

$$\left| \int_{[0,1]^n} f\left(\varphi_n\left(t\right)\right) dt - f\left(\frac{a+b}{2}\right) \right| \leq \varepsilon$$

and the required limit is proved. ∎

Proof. (Theorem 107) Since $f : I \to \mathbb{R}$ is convex on I and $a, b \in \mathring{I}$, f is continuous on $[a,b]$. Now, applying Lemma 13 and observing that

$$\begin{aligned} & \int_{[a,b]^n} f\left(\frac{q_1\left(n\right)x_1 + q_2\left(n\right)x_2 + ... + q_n\left(n\right)x_n}{Q_n}\right) dx_1...dx_n \\ = & \int_a^b ... \int_a^b f\left(\frac{q_1\left(n\right)x_1 + q_2\left(n\right)x_2 + ... + q_n\left(n\right)x_n}{Q_n}\right) dx_1...dx_n, \end{aligned}$$

where the last integral is considered in the Riemann sense, the proof of the theorem is completed. ∎

The following corollary holds [**13**]:

Corollary 32. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on I, $a, b \in \mathring{I}$ with $a < b$ and $q_i\left(n\right) > 0$ for all $i, n \geq 1$. If*

$$\lim_{n\to\infty} \frac{q_1^2\left(n\right) + ... + q_n^2\left(n\right)}{Q_n^2} = 0,$$

then

$$\lim_{n\to\infty} B_n^{(n-1)}\left(f, q, \sigma\right) = ... = B_n^{(1)}\left(f, q, \sigma\right) = f\left(\frac{a+b}{2}\right)$$

for all σ a permutation of the indices $\left(1, 2, ..., n\right)$.

Proof. The proof follows by the above theorem and by the inequality

$$f\left(\frac{a+b}{2}\right) \leq B_n^{(n-1)}(f,q,\sigma) \leq .... \leq B_n^{(1)}(f,q,\sigma) \leq A_n(f,q), \ n \geq 2,$$

which was proved in Theorem 105. ∎

3. Estimation of Some Sequences of Multiple Integrals

For an integrable mapping $f : [a,b] \to \mathbb{R}$, let us define the sequence of functionals by the following multiple integrals:

$$\begin{aligned} L_1(f) &:= \frac{f(a)+f(b)}{2}, \\ L_n(f) &:= \frac{1}{2(b-a)^{n-2}} \int_a^b \cdots \int_a^b \left[f\left(\frac{x_1+\dots+x_{n-1}+b}{n}\right) \right. \\ &\quad \left. + f\left(\frac{x_1+\dots+x_{n-1}+a}{n}\right)\right] dx_1 \dots dx_{n-1} \end{aligned}$$

for $n \geq 2$ and

$$A_n(f) := \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b f\left(\frac{x_1+\dots+x_n}{n}\right) dx_1 \dots dx_n \text{ for } n \geq 1.$$

The following lemma holds [**37**].

Lemma 14. *Let $f : I \subseteq \mathbb{R}$ be a differentiable function on $\mathring{I}$ and $a,b \in \mathring{I}$ with $a<b$. If f' is integrable on $[a,b]$, then we have the equality:*

$$\begin{aligned} & L_n(f) - A_n(f) \\ = & \frac{1}{n} \cdot \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b f'\left(\frac{x_1+\dots+x_n}{n}\right)\left(x_n - \frac{a+b}{2}\right) dx_1 \dots dx_n \end{aligned} \tag{4.8}$$

for all $n \geq 1$.

Proof. For $n=1$, we must prove that

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b f'(x)\left(x - \frac{a+b}{2}\right)dx.$$

Indeed, by an integration by parts, we have that:

$$\begin{aligned} \int_a^b f'(x)\left(x-\frac{a+b}{2}\right)dx &= f(x)\left(x-\frac{a+b}{2}\right)\Big|_a^b - \int_a^b f(x)\,dx \\ &= \frac{(b-a)(f(a)+f(b))}{2} - \int_a^b f(x)\,dx \end{aligned}$$

and the required identity is proved.

Let us prove the equality (4.8) for $n \geq 2$.

By an integration by parts, we have:

$$
\begin{aligned}
& \int_a^b f'\left(\frac{x_1+...+x_n}{n}\right)\left(x_n-\frac{a+b}{2}\right)dx_n \\
= \quad & n\, f\left(\frac{x_1+...+x_n}{n}\right)\left(x_n-\frac{a+b}{2}\right)\Big|_a^b - n\int_a^b f\left(\frac{x_1+...+x_n}{n}\right)dx_n \\
= \quad & n\left\{\frac{b-a}{2}\left[f\left(\frac{x_1+...+x_{n-1}+b}{n}\right)+f\left(\frac{x_1+...+x_{n-1}+a}{n}\right)\right]\right. \\
& \left. -\int_a^b f\left(\frac{x_1+...+x_n}{n}\right)dx_n\right\}.
\end{aligned}
$$

If we integrate on $[a,b]^{n-1}$, we have that:

$$
\begin{aligned}
& \frac{1}{n\,(b-a)^n}\int_a^b ... \int_a^b f'\left(\frac{x_1+...+x_n}{n}\right)\left(x_n-\frac{a+b}{2}\right)dx_1...dx_n \\
= \quad & \frac{1}{(b-a)^n}\int_a^b ... \int_a^b \frac{b-a}{2}\left[f\left(\frac{x_1+...+x_{n-1}+b}{n}\right)\right. \\
& \left. +f\left(\frac{x_1+...+x_{n-1}+a}{n}\right)\right]dx_1...dx_{n-1} \\
& -\frac{1}{(b-a)^n}\int_a^b ... \int_a^b f\left(\frac{x_1+...+x_n}{n}\right)dx_1..dx_n \\
= \quad & \frac{1}{2\,(b-a)^{n-1}}\int_a^b ... \int_a^b \left[f\left(\frac{x_1+...+x_{n-1}+b}{n}\right)\right. \\
& \left. +f\left(\frac{x_1+...+x_{n-1}+a}{n}\right)\right]dx_1...dx_{n-1} \\
& -\frac{1}{(b-a)^n}\int_a^b ... \int_a^b f\left(\frac{x_1+...+x_n}{n}\right)dx_1..dx_n \\
= \quad & L_n(f)-A_n(f)
\end{aligned}
$$

and the identity (4.8) is proved. ∎

By the use of the above lemma, we can point out the following estimation results for the sequences defined above [**37**].

THEOREM 108. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I and $a,b \in \mathring{I}$ with $a<b$. Then we have the inequality:*

$$
0 \le A_n(f) - f\left(\frac{a+b}{2}\right) \le n\left[L_n(f)-A_n(f)\right] \tag{4.9}
$$

for all $n \ge 1$.

PROOF. For $n=1$ we have

$$
\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx,
$$

which is Bullen's inequality.

We can assume, without loss of generality, that f is differentiable on I. Thus, we have the inequality

$$f\left(\frac{a+b}{2}\right)-f\left(\frac{x_1+...+x_n}{n}\right)\geq\left(\frac{a+b}{2}-\frac{x_1+...+x_n}{n}\right)f'\left(\frac{x_1+...+x_n}{n}\right)$$

for all $x_1,...,x_n\in[a,b]$.

Integrating on $[a,b]^n$ and dividing by $(b-a)^n$, we get that

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right)-A_n(f)\\
\geq\ & \frac{1}{(b-a)^n}\int_a^b...\int_a^b\left(\frac{a+b}{2}-\frac{x_1+...+x_n}{n}\right)f'\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n\\
=\ & \frac{1}{(b-a)^n}\int_a^b...\int_a^b\left(\frac{a+b}{2}-x_n\right)f'\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n
\end{aligned}$$

as

$$\begin{aligned}
& \int_a^b...\int_a^b x_1f'\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n\\
=\ & ...=\int_a^b...\int_a^b x_nf'\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n.
\end{aligned}$$

Using Lemma 14, we have that

$$\begin{aligned}
& \frac{1}{(b-a)^n}\int_a^b...\int_a^b\left(\frac{a+b}{2}-x_n\right)f'\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n\\
=\ & -n\left(L_n(f)-A_n(f)\right),
\end{aligned}$$

and from the above inequality we get (4.9). ∎

Another result of this type is embodied in the following theorem [**37**].

THEOREM 109. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function defined on the interval I and $a,b\in\mathring{I}$ with $a<b$. Then we have the inequality:*

$$0\leq A_n(f)-A_{n+1}(f)\leq\frac{n}{n+1}\left[L_n(f)-A_n(f)\right]\tag{4.10}$$

for all $n\geq 1$.

PROOF. For $n=1$ we have to prove that

$$\begin{aligned}
0 &\leq \frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy\\
&\leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right],
\end{aligned}$$

which is known (see Corollary 29, Chapter 3).

We can assume, without loss of generality, that f is differentiable and convex on I. Thus, we have the inequality

$$\begin{aligned} & f\left(\frac{x_1+...+x_{n+1}}{n+1}\right)-f\left(\frac{x_1+...+x_n}{n}\right) \\ \geq & \left(\frac{x_1+...+x_{n+1}}{n+1}-\frac{x_1+...+x_n}{n}\right)f'\left(\frac{x_1+...+x_n}{n}\right) \\ = & \left[\frac{nx_{n+1}-(x_1+...+x_n)}{n(n+1)}\right]f'\left(\frac{x_1+...+x_n}{n}\right), \end{aligned}$$

for all $x_1,...,x_{n+1}\in[a,b]$.

Integrating on $[a,b]^{n+1}$, we get:

$$\begin{aligned} & A_{n+1}(f)-A_n(f) \\ \geq & \frac{1}{(b-a)^{n+1}n(n+1)}\left[\int_a^b...\int_a^b nx_{n+1}f'\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_{n+1}\right. \\ & \left.-\int_a^b...\int_a^b(x_1+...+x_n)f'\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_{n+1}\right] \\ = & \frac{1}{(b-a)^{n+1}n(n+1)}\left[n\cdot\frac{b^2-a^2}{2}\int_a^b...\int_a^b f'\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n\right. \\ & \left.-n(b-a)\int_a^b...\int_a^b x_nf'\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n\right] \\ = & \frac{1}{(b-a)^n(n+1)}\left[\int_a^b...\int_a^b\left(\frac{a+b}{2}-x_n\right)f'\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n\right] \\ = & \frac{n}{n+1}\left[A_n(f)-L_n(f)\right], \end{aligned}$$

and the inequality (4.10) is proved. ∎

Next, we shall point out some estimations for the difference $L_n(f)-A_n(f)$ (see [**37**]).

Theorem 110. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\mathring{I}$ and $a,b\in\mathring{I}$ with $a<b$. If $|f'|^2$ is integrable on $[a,b]$, then we have the inequality:*

$$\begin{aligned} & |L_n(f)-A_n(f)| \\ \leq & \frac{\sqrt{3}(b-a)}{6n\sqrt{n}}\left[\frac{1}{(b-a)^n}\int_a^b...\int_a^b\left|f'\left(\frac{x_1+...+x_n}{n}\right)\right|^2dx_1...dx_n\right]^{\frac{1}{2}} \end{aligned} \tag{4.11}$$

for all $n\geq1$.

Proof. If $|f'|^2$ is integrable on $[a,b]$, then f' is integrable on $[a,b]$ and we have the equality (see Lemma 14):

$$\begin{aligned} & L_n(f)-A_n(f) \\ = & \frac{1}{n}\cdot\frac{1}{(b-a)^n}\int_a^b...\int_a^b f'\left(\frac{x_1+...+x_n}{n}\right)\left(x_n-\frac{a+b}{2}\right)dx_1...dx_n. \end{aligned}$$

On the other hand, it is clear that

$$\begin{aligned}&\int_a^b \cdots \int_a^b f'\left(\frac{x_1+\ldots+x_n}{n}\right) x_n \, dx_1 \ldots dx_n\\ =\ &\int_a^b \cdots \int_a^b f'\left(\frac{x_1+\ldots+x_n}{n}\right)\left(\frac{x_1+\ldots+x_n}{n}\right) dx_1 \ldots dx_n,\end{aligned}$$

and thus

$$\begin{aligned}&L_n(f)-A_n(f)\\ =\ &\frac{1}{n(b-a)^n}\int_a^b \cdots \int_a^b f'\left(\frac{x_1+\ldots+x_n}{n}\right)\left(\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\right) dx_1 \ldots dx_n\end{aligned}$$

for all $n \geq 1$.

If we apply the Cauchy-Buniakowsky-Schwartz integral inequality, we have

$$\begin{aligned}&\left|L_n(f)-A_n(f)\right| \\ \leq\ &\frac{1}{n}\left(\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left|f'\left(\frac{x_1+\ldots+x_n}{n}\right)\right|^2 dx_1 \ldots dx_n\right)^{\frac{1}{2}} \\ &\times\left(\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left|\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\right|^2 dx_1 \ldots dx_n\right)^{\frac{1}{2}}.\end{aligned} \tag{4.12}$$

Let us compute:

$$A:=\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\right)^2 dx_1 \ldots dx_n.$$

We have:

$$\begin{aligned}A\ =\ &\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left[\frac{1}{n^2}\left(x_1^2+\ldots+x_n^2+2\sum_{1\leq i<j\leq n} x_i x_j\right)\right] dx_1 \ldots dx_n\\ &-2\cdot\frac{a+b}{2}\cdot\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(\frac{x_1+\ldots+x_n}{n}\right) dx_1 \ldots dx_n+\left(\frac{a+b}{2}\right)^2.\end{aligned}$$

However, a simple calculation shows us that:

$$\begin{aligned}&\frac{1}{n^2}\cdot\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(x_1^2+\ldots+x_n^2+2\sum_{1\leq i<j\leq n} x_i x_j\right) dx_1 \ldots dx_n\\ =\ &\frac{1}{n^2}\left[n\cdot\frac{1}{(b-a)}\int_a^b x^2 dx+2\cdot\frac{n(n-1)}{2}\left(\frac{1}{b-a}\int_a^b x dx\right)^2\right]\\ =\ &\frac{1}{n}\left[\frac{b^3-a^3}{3(b-a)}+\frac{2(n-1)}{2}\cdot\left(\frac{a+b}{2}\right)^2\right]\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(\frac{x_1+\ldots+x_n}{n}\right) dx_1 \ldots dx_n\\ =\ &\frac{1}{(b-a)}\int_a^b x dx=\frac{a+b}{2}.\end{aligned}$$

Thus,

$$\begin{aligned}
A &= \frac{1}{n}\left[\frac{a^2+ab+b^2}{3}+(n-1)\left(\frac{a+b}{2}\right)^2\right]-\left(\frac{a+b}{2}\right)^2 \\
&= \frac{1}{n}\left[\frac{a^2+ab+b^2}{3}-\left(\frac{a+b}{2}\right)^2\right] \\
&= \frac{1}{12n}(b-a)^2 .
\end{aligned}$$

Using inequality (4.12) with A as above, we easily obtain the inequality (4.11) . We shall omit the details. ∎

COROLLARY 33. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathring{I}$ and $a, b \in \mathring{I}$ with $a < b$. If $M := \sup_{x\in[a,b]} |f'(x)| < \infty$, then we have the inequality:*

$$|L_n(f) - A_n(f)| \le \frac{\sqrt{3}(b-a)M}{6n\sqrt{n}} \tag{4.13}$$

for all $n \ge 1$.

The above corollary allows us to state the following estimation result for convex mappings [**37**].

THEOREM 111. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on I and $a, b \in \mathring{I}$ with $a < b$. If $M := \sup_{x\in[a,b]} |f'(x)| < \infty$, then we have the inequalities:*

$$0 \le A_n(f) - f\left(\frac{a+b}{2}\right) \le \frac{\sqrt{3}(b-a)M}{6\sqrt{n}},\ n \ge 1 \tag{4.14}$$

and

$$0 \le A_n(f) - A_{n+1}(f) \le \frac{\sqrt{3}(b-a)M}{6(n+1)\sqrt{n}},\ n \ge 1. \tag{4.15}$$

Moreover, we have that:

$$\lim_{n\to\infty} n^p\left[A_n(f) - f\left(\frac{a+b}{2}\right)\right] = 0 \text{ for } 0 \le p < \frac{1}{2}$$

and

$$\lim_{n\to\infty} n^q\left[A_n(f) - A_{n+1}(f)\right] = 0 \text{ for } 0 \le q < \frac{3}{2}.$$

Next, we shall point out some estimation results for the weighted sequence

$$A_n(f,q) = \frac{1}{(b-a)^n}\int_a^b \dots \int_a^b f\left(\frac{q_1x_1 + \dots + q_nx_n}{Q_n}\right)dx_1\dots dx_n$$

for all $n \ge 1$, where $q_i > 0,\ i = \overline{1,n}$, [**37**].

THEOREM 112. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on I and $a, b \in \mathring{I}$ with $a < b$. If $\left|f'_+\right|^2$ is integrable on $[a, b]$, then we have the inequality*

$$\begin{aligned}
(4.16)\quad 0 &\leq A_n(f,q) - f\left(\frac{a+b}{2}\right) \\
&\leq \frac{\sqrt{3}\left(\sum_{j=1}^{n} q_j^2\right)^{\frac{1}{2}}(b-a)}{6Q_n} \\
&\quad \times \left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left|f'_+\left(\frac{q_1x_1+\ldots+q_nx_n}{Q_n}\right)\right|^2 dx_1\ldots dx_n\right]^{\frac{1}{2}}
\end{aligned}$$

for $n \geq 1$.

PROOF. By the convexity of f, we can write that

$$\begin{aligned}
&f\left(\frac{q_1x_1+\ldots+q_nx_n}{Q_n}\right) - f\left(\frac{a+b}{2}\right) \\
\leq\; &\left(\frac{q_1x_1+\ldots+q_nx_n}{Q_n} - \frac{a+b}{2}\right) f'_+\left(\frac{q_1x_1+\ldots+q_nx_n}{Q_n}\right)
\end{aligned}$$

for all $x_1, \ldots, x_n \in [a, b]$.

If we integrate over $[a, b]^n$, we obtain

$$\begin{aligned}
(4.17)\quad &A_n(f,q) - f\left(\frac{a+b}{2}\right) \\
\leq\; &\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(\frac{q_1x_1+\ldots+q_nx_n}{Q_n} - \frac{a+b}{2}\right) \\
&\times f'_+\left(\frac{q_1x_1+\ldots+q_nx_n}{Q_n}\right) dx_1\ldots dx_n \\
\leq\; &\left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(\frac{q_1x_1+\ldots+q_nx_n}{Q_n} - \frac{a+b}{2}\right)^2 dx_1\ldots dx_n\right]^{\frac{1}{2}} \\
&\times \left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left|f'_+\left(\frac{q_1x_1+\ldots+q_nx_n}{Q_n}\right)\right|^2 dx_1\ldots dx_n\right]^{\frac{1}{2}}
\end{aligned}$$

on using the Cauchy-Buniakowsky-Schwartz inequality for the last inequality.

Now, denote

$$B := \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(\frac{q_1x_1+\ldots+q_nx_n}{Q_n} - \frac{a+b}{2}\right)^2 dx_1\ldots dx_n,\ n \geq 1.$$

Then we have

$$\begin{aligned} B &= \frac{1}{Q_n^2}\cdot\frac{1}{(b-a)^n} \\ &\times \int_a^b ... \int_a^b \left[q_1^2x_1^2+...+q_n^2x_n^2+2\sum_{1\le i<j\le n} q_ix_iq_jx_j \right] dx_1...dx_n \\ &-2\cdot\frac{a+b}{2}\cdot\frac{1}{(b-a)^n}\int_a^b ... \int_a^b \left(\frac{q_1x_1+...+q_nx_n}{Q_n}\right) dx_1...dx_n \\ &+\left(\frac{a+b}{2}\right)^2 . \end{aligned}$$

However,

$$\begin{aligned} &\frac{1}{Q_n^2}\cdot\frac{1}{(b-a)^n}\int_a^b ... \int_a^b \left[\sum_{j=1}^n q_j^2x_j^2+2\sum_{1\le i<j\le n} q_ix_iq_jx_j \right] dx_1...dx_n \\ &= \frac{1}{Q_n^2}\left[\left(\sum_{j=1}^n q_j^2\right)\cdot\frac{1}{b-a}\int_a^b x^2dx+2\sum_{1\le i<j\le n} q_iq_j\left(\frac{1}{b-a}\int_a^b xdx\right)^2 \right] \\ &= \frac{1}{Q_n^2}\left[\frac{b^2+ab+a^2}{3}\sum_{j=1}^n q_j^2+2\sum_{1\le i<j\le n} q_iq_j\left(\frac{a+b}{2}\right)^2 \right] \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{(b-a)^n}\int_a^b ... \int_a^b \left[\frac{q_1x_1+...+q_nx_n}{Q_n}\right] dx_1...dx_n \\ &= \frac{1}{b-a}\int_a^b xdx=\frac{a+b}{2}. \end{aligned}$$

Then we have

$$\begin{aligned} B &= \frac{1}{Q_n^2}\left[\sum_{j=1}^n q_j^2\left(\frac{b^2+ab+a^2}{3}\right)+2\sum_{1\le i<j\le n} q_iq_j\left(\frac{a+b}{2}\right)^2 \right]-\left(\frac{a+b}{2}\right)^2 \\ &= \frac{1}{Q_n^2}\left[\sum_{j=1}^n q_j^2\left(\frac{b^2+ab+a^2}{3}\right)+2\sum_{1\le i<j\le n} q_iq_j\left(\frac{a+b}{2}\right)^2-Q_n^2\left(\frac{a+b}{2}\right)^2 \right]. \end{aligned}$$

As

$$Q_n^2=\sum_{j=1}^n q_j^2+2\sum_{1\le i<j\le n} q_iq_j,$$

then

$$\begin{aligned} B &= \frac{1}{Q_n^2}\cdot\sum_{j=1}^n q_j^2\left[\frac{b^2+ab+a^2}{3}-\left(\frac{a+b}{2}\right)^2\right] \\ &= \frac{(b-a)^2\sum_{j=1}^n q_j^2}{12Q_n^2}. \end{aligned}$$

Using the inequality (4.17) , we deduce the desired inequality (4.16) . ∎

COROLLARY 34. *With the above assumptions, given that* $M := \sup_{x\in[a,b]} |f'(x)| < \infty$, *we have the inequality:*

$$0 \le A_n(f,q) - f\left(\frac{a+b}{2}\right) \le \frac{\sqrt{3}(b-a)\left(\sum_{j=1}^n q_j^2\right)^{\frac{1}{2}}}{6Q_n} M \tag{4.18}$$

for $n \ge 1$.

REMARK 56. *Note that if* $\lim_{n\to\infty} \frac{\sum_{j=1}^n q_j^2}{Q_n^2} = 0$, *then, from* (4.18), *we recapture the result from Theorem 107.*

The following result also holds [**37**]:

THEOREM 113. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a convex function on* I *and* $a, b \in \mathring{I}$ *with* $a < b$. *If* $\left|f'_+\right|^2$ *is integrable on* $[a,b]$ *and* $q_i > 0$ $(i \in \mathbb{N}^*)$, *then one has the estimation:*

$$\begin{aligned}
0 &\le A_n(f,q) - A_n(f) \\
&\le \frac{\sqrt{3}(b-a)}{6}\left[\sum_{j=1}^n \left(\frac{q_j}{Q_n} - \frac{1}{n}\right)^2\right]^{\frac{1}{2}} \\
&\quad \times \left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left|f'_+\left(\frac{q_1x_1 + \ldots + q_nx_n}{Q_n}\right)\right|^2 dx_1 \ldots dx_n\right]^{\frac{1}{2}}
\end{aligned} \tag{4.19}$$

for all $n \ge 1$, *where* $Q_n := \sum_{i=1}^n q_i$.

PROOF. Using the convexity of f, we have that

$$\begin{aligned}
&f\left(\frac{q_1x_1 + \ldots + q_nx_n}{Q_n}\right) - f\left(\frac{x_1 + \ldots + x_n}{n}\right) \\
&\le \left(\frac{q_1x_1 + \ldots + q_nx_n}{Q_n} - \frac{x_1 + \ldots + x_n}{n}\right) f'_+\left(\frac{q_1x_1 + \ldots + q_nx_n}{Q_n}\right)
\end{aligned}$$

for all $x_1, \ldots, x_n \in [a,b]$.

Integrating on $[a,b]^n$, we obtain

$$\begin{aligned}
&A_n(f,q) - A_n(f) \\
&\le \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(\frac{q_1x_1 + \ldots + q_nx_n}{Q_n} - \frac{x_1 + \ldots + x_n}{n}\right) \\
&\quad \times f'_+\left(\frac{q_1x_1 + \ldots + q_nx_n}{Q_n}\right) dx_1 \ldots dx_n \\
&\le \left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(\frac{q_1x_1 + \ldots + q_nx_n}{Q_n} - \frac{x_1 + \ldots + x_n}{n}\right)^2 dx_1 \ldots dx_n\right]^{\frac{1}{2}} \\
&\quad \times \left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left|f'_+\left(\frac{q_1x_1 + \ldots + q_nx_n}{Q_n}\right)\right|^2 dx_1 \ldots dx_n\right]^{\frac{1}{2}},
\end{aligned} \tag{4.20}$$

by applying the Cauchy-Buniakowsky-Schwartz integral inequality for the last inequality.

Let us define

$$C := \frac{1}{(b-a)^n} \int_a^b ... \int_a^b \left(\frac{q_1 x_1 + ... + q_n x_n}{Q_n} - \frac{x_1 + ... + x_n}{n} \right)^2 dx_1 ... dx_n.$$

Then we have:

$$\begin{aligned}
C &= \frac{1}{Q_n^2 n^2} \cdot \frac{1}{(b-a)^n} \\
&\quad \times \int_a^b ... \int_a^b \left[(nq_1 - Q_n) x_1 + ... + (nq_n - Q_n) x_n \right]^2 dx_1 ... dx_n \\
&= \frac{1}{n^2 Q_n^2} \cdot \frac{1}{(b-a)^n} \int_a^b ... \int_a^b \left[\sum_{j=1}^n (nq_j - Q_n)^2 x_j^2 \right. \\
&\quad \left. + 2 \sum_{1 \le i < j \le n} (nq_i - Q_n)(nq_j - Q_n) x_i x_j \right] dx_1 ... dx_n \\
&= \frac{1}{n^2 Q_n^2} \left[\sum_{j=1}^n (nq_j - Q_n)^2 \frac{1}{b-a} \int_a^b x^2 dx \right. \\
&\quad \left. + 2 \sum_{1 \le i < j \le n} (nq_i - Q_n)(nq_j - Q_n) \left(\frac{1}{b-a} \int_a^b x dx \right)^2 \right] \\
&= \frac{1}{n^2 Q_n^2} \left[\sum_{j=1}^n (nq_j - Q_n)^2 \frac{a^2 + ab + b^2}{3} \right. \\
&\quad \left. + 2 \sum_{1 \le i < j \le n} (nq_i - Q_n)(nq_j - Q_n) \left(\frac{a+b}{2} \right)^2 \right] \\
&= \frac{1}{n^2 Q_n^2} \left[\sum_{j=1}^n (nq_j - Q_n)^2 \left[\frac{a^2 + ab + b^2}{3} - \left(\frac{a+b}{2} \right)^2 \right] \right. \\
&\quad \left. + \left(\frac{a+b}{2} \right)^2 \left[\sum_{j=1}^n (nq_i - Q_n)^2 + 2 \sum_{1 \le i < j \le n} (nq_i - Q_n)(nq_j - Q_n) \right] \right].
\end{aligned}$$

However, it is easy to see that

$$\begin{aligned}
&\sum_{j=1}^n (nq_j - Q_n)^2 + 2 \sum_{1 \le i < j \le n} (nq_i - Q_n)(nq_j - Q_n) \\
&= \left[\sum_{j=1}^n (nq_j - Q_n) \right]^2 = 0.
\end{aligned}$$

Hence,

$$C = \frac{\sum_{j=1}^n (Q_n - nq_j)^2}{Q_n^2 n^2} \cdot \frac{(b-a)^2}{12}.$$

Finally, by using inequality (4.20), we deduce the desired inequality (4.19). ∎

COROLLARY 35. *With the above assumptions, given that* $M := \sup_{x\in[a,b]} |f'(x)| < \infty$, *we have the inequality:*

$$0 \le A_n(f,q) - A_n(f) \le \frac{\sqrt{3}(b-a)M}{6}\left[\sum_{j=1}^{n}\left(\frac{q_j}{Q_n}-\frac{1}{n}\right)^2\right]^{\frac{1}{2}} \tag{4.21}$$

for all $n \ge 1$.

REMARK 57. *If we assume that* $q_i > 0$ $(i \in \mathbb{N}^*)$ *are such that*

$$\lim_{n\to\infty}\frac{\sum_{j=1}^{n}(Q_n - nq_j)^2}{Q_n^2 n^2} = 0,$$

then we have

$$\lim_{n\to\infty}[A_n(f,q) - A_n(f)] = 0.$$

4. Further Generalizations

Now, let T be a nonempty set and m be a natural number, with $m \ge 2$. Suppose that $\alpha_1, ..., \alpha_m : T \to \mathbb{R}$ are m functions with the property

$$\alpha_i(t) \ge 0 \ \left(i = \overline{1,m}\right) \text{ and } \alpha_1(t) + ... + \alpha_m(t) = 1 \text{ for all } t \in T.$$

Consider $f : I \subseteq \mathbb{R} \to \mathbb{R}$, a given convex mapping in the interval I and $a, b \in \mathring{I}$ with $a < b$. We can define the following sequence of mappings [**57**]

$$\begin{aligned}
H_1^{[m]}(t) &:= \frac{1}{b-a}\int_a^b f\left(\alpha_1(t)x_1 + (\alpha_2(t) + ... + \alpha_m(t))\frac{a+b}{2}\right)dx_1\\
H_2^{[m]}(t) &:= \frac{1}{(b-a)^2}\int_a^b\int_a^b f\bigg(\alpha_1(t)x_1 + \alpha_2(t)x_2\\
&\quad + (\alpha_3(t) + ... + \alpha_m(t))\frac{a+b}{2}\bigg)dx_1dx_2\\
&\quad ....\\
H_{m-1}^{[m]}(t) &:= \frac{1}{(b-a)^{m-1}}\int_a^b ... \int_a^b f\bigg(\alpha_1(t)x_1 + ... + \alpha_{m-1}(t)x_{m-1}\\
&\quad + \alpha_m(t)\frac{a+b}{2}\bigg)dx_1...dx_{m-1}
\end{aligned}$$

and

$$H^{[m]}(t) := \frac{1}{(b-a)^m}\int_a^b ... \int_a^b f(\alpha_1(t)x_1 + ... + \alpha_m(t)x_m)\,dx_1...dx_m,$$

where t is in T.

The following theorem holds [**57**]:

THEOREM 114. *Let* f, α_i $\left(i = \overline{1,m}\right)$ *and* m *be as above. Then:*

(i) *We have the inequalities:*

$$f\left(\frac{a+b}{2}\right) \le H_1^{[m]}(t) \le ... \le H_{m-1}^{[m]}(t) \le H^{[m]}(t) \le \frac{1}{b-a}\int_a^b f(x)\,dx \tag{4.22}$$

for all $t \in T$;

(ii) *If there exists a* $t_0 \in T$ *such that* $\alpha_1(t_0) = ... = \alpha_p(t_0) = 0$ $(1 \le p \le m-1)$, *then*

$$\inf_{t \in T} H_l^{[m]}(t) = H_l^{[m]}(t_0) = f\left(\frac{a+b}{2}\right) \text{ for } 1 \le l \le p; \tag{4.23}$$

(iii) *If there exists a* $t_1 \in T$ *such that* $\alpha_p(t_1) = 1$ $(1 \le p \le m-1)$, *then*

$$\sup_{t \in T} H_l^{[m]}(t) = \sup_{t \in T} H^{[m]}(t) = H_l^{[m]}(t_1) = H^{[m]}(t_1) = \frac{1}{b-a}\int_a^b f(x)\,dx \tag{4.24}$$

and

$$\inf_{t \in T} H_q^{[m]}(t) = H_q^{[m]}(t_1) = f\left(\frac{a+b}{2}\right) \tag{4.25}$$

for all $p \le l \le m-1$ *and* $1 \le q \le p-1$;

(iv) *If* T *is a convex subset of a linear space* Y *and the* α_i $\left(i = \overline{1,m}\right)$ *satisfy the condition*

$$\alpha_i(\gamma t_1 + \beta t_2) = \gamma \alpha_i(t_1) + \beta \alpha_i(t_2), \ \left(i = \overline{1,m}\right)$$

for all t_1, t_2 *in* T *and* γ, β *with* $\gamma + \beta = 1$ *and* $\gamma, \beta \ge 0$, *then* $H_l^{[m]}$ $(1 \le l \le m-1)$ *and* $H^{[m]}$ *are convex mappings on* T.

Proof.

(i) By Jensen's integral inequality, we have that

$$\begin{aligned} H_1^{[m]}(t) &\ge f\left(\alpha_1(t) \cdot \frac{1}{b-a}\int_a^b x_1 dx_1 + (\alpha_2(t) + ... + \alpha_m(t))\frac{a+b}{2}\right) \\ &= f\left(\frac{a+b}{2}\right) \end{aligned}$$

for all $t \in T$, which shows the first inequality in (4.22).

Now, suppose that $1 \le l \le m-1$ and $t \in T$. Then, by Jensen's integral inequality, we also have:

$$\begin{aligned} & H_{l+1}^{[m]}(t) \\ = & \frac{1}{(b-a)^{l+1}}\int_a^b ... \int_a^b f\Bigg(\alpha_1(t)x_1 + ... + \alpha_{l+1}(t)x_{l+1} \\ & + (\alpha_{l+2}(t) + ... + \alpha_m(t))\frac{a+b}{2}\Bigg)dx_1...dx_{l+1} \\ \ge & \frac{1}{(b-a)^l}\int_a^b ... \int_a^b f\Bigg(\alpha_1(t)x_1 + ... + \alpha_l(t)x_l \\ & + \alpha_{l+1}(t)\frac{1}{b-a}\int_a^b x_{l+1}dx_{l+1} + (\alpha_{l+2}(t) + ... + \alpha_m(t))\frac{a+b}{2}\Bigg)dx_1...dx_l \\ = & H_l^{[m]}(t), \end{aligned}$$

which shows that the finite sequence $\left\{H_l^{[m]}(t)\right\}_{l=\overline{1,m-1}}$ is monotonic non-decreasing for all $t \in T$.

On the other hand, Jensen's inequality yields that

$$\begin{aligned} & H^{[m]}(t) \\ \geq\ & \frac{1}{(b-a)^{m-1}} \int_a^b \dots \int_a^b f\Bigg(\alpha_1(t)x_1 + \dots + \alpha_{m-1}(t)x_{m-1} \\ & + \alpha_m(t) \frac{1}{b-a} \int_a^b x_m dx_m \Bigg) dx_1 \dots dx_{m-1} \\ =\ & H_{m-1}^{[m]}(t) \end{aligned}$$

for all $t \in T$.
Finally, by the convexity of f, one has:

$$f(\alpha_1(t)x_1 + \dots + \alpha_m(t)x_m) \leq \alpha_1(t) f(x_1) + \dots + \alpha_m(t) f(x_m)$$

for all $t \in T$ and $x_i \in [a,b]$ with $i = \overline{1,m}$. Integrating this inequality on $[a,b]^m$, we derive:

$$\begin{aligned} & \int_a^b \dots \int_a^b f(\alpha_1(t)x_1 + \dots + \alpha_m(t)x_m) dx_1 \dots dx_m \\ \leq\ & \int_a^b \dots \int_a^b [\alpha_1(t) f(x_1) + \dots + \alpha_m(t) f(x_m)] dx_1 \dots dx_m \\ =\ & (b-a)^{m-1} \int_a^b f(x) dx, \end{aligned}$$

which implies the inequality

$$H^{[m]}(t) \leq \frac{1}{b-a} \int_a^b f(x) dx$$

for all $t \in T$, and the statement (4.22) is proved.

(ii) If $\alpha_1(t_0) = \dots = \alpha_p(t_0) = 0 \ (1 \leq p \leq m-1)$, then

$$\alpha_{p+1}(t_0) + \dots + \alpha_m(t_0) = 1.$$

Therefore,

$$\begin{aligned} & H_p^{[m]}(t_0) \\ =\ & \frac{1}{(b-a)^p} \int_a^b \dots \int_a^b f\left[(\alpha_{p+1}(t_0) + \dots + \alpha_m(t_0)) \frac{a+b}{2} \right] dx_1 \dots dx_p \\ =\ & f\left(\frac{a+b}{2}\right). \end{aligned}$$

Since

$$f\left(\frac{a+b}{2}\right) \leq H_1^{[m]}(t_0) \leq \dots \leq H_p^{[m]}(t_0),$$

the statement (ii) is proved.

(iii) If $\alpha_p(t_1) = 1$, then $\alpha_s(t_1) = 0$ for all $s \neq p$ $(1 \leq s \leq m)$. Then for $p \leq l \leq m-1$, we have

$$\begin{aligned}
H_l^{[m]}(t_1) &= \frac{1}{(b-a)^l}\int_a^b \dots \int_a^b f(x_p)\,dx_1 \dots dx_p = \frac{1}{b-a}\int_a^b f(x)\,dx \\
&= \frac{1}{(b-a)^m}\int_a^b \dots \int_a^b f(x_p)\,dx_1 \dots dx_m \\
&= H^{[m]}(t_1).
\end{aligned}$$

If $1 \leq q \leq p-1$, one has

$$H_q^{[m]}(t_1) = \frac{1}{(b-a)^q}\int_a^b \dots \int_a^b f\left(\frac{a+b}{2}\right) dx_1 \dots dx_q = f\left(\frac{a+b}{2}\right).$$

Using the statement (i), we easily obtain the bounds (4.24) and (4.25).

(iv) Follows by the convexity of f. We shall omit the details.

∎

With the assumptions for f, α_i $\left(i = \overline{1,m}\right)$, T and m, we can define another sequence of mappings connected with $H_l^{[m]}$ $(1 \leq l \leq m-1)$ and given by:

$$\begin{aligned}
G_1^{[m]}(t) &:= \alpha_1(t)\frac{1}{b-a}\int_a^b f(x)\,dx + (\alpha_2(t) + \dots + \alpha_m(t))\, f\left(\frac{a+b}{2}\right) \\
G_2^{[m]}(t) &:= (\alpha_1(t) + \alpha_2(t))\frac{1}{b-a}\int_a^b f(x)\,dx + (\alpha_3(t) + \dots + \alpha_m(t)) \\
&\quad \times f\left(\frac{a+b}{2}\right) \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
G_{m-1}^{[m]}(t) &:= (\alpha_1(t) + .. + \alpha_{m-1}(t))\frac{1}{b-a}\int_a^b f(x)\,dx + \alpha_m(t)\, f\left(\frac{a+b}{2}\right)
\end{aligned}$$

for all $t \in T$.

The following theorem also holds [**57**].

THEOREM 115. *With the above assumptions, one has:*

(i) *The inequalities*

$$f\left(\frac{a+b}{2}\right) \leq G_1^{[m]}(t) \leq \dots \leq G_{m-1}^{[m]}(t) \leq \frac{1}{b-a}\int_a^b f(x)\,dx$$

hold for all $t \in T$;

(ii) *For all* $i = \overline{1,m}$, *one has*

$$H_i^{[m]}(t) \leq G_i^{[m]}(t) \ \text{ for all } t \in T;$$

(iii) *If there exists a* $t_0 \in T$ *such that* $\alpha_1(t_0) = \dots = \alpha_p(t_0) = 0$ $(1 \leq p \leq m-1)$, *then:*

$$\inf_{t \in T} G_l^{[m]}(t) = G_l^{[m]}(t_0) = f\left(\frac{a+b}{2}\right) \ \text{ for all } 1 \leq l \leq p;$$

(iv) *If there exists a* $t_1 \in T$ *such that* $\alpha_p(t_1) = 1$, *then*

$$\sup_{t\in T} G_l^{[m]}(t) = G_l^{[m]}(t_1) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

and

$$\inf_{t\in T} G_q^{[m]}(t) = G_q^{[m]}(t_1) = f\left(\frac{a+b}{2}\right),$$

for all $p \le l \le m-1$ *and* $1 \le q \le p-1$;

(v) *If* T *is a convex subset of a linear space* Y, $f\left(\frac{a+b}{2}\right) \ge 0$ *and the* α_i $\left(i = \overline{1,m}\right)$ *are convex mappings on* T, *then the* $G_l^{[m]}$ *are convex on* T *for all* $l = \overline{1,m-1}$.

Proof. The argument of (i), (iii), (iv) and (v) are obvious by the definition of $G_l^{[m]}$ $\left(l = \overline{1,m-1}\right)$. Let us prove the statement (ii).

Since f is convex on I, one has

$$\begin{aligned} H_i^{[m]}(t) &= \frac{1}{(b-a)^i}\int_a^b \cdots \int_a^b f\Bigg(\alpha_1(t)x_1 + \dots + \alpha_i(t)x_i \\ &\quad + (\alpha_{i+1}(t) + .. + \alpha_m(t))\frac{a+b}{2}\Bigg)dx_1...dx_i \\ &\le \frac{1}{(b-a)^i}\int_a^b \cdots \int_a^b \Bigg[\alpha_1(t)f(x_1) + .. + \alpha_i(t)f(x_i) \\ &\quad + (\alpha_{i+1}(t) + \dots + \alpha_m(t))f\left(\frac{a+b}{2}\right)\Bigg]dx_1...dx_i \\ &= G_i^{[m]}(t),\ i = \overline{1,m-1} \end{aligned}$$

for all $t \in T$, which proves the statement (ii). ∎

5. Properties of the Sequence of Mappings H_n

Let $I \subseteq \mathbb{R} \to \mathbb{R}$ be an interval of real numbers and $a, b \in I$ with $a < b$ and $f : I \to \mathbb{R}$ be a mapping with the property that it is integrable on $[a,b]$. Then we can define the following sequence of mappings $H_n : [0,1] \to \mathbb{R}$ given by

$$H_n(t) := \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\left(t \cdot \frac{x_1 + \dots + x_n}{n} + (1-t)\frac{a+b}{2}\right)dx_1...dx_n$$

for $n \ge 1$ and $t \in [0,1]$.

Some properties of this sequence of mappings are embodied in the following theorem [**24**].

Theorem 116. *Let* $f : I \subseteq \mathbb{R}$ *be a convex mapping on* I *and* $a, b \in I$ *with* $a < b$. *Then*

(i) *The* H_n *are convex on* $[0,1]$ *for all* $n \ge 1$;

(ii) *One has the inequalities:*

$$
\begin{aligned}
(4.26)\qquad & f\left(\frac{a+b}{2}\right) \\
\leq\ & H_n(t) \\
\leq\ & \frac{1}{(b-a)^{n+1}}\int_a^b \cdots \int_a^b f\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\,x_{n+1}\right)dx_1\ldots dx_{n+1}
\end{aligned}
$$

and

$$
\begin{aligned}
(4.27)\qquad & H_n(t) \\
\leq\ & t\cdot\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\left(\frac{x_1+\ldots+x_n}{n}\right)dx_1\ldots dx_n \\
& +(1-t)\cdot f\left(\frac{a+b}{2}\right) \\
\leq\ & \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\left(\frac{x_1+\ldots+x_n}{n}\right)dx_1\ldots dx_n
\end{aligned}
$$

for all $t\in[0,1]$;

(iii) *One has the bounds*

$$
\inf_{t\in[0,1]} H_n(t) = f\left(\frac{a+b}{2}\right) = H_n(0)\ \text{ for all } n\geq 1; \tag{4.28}
$$

and

$$
\begin{aligned}
(4.29)\qquad & \sup_{t\in[0,1]} H_n(t) \\
=\ & \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\left(\frac{x_1+\ldots+x_n}{n}\right)dx_1\ldots dx_n \\
=\ & H_n(1)\ \text{ for } n\geq 1;
\end{aligned}
$$

(iv) *The mapping H_n is monotonic nondecreasing on $[0,1]$ for all $n\geq 1$.*

PROOF. (i) Follows by the convexity of f.

(ii) Applying Jensen's integral inequality, we have that

$$
\begin{aligned}
& \frac{1}{b-a}\int_a^b f\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\,x_{n+1}\right)dx_{n+1} \\
\geq\ & f\left[\frac{1}{b-a}\int_a^b \left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\,x_{n+1}\right)dx_{n+1}\right] \\
=\ & f\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\cdot\frac{1}{b-a}\int_a^b x_{n+1}dx_{n+1}\right) \\
=\ & f\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)
\end{aligned}
$$

for all $x_1,\ldots,x_n\in[a,b]$ and $t\in[0,1]$.

Integrating this inequality on $[a,b]^n$ over the variables $x_1,\ldots,x_n$, we deduce the second inequality in (4.26).

By Jensen's integral inequality for multiple integrals, we have

$$\begin{aligned}
& \frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f\left(t\cdot\frac{x_1+\dots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)dx_1\dots dx_n \\
\geq\ & f\left[\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b \left(t\cdot\frac{x_1+\dots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)dx_1\dots dx_n\right] \\
=\ & f\left[t\cdot\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b \left(\frac{x_1+\dots+x_n}{n}\right)dx_1\dots dx_n+(1-t)\cdot\frac{a+b}{2}\right] \\
=\ & f\left(t\cdot\frac{a+b}{2}+(1-t)\cdot\frac{a+b}{2}\right)=f\left(\frac{a+b}{2}\right),
\end{aligned}$$

and the inequality is completely proved.

By the convexity of f on $[a,b]$, we can write that:

$$\begin{aligned}
& f\left(t\cdot\frac{x_1+\dots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right) \\
\leq\ & t\cdot f\left(\frac{x_1+\dots+x_n}{n}\right)+(1-t)\cdot f\left(\frac{a+b}{2}\right)
\end{aligned}$$

for all $x_1,\dots,x_n\in[a,b]$ and $t\in[0,1]$. Integrating this inequality on $[a,b]^n$, we deduce

$$\begin{aligned}
& H_n(t) \\
=\ & \frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f\left(t\cdot\frac{x_1+\dots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)dx_1\dots dx_n \\
\leq\ & t\cdot\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f\left(\frac{x_1+\dots+x_n}{n}\right)dx_1\dots dx_n+(1-t)\cdot f\left(\frac{a+b}{2}\right),
\end{aligned}$$

and the first inequality in (4.27) is proved.

As we know that

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f\left(\frac{x_1+\dots+x_n}{n}\right)dx_1\dots dx_n,$$

(see the inequality (4.2)) we obtain the last part of (4.27).

(iii) The bounds (4.28) and (4.29) follow by the inequalities (4.26) and (4.27). We shall omit the details.

(iv) Let $0<t_1<t_2\leq1$. By the convexity of f, we have that

$$\frac{H_n(t_2)-H_n(t_1)}{t_2-t_1}\geq\frac{H_n(t_1)-H_n(0)}{t_1},$$

but $H_n(t_1)\geq H_n(0)$ (see the first inequality in (4.26)) and then we get that $H_n(t_2)-H_n(t_1)\geq0$ for all $0\leq t_1<t_2\leq1$, which, along with (4.28), shows that the mapping $H_n(\cdot)$ is monotonic nondecreasing on $[0,1]$.

∎

We shall now give another result of monotonicity which, in a sense, completes the above theorem [**24**].

THEOREM 117. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping on I and $a, b \in T$, with $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq H_{n+1}(t) \leq H_n(t) \leq ... \leq H_1(t) = H(t) \tag{4.30}$$

for all $n \geq 1$ and $t \in [0,1]$. That is, the sequence of mappings $(H_n)_{n\geq 1}$ is monotonically nonincreasing.

PROOF. We shall give two arguments.

(1) Let us define the real numbers belonging to $[a, b]$:

$$\begin{aligned} y_1 &: = t \cdot \frac{x_1 + ... + x_n}{n} + (1-t) \cdot \frac{a+b}{2}; \\ y_2 &: = t \cdot \frac{x_2 + x_3 + ... + x_{n+1}}{n} + (1-t) \cdot \frac{a+b}{2}; \\ &\quad \text{..} \\ y_{n+1} &: = t \cdot \frac{x_{n+1} + x_1 + ... + x_{n-1}}{n} + (1-t) \cdot \frac{a+b}{2}, \end{aligned}$$

where $x_1, ..., x_n \in [a, b]$.
Using Jensen's discrete inequality,

$$\frac{1}{n+1}\left[f(y_1) + f(y_2) + ... + f(y_{n+1})\right] \geq f\left(\frac{y_1 + y_2 + ... + y_{n+1}}{n+1}\right)$$

and, taking into account that:

$$\begin{aligned} &\frac{y_1 + y_2 + ... + y_{n+1}}{n+1} \\ = &\frac{1}{n+1}\left[t \cdot \frac{n(x_1 + ... + x_n)}{n} + (n+1)(1-t) \cdot \frac{a+b}{2}\right] \\ = &t \cdot \frac{x_1 + ... + x_{n+1}}{n+1} + (1-t) \cdot \frac{a+b}{2}, \end{aligned}$$

we obtain the inequality:

$$\begin{aligned} &\frac{1}{n+1}\left[f\left(t \cdot \frac{x_1 + ... + x_n}{n} + (1-t) \cdot \frac{a+b}{2}\right)\right. \\ &+ f\left(t \cdot \frac{x_2 + ... + x_{n+1}}{n} + (1-t) \cdot \frac{a+b}{2}\right) + ... \\ &\left.+ f\left(t \cdot \frac{x_{n+1} + x_1 + ... + x_{n-1}}{n} + (1-t) \cdot \frac{a+b}{2}\right)\right] \\ \geq\ &f\left(t \cdot \frac{x_1 + ... + x_{n+1}}{n+1} + (1-t) \cdot \frac{a+b}{2}\right) \end{aligned}$$

for all $t\in[0,1]$ and $x_1,...,x_n\in[a,b]$.
Integrating this inequality on $[a,b]^{n+1}$, we deduce

$$\begin{aligned}
&\frac{1}{n+1}\left[\frac{1}{(b-a)^{n+1}}\int_a^b...\int_a^b f\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)dx_1...dx_{n+1}\right.\\
&+...+\frac{1}{(b-a)^{n+1}}\int_a^b...\int_a^b f\left(t\cdot\frac{x_{n+1}+x_1+...+x_{n-1}}{n}\right.\\
&\left.\left.+(1-t)\cdot\frac{a+b}{2}\right)dx_1...dx_{n+1}\right]\\
\geq\ &\frac{1}{(b-a)^{n+1}}\int_a^b...\int_a^b f\left(t\cdot\frac{x_1+...+x_{n+1}}{n+1}+(1-t)\cdot\frac{a+b}{2}\right)dx_1...dx_{n+1}.
\end{aligned}$$

However, it is easy to see that:

$$\begin{aligned}
&\frac{1}{(b-a)^{n+1}}\int_a^b...\int_a^b f\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)dx_1...dx_{n+1}\\
=\ &......=\frac{1}{(b-a)^{n+1}}\int_a^b...\int_a^b f\left(t\cdot\frac{x_{n+1}+x_1+...+x_{n-1}}{n}\right.\\
&\left.+(1-t)\cdot\frac{a+b}{2}\right)dx_1...dx_{n+1}\\
=\ &\frac{1}{(b-a)^{n}}\int_a^b...\int_a^b f\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)dx_1...dx_n
\end{aligned}$$

and thus, by the above inequality, we deduce

$$H_n(t)\geq H_{n+1}(t)\ \text{ for all } t\in[0,1]\ \text{ and } n\geq 1.$$

The proof is thus completed.

(2) Now, we shall present the second proof for the above inequality.
By the convexity of f, we can state that:

$$\begin{aligned}
&f\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)\\
&-f\left(t\cdot\frac{x_1+...+x_{n+1}}{n+1}+(1-t)\cdot\frac{a+b}{2}\right)\\
\geq\ &f'_+\left(t\cdot\frac{x_1+...+x_{n+1}}{n+1}+(1-t)\cdot\frac{a+b}{2}\right)\\
&\times\left[t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}-t\cdot\frac{x_1+...+x_{n+1}}{n+1}-(1-t)\cdot\frac{a+b}{2}\right]\\
=\ &tf'_+\left(t\cdot\frac{x_1+...+x_{n+1}}{n+1}+(1-t)\cdot\frac{a+b}{2}\right)\left(\frac{x_1+...+x_n-nx_{n+1}}{n(n+1)}\right)
\end{aligned}$$

for all $t \in [0,1]$ and $x_1, ..., x_n \in [a,b]$.
Integrating this inequality on $[a,b]^{n+1}$, we obtain

$$\begin{aligned}
& H_n(t) - H_{n+1}(t) \\
\geq\ & \frac{t}{n(n+1)}\left[\frac{1}{(b-a)^{n+1}}\int_a^b ... \int_a^b f'_+\left(t\cdot\frac{x_1+...+x_{n+1}}{n+1}\right.\right. \\
& \left.\left. +(1-t)\cdot\frac{a+b}{2}\right)(x_1+...+x_n - nx_{n+1})\,dx_1...dx_{n+1}\right] \\
=\ & \frac{t}{n(n+1)}\left[\frac{1}{(b-a)^{n+1}}\int_a^b ... \int_a^b f'_+\left(t\cdot\frac{x_1+...+x_{n+1}}{n+1}\right.\right. \\
& \left. +(1-t)\cdot\frac{a+b}{2}\right)x_1 dx_1...dx_{n+1} + ... \\
& \quad + \frac{1}{(b-a)^{n+1}}\int_a^b ... \int_a^b f'_+\left(t\cdot\frac{x_1+...+x_{n+1}}{n+1}\right. \\
& \quad \left. +(1-t)\cdot\frac{a+b}{2}\right)x_n dx_1...dx_{n+1} \\
& \quad - n\frac{1}{(b-a)^{n+1}}\int_a^b ... \int_a^b f'_+\left(t\cdot\frac{x_1+...+x_{n+1}}{n+1}\right. \\
& \quad \left.\left. +(1-t)\cdot\frac{a+b}{2}\right)x_{n+1} dx_1...dx_{n+1}\right] = 0
\end{aligned}$$

and the second proof is completed.

∎

It is natural to ask what happens with the difference $H_n(t) - f\left(\frac{a+b}{2}\right)$ which is clearly non-negative for all $t \in [0,1]$.

The following theorem contains an upper bound for this difference [**24**].

THEOREM 118. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and f'_+ its right derivative which exists on $\mathring{I}$ and is monotonic nondecreasing on $\mathring{I}$. If $a, b \in \mathring{I}$ with $a < b$, then we have the inequalities*

$$\begin{aligned}
(4.31)\qquad 0 \leq\ & H_n(t) - f\left(\frac{a+b}{2}\right) \\
\leq\ & \frac{t}{(b-a)^n}\int_a^b ... \int_a^b f'_+\left(t\cdot\frac{x_1+...+x_n}{n} + (1-t)\cdot\frac{a+b}{2}\right) \\
& \times\left(x_1 - \frac{a+b}{2}\right)dx_1...dx_n \\
\leq\ & \frac{t}{\sqrt{n}}\cdot\frac{b-a}{2\sqrt{3}}\left[\frac{1}{(b-a)^n}\int_a^b ... \int_a^b \left[f'_+\left(t\cdot\frac{x_1+...+x_n}{n}\right.\right.\right. \\
& \left.\left.\left. +(1-t)\cdot\frac{a+b}{2}\right)\right]^2 dx_1...dx_n\right]^{\frac{1}{2}}
\end{aligned}$$

for all $n \geq 1$ and $t \in [0,1]$.

PROOF. As f is convex on I, we can write:

$$f(x)-f(y)\geq f'_{+}(y)(x-y) \text{ for all } x,y\in \mathring{I}.$$

Choosing in this inequality

$$x=\frac{a+b}{2} \text{ and } y=t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2},$$

we deduce the inequality:

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)-f\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right) \\ \geq\ & tf'_{+}\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)\left(\frac{a+b}{2}-\frac{x_1+...+x_n}{n}\right). \end{aligned}$$

Integrating this inequality on $[a,b]^n$, we derive that

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)-H_n(t) \\ \geq\ & t\left[\frac{1}{(b-a)^n}\int_a^b...\int_a^b\frac{a+b}{2}\cdot f'_{+}\left(t\cdot\frac{x_1+...+x_n}{n}\right.\right. \\ & \left.+(1-t)\cdot\frac{a+b}{2}\right)dx_1...dx_n \\ & -\frac{1}{(b-a)^n}\int_a^b...\int_a^b f'_{+}\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right) \\ & \left.\times\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n\right]. \end{aligned} \tag{4.32}$$

As a simple calculation shows that:

$$\begin{aligned} & \frac{1}{(b-a)^n}\int_a^b...\int_a^b f'_{+}\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right) \\ & \times\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n \\ =\ & \frac{1}{(b-a)^n}\int_a^b...\int_a^b f'_{+}\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)x_1dx_1...dx_n, \end{aligned}$$

by the inequality (4.32), we deduce the second part of (4.31).

Now, let us observe that the second term in the inequality (4.32) is the integral

$$\begin{aligned} I\ =\ & -\frac{t}{(b-a)^n}\int_a^b...\int_a^b f'_{+}\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right) \\ & \times\left(\frac{x_1+...+x_n}{n}-\frac{a+b}{2}\right)dx_1...dx_n. \end{aligned}$$

By the well-known Cauchy-Buniakowsky-Schwartz integral inequality for multiple integrals, we have

$$\begin{aligned} I &\leq t\left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left[f'_+\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)\right]^2 dx_1\ldots dx_n\right]^{\frac{1}{2}} \\ &\quad\times\left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\right)^2 dx_1\ldots dx_n\right]^{\frac{1}{2}}. \end{aligned}$$

As a simple calculation shows us that (see the proof of Theorem 110):

$$\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\right)^2 dx_1\ldots dx_n=\frac{1}{12n}(b-a)^2,$$

we deduce the last part of the inequality (4.31).

The proof of the theorem is thus completed. ∎

Corollary 36. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in \mathring{I}$ with $a < b$. Put $M := \sup_{x\in[a,b]} |f'(x)| < \infty$. Then we have the inequality:*

$$0 \leq H_n(t) - f\left(\frac{a+b}{2}\right) \leq \frac{t(b-a)M}{2\sqrt{3}\sqrt{n}} \tag{4.33}$$

for all $t \in [0,1]$ and $n \geq 1$.
In particular, we have

$$\lim_{n\to\infty} H_n(t) = f\left(\frac{a+b}{2}\right) \text{ uniformly on } [0,1].$$

Proof. The argument is obvious by the above theorem on observing that, under the above assumptions, we have

$$\left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left[f'_+\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)\right]^2 dx_1\ldots dx_n\right]^{\frac{1}{2}} \leq M.$$

∎

The following result also holds [**24**]:

Theorem 119. *With the above assumptions, we have*

$$\begin{aligned} 0 &\leq \frac{t}{(b-a)^n}\int_a^b \cdots \int_a^b f\left(\frac{x_1+\ldots+x_n}{n}\right) dx_1\ldots dx_n \\ &\quad+(1-t)f\left(\frac{a+b}{2}\right)-H_n(t) \end{aligned} \tag{4.34}$$

$$\begin{aligned}
&\leq t(1-t)\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(f'_+\left(\frac{x_1+\ldots+x_n}{n}\right)-f'_+\left(\frac{a+b}{2}\right)\right) \\
&\quad \times\left(\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\right)dx_1\ldots dx_n \\
&= \frac{t(1-t)}{(b-a)^n}\int_a^b \cdots \int_a^b f'_+\left(\frac{x_1+\ldots+x_n}{n}\right)\left(x_1-\frac{a+b}{2}\right)dx_1\ldots dx_n \\
&\leq \frac{t(1-t)(b-a)}{2\sqrt{3}\sqrt{n}} \\
&\quad \times\left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left[f'_+\left(\frac{x_1+\ldots+x_n}{n}\right)\right]^2 dx_1\ldots dx_n\right]^{\frac{1}{2}}
\end{aligned}$$

for all $n\geq 1$ *and* $t\in[0,1]$.

Proof. By the convexity of f we can write

$$\begin{aligned}
&f\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\right)-f\left(\frac{x_1+\ldots+x_n}{n}\right) \\
&\geq f'_+\left(\frac{x_1+\ldots+x_n}{n}\right)\left[t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}-\frac{x_1+\ldots+x_n}{n}\right] \\
&= (1-t)f'_+\left(\frac{x_1+\ldots+x_n}{n}\right)\left(\frac{a+b}{2}-\frac{x_1+\ldots+x_n}{n}\right)
\end{aligned}\tag{4.35}$$

for all $t\in[0,1]$ and $x_1,\ldots,x_n\in[a,b]$.

Similarly, we have:

$$\begin{aligned}
&f\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right) \\
&\geq f'_+\left(\frac{a+b}{2}\right)\left[t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}-\frac{a+b}{2}\right] \\
&= -tf'_+\left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}-\frac{x_1+\ldots+x_n}{n}\right)
\end{aligned}\tag{4.36}$$

for all $t\in[0,1]$ and $x_1,\ldots,x_n\in[a,b]$.

If we multiply the inequality (4.35) by t and (4.36) by $(1-t)$ and add the obtained inequalities, we deduce

$$\begin{aligned}
&f\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right) \\
&-tf\left(\frac{x_1+\ldots+x_n}{n}\right)-(1-t)\cdot f\left(\frac{a+b}{2}\right) \\
&\geq t(1-t)\left[f'_+\left(\frac{x_1+\ldots+x_n}{n}\right)-f'_+\left(\frac{a+b}{2}\right)\right]\left(\frac{a+b}{2}-\frac{x_1+\ldots+x_n}{n}\right).
\end{aligned}$$

That is,

$$\begin{aligned} 0 \leq tf\left(\frac{x_1+...+x_n}{n}\right)+(1-t)\cdot f\left(\frac{a+b}{2}\right) & \\ -f\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right) & \\ \leq\ t(1-t)\left[f'_+\left(\frac{x_1+...+x_n}{n}\right)-f'_+\left(\frac{a+b}{2}\right)\right]\left(\frac{x_1+...+x_n}{n}\right. & \end{aligned}$$

for all $t\in[0,1]$ and $x_1,...,x_n\in[a,b]$.

Integrating this inequality on $[a,b]^n$, we have:

$$\begin{aligned} 0 &\leq tH_n(1)+(1-t)H_n(0)-H_n(t) \\ &\leq t(1-t)\frac{1}{(b-a)^n}\left[\int_a^b...\int_a^b f'_+\left(\frac{x_1+...+x_n}{n}\right)\right. \\ &\quad\times\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n+f'_+\left(\frac{a+b}{2}\right)\cdot\frac{a+b}{2} \\ &\quad-f'_+\left(\frac{a+b}{2}\right)\int_a^b...\int_a^b\frac{x_1+...+x_n}{n}dx_1...dx_n \\ &\quad\left.-\frac{a+b}{2}\int_a^b...\int_a^b f'_+\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n\right] \\ &= t(1-t)\left[\frac{1}{(b-a)^n}\int_a^b...\int_a^b f'_+\left(\frac{x_1+...+x_n}{n}\right)x_1dx_1...dx_n\right. \\ &\quad\left.-\frac{a+b}{2}\cdot\frac{1}{(b-a)^n}\int_a^b...\int_a^b f'_+\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n\right], \end{aligned}$$

as a simple calculation shows us that:

$$\begin{aligned} &\frac{1}{(b-a)^n}\int_a^b...\int_a^b f'_+\left(\frac{x_1+...+x_n}{n}\right)\left(\frac{x_1+...+x_n}{n}\right)dx_1...dx_n \\ =\ &\frac{1}{(b-a)^n}\int_a^b...\int_a^b f'_+\left(\frac{x_1+...+x_n}{n}\right)x_1dx_1...dx_n \end{aligned}$$

and

$$\frac{1}{(b-a)^n}\int_a^b...\int_a^b\frac{x_1+...+x_n}{n}dx_1...dx_n=\frac{a+b}{2}.$$

Thus, the first inequality in (4.34) is proved.

Now, by the Cauchy-Buniakowsky-Schwartz integral inequality, we have that

$$\begin{aligned}
&\left|\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f'_+\left(\frac{x_1+\dots+x_n}{n}\right)\left(x_1-\frac{a+b}{2}\right)dx_1\dots dx_n\right| \\
= &\left|\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f'_+\left(\frac{x_1+\dots+x_n}{n}\right)\left(\frac{x_1+\dots+x_n}{n}-\frac{a+b}{2}\right)dx_1\dots dx_n\right| \\
\le &\left(\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b \left[f'_+\left(\frac{x_1+\dots+x_n}{n}\right)\right]^2 dx_1\dots dx_n\right)^{\frac{1}{2}} \\
&\times\left(\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b \left(\frac{x_1+\dots+x_n}{n}-\frac{a+b}{2}\right)^2 dx_1\dots dx_n\right)^{\frac{1}{2}}
\end{aligned}$$

and as

$$\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b \left(\frac{x_1+\dots+x_n}{n}-\frac{a+b}{2}\right)^2 dx_1\dots dx_n=\frac{(b-a)^2}{12n},$$

the theorem is thus proved. ∎

REMARK 58. *Note that instead of the right membership in the inequality* (4.34) *we can also put the term T given by*

$$\begin{aligned}
T \ :&= \frac{t(1-t)(b-a)}{2\sqrt{3}\sqrt{n}} \\
&\times\left[\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b \left[f'_+\left(\frac{x_1+\dots+x_n}{n}\right)-f'_+\left(\frac{a+b}{2}\right)\right]^2 dx_1\dots dx_n\right]^{\frac{1}{2}}.
\end{aligned}$$

COROLLARY 37. *With the above assumptions and if $M:=\sup_{x\in[a,b]}|f'(x)|<\infty$, then we have the inequality:*

$$0\le tH_n(1)+(1-t)H_n(0)-H_n(t)\le\frac{t(1-t)}{2\sqrt{3}\sqrt{n}}\cdot M \tag{4.37}$$

for all $n\ge1$ and $t\in[0,1]$.
In particular,

$$\lim_{n\to\infty}[tH_n(1)+(1-t)H_n(0)-H_n(t)]=0$$

uniformly on $[0,1]$.

The following corollary is interesting as well.

COROLLARY 38. *With the above assumptions and if there exists a constant $K>0$ such that:*

$$\left|f'_+(x)-f'_+(y)\right|\le K|x-y| \text{ for all } x,y\in[a,b],$$

then we have the inequality:

$$0\le tH_n(1)+(1-t)H_n(0)-H_n(t)\le\frac{Kt(1-t)}{12n}(b-a)^2 \tag{4.38}$$

for all $t\in[0,1]$ and $n\ge1$.

PROOF. The argument follows by the above remark on observing that

$$\begin{aligned}
&\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left[f'_+\left(\frac{x_1+\ldots+x_n}{n}\right)-f'_+\left(\frac{a+b}{2}\right)\right]^2 dx_1\ldots dx_n \\
\leq\;& \frac{K^2}{(b-a)^n}\int_a^b \cdots \int_a^b \left[\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\right]^2 dx_1\ldots dx_n \\
=\;& \frac{K^2(b-a)^2}{12n}.
\end{aligned}$$

∎

In addition, it is natural to ask about an upper bound for the difference $H_n(1)-H_n(t)$, $n\geq 1$ for all $t\in[0,1]$, [**24**].

THEOREM 120. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex mapping on the interval I and $a,b\in\mathring{I}$ with $a<b$. Then we have the inequalities*

$$\begin{aligned}
0 \;&\leq\; \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\left(\frac{x_1+\ldots+x_n}{n}\right)dx_1\ldots dx_n - H_n(t) \\
&\leq\; (1-t)\left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(f'_+\left(\frac{x_1+\ldots+x_n}{n}\right)\right.\right. \\
&\qquad \left.\left.-f'_+\left(\frac{a+b}{2}\right)\right)\left(\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\right)dx_1\ldots dx_n\right] \\
&\leq\; (1-t)\left[\frac{1}{(b-a)^n}\right. \\
&\qquad \left.\times\int_a^b \cdots \int_a^b f'_+\left(\frac{x_1+\ldots+x_n}{n}\right)\left(x_1-\frac{a+b}{2}\right)dx_1\ldots dx_n\right] \\
&\leq\; \frac{(1-t)(b-a)}{2\sqrt{3}\sqrt{n}} \\
&\qquad \times\left(\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left[f'_+\left(\frac{x_1+\ldots+x_n}{n}\right)\right]^2 dx_1\ldots dx_n\right)^{\frac{1}{2}}
\end{aligned} \tag{4.39}$$

for all $t\in[0,1]$ and $n\geq 1$.

PROOF. By the convexity of f, we have that

$$\begin{aligned}
&f\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)-f\left(\frac{x_1+\ldots+x_n}{n}\right) \\
\geq\;& (1-t)f'_+\left(\frac{x_1+\ldots+x_n}{n}\right)\left[\frac{a+b}{2}-\frac{x_1+\ldots+x_n}{n}\right]
\end{aligned}$$

for all $x_1,\ldots,x_n\in[a,b]$ and $t\in[0,1]$.

Now, the argument follows as above and we shall omit the details. ∎

REMARK 59. *In the right membership of the above we also can put the term:*

$$\frac{(1-t)(b-a)}{2\sqrt{3}\sqrt{n}} \times \left(\frac{1}{(b-a)^n} \int_a^b ... \int_a^b \left[f'_+ \left(\frac{x_1 + ... + x_n}{n} \right) - f'_+ \left(\frac{a+b}{2} \right) \right]^2 dx_1 ... dx_n \right)^{\frac{1}{2}}.$$

COROLLARY 39. *With the above assumptions and if $M := \sup_{x \in [a,b]} |f'(x)| < \infty$, then we have the inequality:*

$$0 \leq H_n(1) - H_n(t) \leq \frac{(1-t)(b-a)M}{2\sqrt{3}\sqrt{n}}. \tag{4.40}$$

In particular,

$$\lim_{n \to \infty} [H_n(1) - H_n(t)] = 0$$

uniformly on $[0,1]$.

COROLLARY 40. *With the above assumptions, and if there exists a constant $K > 0$ such that*

$$\left| f'_+(x) - f'_+(y) \right| \leq K |x - y| \text{ for all } x, y \in [a,b],$$

then we have the inequality:

$$0 \leq H_n(1) - H_n(t) \leq \frac{K(1-t)(b-a)^2}{12n} \tag{4.41}$$

for all $n \geq 1$ and $t \in [0,1]$.

We shall point out now an upper bound for the difference $H_n(t) - H_{n+1}(t)$, $n \geq 1$ which is non-negative for all $t \in [0,1]$ (c.f. Theorem 117) (see [**24**]):

THEOREM 121. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I and $a, b \in \mathring{I}$ with $a < b$. Then we have the inequality:*

$$\begin{aligned} 0 &\leq H_n(t) - H_{n+1}(t) \\ &\leq \frac{t}{n+1} \left[\frac{1}{(b-a)^n} \int_a^b ... \int_a^b f'_+ \left(t \cdot \frac{x_1 + ... + x_n}{n} \right.\right. \\ &\quad \left.\left. + (1-t) \cdot \frac{a+b}{2} \right) \left(x_1 - \frac{a+b}{2} \right) dx_1 ... dx_n \right] \\ &\leq \frac{t(b-a)}{2\sqrt{3}\sqrt{n}(n+1)} \left[\frac{1}{(b-a)^n} \int_a^b ... \int_a^b \left[f'_+ \left(t \cdot \frac{x_1 + ... + x_n}{n} \right.\right.\right. \\ &\quad \left.\left.\left. + (1-t) \cdot \frac{a+b}{2} \right) \right]^2 dx_1 ... dx_n \right]^{\frac{1}{2}} \end{aligned} \tag{4.42}$$

for all $t \in [0,1]$ and $n \geq 1$.

PROOF. By the convexity of f, we have that

$$\begin{aligned}
& f\left(t\cdot\frac{x_1+...+x_{n+1}}{n+1}+(1-t)\cdot\frac{a+b}{2}\right)-f\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)\\
\geq\ & f'_+\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)\\
& \times\left[t\cdot\frac{x_1+...+x_{n+1}}{n+1}+(1-t)\cdot\frac{a+b}{2}-t\cdot\frac{x_1+...+x_n}{n}-(1-t)\cdot\frac{a+b}{2}\right]\\
=\ & \frac{t}{n(n+1)}f'_+\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)[nx_{n+1}-(x_1+...+x_n)]
\end{aligned}$$

for all $x_1,...,x_n\in[a,b]$ and $t\in[0,1]$.

Integrating this inequality on $[a,b]^{n+1}$, we derive

$$\begin{aligned}
0\ \leq\ & H_n(t)-H_{n+1}(t)\\
\leq\ & \frac{t}{n(n+1)}\left[\frac{1}{(b-a)^n}\int_a^b...\int_a^b f'_+\left(t\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)\right.\\
& \times(x_1+...+x_n)\,dx_1...dx_n\\
& -n\cdot\frac{a+b}{2}\cdot\frac{1}{(b-a)^n}\\
& \left.\times\int_a^b...\int_a^b f'_+\left(t\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)dx_1...dx_n\right]\\
=\ & \frac{t}{n+1}\left[\frac{1}{(b-a)^n}\int_a^b...\int_a^b f'_+\left(t\cdot\frac{x_1+...+x_n}{n}\right.\right.\\
& \left.+(1-t)\cdot\frac{a+b}{2}\right)x_1dx_1...dx_n\\
& -\frac{a+b}{2}\cdot\frac{1}{(b-a)^n}\int_a^b...\int_a^b f'_+\left(t\cdot\frac{x_1+...+x_n}{n}\right.\\
& \left.\left.+(1-t)\cdot\frac{a+b}{2}\right)dx_1...dx_n\right],
\end{aligned}$$

and the first inequality in (4.42) is proved.

Now, let us observe that

$$\begin{aligned}
& \frac{1}{(b-a)^n}\int_a^b...\int_a^b f'_+\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)\\
& \times\left(x_1-\frac{a+b}{2}\right)dx_1...dx_n\\
=\ & \frac{1}{(b-a)^n}\int_a^b...\int_a^b f'_+\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)\\
& \times\left(\frac{x_1+...+x_n}{n}-\frac{a+b}{2}\right)dx_1...dx_n\leq
\end{aligned}$$

$$\leq \left(\frac{1}{(b-a)^n}\int_a^b ... \int_a^b \left[f'_+\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)\right]^2 dx_1...dx_n\right)^{\frac{1}{2}}$$
$$\times\left(\frac{1}{(b-a)^n}\int_a^b ... \int_a^b \left(\frac{x_1+...+x_n}{n}-\frac{a+b}{2}\right)^2 dx_1...dx_n\right)^{\frac{1}{2}}.$$

However, we showed that:

$$\frac{1}{(b-a)^n}\int_a^b ... \int_a^b \left(\frac{x_1+...+x_n}{n}-\frac{a+b}{2}\right)^2 dx_1...dx_n=\frac{(b-a)^2}{12n},$$

and thus the last inequality has also been proved. ∎

Corollary 41. *With the above assumptions, given that* $M := \sup_{x\in[a,b]}|f'(x)| < \infty$, *then we have*

$$0 \leq H_n(t) - H_{n+1}(t) \leq \frac{Mt(b-a)}{2\sqrt{3}\sqrt{n}(n+1)} \tag{4.43}$$

for all $t \in [0,1]$ *and* $n \geq 1$.
In particular,

$$\lim_{n\to\infty}[H_n(t) - H_{n+1}(t)] = 0$$

uniformly on $[0,1]$.

The following theorem also holds [**24**].

Theorem 122. *With the above assumptions, we also have the bound*

$$\begin{aligned} 0 &\leq H_n(t) - H_{n+1}(t) \\ &\leq \frac{t(b-a)}{2\sqrt{3}\sqrt{n}(n+1)}\left[\frac{1}{(b-a)^n}\int_a^b ... \int_a^b \left[f'_+\left(t\cdot\frac{x_1+...+x_n}{n}\right.\right.\right. \\ &\quad \left.\left.\left. +(1-t)\cdot\frac{a+b}{2}\right)-f'_+\left(\frac{a+b}{2}\right)\right]^2 dx_1...dx_n\right]^{\frac{1}{2}} \end{aligned} \tag{4.44}$$

for all $t \in [0,1]$ *and* $n \geq 1$.

Proof. Let us observe that

$$\frac{1}{(b-a)^n}\int_a^b ... \int_a^b \left[f'_+\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)-f'_+\left(\frac{a+b}{2}\right)\right]$$
$$\times\left(\frac{x_1+...+x_n}{n}-\frac{a+b}{2}\right)dx_1...dx_n$$

$$
\begin{aligned}
&= \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left[f'_+\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)\left(\frac{x_1+\ldots+x_n}{n}\right)\right. \\
&\quad + f'_+\left(\frac{a+b}{2}\right)\cdot\frac{a+b}{2} - f'_+\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)\frac{a+b}{2} \\
&\quad \left. - f'_+\left(\frac{a+b}{2}\right)\left(\frac{x_1+\ldots+x_n}{n}\right)\right] dx_1\ldots dx_n \\
&= \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f'_+\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right) x_1\, dx_1\ldots dx_n \\
&\quad + f'_+\left(\frac{a+b}{2}\right)\cdot\frac{a+b}{2} \\
&\quad - \frac{a+b}{2}\cdot\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f'_+\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right) dx_1\ldots dx_n \\
&\quad - f'_+\left(\frac{a+b}{2}\right)\cdot\frac{a+b}{2} \\
&= \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f'_+\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right) \\
&\quad \times\left(x_1-\frac{a+b}{2}\right) dx_1\ldots dx_n.
\end{aligned}
$$

Now, using Theorem 121 and the Cauchy-Buniakowsky-Schwartz integral inequality, we have that:

$$
\begin{aligned}
0 &\leq H_n(t)-H_{n+1}(t) \\
&\leq \frac{t}{n+1}\cdot\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left[f'_+\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)\right. \\
&\quad \left. - f'_+\left(\frac{a+b}{2}\right)\right]\left(\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\right) dx_1\ldots dx_n \\
&\leq \frac{t}{n+1}\left\{\frac{1}{(b-a)^n}\right. \\
&\quad \left.\times \int_a^b \cdots \int_a^b \left[f'_+\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)\right]^2 dx_1\ldots dx_n\right\}^{\frac{1}{2}} \\
&\quad \times\left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\right)^2 dx_1\ldots dx_n\right]^{\frac{1}{2}} \\
&= \frac{t(b-a)}{2\sqrt{3}\sqrt{n}(n+1)}\left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left[f'_+\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)\right.\right. \\
&\quad \left.\left. - f'_+\left(\frac{a+b}{2}\right)\right]^2 dx_1\ldots dx_n\right]^{\frac{1}{2}}
\end{aligned}
$$

and the theorem is proved. ∎

Corollary 42. *With the above assumptions, given that there exists a $K>0$ such that*

$$\left|f'_{+}(x)-f'_{+}(y)\right| \leq K|x-y| \text{ for all } x,y\in[a,b],$$

then we have the inequality:

$$0 \leq H_n(t)-H_{n+1}(t) \leq \frac{t^2(b-a)^2K^2}{12n(n+1)} \tag{4.45}$$

for all $t\in[0,1]$ and $n\geq 1$.

Proof. By (4.44), we can state that

$$\begin{aligned} & \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left[f'_{+}\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)\right. \\ & \left. -f'_{+}\left(\frac{a+b}{2}\right)\right]^2 dx_1\ldots dx_n \\ \leq\; & K^2\frac{t^2}{(b-a)^n}\int_a^b \cdots \int_a^b \left(\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\right)^2 dx_1\ldots dx_n, \end{aligned}$$

and the corollary is proved. ∎

Finally, note that, by a similar argument to that in the proof of Theorem 122, we can give the following result which completes, in a sense, the estimation in Theorem 118 (see also [**24**]).

Theorem 123. *With the above assumptions, one has the inequality:*

$$\begin{aligned} 0 &\leq H_n(t)-f\left(\frac{a+b}{2}\right) \\ &\leq \frac{t(b-a)}{2\sqrt{3}\sqrt{n}}\left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left[f'_{+}\left(t\cdot\frac{x_1+\ldots+x_n}{n}\right.\right.\right. \\ &\qquad \left.\left.\left.+(1-t)\cdot\frac{a+b}{2}\right)-f'_{+}\left(\frac{a+b}{2}\right)\right]^2 dx_1\ldots dx_n\right]^{\frac{1}{2}} \end{aligned} \tag{4.46}$$

for all $t\in[0,1]$ and $n\geq 1$.

Proof. Using Theorem 118, we have:

$$\begin{aligned} 0 &\leq H_n(t)-H_{n+1}(t) \\ &\leq \frac{t}{(b-a)^n}\int_a^b \cdots \int_a^b f'_{+}\left(t\cdot\frac{x_1+\ldots+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right) \\ &\quad \times\left(x_1-\frac{a+b}{2}\right)dx_1\ldots dx_n. \end{aligned}$$

However, we know, by Theorem 122, that

$$
\begin{aligned}
&\frac{1}{(b-a)^n}\int_a^b ... \int_a^b f'_+\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)\\
&\times\left(x_1-\frac{a+b}{2}\right)dx_1...dx_n\\
= &\frac{1}{(b-a)^n}\int_a^b ... \int_a^b \left[f'_+\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)-f'_+\left(\frac{a+b}{2}\right)\right]\\
&\times\left(\frac{x_1+...+x_n}{n}-\frac{a+b}{2}\right)dx_1...dx_n.
\end{aligned}
$$

Applying the Cauchy-Buniakowsky-Schwartz integral inequality, we deduce the desired result. ∎

Corollary 43. *With the above assumptions, given that there exists a $K>0$ such that*

$$\left|f'_+(x)-f'_+(y)\right|\le K\left|x-y\right| \text{ for all } x,y\in[a,b],$$

we have the inequality:

$$0\le H_n(t)-f\left(\frac{a+b}{2}\right)\le\frac{t^2(b-a)^2K}{12n}$$

for all $t\in[0,1]$ and $n\ge1$.

6. Applications for Special Means

(1) Let $0\le a<b$ and $p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$. Let us define the sequence of mappings:

$$h_{p,n}(t):=\frac{1}{(b-a)^n}\int_a^b ... \int_a^b\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot\frac{a+b}{2}\right)^p dx_1...dx_n,$$

where $n\ge1,\ n\in\mathbb{N}$, and $t\in[0,1]$.

By the use of the above results, we can state the following properties:

(i) $h_{p,n}(t)$ are convex and monotonic nondecreasing on $[0,1]$;

(ii) $h_{p,n}(t)\ge h_{p,n+1}(t)$ for all $n\ge1$ and $t\in[0,1]$;

(iii) One has the inequalities

$$
\begin{aligned}
&[A(a,b)]^p\le h_{p,n}(t)\\
\le &\frac{1}{(b-a)^{n+1}}\int_a^b ... \int_a^b\left(t\cdot\frac{x_1+...+x_n}{n}+(1-t)\cdot x_{n+1}\right)^p dx_1...dx_{n+1}
\end{aligned}
$$

and

$$
\begin{aligned}
h_{p,n}(t) &\le t\cdot\frac{1}{(b-a)^n}\int_a^b ... \int_a^b\left(\frac{x_1+...+x_n}{n}\right)^p dx_1...dx_n\\
&\quad+(1-t)\cdot[A(a,b)]^p\\
&\le \frac{1}{(b-a)^n}\int_a^b ... \int_a^b\left(\frac{x_1+...+x_n}{n}\right)^p dx_1...dx_n
\end{aligned}
$$

for all $n\ge1$ and $t\in[0,1]$;

(iv) If $p \geq 1$, then one has the inequalities

$$0 \leq h_{p,n}(t) - [A(a,b)]^p \leq \frac{t(b-a)pb^{p-1}}{2\sqrt{3}\sqrt{n}}$$

and

$$\begin{aligned} 0 &\leq \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \left(\frac{x_1+\dots+x_n}{n}\right)^p dx_1 \dots dx_n - h_{p,n}(t) \\ &\leq \frac{(1-t)(b-a)pb^{p-1}}{2\sqrt{3}\sqrt{n}} \end{aligned}$$

for all $n \geq 1$ and $t \in [0,1]$.

(v) If $p \geq 1$, one has the inequalities

$$0 \leq t \cdot h_{p,n}(1) + (1-t) \cdot h_{p,n}(0) - h_{p,n}(t) \leq \frac{t(1-t)pb^{p-1}}{2\sqrt{3}\sqrt{n}}$$

and

$$0 \leq h_{p,n}(t) - h_{p,n+1}(t) \leq \frac{t(b-a)pb^{p-1}}{2\sqrt{3}\sqrt{n}(n+1)}$$

for all $n \geq 1$ and $t \in [0,1]$.

(vi) If $p \geq 2$, then one has the inequalities

$$0 \leq h_{p,n}(t) - [A(a,b)]^p \leq \frac{t^2(b-a)^2 p(p-1)b^{p-2}}{12n}$$

and

$$0 \leq h_{p,n}(t) - h_{p,n+1}(t) \leq \frac{t^2(b-a)^2 p(p-1)b^{p-2}}{12n(n+1)}$$

and

$$\begin{aligned} 0 &\leq t \cdot h_{p,n}(1) + (1-t) \cdot h_{p,n}(0) - h_{p,n}(t) \\ &\leq \frac{t(1-t)(b-a)^2 p(p-1)b^{p-2}}{12n} \end{aligned}$$

and

$$0 \leq h_{p,n}(1) - h_{p,n}(t) \leq \frac{(1-t)(b-a)^2 p(p-1)b^{p-2}}{12n}$$

for all $n \geq 1$ and $t \in [0,1]$.

CHAPTER 5

The $H.-H.$ Inequality for Different Kinds of Convexity

1. Integral Inequalities of $H.-H.$ Type for Log-Convex Functions

In what follows, I will be used to denote an interval of real numbers.

A function $f: I \to [0,\infty)$ is said to be *log-convex* or *multiplicatively convex* if $\log t$ is convex, or, equivalently, if for all $x, y \in I$ and $t \in [0,1]$ one has the inequality [**147**, p. 7]:

$$f(tx+(1-t)y) \leq [f(x)]^t [f(y)]^{1-t}. \tag{5.1}$$

We note that if f and g are convex and g is increasing, then $g \circ f$ is convex; moreover, since $f = \exp(\log f)$, it follows that a log-convex function is convex, but the converse may not necessarily be true [**147**, p. 7]. This follows directly from (5.1) because, by the arithmetic-geometric mean inequality, we have

$$[f(x)]^t [f(y)]^{1-t} \leq tf(x) + (1-t) f(y)$$

for all $x, y \in I$ and $t \in [0,1]$.

Let us recall the Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{5.2}$$

where $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex map on the interval I, $a, b \in I$ and $a < b$.

Note that if we apply the above inequality for the log-convex functions $f: I \to (0,\infty)$, we have that

$$\ln\left[f\left(\frac{a+b}{2}\right)\right] \leq \frac{1}{b-a}\int_a^b \ln f(x)\,dx \leq \frac{\ln f(a) + \ln f(b)}{2},$$

from which we get

$$f\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \leq \sqrt{f(a) f(b)}, \tag{5.3}$$

which is an inequality of Hadamard's type for log-convex functions.

Let us denote by $A(a,b)$ the *arithmetic mean* of the nonnegative real numbers, and by $G(a,b)$ the *geometric mean* of the same numbers.

Note that, by the use of these notations, Hadamard's inequality (5.2) can be written in the form:

$$f(A(a,b)) \leq \frac{1}{b-a}\int_a^b A(f(x), f(a+b-x))\,dx \leq A(f(a), f(b)). \tag{5.4}$$

It is easy to see this as

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$$

We now prove a similar result for log-convex mappings and geometric means [**59**].

THEOREM 124. *Let $f : I \to [0,\infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequality:*

$$f(A(a,b)) \le \frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))\,dx \le G(f(a), f(b)). \tag{5.5}$$

PROOF. Since f is log-convex, we have that

$$f(ta+(1-t)b) \le [f(a)]^t [f(b)]^{1-t}$$

for all $t \in [0,1]$ and

$$f((1-t)a+tb) \le [f(a)]^{1-t}[f(b)]^t$$

for all $t \in [0,1]$.

If we multiply the above inequalities and take square roots, we obtain

$$G(f(ta+(1-t)b), f((1-t)a+tb)) \le G(f(a), f(b)). \tag{5.6}$$

for all $t \in [0,1]$.

Integrating this inequality on $[0,1]$ over t, we get

$$\int_0^1 G(f(ta+(1-t)b), f((1-t)a+tb))\,dt \le G(f(a), f(b)).$$

If we change the variable $x := ta+(1-t)b$, $t \in [0,1]$, we obtain

$$\begin{aligned}&\int_0^1 G(f(ta+(1-t)b), f((1-t)a+tb))\,dt \\ = \;& \frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))\,dx\end{aligned}$$

and the second inequality in (5.5) is proved.

Now, by (5.1), for $t = \frac{1}{2}$, we have that

$$f\left(\frac{x+y}{2}\right) \le \sqrt{f(x) f(y)} \text{ for all } x, y \in I.$$

If we choose $x = ta+(1-t)b$, $y = (1-t)a+tb$, we get the inequality

$$f\left(\frac{a+b}{2}\right) \le G(f(ta+(1-t)b), f((1-t)a+tb)) \tag{5.7}$$

for all $t \in [0,1]$.

Integrating this inequality on $[0,1]$ over t, we obtain the first inequality in (5.5). This proves the theorem. ∎

COROLLARY 44. *With the above assumptions, $a \ge 0$ and f nondecreasing on I, we have the inequality:*

$$\begin{aligned} f(G(a,b)) &\le \frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))\,dx \\ &\le G(f(a), f(b)). \end{aligned} \tag{5.8}$$

The following result offers another inequality of Hadamard type for convex functions.

Corollary 45. *Let $f : I \to \mathbb{R}$ be a convex function on the interval I of the real numbers and $a, b \in I$ with $a < b$. Then one has the inequalities:*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \ln\left[\frac{1}{b-a}\int_a^b \exp\left[\frac{f(x)+f(a+b-x)}{2}\right]dx\right] \\
&\leq \frac{f(a)+f(b)}{2}.
\end{aligned}
\tag{5.9}
$$

Proof. Define the mapping $g : I \to (0,\infty)$, $g(x) = \exp f(x)$, which is clearly log-convex on I.

Now, if we apply Theorem 124, we obtain

$$
\begin{aligned}
\exp f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b \sqrt{\exp f(x)\cdot \exp f(a+b-x)}dx \\
&\leq \sqrt{\exp f(a)\cdot \exp f(b)},
\end{aligned}
$$

which implies (5.9). ∎

The following theorem for log-convex functions also holds [**59**].

Theorem 125. *Let $f : I \to (0,\infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequalities:*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \\
&\leq \frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))\,dx \\
&\leq \frac{1}{b-a}\int_a^b f(x)\,dx \\
&\leq L(f(a), f(b)),
\end{aligned}
\tag{5.10}
$$

where $L(p,q)$ is the logarithmic mean of the strictly positive real numbers p, q, i.e.,

$$
L(p,q) = \frac{p-q}{\ln p - \ln q} \text{ if } p \neq q \text{ and } L(p,p) = p.
$$

Proof. The first inequality from (5.10) was proved before.

We now have that

$$
G(f(x), f(a+b-x)) = \exp\left[\ln\left(G(f(x), f(a+b-x))\right)\right]
$$

for all $x \in [a,b]$.

Integrating this equality on $[a,b]$ and using the well-known Jensen's integral inequality for the convex mapping $\exp(\cdot)$, we have that

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b G\left(f\left(x\right),f\left(a+b-x\right)\right)dx\\
&= \frac{1}{b-a}\int_a^b \exp\left[\ln\left(G\left(f\left(x\right),f\left(a+b-x\right)\right)\right)\right]dx\\
&\geq \exp\left[\frac{1}{b-a}\int_a^b \ln\left[G\left(f\left(x\right),f\left(a+b-x\right)\right)\right]\right]dx\\
&= \exp\left[\frac{1}{b-a}\int_a^b \left(\frac{\ln f\left(x\right)+\ln f\left(a+b-x\right)}{2}\right)\right]dx\\
&= \exp\left[\frac{1}{b-a}\int_a^b \ln f\left(x\right)dx\right].
\end{aligned}$$

As it is clear that

$$\int_a^b \ln f\left(x\right)dx=\int_a^b \ln f\left(a+b-x\right)dx,$$

the second inequality in (5.10) is proved.

By the arithmetic mean-geometric mean inequality, we have that

$$G\left(f\left(x\right),f\left(a+b-x\right)\right)\leq\frac{f\left(x\right)+f\left(a+b-x\right)}{2},\ x\in[a,b]$$

from which we get, by integration, that

$$\frac{1}{b-a}\int_a^b G\left(f\left(x\right),f\left(a+b-x\right)\right)dx\leq\frac{1}{b-a}\int_a^b f\left(x\right)dx$$

and the third inequality in (5.10) is proved.

To prove the last inequality, we observe, by the log-convexity of f, that

$$f\left(ta+\left(1-t\right)b\right)\leq\left[f\left(a\right)\right]^t\left[f\left(b\right)\right]^{1-t} \tag{5.11}$$

for all $t\in[a,b]$.

Integrating (5.11) over t in $[0,1]$, we have

$$\int_0^1 f\left(ta+\left(1-t\right)b\right)dt\leq\int_0^1\left[f\left(a\right)\right]^t\left[f\left(b\right)\right]^{1-t}dt.$$

As

$$\int_0^1 f\left(ta+\left(1-t\right)b\right)dt=\frac{1}{b-a}\int_a^b f\left(x\right)dx$$

and

$$\int_0^1\left[f\left(a\right)\right]^t\left[f\left(b\right)\right]^{1-t}dt=L\left[f\left(a\right),f\left(b\right)\right],$$

the theorem is proved. ∎

Corollary 46. *Let $g : I \to \mathbb{R}$ be a convex function on the interval I of the real numbers and $a, b \in I$ with $a < b$. Then one has the inequality:*

$$
\begin{aligned}
\exp g\left(\frac{a+b}{2}\right) &\leq \exp\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \\
&\leq \frac{1}{b-a}\int_a^b \exp\left[\frac{g(x)+g(a+b-x)}{2}\right]dx \\
&\leq \frac{1}{b-a}\int_a^b \exp g(x)\,dx \leq E(g(a), g(b)),
\end{aligned}
\tag{5.12}
$$

where E is the exponential mean, i.e.,

$$E(p,q) = \frac{\exp p - \exp q}{p-q} \text{ for } p \neq q \text{ and } E(p,p) = p.$$

Remark 60. *Note that the inequality*

$$
\begin{aligned}
\exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] &\leq \frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))\,dx \\
&\leq \frac{1}{b-a}\int_a^b f(x)\,dx
\end{aligned}
$$

holds for every strictly positive and integrable mapping $f : [a,b] \to \mathbb{R}$ and the inequality

$$
\begin{aligned}
\exp\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) &\leq \frac{1}{b-a}\int_a^b \exp\left[\frac{f(x)+f(a+b-x)}{2}\right]dx \\
&\leq \frac{1}{b-a}\int_a^b \exp f(x)\,dx
\end{aligned}
$$

holds for every $f : [a,b] \to \mathbb{R}$ an integrable mapping on $[a,b]$.

Taking into account that the above two inequalities hold, we can assert that for every $f : [a,b] \to (0,\infty)$ an integrable map on $[a,b]$ we have the inequalities:

$$
\begin{aligned}
&\exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \\
&\leq \frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))\,dx \\
&\leq \frac{1}{b-a}\int_a^b f(x)\,dx \\
&\leq \ln\left[\frac{1}{b-a}\int_a^b \exp A(f(x), f(a+b-x))\,dx\right] \\
&\leq \ln\left[\frac{1}{b-a}\int_a^b \exp f(x)\,dx\right],
\end{aligned}
\tag{5.13}
$$

which is of interest in itself.

The following results improving the $H.-H.$ inequality for differentiable log-convex functions also hold [**39**].

THEOREM 126. *Let $f : I \to (0,\infty)$ be a differentiable log-convex function on the interval of real numbers $\mathring{I}$ (the interior of I) and $a,b \in \mathring{I}$ with $a<b$. Then the following inequalities hold:*

$$\begin{aligned}(5.14)\qquad & \frac{\frac{1}{b-a}\int_a^b f(x)\,dx}{f\left(\frac{a+b}{2}\right)} \\ \geq\ & L\left(\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(\frac{b-a}{2}\right)\right], \exp\left[-\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(\frac{b-a}{2}\right)\right]\right) \geq 1.\end{aligned}$$

PROOF. Since f is differentiable and log-convex on $\mathring{\mathrm{I}}$, we have that

$$\log f(x) - \log f(y) \geq \frac{d}{dt}(\log f)(y)(x-y)$$

for all $x,y \in \mathring{\mathrm{I}}$, which gives that

$$\log\left[\frac{f(x)}{f(y)}\right] \geq \frac{f'(y)}{f(y)}(x-y)$$

for all $x,y \in \mathring{\mathrm{I}}$. That is,

$$f(x) \geq f(y)\exp\left[\frac{f'(y)}{f(y)}(x-y)\right] \quad \text{for all } x,y \in \mathring{\mathrm{I}}. \tag{5.15}$$

Now, if we choose $y = \frac{a+b}{2}$, we obtain:

$$\frac{f(x)}{f\left(\frac{a+b}{2}\right)} \geq \exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right],\ x\in[a,b]. \tag{5.16}$$

Integrating this inequality over x on $[a,b]$ and using Jensen's integral inequality, we deduce that:

$$\begin{aligned}(5.17)\ \frac{\frac{1}{b-a}\int_a^b f(x)\,dx}{f\left(\frac{a+b}{2}\right)} &\geq \frac{1}{b-a}\int_a^b \exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx \\ &\geq \exp\left[\frac{1}{b-a}\int_a^b\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx\right] = 1.\end{aligned}$$

Now, as for $\alpha \neq 0$ we have that

$$\begin{aligned}\frac{1}{b-a}\int_a^b \exp(\alpha x)\,dx &= \frac{\exp(\alpha b)-\exp(\alpha a)}{\alpha(b-a)} \\ &= L[\exp(\alpha b),\exp(\alpha a)],\end{aligned}$$

where $L(\cdot,\cdot)$ is the usual logarithmic mean, then

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \exp\left[\alpha\left(x-\frac{a+b}{2}\right)\right]dx \\ =\ & \frac{\exp\left[\alpha\left(\frac{b-a}{2}\right)\right]-\exp\left[-\alpha\left(\frac{b-a}{2}\right)\right]}{\alpha\left[\left(\frac{b-a}{2}\right)-\left(-\left(\frac{b-a}{2}\right)\right)\right]} \\ =\ & L\left(\exp\left[\alpha\left(\frac{b-a}{2}\right)\right],\exp\left[-\alpha\left(\frac{b-a}{2}\right)\right]\right).\end{aligned}$$

Using the above equality for $\alpha=\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}$ the inequality (5.17) gives the desired result (5.14). ∎

The following corollary holds.

Corollary 47. *Let $g:I\to\mathbb{R}$ be a differentiable convex function on $\mathring{I}$ and $a,b\in\mathring{I}$ with $a<b$. Then we have the inequality:*

$$\begin{aligned}(5.18)\qquad & \frac{\frac{1}{b-a}\int_a^b\exp\left(g\left(x\right)\right)dx}{\exp g\left(\frac{a+b}{2}\right)}\\ \geq\; & L\left(\exp\left[g'\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right)\right],\exp\left[-g'\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right)\right]\right)\geq 1.\end{aligned}$$

The following theorem also holds [**39**].

Theorem 127. *Let $f:I\to\mathbb{R}$ be as in Theorem 126. Then we have the inequality:*

$$\begin{aligned}(5.19)\qquad \frac{\frac{f(a)+f(b)}{2}}{\frac{1}{b-a}\int_a^b f\left(x\right)dx} &\geq 1+\log\left[\frac{\int_a^b f\left(x\right)dx}{\int_a^b f\left(x\right)\exp\left[\frac{f'(x)}{f(x)}\left(\frac{a+b}{2}-x\right)\right]dx}\right]\\ &\geq 1+\log\left[\frac{\frac{1}{b-a}\int_a^b f\left(x\right)dx}{f\left(\frac{a+b}{2}\right)}\right]\geq 1.\end{aligned}$$

Proof. From the inequality (5.15) we have

$$f\left(\frac{a+b}{2}\right)\geq f\left(y\right)\exp\left[\frac{f'\left(y\right)}{f\left(y\right)}\left(\frac{a+b}{2}-y\right)\right],$$

for all $y\in[a,b]$.

Integrating over y and using Jensen's integral inequality for $\exp\left(\cdot\right)$ functions, we have

$$\begin{aligned}\left(b-a\right)f\left(\frac{a+b}{2}\right) &\geq \int_a^b f\left(y\right)\exp\left[\frac{f'\left(y\right)}{f\left(y\right)}\left(\frac{a+b}{2}-y\right)\right]dy\\ &\geq \int_a^b f\left(y\right)dy\cdot\exp\left(\frac{\int_a^b f\left(y\right)\left[\frac{f'(y)}{f(y)}\left(\frac{a+b}{2}-y\right)\right]dy}{\int_a^b f\left(y\right)dy}\right)\\ &= \int_a^b f\left(y\right)dy\cdot\exp\left(\frac{\int_a^b f'\left(y\right)\left(\frac{a+b}{2}-y\right)dy}{\int_a^b f\left(y\right)dy}\right).\end{aligned}$$

A simple integration by parts gives

$$\int_a^b f'\left(y\right)\left(\frac{a+b}{2}-y\right)dy=\int_a^b f\left(y\right)dy-\frac{f\left(a\right)+f\left(b\right)}{2}\left(b-a\right).$$

Then we have

$$\begin{aligned}\exp\left[1-\frac{\frac{f(a)+f(b)}{2}(b-a)}{\int_a^b f(x)\,dx}\right] &\leq \frac{\int_a^b f(y)\exp\left[\frac{f'(y)}{f(y)}\left(\frac{a+b}{2}-y\right)\right]dy}{\int_a^b f(y)\,dy} \\ &\leq \frac{(b-a)\,f\left(\frac{a+b}{2}\right)}{\int_a^b f(y)\,dy},\end{aligned}$$

which is equivalent to

$$\begin{aligned}1-\frac{\frac{f(a)+f(b)}{2}(b-a)}{\int_a^b f(x)\,dx} &\leq \log\left[\frac{\int_a^b f(y)\exp\left[\frac{f'(y)}{f(y)}\left(\frac{a+b}{2}-y\right)\right]dy}{\int_a^b f(y)\,dy}\right] \\ &\leq \log\left[\frac{f\left(\frac{a+b}{2}\right)}{\frac{1}{b-a}\int_a^b f(x)\,dx}\right]\end{aligned}$$

from where we get the desired inequality. ∎

The following corollary is a natural consequence of the above theorem.

COROLLARY 48. *Let $g : I \to \mathbb{R}$ be as in Corollary 47. Then we have the inequality:*

$$\begin{aligned}\frac{\frac{\exp g(a)+\exp g(b)}{2}}{\frac{1}{b-a}\int_a^b \exp g(x)\,dx} &\geq 1+\log\left[\frac{\int_a^b \exp g(x)\,dx}{\int_a^b \exp\left[g(x)-\left(x-\frac{a+b}{2}\right)g'(x)\right]dx}\right] \\ &\geq 1+\log\left[\frac{\frac{1}{b-a}\int_a^b \exp g(x)\,dx}{\exp g\left(\frac{a+b}{2}\right)}\right]\geq 1.\end{aligned}$$

1.1. Examples. The function $f(x)=\frac{1}{x}$, $x\in(0,\infty)$ is log-convex on $(0,\infty)$. Then we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b \frac{dx}{x} &= L^{-1}(a,b), \\ f\left(\frac{a+b}{2}\right) &= A^{-1}(a,b), \\ \frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)} &= -\frac{1}{A}.\end{aligned}$$

Now, applying the inequality (5.14) for the function $f(x)=\frac{1}{x}$, we get the inequality:

$$\frac{A(a,b)}{L(a,b)}\geq L\left(\exp\left(-\frac{b-a}{2A}\right),\exp\left(\frac{b-a}{2A}\right)\right)\geq 1, \tag{5.20}$$

which is a refinement of the well-known inequality

$$A(a,b)\geq L(a,b), \tag{5.21}$$

where $A(a,b)$ is the *arithmetic mean* and $L(a,b)$ is the *logarithmic mean* of a,b, that is, $A(a,b)=\frac{a+b}{2}$, and $L(a,b)=\frac{a-b}{\ln a-\ln b}$.

For $f(x) = \frac{1}{x}$, we also get

$$\frac{f(a) + f(b)}{2} = H^{-1}(a, b),$$

where $H(a, b) := \frac{1}{\frac{1}{a} + \frac{1}{b}}$ is the *harmonic mean* of a, b. Now, using the inequality (5.19) we obtain another interesting inequality:

$$\frac{L(a, b)}{H(a, b)} \geq 1 + \log\left[\frac{A(a, b)}{L(a, b)}\right] \geq 1, \tag{5.22}$$

which is a refinement of the following well-known inequality

$$L(a, b) \geq H(a, b). \tag{5.23}$$

Similar inequalities may be stated for the log-convex functions $f(x) = x^x$, $x > 0$ or $f(x) = e^x + 1$, $x \in \mathbb{R}$, etc. We omit the details.

2. The $H. - H.$ Inequality for $r-$Convex Functions

2.1. Introduction. Recall that a positive function f is *log-convex* on a real interval $[a, b]$ if for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$ we have

$$f(\lambda x + (1 - \lambda) y) \leq f(x)^{\lambda} f(y)^{1-\lambda}. \tag{5.24}$$

If the reverse inequality holds, f is said to be *log-concave.*

In addition, the power mean $M_r(x, y; \lambda)$ of order r of positive numbers x, y is defined by

$$M_r(x, y; \lambda) = \begin{cases} (\lambda x^r + (1 - \lambda) y^r)^{\frac{1}{r}}, & \text{if } r \neq 0 \\ \\ x^{\lambda} y^{1-\lambda}, & \text{if } r = 0. \end{cases}$$

In the special case $\lambda = \frac{1}{2}$, we contract this notation to $M_r(x, y)$.

In view of the above, a natural generalising concept is that of $r-$convexity. A positive function f is $r-$*convex* on $[a, b]$ (see [**80**]) if for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$

$$f(\lambda x + (1 - \lambda) y) \leq M_r(f(x), f(y); \lambda). \tag{5.25}$$

The definition of $r-$convexity naturally complements the concept of $r-$*concavity,* in which the inequality is reversed (cf. Uhrin [**180**]) and which plays an important role in statistics. We have that $0-$convex functions are simply log-convex functions and $1-$convex functions are ordinary convex functions. For the latter, the requirement that an $r-$ convex function be positive clearly can be relaxed.

Again, in all of the above, we may take a real linear space X in place of the real line. The condition $x, y \in [a, b]$ then becomes $x, y \in U$ for U a convex set in X.

We shall develop Hermite-Hadamard-type inequalities for log-convex functions (in Subsection 2.2) and more generally for $r-$convex functions (in Subsection 2.3). It is convenient to separate off the proof of the former special case as the functional representations differ in points of detail from those of the general case.

It will be convenient to invoke the logarithmic mean $L(x, y)$ of tow positive numbers x, y, which is given by

$$L(x, y) = \begin{cases} \frac{x-y}{\ln x - \ln y}, & x \neq y \\ \\ x, & x = y \end{cases}$$

and the generalised logarithmic means of order r of positive numbers x, y, defined by

$$L_r(x,y)=\begin{cases}\frac{r}{r+1}\cdot\frac{x^{r+1}-y^{r+1}}{x^r-y^r}, & r\neq 0,-1,\ \ x\neq y\\ \\ \frac{x-y}{\ln x-\ln y}, & r=0,\ \ x\neq y\\ \\ \frac{\ln x-\ln y}{x-y}, & r=-1,\ \ x\neq y\\ \\ x, & x=y.\end{cases}$$

Finally, in Subsection 2.4, we present generalisations of two recent results in the literature, one for log-convex and the other for log-concave functions.

2.2. Results for Log-convex Functions. In the previous section, we proved the following result (see also [**80**, Theorem 2.6]).

THEOREM 128. *Let f be a positive, log-convex function on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(t)\,dt\leq L(f(a),f(b)).$$

For f a positive log-concave functions, the inequality is reversed.

A similar proof to that in the second part of Theorem 125 gives the following generalisation (see [**80**, Theorem 2.2]).

THEOREM 129. *Let f be a positive, log-convex function on a convex set $U\subset X$, where X is a linear vector space. Then for $a,b\in U$,*

$$\int_0^1 f(sa+(1-s)b)\,ds\leq L(f(a),f(b)).$$

Some tighter inequalities may be derived by way of corollaries to Theorems 129 and 128

COROLLARY 49. *Let f be a positive log-convex function on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq \min_{x\in[a,b]}\left[\frac{(x-a)L(f(a),f(x))+(b-x)L(f(x),f(b))}{b-a}\right]. \tag{5.26}$$

If f is a positive log-concave function, then

$$\frac{1}{b-a}\int_a^b f(t)\,dt \geq \max_{x\in[a,b]}\left[\frac{(x-a)L(f(a),f(x))+(b-x)L(f(x),f(b))}{b-a}\right]. \tag{5.27}$$

PROOF. Let f be a positive log-convex function. Then by Theorem 128, we have that

$$\begin{aligned}\int_a^b f(t)\,dt &= \int_a^x f(t)\,dt+\int_x^b f(t)\,dt\\ &\leq (x-a)L(f(a),f(x))+(b-x)L(f(x),f(b))\end{aligned}$$

for all $x\in[a,b]$, whence (5.26). Similarly, we can prove (5.27). ∎

Corollary 50. *([**80**]) Let f be a positive log-convex function on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{n}\sum_{i=1}^n L\left(f\left(a+\frac{i-1}{n}(b-a)\right), f\left(a+\frac{i}{n}(b-a)\right)\right).$$

If f is a positive log-concave function, the inequality is reversed.

Proof. The result follows by applying Theorem 128 to the integrals on the right in

$$\int_a^b f(t)\,dt = \sum_{i=1}^n \int_{a+\frac{(i-1)(b-a)}{n}}^{a+\frac{i(b-a)}{n}} f(t)\,dt.$$

∎

Corollary 51.

(a) *If $f:[a,b]\to\mathbb{R}_+$ is log-convex, then*

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le M_{\frac{1}{3}}(f(a), f(b)),$$

while, if f is log-concave, then

$$\frac{1}{b-a}\int_a^b f(t)\,dt \ge \sqrt{f(a)f(b)}. \tag{5.28}$$

(b) *If $f:U\to\mathbb{R}_+$ $(U\subset X)$ is log-convex, then for $a,b\in U$,*

$$\int_0^1 f(sa+(1-s)b)\,ds \le M_{\frac{1}{3}}(f(a), f(b)),$$

while if f is log-concave, then

$$\int_0^1 f(sa+(1-s)b)\,ds \ge \sqrt{f(a)f(b)}.$$

Proof. Part (a) follows from Theorem 128 and the inequalities

$$G(a,b) \le L(a,b) \le M_{\frac{1}{3}}(a,b)$$

for logarithmic means (cf. [**98**], [**134**], [**148**]). ∎

2.3. Inequalities of Hadamard Type for $r-$Convex Functions. The following result holds.

Theorem 130. *([**80**]) Suppose f is a positive r-convex function on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le L_r(f(a), f(b)).$$

If f is a positive $r-$concave function, then the inequality is reversed.

Proof. The case $r=0$ has been dealt with as Theorem 128. Suppose that $r\neq 0,-1$. First assume that $f(a)\neq f(b)$. By (5.25) we have

$$\begin{aligned}\int_a^b f(t)\,dt &= (b-a)\int_0^1 f(sb+(1-s)a)\,ds\\ &\leq (b-a)\int_0^1 \{sf^r(b)+(1-s)f^r(a)\}^{\frac{1}{r}}\,ds\\ &= (b-a)\int_{f^r(a)}^{f^r(b)} \frac{t^{\frac{1}{r}}dt}{f^r(b)-f^r(a)}\\ &= (b-a)\frac{r}{r+1}\cdot\frac{f^{r+1}(b)-f^{r+1}(a)}{f^r(b)-f^r(a)}\\ &= (b-a)L_r(f(a),f(b)).\end{aligned}$$

For $f(a)=f(b)$, we have similarly

$$\begin{aligned}\int_a^b f(t)\,dt &\leq (b-a)\int_0^1 \{sf^r(a)+(1-s)f^r(a)\}^{\frac{1}{r}}\,ds\\ &= (b-a)f(a)\\ &= (b-a)L_r(f(a),f(a)).\end{aligned}$$

Finally, let $r=-1$. For $f(a)\neq f(b)$ we have again

$$\begin{aligned}\int_a^b f(t)\,dt &\leq (b-a)\int_0^1 \left\{sf^{-1}(b)+(1-s)f^{-1}(a)\right\}^{-1}ds\\ &= \frac{b-a}{\frac{1}{f(b)}-\frac{1}{f(a)}}\int_{\frac{1}{f(a)}}^{\frac{1}{f(b)}} t^{-1}dt\\ &= \frac{b-a}{\frac{1}{f(b)}-\frac{1}{f(a)}}\left(\ln\frac{1}{f(b)}-\ln\frac{1}{f(a)}\right)\\ &= (b-a)f(a)f(b)\frac{\ln f(a)-\ln f(b)}{f(a)-f(b)}\\ &= (b-a)L_{-1}(f(a),f(b)).\end{aligned}$$

The proof when $f(a)=f(b)$ is similar. ∎

2.4. On Some Inequalities of Fink, Mond and Pečarić. Recently, Fink [**77**] showed that

$$\frac{\pi}{4}\int_{-1}^1 f(x+vt)\cos\frac{\pi t}{2}dt\leq\frac{f(x+v)+f(x-v)}{2} \tag{5.29}$$

for a positive, log-convex function $f: mathbbR\to(0,\infty)$. He gave also a reverse and rather more complicated inequality for positive, log-concave functions. Mond and Pečarić [**115**] established the inequality

$$\frac{\pi}{4}\int_{-1}^1 f(x+vt)\cos\frac{\pi t}{2}dt\geq\sqrt{f(x+v)f(x-v)} \tag{5.30}$$

for a positive, log-concave function.

We now consider some generalisation of these inequalities. We shall invoke a useful result due to Féjer [**72**].

LEMMA 15. *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $p:[a,b]\to\mathbb{R}$ a positive, integrable function such that*

$$p\left(a+t\right)=p\left(b-t\right),\ \ 0\leq t\leq\frac{1}{2}\left(b-a\right).$$

Then

$$f\left(\frac{a+b}{2}\right)\leq\frac{\int_a^b p\left(t\right)f\left(t\right)dt}{\int_a^b p\left(t\right)dt}\leq\frac{f\left(a\right)+f\left(b\right)}{2}.$$

THEOREM 131. *Let p be a nonnegative, integrable, even function.*

(a) *If f is a positive log-convex function, then*

$$\frac{\int_{-1}^1 p\left(t\right)f\left(x+vt\right)dt}{\int_{-1}^1 p\left(t\right)dt}\leq\frac{f\left(x+v\right)+f\left(x-v\right)}{2}. \tag{5.31}$$

(b) *If f is a positive log-concave function, then*

$$\frac{\int_{-1}^1 p\left(t\right)f\left(x+vt\right)dt}{\int_{-1}^1 p\left(t\right)dt}\geq\sqrt{f\left(x+v\right)f\left(x-v\right)}. \tag{5.32}$$

PROOF. (a) Since f is log-convex, we have

$$f\left(x+vt\right)\leq\left[f\left(x+v\right)\right]^{\frac{(1+t)}{2}}\left[f\left(x-v\right)\right]^{\frac{(1-t)}{2}}\quad\left(-1\leq t\leq1\right). \tag{5.33}$$

Set $B=\left(\frac{f(x+v)}{f(x-v)}\right)^{\frac{1}{2}}$. Then integration yields

$$\frac{\int_{-1}^1 p\left(t\right)f\left(x+vt\right)dt}{\int_{-1}^1 p\left(t\right)dt}\leq\left(f\left(x+v\right)f\left(x-v\right)\right)^{\frac{1}{2}}\frac{\int_{-1}^1 p\left(t\right)B^t dt}{\int_{-1}^1 p\left(t\right)dt}. \tag{5.34}$$

Since the map $:t\to B^t$ is convex and p satisfies

$$p\left(-1+t\right)=p\left(1-t\right),$$

we can apply Lemma 15 to derive

$$1\leq\frac{\int_{-1}^1 p\left(t\right)B^t dt}{\int_{-1}^1 p\left(t\right)dt}\leq\frac{B+B^{-1}}{2}. \tag{5.35}$$

By the second inequality, (5.34) becomes

$$\begin{aligned}\frac{\int_{-1}^1 f\left(x+vt\right)p\left(t\right)dt}{\int_{-1}^1 p\left(t\right)dt} &\leq \frac{B+B^{-1}}{2}\left(f\left(x+v\right)f\left(x-v\right)\right)^{\frac{1}{2}}\\ &= \frac{f\left(x+v\right)+f\left(x-v\right)}{2}\end{aligned}$$

and we have (5.31).

(b) If f is log-concave, the inequality in (5.33) and so also (5.34) is reversed. The first inequality in (5.35) gives (5.32) at once.

∎

REMARK 61. *The inequality (5.32) is a weighted generalisation of (5.28).*

REMARK 62. *Let f be a positive log-convex function. Then it is also convex, so using Lemma 15, we have (5.31). That is, we have*

$$f(x) \leq \frac{\int_{-1}^{1} p(t) f(x+vt)\,dt}{\int_{-1}^{1} p(t)\,dt} \leq \frac{f(x+v)+f(x-v)}{2}.$$

Thus we see that (5.29) follows from Féjer's generalisation of the Hadamard inequality. Inequality (5.30) does not appear to follow in this way.

By the same argument, we have the following result.

THEOREM 132. *Let p be a nonnegative, integrable, even function and U a convex set from a linear vector space X. Suppose $x+v, x-v \in U$ and $x, v \in X$. If f is a positive, log-convex function, then (5.31) holds, while if f is a positive, log-concave function, then (5.32) holds.*

THEOREM 133. *Let p be as in Theorem 132 and f a positive function.*

(a) *Suppose f is $r-$convex. Then if $r \leq 1$,*

$$\frac{\int_{-1}^{1} f(x+vt)\,p(t)\,dt}{\int_{-1}^{1} p(t)\,dt} \leq \frac{f(x+v)+f(x-v)}{2}, \tag{5.36}$$

while if $r \geq 1$

$$\frac{\int_{-1}^{1} f(x+vt)\,p(t)\,dt}{\int_{-1}^{1} p(t)\,dt} \leq M_r\left(f(x+v), f(x-v)\right). \tag{5.37}$$

(b) *Suppose f is $r-$concave. Then if $r \leq 1$, (5.37) holds with the inequality reversed. If $r \geq 1$, then the inequality (5.36) is reversed.*

PROOF. For $r=0$ the theorem reduces to Theorem 132, so we may suppose that $r \neq 0$.

If f is an $r-$convex function, then by definition we have for $|t| \leq 1$ that

$$f\left(\frac{1+t}{2}(x+v)+\frac{1-t}{2}(x-v)\right) \leq \left[\frac{1+t}{2} f^r(x+v)+\frac{1-t}{2} f^r(x-v)\right]^{\frac{1}{r}},$$

that is,

$$f(x+tv) \leq h(t),$$

where

$$h(t) := \left[\frac{1+t}{2} f^r(x+v)+\frac{1-t}{2} f^r(x-v)\right]^{\frac{1}{r}},$$

and so

$$\frac{\int_{-1}^{1} f(x+vt)\,p(t)\,dt}{\int_{-1}^{1} p(t)\,dt} \leq \frac{\int_{-1}^{1} h(t)\,p(t)\,dt}{\int_{-1}^{1} p(t)\,dt}. \tag{5.38}$$

Now h is convex on $[-1,1]$ for $r \leq 1$ and concave for $r \geq 1$. So for $r \leq 1$, we have by Lemma 15 that

$$\begin{aligned} M_r\left(f(x+v), f(x-v)\right) &\leq \frac{\int_{-1}^{1} h(t)\,p(t)\,dt}{\int_{-1}^{1} p(t)\,dt} \\ &\leq \frac{f(x+v)+f(x-v)}{2}, \end{aligned} \tag{5.39}$$

while for $r \geq 1$ we have the reverse inequality.

Therefore (5.38) gives for $r \leq 1$ that (5.36) holds and for $r \geq 1$ that (5.37) holds.

Now, let f be an $r-$concave function. Then for $r \neq 0$, the inequality (5.38) is reversed. So, using (5.39), we have for $r \leq 1$ the reverse inequality to (5.37) and for $r \geq 1$ the reverse inequality in (5.36), and we are done. ∎

REMARK 63. *Inequality (5.36) can be obtained from the fact that an $r-$convex function is convex for $r \leq 1$. Similarly, an $r-$concave function is concave for $r \geq 1$.*

3. Stolarsky Means and $H. - H.$'s Inequality

3.1. Introduction. Recall the second part of the $H. - H.$ inequality. This states that if $f : [a, b] \to \mathbb{R}$ is convex, then

$$\frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2}.$$

The $H. - H.$ inequality has recently been extended in two quite different ways. Recall that the integral power mean M_p of a positive function f on $[a, b]$ is a functional given by

$$M_p(f) = \begin{cases} \left[\frac{1}{b-a}\int_a^b f(t)^p dt\right]^{1/p}, & p \neq 0, \\ \exp\left[\frac{1}{b-a}\int_a^b \ln f(t)dt\right], & p = 0. \end{cases} \tag{5.40}$$

Further, the extended logarithmic mean L_p of two positive numbers a, b is given for $a = b$ by $L_p(a, a) = a$ and for $a \neq b$ by

$$L_p(a,b) = \begin{cases} \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p \neq -1, 0, \\ \dfrac{b-a}{\ln b - \ln a}, & p = -1, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & p = 0. \end{cases}$$

The second part of the $H. - H.$ inequality may now be recast as a relationship

$$M_1(f) \leq L_1(f(a), f(b))$$

between integral power means and extended logarithmic means. In [**134**] the following extension is derived for this suggestive result (see also [**138**]).

THEOREM 134. *If $f : [a, b] \to \mathbb{R}$ is positive, continuous and convex, then*

$$M_p(f) \leq L_p(f(a), f(b)), \tag{5.41}$$

while if f is concave, (5.41) is reversed.

REMARK 64. *We note that $L_{-1}(a, b)$ is the well–known logarithmic mean $L(a, b)$ and $L_0(a, b)$ is the identric mean $I(a, b)$.*

The second extension involves the power mean $M_r(x,y;\lambda)$ of order r of positive numbers x, y, which is defined by

$$M_r(x,y;\lambda) = \begin{cases} (\lambda x^r + (1-\lambda)y^r)^{1/r}, & \text{if } r \neq 0, \\ x^\lambda y^{1-\lambda}, & \text{if } r = 0. \end{cases}$$

In the special case $\lambda = \frac{1}{2}$ this notation is contracted to $M_r(x,y)$.

It involves also the alternative extended logarithmic mean $F_r(x,y)$ of two positive numbers x, y, which is prescribed by $F_r(x,x) = x$ and for $x \neq y$ by

$$F_r(x,y) = \begin{cases} \frac{r}{r+1} \cdot \frac{x^{r+1} - y^{r+1}}{x^r - y^r}, & r \neq 0, -1, \\ \frac{x-y}{\ln x - \ln y}, & r = 0, \\ xy \frac{\ln x - \ln y}{x - y}, & r = -1. \end{cases},$$

This includes the usual logarithmic mean as the special case $r = 0$.

An idea of r–convexity may be introduced *via* power means (see also [**138**]).

Definition 1. *A positive function f is said to be r–convex on an interval $[a,b]$ if, for all $x, y \in [a,b]$ and $\lambda \in [0,1]$,*

$$f(\lambda x + (1-\lambda)y) \leq M_r(f(x), f(y); \lambda). \tag{5.42}$$

This definition of r–convexity naturally complements the concept of r–concavity in which the inequality is reversed (see [**180**]). This concept plays an important role in statistics.

The definition of r–convexity can be expanded as the condition that

$$f^r(\lambda x + (1-\lambda)y) \leq \begin{cases} \lambda f^r(x) + (1-\lambda) f^r(y), & \text{if } r \neq 0, \\ f^\lambda(x) f^{1-\lambda}(y), & \text{if } r = 0. \end{cases}$$

For a positive function f, it is applicable for nonintegral values of r. Also, suppose as is usual that f is nonnegative and possesses a second derivative. If $r \geq 2$, then

$$\frac{d^2}{dx^2} f^r = r(r-1) f^{r-2} (f')^2 + r f^{r-1} f'',$$

which is nonnegative if $f'' \geq 0$. Hence under the restrictions noted, ordinary convexity implies r–convexity. The reverse implication is not the case, as is shown by the function $f(x) = x^{1/2}$ for $x > 0$.

We note that the standard definition of r–convexity (see [**114**, Chapter 1, Section 6]) is quite different. Recall that when the derivative $f^{(r)}$ exists, f is r–convex if and only if $f^{(r)} \geq 0$ (see [**114**, Chapter 1, Theorem 1]). Consider the function

$$f(x) := x(x^3 - x^2 + 1)$$

on $I = \left(\frac{1}{4}, \frac{1}{2}\right)$. For $x \in I$, we have $f^{(2)} < 0$ but $f^{(3)} > 0$, so f is 3–convex but not convex. The function $g = -f$ on the same domain is a function which is convex but not 3–convex.

After this lengthy aside, we are ready to state the second extension of Hadamard's inequality, which was established recently in [**80**]. This relaxes the assumption of convexity to one of r–convexity.

THEOREM 135. *Suppose f is a positive function on $[a,b]$. If f is r–convex, then*

$$\frac{1}{b-a}\int_a^b f(t)dt \le F_r(f(a),f(b)), \tag{5.43}$$

while if f is r–concave, the inequality is reversed.

In Subsection 3.3 we prove a result which subsumes Theorems 134 and 135 as special cases. The relevant generalization of L_p and F_r turns out to be the well–known Stolarsky mean $E(x,y;r,s)$ (see Stolarsky [**172**]). This is given by $E(x,x;r,s)=x$ if $x=y>0$ and for distinct positive numbers x, y by

$$E(x,y;r,s)=\left[\frac{r}{s}\frac{y^s-x^s}{y^r-x^r}\right]^{\frac{1}{s-r}}\ ,\ r\neq s \quad \text{and} \quad r,s\neq 0,$$

$$E(x,y;r,0)=E(x,y;0,r)=\left[\frac{1}{r}\frac{y^r-x^r}{\ln y-\ln x}\right]^{\frac{1}{r}}\ ,\ r\neq 0,$$

$$E(x,y;r,r)=e^{-\frac{1}{r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{\frac{1}{x^r-y^r}}\ ,\ r\neq 0,$$

$$E(x,y;0,0)=\sqrt{xy}\ .$$

Clearly $E(x,y;1,p+1)=L_p(x,y)$ and $E(x,y;r,r+1)=F_r(x,y)$.

The key to our proof is a new integral representation of Stolarsky's mean. This is of some interest in its own right and is presented in Subection 3.2.

In Subsection 3.4 we establish a related generalization of the Fink–Mond–Pečarić inequalities [**77, 115**].

3.2. Integral Representations. Carlson [**14**] has established the integral representation

$$L(x,y)=\left[\int_0^1\frac{dt}{tx+(1-t)y}\right]^{-1}, \tag{5.44}$$

while Neuman [**120**] has given the alternative integral representation

$$L(x,y)=\int_0^1 x^t y^{1-t}dt. \tag{5.45}$$

Let $\bar{M}_p(f)$ denote the integral mean (5.40) for $a=0$, $b=1$ and put $e_{x,y}(t):=tx+(1-t)y$. A simple evaluation of the right–hand side shows that

$$L_p(x,y)=\bar{M}_p(e_{x,y}),$$

which provides a generalization of the integral representation (5.44) for the extended logarithmic means $L_p(x,y)$. Similarly we can derive

$$F_r(x,y)=\int_0^1 M_r(x,y;t)dt$$

as a natural extension of (5.45).

In the above we regard $M_r(x,y;t)$ as a function of the parameter t. Set

$$m_{r,x,y}(t):=M_r(x,y;t).$$

Then we may evaluate the integrals on the right–hand side of

$$\bar{M}_{s-r}(m_{r,x,y}) = \begin{cases} \left[\int_0^1 (M_r(x,y;t))^{s-r}\, dt\right]^{1/(s-r)}, & s \neq r \\ \exp\left[\int_0^1 \ln M_r(x,y;t)dt\right], & s = r \end{cases} \tag{5.46}$$

to give a simple derivation of the representation

$$E(x,y;r,s) = \bar{M}_{s-r}(m_{r,x,y}). \tag{5.47}$$

3.3. Hadamard's Inequality for r–Convex Functions. The following result holds [**138**].

THEOREM 136. *Suppose f is a positive function on $[a,b]$. Then if f is r–convex,*

$$M_p(f) \leq E(f(a), f(b); r, p+r), \tag{5.48}$$

while if f is r–concave, the inequality is reversed.

PROOF. First we suppose r–convexity. Let $p \neq 0$. Then

$$\begin{aligned} M_p(f) &= \left[\frac{1}{b-a}\int_a^b f^p(t)dt\right]^{1/p} = \left[\int_0^1 f^p(sb+(1-s)a)ds\right]^{1/p} \\ &\leq \left[\int_0^1 M_r^p(f(b), f(a); s)ds\right]^{1/p}. \end{aligned}$$

From (5.46) and (5.47) we deduce that

$$M_p(f) \leq \bar{M}_p\left(m_{r,f(b),f(a)}\right) = E\left(f(a), f(b); r, r+p\right).$$

Similarly, for $p = 0$, we have

$$\begin{aligned} M_0(f) &= \exp\left[\frac{1}{b-a}\int_a^b \ln f(t)dt\right] = \exp\left[\int_0^1 \ln f(sb+(1-s)a)ds\right] \\ &\leq \exp\left[\int_0^1 \ln M_r(f(b), f(a); s)ds\right] \end{aligned}$$

and again from (5.46) and (5.47) we derive

$$M_0(f) \leq \bar{M}_0\left(m_{r,f(b),f(a)}\right) = E\left(f(a), f(b); r, r\right).$$

The proof in the case of r–concavity is exactly similar. ∎

REMARK 65. *([**138**]) For $p = 1$, (5.48) becomes*

$$M_1(f) \leq E(f(a), f(b); r, r+1),$$

that is (5.43), while for $r = 1$ we have

$$M_p(f) \leq E(f(a), f(b); 1, p+1),$$

which is (5.41). Thus our result subsumes Theorems 134 and 135.

We observe that an r-convex function f can be defined on a convex set U in a real linear space X with (5.42) holding whenever $x, y \in U$ and $\lambda \in [0,1]$. This leads to the following result (see also [**138**]).

THEOREM 137. *Suppose f is a positive function on $U(\subset X)$, where U is convex and X a linear space. Then if f is r–convex,*

$$\bar{M}_p(f(e_{a,b})) \leq E(f(a), f(b); r, p+r),$$

while if f is r–concave, the inequality is reversed.

3.4. A Further Generalization of the Fink–Mond–Pečarić Inequalities. The following generalization of Fink-Mond-Pečarić inequalities ([**77, 115**]) was obtained in [**80**].

THEOREM 138. *Let w be a nonnegative, integrable, even function on $[-1,1]$ with positive integral and let f be a positive function.*

a) *If f is r–convex function, then for $r \leq 1$,*

$$\frac{\int_{-1}^{1} f(x+vt)w(t)dt}{\int_{-1}^{1} w(t)dt} \leq \frac{f(x+v)+f(x-v)}{2}, \tag{5.49}$$

while if $r \geq 1$,

$$\frac{\int_{-1}^{1} f(x+vt)w(t)dt}{\int_{-1}^{1} w(t)dt} \leq M_r(f(x+v), f(x-v)). \tag{5.50}$$

b) *Suppose f is r–concave. Then if $r \leq 1$, the inequality (5.50) is reversed, while if $r \geq 1$, the inequality (5.49) is reversed.*

We extend these results to allow a power mean of order p on the left–hand sides of (5.49) and (5.50) in place of an arithmetic integral mean. The power mean of order p is defined by

$$\tilde{M}_p(f,w) = \begin{cases} \left[\frac{\int_{-1}^{1} f^p(t)w(t)dt}{\int_{-1}^{1} w(t)dt}\right]^{1/p}, & p \neq 0 \\ \exp\left\{\frac{\int_{-1}^{1} w(t)\ln f(t)dt}{\int_{-1}^{1} w(t)dt}\right\}, & p = 0. \end{cases}$$

We shall need the following useful result due to Fejér [**72**].

LEMMA 16. *Suppose $h : [a,b] \to \mathbb{R}$ is convex and $w : [a,b] \to \mathbb{R}$ a nonnegative, integrable function with positive integral and such that*

$$w(a+t) = w(b-t)\ , \quad 0 \leq t \leq \frac{1}{2}(b-a). \tag{5.51}$$

Then

$$h\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b w(t)h(t)dt}{\int_a^b w(t)dt} \leq \frac{h(a)+h(b)}{2}. \tag{5.52}$$

The inequality is reversed if h is concave.

We shall derive the following generalization of Theorem 138 (cf. [**138**]).

THEOREM 139. *Let w be a nonnegative, integrable, even function with positive integral over $[-1,1]$, and let f be a positive function. Put $\tilde{f}(t) := f(x+vt)$ for $t \in [-1,1]$.*

a) *If f is r–convex and $m = \max\{r,p\}$, then*

$$\tilde{M}_p(\tilde{f}, w) \leq M_m(f(x+v), f(x-v)). \tag{5.53}$$

b) *If f is r-concave and $m = \min\{r, p\}$, then the inequality is reversed.*

PROOF. The Proof is as follows.

a) Take f to be positive and r–convex. First suppose that $p, r \neq 0$. Since f is r–convex, we have

$$f\left(\frac{1+t}{2}(x+v)+\frac{1-t}{2}(x-v)\right) \leq \left[\frac{1+t}{2}f^r(x+v)+\frac{1-t}{2}f^r(x-v)\right]^{1/r}.$$

Therefore

$$\begin{aligned}(5.54)\quad \tilde{M}_p(\tilde{f}, w) &= \left[\frac{\int_{-1}^1 f^p(x+vt)w(t)dt}{\int_{-1}^1 w(t)dt}\right]^{1/p} \\ &\leq \left[\frac{\int_{-1}^1 \left[\frac{1+t}{2}f^r(x+vt)+\frac{1-t}{2}f^r(x-v)\right]^{p/r} w(t)dt}{\int_{-1}^1 w(t)dt}\right]^{1/p} \\ &= \left[\frac{\int_{-1}^1 h(t)w(t)dt}{\int_{-1}^1 w(t)dt}\right]^{1/p},\end{aligned}$$

where $h(t) = \left[\frac{1+t}{2}f^r(x+v)+\frac{1-t}{2}f^r(x-v)\right]^{p/r}$.

We have that $h(t)$ is convex on $[-1, 1]$ for $r \leq p$ and concave for $r \geq p$. Since w is even, $w(-1+t) = w(1-t)$, so that (5.51) holds for $a = -1$, $b = 1$. Hence by Lemma 16

$$\left(\frac{f^r(x+v)+f^r(x-v)}{2}\right)^{p/r} \leq \frac{\int_{-1}^1 w(t)h(t)dt}{\int_{-1}^1 w(t)dt} \leq \frac{f^p(x+v)+f^p(x-v)}{2}$$

applies if $r \leq p$ and the reverse inequality holds for $r \geq p$.
If $r \leq p$ with $p > 0$, we may take p–th roots to derive

$$\begin{aligned}(5.55)\qquad M_r(f(x+v), f(x-v)) &\leq \left(\frac{\int_{-1}^1 w(t)h(t)dt}{\int_{-1}^1 w(t)dt}\right)^{1/p} \\ &\leq M_p(f(x+v), f(x-v)).\end{aligned}$$

The same conclusion holds if $r \geq p$ with $p < 0$. The inequalities in (5.55) are reversed if $r \leq p$ with $p < 0$ or $r \geq p$ with $p > 0$. Coupling these results with (5.54) gives part (a) of the enunciation for the case $p, r \neq 0$. Now suppose $r = 0$ and $p \neq 0$. Then we have

$$\begin{aligned}(5.56)\qquad \tilde{M}_p(\tilde{f}, w) &= \left[\frac{\int_{-1}^1 f^p(x+vt)w(t)dt}{\int_{-1}^1 w(t)dt}\right]^{1/p} \\ &\leq \left[\frac{\int_{-1}^1 f(x+v)^{p\frac{1+t}{2}} f(x-v)^{p\frac{1-t}{2}} w(t)dt}{\int_{-1}^1 w(t)dt}\right]^{1/p}.\end{aligned}$$

Put

$$h(t) = f(x+v)^{p\frac{1+t}{2}} f(x-v)^{p\frac{1-t}{2}}.$$

Then h is convex, and by (5.52)

$$\left(\sqrt{f(x+v)f(x-v)}\right)^p \leq \frac{\int_{-1}^1 w(t)h(t)dt}{\int_{-1}^1 w(t)dt} \leq \frac{f^p(x+v)+f^p(x-v)}{2}.$$

For $p>0$, we get (5.55) with $r=0$. For $p<0$ the inequalities are reversed. So by (5.56), we again have (5.53).

Suppose $r \neq 0$ and $p=0$. Then we have

$$\begin{aligned} &\tilde{M}_0(\tilde{f},w) \\ = \quad & \exp\left[\frac{\int_{-1}^1 w(t)\ln f(x+vt)dt}{\int_{-1}^1 w(t)dt}\right] \\ \leq \quad & \exp\left[\frac{\int_{-1}^1 w(t)\ln\left[\frac{1+t}{2}f^r(x+v)+\frac{1-t}{2}f^r(x-v)\right]^{1/r}dt}{\int_{-1}^1 w(t)dt}\right]. \end{aligned} \tag{5.57}$$

The function

$$h(t) := \ln\left[\frac{1+t}{2}f^r(x+v)+\frac{1-t}{2}f^r(x-v)\right]^{1/r}$$

is convex for $r<0$ and concave for $r>0$. So, for $r<0$, (5.52) gives

$$\begin{aligned} \ln M_r(f(x+v),f(x-v)) \quad &\leq \quad \frac{\int_{-1}^1 w(t)h(t)dt}{\int_{-1}^1 w(t)dt} \\ &\leq \quad \ln M_0(f(x+v),f(x-v)). \end{aligned} \tag{5.58}$$

The inequalities are reversed if $r>0$. By (5.57) and (5.58) we again have (5.53).

Finally, suppose $r=0$, $p=0$. We have

$$\begin{aligned} \tilde{M}_0(\tilde{f},w) \quad &= \quad \exp\left[\frac{\int_{-1}^1 w(t)\ln f(x+vt)dt}{\int_{-1}^1 w(t)dt}\right] \\ &\leq \quad \exp\left[\frac{\int_{-1}^1 w(t)\ln f(x+v)^{\frac{1+t}{2}}f(x-v)^{\frac{1-t}{2}}dt}{\int_{-1}^1 w(t)dt}\right] \\ &= \quad M_0[f(x+v),f(x-v)]. \end{aligned}$$

Thus (a) is established in all cases.

b) The proof is similar.

∎

REMARK 66. *([**138**]) Suppose U is a convex set in a real linear space X. Then the conclusion of Theorem 139 holds when $x, v \in X$ are such that $x+v$, $x-v \in U$ and $f: U \to \mathbb{R}_+$ is r–convex (r–concave) on U.*

4. Functional Stolarsky Means and $H.-H.$ Inequality

4.1. Introduction. As a response to the needs of diverse applications, a considerable variety of particular means of sets of numbers have been proposed and studied in the literature. See, for example, the compendious treatment of Bullen, Mitrinović and Vasić [**12**]. Valuable work has been done in systematising and unifying this area *via* the judicious introduction of parameters.

A helpful paradigm is due to Stolarsky [**172**]. See also Tobey [**179**]. The Stolarsky mean $E_{r,s}(x,y)$ of two positive numbers x and y is given by $E_{r,s}(x,x)=x$ when the numbers coincide and otherwise by

$$E_{r,s}(x,y)=\begin{cases}\left[\dfrac{r}{s}\cdot\dfrac{y^s-x^s}{y^r-x^r}\right]^{1/(s-r)}, & r\neq s,\ r,s\neq 0;\\[2ex] \left[\dfrac{1}{r}\cdot\dfrac{y^r-x^r}{\ln y-\ln x}\right]^{1/r}, & r\neq 0,\ s=0;\\[2ex] e^{-1/r}\left(\dfrac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)}, & s=r\neq 0;\\[2ex] \sqrt{xy}, & r=s=0.\end{cases} \tag{5.59}$$

For various choices of the parameters r, s, this subsumes a number of commonly employed means as special cases. Apart from direct application, it has theoretical interest. Thus there is a comparison theorem prescribing for which pairs (r,s), (u,v) the inequality $E_{r,s}(x,y)<E_{u,v}(x,y)$ holds for all $x\neq y$ (see Leach and Sholander [**90**] and Páles [**131, 132**]). A trivial special case is the familar inequality between the geometric and arithmetic means of a pair of distinct positive numbers.

An interesting representation has been found [**138**] linking the Stolarsky mean with power means and integral means. The power mean $m_r(x,y;t)$ of order r and weights t and $1-t$ (for $t\in[0,1]$) of positive numbers x,y is defined by

$$m_r(x,y;t)=\begin{cases}\left(tx^r+(1-t)y^r\right)^{1/r} & \text{if } \ r\neq 0,\\[1ex] x^ty^{1-t}, & \text{if } \ r=0,\end{cases} \tag{5.60}$$

whilst the integral mean over $[0,1]$ of a positive function f is

$$M_r(f)=\begin{cases}\left[\int_0^1(f(t))^r dt\right]^{1/r}, & \text{if } \ r\neq 0,\\[1ex] \exp\left(\int_0^1 \ln f(t)dt\right), & \text{if } \ r=0.\end{cases}$$

It can be verified readily that

$$E_{r,s}(x,y)=M_{s-r}(m_r),$$

where $m_r(t):=m_r(x,y;t)$.

This suggests that a natural way to generalize the Stolarsky mean is to replace the role of a power mean in this relation by a quasiarithmetic mean. In this section, following [**137**], we develop such a generalization, which is seen to subsume and unify some recently proposed functional means.

In Subsection 4.2 we define a general class of weighted functional Stolarsky means and establish a basic comparison theorem. In Subsection 4.3 we generalize some Hadamard–type results from [**138**] for r–convex functions. We conclude in Subsection 4.4 by addressing multidimensional generalizations.

4.2. Functional Stolarsky Means.

Definition 2. [**137**] *Let $g(\cdot)$ be strictly monotone and continuous function on an interval I, and let f be strictly monotone and continuous on the range of g^{-1}. Suppose μ is a probability measure on $[0,1]$. Then the weighted functional Stolarsky mean of two real numbers $x,\ y \in I$ is given by*

$$\phi_{f,g}(x,y;\mu) = f^{-1}\left\{\int_0^1 f\left[g^{-1}(ug(y)+(1-u)g(x))\right]d\mu(u)\right\}.$$

We have trivially that $\phi_{f,g}(x,x;\mu) = x$. Our definition subsumes a number of means extant in the literature. Thus for $g(x) := x,\ f : (0,\infty) \to \mathbb{R}$, we have a functional mean considered in [**17**]. If $\mu(u) := u$, we suppress μ from the notation for ϕ and write $\phi_{f,g}(x,y)$. For $f(x) = x^{s-r}$ and $g(x) = x^r$, $\phi_{f,g}(x,y)$ reduces to the classical Stolarsky mean $E_{r,s}(x,y)$ given by (5.59).

For $x \neq y$, set $t = u[g(y)-g(x)]+g(x)$. Under this change of variable we derive

$$\phi_{f,g}(x,y) = f^{-1}\left\{\frac{1}{g(y)-g(x)}\int\limits_{g(x)}^{g(y)} f\left(g^{-1}(t)\right)dt\right\}.$$

For $f(x) := x$, this reduces to the mean considered in [**161**].

The general functional Stolarsky mean admits the following comparison theorem [**137**].

Theorem 140. *Suppose f,g satisfy the conditions of Definition 2 and similarly for F,G. If*

(i) *$F\circ f^{-1}$ is convex and F increasing, or $F\circ f^{-1}$ concave and F decreasing, and*

(ii) *either $G\circ g^{-1}$ is convex and G increasing, or $G\circ g^{-1}$ concave and G decreasing,*

then

$$\phi_{f,g}(x,y;\mu) \le \phi_{F,G}(x,y;\mu).$$

If

(iii) *either $F\circ f^{-1}$ is convex and F decreasing, or $F\circ f^{-1}$ concave and F increasing, and*

(iv) *either $G\circ g^{-1}$ is convex and G decreasing, or $G\circ g^{-1}$ concave and G increasing,*

then

$$\phi_{f,g}(x,y;\mu) \ge \phi_{F,G}(x,y;\mu).$$

Proof. Suppose $G\circ g^{-1}$ is convex. Then the discrete Jensen inequality gives for X, Y in the domain of $G\circ g^{-1}$ that

$$(G\circ g^{-1})(tX+(1-t)Y) \le t((G\circ g^{-1})(X)+(1-t)(G\circ g^{-1})(Y).$$

For $X = g(x)$ and $Y = g(y)$, this is equivalent to

$$G\{g^{-1}[tg(x)+(1-t)g(y)]\} \le tG(x)+(1-t)G(y).$$

If G is increasing, we have consequently that

$$g^{-1}[tg(x)+(1-t)g(y)] \le G^{-1}[tG(x)+(1-t)G(y)]. \tag{5.61}$$

Similarly, we can prove that (5.61) holds if $G \circ g^{-1}$ is concave and G decreasing, and that the inequality is reversed if either $G \circ g^{-1}$ is convex and G decreasing, or $G \circ g^{-1}$ concave and G increasing.

Moreover, by the integral Jensen inequality for a convex function $F \circ f^{-1}$, we have for H integrable that

$$(F \circ f^{-1}) \left[\int_0^1 H(t) d\mu(t) \right] \leq \int_0^1 (F \circ f^{-1})(H(t)) d\mu(t),$$

which for $H(t) = f(h(t))$ becomes

$$F \left\{ f^{-1} \left[\int_0^1 f(h(t)) d\mu(t) \right] \right\} \leq \int_0^1 F(h(t)) d\mu(t).$$

If F is also increasing we get

$$f^{-1} \left[\int_0^1 f(h(t)) d\mu(t) \right] \leq F^{-1} \left[\int_0^1 F(h(t)) d\mu(t) \right]. \tag{5.62}$$

Similarly, we can prove that (5.62) applies if $F \circ f^{-1}$ is concave and F decreasing, and that (5.62) is reversed if either $F \circ f^{-1}$ is convex and F decreasing, or $F \circ f^{-1}$ concave and F increasing. We have that f and f^{-1} are either both increasing or both decreasing. Therefore if $h_1(t) \leq h_2(t)$, we have

$$f^{-1} \left[\int_0^1 f(h_1(t)) d\mu(t) \right] \leq f^{-1} \left[\int_0^1 f(h_2(t)) d\mu(t) \right].$$

Now let $x \neq y$ and suppose the conditions for (5.61) and (5.62) are satisfied. Then

$$\begin{aligned} \phi_{f,g}(x, y; \mu) &= f^{-1} \left\{ \int_0^1 f\left[g^{-1}(ug(y) + (1-u)g(x)) \right] d\mu(u) \right\} \\ &\leq f^{-1} \left\{ \int_0^1 f[G^{-1}(uG(y) + (1-u)G(x))] d\mu(u) \right\} \\ &\leq F^{-1} \left\{ \int_0^1 F[G^{-1}(uG(y) + (1-u)G(x))] d\mu(u) \right\} \\ &= \phi_{F,G}(x, y; \mu). \end{aligned} \tag{5.63}$$

If the conditions apply for the inequalities in (5.61) and (5.62) to be reversed, we have the reverse inequalities in (5.63) too. ∎

In the special case $g(x) = G(x) = x$ with f and F strictly increasing on $(0, \infty)$, this reduces to [**17**, Theorem 1.3].

4.3. Inequalities of Hadamard Type for g-Convex Functions. In [**138**] the following definition was given (see also [**137**]).

Definition 3. *Let f be a real–valued function on an interval $[a, b]$ and g a strictly monotone continuous function on the range of f. We say that f is g–convex if, for all x and $y \in [a, b]$ and $\lambda \in [0, 1]$,*

$$f(\lambda x + (1-\lambda)y) \leq g^{-1}[\lambda(g \circ f)(x) + (1-\lambda)(g \circ f)(y)].$$

We say that f is g–concave if the reverse inequality holds.

THEOREM 141. *([**137**]) Suppose f is defined on $[a,b]$ and let F be a strictly monotone continuous function defined on the range of f. If f is G–convex, then*

$$F^{-1}\left[\frac{1}{b-a}\int_a^b F(f(x))dx\right] \le \phi_{F,G}(f(a),f(b)).$$

If f is G–concave then the reverse inequality applies.

PROOF. We have for a G–convex function f that

$$\begin{aligned} F^{-1}\left[\frac{1}{b-a}\int_a^b F(f(x))dx\right] &= F^{-1}\left[\int_0^1 F(f(ub+(1-u)a))du\right] \\ &\le F^{-1}\left[\int_0^1 F\circ G^{-1}[uG(f(b))+(1-u)G(f(a))]du\right] \\ &= \phi_{F,G}(f(a),f(b)). \end{aligned}$$

The second part follows similarly. ∎

For $F(x)=x^p$ and $G(x)=x^r$, this reduces to [**138**, Theorem 3.1] and for $F(x)=x$ to a result from [**161**].

THEOREM 142. *([**137**]) Suppose $f:[a,b]\to\mathbb{R}$ is continuous. Let F be a strictly monotone continuous function defined on the range of f and $w:[a,b]\to\mathbb{R}$ an integrable positive function. If either*

(i) f is g-convex, $F\circ g^{-1}$ is convex and F decreasing, or
(ii) f is g-concave, $F\circ g^{-1}$ is concave and F increasing,
then

$$F^{-1}\left\{\frac{\int_a^b w(x)F(f(x))dx}{\int_a^b w(x)dx}\right\} \le g^{-1}\{\alpha^*(g\circ f)(b)+(1-\alpha^*)(g\circ f)(a)\}, \tag{5.64}$$

where

$$\alpha(x)=\frac{x-a}{b-a},\quad \alpha^*=\frac{\int_a^b \alpha(x)w(x)dx}{\int_a^b w(x)dx}.$$

The inequality in (5.64) is reversed if either
(iii) f is g-concave, $F\circ g^{-1}$ is convex and F increasing, or
(iv) f is g-concave, $F\circ g^{-1}$ is concave and F decreasing.
Moreover, if either
(v) f is g-convex, $F\circ g^{-1}$ is convex and F increasing, or
(vi) f is g-convex, $F\circ g^{-1}$ is concave and F decreasing,
then

$$F^{-1}\left[\frac{\int_a^b w(x)F(f(x))dx}{\int_a^b w(x)dx}\right] \le F^{-1}[\alpha^*(F\circ f)(b)+(1-\alpha^*)(F\circ f)(a)]. \tag{5.65}$$

The inequality in (5.65) is reversed if either
(vii) f is g-concave, $F\circ g^{-1}$ convex and F decreasing, or
(viii) f is g-concave, $F\circ g^{-1}$ concave and F increasing.

PROOF. Let f be g-convex (respectively g-concave). We have

$$(5.66)\quad F^{-1}\left[\frac{\int_a^b w(x)F(f(x))dx}{\int_a^b w(x)dx}\right]$$
$$= F^{-1}\left\{\frac{\int_a^b w(x)F[f(\alpha(x)b+(1-\alpha(x))a)]dx}{\int_a^b w(x)dx}\right\}$$
$$\underset{(\le)}{\ge} F^{-1}\left\{\frac{\int_a^b w(x)F[g^{-1}\{\alpha(x)(g\circ f)(b)+(1-\alpha(x))(g\circ f)(a)\}]dx}{\int_a^b w(x)dx}\right\}.$$

On the other hand, by Jensen's integral inequality we have that if $F\circ g^{-1}$ is convex (concave) then

$$(5.67)\quad \frac{\int_a^b w(x)F\{g^{-1}[\alpha(x)(g\circ f)(b)+(1-\alpha(x))(g\circ f)(a)]\}dx}{\int_a^b w(x)dx}$$
$$\underset{(\le)}{\ge} F\left\{g^{-1}\left[\frac{\int_a^b w(x)[\alpha(x)(g\circ f)(b)+(1-\alpha(x))(g\circ f)(a)]dx}{\int_a^b w(x)dx}\right]\right\}$$
$$= F\left\{g^{-1}[\alpha^*(g\circ f)(b)+(1-\alpha^*)(g\circ f)(a)]\right\}.$$

From (5.66) and (5.67) we get (5.64) (the reverse inequality).

Moreover, by Jensen's discrete inequality, if $F\circ f^{-1}$ is convex (concave), we have that

$$(5.68)\quad \frac{\int_a^b w(x)F\{g^{-1}[\alpha(x)(g\circ f)(b)+(1-\alpha(x))(g\circ f)(a)]\}dx}{\int_a^b w(x)dx}$$
$$\underset{(\le)}{\ge} \frac{\int_a^b w(x)\{\alpha(x)(F\circ f)(b)+(1-\alpha(x))(F\circ f)(a)]\}dx}{\int_a^b w(x)dx}$$
$$= \alpha^*(F\circ f)(b)+(1-\alpha^*)(F\circ f)(a).$$

From (5.66) and (5.68) we get (5.65) (its reverse inequality). ∎

Let $F(x)=x$ and suppose w is symmetric on $[a,b]$, that is,

$$w(a+t)=w(b-t),\quad 0\le t\le \frac{1}{2}(b-a).$$

Then $\alpha^*=1/2$ and we have a result obtained in [**143**].

DEFINITION 4. *([**137**]) A positive function f is said to be r–convex on an interval $[a,b]$ if, for all $x,y\in[a,b]$ and $\lambda\in[0,1]$,*

$$f(\lambda x+(1-\lambda)y)\le m_r(f(x),f(y);\lambda),$$

where m_r is defined by (5.60).

COROLLARY 52. *([**137**]) Let $f:[a,b]\to\mathbb{R}$ be a positive continuous function and w an integrable positive function.*

(a) *If f is r-convex and $\ell=\max\{r,p\}$, then*

$$\left[\frac{\int_a^b w(x)(f(x))^p dx}{\int_a^b w(x)}\right]^{1/p}\le m_\ell(f(b),f(a);\lambda).$$

(b) *If f is r-concave and $\ell = \min\{r, p\}$, then the inequality is reversed.*

For $p = 1$ and $r = 0$, that is, when f is a log-convex function, we have

$$[f(b)]^{\lambda}[f(a)]^{1-\lambda} \leq \frac{\int_a^b w(x)f(x)dx}{\int_a^b w(x)dx} \leq \lambda f(b) + (1-\lambda)f(a).$$

(see Fink [**75**] and Pečarić and Čuljak [**143**]). For some related results see also [**77, 80, 115, 134**].

4.4. Multidimensional Functional Stolarsky–Tobey Means.

Definition 5. *([**137**]) Let $E_{n-1} \subset \mathbb{R}^{n-1}$ represent the simplex*

$$E_{n-1} = \left\{ (u_1, \ldots, u_{n-1}) : u_i \geq 0 \ (1 \leq i \leq n-1), \ \sum_{j=1}^{n-1} u_j \leq 1 \right\}$$

and set $u_n = 1 - \sum\limits_{j=1}^{n-1} u_j$. With $\mathbf{u} = (u_1, \ldots, u_n)$, let $\mu(\mathbf{u})$ be a probability measure on E_{n-1}.

For $\mathbf{u} \in E_{n-1}$, $r \in \mathbb{R}$ and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$, the power mean of order r of $x_1, \ldots, x_n$ is defined by

$$m_r(\mathbf{x}; \mathbf{u}) := \begin{cases} \left(\sum\limits_{i=1}^{n} u_i x_i^r \right)^{1/r}, & \text{if } r \neq 0, \\ \prod\limits_{i=1}^{n} x_i^{u_i} & \text{if } r = 0. \end{cases}$$

The integral power mean $\overline{M}_t$ of order $t \in \mathbb{R}$ of a positive function f on E_{n-1} with probability measure μ is defined by

$$\overline{M}_t(f; \mu) := \begin{cases} \left[\int_{E_{n-1}} \{f(\mathbf{u})\}^t \, d\mu(\mathbf{u}) \right]^{1/t}, & \text{if } t \neq 0, \\ \exp\left[\int_{E_{n-1}} \ln(f(\mathbf{u})) d\mu(\mathbf{u}) \right], & \text{if } t = 0, \end{cases}$$

assuming that the expressions involved are well–defined (see [**84**, Chapter 3]).

Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ and $r, t \in \mathbb{R}$. Tobey [**179**] has studied the two–dimensional homogeneous mean

$$L_{r,t}(\mathbf{x}; \mu) := \overline{M}_t(m_r(\mathbf{x}; \cdot); \mu)$$

of $x_1, \ldots, x_n$.

Now let I be a real interval and $x_i \in I$ $(1 \leq i \leq n)$ and suppose f, g are two strictly monotone continuous functions on I. We say that $\phi_{f,g}(\mathbf{x}; \mu)$ is a *functional Stolarsky–Tobey mean* if

$$\phi_{f,g}(\mathbf{x}; \mu) = f^{-1} \left\{ \int_{E_{n-1}} f \left[g^{-1} \left(\sum_{i=1}^{n} u_i g(x_i) \right) \right] d\mu(\mathbf{u}) \right\},$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

Special cases of the above means are given in [**17, 120, 145, 161**]. For example, for $g(x) := x$ we have a functional mean considered in [**17**]. Tobey's homogeneous mean is subsumed under $f(y) = y^t$, $g(y) = y^r$ and $I = \mathbb{R}$.

The proof of the following theorem follows closely that of Theorem 140.

THEOREM 143. *([**137**]) If*

(i) either $F \circ f^{-1}$ is convex and F increasing, or $F \circ f^{-1}$ concave and F decreasing, and

(ii) either $G \circ g^{-1}$ is convex and G increasing, or $G \circ g^{-1}$ concave and G decreasing,

then

$$\phi_{f,g}(\mathbf{x};\mu) \leq \phi_{F,G}(\mathbf{x};\mu).$$

If

(iii) either $F \circ f^{-1}$ is convex and F decreasing, or $F \circ f^{-1}$ concave and F increasing, and

(iv) either $G \circ g^{-1}$ is convex and G is decreasing, or $G \circ g^{-1}$ is concave and G is increasing,

then

$$\phi_{f,g}(\mathbf{x};\mu) \geq \phi_{F,G}(\mathbf{x};\mu).$$

Denote by

$$w_i = \int_{E_{n-1}} u_i d\mu(\mathbf{u})$$

the i-th weight associated with the probability measure μ on E_{n-1}. Then $w_i > 0$ $(1 \leq i \leq n)$ and $w_1 + \cdots + w_n = 1$.

We have

$$\phi_{f,f}(\mathbf{x};\mu) = f^{-1}\left\{\sum_{i=1}^{n} w_i f(x_i)\right\},$$

which is just the quasiarithmetic mean of the numbers $\{x_i\}$ with weights $\{w_i\}$ for the function f.

THEOREM 144. *([**137**]) If either*

(i) $f \circ g^{-1}$ is convex and f increasing, or

(ii) $f \circ g^{-1}$ is concave and f decreasing,

then

$$\phi_{g,g}(\mathbf{x};\mu) \leq \phi_{f,g}(\mathbf{x};\mu) \leq \phi_{f,f}(\mathbf{x};\mu). \tag{5.69}$$

If either

(iii) $f \circ g^{-1}$ is convex and f decreasing, or

(iv) $f \circ g^{-1}$ is concave and f increasing,

then the inequality is reversed.

PROOF. By Jensen's integral inequality we have that if $f \circ g^{-1}$ is convex,

$$\begin{aligned}
&\int_{E_{n-1}} f\left[g^{-1}\left(\sum_{i=1}^{n} u_i g(x_i)\right)\right] d\mu(\mathbf{u}) \\
\geq\ & f\left\{g^{-1}\left[\int_{E_{n-1}}\left(\sum_{i=1}^{n} u_i g(x_i)\right) d\mu(\mathbf{u})\right]\right\} = f\left\{g^{-1}\left[\sum_{i=1}^{n} w_i g(x_i)\right]\right\} \\
=\ & f(\phi_{g,g}(\mathbf{x};\mu)).
\end{aligned}$$

By Jensen's discrete inequalitiy we have that if $f \circ g^{-1}$ is convex, then

$$\begin{aligned}(5.70)\qquad & \int_{E_{n-1}} f\left[g^{-1}\left(\sum_{i=1}^n u_i g(x_i)\right)\right] d\mu(\mathbf{u}) \\ \leq \; & \int_{E_{n-1}} \sum_{i=1}^n u_i f(x_i) d\mu(\mathbf{u}) = \sum_{i=1}^n w_i f(x_i) = f(\phi_{f,f}(\mathbf{x};\mu)).\end{aligned}$$

If f is increasing, (5.69) now follows from (5.70) and (5.70). The other cases are derived similarly. ∎

Our next theorem considers unweighted functional Stolarsky–Tobey means, when μ reduces to Lebesgue measure

$$d\mu(\mathbf{u}) = (n-1)! du_1 \dots du_{n-1} = (n-1)! d\mathbf{u}.$$

An easy calculation gives

$$\int_{E_{n-1}} du_1 \dots du_{n-1} = \frac{1}{(n-1)!}$$

and

$$w_i = \int_{E_{n-1}} u_i d\mu(\mathbf{u}) = \frac{1}{n}.$$

We write $\phi_{f,g}(\mathbf{x})$ for $\phi_{f,g}(\mathbf{x},\mu)$ in this case.

THEOREM 145. *([**137**]) Suppose $x_i \neq x_j$ for $i \neq j$ and let $H(t)$ be such that $H^{(n-1)} = f \circ g^{-1}$. Then*

$$\phi_{f,g}(\mathbf{x}) = f^{-1}\left[(n-1)! \sum_{i=1}^n \frac{(H \circ g)(x_i)}{\prod_{j \in A(i)} (g(x_i) - g(x_j))}\right],$$

where $A(i) := \{1, 2, \dots, n\} \setminus \{i\}$.

PROOF. We use the well–known relation

$$[t_1, \dots, t_n] f = \sum_{i=1}^n \frac{f(t_i)}{\prod_{j \in A(i)} (t_i - t_j)} = \int_{E_{n-1}} f^{(n-1)}\left(\sum_{i=1}^n u_i t_i\right) d\mathbf{u},$$

where $[t_1, \dots, t_n] f$ stands for the divided differences of order $n-1$ of t with knots at $t_1, \dots, t_n$ and $t \in C^{n-1}(a,b)$, $a = \min(t_i)$, $b = \max(t_i)$, $1 \leq i \leq n$. So we have

$$\begin{aligned}\phi_{f,g}(\mathbf{x}) &= f^{-1}\left\{(n-1)! \int_{E_{n-1}} (f \circ g^{-1})\left(\sum_{i=1}^n u_i g(x_i)\right) d\mathbf{u}\right\} \\ &= f^{-1}\left\{(n-1)! \int_{E_{n-1}} H^{(n-1)}\left(\sum_{i=1}^n u_i g(x_i)\right) d\mathbf{u}\right\},\end{aligned}$$

whence the desired result. ∎

The above gives as special cases results obtained in [**120, 145, 161**].

Theorems 144 and 145 give the following.

COROLLARY 53. *([**137**]) If either (i) or (ii) of Theorem 144 holds and H is as in Theorem 145, then*

$$\begin{aligned} g\left(\frac{1}{n}\sum_{i=1}^{n} g(x_i)\right) &\leq f^{-1}\left[(n-1)!\sum_{i=1}^{n}\frac{(H\circ g)(x_i)}{\prod_{j\in A(i)}(g(x_i)-g(x_j))}\right] \\ &\leq f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right). \end{aligned}$$

If either (iii) or (iv) from Theorem 144 applies, then the inequalities are reversed.

5. Generalization of $H. - H.$ Inequality for G-Convex Functions

5.1. Introduction. Recall that a positive function f is said to be r-convex on an interval $[a,b]$ if, for all $x,y\in[a,b]$ and $\lambda\in[0,1]$,

$$f\left(\lambda x+(1-\lambda y)\right)\leq\begin{cases}\left[\lambda f^{r}\left(x\right)+\left(1-\lambda y\right)f^{r}\left(y\right)\right]^{\frac{1}{r}}, & \text{if}\quad r\neq 0\\ f^{\lambda}\left(x\right)f^{1-\lambda}\left(y\right), & \text{if}\quad r=0.\end{cases}\tag{5.71}$$

If the inequality (5.71) is reversed, then is said to be r-concave ([**80**]). This concept plays an important role in statistics. By the concept of r-convexity, the authors in [**80**] proved that:

THEOREM 146. *([**80**]). Suppose that f is a positive r-convex function on $[a,b]$. Then*

$$\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\leq L_{r}\left(f\left(a\right),f\left(b\right)\right).\tag{5.72}$$

If f is a positive r-concave function, then the inequality (5.72) is reversed, where

$$L_{r}\left(f\left(a\right),f\left(b\right)\right)=\begin{cases}\frac{r}{r+1}\frac{f^{r+1}(a)-f^{r+1}(b)}{f^{r}(a)-f^{r}(b)}, & r\neq 0,-1, \quad f\left(a\right)\neq f\left(b\right)\\ \frac{f(a)-f(b)}{\ln f(a)-\ln f(b)}, & r=0, \quad f\left(a\right)\neq f\left(b\right)\\ f\left(a\right)f\left(b\right)\frac{\ln f(a)-\ln f(b)}{f(a)-f(b)}, & r=-1, \quad f\left(a\right)\neq f\left(b\right)\\ f\left(a\right), & \quad f\left(a\right)=f\left(b\right).\end{cases}$$

The authors in [**138**] established a relationship between power mean of f and Stolarsky mean of $f(a)$ and $f(b)$, which generalize (5.72) and is stated as the following :

THEOREM 147. *([**138**]). Suppose that f is a positive r-convex function on $[a,b]$, then*

$$M_{p}\left(f\right)\leq E\left(f\left(a\right),\,f\left(b\right);r,p+r\right).\tag{5.73}$$

If f is r-concave, the inequality (5.73) is reversed, where

$$M_{p}\left(f\right)=\begin{cases}\left[\frac{1}{b-a}\int_{a}^{b}f^{p}\left(t\right)dt\right]^{\frac{1}{p}}, & p\neq 0,\\ \exp\left[\frac{1}{b-a}\int_{a}^{b}\ln f\left(t\right)dt\right], & p=0,\end{cases}$$

$$E\left(f\left(a\right),\, f\left(b\right);r,p+r\right) = \left[\frac{r}{p+r}\cdot\frac{f^{r+1}\left(a\right)-f^{r+1}\left(b\right)}{f^{r}\left(a\right)-f^{r}\left(b\right)}\right]^{\frac{1}{p}},$$
$$\left(r,p+r\right)\neq 0,\ p\neq 0,\ f\left(a\right)\neq f\left(b\right)$$

$$\begin{aligned}
E\left(f\left(a\right),\, f\left(b\right);0,r\right) &= E\left(f\left(a\right),\, f\left(b\right);r,0\right)\\
&= \left[\frac{1}{r}\cdot\frac{f^{r}\left(a\right)-f^{r}\left(b\right)}{\ln f\left(a\right)-\ln f\left(b\right)}\right]^{\frac{1}{r}},\ r\neq 0,\ f\left(a\right)\neq f\left(b\right)\\
E\left(f\left(a\right),\, f\left(b\right);r,r\right) &= e^{\frac{-1}{r}}\left(\frac{f\left(a\right)^{f^{r}\left(a\right)}}{f\left(b\right)^{f^{r}\left(b\right)}}\right)^{\frac{1}{f^{r}\left(a\right)-f^{r}\left(b\right)}},\ r\neq 0,\ f\left(a\right)\neq f\left(b\right)\\
E\left(f\left(a\right),\, f\left(b\right);0,0\right) &= \sqrt{f\left(a\right)\, f\left(b\right)},\ \ f\left(a\right)\neq f\left(b\right)\\
E\left(f\left(a\right),\, f\left(b\right);r,p+r\right) &= f\left(a\right),
\end{aligned}$$

Fejèr in [**72**] had a weighted generalization of Hadamard's inequality :

THEOREM 148. *([**72**]). If $f:[a,b]\to\mathbb{R}$ is convex and $w:[a,b]\to\mathbb{R}$ is a nonnegative and integrable function such that w is symmetric to $x=\frac{a+b}{2}$, then*

$$f\left(\frac{a+b}{2}\right)\int_{a}^{b}w\left(t\right)dt\leq\int_{a}^{b}f\left(t\right)w\left(t\right)dt\leq\frac{f\left(a\right)+f\left(b\right)}{2}\int_{a}^{b}w\left(t\right)dt. \tag{5.74}$$

If f is concave, then the inequality (5.74) is reversed.

The Fejèr type inequality is much less represented in the literature. In [**138**], authors proved the following :

THEOREM 149. *([**138**]). Let w be a nonnegative, integrable, even function with positive integral over $[1,-1]$, and let f be a positive function. Put $\tilde{f}\left(t\right)=f\left(x+vt\right)$ for $t\in[-1,1]$.*

(a) *If f is r-convex and $m=\max\{r,p\}$, then*

$$\tilde{M}_{p}\left(\tilde{f},w\right)\leq M_{m}\left(f\left(x+v\right),f\left(x-v\right)\right) \tag{5.75}$$

(b) *If f is r-concave and $m=min\{r,p\}$, then the inequality (5.75) is reversed, where*

$$\tilde{M}_{p}\left(\tilde{f},w\right)=\begin{cases}\left[\dfrac{\int_{-1}^{1}f^{p}\left(x+vt\right)w\left(t\right)dt}{\int_{-1}^{1}w\left(t\right)dt}\right]^{\frac{1}{p}}, & p\neq 0\\[2ex] \exp\left[\dfrac{\int_{-1}^{1}w\left(t\right)\ln f\left(x+vt\right)dt}{\int_{-1}^{1}w\left(t\right)dt}\right], & p=0,\end{cases}$$

and

$$M_{m}\left(f\left(x+v\right),f\left(x-v\right)\right)=\begin{cases}\left[\dfrac{f^{m}\left(x+v\right)+f^{m}\left(x-v\right)}{2}\right]^{\frac{1}{m}}, & m\neq 0\\[2ex] \sqrt{f\left(x+v\right)f\left(x-v\right)}, & m=0.\end{cases}$$

For other Fejèr type inequalities see [**77**], [**115**], and [**138**]. In this section we shall establish some general Fejèr type inequalities and we shall show that the above theorems follow from these inequalities. For this purpose we introduce a generalization of the concept of r-convexity (r-concavity) (cf. [**97**]):

Let $f:[a,b]\to\mathbb{R}$ be a real-valued function and $g:J\to\mathbb{R}$ be a strictly monotonically continuous function on interval J such that $f([a,b])\subset J$. We say that f is g-convex if for all x and y in $[a,b]$ and $\lambda\in[0,1]$ (see [**97**])

$$f(\lambda x+(1-\lambda)y)\leq g^{-1}\left[\lambda(g\circ f)(x)+(1-\lambda)(g\circ f)(y)\right]. \tag{5.76}$$

We say that f is g-concave if (5.76) is reversed . This definition is quite different from the standard definition of *g-convex* (see Definition 1.21, [**147**]).

REMARK 67. (a) *([**97**]) If $f:[a,b]\to\mathbb{R}$ is convex, then f is g-convex with $g(x)=x$.*

(b) *If $f:[a,b]\to\mathbb{R}^+$ is log-convex, then f is g-convex with $g(x)=\ln x$.*

(c) *If $f:[a,b]\to\mathbb{R}^+$ is r-convex, $r\neq 0$, then f is g-convex with $g(x)=x^r$, $r\neq 0$.*

5.2. Weighted Generalization of Hadamard's Inequality for g-Convex Functions. Let $f:[a,b]\to\mathbb{R}$ be an integrable function and $g:J\to\mathbb{R}$ be a strictly monotonically continuous function with $f([a,b])\subset J$. Define G, H by (see [**97**])

$$\begin{aligned} G(x) &= g^{-1}\left[\frac{a+b-2x}{b-a}(g\circ f)(a)+\frac{2x-2a}{b-a}(g\circ f)\left(\frac{a+b}{2}\right)\right] \\ &\quad +g^{-1}\left[\frac{2x-2a}{b-a}(g\circ f)\left(\frac{a+b}{2}\right)+\frac{a+b-2x}{b-a}(g\circ f)(b)\right], \\ H(x) &= g^{-1}\left[\frac{b-x}{b-a}(g\circ f)(a)+\frac{x-a}{b-a}(g\circ f)(b)\right] \\ &\quad +g^{-1}\left[\frac{x-a}{b-a}(g\circ f)(a)+\frac{b-x}{b-a}(g\circ f)(b)\right], \end{aligned}$$

for all $x\in[a,b]$, then we have the following theorem (cf. [**97**]).

THEOREM 150. *Suppose that f,g,G,H are defined as above, let f be a g-convex function and let $w:[a,b]\to\mathbb{R}$ be a nonnegative integrable function such that w is symmetric to $t=\frac{a+b}{2}$.*

(a) *If g is strictly increasing concave or strictly decreasing convex then*

$$\begin{aligned} \int_a^b f(t)\,w(t)\,dt &\leq \int_a^{\frac{a+b}{2}} G(t)\,w(t)\,dt\leq\int_a^{\frac{a+b}{2}} H(t)\,w(t)\,dt \\ &\leq \frac{f(a)+f(b)}{2}\int_a^b w(t)\,dt. \end{aligned} \tag{5.77}$$

(b) *If g is strictly increasing convex or strictly decreasing concave then*

$$\begin{aligned} \int_a^b f(t)\,w(t)\,dt &\leq \int_a^{\frac{a+b}{2}} G(t)\,w(t)\,dt\leq\int_a^{\frac{a+b}{2}} H(t)\,w(t)\,dt \\ &\leq g^{-1}\left[\frac{(g\circ f)(a)+(g\circ f)(b)}{2}\right]\int_a^b w(t)\,dt. \end{aligned} \tag{5.78}$$

Further, in case f is g-concave. If g is strictly increasing convex or strictly decreasing concave on J, then the inequality (5.77) is reversed; if g is strictly increasing concave or strictly decreasing convex on J then the inequality (5.78) is reversed.

PROOF. The proof is as follows.

(a) Suppose that g is strictly increasing concave, since f is g-convex, then $g\circ f$ is convex and g^{-1} is strictly increasing convex . By using the symmetrical properties of w and changing variables,

$$\begin{aligned}
&\int_a^b f(t)\,w(t)\,dt \\
=\;& \int_a^{\frac{a+b}{2}} f(t)\,w(t)\,dt + \int_{\frac{a+b}{2}}^b f(t)\,w(t)\,dt \\
=\;& \int_a^{\frac{a+b}{2}} [f(t) + f(a+b-t)]\,w(t)\,dt \\
=\;& \int_a^{\frac{a+b}{2}} \left[g^{-1}((g\circ f)(t)) + g^{-1}((g\circ f)(a+b-t))\right] w(t)\,dt \\
\le\;& \int_a^{\frac{a+b}{2}} \left(g^{-1}\left[\frac{a+b-2t}{b-a}(g\circ f)(a) + \frac{2t-2a}{b-a}(g\circ f)\left(\frac{a+b}{2}\right)\right]\right. \\
&\left. + g^{-1}\left[\frac{2t-2a}{b-a}(g\circ f)\left(\frac{a+b}{2}\right) + \frac{a+b-2t}{b-a}(g\circ f)(b)\right]\right) w(t)\,dt \\
=\;& \int_a^{\frac{a+b}{2}} G(t)\,w(t)\,dt \\
\le\;& \int_a^{\frac{a+b}{2}} \left(g^{-1}\left[\frac{b-t}{b-a}(g\circ f)(a) + \frac{t-a}{b-a}(g\circ f)(b)\right]\right. \\
&\left. + g^{-1}\left[\frac{t-a}{b-a}(g\circ f)(a) + \frac{b-t}{b-a}(g\circ f)(b)\right]\right) w(t)\,dt \\
=\;& \int_a^{\frac{a+b}{2}} H(t)\,w(t)\,dt \le \frac{f(a)+f(b)}{2}\int_a^b w(t)\,dt.
\end{aligned}$$

Next, suppose that g is strictly decreasing convex, then $g\circ f$ is concave and g^{-1} is strictly decreasing convex. By using the same method, we see that (5.77) also holds.

(b) Suppose that g is strictly increasing convex, then $g\circ f$ is convex and g^{-1} is strictly increasing concave and

$$\begin{aligned}
\int_a^b f(t)\,w(t)\,dt \;\le\;& \int_a^{\frac{a+b}{2}} G(t)\,w(t)\,dt \le \int_a^{\frac{a+b}{2}} H(t)\,w(t)\,dt \\
=\;& \int_a^{\frac{a+b}{2}} \left(g^{-1}\left[\frac{b-t}{b-a}(g\circ f)(a) + \frac{t-a}{b-a}(g\circ f)(b)\right]\right. \\
&\left. + g^{-1}\left[\frac{t-a}{b-a}(g\circ f)(a) + \frac{b-t}{b-a}(g\circ f)(b)\right]\right) w(t)\,dt \\
\le\;& \int_a^{\frac{a+b}{2}} 2g^{-1}\left[\frac{(g\circ f)(a) + (g\circ f)(b)}{2}\right] w(t)\,dt \\
=\;& g^{-1}\left[\frac{(g\circ f)(a) + (g\circ f)(b)}{2}\right]\int_a^b w(t)\,dt.
\end{aligned}$$

Next, suppose that g is strictly decreasing concave, then $g\circ f$ is concave and g^{-1} is strictly decreasing concave and (5.78) still holds.

Further, in case f is g-concave, we can use the same method to prove that the statement is true and the proof is completed. ∎

REMARK 68. *([**97**]) If f is convex in $[a,b]$, then by Remark 67, f is g-convex with $g(x)=x$ and by Theorem 150 we have*

$$\begin{aligned} &\int_a^b f(t)\,w(t)\,dt \le \int_a^{\frac{a+b}{2}} G(t)\,w(t)\,dt \\ = &\int_a^{\frac{a+b}{2}} 2\left[\frac{2t-2a}{b-a} f\left(\frac{a+b}{2}\right)+\frac{a+b-2t}{b-a}\cdot\frac{f(a)+f(b)}{2}\right] w(t)\,dt \\ \le &\frac{f(a)+f(b)}{2}\int_a^b w(t)\,dt. \end{aligned}$$

Hence Theorem 150 refines and generalizes the second Fejèr inequality .

REMARK 69. *([**97**]) If f is convex in $[a,b]$, then by Remark 67, f is g-convex with $g(x)=x$. Let $w(x)=1$, $x\in[a,b]$. Then by Theorem 150 we have*

$$\begin{aligned} \int_a^b f(t)\,dt &\le \int_a^{\frac{a+b}{2}} G(t)\,dt = \frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \\ &\le \frac{f(a)+f(b)}{2}(b-a) \end{aligned}$$

which implies that

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{b-a}\int_a^{\frac{a+b}{2}} G(t)\,dt \\ = &\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \le \frac{f(a)+f(b)}{2}, \end{aligned}$$

*this is the Bullen's inequality [**77**].*

REMARK 70. *Let f be a r-convex on $[a,b]$, then by Remark 67 f is g-convex with*

$$g(x)=\begin{cases} x^r, & r\neq 0 \\ \ln x, & r=0 \end{cases}, \quad x\in(0,\infty).$$

Let $w(x)=1$, $x\in[a,b]$. Then by Theorem 150,

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(t)\,dt &\le \frac{1}{b-a}\int_a^{\frac{a+b}{2}} H(t)\,dt = L_r(f(a),f(b)) \\ &\le \begin{cases} \dfrac{f(a)+f(b)}{2}, & r\le 1 \\ \left[\dfrac{f^r(a)+f^r(b)}{2}\right]^{\frac{1}{r}}, & r>1. \end{cases} \end{aligned}$$

Further, if f is positive r-concave, then the above inequality is reversed. Hence Theorem 150 generalizes Theorem 146.

5.3. Further Generalisations. Let $f:[a,b]\to\mathbb{R}$ be an integrable g-convex (g-concave) function and let $g:J\to\mathbb{R}$, $h:I\to\mathbb{R}$ be strictly increasing and continuous functions with $f([a,b])\subset J$. Define M, N by (see [**97**])

$$\begin{aligned}
M(x) &= \left(h\circ g^{-1}\right)\left[\frac{a+b-2x}{b-a}(g\circ f)(a)+\frac{2x-2a}{b-a}(g\circ f)\left(\frac{a+b}{2}\right)\right] \\
&\quad + \left(h\circ g^{-1}\right)\left[\frac{x-a}{b-a}(g\circ f)(a)+\frac{x-a}{b-a}(g\circ f)(b)\right], \\
N(x) &= \left(h\circ g^{-1}\right)\left[\frac{b-x}{b-a}(g\circ f)(a)+\frac{x-a}{b-a}(g\circ f)(b)\right] \\
&\quad + \left(h\circ g^{-1}\right)\left[\frac{x-a}{b-a}(g\circ f)(a)+\frac{b-x}{b-a}(g\circ f)(b)\right],
\end{aligned}$$

for all $x\in[a,b]$, then we have the following results.

THEOREM 151. *([**97**]) Let f,g,h,M,N be functions defined as above and let $w:[a,b]\to\mathbb{R}$ be integrable on $[a,b]$ with positive integral and symmetric to $x=\frac{a+b}{2}$. Suppose that h is strictly increasing.*

(a) *If f is g-convex and $h\circ g^{-1}$ is convex then*

$$\begin{aligned}
\int_a^b (h\circ f)(t)\,w(t)\,dt &\le \int_a^{\frac{a+b}{2}} M(t)\,w(t)\,dt \le \int_a^{\frac{a+b}{2}} N(t)\,w(t)\,dt \\
&\le \frac{(h\circ f)(a)+(h\circ f)(b)}{2}\int_a^b w(t)\,dt.
\end{aligned} \tag{5.79}$$

(b) *If f is g-concave and $h\circ g^{-1}$ is concave then the inequality (5.79) is reversed.*

(c) *If f is g-convex and $h\circ g^{-1}$ is concave, then*

$$\begin{aligned}
&\int_a^b (h\circ f)(t)\,w(t)\,dt \\
&\le \int_a^{\frac{a+b}{2}} M(t)\,w(t)\,dt \le \int_a^{\frac{a+b}{2}} N(t)\,w(t)\,dt \\
&\le \left(h\circ g^{-1}\right)\left[\frac{(g\circ f)(a)+(g\circ f)(b)}{2}\right]\int_a^b w(t)\,dt.
\end{aligned} \tag{5.80}$$

(d) *If f is g-concave and $h\circ g^{-1}$ is convex, then the inequality (5.80) is reversed.*

PROOF. The proof is as follows.

(a) Suppose that g is strictly increasing, since f is g-convex then $g\circ f$ is convex and $h\circ g^{-1}$ is strictly increasing convex. By using the symmetrical properties of w and changing variables we obtain

$$\begin{aligned}
&\int_a^b (h\circ f)(t)\,w(t)\,dt = \int_a^{\frac{a+b}{2}} (h\circ f)(t)\,w(t)\,dt + \int_{\frac{a+b}{2}}^b (h\circ f)(t)\,w(t)\,dt \\
= &\int_a^{\frac{a+b}{2}} \left[(h\circ f)(t) + (h\circ f)(a+b-t)\right] w(t)\,dt \\
= &\int_a^{\frac{a+b}{2}} \left[\left(h\circ g^{-1}\right)(g\circ f)(t) + \left(h\circ g^{-1}\right)(g\circ f)(a+b-t)\right] w(t)\,dt \\
\leq &\int_a^{\frac{a+b}{2}} \Bigg(\left(h\circ g^{-1}\right)\left[\frac{a+b-2t}{b-a}(g\circ f)(a) + \frac{2t-2a}{b-a}(g\circ f)\left(\frac{a+b}{2}\right)\right] \\
&+ \left(h\circ g^{-1}\right)\left[\frac{2t-2a}{b-a}(g\circ f)\left(\frac{a+b}{2}\right) + \frac{a+b-2t}{b-a}(g\circ f)(b)\right]\Bigg) w(t)\,dt \\
= &\int_a^{\frac{a+b}{2}} M(t)\,w(t)\,dt \\
\leq &\int_a^{\frac{a+b}{2}} \Bigg(\left(h\circ g^{-1}\right)\left[\frac{b-t}{b-a}(g\circ f)(a) + \frac{t-a}{b-a}(g\circ f)(b)\right] \\
&+ \left(h\circ g^{-1}\right)\left[\frac{t-a}{b-a}(g\circ f)(a) + \frac{b-t}{b-a}(g\circ f)(b)\right]\Bigg) w(t)\,dt \\
= &\int_a^{\frac{a+b}{2}} N(t)\,w(t)\,dt \leq \int_a^{\frac{a+b}{2}} \left[(h\circ f)(a) + (h\circ f)(b)\right] w(t)\,dt \\
= &\frac{(h\circ f)(a) + (h\circ f)(b)}{2}\int_a^b w(t)\,dt.
\end{aligned}$$

Next, suppose that g is strictly decreasing, then $g\circ f$ is concave and $h\circ g^{-1}$ is strictly decreasing convex and the inequality (5.79) also holds.

(b) Suppose that g is strictly increasing, since f is g-concave and $h\circ g^{-1}$ is concave, then $g\circ f$ is concave and $h\circ g^{-1}$ is strictly increasing concave.

By using the method as in (a) we have

$$\begin{aligned}
(5.81)\qquad & \int_a^b (h\circ f)(t)\,w(t)\,dt \\
\geq\ & \int_a^{\frac{a+b}{2}} \left(\left(h\circ g^{-1}\right)\left[\frac{a+b-2t}{b-a}(g\circ f)(a)+\frac{2t-2a}{b-a}(g\circ f)\left(\frac{a+b}{2}\right)\right]\right. \\
& \left. + \left(h\circ g^{-1}\right)\left[\frac{2t-2a}{b-a}(g\circ f)\left(\frac{a+b}{2}\right)+\frac{a+b-2t}{b-a}(g\circ f)(b)\right]\right) w(t)\,dt \\
=\ & \int_a^{\frac{a+b}{2}} M(t)\,w(t)\,dt \\
\geq\ & \int_a^{\frac{a+b}{2}} \left(\left(h\circ g^{-1}\right)\left[\frac{b-t}{b-a}(g\circ f)(a)+\frac{t-a}{b-a}(g\circ f)(b)\right]\right. \\
& \left. + \left(h\circ g^{-1}\right)\left[\frac{t-a}{b-a}(g\circ f)(a)+\frac{b-t}{b-a}(g\circ f)(b)\right]\right) w(t)\,dt \\
=\ & \int_a^{\frac{a+b}{2}} N(t)\,w(t)\,dt \geq \int_a^{\frac{a+b}{2}} \left[(h\circ f)(a)+(h\circ f)(b)\right] w(t)\,dt \\
=\ & \frac{(h\circ f)(a)+(h\circ f)(b)}{2}\int_a^b w(t)\,dt.
\end{aligned}$$

Next, suppose that g is strictly decreasing, since f is g-concave and $h\circ g^{-1}$ is concave, then $g\circ f$ is convex and $h\circ g^{-1}$ is strictly decreasing concave and (5.81) also holds.

(c) Suppose that g is strictly increasing, since f is g-convex and $h\circ g^{-1}$ is concave, then $g\circ f$ is convex and $h\circ g^{-1}$ is strictly increasing concave. By using the same method as in (a) we have

$$\begin{aligned}
& \int_a^b (h\circ f)(t)\,w(t)\,dt \\
\leq\ & \int_a^{\frac{a+b}{2}} \left(\left(h\circ g^{-1}\right)\left[\frac{a+b-2t}{b-a}(g\circ f)(a)+\frac{2t-2a}{b-a}(g\circ f)\left(\frac{a+b}{2}\right)\right]\right. \\
& \left. + \left(h\circ g^{-1}\right)\left[\frac{2t-2a}{b-a}(g\circ f)\left(\frac{a+b}{2}\right)+\frac{a+b-2t}{b-a}(g\circ f)(b)\right]\right) w(t)\,dt \\
=\ & \int_a^{\frac{a+b}{2}} M(t)\,w(t)\,dt \\
\leq\ & \int_a^{\frac{a+b}{2}} \left(\left(h\circ g^{-1}\right)\left[\frac{b-t}{b-a}(g\circ f)(a)+\frac{t-a}{b-a}(g\circ f)(b)\right]\right. \\
& \left. + \left(h\circ g^{-1}\right)\left[\frac{t-a}{b-a}(g\circ f)(a)+\frac{b-t}{b-a}(g\circ f)(b)\right]\right) w(t)\,dt \\
=\ & \int_a^{\frac{a+b}{2}} N(t)\,w(t)\,dt \leq \int_a^{\frac{a+b}{2}} 2\left(h\circ g^{-1}\right)\left[\frac{(g\circ f)(a)+(g\circ f)(b)}{2}\right] w(t)\,dt \\
=\ & \left(h\circ g^{-1}\right)\left[\frac{(g\circ f)(a)+(g\circ f)(b)}{2}\right]\int_a^{\frac{a+b}{2}} w(t)\,dt.
\end{aligned}$$

Next, suppose that g is strictly decreasing, since f is g-convex and $h\circ g^{-1}$ is concave, then $g\circ f$ is concave and $h\circ g^{-1}$ is strictly decreasing and the inequality (5.80) also holds.

(d) Suppose that g is increasing, since f is g-concave and $h\circ g^{-1}$ is convex, then $g\circ f$ is concave and $h\circ g^{-1}$ is strictly increasing convex. By using the same method as used in (a), we have

$$
\begin{aligned}
(5.82)\qquad & \int_a^b (h\circ f)(t)\,w(t)\,dt \\
\geq & \int_a^{\frac{a+b}{2}} \Bigg(\left(h\circ g^{-1}\right)\left[\frac{a+b-2t}{b-a}(g\circ f)(a)+\frac{2t-2a}{b-a}(g\circ f)\left(\frac{a+b}{2}\right)\right] \\
& + \left(h\circ g^{-1}\right)\left[\frac{2t-2a}{b-a}(g\circ f)\left(\frac{a+b}{2}\right)+\frac{a+b-2t}{b-a}(g\circ f)(b)\right]\Bigg) w(t)\,dt \\
= & \int_a^{\frac{a+b}{2}} M(t)\,w(t)\,dt \\
\geq & \int_a^{\frac{a+b}{2}} \Bigg(\left(h\circ g^{-1}\right)\left[\frac{b-t}{b-a}(g\circ f)(a)+\frac{t-a}{b-a}(g\circ f)(b)\right] \\
& + \left(h\circ g^{-1}\right)\left[\frac{t-a}{b-a}(g\circ f)(a)+\frac{b-t}{b-a}(g\circ f)(b)\right]\Bigg) w(t)\,dt \\
= & \int_a^{\frac{a+b}{2}} N(t)\,w(t)\,dt \geq \int_a^{\frac{a+b}{2}} 2\left(h\circ g^{-1}\right)\left[\frac{(g\circ f)(a)+(g\circ f)(b)}{2}\right] w(t)\,dt \\
= & \left(h\circ g^{-1}\right)\left[\frac{(g\circ f)(a)+(g\circ f)(b)}{2}\right]\int_a^{\frac{a+b}{2}} w(t)\,dt.
\end{aligned}
$$

Next, suppose that g is strictly decreasing, since f is g-concave and $h\circ g^{-1}$ is convex, then $g\circ f$ is convex and $h\circ g^{-1}$ is strictly decreasing convex, hence the inequality (5.82) also holds.

The proof is completed. ∎

A similar proof gives the following theorem:

THEOREM 152. *([**97**]) Let f,g,h,M,N,w be functions as in Theorem 151. Suppose that h is strictly decreasing on I.*

(a) *If f is g-concave and $h\circ g^{-1}$ is convex, then the inequality (5.79) holds.*
(b) *If f is g-convex and $h\circ g^{-1}$ is convex, then the inequality (5.79) reversed.*
(c) *If f is g-concave and $h\circ g^{-1}$ is convex, then the inequality (5.80) holds.*
(d) *If f is g-convex and $h\circ g^{-1}$ is convex, then the inequality (5.80) reversed.*

COROLLARY 54. *Let $f:[a,b]\to\mathbb{R}^+$ be a positive integrable function and let $w:[a,b]\to\mathbb{R}$ be a nonnegative integrable function with positive integral such that w is symmetric to $x=\frac{a+b}{2}$.*

(a) *If f is r-convex and $m=\max\{r,p\}$, then*

$$
(5.83)\qquad M_p(f,w)\leq M_m(f(a),f(b)).
$$

(b) *If f is r-concave and $m=\min\{r,p\}$, then the inequality (5.83) is reversed.*

PROOF. The proof is as follows.

(a) Suppose that f is r-convex, let g, h be functions defined by

$$h(x) = \begin{cases} x^p, & p \neq 0 \\ \ln x, & p = 0 \end{cases} ; \quad g(x) = \begin{cases} x^r, & r \neq 0 \\ \ln x, & r = 0 \end{cases}$$

for all $x \in (0, \infty)$. Then f is a g-convex function and

$$\left(h \circ g^{-1}\right)(x) = \begin{cases} x^{\frac{p}{r}}, & r \neq 0,\ p \neq 0; \\ e^{px} & r = 0,\ p \neq 0; \\ \frac{1}{r} \ln x & r \neq 0,\ p = 0; \\ x & r = 0,\ p = 0. \end{cases}$$

By using Theorems 151 and 152, we can state

(1) If $p > 0$, then h is strictly increasing, $h \circ g^{-1}$ is convex for $r \leq p$ and $h \circ g^{-1}$ is concave for $r > p$, hence

$$M_p(f, w) = \left[\frac{\int_a^b f^p(t)\, w(t)\, dt}{\int_a^b w(t)\, dt}\right]^{\frac{1}{p}} \leq \begin{cases} \left[\dfrac{f^p(a) + f^p(b)}{2}\right]^{\frac{1}{p}}, & r \leq p; \\ \left[\dfrac{f^r(a) + f^r(b)}{2}\right]^{\frac{1}{r}}, & r > p. \end{cases}$$

(2) If $p < 0$, then h is strictly decreasing, $h \circ g^{-1}$ is convex for $p \leq r$ and $h \circ g^{-1}$ is concave for $r < p$, hence

$$M_p(f, w) \leq \begin{cases} \left[\dfrac{f^p(a) + f^p(b)}{2}\right]^{\frac{1}{p}}, & r < p; \\ \left[\dfrac{f^r(a) + f^r(b)}{2}\right]^{\frac{1}{r}}, & p \leq r \neq 0; \\ \sqrt{f(a)\, f(b)}, & p < r = 0. \end{cases}$$

(3) If $p = 0$ then h is strictly increasing, $h \circ g^{-1}$ is convex for $r \leq 0$, and $h \circ g^{-1}$ is concave for $r > 0$, hence

$$M_p(f, w) = \exp\left(\frac{\int_a^b w(t) \ln f(t)\, dt}{\int_a^b w(t)\, dt}\right) \leq \begin{cases} \sqrt{f(a)\, f(b)}, & r \leq 0 \\ \left[\dfrac{f^r(a) + f^r(b)}{2}\right]^{\frac{1}{r}}, & r > 0. \end{cases}$$

By (1), (2), (3), if $m = \max\{r, p\}$, then

$$M_p(f, w) \leq \begin{cases} \left[\dfrac{f^m(a) + f^m(b)}{2}\right]^{\frac{1}{m}}, & m \neq 0 \\ \sqrt{f(a)\, f(b)}, & m = 0 \end{cases} = M_m(f(a), f(b)).$$

(b) The proof is similar to (a).

∎

REMARK 71. *([**97**]) Let f be positive r-convex on $[a,b]$, then f is g-convex with*

$$g(t)=\begin{cases} t^r, & r\neq 0 \\ \\ \ln t, & r=0 \end{cases} \qquad t\in(0,\infty).$$

Let

$$h(t)=\begin{cases} t^p, & p\neq 0 \\ \\ \ln t, & p=0 \end{cases}$$

and $w(t)=1$, for all $x\in[a,b]$. Then

$$\left(h\circ g^{-1}\right)(t)=\begin{cases} t^{\frac{p}{r}}, & r\neq 0,\ p\neq 0; \\ \\ e^{pt} & r=0,\ p\neq 0; \\ \\ \frac{1}{r}\ln t & r\neq 0,\ p=0; \\ \\ t & r=0,\ p=0. \end{cases}$$

By Theorems 151 and 152, we have

$$h^{-1}\left[\frac{1}{b-a}\int_a^b (h\circ f)(t)\,dt\right]\leq h^{-1}\left[\frac{1}{b-a}\int_a^{\frac{a+b}{2}} N(t)\,dt\right]=E\left(f(a),f(b),r,p+r\right),$$

hence

$$M_p(f)\leq E\left(f(a),f(b),r,p+r\right).$$

Similarly, if f is positive r-concave, then

$$M_p(f)\geq E\left(f(a),f(b),r,p+r\right).$$

This is Theorem 147.

REMARK 72. *([**97**]) In Corollary 54, let $a=-1$, $b=1$ and let $f(t)$ be replaced by $f(x+tv)$. Then $\tilde{M}_p\left(\tilde{f},w\right)\leq M_m\left(f(x-v),f(x+v)\right)$ for positive r-convex f and the inequality is reversed for positive r-concave f. This is Theorem 149.*

6. $H.-H.$ Inequality for the Godnova-Levin Class of Functions

In 1985, E. K. Godnova and V. I. Levin (see [**67**] or [**114**, pp. 410-433]) introduced the following class of functions:

A map $f:I\to\mathbb{R}$ is said to belong to the class $Q(I)$ if it is nonnegative and for all $x,y\in I$ and $\lambda\in(0,1)$, satisfies the inequality

$$f(\lambda x+(1-\lambda)y)\leq\frac{f(x)}{\lambda}+\frac{f(y)}{1-\lambda}. \tag{5.84}$$

They also noted that all nonnegative monotonic and nonnegative convex functions belong to this class and also proved the following motivating result:

If $f\in Q(I)$ and $x,y,z\in I$, then

$$f(x)(x-y)(x-z)+f(y)(y-x)(y-z)+f(z)(z-x)(z-y)\geq 0. \tag{5.85}$$

In fact (5.85) is even equivalent to (5.84) so it can alternatively be used in the definition of the class $Q(I)$.

For the case $f(x)=x^r$, $r\in\mathbb{R}$, the inequality (5.85) obviously coincides with the well-known Schur inequality.

The following result of Hermite-Hadamard type holds [**67**].

THEOREM 153. *Let* $f \in Q(I)$, $a, b \in I$ *with* $a < b$ *and* $f \in L_1[a,b]$. *Then one has the inequalities:*

$$f\left(\frac{a+b}{2}\right) \le \frac{4}{b-a}\int_a^b f(x)\,dx \tag{5.86}$$

and

$$\frac{1}{b-a}\int_a^b p(x) f(x)\,dx \le \frac{f(a)+f(b)}{2}, \tag{5.87}$$

where $p(x) = \frac{(b-x)(x-a)}{(b-a)^2}$, $x \in [a,b]$.
The constant 4 *in* (5.86) *is the best possible.*

PROOF. Since $f \in Q(I)$, we have, for all $x, y \in I$ (with $\lambda = \frac{1}{2}$ in (5.84)) that

$$2(f(x)+f(y)) \ge f\left(\frac{x+y}{2}\right),$$

i.e., with $x = ta + (1-t)b$, $y = (1-t)a + tb$,

$$2(f(ta+(1-t)b) + f((1-t)a+tb)) \ge f\left(\frac{a+b}{2}\right).$$

By integrating, we therefore have that

$$2\left[\int_0^1 f(ta+(1-t)b)\,dt + \int_0^1 f((1-t)a+tb)\,dt\right] \ge f\left(\frac{a+b}{2}\right). \tag{5.88}$$

Since

$$\begin{aligned}\int_0^1 f(ta+(1-t)b)\,dt &= \int_0^1 f((1-t)a+tb)\,dt \\ &= \frac{1}{b-a}\int_a^b f(x)\,dx,\end{aligned}$$

we get the inequality (5.86) from (5.88).

For the proof of (5.87), we first note that if $f \in Q(I)$, then for all $a, b \in I$ and $\lambda \in [0,1]$, it yields

$$\lambda(1-\lambda) f(\lambda a + (1-\lambda)b) \le (1-\lambda) f(a) + \lambda f(b)$$

and

$$\lambda(1-\lambda) f((1-\lambda)a + \lambda b) \le \lambda f(a) + (1-\lambda) f(b).$$

By adding these inequalities and integrating, we find that

$$\begin{aligned}&\int_0^1 \lambda(1-\lambda)[f(\lambda a+(1-\lambda)b) + f((1-\lambda)a+\lambda b)]\,d\lambda \\ \le\ & f(a)+f(b).\end{aligned} \tag{5.89}$$

Moreover,

$$\begin{aligned}(5.90)\qquad & \int_0^1 \lambda(1-\lambda) f(\lambda a+(1-\lambda) b)\, d\lambda \\ =\ & \int_0^1 \lambda(1-\lambda) f((1-\lambda) a+\lambda b)\, d\lambda \\ =\ & \frac{1}{b-a} \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} f(x)\, dx.\end{aligned}$$

We get (5.87) by combining (5.89) with (5.90) and the proof is complete.

The constant 4 in (5.86) is the best possible because this inequality obviously reduces to an equality for the function

$$f(x)=\begin{cases}1, & a \leq x<\frac{a+b}{2} \\ 4, & x=\frac{a+b}{2} \\ 1, & \frac{a+b}{2}<x \leq b.\end{cases}$$

Additionally, this function is in the class $Q(I)$ because

$$\begin{aligned}\frac{f(x)}{\lambda}+\frac{f(y)}{1-\lambda} & \geq \frac{1}{\lambda}+\frac{1}{1-\lambda}=g(\lambda) \\ & \geq \min_{0<\lambda<1} g(\lambda)=g\left(\frac{1}{2}\right) \\ & = 4 \geq f(\lambda x+(1-\lambda) y)\end{aligned}$$

for all $x, y \in[a, b]$ and $\lambda \in(0,1)$.

The proof is thus complete. ∎

Next, we shall restrict the Godnova-Levin class of functions and point out a sharp version of Hadamard's inequality in this class. More precisely, we say that a map $f: I \rightarrow \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and, for all $x, y \in I$ and $\lambda \in[0,1]$, satisfies the following inequality

$$f(\lambda x+(1-\lambda) y) \leq f(x)+f(y). \tag{5.91}$$

Obviously, $Q(I) \supset P(I)$ and for applications it is important to note that $P(I)$ also consists only of nonnegative monotonic, convex and quasi-convex functions, i.e., nonnegative functions satisfying

$$f(\lambda x+(1-\lambda) y) \leq \max \{f(x), f(y)\}.$$

The following result of Hermite-Hadamard type holds [**67**]:

Theorem 154. *Let $f \in P(I)$, $a, b \in I$ with $a<b$ and $f \in L_1[a, b]$. Then one has the inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_a^b f(x)\, dx \leq 2(f(a)+f(b)). \tag{5.92}$$

Both inequalities are the best possible.

PROOF. According to (5.91) with $x = ta + (1-t)\,b,\ y = (1-t)\,a + tb$ and $\lambda = \frac{1}{2}$, we find that

$$f\left(\frac{a+b}{2}\right) \leq f\left(ta + (1-t)\,b\right) + f\left((1-t)\,a + tb\right)$$

for all $t \in [0,1]$. Thus, by integrating on $[0,1]$, we obtain

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \int_0^1 \left[f\left(ta + (1-t)\,b\right) + f\left((1-t)\,a + tb\right)\right] dt \\ &= \frac{2}{b-a}\int_a^b f(x)\,dx \end{aligned}$$

and the first inequality is proved.

The proof of the second inequality follows by using (5.91) with $x = a$ and $y = b$ and integrating with respect to λ over $[0,1]$.

The first inequality in (5.92) reduces to an equality for the nondecreasing function

$$f(x) = \begin{cases} 0, & a \leq x < \frac{a+b}{2} \\ \\ 1, & \frac{a+b}{2} \leq x \leq b \end{cases}$$

and the second inequality reduces to an equality for the nondecreasing function

$$f(x) = \begin{cases} 0, & x = a \\ 1, & a < x \leq b. \end{cases}$$

The proof is thus complete. ∎

6.1. Inequalities for Positive Functionals. In paper [**113**], D.S. Mitrinović and J.E. Pečarić prove the following Jensen type inequality for functions of $Q(I)$ type.

THEOREM 155. *Suppose that $f \in Q(I)$; $x \in I^n$ $(n \geq 2)$ and $w = (w_1, w_2, \ldots, w_n)$ is a positive n-tuple. Then*

$$f\left(\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right) \leq W_n \sum_{i=1}^n \frac{f(x_i)}{w_i}, \tag{5.93}$$

where $W_n := \sum_{i=1}^n w_i$.

Some reverses of this inequality are also pointed out.

In this subsection, we give some inequalities of Hadamard type for the functions $f \in Q(I)$ and for normalized isotonic linear functionals. Some applications to elementary inequalities are also noted.

Let T be a nonempty set and L a linear class of real valued functions $f, g : T \to \mathbb{R}$ having the properties

(L1) $f, g \in L$ implies $\alpha f + \beta g \in L$ for all $\alpha, \beta \in \mathbb{R}$.
(L2) $\mathbf{1} \in L$, where $\mathbf{1}(t) = 1$, for all $t \in T$.

We also consider *isotonic linear functionals* $A : L \to \mathbb{R}$. That is, we suppose that

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$;
(A2) $f \in L$, $f \geq 0$ (i.e. $f(t) \geq 0$ for all $t \in T$) implies $A(f) \geq 0$ (i.e., A is isotonic).

The following result is similar to the first inequality in the $H. - H.$ inequality.

THEOREM 156. *Let $f \in Q(I)$, $a, b \in I$ and $h : T \to [0,1]$ so that the maps $f \circ (ah + b(\mathbf{1} - h))$ and $f \circ (a(\mathbf{1} - h) + bh)$ belong to L. Then for all A, an isotonic linear functional with $A(\mathbf{1}) = 1$, one has the inequality*

$$f\left(\frac{a+b}{2}\right) \leq 2\left[A(f \circ (ah + b(1-h))) + A(f(a(1-h)+bh))\right]. \tag{5.94}$$

PROOF. Since$f \in \mathbb{Q}(I)$, then for all $x, y \in I$ one has

$$2(f(x) + f(y)) \geq f\left(\frac{x+y}{2}\right). \tag{5.95}$$

Let $t \in T$. If we choose $x = h(t)a + (1 - h(t))b$, $y = (1 - h(t))a + bh(t)$, we observe that $x, y \in I$ and, by (5.95),

$$\begin{aligned} &2[f(h(t)a + (1-h(t))b) + f((1-h(t))a + h(t)b)] \\ \geq\ & f\left(\frac{h(t)a + (1-h(t))b + (1-h(t))a + bh(t)}{2}\right) = f\left(\frac{a+b}{2}\right). \end{aligned}$$

This shows that we have, in the order of L,

$$2\left[f \circ (ah + b(\mathbf{1} - h)) + f \circ ((\mathbf{1} - h)a + bh)\right] \geq f\left(\frac{a+b}{2}\right).$$

Applying A and using properties (A1) and (A2), we obtain the desired inequality (5.94). ∎

The inequality (5.86) can be recaptured as follows [**67**].

COROLLARY 55. *Let $f \in Q(I)$, $a, b \in I$ with $a < b$ and f integrable in $[a, b]$. Then one has the inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{4}{b-a}\int_a^b f(x)\,dx \tag{5.96}$$

PROOF. Applying the above theorem for $A = \int_0^1$ and for $h : [0,1] \to [0,1]$ with $h(t) = t$, we have

$$f\left(\frac{a+b}{2}\right) \leq 2\int_0^1 (f(ta + (1-t)b) + f((1-t)a + tb))dt.$$

However, a simple calculation shows that

$$\int_0^1 f(ta + (1-t)b)dt = \int_0^1 f((1-t)a + tb)dt = \frac{1}{b-a}\int_a^b f(x)\,dx$$

and the inequality is thus proved. ∎

REMARK 73. *Since every nonnegative monotone function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ and belongs to $Q([a, b])$, hence (5.96) holds for this class of functions.*

The following discrete inequality also holds:

COROLLARY 56. *Let $f \in Q(I)$, $a, b \in I$, $p_i \geq 0$ $(i = \overline{1, n})$ with $P_n = \sum_{i=1}^n p_i > 0$ and $t_i \in [0, 1]$ $(i = \overline{1, n})$. Then one has the inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{P_n}\sum_{i=1}^n p_i[f(t_i a + (1-t_i)b) + f((1-t_i)a + t_i b)]. \tag{5.97}$$

A variant of the second Hadamard inequality for functions in the class $Q(I)$ is embodied in the following.

THEOREM 157. *Let $f \in Q(I)$, $a, b \in I$ and $h : T \to [0,1]$ so that $h(\mathbf{1}-h) \cdot f \circ (ah + b(\mathbf{1}-h))$, $h(\mathbf{1}-h) \cdot f \circ (a(\mathbf{1}-h) + bh)$ belong to L. Then for all A an isotonic linear functional so that $A(\mathbf{1}) = 1$, one has the inequality*

$$A\left(\frac{h(\mathbf{1}-h)}{2} \cdot f \circ (ah + b(\mathbf{1}-h))\right) + A\left(\frac{h(\mathbf{1}-h)}{2} \cdot f \circ (a(\mathbf{1}-h) + bh)\right) \leq \frac{f(a)+f(b)}{2}. \tag{5.98}$$

PROOF. If $f \in Q(I)$, then for all $x, y \in I$ and $\lambda \in [0,1]$

$$\lambda(1-\lambda) f(\lambda x + (1-\lambda) y) \leq (1-\lambda) f(x) + \lambda f(y)$$

and

$$\lambda(1-\lambda) f((1-\lambda) x + \lambda y) \leq \lambda f(x) + (1-\lambda) f(y)$$

Adding these inequalities, we get

$$\lambda(1-\lambda)[f(\lambda x + (1-\lambda) y + f((1-\lambda) x + \lambda y)] \leq f(x) + f(y)$$

for all $\lambda \in [0,1]$ and $x, y \in I$. Let $t \in T$. If we choose $\lambda = h(t)$, $x = a$, $y = b$, then we have

$$\frac{h(t)(1-h(t))}{2} f(ah(t) + b(1-h(t))) + \frac{h(t)(1-h(t))}{2} f(a(1-h(t)) + bh(t)) \leq \frac{f(a)+f(b)}{2}$$

which gives in the order of L:

$$\frac{h(1-h)}{2}[f \circ (ah + b(\mathbf{1}-h))] + \frac{h(1-h)}{2}[f \circ (a(\mathbf{1}-h) + bh)] \leq \frac{f(a)+f(b)}{2} \cdot \mathbf{1}.$$

Applying to this inequality the functional A and using the properties (A1) and (A2), we get the desired inequality (5.98). ∎

The inequality (5.87) can be recaptured as follows [**67**].

COROLLARY 57. *Let $f \in Q(I)$, $a, b \in I$ with $a < b$ and f integrable in $[a,b]$. Then one has the inequality*

$$\frac{1}{b-a}\int_a^b p(t) f(t) dt \leq \frac{f(a)+f(b)}{2}, \tag{5.99}$$

where

$$p(t) = \frac{(b-t)(t-a)}{(b-a)^2}, \ t \in [a,b].$$

PROOF. If we apply the above theorem for $A = \int_0^1$ and $h : [0,1] \to [0,1]$ with $h(t) = t$, we get

$$\frac{f(a)+f(b)}{2} \geq \frac{1}{2}\int_0^1 t(1-t)[f(ta + (1-t)b) + f((1-t)a + tb)] dt.$$

But a simple calculation shows that

$$\begin{aligned}\int_0^1 t(1-t)f(ta+(1-t)b)dt &= \int_0^1 t(1-t)f((1-t)a+tb)dt \\ &= \frac{1}{b-a}\int_a^b \frac{(b-x)(x-a)}{(b-a)^2}f(x)dx\end{aligned}$$

and thus the inequality (5.99) is proved. ∎

REMARK 74. *The inequality (5.99) holds for every nonnegative monotonic function on $[a,b]$.*

The discrete variant of (5.98) is the following:

COROLLARY 58. *With the assumptions of Corollary 56, one has the inequalities*

$$\begin{aligned}&\frac{1}{P_n}\sum_{i=1}^n p_i\cdot\frac{t_i(1-t_i)}{2}[f(t_ia+(1-t_i)b)+f((1-t_i)a+t_ib)] \\ \leq\ &\frac{f(a)+f(b)}{2}.\end{aligned} \tag{5.100}$$

6.2. Applications.

(1) If $f(x)=x^p$ $(p\geq 1)$, $x\in[0,\infty)$ and $0\leq a<b$, then by the well-known $H.-H.$ inequality, we have:

$$\left(\frac{a+b}{2}\right)^p\leq\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\leq\frac{a^p+b^p}{2}.$$

If $r\in(0,1)$, then $f(x)=x^r$ is concave on $[0,\infty)$ and by the same inequality, we have

$$\left(\frac{a+b}{2}\right)^r\geq\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\geq\frac{a^r+b^r}{2}. \tag{5.101}$$

Since $f(x)=x^r$ is nonnegative and monotonic nondecreasing on $[a,b]\subset[0,\infty)$, we can apply (5.96), i.e, we have the inequality [**67**]

$$\left(\frac{a+b}{2}\right)^r\leq 4\cdot\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}. \tag{5.102}$$

In conclusion, by the inequalities (5.101) and (5.102), we can state that [**67**]

$$\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\leq\left(\frac{a+b}{2}\right)^r\leq 4\cdot\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\text{ for all } r\in(0,1].$$

If we use (5.99) for the mapping $f(x)=x^r$ $(r\in(0,1))$ then we have

$$\frac{1}{b-a}\int_a^b p(t)t^r dt\leq\frac{a^r+b^r}{2}$$

where

$$p(t)=\frac{(b-t)(t-a)}{(b-a)^2},\ t\in[a,b]\subset(0,\infty)$$

which gives a converse for the inequality (5.101).

(2) Let p$\neq$ 0 and a, b > 0. Denote the power mean

$$A_p(a,b)=\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\quad (p\neq 0)$$

with $A(a,b)=A_1(a,b)$ and $A_0(a,b)=G(a,b)=\sqrt{ab}$. Also, consider the identric mean defined for $a<b$

$$I(a,b)=e^{-1}\left(\frac{b^b}{a^a}\right)^{\frac{1}{(b-a)}}.$$

In [**148**], A.O. Pittenger proved that

(5.103) $$A_{\frac{2}{3}}<I<A_{\ln 2}\quad (\ln \text{ denotes } \log_e)$$

and the indices are sharp, i.e., I and A_p are not comparable for $p\in\left(\frac{2}{3},\ln 2\right)$.

If we apply inequality (5.96) for the nonnegative monotonically increasing function $f(x)=\ln x$, $x>1$, we have

$$\begin{aligned}\ln\left(\frac{a+b}{2}\right) &\leq \frac{4}{b-a}\int_a^b \ln x dx=4\left(\frac{b\ln b-a\ln a}{b-a}-1\right)\\ &= 4\ln\left[e^{-1}\left(\frac{b^b}{a^a}\right)^{\frac{1}{(b-a)}}\right]=\ln(I^4(a,b))\quad (a,b>1)\end{aligned}$$

which gives

(5.104) $$A\leq I^4\quad (a,b>1).$$

Thus by (5.103) and (5.104) we have

$$A_{\frac{2}{3}}<I<A_{\ln 2}<A<I^4\ \text{ for }\ a,b>1.$$

If we apply inequality (5.99) for the mapping $f:(1,\infty)\to\mathbb{R}$, $f(x)=\ln(x)$, we get

$$\frac{1}{b-a}\int_a^b p(t)\ln(t)\,dt\leq\frac{\ln a+\ln b}{2}=\ln G(a,b)\quad (a,b>1)$$

i.e.,

$$\exp\left[\int_a^b\frac{(b-t)(t-a)}{(b-a)^3}\ln(t)\,dt\right]\leq G(a,b)\quad\text{for all }\ a,b>1.$$

(3) If we apply the discrete inequalities (5.97), (5.100) and (5.101) for the mapping $f(x)=x^r$ $(r\in(0,1])$, $x\in[0,\infty)$, we deduce the inequality:

$$\begin{aligned}&\frac{1}{P_n}\sum_{i=1}^n p_i\frac{t_i(1-t_i)}{2}[(t_ia+(1-t_i)b)^r+((1-t_i)a+t_ib)^r]\\ \leq\ & \frac{a^r+b^r}{2}\leq\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\leq\left(\frac{a+b}{2}\right)^r\\ \leq\ & \frac{2}{P_n}\sum_{i=1}^n p_i[(t_ia+(1-t_i)b)^r+((1-t_i)a+t_ib)^r]\end{aligned}$$

for all $t_i\in[0,1]$, $p_i\geq 0$ $(i=\overline{1,n})$ with $P_n>0$.

(4) If we apply the discrete inequalities (5.97) and (5.100) for the mapping ln on $[1,\infty)$, we have the following inequalities related to the well-known arithmetic mean-geometric mean inequality $G(a,b) \leq A(a,b)$,

$$\begin{aligned} & \left[\prod_{i=1}^{n}\left[(t_i a+(1-t_i)b)((1-t_i)a+t_i b)\right]^{\frac{p_i t_i(1-t_i)}{2}}\right]^{\frac{1}{P_n}} \\ \leq\ & G(a,b) \leq A(a,b) \\ \leq\ & \left[\prod_{i=1}^{n}\left[(t_i a+(1-t_i)b)((1-t_i)a+t_i b)\right]^{2p_i}\right]^{\frac{1}{P_n}}, \end{aligned}$$

where $t_i \in [0,1]$, $p_i \geq 0$ $(i=\overline{1,n})$ with $P_n > 0$.

7. The $H.-H.$ Inequality for Quasi-Convex Functions

We shall start with the following definition.

DEFINITION 6. *The mapping $f : I \to \mathbb{R}$ is said to be Jensen or J-quasi-convex if*

$$f\left(\frac{x+y}{2}\right) \leq \max\left\{f(x), f(y)\right\} \tag{5.105}$$

for all $x, y \in I$.

Note that the class $JQC(I)$ of *J-quasi-convex* functions on I contains the class $J(I)$ if *J-convex* functions on I. In other words, functions satisfying the condition

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \text{ for all } x, y \in I. \tag{5.106}$$

The following inequality of Hermite-Hadamard type holds [**62**].

THEOREM 158. *Suppose $a, b \in I \subseteq \mathbb{R}$ and $a < b$. If $f \in JQC(I) \cap L_1[a,b]$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx + I(a,b), \tag{5.107}$$

where

$$I(a,b) := \frac{1}{2(b-a)}\int_a^b |f(x) - f(a+b-x)|\,dx$$

Furthermore, $I(a,b)$ satisfies the inequalities:

$$\begin{aligned} 0 &\leq I(a,b) \\ &\leq \frac{1}{b-a}\min\left\{\int_a^b |f(x)|\,dx\ ,\right. \\ &\qquad \left.\frac{1}{\sqrt{2}}\left((b-a)\int_a^b f^2(x)\,dx - J(a,b)\right)^{\frac{1}{2}}\right\}, \end{aligned} \tag{5.108}$$

where

$$J(a,b) := (b-a)\int_a^b f(x)\,f(a+b-x)\,dx.$$

PROOF. Since f is J-quasi-convex on I, we have, for all $x,y\in I$:

$$f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)+|f(x)-f(y)|}{2}.$$

For $t\in[0,1]$, put $x=ta+(1-t)b,\ y=(1-t)a+tb\in I$. Then

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f(ta+(1-t)b)+f((1-t)a+tb)\right.\\&\quad\left.+|f(ta+(1-t)b)-f((1-t)a+tb)|\right].\end{aligned}$$

Integrating this inequality over $[0,1]$ gives

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[\int_0^1 f(ta+(1-t)b)\,dt+\int_0^1 f((1-t)a+tb)\,dt\right]\\&\quad+\frac{1}{2}\int_0^1|f(ta+(1-t)b)-f((1-t)a+tb)|\,dt.\end{aligned}$$

Since

$$\int_0^1 f(ta+(1-t)b)\,dt=\int_0^1 f((1-t)a+tb)\,dt=\frac{1}{b-a}\int_a^b f(x)\,dx,$$

using the change of variable $x=ta+(1-t)b$, we have

$$\begin{aligned}&\frac{1}{2}\int_0^1|f(ta+(1-t)b)-f((1-t)a+tb)|\,dt\\&=\frac{1}{2(b-a)}\int_a^b|f(x)-f(a+b-x)|\,dx\end{aligned}$$

and the inequality (5.107) is proved.

We now observe that

$$\begin{aligned}0 &\leq I(a,b)\leq\frac{1}{2(b-a)}\left[\int_a^b|f(x)|\,dx+\int_a^b|f(a+b-x)|\,dx\right]\\&=\frac{1}{b-a}\int_a^b|f(x)|\,dx\end{aligned}$$

On the other hand, by the Cauchy-Buniakowsky-Schwartz inequality, we have

$$\begin{aligned}&\frac{1}{2(b-a)}\int_a^b|f(x)-f(a+b-x)|\,dx\\&\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b|f(x)-f(a+b-x)|^2\,dx\right]^{\frac{1}{2}}\\&=\frac{1}{2}\left[\frac{1}{b-a}\int_a^b\left(f^2(x)-2f(x)f(a+b-x)+f^2(a+b-x)\right)dx\right]^{\frac{1}{2}}\\&=\frac{1}{2}\left[\frac{2}{b-a}\int_a^b f^2(x)\,dx-\frac{2}{b-a}\int_a^b f(x)f(a+b-x)\,dx\right]^{\frac{1}{2}}\\&=\frac{\sqrt{2}}{2(b-a)}\left[(b-a)\int_a^b f^2(x)\,dx-(b-a)\int_a^b f(x)f(a+b-x)\,dx\right]^{\frac{1}{2}}\end{aligned}$$

and the inequality (5.108) is proved. ∎

Remark 75. *If $f : I \to \mathbb{R}$ is quasi-convex and nonnegative, then f is J-quasi-convex and thus satisfies*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx + I(a,b) \leq \frac{2}{b-a}\int_a^b f(x)\,dx,$$

which improves the first inequality from (5.92) *for quasi-convex functions.*

E. M. Wright introduced an interesting class of functions in [**187**].

We say $f : I \to \mathbb{R}$ is *Wright-convex* function on $I \subseteq \mathbb{R}$ if, for each $y > x$ and $\delta > 0$ with $y + \delta,\ x \in I$ we have

$$f(x+\delta) - f(x) \leq f(y+\delta) - f(y). \tag{5.109}$$

The following characterisation holds for $W-$convex functions [**62**].

Proposition 58. *Suppose $I \subseteq \mathbb{R}$. Then the following statements are equivalent for a function $f : I \to \mathbb{R}$*

(i) *f is $W-$convex on I;*
(ii) *For all $a, b \in I$ and $t \in [0,1]$, we have the inequality:*

$$f((1-t)a + tb) + f(ta + (1-t)b) \leq f(a) + f(b). \tag{5.110}$$

Proof. For "(i)⇒(ii)", let $a, b \in I$ and $t \in [0,1)$. Firstly, suppose $a < b$. If f is $W-$convex on I, then for all $y > x$ and $\delta > 0$ with $y + \delta, x \in I$ we have

$$f(x+\delta) - f(x) \leq f(y+\delta) - f(y). \tag{5.111}$$

Choose $x = a,\ y = ta + (1-t)b$ and $\delta := b - (ta + (1-t)b) > 0$. Then $x + \delta = (1-t)a + tb,\ y + \delta = b$ and thus, by (5.111), we obtain

$$f((1-t)a + tb) - f(a) \leq f(b) - f(ta + (1-t)b)$$

whence we have (5.110).

The proof is similar for the case $a > b$.

For "(ii)⇒(i)", let $y > x$ and $\delta > 0$ with $y + \delta, x \in I$. In (5.111) choose $a = x,\ b > a$ and $t \in [0,1)$ with $ta + (1-t)b = y$ and $b - (ta + (1-t)b) = \delta$. We have $y + \delta = b \in I,\ x \in I$ and $x + \delta = (1-t)a + tb$. From (5.110) we derive

$$f(x) + f(y+\delta) \geq f(y) + f(x+\delta),$$

which shows that the map is $W-$convex on I. ∎

The equivalence motivates the introduction of the following class of functions [**62**].

Definition 7. *For $I \subseteq \mathbb{R}$, the mapping $f : I \to \mathbb{R}$ is Wright-quasi-convex if, for all $x, y \in I$ and $t \in [0,1]$, one has the inequality*

$$\frac{1}{2}\left[f(tx + (1-t)y) + f((1-t)x + ty)\right] \leq \max\{f(x), f(y)\}, \tag{5.112}$$

or, equivalently,

$$\frac{1}{2}\left[f(y) + f(\delta)\right] \leq \max\{f(x), f(y+\delta)\}$$

for every $x, y + \delta \in I$ with $x < y$ and $\delta > 0$.

We show that the following inequality of Hermite-Hadamard type holds [**62**].

THEOREM 159. *Let $f : I \to \mathbb{R}$ be a W-quasi-convex map on I and suppose $a, b \in I \subseteq \mathbb{R}$ with $a < b$ and $f \in L_1[a, b]$. Then we have the inequality*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \max\{f(a), f(b)\}. \tag{5.113}$$

PROOF. For all $t \in [0, 1]$ we have

$$\frac{1}{2}\left[f(ta + (1-t)b) + f((1-t)a + tb)\right] \leq \max\{f(a), f(b)\}.$$

On integrating this inequality over $[0, 1]$ and using

$$\int_0^1 f(ta + (1-t)b)\,dt = \int_0^1 f((1-t)a + tb)\,dt = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

we obtain the desired inequality. ∎

REMARK 76. *If f is quasi-convex and nonnegative, then*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \max\{f(a), f(b)\} \leq f(a) + f(b),$$

which improves the second inequality in (5.92) *for quasi-convex and nonnegative functions.*

We now introduce the notion of a *quasi-monotone* function.

DEFINITION 8. *For $I \subseteq \mathbb{R}$, the mapping $f : I \to \mathbb{R}$ is quasi-monotone on I if it is either monotone on $I = [c, d]$ or monotone nonincreasing on a proper subinterval $[c, c'] \subset I$ and monotone nondecreasing on $[c', d]$.*

The class $QM(I)$ of quasi-monotone functions on I provides an immediate characterisation of quasi-convex functions [**62**].

PROPOSITION 59. *Suppose $I \subseteq \mathbb{R}$. Then the following statements are equivalent for a function $f : I \to \mathbb{R}$.*

(a) $f \in QM(I)$;
(b) *On any subinterval of I, f achieves a supremum at an end point;*
(c) $f \in QC(I)$.

PROOF. That (a) implies (b) is immediate from the definition of quasi-monotonicity. For the reverse implication, suppose it is possible that (b) holds but $f \notin QM(I)$. Then there must exist points $x, y, z \in I$ with $x < y < z$ and $f(y) > \max\{f(x), f(z)\}$, contradicting (b) for the subinterval $[x, z]$. The equivalence of (b) and (c) is simply the definition of quasi-convexity. ∎

The following inclusion results hold [**62**].

THEOREM 160. *Let $WQC(I)$ denote the class of Wright-quasi-convex functions on $I \subseteq \mathbb{R}$. Then*

$$QC(I) \subset WQC(I) \subset JQC(I). \tag{5.114}$$

Both inclusions are proper.

PROOF. Let $f \in QC(I)$. Then, for all $x, y \in I$ and $t \in [0, 1]$ we have

$$f(tx + (1-t)y) \leq \max\{f(x), f(y), f((1-t)x + ty)\} \leq \max\{f(x), f(y)\}$$

which gives by addition that

$$\frac{1}{2}\left[f\left(tx+(1-t)\,y\right)+f\left((1-t)\,x+ty\right)\right]\leq\max\left\{f\left(x\right),f\left(y\right)\right\} \tag{5.115}$$

for all $x,y\in I$ and $t\in[0,1]$, i.e., $f\in WQC\left(I\right)$. The second inclusion becomes obvious on choosing $t=\frac{1}{2}$ in (5.115).

Let H be a Hamel basis over the rationals. Then each real number u has a unique representation

$$u=\sum_{h\in H}r_{u,h}\cdot h$$

in which only finitely many of the coefficients $r_{u,h}$ are nonzero. Define a mapping $f:I\to\mathbb{R}$ by

$$f\left(u\right)=\sum_{h\in H}r_{u,h}.$$

Then

$$\begin{aligned}\frac{1}{2}\left[f\left(y\right)+f\left(x+\delta\right)\right] &= \frac{1}{2}\left[\sum_{h}r_{y,h}+\sum_{h}\left(r_{x,h}+r_{\delta,h}\right)\right]\\ &= \frac{1}{2}\left[\sum_{h}r_{x,h}+\sum_{h}\left(r_{y,h}+r_{\delta,h}\right)\right]\\ &\leq \max\left[\sum_{h}r_{x,h},\sum_{h}\left(r_{y,h}+r_{\delta,h}\right)\right]\\ &= \max\left\{f\left(x\right),f\left(y+\delta\right)\right\}\end{aligned}$$

so that $f\in WQC\left(I\right)$.

We now demonstrate that H can be selected so that $f\notin QC\left(I\right)$. Choose $\delta>0$ and $x\neq0$ to be rational and $y+\delta$ to be irrational. We may choose H such that $y+\delta,-\left|x\right|\in H$. Then $f\left(\delta\right)<0,\ f\left(x\right)=-sgn\left(x\right)$ and $f\left(y+\delta\right)=1$. The map f is additive, so that

$$f\left(y\right)=f\left(y+\delta\right)-f\left(\delta\right)>f\left(y+\delta\right)=1=\max\left\{f\left(x\right),f\left(y+\delta\right)\right\}.$$

Hence, $f\notin QC\left(I\right)$.

For the second inequality in (5.114), consider the Dirichlet map $f:I\to\mathbb{R}$ defined by

$$f\left(u\right)=\begin{cases}1 & \text{for } u \text{ irrational}\\ 0 & \text{for } u \text{ rational.}\end{cases}$$

If x and y are both rational, then so is $\frac{(x+y)}{2}$, so that, in this case

$$f\left(\frac{x+y}{2}\right)=\max\left\{f\left(x\right),f\left(y\right)\right\}. \tag{5.116}$$

If one of x,y is rational and the other irrational, then $\frac{(x+y)}{2}$ is irrational and so, again, (5.116) holds. If both x and y are irrational, then $\max\left\{f\left(x\right),f\left(y\right)\right\}=1$, so that

$$f\left(\frac{x+y}{2}\right)\leq\max\left\{f\left(x\right),f\left(y\right)\right\}.$$

Hence $f \in JQC(I)$. However, if x and y are distinct rationals, there are uncountably many values of $t \in (0,1)$ for which $tx + (1-t)y$ and $(1-t)x + ty$ are both irrational. For each such t

$$\frac{1}{2}\left[f(tx+(1-t)y) + f((1-t)x+ty)\right] > \max\{f(x), f(y)\}$$

so that $f \notin WQC(I)$. Hence $WQC(I)$ is a proper subset of $JQC(I)$. ∎

We also have the following result [**62**].

Theorem 161. *We have the inclusions*

$$W(I) \subset WQC(I),\ C(I) \subset QC(I),\ J(I) \subset JQC(I).$$

Each inclusion is proper. Note that $C(I)$, $W(I)$ *and* $J(I)$ *are the sets of convex,* $W-$*convex and* $J-$*convex functions on* I *respectively.*

Proof. By Proposition 58, we have for $f \in W(I)$ that

$$\frac{1}{2}\left[f((1-t)a+tb) + f(ta+(1-t)b)\right] \leq \frac{f(a)+f(b)}{2}$$

for all $a, b \in I$ and $t \in [0,1]$.

Since

$$\frac{f(a)+f(b)}{2} \leq \max\{f(a), f(b)\} \quad \text{for all } a, b \in I,$$

the inequality (5.112) is satisfied, that is, $f \in WQC(I)$ and the first inclusion is thus proved.

Similar proofs hold for the other two.

As

$$C(I) \subset W(I) \subset J(I) \tag{5.117}$$

and each inclusion is proper ([**91**] and [**93**]) and by the relation (5.114), for each inclusion to be proper, it is sufficient that there should exist a function f with $f \in QC(I)$ but $f \notin J(I)$. Clearly, any strictly concave monotonic function suffices. ∎

Remark 77. *In view of the results of the foregoing theorem, the fact that there are functions in* $QC(I)$ *which are not in* $J(I)$ *makes it tempting to try to cocatenate the set inclusions* (5.117) *and* (5.114). *However, no result of this sort appears to exist without the imposition of further assumptions. Thus, for example, by the use if the Hamel basis, solutions to* (5.112) *may be constructed which are unbounded on every subinterval, whereas all members of* $QC(I)$ *are bounded on every finite interval. Hence, it is not the case that* $WC(I) \subset QC(I)$.

We now show that the three classes of quasi-convex functions in Theorem 161 collapse into one under the additional constraint of continuity. We denote by $QM_0(I)$ the class of quasi-monotone functions under this constraint, with similar notation for the other classes involved in Theorems 159 and 161, [**62**].

Theorem 162. *For a given interval* $I \subseteq \mathbb{R}$,

$$QC_0(I) = WQC_0(I) = JQC_0(I).$$

Proof. The proofs of the basic inclusion results of Theorem 160 do not involve continuity, so that

$$QC_0(I) \subset WQC_0(I) \subset JQC_0(I).$$

For the same reason, by Proposition 59, we have $QM_0(I) = QC_0(I)$. Hence, it suffices to prove that $JQC_0(I) \subset QM_0(I)$. We proved by *reducio ad absurdum.*

Suppose it is possible that $f \in JQC_0(I)$ but $f \notin QM_0(I)$. Then there must exist points $x, y, z \in I$ with $x < z < y$ and $f(z) > f(x) = f(y)$. Let $|y - x| = d$. By continuity there exists a given interval $I_0 \subset [x, y]$ of length $d_0 > 0$ with $z \in I_0$ and f strictly exceeding $f(x)$ on I_0. Since $f \in JQC(I)$, we have $\frac{x+y}{2} \notin I_0$, so that I_0 is properly contained in either $\left(x, \frac{x+y}{2}\right)$ or $\left(\frac{x+y}{2}, y\right)$.

Invoking continuity again, there must be, according to which of these two cases holds, either a point $x' \in \left(x, \frac{x+y}{2}\right]$ with $f(x') = f(x)$ and $I_0 \subset (x, x')$ or a point $y' \in \left[\frac{x+y}{2}, y\right)$ with $f(y') = f(y)$ and $I_0 \subset (y', y)$. Call this interval (x', y').

The previous argument may be repeated to show that there exists x'', y'' with $f(x'') = f(y'') \leq \frac{d}{4}$ and $\frac{d}{2^n} > d_0$ for all $n \geq 1$, which is impossible. ∎

Remark 78. *Theorem 161 does not extend in this way. Thus, for example, if f is continuous, strictly concave and monotonic, we have $f \in QM_0$ but $f \notin W_0$.*

8. $P-$functions, Quasiconvex Functions and $H. - H.$ Type Inequalities

8.1. Introduction. A nonnegative function p defined on the segment S is said to be *a function of P type* ([**67**]) (or simply, a *P-function*) if

$$p(\lambda x + (1-\lambda)y) \leq p(x) + p(y); \quad x, y \in S, \ 0 \leq \lambda \leq 1.$$

Let $S = [a, b]$ and let $\mathcal{P}_S$ be the class of P-functions defined on S. It has been proved in [**67**] that for an integrable function $f \in \mathcal{P}_S$, we have the $H. - H.$-type inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_a^b f(x)dx \leq 2(f(a) + f(b)).$$

In this section, following [**139**], we consider the following generalization of the left side of this inequality. Assume for the sake of simplicity that $[a, b] = [0, 1]$. If f is an integrable P-function and $u \in (0, 1)$, then

$$f(u) \leq \frac{1}{\min(u, 1-u)} \int_0^1 f(x)dx.$$

In fact we present a version of this inequality for an integral with respect to an atomless probability measure μ defined on the Borel σ-algebra of subsets of the segment $[0, 1]$. For nonnegative quasiconvex functions we show that this also holds for an arbitrary (not necessarily atomless) probability measure. These results are the subject of Subection 8.4.

More generally, we study links between P- functions and nonnegative quasiconvex functions, which form an important class of generalized convex functions (see, for example, [**151**]). It is well-known that the sum of quasiconvex functions is not necessarily quasiconvex. The cone hull of the set of all quasiconvex functions defined on a segment S is a very large set, containing for example all functions of bounded variation. The cone hull of the set Q_+ of all nonnegative quasiconvex functions is also very broad, but one can find nonnegative functions of bounded variation which do not belong to this set. The pointwise supremum of a family of elements of Q_+ is again an element of Q_+. One of the important problems of the theory of quasiconvex functions is to describe the least cone containing Q_+ which is closed in the topology of pointwise convergence and contains pointwise suprema of all families of its elements. In Subection 8.3 we show that this cone coincides with the set $\mathcal{P}_S$ of all P-functions defined on S.

We use methods of abstract convexity (see for example [**96, 133, 171**]). The approach is based on the description of small supremal generators of the sets under consideration.

8.2. Preliminaries. First we recall some definitions from abstract convexity. Let $\mathbb{R} = (-\infty, +\infty)$ be a real line and $\mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$. Consider a set X and a set H of functions $h : X \to \mathbb{R}$ defined on X. A function $f : X \to \mathbb{R}_{+\infty}$ is called *abstract convex* with respect to H (or H- *convex*) if there exists a set $U \subset H$ such that $f(x) = \sup\{h(x) : h \in U\}$. The set

$$\mathrm{s}(f, H) = \{h \in H : \ h(x) \leq f(x) \quad \text{for all} \quad x \in X\}$$

is called the *support set* of a function f with respect to H. Clearly f is H-convex if and only if $f(x) = \sup\{h(x) : h \in \mathrm{s}(f, H)\}$ for all $x \in X$.

Let Y be a set of functions $f : X \to \mathbb{R}_{+\infty}$. A set $H \subset Y$ is called a *supremal generator* of the set Y if each function $f \in Y$ is abstract convex with respect to H.

We consider only nonnegative functions defined on the real line $\mathbb{R}$ and mapping into $[0, +\infty]$. Recall that a function f defined on $\mathbb{R}$ is called *quasiconvex* if $f(\alpha x + (1-\alpha)y) \leq \max(f(x), f(y))$ for all $x, y \in \mathbb{R}$ and $\alpha \in (0, 1)$. A function f is quasiconvex if and only if its lower level sets $\{x : f(x) \leq c\}$ are segments for all $c \in \mathbb{R}$.

Let us give some examples.

EXAMPLE 8. *(see for instance [**159**]). Let H_1 be a set of two-step functions h of the form*

$$h(x) = \begin{cases} c & vx \geq d \\ 0 & vx < d \end{cases}$$

with $v \in \{1, -1\}$, $c \geq 0$, $d \in \mathbb{R}$. Then H_1 is a supremal generator of the set Q_+ of all nonnegative quasiconvex functions. Indeed $H_1 \subset Q_+$; since the pointwise supremum of a family of quasiconvex functions is again quasiconvex, it follows that each H_1-convex function belongs to Q_+. Consider now a function $q \in Q_+$ and the family of level sets $S_c = \{x : q(x) \leq c\}$ with $c \geq 0$. Since q is quasiconvex it follows that S_c is a segment for each $c \geq 0$. Let $x_o \in \mathbb{R}$. Assume for the sake of definiteness that $q(x_o) > 0$ and let $0 < c < q(x_o)$. Since $x_o \notin S_c$ it follows that there exists $v \in \{-1, 1\}$ such that $vx_o > vx$ for all $x \in S_c$. Let $d = vx_o$. Then $\bar{c} := \inf_{vx \geq d} q(x) \leq q(x_o)$. Since $S_c \subset \{x : vx < d\}$ it follows that the inequality $vx \geq d$ implies $q(x) > c$. Hence $c \leq \bar{c}$. Let

$$h(x) = \begin{cases} \bar{c} & vx \geq d \\ 0 & vx < d. \end{cases}$$

Clearly $h \leq q$ and $h(x_o) \geq \bar{c} \geq c$. Since c is an arbitrary number such that $0 < c < q(x_o)$ it follows that $q(x) = \sup\{h(x) : \ h \in s(f, H_1)\}$ for all $x \in IR$.

EXAMPLE 9. *(see for example [**159**]). Let H_2 be the set of all two-step functions of the form*

$$h(x) = \begin{cases} c & vx > d \\ 0 & vx \leq d \end{cases}$$

with the same v, c and d as in Example 8. Then H_2 is a supremal generator of the set Q^l_+ of all lower semicontinuous nonnegative quasiconvex functions.

EXAMPLE 10. *([**139**]) Let H_3 be the set of all functions h of the form*

$$h(x) = \begin{cases} c & x = u \\ 0 & x \neq u \end{cases}$$

with $u \in \mathbb{R}$, $c \geq 0$. Then H_3 is a supremal generator of the set of all nonnegative functions defined on $\mathbb{R}$.

EXAMPLE 11. *([**139**]) Let H_4 be the set of all Urysohn peaks on $\mathbb{R}$, that is, continuous functions $g : \mathbb{R} \to \mathbb{R}_+$ of the form*

$$g(x) = \begin{cases} 0 & |x-u| \geq \delta \\ c & x = u \\ \text{affine} & u-\delta < x < u \\ \text{affine} & u < x < x+\delta, \end{cases}$$

where $u \in \mathbb{R}$, $c \geq 0$ and $\delta > 0$. It is easy to check that H_4 is a supremal generator of the set of all functions that are lower semicontinuous on $\mathbb{R}$.

8.3. P-functions. A function $p : \mathbb{R} \to [0, +\infty]$ is called a function of type P ([**67**]) (or P-function) if

$$p(\lambda x + (1-\lambda)y) \leq p(x) + p(y) \quad \text{for all} \quad \lambda \in (0,1) \quad \text{and} \quad x, y \in \mathbb{R}. \tag{5.118}$$

Denote by $\mathcal{P}$ the set of all P-functions.

Let us point out some properties of a function $f \in \mathcal{P}$ (cf. [**139**]).

(1) If $\lambda_i > 0$ $(i = 1, \ldots, m)$ and $\sum_{i=1}^m \lambda_i = 1$, then $f(\sum_{i=1}^m \lambda_i x_i) \leq \sum_{i=1}^m f(x_i)$. This can be proved by induction.
(2) The set $\operatorname{dom} f = \{x \in \mathbb{R} : f(x) < +\infty\}$ is a segment. Indeed let $x_- = \inf \operatorname{dom} f$, $x_+ = \sup \operatorname{dom} f$. Suppose $x_- < x < x_+$. Then there exist points $x_1, x_2 \in \operatorname{dom} f$ such that $x \in (x_1, x_2)$. It follows from the definition of $\mathcal{P}$ that $f(x) < +\infty$, that is, $x \in \operatorname{dom} f$.
(3) If the set $\{x : f(x) = 0\}$ is nonempty, then it is clearly a segment.

Let $S \subset \mathbb{R}$ be a segment and $\mathcal{P}_S$ the set of all P-functions defined on the segment S and mapping into $[0, +\infty]$. We have $\mathcal{P} = \mathcal{P}_S$ with $S = \mathbb{R}$. Let $S \neq \mathbb{R}$. For each function f defined on S, consider its extension $f_{+\infty}$ defined by

$$f_{+\infty}(x) = \begin{cases} f(x) & x \in S \\ +\infty & x \notin S. \end{cases} \tag{5.119}$$

Clearly $f \in \mathcal{P}_S$ if and only if $f_{+\infty} \in \mathcal{P}$.

Let S be a segment. It is easy to check that the class $\mathcal{P}_S$ enjoys the following properties (cf [**139**]).

(1) $\mathcal{P}_S$ is a cone: if $f_1, f_2 \in \mathcal{P}_S$, then $f_1 + f_2 \in \mathcal{P}_S$; if $\lambda > 0, f \in \mathcal{P}_S$, then $\lambda f \in \mathcal{P}_S$;
(2) $\mathcal{P}_S$ is a complete upper semilattice: if $(f_\alpha)_{\alpha \in A}$ is a family of functions from $\mathcal{P}_S$ and $f(x) = \sup_{\alpha \in A} f_\alpha(x)$, then $f \in \mathcal{P}_S$;
(3) $\mathcal{P}_S$ is closed under pointwise convergence.

The classes $\mathcal{P}_S$ are extremely broad. We now describe some subclasses of $\mathcal{P}_S$.

(1) Each quasiconvex nonnegative function defined on S belongs to $\mathcal{P}_S$. In particular nonnegative convex, increasing and decreasing functions defined on S belong to $\mathcal{P}_S$.

(2) Let h be a bounded function defined on S. Then there exists a number $c > 0$ such that the function $f(x) = h(x) + c$ belongs to $\mathcal{P}_S$. Indeed, let $c = \sup_{x,y\in S,\ z\in[x,y]} h(z) - h(x) - h(y)$. We have for each $x, y \in S$ and $\alpha \in [0, 1]$ that

$$h(\alpha x + (1-\alpha)y) \leq h(x) + h(y) + c.$$

Let $f(x) = h(x) + c$. Then

$$f(\alpha x + (1-\alpha)y) = h(\alpha x + (1-\alpha)y) + c \leq (h(x) + c) + (h(y) + c) = f(x) + f(y),$$

that is, $f \in \mathcal{P}_S$.

We now describe a small supremal generator of the set $\mathcal{P}$. Applying this generator and the extension defined by (5.119), we can easily describe a supremal generator of the class $\mathcal{P}_S$ for a segment $S \in \mathbb{R}$.

Let T be the set of all collections $t = \{u; c_1, c_2\}$ with $u \in \mathbb{R}$ and nonnegative c_1, c_2. For $t = \{u; c_1, c_2\} \in T$, consider the function h_t defined on $\mathbb{R}$ by (see [**139**])

$$h_t(x) = \begin{cases} c_1 & x < u \\ c_1 + c_2 & x = u \\ c_2 & x > u. \end{cases} \tag{5.120}$$

It is easy to check that $h_t \in \mathcal{P}$ for all $t \in T$. Let H be the set of all functions of the form h_t with $t \in T$. Clearly H is a conic set, that is, if $h \in H$ and $\lambda > 0$, then $\lambda h \in H$.

The following statement describes a certain extremal property of elements $h \in H$ (cf. [**139**]).

PROPOSITION 60. *Let $h \in H$, $h = h_t$ with $t = \{u; c_1, c_2\}$. If $f \in \mathcal{P}$, $f \leq h$ and $f(u) = h(u)$, then $f = h$.*

PROOF. Let $f \in \mathcal{P}$, $f \leq h$ and $f(u) = h(u)$. Take a point $y < u$ and find a point $z > u$ and a number $\lambda \in (0, 1)$ such that $u = \lambda y + (1-\lambda)z$. Since $f \in \mathcal{P}$ it follows that

$$c_1 + c_2 = h(u) = f(u) \leq f(y) + f(z) \leq h(y) + h(z) = c_1 + c_2.$$

Hence $f(y) + f(z) = c_1 + c_2$. Since $f(y) \leq c_1$ and $f(z) \leq c_2$ it follows that $f(y) = c_1$. In the same manner, we can show that $f(v) = c_2$ for an arbitrary point $v > u$. ∎

PROPOSITION 61. *([**139**]) H is a supremal generator of $\mathcal{P}$.*

PROOF. Let $f \in \mathcal{P}$ and $u \in \mathbb{R}$. First assume that $u \in \operatorname{dom} f$. Let $\varepsilon > 0$ and $c_1' = \inf\limits_{x<u} f(x)$, $c_2' = \inf_{x>u} f(x)$. We now check that $f(u) - 2\varepsilon \leq c_1' + c_2'$. Let points $x_1 < u$ and $x_2 > u$ be such that $f(x_1) \leq c_1' + \varepsilon$ and $f(x_2) \leq c_2' + \varepsilon$ respectively. Then $f(u) - 2\varepsilon \leq f(x_1) + f(x_2) - 2\varepsilon \leq c_1' + c_2'$. Take nonnegative numbers c_1 and c_2 such that $c_1 \leq c_1', c_2 \leq c_2'$ and $c_1 + c_2 = f(u) - 2\varepsilon$. Consider the function h_t with $t = \{u; c_1, c_2\} \in T$. It follows from the definition of t that $h_t \leq f$ and $h_t(u) = f(u) - 2\varepsilon$. Thus $f(u) = \sup\{h(u) : h \in H, h \leq f\}$ for all $u \in \operatorname{dom} f$.

Consider now a point $u \notin \operatorname{dom} f$. Assume for the sake of the definiteness that $u \leq \inf \operatorname{dom} f$. Let $t = \{u; c_1, c_2\} \in T$, where $c_2 = \inf_{x\in\mathbb{R}} f(x)$ and c_1 is an arbitrary positive number. Then $h_t \leq f$. So $f(u) = +\infty = \sup\limits_{h\in H, h\leq f} h(u)$. ∎

REMARK 79. *([**139**]) This proposition may be compared with the examples of Subsection 8.2. Indeed we can consider H as a certain mixture of two-step functions from Example 8 and pointed functions from Example 10.*

Clearly the function h_t is a function of bounded variation for each $t \in T$. Therefore h_t can be represented as the sum of increasing and decreasing functions. We now show that for each $t \in T$ the function h_t can be represented as the sum of nonnegative increasing and decreasing functions (cf. [**139**]).

PROPOSITION 62. *([**139**]) Let $t = \{u; c_1, c_2\} \in T$. Then there exist a nonnegative increasing function h_t^1 and a nonnegative decreasing function h_t^2 such that $h_t = h_t^1 + h_t^2$.*

PROOF. Let

$$h_t^1(x) = \begin{cases} 0 & x < u \\ c_2 & x \geq u \end{cases}; \qquad h_t^2(x) = \begin{cases} c_1 & x \leq u \\ 0 & x > u \end{cases}.$$

It is easy to check that $h_1^t + h_t^2 = h_t$, h_t^1 is an increasing function and h_t^2 is a decreasing function. ∎

Let $c(Q_+)$ be the cone hull of the set Q_+ of all nonnegative quasiconvex functions, that is, the set of all functions f of the form $f = q_1 + q_2$ where $q_1, q_2 \in Q_+$. It follows from Proposition 62 that $H \subset c(Q_+)$.

Let $\tilde{H}$ be the upper semilattice generated by H, that is, the set of all functions $\tilde{h}$ of the form

$$\tilde{h}(x) = \max_{i=1,\dots,m} h_i(x), \qquad h_i \in H,\ i = 1, \dots, m; \qquad m = 1, 2, \dots.$$

and $\bar{H}$ be the upper semilattice generated by $c(Q_+)$, that is, the set of all functions $\bar{h}$ of the form

$$\bar{h}(x) = \max_{i=1,\dots,m} \tilde{q}_i(x) \qquad \tilde{q}_i \in c(Q_+),\ i = 1, \dots, m; \qquad m = 1, 2, \dots.$$

PROPOSITION 63. *([**139**]) $\mathcal{P} = cl\,\tilde{H} = cl\,\bar{H}$, where $cl\,A$ is the closure of the set A in the topology of pointwise convergence.*

PROOF. Let $f \in \mathcal{P}$ and $\mathrm{s}(f, \tilde{H}) = \{\tilde{h} \in \tilde{H} : \tilde{h} \leq f\}$ be the support set of f with respect to $\tilde{H}$. We can consider $\mathrm{s}(f, \tilde{H})$ as a directed set with respect to the natural order relation: $\tilde{h}_1 \geq \tilde{h}_2$ if $\tilde{h}_1(x) \geq \tilde{h}_2(x)$ for all $x \in \mathbb{R}$. Since

$$f(x) = \sup\{h(x) : h \in \mathrm{s}(f, H)\} = \sup\{\tilde{h}(x) : \tilde{h} \in \mathrm{s}(f, \tilde{H})\} \qquad (x \in \mathbb{R})$$

and the generalized sequence $\{\tilde{h} : \tilde{h} \in \mathrm{s}(f, \tilde{H})\}$ is increasing, it follows that $f(x)$ is pointwise limit of this generalized sequence. Thus

$$\mathcal{P} \subset \mathrm{cl}\,\tilde{H}. \tag{5.121}$$

Since $H \subset c(Q_+)$ it follows that $\tilde{H} \subset \bar{H}$. As $\mathcal{P}$ is a cone and an upper semilattice and the set Q_+ is contained in $\mathcal{P}$ it follows that $\bar{H} \subset \mathcal{P}$. Since $\mathcal{P}$ is closed in the topology of pointwise convergence, it follows that $\mathrm{cl}\,\tilde{H} \subset \mathrm{cl}\,\bar{H} \subset \mathcal{P}$. The desired result follows from this inclusion and (5.121). ∎

REMARK 80. *([**139**]) We have proved that a function f belongs to $\mathcal{P}$ if and only if this function can be represented as the pointwise limit of a generalized sequence (f_α) where each f_α is a finite maximum of the functions represented as the sum of two nonnegative quasiconvex functions. It follows also from Propositions 61* *and 62 that each $f \in \mathcal{P}$ can be represented as the supremum of a family of functions belonging to $c(Q_+)$. Since $\mathcal{P}$ is a cone and a complete upper semilattice and $Q_+ \subset \mathcal{P}$, it follows that $\mathcal{P}$ coincides with the set of all functions which can be represented in such a form.*

Let $\mathcal{P}_l$ be the set of all l.s.c functions belonging to $\mathcal{P}$. We now describe a supremal generator of $\mathcal{P}_l$ consisting of continuous functions. Consider the set S of all collections $s = \{u; c_1, c_2; \delta\}$, where $\{u; c_1, c_2\} \in T$ and $\delta > 0$. For $s \in S$ define the continuous function l_s by

$$l_s(x) = c_1 \quad (x \le u-\delta), \qquad l_s(u) = c_1 + c_2, \qquad l_s(x) = c_2 \ (x \ge u+\delta),$$
$$l_s \quad \text{is affine on segments} \quad [u-\delta, u] \quad \text{and} \quad [u, u+\delta].$$

Denote the set of all functions l_s with $s \in S$ by L.

PROPOSITION 64. *([**139**]) L is a supremal generator of $\mathcal{P}_l$.*

PROOF. Let $f \in \mathcal{P}_l$ and $u \in \operatorname{dom} f$. Since f is l.s.c, it follows that for each $\varepsilon > 0$ there exists $\delta > 0$ such that $f(x) > f(u) - \varepsilon$ if $|x-u| < \delta$. There exist numbers $c_1 \le \inf_{x<u-\delta} f(x)$ and $c_2 \le \inf_{x>u+\delta} f(x)$ such that $f(u) - \varepsilon = c_1 + c_2$, for the same reasons as those given in the proof of Proposition 61. Let $s = \{u; c_1, c_2; \delta\}$. It follows from the definition of the numbers c_1, c_2 and δ that $l_s \le f$. Since $l_s(u) = f(u) - \varepsilon$ it follows that $f(u) = \sup\{l(u) : l \in L, l \le f\}$. It is easy to check that this equality also holds for points $u \notin \operatorname{dom} f$. ∎

PROPOSITION 65. *([**139**]) Each function $l \in L$ can be represented as the sum of increasing and decreasing continuous functions.*

PROOF. Let $l = l_s$ with $s = \{u; c_1, c_2; \delta\}$. A simple calculation shows that $l = l_1 + l_2$ where

$$l_1 = \begin{cases} 0 & x \le u-\delta \\ \frac{c_2}{\delta}(x-u+\delta) & x \in (u-\delta, u) \\ c_2 & x > u \end{cases}, \quad l_2 = \begin{cases} c_1 & x \le u \\ \frac{c_1}{\delta}(u+\delta-x) & x \in (u, u+\delta) \\ 0 & x > u+\delta \end{cases}.$$

It is easy to check that l_1, l_2 are continuous, l_1 is increasing and l_2 is decreasing. ∎

For the same reasons as those in the proof of Proposition 63, it follows that $\mathcal{P}_l \subset \operatorname{cl} \tilde{L} = \operatorname{cl} \bar{L}$ and $\operatorname{cl} \mathcal{P}_l = \operatorname{cl} \tilde{L} = \operatorname{cl} \bar{L}$ where $\tilde{L}$ is the set of all functions that can be presented as the maximum of a finite family of elements of L and $\bar{L}$ is the set of all functions that can be presented as the maximum of the finite family of the sum of two continuous quasiconvex functions.

8.4. Inequalities of $H.-H.$ type. We begin with the following *Principle of Preservation of Inequalities* ([**96**]).

PROPOSITION 66. *Let Y be a set of functions defined on a set X and equipped with the natural order relation. Let H be a supremal generator of Y. Further, let a be an increasing functional defined on Y and $u \in X$. Then*

$$h(u) \le a(h) \quad \textit{for all} \quad h \in H \quad \textit{if and only if} \quad f(u) \le a(f) \quad \textit{for all} \quad f \in Y.$$

PROOF. We have

$$\begin{aligned} f(u) &= \sup\{h(u) : h \in \mathrm{s}(f,H)\} \le \sup\{a(h) : h \in \mathrm{s}(f,H)\} \\ &\le a(\sup\{h : h \in \mathrm{s}(f,H)\}) = a(f). \end{aligned}$$

∎

We now establish some inequalities of $H.-H.$ type for P-functions by applying the principle of preservation of inequalities.

Consider the Borel σ-algebra Σ of subsets of the segment $[0,1]$ and a measure μ, that is, a nonnegative σ-additive function defined on Σ. Assume that $\mu([0,1])=1$. Let $\mathcal{P}_o$ be the set of all measurable (with respect to Σ) functions $f\in\mathcal{P}$ such that $\operatorname{dom} f=[0,1]$. Since H consists of Borel- measurable functions on $\mathbb{R}$ and H is a supremal generator of $\mathcal{P}$, it follows that H is a supremal generator of $\mathcal{P}_o$ as well. Let $I:\mathcal{P}_o\to[0,+\infty]$ be the functional defined by

$$I(f)=\int_0^1 f d\mu.$$

For $y\in[0,1)$ consider the functions

$$e_y^1(x)=\begin{cases}1 & x\le y\\ 0 & x>y\end{cases};\qquad e_y^2(x)=\begin{cases}0 & x\le y\\ 1 & x>y.\end{cases}$$

Let

$$g_1(y)=\int_0^1 e_y^1 d\mu=\mu([0,y]),\qquad g_2(y)=\int_0^1 e_y^2 d\mu=\mu((y,1]),\qquad y\in[0,1).$$

By definition, set $g_1(1)=1$, $g_2(1)=0$. Clearly g_1 is increasing, g_2 is decreasing and $g_1(y)+g_2(y)=\mu([0,1])=1$ for all $y\in[0,1]$.

Let us calculate $I(h_t)$ for $h_t\in H$.

Lemma 17. *([**139**]) Let μ be an atomless measure, that is, $\mu(\{x\})=0$ for each $x\in[0,1]$. Let $h_t\in H$ be the function corresponding to a collection $t=\{y;c_1,c_2\}$. Then $I(h_t)=c_1g_1(y)+c_2g_2(y)$.*

Proof. Consider the function $\tilde{h}_t$ given by

$$\tilde{h}_t(x)=\begin{cases}c_1 & x\le y\\ c_2 & x>y\end{cases}=\begin{cases}c_1e_y^1(x) & x\le y\\ c_2e_y^2(x) & x>y\end{cases}.$$

Since μ is atomless, it follows that $I(h_t)=I(\tilde{h}_t)$. We have

$$I(h_t)=\int_0^1 h_t d\mu=\int_0^1\tilde{h}_t d\mu=\int_0^y c_1e_y^1 d\mu+\int_y^1 c_2e_y^2 d\mu=c_1g_1(y)+c_2g_2(y).$$

∎

For $u\in(0,1)$, consider the number

$$\gamma_u=\min_{c_1\ge0,\,c_2\ge0}\frac{c_1g_1(u)+c_2g_2(u)}{c_1+c_2}. \tag{5.122}$$

It easy to check that

$$\gamma_u=\min(g_1(u),g_2(u)). \tag{5.123}$$

Indeed if $g_1(u)\ge g_2(u)$, then $\gamma_u=g_2(u)$ and if $g_1(u)\le g_2(u)$, then $\gamma_u=g_1(u)$. Thus (5.123) holds.

Theorem 163. *([**139**]) Let μ be an atomless measure and $u\in(0,1)$. Then*

$$f(u)\le\frac{1}{\min(g_1(u),g_2(u))}\int_0^1 f d\mu \tag{5.124}$$

for all $f\in\mathcal{P}_o$.

PROOF. Clearly I is an increasing functional defined on the set $\mathcal{P}_o$. First we check that (5.124) holds for all $h \in H$. We consider separately functions h_t which are defined by collections $t = (u; c_1, c_2)$ and by collections $t = (y; c_1, c_2)$ with $y \neq u$.

Let $t = \{u; c_1, c_2\}$. It follows directly from (5.122), (5.123), Lemma 17 and the equality $h_t(u) = c_1 + c_2$ that

$$\begin{aligned} h_t(u) &= c_1 + c_2 \leq \frac{1}{\gamma_u}(c_1 g_1(u) + c_2 g_2(u)) \\ &= \frac{1}{\gamma_u}\int_0^1 h_t d\mu = \frac{1}{\min(g_1(u), g_2(u))}\int_0^1 h_t d\mu. \end{aligned} \tag{5.125}$$

Assume now that $t = \{y; c_1, c_2\}$ with $y \neq u$. It follows from Lemma 17 that

$$I(h_t) = \int_0^1 h_t(x)dx = c_1 g_1(y) + c_2 g_2(y).$$

Let us calculate $h_t(u)$. Since $y \neq u$ it follows that either $h_t(u) = \min(c_1, c_2)$ or $h_t(u) = \max(c_1, c_2)$. In the first case we have, taking into account that $g_1(y) \geq 0$, $g_2(y) \geq 0$ and $g_1(y) + g_2(y) = 1$, that

$$h_t(u) = \min(c_1, c_2) \leq c_1 g_1(y) + c_2 g_2(y) = I(h_t). \tag{5.126}$$

Since $g_1(u) \leq 1$, $g_2(u) \leq 1$, it follows that

$$h_t(u) \leq I(h_t) \leq \frac{1}{\min(g_1(u), g_2(u))} I(h_t).$$

Assume now that $h_t(u) = \max(c_1, c_2)$. If $c_1 \geq c_2$ then $h_t(u) = c_1$ and $y > u$, the latter following directly from the definition of the function h_t. Since g_1 is an increasing function, we have for $y > u$ that

$$g_1(u)h_t(u) \leq g_1(y)c_1 \leq g_1(y)c_1 + g_2(y)c_2 = I(h_t). \tag{5.127}$$

Thus

$$h_t(u) \leq \frac{1}{g_1(u)} I(h_t) \leq \frac{1}{\min(g_1(u), g_2(u))} I(h_t).$$

If $c_2 \geq c_1$, then $h_t(u) = c_2$ and $y < u$. In the same manner we have

$$h_t(u) \leq \frac{1}{g_2(u)} I(h_t) \leq \frac{1}{\min(g_1(u), g_2(u))} I(h_t). \tag{5.128}$$

Thus we have verified that the desired inequality (5.124) holds for all $h \in H$. Since H is a supremal generator of $\mathcal{P}_o$ and I is an increasing functional, we can conclude, by applying the principle of preservation of inequalities, that (5.124) holds for all $f \in \mathcal{P}_o$. ∎

REMARK 81. *([**139**]) Let $t = \{u; c_1, c_2\}$, where $\gamma_u(c_1 + c_2) = c_1 g_1(u) + c_2 g_2(u)$. It follows from (5.125) that the equality*

$$h_t(u) = \frac{1}{\min(g_1(u), g_2(u))}$$

holds. Thus the inequality (5.124) cannot be improved for all P- functions.

Let us give an example.

EXAMPLE 12. *([**139**]) Let μ be the Lebesgue measure, that is, $I(f)=\int_0^1 f(x)dx$. Then $g_1(y)=y$, $g_2(y)=1-y$. It follows from Theorem 163 that*

$$f(u)\leq \frac{1}{\min(u,1-u)}\int_0^1 f(x)dx$$

for all $f\in\mathcal{P}_o$. In particular we have

$$f\left(\frac{1}{2}\right)\leq 2\int_0^1 f(x)dx \qquad (f\in\mathcal{P}_o). \tag{5.129}$$

This result was established in [**67**].

REMARK 82. *([**139**]) In a similar manner we can prove that (5.124) holds for each nonnegative quasiconvex function and for each (not necessarily atomless) nonnegative measure μ such that $\mu([0,1])=1$.*

REMARK 83. *([**139**]) We can use for this purpose the supremal generator described in Example 8. Indeed the atomlessness of the measure μ has been used only for eliminating the special value of the function h_t with $t=\{u;c_1,c_2\}$ at the point u. We do not need to eliminate this value in the case under consideration.*

9. Convexity According to the Geometric Mean

9.1. Introduction. The usual definition of a convex function (of one variable) depends on the structure of $\mathbb{R}$ as an ordered vectorial space. As $\mathbb{R}$ is actually an ordered field, it is natural to ask what happens when addition is replaced by multiplication and the arithmetic mean is replaced by the geometric mean. A moment's reflection reveals an entire new world of beautiful inequalities, involving a broad range of functions from the elementary ones, such as sin, cos, exp, to the special ones, such as Γ, Psi, L (Lobacevski's function), Si (the integral sine), etc. (cf. [**128**]).

Depending on which type of mean, arithmetic (A), or geometric (G), we consider respectively on the domain and the codomain of definition, we shall encounter one of the following four classes of functions [**128**]:

AA – *convex functions*, the usual convex functions
AG – *convex functions*
GA – *convex functions*
GG – *convex functions.*

It is worth noticing that while (A) makes no restriction about the interval I where it applies (it is so because $x,y\in I$, $\lambda\in[0,1]$ implies that $(1-\lambda)x+\lambda y\in I$), the use of ($G$) forces us to restrict to the subintervals J of $(0,\infty)$ in order to assure that

$$x,y\in J,\ \lambda\in[0,1]\Longrightarrow x^{1-\lambda}y^{\lambda}\in J.$$

To be more specific, the $AG-$*convex functions* (usually known as $\log-$*convex functions*) are those functions $f:I\in(0,\infty)$ for which

$$x,y\in J,\ \lambda\in[0,1]\Longrightarrow f((1-\lambda)x+\lambda y)\leq f(x)^{1-\lambda}f(y)^{\lambda}, \tag{AG}$$

i.e., for which $\log f$ is convex.

The $GG-$*convex functions* (called in what follows *multiplicatively convex functions*) are those functions $f:I\to J$ (acting on subintervals of $(0,\infty)$) such that (see [**128**])

$$(GG) \qquad x,y\in J,\ \lambda\in[0,1]\Longrightarrow f\left(x^{1-\lambda}y^{\lambda}\right)\leq f\left(x\right)^{1-\lambda}f\left(y\right)^{\lambda}.$$

Due to the following form of the $AM-GM$ Inequality,

$$a,b\in(0,\infty)\,,\ \lambda\in[0,1]\Longrightarrow a^{1-\lambda}b^{\lambda}\leq(1-\lambda)\,a+\lambda b, \tag{5.130}$$

every log −convex function is also convex. The most notable example of such a functions is Euler's *gamma function,*

$$\Gamma\left(x\right)=\int_0^\infty t^{x-1}e^{-t}dt,\ \ x>0.$$

In fact,

$$\frac{d^2}{dx^2}\log\Gamma\left(x\right)=\sum_{n=0}^{\infty}\frac{1}{\left(x+n\right)^2}\ \text{ for }\ x>0\ \ (\text{see } [\mathbf{185}]).$$

As noticed by H. Bohr and J. Mollerup ([**10**], see also [**4**]), the gamma function is the only function $f:(0,\infty)\to(0,\infty)$ with the following three properties:

(Γ1) f is log −convex;
(Γ2) $f\left(x+1\right)=xf\left(x\right)$ for every $x>0$;
(Γ3) $f\left(n+1\right)=n!$ for every $n\in\mathbb{N}$.

The class of all $GA-$*convex* functions is constituted by all functions $f:I\to\mathbb{R}$ (defined on subintervals of $(0,\infty)$) for which

$$(GA) \qquad x,y\in I\text{ and }\lambda\in[0,1]\Longrightarrow f\left(x^{1-\lambda}y^{\lambda}\right)\leq(1-\lambda)\,f\left(x\right)+\lambda f\left(y\right).$$

In the context of twice differentiable functions $f:I\to\mathbb{R}$, $GA-$convexity means $x^2f''+xf'\geq0$, so that all twice differentiable nondecreasing convex functions are also $GA-$convex. Notice that the inequality (5.130) above is of this nature.

The aim of this section, following [**128**], is to investigate the class of multiplicatively convex functions as a source of inequalities. We shall develop a parallel to the classical theory of convex functions based on the following remark, which relates the two classes of functions:

Suppose that I is a subinterval of $(0,\infty)$ and $f:I\to(0,\infty)$ is a multiplicatively convex function. Then (see [**128**]*)*

$$F=\log\circ f\circ\exp:\log\left(I\right)\to\mathbb{R}$$

is a convex function. Conversely, if J is an interval (for which $\exp(J)$ is a subinterval of $(0,\infty)$) and *$F:J\to\mathbb{R}$ is a convex function, then (see* [**128**]*)*

$$f=\exp\circ F\circ\log:\exp\left(J\right)\to(0,\infty)$$

is a convex function.

Equivalently, f is multiplicatively convex if and only if, $\log f\left(x\right)$ is a convex function of $\log x$. See Lemma 18 below. Modulo this characterisation, the class of all multiplicatively convex functions was first considered by P. Montel [**117**], in a well written paper discussing the analogues of the notion of convex functions in n variables. However, the roots of the research in this area can be traced back to long before his time. Let us mention two such results here (see also [**128**]).

THEOREM 164 (Hadamard's Three Circles Theorem). *Let f be an analytical function in the annulus $a < |z| < b$. Then $\log M(r)$ is a convex function of $\log r$, where*

$$M(r) = \sup_{|z|=r} |f(z)|.$$

THEOREM 165 (G.H. Hardy's Mean Value Theorem). *Let f be an analytical function in the annulus $a < |z| < b$ and let $p \in [1, \infty)$. Then $\log M_p(r)$ is a convex function of $\log r$, where*

$$M_p(r) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^p d\theta \right)^{\frac{1}{p}}.$$

As $\lim_{n\to\infty} M_n(r) = M(r)$, Hardy's aforementioned result implies Hadamard's. As is well known, Hadamard's result is instrumental in deriving the celebrated Riesz-Thorin Interpolation Theorem (see [**84**]).

Books like those of Hardy, Littlewood and Polya [**84**] and A.W. Roberts and D.E. Varberg [**158**] make some peripheric references to the functions f for which $\log f(x)$ is a convex function of $\log x$. Nowadays, the subject of multiplicative convexity seems to be even forgotten, which is a pity because of its richness. What we attempt to do in this section is not only to call attention to the abundance of beautiful inequalities falling in the realm of multiplicative convexity, but also to prove that many classical inequalities such as the $AM - GM$ Inequality can benefit from a better understanding via the multiplicative approach of convexity.

9.2. Generalities on Multiplicatively Convex Functions. The class of multiplicatively convex functions can be easily described as being constituted by those functions f (acting on subintervals of $(0, \infty)$) such that $\log f(x)$ is a convex function of $\log x$ (see [**128**]):

LEMMA 18. *Suppose that I is a subinterval of $(0, \infty)$. A function $f : I \to (0, \infty)$ is multiplicatively convex if and only if:*

$$\begin{vmatrix} 1 & \log x_1 & \log f(x_1) \\ 1 & \log x_2 & \log f(x_2) \\ 1 & \log x_3 & \log f(x_3) \end{vmatrix} \geq 0 \tag{5.131}$$

for every $x_1 \leq x_2 \leq x_3$ in I; equivalently, if and only if:

$$f(x_1)^{\log x_3} f(x_2)^{\log x_1} f(x_3)^{\log x_2} \geq f(x_1)^{\log x_2} f(x_2)^{\log x_3} f(x_3)^{\log x_1} \tag{5.132}$$

for every $x_1 \leq x_2 \leq x_3$ in I.

PROOF. The proof follows directly from the definition of multiplicative convexity, taking logarithms and noticing that any point between x_1 and x_3 is of the form $x_1^{1-\lambda} x_3^{\lambda}$, for some $\lambda \in (0, 1)$. ∎

COROLLARY 59. *([**128**]) Every multiplicatively convex function $f : I \to (0, \infty)$ has finite lateral derivatives at each interior point of I. Moreover, the set of all points where f is not differentiable is at most countable.*

An example of a multiplicatively convex function which is not differentiable at countably many points is

$$\exp\left(\sum_{n=0}^{\infty}\frac{|\log x-n|}{2^n}\right).$$

By Corollary 59, every multiplicatively convex function is continuous in the interior of its domain of definition. Under the presence of continuity, the multiplicative convexity can be restated in terms of the geometric mean:

THEOREM 166. *([**128**]) Suppose that I is a subinterval of $(0,\infty)$. A continuous function $f: I \to [0,\infty)$ is multiplicatively convex if and only if:*

$$x, y \in I \Longrightarrow f\left(\sqrt{xy}\right) \leq \sqrt{f(x)\, f(y)}. \tag{5.133}$$

PROOF. The necessity is clear. The sufficiency part follows from the connection between the multiplicative convexity and the usual convexity (as noticed in the Introduction of this section) and the well known fact that mid-convexity (i.e., Jensen convexity) is equivalent to convexity under the presence of continuity. See [**84**]. ∎

Theorem 166 above reveals the essence of multiplicative convexity as being the *convexity according to the geometric mean;* in fact, under the presence of continuity, the multiplicatively convex functions are precisely those functions $f: I \to [0,\infty)$ for which

$$x_1,\ldots,x_n \in I \Longrightarrow f\left(\sqrt[n]{x_1\ldots x_n}\right) \leq \sqrt[n]{f(x_1)\ldots f(x_n)}. \tag{5.134}$$

In this respect, it is natural to say that a function $f: I \to (0,\infty)$ is *multiplicatively concave* if $\frac{1}{f}$ is multiplicative convex and *multiplicatively affine* if f is of the form Cx^{α} for some $C>0$ and some $\alpha \in \mathbb{R}$.

A refinement of the notion of multiplicative convexity is that of *strict multiplicative convexity*, which in the context of continuity will mean

$$f\left(\sqrt[n]{x_1\ldots x_n}\right) < \sqrt[n]{f(x_1)\ldots f(x_n)}$$

unless $x_1 = \cdots = x_n$. Clearly, our remark concerning the connection between the multiplicatively convex functions and the usual convex functions has a "strict" counterpart.

A large class of strictly multiplicatively convex functions, is indicated by the following result, which developed from [**84**], Theorem 177, page 125:

PROPOSITION 67. *([**128**]) Every polynomial $P(x)$ with nonnegative coefficients is a multiplicatively convex function on $[0,\infty)$. More generally, every real analytic function $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with nonnegative coefficients is a multiplicatively convex functions on $(0,R)$, where R denotes the radius of convergence.*
Moreover, except for the case of functions Cx^n (with $C>0$ and $n\in\mathbb{N}$), the above examples are strictly multiplicatively convex functions.

PROOF. By continuity, it suffices to prove only the first assertion. For, suppose that $P(x) = \sum_{n=0}^{N} c_n x^n$. According the Theorem 166, we have to prove that

$$x, y > 0 \Longrightarrow \left(P\left(\sqrt{xy}\right)\right)^2 \leq P(x)\, P(y),$$

equivalently,

$$x, y > 0 \Longrightarrow \left(P(xy)\right)^2 \leq P\left(x^2\right) P\left(y^2\right).$$

Or, the latter is an easy consequence of the Cauchy-Schwartz inequality. ∎

Examples of such real analytic functions are:

$$\begin{aligned}&\exp,\ \sinh,\ \cosh\ \text{ on }\ (0,\infty)\\ &\tan,\ \sec,\ \csc,\ \frac{1}{x}-\cot x\ \text{ on }\ \left(0,\frac{\pi}{2}\right)\\ &\arcsin\ \text{ on }(0,1]\\ &-\log\left(1-x\right),\ \frac{1+x}{1-x}\ \text{ on }\ (0,1)\,.\end{aligned}$$

See the table of series of I.S. Gradshteyn and I.M. Ryzhik [**82**].

REMARK 84. i) *([**128**]) If a function f is multiplicatively convex, then so is $x^{\alpha}f^{\beta}(x)$ (for all $\alpha\in\mathbb{R}$ and all $\beta>0$).*

ii) *If f is continuous, and one of the functions $[f(x)]^{x}$ and $f\left(e^{\frac{1}{\log x}}\right)$ is multiplicatively convex, then so is the other.*

REMARK 85. *([**128**]) S. Saks [**162**] noticed that for a continuous function $f: I\to(0,\infty)$, $\log f(x)$ is a convex function of $\log f$ if and only if for every $\alpha>0$ and every compact subinterval J of I, $x^{\alpha}f(x)$ should attain its maximum in J at one of the ends of J.*

APPLICATIONS 1. *([**128**]) Proposition 67 is the source of many interesting inequalities. Here are several elementary examples, obtained via Theorem 166:*

a) *(See D. Mihet [**110**]). If P is a polynomial with nonnegative coefficients then*

$$P(x_1)\dots P(x_n)\geq P\left(\sqrt[n]{x_1\dots x_n}\right)^n \text{ for every } x_1,\dots,x_n\geq 0.$$

*This inequality extends the classical inequality of Huygens (which corresponds to the case where $P(x)=1+x$) and complements a remark made by C.H. Kimberling [**92**] to Chebyshev's inequality, namely,*

$$(P(1))^{n-1}P(x_1\dots x_n)\geq P(x_1)\dots P(x_n)$$

if all x_k are either in $[0,1]$ or in $[1,\infty)$.
A similar conclusion is valid for every real analytic function as in Proposition 67 above.

b) *The $AM-GM$ Inequality is an easy consequence of the strict multiplicative convexity of e^{x} on $[0,\infty)$. A strengthened version of this will be presented in Subsection 9.5 below.*

c) *Because $\frac{1+x}{1-x}$ is strictly multiplicatively convex on $(0,1)$,*

$$\prod_{k=1}^{n}\frac{1+x_k}{1-x_k}>\left(\frac{1+\left(\prod x_k\right)^{\frac{1}{n}}}{1-\left(\prod x_k\right)^{\frac{1}{n}}}\right)^n \text{ for every } x_1,\dots,x_n\in[0,1)$$

unless $x_1=\dots=x_n$.

d) *Because arcsin is a strictly multiplicatively convex function on $(0,1]$, in any triangle (with the exception of equilaterals) the following inequality*

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}<\left(\sin\left(\frac{1}{2}\sqrt[3]{ABC}\right)\right)^3$$

holds. That improves on a well known fact, namely

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}<\frac{1}{8}$$

unless $A = B = C$ (which is a consequence of the strict log-concavity of the function sine). In a similar way, one can argue that

$$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} < \left(\sin\left(\frac{1}{2}\sqrt[3]{(\pi - A)(\pi - B)(\pi - C)}\right)\right)^3$$

unless $A = B = C$.

e) *As tan is a strictly multiplicatively convex function on $\left(0, \frac{\pi}{2}\right)$, in any triangle we have*

$$\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} > \left(\tan\left(\frac{1}{2}\sqrt[3]{ABC}\right)\right)^3$$

unless $A = B = C$.

The next example provides an application of Proposition 67 via Lemma 18:

f) *If $0 < a < b < c$ (or $0 < b < c < a$, or $0 < c < a < b$), then*

$$P(a)^{\log c} P(b)^{\log a} P(c)^{\log b} > P(a)^{\log b} P(b)^{\log c} P(c)^{\log a}$$

for every polynomial P with nonnegative coefficients and positive degree (and, more generally, for every strictly multiplicatively convex function). That complements the conclusion of the standard rearrangement inequalities (cf. [**71**, p. 167]*): If $0 < a < b < c$, and $P > 0$, then*

$$\begin{aligned}
P(a)^{\log c} P(b)^{\log b} P(c)^{\log a} &= \inf_{\sigma}\left[P(a)^{\log\sigma(a)} P(b)^{\log\sigma(b)} P(c)^{\log\sigma(c)}\right], \\
P(a)^{\log a} P(b)^{\log b} P(c)^{\log c} &= \sup_{\sigma}\left[P(a)^{\log\sigma(a)} P(b)^{\log\sigma(b)} P(c)^{\log\sigma(c)}\right],
\end{aligned}$$

where σ runs the set of all permutations of $\{a, b, c\}$.

The integral characterization of multiplicatively convex functions is another source of inequalities. We leave the (straightforward) details to the interested reader.

9.3. The Analogue of Popoviciu's Inequality. The technique of majorisation, which dominates the classical study of convex functions, can be easily adapted in the context of multiplicatively convex functions via the correspondence between two classes of functions. Here we shall restrict ourselves to the multiplicative analogue of a famous inequality due to Hardy, Littlewood and Polya [**84**]:

PROPOSITION 68. *(*[**128**]*) Suppose that $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ are two families of numbers in a subinterval I of $(0, \infty)$ such that*

$$\begin{aligned}
x_1 &\geq y_1 \\
x_1 x_2 &\geq y_1 y_2 \\
&\cdots \\
x_1 x_2 \ldots x_{n-1} &\geq y_1 y_2 \ldots y_{n-1} \\
x_1 x_2 \ldots x_n &\geq y_1 y_2 \ldots y_n.
\end{aligned}$$

Then

$$f(x_1) f(x_2) \ldots f(x_n) \geq f(y_1) f(y_2) \ldots f(y_n) \tag{5.135}$$

for every multiplicatively convex function $f : I \to (0, \infty)$.

A result due to H. Weyl [**184**] (see also [**106**, p. 231]) gives us the basic example of a pair of sequences satisfying the hypothesis of Proposition 68: *Given any matrix $A \in M_n(\mathbb{C})$ having the eigenvalues $\lambda_1, \dots, \lambda_n$ and the singular values $s_1, \dots, s_n$, they can be rearranged such that*

$$\begin{aligned} |\lambda_1| &\geq \cdots \geq |\lambda_n|, \ s_1 \geq \cdots \geq s_n \\ \left|\prod_{k=1}^{m} \lambda_k\right| &\leq \prod_{k=1}^{m} s_k \ \text{ for } \ k = 1, \dots, n-1 \ \text{ and } \left|\prod_{k=1}^{n} \lambda_k\right| = \prod_{k=1}^{n} s_k. \end{aligned}$$

Recall that the *singular values* of A are precisely the eigenvalues of its modulus, $|A| = (A^*A)^{\frac{1}{2}}$. The spectral mapping theorem assures that $s_k = |\lambda_k|$ when A is self-adjoint. One could suppose that for an arbitrary matrix, $|\lambda_k| \leq s_k$ for all k. However, this is not true. A counterexample is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$$

whose eigenvalues are $\lambda_1 = 2 > \lambda_2 = -1$ and the singular values are $s_1 = 4 > s_2 = 1$.

As noticed by A. Horn [**86**] (see also [**106**, p. 233]), the converse of Weyl's aforementioned result is also true, i.e., all the families of numbers which fulfill the hypotheses of Proposition 68 are derived in that manner.

According to the above discussion, the following result holds:

PROPOSITION 69. *([**128**]) Let $A \in M_n(\mathbb{C})$ be any matrix having the eigenvalues $\lambda_1, \dots, \lambda_n$ and the singular values $s_1, \dots, s_n$, listed such that $|\lambda_1| \geq \cdots \geq [\lambda_n]$ and $s_1 \geq \cdots \geq s_n$. Then*

$$\prod_{k=1}^{n} f(s_k) \geq \prod_{k=1}^{n} f(|\lambda_k|)$$

for every multiplicatively convex function f which is continuous on $[0, \infty)$.

We shall give another application of Proposition 68, which seems to be new even for polynomials with nonnegative coefficients (see also [**128**]).

THEOREM 167. *(The multiplicative analogue of Popoviciu's Inequality [**150**]). Suppose that $f : I \to (0, \infty)$ is a multiplicatively convex function. Then*

$$f(x) f(y) f(z) f^3\left(\sqrt[3]{xyz}\right) \geq f^2\left(\sqrt{xy}\right) f^2\left(\sqrt{yz}\right) f^2\left(\sqrt{zx}\right) \tag{5.136}$$

for every $x, y, z \in I$. Moreover, for the strictly multiplicatively convex functions the equality occurs only when $x = y = z$.

PROOF. Without loss of generality we may assume that $x \geq y \geq z$. Then

$$\sqrt{xy} \geq \sqrt{zx} \geq \sqrt{yz} \ \text{ and } \ x \geq \sqrt[3]{xyz} \geq z.$$

If $x \geq \sqrt[3]{xyz} \geq y \geq z$, the desired conclusion follows from Proposition 68 applied to

$$\begin{aligned} x_1 &= x, \ x_2 = x_3 = x_4 = \sqrt[3]{xyz}, \ x_5 = y, \ x_6 = z \\ y_1 &= y_2 = \sqrt{xy}, \ y_3 = y_4 = \sqrt{xz}, \ y_5 = y_6 = \sqrt{yz} \end{aligned}$$

while in the case $x \geq y \geq \sqrt[3]{xyz} \geq z$, we have to consider

$$\begin{aligned} x_1 &= x, \ x_2 = y, \ x_3 = x_4 = x_5 = \sqrt[3]{xyz}, \ x_6 = z \\ y_1 &= y_2 = \sqrt{xy}, \ y_3 = y_4 = \sqrt{xz}, \ y_5 = y_6 = \sqrt{yz}. \end{aligned}$$

∎

According to Theorem 167 (applied to $f(x) = e^x$), for every $x, y, z > 0$ we have

$$\frac{x+y+z}{3} + \sqrt[3]{xyz} > \frac{2}{3}\left(\sqrt{xy} + \sqrt{yz} + \sqrt{zx}\right)$$

unless $x = y = z$.

9.4. Multiplicative Convexity of Special Functions . We begin this subsection by recalling the following result:

Proposition 70. *(P. Montel [**117**]) Let $f : [0, a) \to [0, \infty)$ be a continuous function, which is multiplicatively convex on $(0, a)$. Then*

$$F(x) = \int_0^x f(t)\, dt$$

is also continuous on $[0, a)$ and multiplicatively convex on $(0, a)$.

Proof. Montel's original argument was based on the fact that under the presence of continuity, f is multiplicatively convex if and only if:

$$2f(x) \le k^{\alpha} f(kx) + k^{-\alpha} f\left(\frac{x}{k}\right),$$

for every $x \in I$ and every $k > 0$ such that kx and $\frac{x}{k}$ both belong to I.

Actually, due to the continuity of F, it suffices to show that

$$\left(F\left(\sqrt{xy}\right)\right)^2 \le F(x)\, F(y) \quad \text{for every} \quad x, y \in [0, a),$$

which is a consequence of the corresponding inequality at the level if integral sums,

$$\left[\frac{\sqrt{xy}}{n}\sum_{k=0}^{n-1} f\left(k\frac{\sqrt{xy}}{n}\right)\right]^2 \le \left[\frac{x}{n}\sum_{k=0}^{n-1} f\left(k\frac{x}{n}\right)\right]\left[\frac{y}{n}\sum_{k=0}^{n-1} f\left(k\frac{y}{n}\right)\right],$$

i.e., of

$$\left[\sum_{k=0}^{n-1} f\left(k\frac{\sqrt{xy}}{n}\right)\right]^2 \le \left[\sum_{k=0}^{n-1} f\left(k\frac{x}{n}\right)\right]\left[\sum_{k=0}^{n-1} f\left(k\frac{y}{n}\right)\right].$$

To see that the latter inequality holds, notice that

$$\left[f\left(k\frac{\sqrt{xy}}{n}\right)\right]^2 \le \left[f\left(k\frac{x}{n}\right)\right]\left[f\left(k\frac{y}{n}\right)\right]$$

and then apply the Cauchy-Schwartz inequality. ∎

As tan is continuous on $\left[0, \frac{\pi}{2}\right)$ and multiplicatively convex on $\left(0, \frac{\pi}{2}\right)$, a repeated application of Proposition 70 shows us that the *Lobacevski function,*

$$L(x) = -\int_0^x \log\cos t\, dt$$

is multiplicatively convex on $\left(0, \frac{\pi}{2}\right)$.

Starting with $\frac{t}{\sin t}$ and then switching to $\frac{\sin t}{t}$, which is multiplicatively concave, a similar argument leads us to the fact that the *integral sine,*

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t} dt,$$

is multiplicatively concave on $\left(0, \frac{\pi}{2}\right)$.

Another striking example is the following.

Proposition 71. *([**128**]) Γ is a strictly multiplicatively convex function on* $[1,\infty)$.

Proof. In fact, $\log\Gamma(1+x)$ is strictly convex and increasing on $(1,\infty)$. Or, an increasing strictly convex function of a strictly convex function is also strictly convex. Thus, $F(x)=\log\Gamma(1+e^x)$ is strictly convex on $(0,\infty)$ and hence

$$\Gamma(1+x)=e^{F(\log x)}$$

is strictly multiplicatively convex on $[1,\infty)$. As $\Gamma(1+x)=x\Gamma(x)$, we conclude that Γ itself is strictly multiplicatively convex on $[1,\infty)$. ∎

According to Proposition 71,

$$\Gamma^3\left(\sqrt[3]{xyz}\right)<\Gamma(x)\Gamma(y)\Gamma(z)\quad\text{for every } x,y,z\geq 1$$

except the case where $x=y=z$.

On the other hand, by Theorem 167, we infer that:

$$\Gamma(x)\Gamma(y)\Gamma(z)\Gamma^3\left(\sqrt[3]{xyz}\right)\geq\Gamma^2\left(\sqrt{xy}\right)\Gamma^2\left(\sqrt{yz}\right)\Gamma^2\left(\sqrt{zx}\right)$$

for every $x,y,z\geq 1$; the equality occurs only for $x=y=z$.

Another applications of Proposition 71 is the fact that the function $\frac{\Gamma(2x+1)}{\Gamma(x+1)}$ is strictly multiplicatively convex on $[1,\infty)$. In fact, it suffices to recall the *Gauss-Legendre duplication formula,*

$$\frac{\Gamma(2x+1)}{\Gamma(x+1)}=\frac{2^{2x}\Gamma\left(x+\frac{1}{2}\right)}{\sqrt{\pi}}.$$

In order to present further inequalities involving the gamma function we shall need the following criteria of multiplicative convexity for differentiable functions.

Proposition 72. *([**128**]) Let $f:I\to(0,\infty)$ be a differentiable function defined on a subinterval of $(0,\infty)$. Then the following assertions are equivalent:*

i) *f is multiplicatively convex;*
ii) *The function $\frac{xf'(x)}{f(x)}$ is nondecreasing;*
iii) *f verifies the inequality*

$$\frac{f(x)}{f(y)}\geq\left(\frac{x}{y}\right)^{y\cdot\frac{f'y}{f(y)}}\quad\textit{for every } x,y\in I. \tag{5.137}$$

Moreover, if f is twice differentiable, then f is multiplicatively convex if and only if

$$x\left[f(x)f''(x)-f'^2(x)\right]+f(x)f'(x)\geq 0\quad\textit{for every } x>0. \tag{5.138}$$

The corresponding variants for the strictly multiplicatively convex functions also work.

Proof. As a matter of fact, according to a remark in the Introduction of [**128**], a function $f:I\to(0,\infty)$ is multiplicatively convex if and only if the function $F:\log(I)\to\mathbb{R}$, $F(x)=\log f(e^x)$ is convex. Taking into account that the differentiability is preserved under the above correspondence, the statement to be proved is simply a translation of the usual criteria of convexity (as known in the differentiability framework) into criteria of multiplicative convexity. ∎

Directly related to the gamma function is the *psi function,*

$$Psi\,(x) = \frac{d}{dx}\log\Gamma\,(x) = \frac{\Gamma'\,(x)}{\Gamma\,(x)},\ x > 0$$

also known as the digamma function. It satisfies the functional equation $\psi\,(x+1) = \psi\,(x) + \frac{1}{x}$ and can also be represented as

$$Psi\,(x) = -\gamma - \int_0^1 \frac{t^{x-1}-1}{1-t}dt,$$

where $\psi = 0.5772$ is Euler's constant. See [**4**].

By combining Propositions 71 and 72 above, we obtain the inequality:

$$\frac{\Gamma\,(x)}{\Gamma\,(y)} \geq \left(\frac{x}{y}\right)^{y\cdot Psi(y)} \quad \text{for every } x, y \geq 1,$$

as well as the fact that $x\mathrm{Psi}(x)$ is increasing for $x \geq 1$.

The latter inequality can be used to estimate Γ from below on $[1,2]$. The interest comes from the fact that Γ is convex and attains its global minimum in that interval because $\Gamma\,(1) = \Gamma\,(2)$; more precisely, the minimum is attained near 1.46. Taking $y = 1$ and then $y = \frac{3}{2}$ in (Psi), we get

$$\Gamma\,(x) \geq \max\left\{x^{-\gamma}, \frac{1}{2}\sqrt{\pi}\left(\frac{2x}{3}\right)^{\frac{3}{2}(2-\gamma-2\ln 2)}\right\} \quad \text{for every } x \in [1,2]\,.$$

9.5. An Estimate of the $AM-GM$ **Inequality.** Suppose that I is a subinterval of $(0,\infty)$ and that $f : I \to (0,\infty)$ is a twice differentiable function. We are interested in determining the values for which $\alpha \in \mathbb{R}$ the function

$$\varphi\,(x) = f\,(x)\cdot x^{\left(-\frac{\alpha}{2}\right)\log x}$$

is multiplicatively convex on I, or equivalently, for what values $\alpha \in \mathbb{R}$ the function

$$\Phi\,(x) = \log\varphi\,(e^x) = \log f\,(e^x) - \frac{\alpha x^2}{2},$$

is convex on $\log\,(I)$. By using the fact that the convexity of a twice differentiable function Φ is equivalent to $\Phi'' \geq 0$, we get a quick answer to the aforementioned problem:

$$\alpha \leq A\,(f)\,,$$

where

$$\begin{aligned} A\,(f) &= \inf_{x\in\log(I)} \frac{d^2}{dx^2}\log f\,(e^x) \\ &= \inf_{x\in\log(I)} \frac{x^2\left[f\,(x)\,f''\,(x) - \left(f'\,(x)\right)^2\right] + xf\,(x)\,f'\,(x)}{\left(f\,(x)\right)^2}. \end{aligned}$$

By considering also

$$B\,(f) = \sup_{x\in\log(I)} \frac{d^2}{dx^2}\log f\,(e^x)\,,$$

we arrive at the following result: *Under the above hypotheses,*

$$\begin{aligned}\exp\left(\frac{A(f)}{2n^2}\sum_{j<k}(\log x_j - \log x_k)\right)^2 &\leq \frac{\left(\prod_{k=1}^{n} f(x_k)\right)^{\frac{1}{n}}}{f\left(\left(\prod_{k=1}^{n} x_k\right)^{\frac{1}{n}}\right)} \\ &\leq \exp\left(\frac{B(f)}{2n^2}\sum_{j<k}(\log x_j - \log x_k)\right)^2\end{aligned}$$

for every $x_1, \ldots, x_n \in I$.

In particular, for $f(x) = e^x$, $x \in [A, B]$ (where $0 < A \leq B$), we have $A(f) = A$ and $B(f) = B$ and we are led to the following improvement upon the $AM - GM$ Inequality.

THEOREM 168. *([**128**]). Suppose that* $0 < A \leq B$. *Then*

$$\begin{aligned}\frac{A}{2n^2}\sum_{j<k}(\log x_j - \log x_k)^2 &\leq \frac{1}{n}\sum x_k - \left(\prod_{k=1}^{n} x_k\right)^{\frac{1}{n}} \\ &\leq \frac{B}{2n^2}\sum_{j<k}(\log x_j - \log x_k)^2\end{aligned} \tag{5.139}$$

for every $x_1, \ldots, x_n \in [A, B]$.

As

$$\frac{1}{2n^2}\sum_{j<k}(\log x_j - \log x_k)^2$$

represents the variance of the random variable whose distribution is

$$\begin{pmatrix} \log x_1 & \log x_2 & \cdots & \log x_k \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix},$$

Theorem 168 reveals the probabilistic character of the $AM - GM$ Inequality. Using the technique of approximating the integrable functions by step functions, one can immediately derive from Theorem 168 the following general result.

THEOREM 169. *([**128**]) Let* (Ω, Σ, P) *be a probability space and let* X *be a random variable on this space, taking values in the interval* $[A, B]$, *where* $0 < A \leq B$. *Then*

$$A \leq \frac{M(X) - e^{M(\log X)}}{D^2(\log X)} \leq B.$$

9.6. Integral Means. In the standard approach, the *mean value* of an integrable function $f : [a, b] \to \mathbb{R}$ is defined by

$$M(f) = \frac{1}{b-a}\int_a^b f(t)\,dt$$

and the discussion above motivates for it the alternative notation $M_{AA}(f)$, as it represents the average value of f according to the arithmetic mean.

Taking into account Lemma 1 in [**124**], the *multiplicative mean value* of a function $f : [a, b] \to (0, \infty)$ (where $0 < a < b$) will be defined by the formula (see [**124**])

$$M_{GG}(f) = \exp\left(\frac{1}{\log b - \log a}\int_{\log a}^{\log b} \log f(e^t)dt\right)$$

equivalently,

$$\begin{aligned} M_{GG}(f) &= \exp\left(\frac{1}{\log b - \log a}\int_a^b \log f(t)\,\frac{dt}{t}\right) \\ &= \exp\left(L(a,b)\,M\left(\frac{\log f(t)}{t}\right)\right) \end{aligned}$$

where

$$L(a,b) = \frac{b-a}{\log b - \log a}$$

represents the *logarithmic mean* of a and b.

In what follows, we shall adopt for the multiplicative mean value of a function f the (more suggestive) notation $M_*(f)$ (see [**124**]).

The main properties of the multiplicative mean are listed below (cf. [**124**]):

$$M_*(1) = 1$$
$$m \le f \le M \;\Rightarrow m \le M_*(f) \le M$$
$$M_*(fg) = M_*(f)\,M_*(g).$$

It is worth noticing that similar schemes can be developed for other pairs of types of convexity, attached to different averaging devices (see [**127**]). We shall not enter the details here, but the reader can verify easily that many other mean values come this way. For example, the *geometric mean* of a function f,

$$\exp\left(\frac{1}{b-a}\int_a^b \log f(t)dt\right)$$

is nothing but the mean value $M_{AG}(f)$, corresponding to the pair (A)$-$(G). The geometric mean of the identity of $[a, b]$,

$$I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}.$$

(usually known as the *identric mean* of a and b) appears many times in computing the multiplicative mean value of some concrete functions.

Notice that the multiplicative mean value introduced here escapes the classical theory of integral $f-$means. In fact, it illustrates, in a special case, the usefulness of extending that theory for normalized weighted measures.

The aim of the next two subsections is to show that two major inequalities in convex function theory, namely the Jensen inequality and the Hermite-Hadamard inequality, have multiplicative counterparts (cf [**124**]). As a consequence we obtain several new inequalities, which are quite delicate outside the framework of multiplicative convexity.

9.7. The Multiplicative Analogue of Jensen's Inequality. In what follows we shall be concerned only with the integral version of the Jensen Inequality (see [**124**]).

THEOREM 170. *Let $f : [a, b] \to (0, \infty)$ be a continuous function defined on a subinterval of $(0, \infty)$ and let $\varphi : J \to (0, \infty)$ be a multiplicatively convex continuous function defined on an interval J which includes the image of f. Then*

$$\varphi\left(M_*(f)\right) \leq M_*(\varphi \circ f). \tag{5.140}$$

PROOF. In fact, using constant step divisions of $[a, b]$ we have

$$\begin{aligned} M_*(f) &= \exp\left(\frac{1}{\log b - \log a}\int_a^b \log f(t)\,\frac{dt}{t}\right) \\ &= \lim_{n\to\infty}\exp\left(\sum_{k=1}^{n}\log f(t_k)\frac{\log t_{k+1} - \log t_k}{\log b - \log a}\right) \end{aligned}$$

which yields, by the multiplicative convexity of φ,

$$\begin{aligned} \varphi\left(M_*(f)\right) &= \lim_{n\to\infty}\varphi\left(\exp\left(\sum_{k=1}^{n}\log f(t_k)\frac{\log t_{k+1} - \log t_k}{\log b - \log a}\right)\right) \\ &\leq \lim_{n\to\infty}\left(\exp\left(\sum_{k=1}^{n}\log(\varphi\circ f)(t_k)\frac{\log t_{k+1} - \log t_k}{\log b - \log a}\right)\right) \\ &= M_*(\varphi\circ f). \end{aligned}$$

∎

The multiplicative analogue of Jensen's Inequality is the source of many interesting inequalities. We notice here only a couple of them. First, letting $\varphi = \exp t^{\alpha}$ $(\alpha \in \mathbb{R})$, we are led to the following concavity type property of the log function:

$$\left(\frac{1}{\log b - \log a}\int_a^b \log f(t)\,\frac{dt}{t}\right)^{\alpha} \leq \log\left(\frac{1}{\log b - \log a}\int_a^b f^{\alpha}(t)\,\frac{dt}{t}\right)$$

for every $\alpha \in \mathbb{R}$ and every function f as in the statement of Theorem 170 above.

Particularly, for $f = e^t$, we have

$$L(a,b)^{\alpha} \leq \log\left(\frac{1}{\log b - \log a}\int_a^b e^{\alpha\, t}\,\frac{dt}{t}\right)$$

whenever $\alpha \in \mathbb{R}$.

Our second illustration of Theorem 170 concerns the pair $\varphi = \log t$ and $f = e^t$; φ is multiplicatively concave on $(1, \infty)$, which is a consequence of the AM-GM Inequality. The multiplicative mean of $f = e^t$ is $\exp\left(\frac{b-a}{\log b - \log a}\right)$, so that we have

$$L(a,b) \geq \exp\left(\frac{1}{\log b - \log a}\int_a^b \log\log t\,\frac{dt}{t}\right) = I(\log a, \log b)$$

for every $1 < a < b$. However, a direct application of the Hermite-Hadamard inequality gives us (in the case of the exp function) a better result:

$$L(a,b) > \sqrt{ab} > \log\sqrt{ab} > I(\log a, \log b).$$

The problem of estimating from above the difference of the two sides in Jensen's Inequality,

$$M_*(\varphi \circ f) - \varphi\left(M_*(f)\right)$$

can be discussed adapting the argument in [**126**]. We leave the details to the reader.

9.8. The Multiplicative Analogue of the Hermite-Hadamard Inequality. The classical Hermite-Hadamard Inequality states that if $f : [a, b] \to \mathbb{R}$ is a convex function then

$$(HH) \qquad f\left(\frac{a+b}{2}\right) \leq M(f) \leq \frac{f(a)+f(b)}{2} \ ,$$

which follows easily from the midpoint and trapezoidal approximation to the middle term. Moreover, under the presence of continuity, equality occurs (in either side) only for linear functions.

The next result represents the multiplicative analogue of the Hermite-Hadamard Inequality (cf. [**124**]):

THEOREM 171. *Suppose that* $0 < a < b$ *and let* $f : [a, b] \to (0, \infty)$ *be a continuous multiplicatively convex function. Then*

$$(*HH) \qquad f(\sqrt{ab}) \leq M_*(f) \leq \sqrt{f(a)f(b)}.$$

The left side inequality is strict unless f *is multiplicatively affine, while the right side inequality is strict unless* f *is multiplicatively affine on each of the subintervals* $[a, \sqrt{ab}]$ *and* $[\sqrt{ab}, b]$.

As noticed L. Fejér [**72**], the classical Hermite-Hadamard Inequality admits a weighted extension by replacing dx by $p(t)dt$, where p is a non-negative function whose graph is symmetric with respect to the center $(a+b)/2$. Of course, this fact has a counterpart in $(*HH)$, where dt/t can be replaced by $p(t)\,dt/t$, with p a non-negative function such that $p(t/\sqrt{ab}) = p(\sqrt{ab}/t)$.

In the additive framework, the mean value verifies the equality

$$M(f) = \frac{1}{2}\left(M(f \,|\, [a, \frac{a+b}{2}]) + M(f \,|\, [\frac{a+b}{2}, b])\right)$$

which can be checked by an immediate computation; in the multiplicative setting it reads as follows ([**124**]):

LEMMA 19. *Let* $f : [a, b] \to (0, \infty)$ *be an integrable function, where* $0 < a < b$. *Then*

$$M_*(f)^2 = M_*(f \,|\, [a, \sqrt{ab}]) \cdot M_*(f \,|\, [\sqrt{ab}, b]).$$

COROLLARY 60. *(*[**124**]*) The multiplicative analogue of the Hermite-Hadamard Inequality can be improved upon*

$$(5.141) \qquad \begin{aligned} f(a^{1/2}b^{1/2}) \quad &< \quad \left(f(a^{3/4}b^{1/4})f(a^{1/4}b^{3/4})\right)^{1/2} < M_*(f) \\ &< \quad \left(f(a^{1/2}b^{1/2})\right)^{1/2} f(a)^{1/4}f(b)^{1/4} \\ &< \quad \left(f(a)f(b)\right)^{1/2}. \end{aligned}$$

A moment's reflection shows that by iterating Corollary 60 one can exhibit approximations of $M_*(f)$ from below (or from above) in terms of (G)-convex combinations of the values of f at the *multiplicatively dyadic* points $a^{(2^n-k)/2^n}b^{k/2^n}$, $k = 0, ..., 2^n$, $n \in \mathbb{N}$.

For $f = \exp|[a,b]$ (where $0 < a < b$) we have $M_*(f) = \exp\left(\frac{b-a}{\log b - \log a}\right)$. According to the Corollary 60 above we obtain the inequalities

$$\frac{a^{3/4}b^{1/4} + a^{1/4}b^{3/4}}{2} < \frac{b-a}{\log b - \log a} < \frac{1}{2}\left(\frac{a+b}{2} + \sqrt{ab}\right),$$

first noticed by J. Sándor [**167**].

For $f = \Gamma\,|\,[a,b]$ (where $1 \le a < b$) we obtain the inequalities

$$(\Gamma) \qquad \log\Gamma\left(a^{1/2}b^{1/2}\right) < \frac{1}{\log b - \log a}\int_a^b \frac{\log\Gamma(x)}{x}dx < \frac{1}{2}\,\log\Gamma(a)\Gamma(b)$$

which can be strenghtened via Corollary 60.

The middle term can be evaluated by Binet's formula (see [**185**, p. 249]), which leads us to

$$\frac{\log\Gamma(x)}{x} = \log x - 1 - \frac{1}{2}\cdot\frac{\log x}{x} + \frac{\log\sqrt{2\pi}}{x} + \frac{\theta(x)}{x}$$

where θ is a decresing function with $\lim_{x\to\infty}\theta(x) = 0$. In fact,

$$\begin{aligned}\theta(x) &= \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)e^{-xt}\frac{1}{t}\,dt \\ &= \sum_{k=1}^\infty \frac{B_{2k}}{2k(2k-1)x^{2k}} \\ &= \frac{1}{12x^2} - \frac{1}{360x^4} + \frac{1}{1260x^6} - \dots\end{aligned}$$

where the B_{2k}'s denote the Bernoulli numbers. Then

$$\begin{aligned}\log M_*(\Gamma\,|\,[a,b]) &= \frac{1}{\log b - \log a}\int_a^b \frac{\log\Gamma(t)}{t}dt \\ &= \frac{-2(b-a)}{\ln b - \ln a} - \frac{1}{4}\ln ab + \ln\sqrt{2\pi} + \frac{(\ln b)\,b - (\ln a)\,a}{\ln b - \ln a} \\ &\quad + \frac{1}{\log b - \log a}\int_a^b \frac{\theta(x)}{x}dx \\ &= -L(a,b) - \frac{1}{4}\ln ab + \ln\sqrt{2\pi} + L(a,b)\log I(a,b) + \theta(c)\end{aligned}$$

for a suitable $c \in (a,b)$.

We pass now to the problem of estimating the precision in the Hermite-Hadamard Inequality. For, we shall need a preparation.

Given a function $f : I \to (0,\infty)$ (with $I \subset (0,\infty)$) we shall say that f is *multiplicatively Lipschitzian* provided there exist a constant $L > 0$ such that

$$\max\left\{\frac{f(x)}{f(y)}, \frac{f(y)}{f(x)}\right\} \le \left(\frac{y}{x}\right)^L$$

for all $x < y$ in I; the smallest L for which the above inequality holds constitutes the *multiplicative Lipschitzian* (see [**124**]) constant of f and it will be denoted by $||f||_{^\star Lip}$.

REMARK 86. ([**124**]) *Though the family of multiplicatively Lipschitz functions is large enough* (*to deserve attention in its own*), *we know the exact value of the multiplicative Lipschitz constant only in few cases:*

i) *If f is of the form $f(x) = x^\alpha$, then $||f||_{^\star Lip} = \alpha$.*

ii) *If* $f = \exp|[a,b]$ *(where* $0 < a < b$*), then* $||f||_{\star Lip} = b$.

iii) *Clearly,* $||f||_{\star Lip} \leq 1$ *for every non-decreasing functions* f *such that* $f(x)/x$ *is non-increasing. For example, this is the case of the functions* sin *and* sec *on* $(0, \pi/2)$.

iv) *If* f *and* g *are two multiplicatively Lipschitzian functions (defined on the same interval) and* $\alpha, \beta \in \mathbb{R}$*, then* $f^\alpha g^\beta$ *is multiplicatively Lipschitzian too. Moreover,*

$$||f^\alpha g^\beta||_{\star Lip} \leq |\alpha| \cdot ||f||_{\star Lip} + |\beta| \cdot ||g||_{\star Lip}.$$

The following result can be easily derived from the standard form of the Ostrowski Inequality for Lipschitzian functions as stated in [**18**, Corollary 2, p. 345]:

Theorem 172. *([**124**]) Let* $f : [a,b] \to (0,\infty)$ *be a multiplicatively convex continuous function. Then*

$$f(\sqrt{ab}) \leq M_*(f) \leq f(\sqrt{ab}) \left(\frac{b}{a}\right)^{||f||_{\star Lip}/4} \tag{5.142}$$

and

$$M_*(f) \leq \sqrt{f(a)f(b)} \leq M_*(f) \left(\frac{b}{a}\right)^{||f||_{\star Lip}/4}. \tag{5.143}$$

A generalization of the second part of this result, based on Theorem 171 above, will make the subject of the next section.

For $f = \exp|[a,b]$ (where $0 < a < b$), we have $M_*(f) = \exp\left(\frac{b-a}{\log b - \log a}\right)$ and $||f||_{\star Lip} = b$. By Theorem 172, we infer the inequalities

$$\begin{aligned} 0 \quad &< \quad \frac{b-a}{\log b - \log a} - \sqrt{ab} < \frac{b}{4}(\log b - \log a) \\ 0 \quad &< \quad \frac{a+b}{2} - \frac{b-a}{\log b - \log a} < \frac{b}{4}(\log b - \log a). \end{aligned}$$

For $f = \sec$ (restricted to $(0,\pi/2)$ we have $||f||_{\star Lip} = 1$ and

$$\begin{aligned} &M_*(\sec|[a,b]) \\ = \quad &\exp\left(\frac{-1}{\log b - \log a}\int_a^b \frac{\ln\cos x}{x}dx\right) \\ = \quad &\exp\left(\frac{1}{\log b - \log a}\int_a^b \left(\frac{1}{2}x + \frac{1}{12}x^3 + \frac{1}{45}x^5 + \frac{17}{2520}x^7 + ...\right)dx\right) \\ = \quad &\exp\left(\frac{1}{\log b - \log a}\left(\frac{b^2-a^2}{4} + \frac{b^4-a^4}{48} + \frac{b^6-a^6}{270} + ...\right)\right) \end{aligned}$$

for every $0 < a < b < \pi/2$. According to Theorem 172, we have

$$\sec(\sqrt{ab}) < M_*(\sec|[a,b]) < \sec(\sqrt{ab}) \cdot \left(\frac{b}{a}\right)^{1/4}$$

and

$$M_*(\sec|[a,b]) < \sqrt{\sec a \sec b} < M_*(\sec|[a,b]) \cdot \left(\frac{b}{a}\right)^{1/4}$$

for every $0 < a < b < \pi/2$.

9.9. Approximating $M_*(f)$ by Geometric Means. As the reader already noticed, computing (in a compact form) the multiplicative mean value is not an easy task. However, it can be nicely approximated. The following result, inspired by a recent paper of K. Jichang [**89**], outlines the possibility to approximate $M_*(f)$ (from above) by products $\left(\prod_{k=1}^{n} f(x_k)\right)^{1/n}$ for a large range of functions (cf. [**124**]):

THEOREM 173. *([**124**]) Let $f : I \to (0,\infty)$ be a function which is multiplicatively convex or multiplicatively concave.*

If $I = [1,a]$ (with $a > 1$) and f is strictly increasing, then

$$\left(\prod_{k=1}^{n} f(a^{k/n})\right)^{1/n} > \left(\prod_{k=1}^{n+1} f(a^{k/(n+1)})\right)^{1/(n+1)} > M_*(f) \tag{5.144}$$

for every $n = 1,2,3,...$

The conclusion remains valid for $I = [a,1]$ (with $0 < a < 1$) and f a strictly decreasing function as above.

The inequalities (5.144) should be reversed in each of the following two cases:

$I = [1,a]$ (with $a > 1$) and f is strictly decreasing;

$I = [a,1]$ (with $0 < a < 1$) and f a strictly increasing

PROOF. Let us consider first the case of strictly increasing multiplicatively convex functions. In this case, for each $k \in \{1,...,n\}$ we have

$$\begin{aligned} f(a^{k/(n+1)}) &= f(a^{kn^2/(n+1)n^2}) < f(a^{(nk-k+1)/n^2}) \\ &= f(a^{\frac{k-1}{n}\cdot\frac{k-1}{n}+(1-\frac{k-1}{n})\cdot\frac{k}{n}}) \\ &\leq \left(f(a^{(k-1)/n})\right)^{(k-1)/n}\left(f(a^{k/n})\right)^{1-(k-1)/n}. \end{aligned}$$

By multiplying them side by side we get

$$\begin{aligned} \prod_{k=1}^{n} f(a^{k/(n+1)}) &< \prod_{k=1}^{n}\left(\left(f(a^{(k-1)/n})\right)^{(k-1)/n}\left(f(a^{k/n})\right)^{1-(k-1)/n}\right) \\ &= \frac{\left(\prod_{k=1}^{n} f(a^{k/n})\right)^{(n+1)/n}}{f(a)}, \end{aligned}$$

i.e., the left hand inequality in the statement of our theorem.

Consider now the case where f is strictly increasing multiplicatively concave. Then

$$\begin{aligned} f(a^{k/n}) &= f(a^{k(n+1)^2/n(n+1)^2}) > f(a^{k(n+2)/(n+1)^2}) \\ &= f(a^{\frac{k}{n+1}\cdot\frac{k+1}{n+1}+(1-\frac{k}{n+1})\cdot\frac{k}{n+1}}) \\ &\geq \left(f(a^{(k+1)/(n+1)})\right)^{k/(n+1)}\left(f(a^{k/(n+1)})\right)^{1-k/(n+1)} \end{aligned}$$

for each $k \in \{1, ..., n\}$, which leads to

$$\begin{aligned}\prod_{k=1}^{n} f(a^{k/n}) &> \prod_{k=1}^{n} \left(\left(f(a^{(k+1)/(n+1)}) \right)^{k/(n+1)} \left(f(a^{k/(n+1)}) \right)^{1-k/(n+1)} \right) \\ &= \prod_{k=1}^{n} \left(\left(f(a^{(k+1)/(n+1)}) \right)^{k/(n+1)} \left(f(a^{k/(n+1)}) \right)^{(n-k+1)/(n+1)} \right) \\ &= (f(a))^{n/(n+1)} \cdot \left(\prod_{k=1}^{n} f(a^{k/(n+1)}) \right)^{n/(n+1)} \\ &= \left(\prod_{k=1}^{n+1} f(a^{k/(n+1)}) \right)^{n/(n+1)}\end{aligned}$$

i.e., again to the left hand inequality in (5.144).

To end the proof of the first part of the theorem, note that

$$\begin{aligned}\lim_{n\to\infty} \left(\prod_{k=1}^{n} f(a^{k/n}) \right)^{1/n} &= \exp\left(\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log f(a^{k/n}) \right) \\ &= \exp\left(\frac{1}{\log a} \int_0^{\log a} \log f(e^t) dt \right) \\ &= M_*(f).\end{aligned}$$

As the sequence $\pi_n = \left(\prod_{k=1}^{n} f(a^{k/n}) \right)^{1/n}$ is strictly decreasing we conclude that $\pi_n > M_*(f)$ for every $n = 1, 2, 3, ...$

The remainder of the proof follows by a careful inspection of the argument above. ∎

As was noticed in [**128**, p. 163], Γ is strictly multiplicatively convex on $[1, \infty)$. According to Theorem 173, for each $a > 1$ and each natural number n we have

$$\left(\prod_{k=1}^{n} \Gamma(a^{k/n}) \right)^{1/n} > \left(\prod_{k=1}^{n+1} \Gamma(a^{k/(n+1)}) \right)^{1/(n+1)} > \exp\left(\frac{1}{\log a} \int_1^a \frac{\log \Gamma(t)}{t} \, dt \right).$$

The same argument, applied to the multiplicatively concave functions $\sin \frac{\pi x}{2}$ and $\cos \frac{\pi x}{2}$ (cf. [**128**, p. 159]) gives us

$$\begin{aligned}\left(\prod_{k=1}^{n} \sin(\frac{\pi}{2} \, a^{k/n}) \right)^{1/n} &< \left(\prod_{k=1}^{n+1} \sin(\frac{\pi}{2} \, a^{k/(n+1)}) \right)^{1/(n+1)} \\ &< \exp\left(\frac{1}{\log a} \int_1^a \frac{\log \sin(\pi t/2)}{t} \, dt \right)\end{aligned}$$

and

$$\begin{aligned}\left(\prod_{k=1}^{n} \cos(\frac{\pi}{2} \, a^{k/n}) \right)^{1/n} &> \left(\prod_{k=1}^{n+1} \cos(\frac{\pi}{2} \, a^{k/(n+1)}) \right)^{1/(n+1)} \\ &> \exp\left(\frac{1}{\log a} \int_1^a \frac{\log \cos(\pi t/2)}{t} \, dt \right)\end{aligned}$$

for every $a \in (0,1)$; they should be added to a number of other curiosities noticed recently by G. J. Tee [**174**].

The following result answers the question how fast is the convergence which makes the subject of Theorem 173 above:

PROPOSITION 73. *([**124**]) Let $f:[a,b]\to(0,\infty)$ be a strictly multiplicatively convex continuous function. Then*

$$\left(\frac{f(b)}{f(a)}\right)^{1/(2n)} < \left(\prod_{k=1}^{n} f(x_k)\right)^{1/n} / M_*(f) < \left(\frac{b}{a}\right)^{\|f\|_{*Lip}/(2n)} \tag{5.145}$$

where $x_k = a^{1-k/n} b^{k/n}$ for $k = 1, ..., n$

PROOF. According to $(*HH)$, for each $k = 1, ..., n$, we have

$$f(\sqrt{x_{k-1}x_k}) < \exp\left(\frac{n}{\log(b/a)}\int_{x_{k-1}}^{x_k}\log f\,dt\right) < \sqrt{f(x_k)f(x_{k+1})}$$

which yields

$$\left(\prod_{k=1}^{n} f(\sqrt{x_{k-1}x_k})\right)^{1/n} < M_*(f) < \left(\prod_{k=1}^{n} f(x_k)\right)^{1/n} \Bigg/ \left(\frac{f(b)}{f(a)}\right)^{1/(2n)}$$

i.e.,

$$\begin{aligned}\left(\frac{f(b)}{f(a)}\right)^{1/(2n)} &< \left(\prod_{k=1}^{n} f(x_k)\right)^{1/n} / M_*(f) < \\ &< \left(\prod_{k=1}^{n} f(x_k)/f(\sqrt{x_{k-1}x_k})\right)^{1/n}.\end{aligned}$$

Or,

$$\left(\prod_{k=1}^{n} f(x_k)/f(\sqrt{x_k x_{k+1}})\right)^{1/n} \le \prod_{k=1}^{n}\left(\frac{x_k}{x_{k-1}}\right)^{\|f\|_{*Lip}/(2n)}.$$

∎

For $f(x) = e^x$, $x \in [1,a]$, the last result gives ([**124**])

$$\frac{a-1}{2n} < \frac{1}{n}\sum_{k=1}^{n} a^{k/n} < \frac{a}{\log a} + \frac{a}{2n}$$

for all $n = 1, 2, 3, ...$.

10. The $H.-H.$ Inequality of $s-$Convex Functions in the First Sense

The following concept was introduced by Ozlicz in the paper [**129**] and was used in the theory of Ozlicz spaces ([**107**], [**119**]):

Let $0 < s \le 1$. A function $f:\mathbb{R}_+\to\mathbb{R}$ where $\mathbb{R}_+ := [0,\infty)$, is said to be *$s-$convex in the first sense* if:

$$f(\alpha u + \beta v) \le \alpha^s f(u) + \beta^s f(v) \tag{5.146}$$

for all $u, v \in \mathbb{R}_+$ and $\alpha, \beta \ge 0$ with $\alpha^s + \beta^s = 1$. We denote this class of real functions by K_s^1.

We shall present some results from the paper [**87**] referring to the $s-$convex functions in the first sense.

THEOREM 174. *([**87**]) Let $0<s<1$. If $f\in K_s^1$, then f is nondecreasing on $(0,\infty)$ and $\lim_{u\to 0^+} f(u)\le f(0)$.*

PROOF. We have, for $u>0$ and $\alpha\in[0,1]$,

$$f\left[\left(\alpha^{\frac{1}{s}}+(1-\alpha)^{\frac{1}{s}}\right)u\right]\le \alpha f(u)+(1-\alpha)f(u)=f(u).$$

The function

$$h(\alpha)=\alpha^{\frac{1}{s}}+(1-\alpha)^{\frac{1}{s}}$$

is continuous on $[0,1]$, decreasing on $\left[0,\frac{1}{2}\right]$, increasing on $\left[\frac{1}{2},1\right]$ and $h([0,1])=\left[h\left(\frac{1}{2}\right),h(1)\right]=\left[2^{1-\frac{1}{s}},1\right]$. This yields that

$$f(tu)\le f(u)\text{ for all }u>0,\ t\in\left[2^{1-\frac{1}{s}},1\right]. \tag{5.147}$$

If now $t\in\left[2^{1-\frac{1}{s}},1\right]$, then $t^{\frac{1}{2}}\in\left[2^{1-\frac{1}{s}},1\right]$, and therefore, by the fact that (5.147) holds for all $u>0$, we get

$$f(tu)=f\left(t^{\frac{1}{2}}\left(t^{\frac{1}{2}}u\right)\right)\le f\left(t^{\frac{1}{2}}u\right)\le f(u)$$

for all $u>0$. By induction, we therefore obtain that

$$f(tu)\le f(u)\text{ for all }u>0,\ t\in(0,1]. \tag{5.148}$$

Hence, by taking $0<u\le v$ and applying (5.148), we get

$$f(u)=f\left(\frac{u}{v}\cdot v\right)\le f(v),$$

which means that f is non-decreasing on $(0,\infty)$.

The second part can be proved in the following manner. For $u>0$ we have

$$f(\alpha u)=f(\alpha u+\beta 0)\le \alpha^s f(u)+\beta^s f(0)$$

and taking $u\to 0^+$, we obtain

$$\lim_{u\to 0+} f(u)\le \lim_{u\to 0+} f(\alpha u)\le \alpha^s\lim_{u\to 0+} f(u)+\beta^s f(0)$$

and hence

$$\lim_{u\to 0+} f(u)\le f(0).$$

∎

REMARK 87. *([**87**]) The above results generally do not hold in the case of convex functions, i.e., when $s=1$. This is because a convex function $f:\mathbb{R}_+\to\mathbb{R}$ may not necessarily be non-decreasing on $(0,\infty)$.*

REMARK 88. *([**87**]) If $0<s<1$, then the function $f\in K_s^1$ is nondecreasing on $(0,\infty)$ but not necessarily on $[0,\infty)$.*

EXAMPLE 13. *([**87**]) Let $0<s<1$ and $a,b,c,\in\mathbb{R}$. Defining for $u\in\mathbb{R}_+$*

$$f(u)=\begin{cases} a & \text{if}\quad u=0\\ bu^s+c & \text{if}\quad u>0\end{cases},$$

we have:

(i) *If $b\ge 0$ and $c\le a$, then $f\in K_s^1$.*
(ii) *If $b\ge 0$ and $c<a$, then f is non-decreasing on $(0,\infty)$ but not on $[0,\infty)$.*

From the known examples of the $s-$convex functions we can build up other $s-$convex functions using the following composition property [**87**].

THEOREM 175. *Let $0 < s \leq 1$. If $f, g \in K_s^1$ and if $F : \mathbb{R}^2 \to \mathbb{R}$ is an non-decreasing convex function, then the function $h : \mathbb{R}_+ \to \mathbb{R}$ defined by $h(u) := F(f(u), g(u))$ is $s-$convex. In particular, if $f, g \in K_s^1$, then $f+g, \max(f, g) \in K_s^1$.*

PROOF. If $u, v \in \mathbb{R}_+$, then for all $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$ we have

$$\begin{aligned} h(\alpha u + \beta v) &= F(f(\alpha u + \beta v), g(\alpha u + \beta v)) \\ &\leq F(\alpha^s f(u) + \beta^s f(v), \alpha^s g(u) + \beta^s g(v)) \\ &\leq \alpha^s F(f(u), g(u)) + \beta^s F(f(v), g(v)) \\ &= \alpha^s h(u) + \beta^s h(v). \end{aligned}$$

Since $F(u, v) = u + v$ and $F(u, v) = \max(u, v)$ are particular examples of non-decreasing convex functions on $\mathbb{R}^2$, we get particular cases of our theorems. ∎

It is important to know when the condition $\alpha^s + \beta^s = 1$ in the definition of K_s^1 can be equivalently replaced by the condition $\alpha^s + \beta^s \leq 1$, [**87**].

THEOREM 176. *Let $f \in K_s^1$. Then inequality* (5.146) *holds for all $u, v \in \mathbb{R}_+$, and all $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s \leq 1$ if and only if $f(0) \leq 0$.*

PROOF. Necessity is obvious by taking $u = v = 0$ and $\alpha = \beta = 0$. Therefore, assume that $u, v \in \mathbb{R}_+$, $\alpha, \beta \geq 0$ and $0 < \gamma = \alpha^s + \beta^s < 1$. Put $a = \alpha\gamma^{-\frac{1}{s}}$ and $b = \beta\gamma^{-\frac{1}{s}}$. Then $a^s + b^s = \frac{\alpha^s}{\gamma} + \frac{\beta^s}{\gamma} = 1$ and hence

$$\begin{aligned} f(\alpha u + \beta v) &= f\left(a\gamma^{\frac{1}{s}} u + b\gamma^{\frac{1}{s}} v\right) \\ &\leq a^s f\left(\gamma^{\frac{1}{s}} u\right) + b^s f\left(\gamma^{\frac{1}{s}} v\right) \\ &= a^s f\left[\gamma^{\frac{1}{s}} u + (1-\gamma)^{\frac{1}{s}} 0\right] + b^s f\left[\gamma^{\frac{1}{s}} v + (1-\gamma)^{\frac{1}{s}} 0\right] \\ &\leq a^s [\gamma f(u) + (1-\gamma) f(0)] + b^s [\gamma f(v) + (1-\gamma) f(0)] \\ &= a^s \gamma f(u) + b^s \gamma f(v) + (1-\gamma) f(0) \\ &\leq a^s f(u) + b^s f(v). \end{aligned}$$

∎

Using the above theorem we can compare both definitions of the $s-$convexity [**87**].

THEOREM 177. *Let $0 < s_1 \leq s_2 \leq 1$. If $f \in K_{s_2}^1$ and $f(0) \leq 0$, then $f \in K_{s_1}^1$.*

PROOF. Assume that $f \in K_{s_2}^1$ and $u, v \geq 0, \alpha, \beta \geq 0$ with $\alpha^{s_1} + \beta^{s_1} = 1$. Then $\alpha^{s_2} + \beta^{s_2} \leq \alpha^{s_1} + \beta^{s_1} = 1$ and, according to Theorem 176, we have

$$f(\alpha u + \beta v) \leq \alpha^{s_2} f(u) + \beta^{s_2} f(v) \leq \alpha^{s_1} f(u) + \beta^{s_1} f(v),$$

which means that $f \in K_{s_1}^1$. ∎

Let us note first that if f is a non-negative function from K_s^1 and $f(0) = 0$, then f is right continuous at 0, i.e., $f(0+) = f(0) = 0$.

We now prove the following theorem containing some interesting examples of $s-$convex functions [**87**].

THEOREM 178. *Let* $0 < s < 1$ *and let* $p : \mathbb{R}_+ \to \mathbb{R}_+$ *be a nondecreasing function. Then the function* f *defined for* $u \in \mathbb{R}_+$ *by*

$$f(u) = u^{\frac{s}{(1-s)}} p(u) \tag{5.149}$$

belongs to K_s^1.

PROOF. Let $v \geq u \geq 0$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$. We shall consider two cases.

(1) Let $\alpha u + \beta v \leq u$. Then

$$f(\alpha u + \beta v) \leq f(u) = (\alpha^s + \beta^s) f(u) \leq \alpha^s f(u) + \beta^s f(v).$$

(2) Let $\alpha u + \beta v > u$. This yields $\beta v > (1-\alpha) u$ and so $\beta > 0$.
Since $\alpha \leq \alpha^s$ for $\alpha \in [0,1]$, we obtain $\alpha - \alpha^{s+1} \leq \alpha^s - \alpha^{s+1}$ and then

$$\frac{\alpha}{(1-\alpha)} \leq \frac{\alpha^s}{(1-\alpha^s)} = \frac{(1-\beta^s)}{\beta^s}.$$

That is,

$$\frac{\alpha\beta}{(1-\alpha)} \leq \beta^{1-s} - \beta. \tag{5.150}$$

We also have

$$\alpha u + \beta v \leq (\alpha + \beta) v \leq (\alpha^s + \beta^s) v = v$$

and, in view of (5.150),

$$\alpha u + \beta v \leq \frac{\alpha\beta v}{(1-\alpha)} + \beta v \leq \left(\beta^{1-s} - \beta\right) v + \beta v = \beta^{1-s} v,$$

whence

$$(\alpha u + \beta v)^{\frac{s}{(1-s)}} \leq \beta^s v^{\frac{s}{(1-s)}}. \tag{5.151}$$

Applying (5.151) and the monotonicity of p, we arrive at

$$\begin{aligned} f(\alpha u + \beta v) &= (\alpha u + \beta v)^{\frac{s}{(1-s)}} p(\alpha u + \beta v) \\ &\leq \beta^s v^{\frac{s}{(1-s)}} p(\alpha u + \beta v) \leq \beta^s v^{\frac{s}{(1-s)}} p(v) \\ &= \beta^s f(v) \leq \alpha^s f(u) + \beta^s f(v). \end{aligned}$$

The proof is thus completed.

∎

The following theorem contains some other examples of $s-$convex functions in the first sense [**87**]

THEOREM 179. *Let* $f \in K_{s_1}^1$ *and* $g \in K_{s_2}^1$, *where* $0 < s_1, s_2 \leq 1$.

a) *If* f *is a nondecreasing function and* g *is a nonnegative function such that* $f(0) \leq 0 = g(0)$, *then the composition* $f \circ g$ *of* f *and* g *belongs to* K_s^1, *where* $s = s_1 \cdot s_2$.
b) *Assume that* $0 < s_1, s_2 < 1$. *If* f *and* g *are nonnegative functions such that either* $f(0) = 0$ *or* $g(0) = 0$, *then the product* $f \cdot g$ *of* f *and* g *belongs to* K_s^1, *where* $s = \min(s_1, s_2)$.

PROOF. a) Let $u,v\in\mathbb{R}_+$ and $\alpha,\beta\geq 0$ with $\alpha^s+\beta^s=1$, where $s=s_1\cdot s_2$. Since $\alpha^{s_i}+\beta^{s_i}\leq\alpha^{s_1s_2}+\beta^{s_1s_2}=1$ for $i=1,2$, then, by Theorem 176 and the above assumptions, we have

$$\begin{aligned}(f\circ g)(\alpha u+\beta v) &= f(g(\alpha u+\beta v))\leq f(\alpha^{s_2}g(u)+\beta^{s_2}g(v))\\ &\leq \alpha^{s_1s_2}f(g(u))+\beta^{s_1s_2}f(g(u))\\ &\leq \alpha^s(f\circ g)(u)+\beta^s(f\circ g)(v),\end{aligned}$$

which means that $f\circ g\in K_s^1$.

b) According the Theorem 174, both functions f and g are non-decreasing on $(0,\infty)$. Therefore

$$(f(u)-f(v))(g(u)-g(v))\geq 0,$$

or, equivalently

$$f(u)g(v)+f(v)g(u)\leq f(u)g(u)+f(v)g(v) \tag{5.152}$$

for all $v\geq u>0$. If $v>u=0$, then inequality (5.152) is still valid as f and g are non-negative and $f(0)=g(0)=0$.

Now, let $u,v\in\mathbb{R}_+$ and $\alpha,\beta\geq 0$ with $\alpha^s+\beta^s=1$, where $s=\min(s_1,s_2)$. Then $\alpha^{s_i}+\beta^{s_i}\leq\alpha^s+\beta^s=1$ for $i=1,2$, and by Theorem 176 and inequality (5.152), we have

$$\begin{aligned}& f(\alpha u+\beta v)g(\alpha u+\beta v)\\ \leq\ & (\alpha^{s_1}f(u)+\beta^{s_1}f(v))(\alpha^{s_2}g(u)+\beta^{s_2}g(v))\\ =\ & \alpha^{s_1+s_2}f(u)g(u)+\alpha^{s_1}\beta^{s_2}f(u)g(v)+\alpha^{s_2}\beta^{s_1}f(v)g(u)+\beta^{s_1+s_2}f(v)g(v)\\ \leq\ & \alpha^{2s}f(u)g(u)+\alpha^s\beta^s(f(u)g(v)+f(v)g(u))+\beta^{2s}f(v)g(v)\\ \leq\ & \alpha^s f(u)g(u)+\beta^s f(v)g(v)\end{aligned}$$

which means that $f\cdot g\in K_s^1$.

∎

COROLLARY 61. *If ϕ is a convex $\psi-$function, i.e., $\phi(0)=0$ and ϕ is non-decreasing and continuous on $[0,\infty)$, and g is a $\psi-$function from K_s^1, then the composition $\phi\circ g$ belongs to K_s^1. In particular, the $\psi-$function $h(u)=\phi(u^s)$ belongs to K_s^1.*

Finally, we also have [**87**]:

THEOREM 180. *Let f be a $\psi-$function and $f\in K_s^1$ $(0<s<1)$. Then there exists a convex $\psi-$function Φ such that the $\psi-$function Ψ defined for $u\geq 0$ by $\Psi(u)=\Phi(u^s)$ is equivalent to f.*

PROOF. By the $s-$convexity of the function f and by $f(0)=0$, we obtain $f(\alpha u)\leq\alpha^s f(u)$ for all $u\geq 0$ and $\alpha\in[0,1]$.

Assume now that $v>u>0$. Then $f\left(u^{\frac{1}{s}}\right)=f\left(\left(\frac{u}{v}\right)^{\frac{1}{s}}v^{\frac{1}{s}}\right)\leq\left(\frac{u}{v}\right)f\left(v^{\frac{1}{s}}\right)$. That is,

$$\frac{f\left(u^{\frac{1}{s}}\right)}{u}\leq\frac{f\left(v^{\frac{1}{s}}\right)}{v}. \tag{5.153}$$

Inequality (5.153) means that the function $\frac{f\left(u^{\frac{1}{s}}\right)}{u}$ is a non-decreasing function on $(0, \infty)$. Define

$$\Phi(u) := \begin{cases} 0 & \text{for} \quad u = 0 \\ \int_0^u \frac{f\left(t^{\frac{1}{s}}\right)}{t} dt & \text{for} \quad u > 0. \end{cases}$$

Then Φ is a convex $\psi-$function and

$$\Phi(u^s) = \int_0^{u^s} \frac{f\left(t^{\frac{1}{s}}\right)}{t} dt \leq \left[\frac{f\left[(u^s)^{\frac{1}{s}}\right]}{u^s}\right] u^s = f(u),$$

$$\Phi(u^s) \geq \int_{\frac{u^s}{2}}^{u^s} \frac{f\left(t^{\frac{1}{s}}\right)}{t} dt \geq \left[\frac{f\left[\left(\frac{u^s}{2}\right)^{\frac{1}{s}}\right]}{\left(\frac{u^s}{2}\right)}\right] \frac{u^s}{2} = f\left(2^{-\frac{1}{s}} u\right).$$

Therefore,

$$f\left(2^{-\frac{1}{s}} u\right) \leq \Phi(u^s) \leq f(u)$$

for all $u \geq 0$, which means that ψ is equivalent to f (this sense of equivalence is taken from the theory of Ozlicz spaces [**119**]), and the proof is complete. ∎

Now, we will be able to point out some inequalities of Hermite-Hadamard type for the $s-$convex functions in the first sense [**52**].

THEOREM 181. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a $s-$convex mapping in the first sense with $s \in (0, 1)$. If $a, b \in \mathbb{R}_+$ with $a < b$, then one has the inequality:*

$$f\left(\frac{a+b}{2^{\frac{1}{s}}}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx. \tag{5.154}$$

PROOF. If we choose in the definition of $s-$convex mappings $\alpha = \frac{1}{2^{\frac{1}{s}}}$, $\beta = \frac{1}{2^{\frac{1}{s}}}$, we have that $\alpha^s + \beta^s = 1$ and then for all $x, y \in [0, \infty)$

$$f\left(\frac{x+y}{2^{\frac{1}{s}}}\right) \leq \frac{f(x) + f(y)}{2}.$$

If we choose $x = ta + (1-t) b, y = (1-t) a + tb$, $t \in [0, 1]$, we derive that

$$\begin{aligned} & f\left(\frac{a+b}{2^{\frac{1}{s}}}\right) \\ \leq \; & \frac{1}{2} \left[f(ta + (1-t) b) + f((1-t) a + tb)\right] \text{ for all } t \in [0, 1]. \end{aligned}$$

As f is monotonic nondecreasing on $[0, \infty)$, it is integrable on $[a, b]$. Thus, we can integrate over t in the above inequality. Taking into account that

$$\int_0^1 f(ta + (1-t) b)\, dt = \int_0^1 f((1-t) a + tb)\, dt = \frac{1}{b-a} \int_a^b f(x)\, dx,$$

the inequality (5.154) is proved. ∎

The second result, which is similar, in a sense, with the second part of the Hermite-Hadamard inequality for general convex mappings, is embodied in the next theorem [**52**].

THEOREM 182. *With the above assumptions for f and s, one has the inequality:*

$$\int_0^1 f\left(ta+(1-t^s)^{\frac{1}{s}}b\right)\psi(t)\,dt \le \frac{f(a)+f(b)}{2} \tag{5.155}$$

where

$$\psi(t) := \frac{1}{2}\left[1+(1-t^s)^{\frac{1}{s}-1}t^{s-1}\right],\ t\in(0,1].$$

PROOF. If we choose in the definition of $s-$convex mappings in the first sense $\alpha = t, \beta = (1-t^s)^{\frac{1}{s}}, t\in[0,1]$, we have that $\alpha^s+\beta^s = 1$ for all $t\in[0,1]$ and

$$f\left(ta+(1-t^s)^{\frac{1}{s}}b\right) \le t^s f(a)+(1-t^s)f(b)$$

for all $t\in[0,1]$, and similarly

$$f\left((1-t^s)^{\frac{1}{s}}a+tb\right) \le (1-t^s)f(a)+t^s f(b)$$

for all $t\in[0,1]$.

If we add the above two inequalities, we have that

$$\frac{1}{2}\left[f\left(ta+(1-t^s)^{\frac{1}{s}}b\right)+f\left((1-t^s)^{\frac{1}{s}}a+tb\right)\right] \le \frac{f(a)+f(b)}{2}.$$

for all $t\in[0,1]$.

If we integrate this inequality over t on $[0,1]$, we get that

$$\begin{aligned}
&\frac{1}{2}\left[\int_0^1 f\left(ta+(1-t^s)^{\frac{1}{s}}b\right)dt+\int_0^1 f\left((1-t^s)^{\frac{1}{s}}a+tb\right)dt\right] \\
\le\ & \frac{f(a)+f(b)}{2}.
\end{aligned} \tag{5.156}$$

Let us denote $u := (1-t^s)^{\frac{1}{s}}, t\in[0,1]$. Then $t = (1-u^s)^{\frac{1}{s}}$ and $dt = -(1-u^s)^{\frac{1}{s}-1}u^{s-1}$, $u\in(0,1]$ and then we have the change of variable

$$\begin{aligned}
& \int_0^1 f\left((1-t^s)^{\frac{1}{s}}a+tb\right)dt \\
=\ & -\int_1^0 f\left(ua+(1-u^s)^{\frac{1}{s}}b\right)(1-u^s)^{\frac{1}{s}-1}u^{s-1}du \\
=\ & \int_0^1 f\left(ta+(1-t^s)^{\frac{1}{s}}b\right)(1-t^s)^{\frac{1}{s}-1}t^{s-1}dt.
\end{aligned}$$

Using the inequality (5.156), we deduce that

$$\int_0^1 f\left(ta+(1-t^s)^{\frac{1}{s}}b\right)\left[\frac{1+(1-t^s)^{\frac{1}{s}-1}t^{s-1}}{2}\right]dt \le \frac{f(a)+f(b)}{2}$$

and the inequality (5.155) is proved. ∎

Another result of Hermite-Hadamard type holds [**52**].

THEOREM 183. *With the above assumptions, we have the inequality:*

$$f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \leq \int_0^1 f\left(\frac{a+b}{2^{\frac{1}{s}}}\left[t+(1-t^s)^{\frac{1}{s}}\right]\right)dt \leq \int_0^1 f\left(ta+(1-t^s)^{\frac{1}{s}}b\right)\psi(t)\,dt, \tag{5.157}$$

where ψ is as defined in Theorem 182.

PROOF. As $\frac{1}{s}>1$, we have, by the convexity of the mapping $g:[0,\infty)\to\mathbb{R}$, $g(x)=x^{\frac{1}{s}}$ that

$$\frac{(t^s)^{\frac{1}{s}}+(1-t^s)^{\frac{1}{s}}}{2} \geq \left(\frac{t^s+1-t^s}{2}\right)^{\frac{1}{s}} = \frac{1}{2^{\frac{1}{s}}}$$

and then

$$\frac{a+b}{2^{\frac{1}{s}}}\cdot\frac{t+(1-t^s)^{\frac{1}{s}}}{2} \geq \frac{a+b}{2^{\frac{1}{s}}}\cdot\frac{1}{2^{\frac{1}{s}}}$$

from where we obtain

$$\frac{a+b}{2^{\frac{1}{s}}}\cdot\left[t+(1-t^s)^{\frac{1}{s}}\right] \geq \frac{a+b}{2^{\frac{2}{s}-1}}.$$

As the mapping f is monotonic nondecreasing on $(0,\infty)$, we get

$$f\left(\frac{a+b}{2^{\frac{1}{s}}}\left[t+(1-t^s)^{\frac{1}{s}}\right]\right) \geq f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \text{ for all } t\in[0,1],$$

which gives, by integration on $[0,1]$, the first inequality in (5.157).

As f is $s-$convex in the first sense, we have that

$$f\left(\frac{x+y}{2^{\frac{1}{s}}}\right) \leq \frac{f(x)+f(y)}{2}$$

for all $x,y\in[0,\infty)$.

Let us put $x=ta+(1-t^s)^{\frac{1}{s}}b$, $y=(1-t^s)^{\frac{1}{s}}a+tb$, $t\in[0,1]$. Then we have the inequality

$$\begin{aligned}&\frac{1}{2}\left[f\left(ta+(1-t^s)^{\frac{1}{s}}b\right)+f\left((1-t^s)^{\frac{1}{s}}a+tb\right)\right]\\ \geq\;& f\left(\frac{a+b}{2^{\frac{1}{s}}}\left[t+(1-t^s)^{\frac{1}{s}}\right]\right)\end{aligned}$$

for all $t\in[0,1]$.

If we integrate this inequality on $[0,1]$ over t and take into account the change of variable we used in the proof of the previous theorem, we obtain the desired inequality (5.157). ∎

Some other inequalities of $H.-H.$-type for $s-$convex mappings in the first sense are embodied in the following theorem [**52**].

THEOREM 184. *Let $f:[0,\infty)\to\mathbb{R}$ be a $s-$convex mapping in the first sense with $s\in(0,1)$. If $a,b\in\mathbb{R}_+$ with $a<b$, then one has the inequality:*

$$
\begin{aligned}
f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) &\le \int_0^1 f\left(\frac{a+b}{2^{\frac{1}{s}}}\left[t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right]\right)dt \\
&\le \int_0^1 f\left(at^{\frac{1}{s}}+b(1-t)^{\frac{1}{s}}\right)dt \\
&\le \frac{f(a)+f(b)}{2}.
\end{aligned}
\tag{5.158}
$$

PROOF. By the convexity of the mapping $g(x)=x^{\frac{1}{s}}, s\in(0,1)$, we have that

$$\frac{t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}}{2}\ge\left(\frac{t+1-t}{2}\right)^{\frac{1}{s}}=\frac{1}{2^{\frac{1}{s}}}\quad\text{for all } t\in[0,1].$$

Using the monotonicity of f we have that

$$f\left(\frac{a+b}{2^{\frac{1}{s}}}\left[t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right]\right)\ge f\left(\frac{a+b}{2^{\frac{1}{s}}}\cdot\frac{2}{2^{\frac{1}{s}}}\right)=f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)$$

for all $t\in[0,1]$, from where we get the first inequality in (5.158).

As f is $s-$convex in the first sense, then

$$
\begin{aligned}
&\frac{1}{2}\left[f\left(at^{\frac{1}{s}}+b(1-t)^{\frac{1}{s}}\right)+f\left(a(1-t)^{\frac{1}{s}}+bt^{\frac{1}{s}}\right)\right] \\
\ge\ & f\left(\frac{a+b}{2^{\frac{1}{s}}}\left[t^{s}+(1-t)^{\frac{1}{s}}\right]\right)
\end{aligned}
$$

for all $t\in[0,1]$.

If we integrate on $[0,1]$ over t, we obtain

$$
\begin{aligned}
&\frac{1}{2}\left[\int_0^1 f\left(at^{\frac{1}{s}}+b(1-t)^{\frac{1}{s}}\right)dt+\int_0^1 f\left(a(1-t)^{\frac{1}{s}}+bt^{\frac{1}{s}}\right)dt\right] \\
\ge\ & \int_0^1 f\left(\frac{a+b}{2^{\frac{1}{s}}}\left[t^{s}+(1-t)^{\frac{1}{s}}\right]\right)dt.
\end{aligned}
$$

Using the change of variable $u=1-t, t\in[0,1]$, we get that

$$
\begin{aligned}
\int_0^1 f\left(a(1-t)^{\frac{1}{s}}+bt^{\frac{1}{s}}\right)dt &= -\int_1^0 f\left(au^{\frac{1}{s}}+b(1-u)^{\frac{1}{s}}\right)du \\
&= \int_0^1 f\left(at^{\frac{1}{s}}+b(1-t)^{\frac{1}{s}}\right)dt
\end{aligned}
$$

and the second inequality in (5.158) also holds.

By the $s-$convexity of f on $[0,\infty)$, we have that

$$
\begin{aligned}
&f\left(t^{\frac{1}{s}}a+(1-t)^{\frac{1}{s}}b\right) \\
\le\ & tf(a)+(1-t)f(b)\quad\text{for all } t\in[0,1].
\end{aligned}
$$

If we integrate this inequality over t in $[0,1]$, we deduce that

$$
\begin{aligned}
&\int_0^1 f\left(t^{\frac{1}{s}}a+(1-t)^{\frac{1}{s}}b\right)dt \\
\le\ & f(a)\int_0^1 tdt+f(b)\int_0^1(1-t)\,dt=\frac{f(a)+f(b)}{2}
\end{aligned}
$$

and the theorem is proved. ∎

Finally, we have the following result which gives an upper bound for the integral mean $\frac{1}{b-a}\int_a^b f(x)\,dx$ which is different from the one embodied in the Hermite-Hadamard inequality that holds for general convex mappings [**52**].

THEOREM 185. *Let $f:[0,\infty)\to\mathbb{R}_+$ be a $s-$convex mapping in the first sense with $s\in(0,1)$. If $0<a<b$ and the integral*

$$\int_a^\infty x^{\frac{s+1}{s-1}} f(x)\,dx$$

is finite, then one has the inequality

$$\begin{aligned}(5.159)\qquad &\frac{1}{b-a}\int_a^b f(x)\,dx\\ &\le \frac{s}{1-s}\left[a^{\frac{2s}{1-s}}\int_a^\infty x^{\frac{s+1}{s-1}} f(x)\,dx + b^{\frac{2s}{1-s}}\int_b^\infty x^{\frac{s+1}{s-1}} f(x)\,dx\right].\end{aligned}$$

PROOF. By the $s-$convexity of f on $[0,\infty)$, we have that

$$f\left(u^{\frac{1}{s}}z+(1-u)^{\frac{1}{s}}y\right)\le uf(z)+(1-u)f(y)$$

for all $u\in[0,1]$ and $z,y\ge 0$.

Let $z=u^{1-\frac{1}{s}}a,\ u\in(0,1]$ and $y=(1-u)^{1-\frac{1}{s}}b,\ \ u\in[0,1)$. Then we get the inequality:

$$(5.160)\qquad f(ua+(1-u)b)\le uf\left(u^{1-\frac{1}{s}}a\right)+(1-u)f\left((1-u)^{1-\frac{1}{s}}b\right)$$

for all $u\in(0,1)$.

Now, let us observe that the integral

$$\int_0^1 (1-u)f\left((1-u)^{1-\frac{1}{s}}b\right)du$$

becomes, by the change of variable $t=1-u$, $u\in[0,1)$, the integral

$$\int_0^1 tf\left(t^{1-\frac{1}{s}}b\right)dt.$$

We shall now show that the integral $\int_0^1 uf\left(u^{1-\frac{1}{s}}a\right)du$ is finite too.

If we change the variable $x=u^{1-\frac{1}{s}}a$, $u\in(0,1]$, we get

$$u=\left(\frac{x}{a}\right)^{\frac{1}{1-\frac{1}{s}}}=\left(\frac{x}{a}\right)^{\frac{s}{s-1}}=\frac{x^{\frac{s}{s-1}}}{a^{\frac{s}{s-1}}}$$

and

$$du=\frac{s}{s-1}\cdot\frac{1}{a^{\frac{s}{s-1}}}x^{\frac{s}{s-1}-1}dx=\frac{s}{s-1}\cdot\frac{1}{a^{\frac{s}{s-1}}}x^{\frac{1}{s-1}}dx.$$

Then, we have the equality

$$\begin{aligned}\int_0^1 uf\left(u^{1-\frac{1}{s}}a\right)du &= \int_\infty^a\left[\frac{x^{\frac{s}{s-1}}}{a^{\frac{s}{s-1}}}\cdot\frac{s}{s-1}\cdot\frac{x^{\frac{1}{s-1}}}{a^{\frac{s}{s-1}}}f(x)\right]dx\\ &= \frac{s}{1-s}\cdot a^{\frac{2s}{1-s}}\int_a^\infty x^{\frac{s+1}{s-1}}f(x)\,dx<\infty\end{aligned}$$

and similarly,

$$\int_0^1 tf\left(t^{1-\frac{1}{s}}b\right)dt=\frac{s}{1-s}\cdot b^{\frac{2s}{1-s}}\int_b^\infty x^{\frac{s+1}{s-1}}f(x)\,dx<\infty.$$

Now, if we integrate the inequality (5.160) on $(0,1)$ over u, taking into account that

$$\int_0^1 f(ua+(1-u)b)\,dt=\frac{1}{b-a}\int_a^b f(x)\,dx$$

and

$$\begin{aligned}\int_0^1 uf\left(u^{1-\frac{1}{s}}a\right)du &= \frac{s}{1-s}\cdot a^{\frac{2s}{1-s}}\int_a^\infty x^{\frac{s+1}{s-1}}f(x)\,dx,\\ \int_0^1 (1-u)f\left((1-u)^{1-\frac{1}{s}}b\right)du &= \frac{s}{1-s}\cdot b^{\frac{2s}{1-s}}\int_b^\infty x^{\frac{s+1}{s-1}}f(x)\,dx\end{aligned}$$

respectively, we deduce (5.159). ∎

11. The Case for $s-$Convex Functions in the Second Sense

In the paper [**87**], H. Hudzik and L. Maligranda considered, among others, the following class of functions:

Definition 9. *A function $f:\mathbb{R}_+\to\mathbb{R}$ is said to be $s-$convex in the second sense if*

$$f(\alpha u+\beta v)\le\alpha^s f(u)+\beta^s f(v) \tag{5.161}$$

for all $u,v\ge 0$ and $\alpha,\beta\ge 0$ with $\alpha+\beta=1$ and s fixed in $(0,1]$. They denoted this by $f\in K_s^2$.

Now, we shall point out some results from [**87**] that are connected with $s-$convex functions in the second sense.

Proposition 74. *If $f\in K_s^2$, then f is non-negative on $[0,\infty)$.*

Proof. We have, for $u\in\mathbb{R}_+$

$$f(u)=f\left(\frac{u}{2}+\frac{u}{2}\right)\le\frac{f(u)}{2^s}+\frac{f(u)}{2^s}=2^{1-s}f(u).$$

Therefore, $\left(2^{1-s}-1\right)f(u)\ge 0$ and so $f(u)\ge 0$. ∎

Example 14. [**87**]. *Let $0<s<1$ and $a,b,c\in\mathbb{R}$. Defining for $u\in\mathbb{R}_+$*

$$f(u):=\begin{cases} a & \text{if} \quad u=0\\ bu^s+c & \text{if}\quad u>0\end{cases}$$

we have

(i) *If $b\ge 0$ and $0\le c\le a$, then $f\in K_s^2$;*
(ii) *If $b>0$ and $c<0$, then $f\notin K_s^2$.*

It is important to know where the condition $\alpha+\beta=1$ in the definition of K_s^2 can be equivalently replaced by the condition $\alpha+\beta\le 1$.

The following theorem holds [**87**].

Theorem 186. *Let $f\in K_s^2$. Then inequality (5.161) holds for all $u,v\in\mathbb{R}_+$ and $\alpha,\beta\ge 0$ with $\alpha+\beta\le 1$ if and only if $f(0)=0$.*

PROOF. *Necessity.* Taking $u = v = \alpha = \beta = 0$, we obtain $f(0) \leq 0$ and as $f(0) \geq 0$ (Proposition 74), we get $f(0) = 0$.

Sufficiency. Let $u, v \in \mathbb{R}_+$ and $\alpha, \beta \geq 0$ with $0 < \gamma = \alpha + \beta \leq 1$. Put $a = \frac{\alpha}{\gamma}$ and $b = \frac{\beta}{\gamma}$. Then $a + b = \frac{\alpha}{\gamma} + \frac{\beta}{\gamma} = 1$ and so

$$\begin{aligned} f(\alpha u + \beta v) &= f(\alpha\gamma u + \beta\gamma v) \leq a^s f(\gamma u) + b^s f(\gamma v) \\ &= a^s f(\gamma u + (1-\gamma)\,0) + b^s f(\gamma v + (1-\gamma)\,0) \\ &\leq a^s\left[\gamma^s f(u) + (1-\gamma)^s f(0)\right] + b^s\left[\gamma^s f(v) + (1-\gamma)^s f(0)\right] \\ &= a^s\gamma^s f(u) + b^s\gamma^s f(v) + (a^s + b^s)(1-\gamma)^s f(0) \\ &= \alpha^s f(u) + \beta^s f(v). \end{aligned}$$

∎

Using the above theorem and Theorem 176, which is a similar variant for K_s^1, we can compare both definitions of the $s-$convexity [**87**].

THEOREM 187. a) *Let $0 < s \leq 1$. If $f \in K_s^2$ and $f(0) = 0$, then $f \in K_s^1$.*
b) *Let $0 < s_1 \leq s_2 \leq 1$. If $f \in K_{s_2}^2$ and $f(0) = 0$, then $f \in K_{s_1}^2$.*

PROOF. a) Assume that $f \in K_s^2$ and $f(0) = 0$. For $u, v \in \mathbb{R}_+$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$, we have $\alpha + \beta \leq \alpha^s + \beta^s = 1$, and by Theorem 186 we obtain

$$f(\alpha u + \beta v) \leq \alpha^s f(u) + \beta^s f(v),$$

which means that $f \in K_s^1$.

b) Assume that $f \in K_{s_2}^2$ and that $u, v \geq 0, \alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then we have

$$\begin{aligned} f(\alpha u + \beta v) &\leq \alpha^{s_2} f(u) + \beta^{s_2} f(v) \\ &\leq \alpha^{s_1} f(u) + \beta^{s_1} f(v) \end{aligned}$$

which means that $f \in K_{s_1}^2$.

∎

Using a similar argument as that in the proof of Theorem 179, one can state the following theorem as well [**87**].

THEOREM 188. *Let f be a nondecreasing function in K_s^2 and g be a non-negative convex function on $[0, \infty)$. Then the composition $f \circ g$ of f with g belongs to K_s^2.*

The following corollary for $\psi-$functions, i.e., we recall that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *$\psi-$function* if $f(0) = 0$ and f is nondecreasing and continuous, also holds.

COROLLARY 62. *If ϕ is a convex $\psi-$function and f is a $\psi-$function from K_s^2, then the composition $f \circ \phi$ belongs to K_s^2. In particular, the $\psi-$function $h(u) = [\phi(u)]^s$ belongs to K_s^2.*

REMARK 89. [**87**]. *Let $0 < s < 1$. Then there exists a $\psi-$function f in the class K_s^2 which is neither of the form $\phi(u^s)$ nor $[\phi(u)]^s$, with a convex $\psi-$function ϕ.*

The following inequality is the variant of the Hermite-Hadamard result for $s-$convex functions in the second sense [**53**].

THEOREM 189. *Suppose that $f:\mathbb{R}_+\to\mathbb{R}_+$ is a $s-$convex mapping in the second sense, $s\in(0,1)$ and $a,b\in\mathbb{R}_+$ with $a<b$. If $f\in L_1[a,b]$, then one has the inequalities:*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{s+1}. \tag{5.162}$$

PROOF. As f is $s-$convex in the second sense, we have, for all $t\in[0,1]$

$$f(ta+(1-t)b)\leq t^s f(a)+(1-t)^s f(b).$$

Integrating this inequality on $[0,1]$, we get

$$\begin{aligned}\int_0^1 f(ta+(1-t)b)\,dt &\leq f(a)\int_0^1 t^s dt+f(b)\int_0^1(1-t)^s\,dt\\ &= \frac{f(a)+f(b)}{s+1}.\end{aligned}$$

As the change of variable $x=ta+(1-t)b$ gives us that

$$\int_0^1 f(ta+(1-t)b)\,dt=\frac{1}{b-a}\int_a^b f(x)\,dx,$$

the second inequality in (5.162) is proved.

To prove the first inequality in (5.162), we observe that for all $x,y\in I$ we have

$$f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)}{2^s}. \tag{5.163}$$

Now, let $x=ta+(1-t)b$ and $y=(1-t)a+tb$ with $t\in[0,1]$. Then we get by (5.163) that:

$$f\left(\frac{a+b}{2}\right)\leq\frac{f(ta+(1-t)b)+f((1-t)a+tb)}{2^s}\quad\text{for all } t\in[0,1].$$

Integrating this inequality on $[0,1]$, we deduce the first part of (5.162). ∎

REMARK 90. *The constant $k=\frac{1}{s+1}$ for $s\in(0,1]$ is the best possible in the second inequality in* (5.162).
Indeed, as the mapping $f:[0,1]\to[0,1]$ given by $f(x)=x^s$ is $s-$convex in the second sense (see Corollary 62) and

$$\int_0^1 x^s dx=\frac{1}{s+1}\quad\textit{and}\quad\frac{f(0)+f(1)}{s+1}=\frac{1}{s+1}.$$

Now, suppose that f is Lebesgue integrable on $[a,b]$ and consider the mapping $H:[0,1]\to\mathbb{R}$ given by

$$H(t):=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx.$$

We are interested in pointing out some properties of this mapping as in the case of the classical convex mappings.

The following theorem holds [**53**].

THEOREM 190. *Let $f:I\subseteq\mathbb{R}_+\to\mathbb{R}$ be a $s-$convex mapping in the second sense on I, $s\in(0,1]$ and Lebesgue integrable on $[a,b]\subset I$, $a<b$. Then:*

(*i*) *H is $s-$convex in the second sense on $[0,1]$;*

(ii) *We have the inequality:*

$$H(t) \geq 2^{s-1} f\left(\frac{a+b}{2}\right) \quad \text{for all } t \in [0,1]. \tag{5.164}$$

(iii) *We have the inequality:*

$$H(t) \leq \min\{H_1(t), H_2(t)\}, t \in [0,1] \tag{5.165}$$

where

$$H_1(t) = t^s \cdot \frac{1}{b-a} \int_a^b f(x)\,dx + (1-t)^s f\left(\frac{a+b}{2}\right)$$

and

$$H_2(t) = \frac{f\left(ta + (1-t)\frac{a+b}{2}\right) + f\left(tb + (1-t)\frac{a+b}{2}\right)}{s+1}$$

and $t \in (0,1]$;

(iv) *If* $\tilde{H}(t) := \max\{H_1(t), H_2(t)\}$, $t \in [0,1]$, *then*

$$\tilde{H}(t) \leq t^s \cdot \frac{f(a)+f(b)}{s+1} + (1-t)^s \cdot \frac{2}{s+1} f\left(\frac{a+b}{2}\right), \ t \in [0,1] \tag{5.166}$$

Proof. (i) Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. We have successively

$$\begin{aligned}
& H(\alpha t_1 + \beta t_2) \\
= & \frac{1}{b-a} \int_a^b f\left((\alpha t_1 + \beta t_2)x + [1-(\alpha t_1 + \beta t_2)]\frac{a+b}{2}\right) dx \\
= & \frac{1}{b-a} \int_a^b f\left(\alpha\left[t_1 x + (1-t_1)\frac{a+b}{2}\right] + \beta\left[t_2 x + (1-t_2)\frac{a+b}{2}\right]\right) dx \\
\leq & \frac{1}{b-a} \int_a^b \left[\alpha^s f\left(t_1 x + (1-t_1)\frac{a+b}{2}\right) + \beta^s f\left(\left[t_2 x + (1-t_2)\frac{a+b}{2}\right]\right)\right] dx \\
= & \alpha^s H(t_1) + \beta^s H(t_2),
\end{aligned}$$

which shows that H is $s-$convex in the second sense on $[0,1]$.

(ii) Suppose that $t \in (0,1]$. Then a simple change of variable $u = tx + (1-t)\frac{a+b}{2}$ gives us

$$H(t) = \frac{1}{t(b-a)} \int_{ta+(1-t)\frac{a+b}{2}}^{tb+(1-t)\frac{a+b}{2}} f(u)\,du = \frac{1}{p-q} \int_q^p f(u)\,du$$

where $p = tb + (1-t)\frac{a+b}{2}$ and $q = ta + (1-t)\frac{a+b}{2}$.
Applying the first Hermite-Hadamard inequality, we get:

$$\frac{1}{p-q} \int_q^p f(u)\,du \geq 2^{s-1} f\left(\frac{p+q}{2}\right) = 2^{s-1} f\left(\frac{a+b}{2}\right)$$

and the inequality (5.164) is obtained.
If $t = 0$, we have to prove that

$$f\left(\frac{a+b}{2}\right) \geq 2^{s-1} f\left(\frac{a+b}{2}\right),$$

which is also true.

(iii) Applying the second Hermite-Hadamard inequality, we also have

$$\begin{aligned}\frac{1}{p-q}\int_q^p f(u)\,du &\leq \frac{f(p)+f(q)}{r+1}\\ &= \frac{f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}\right)}{r+1}\\ &= H_2(t)\end{aligned}$$

for all $t\in[0,1]$.

Note that if $t=0$, then the required inequality

$$f\left(\frac{a+b}{2}\right)=H(0)\leq H_2(0)=\frac{2}{r+1}\cdot f\left(\frac{a+b}{2}\right)$$

is true as it is equivalent with

$$(r-1)\,f\left(\frac{a+b}{2}\right)\leq 0$$

and we know that for $r\in(0,1)$, $f\left(\frac{a+b}{2}\right)\geq 0$.

On the other hand, it is obvious that

$$f\left(tx+(1-t)\frac{a+b}{2}\right)\leq t^s f(x)+(1-t)^s f\left(\frac{a+b}{2}\right)$$

for all $t\in[0,1]$ and $x\in[a,b]$.

Integrating this inequality on $[a,b]$ we get (5.165) for $H_1(t)$, and the statement is proved.

(iv) We have

$$\begin{aligned}H_2(t) &\leq \frac{t^s f(a)+(1-t)^s f\left(\frac{a+b}{2}\right)+t^s f(b)+(1-t)^s f\left(\frac{a+b}{2}\right)}{s+1}\\ &= t^s\cdot\frac{f(a)+f(b)}{s+1}+(1-t)^s\cdot\frac{2}{s+1}\cdot f\left(\frac{a+b}{2}\right)\end{aligned}$$

for all $t\in[0,1]$.

On the other hand, we know that

$$\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{s+1}$$

and

$$(1-t)^s f\left(\frac{a+b}{2}\right)\leq(1-t)^s\cdot\frac{2}{s+1}\cdot f\left(\frac{a+b}{2}\right),\ t\in[0,1],$$

which gives us that

$$H_1(t)\leq t^s\cdot\frac{f(a)+f(b)}{s+1}+(1-t)^s\cdot\frac{2}{s+1}\cdot f\left(\frac{a+b}{2}\right)$$

and the theorem is proved.

∎

REMARK 91. *For $s=1$, we get the inequalities:*

$$\begin{aligned} H(t) &\leq \min\left\{t\cdot\frac{1}{b-a}\int_a^b f(x)\,dx+(1-t)\cdot f\left(\frac{a+b}{2}\right),\right. \\ &\qquad \left.\frac{f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}\right)}{2}\right\} \end{aligned}$$

and

$$\tilde{H}(t)\leq t\cdot\frac{f(a)+f(b)}{2}+(1-t)\cdot f\left(\frac{a+b}{2}\right)$$

for all $t\in[0,1]$, which complements, in a sense, the results from Section 5 of Chapter III.

Now, assume that $f:[a,b]\to\mathbb{R}$ is Lebesgue integrable on $[a,b]$.

Consider the map

$$F(t):=\frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy,\ t\in[0,1].$$

The following theorem contains the main properties of this mapping [**53**].

THEOREM 191. *Let $f:I\subseteq\mathbb{R}_+\to\mathbb{R}_+$ be a $s-$convex mapping in the second sense, $s\in(0,1]$, $a,b\in I$ with $a<b$ and f Lebesgue integrable on $[a,b]$. Then:*

(i) $F\left(s+\frac{1}{2}\right)=F\left(\frac{1}{2}-s\right)$ *for all* $s\in\left[0,\frac{1}{2}\right]$ *and* $F(t)=F(1-t)$ *for all* $t\in[0,1]$;

(ii) F *is $s-$convex in the second sense on* $[0,1]$;

(iii) We have the inequality:

$$2^{1-s}F(t)\geq F\left(\frac{1}{2}\right)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy,\ t\in[0,1]. \tag{5.167}$$

(iv) We have the inequality

$$F(t)\geq 2^{s-1}H(t)\geq 4^{s-1}f\left(\frac{a+b}{2}\right)\quad \textit{for all } t\in[0,1] \tag{5.168}$$

(v) We have the inequality:

$$\begin{aligned} F(t) &\leq \min\left\{\left[t^s+(1-t)^s\right]\frac{1}{b-a}\int_a^b f(x)\,dx,\right. \\ &\qquad \left.\frac{f(a)+f(ta+(1-t)b)+f((1-t)a+tb)+f(b)}{(s+1)^2}\right\} \end{aligned} \tag{5.169}$$

for all $t\in[0,1]$.

PROOF. (i) It is obvious.

(ii) Goes likewise to the proof of Theorem 190.

(iii) By the fact that f is $s-$convex in the second sense on I, we have

$$\frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2^s}\geq f\left(\frac{x+y}{2}\right)$$

for all $t\in[0,1]$ and $x,y\in[a,b]$. Integrating this inequality on $[a,b]^2$ we get

$$\begin{aligned}&\frac{1}{2^s}\left[\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy+\int_a^b\int_a^b f((1-t)x+ty)\,dxdy\right]\\ \geq\ &\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy.\end{aligned}$$

Since

$$\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy=\int_a^b\int_a^b f((1-t)x+ty)\,dxdy,$$

the above inequality gives us the desired result (5.167).

(iv) First of all, let us observe that

$$F(t)=\frac{1}{b-a}\int_a^b\left[\frac{1}{b-a}\int_a^b f(tx+(1-t)y)\,dx\right]dy.$$

Now, for y fixed in $[a,b]$, we can consider the map $H_y:[0,1]\to\mathbb{R}$ given by

$$H_y(t):=\frac{1}{b-a}\int_a^b f(tx+(1-t)y)\,dx.$$

As shown in the proof of Theorem 190, for $t\in[0,1]$ we have the equality

$$H_y(t)=\frac{1}{p-q}\int_q^p f(u)\,du$$

where $p=tb+(1-t)y,\ \ q=ta+(1-t)y$. Applying the Hermite-Hadamard inequality we get that

$$\frac{1}{p-q}\int_q^p f(u)\,du\geq 2^{s-1}f\left(\frac{p+q}{2}\right)=2^{s-1}f\left(t\cdot\frac{a+b}{2}+(1-t)y\right)$$

for all $t\in(0,1)$ and $y\in[a,b]$. Integrating on $[a,b]$ over y, we easily deduce

$$F(t)\geq 2^{s-1}H(1-t)\quad\text{for all } t\in(0,1).$$

As $F(t)=F(1-t)$, the inequality (5.168) is proved for $t\in(0,1)$.
If $t=0$ or $t=1$, the inequality (5.168) also holds. We shall omit the details.

(v) By the definition of $s-$convex mappings in the second sense, we have

$$f(tx+(1-t)y)\leq t^sf(x)+(1-t)^sf(y)$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. Integrating this inequality on $[a,b]^2$, we deduce the first part of the inequality (5.169).
Now, let us observe, by the second part of the Hermite-Hadamard inequality, that

$$H_y(t)=\frac{1}{p-q}\int_q^p f(u)\,du\leq\frac{f(tb+(1-t)y)+f(ta+(1-t)y)}{s+1},$$

where $p = tb + (1-t)\,y$ and $q = ta + (1-t)\,y,\ t \in [0,1]$.

Integrating this inequality on $[a,b]$ over y, we deduce

$$\begin{aligned} & F(t) \\ \leq\ & \frac{1}{s+1}\left[\frac{1}{b-a}\int_a^b f(tb+(1-t)\,y)\,dy + \frac{1}{b-a}\int_a^b f(ta+(1-t)\,y)\,dy\right]. \end{aligned}$$

A simple calculation shows that

$$\begin{aligned} & \frac{1}{b-a}\int_a^b f(tb+(1-t)\,y)\,dy \\ =\ & \frac{1}{r-l}\int_l^r f(u)\,du \leq \frac{f(r)+f(l)}{s+1} \\ =\ & \frac{f(b)+f(tb+(1-t)\,a)}{s+1}, \end{aligned}$$

where $r = b,\ l = tb + (1-t)\,a,\ t \in (0,1)$; and similarly,

$$\frac{1}{b-a}\int_a^b f(ta+(1-t)\,y)\,dy \leq \frac{f(a)+f(ta+(1-t)\,b)}{s+1},\ t \in (0,1),$$

which gives, by addition, the second inequality in (5.169).

If $t = 0$ or $t = 1$, then this inequality also holds.

We shall omit the details.

∎

12. Inequalities for $m-$Convex and $(\alpha,m)-$Convex Functions

In the paper [**178**], G. H. Toader defines the $m-$convexity, an intermediate between the usual convexity and starshaped property.

In the first part of this section we shall present properties of $m-$convex functions in a similar manner to convex functions.

The following concept has been introduced in [**178**] (see also [**177**] and [**69**]).

Definition 10. *The function $f : [0,b] \to \mathbb{R}$ is said to be m-convex, where $m \in [0,1]$, if for every $x, y \in [0,b]$ and $t \in [0,1]$ we have:*

$$f(tx + m(1-t)\,y) \leq tf(x) + m(1-t)\,f(y). \tag{5.170}$$

Denote by $K_m(b)$ the set of the $m-$convex functions on $[0,b]$ for which $f(0) \leq 0$.

Remark 92. *For $m = 1$, we recapture the concept of convex functions defined on $[0,b]$ and for $m = 0$ we get the concept of starshaped functions on $[0,b]$. We recall that $f : [0,b] \to \mathbb{R}$ is starshaped if*

$$f(tx) \leq tf(x) \quad \text{for all } t \in [0,1] \quad \text{and} \quad x \in [0,b]. \tag{5.171}$$

The following lemmas hold [**177**].

Lemma 20. *If f is in the class $K_m(b)$, then it is starshaped.*

Proof. For any $x \in [0,b]$ and $t \in [0,1]$, we have:

$$f(tx) = f(tx + m(1-t)\cdot 0) \leq tf(x) + m(1-t)\,f(0) \leq tf(x).$$

∎

Lemma 21. *If f is $m-$convex and $0 \leq n < m \leq 1$, then f is $n-$convex.*

PROOF. If $x, y \in [0, b]$ and $t \in [0, 1]$, then

$$\begin{aligned} f(tx + n(1-t)y) &= f\left(tx + m(1-t)\left(\frac{n}{m}\right)y\right) \\ &\leq tf(x) + m(1-t)f\left(\left(\frac{n}{m}\right)y\right) \\ &\leq tf(x) + m(1-t)\ \frac{n}{m}\ f(y) \\ &= tf(x) + n(1-t)f(y) \end{aligned}$$

and the lemma is proved. ∎

As in paper [**109**] due to V. G. Miheşan, for a mapping $f \in K_m(b)$ consider the function

$$p_{a,m}(x) := \frac{f(x) - mf(a)}{x - m}$$

defined for $x \in [0, b] \setminus \{ma\}$, for fixed $a \in [0, b]$, and

$$r_m(x_1, x_2, x_3) := \frac{\begin{vmatrix} 1 & 1 & 1 \\ mx_1 & x_2 & x_3 \\ mf(x_1) & f(x_2) & f(x_3) \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ mx_1 & x_2 & x_3 \\ m^2x_1^2 & x_2^2 & x_3^2 \end{vmatrix}},$$

where $x_1, x_2, x_3 \in [0, b]$, $(x_2 - mx_1)(x_3 - mx_1) > 0$, $x_2 \neq x_3$.

The following theorem holds [**109**].

THEOREM 192. *The following assertions are equivalent:*

$1°$. $f \in K_m(b)$;
$2°$. $p_{a,m}$ *is increasing on the intervals* $[0, ma)$, $(ma, b]$ *for all* $a \in [0, b]$;
$3°$. $r_m(x_1, x_2, x_3) \geq 0$.

PROOF. $1° \Rightarrow 2°$. Let $x, y \in [0, b]$. If $ma < x < y$, then there exists $t \in (0, 1)$ such that

$$x = ty + m(1-t)a. \tag{5.172}$$

We thus have

$$\begin{aligned} p_{a,m}(x) &= \frac{f(x) - mf(a)}{x - ma} \\ &= \frac{f(ty + m(1-t)a) - mf(a)}{ty + m(1-t)a - ma} \\ &\leq \frac{tf(y) + m(1-t)f(a) - mf(a)}{t(y - ma)} \\ &= \frac{f(y) - mf(a)}{y - ma} \\ &= p_{a,m}(y). \end{aligned}$$

If $y < x < ma$, there also exists $t \in (0, 1)$ for which (5.172) holds.

Then we have:

$$\begin{aligned} p_{a,m}(x) &= \frac{f(x) - mf(a)}{x - ma} \\ &= \frac{mf(a) - f(ty + m(1-t)a)}{ma - ty - m(1-t)a} \\ &\geq \frac{mf(a) - tf(y) + m(1-t)f(a)}{t(ma - y)} \\ &= \frac{f(y) - mf(a)}{y - ma} \\ &= p_{a,m}(y). \end{aligned}$$

$2° \Rightarrow 3°$. A simple calculation shows that

$$r_m(x_1, x_2, x_3) = \frac{p_{x_1,m}(x_3) - p_{x_1,m}(x_2)}{x_3 - x_2}.$$

Since $p_{x_1,m}$ is increasing on the intervals $[0, mx_1), (mx_1, b]$, one obtains $r_m(x_1, x_2, x_3) \geq 0$.

$3° \Rightarrow 1°$. Let $x_1, x_3 \in [0, b]$ and let $x_2 = tx_3 + m(1-t)x_1,\ t \in (0,1)$. Obviously $mx_1 < x_2 < x_3$ or $x_3 < x_2 < mx_1$, hence

$$r_m(x_1, x_2, x_3) = \frac{tf(x_3) + m(1-t)f(x_1) - f(tx_3 + m(1-t)x_1)}{t(1-t)(x_3 - mx_1)^2}$$

from where we obtain (5.170), i.e., $f \in K_m(b)$. ∎

The following corollary holds for starshaped functions.

Corollary 63. *Let $f : [0, b] \to \mathbb{R}$. The following statements are equivalent*

(i) *f is starshaped;*

(ii) *The mapping $p(x) := \frac{f(x)}{x}$ is increasing on $(0, b]$.*

The following lemma is also interesting in itself.

Lemma 22. *If f is differentiable on $[0, b]$, then $f \in K_m(b)$ if and only if:*

$$f'(x) \geq \frac{f(x) - mf(y)}{x - my} \tag{5.173}$$

for $x > my,\ y \in (0, b]$.

Proof. The mapping $p_{y,m}$ is increasing on $(my, b]$ iff $p'_{y,m}(x) \geq 0$, which is equivalent with the condition (5.173). ∎

Corollary 64. *If f is differentiable in $[0, b]$, then f is starshaped iff $f'(x) \geq \frac{f(x)}{x}$ for all $x \in (0, b]$.*

The following inequalities of Hermite-Hadamard type for $m-$convex functions hold [**69**].

Theorem 193. *Let $f : [0, \infty) \to \mathbb{R}$ be a $m-$convex function with $m \in (0, 1]$. If $0 \leq a < b < \infty$ and $f \in L_1[a, b]$, then one has the inequality:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \min\left\{\frac{f(a) - mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) - mf\left(\frac{a}{m}\right)}{2}\right\}. \tag{5.174}$$

PROOF. Since f is $m-$convex, we have

$$f\left(tx+m\left(1-t\right)y\right)\leq tf\left(x\right)+m\left(1-t\right)f\left(y\right),\text{ for all }x,y\geq 0,$$

which gives:

$$f\left(ta+\left(1-t\right)b\right)\leq tf\left(a\right)+m\left(1-t\right)f\left(\frac{b}{m}\right)$$

and

$$f\left(tb+\left(1-t\right)b\right)\leq tf\left(b\right)+m\left(1-t\right)f\left(\frac{a}{m}\right)$$

for all $t\in[0,1]$. Integrating on $[0,1]$ we obtain

$$\int_0^1 f\left(ta+\left(1-t\right)b\right)dt\leq\frac{\left[f\left(a\right)+mf\left(\frac{b}{m}\right)\right]}{2}$$

and

$$\int_0^1 f\left(tb+\left(1-t\right)b\right)dt\leq\frac{\left[f\left(b\right)+mf\left(\frac{a}{m}\right)\right]}{2}.$$

However,

$$\int_0^1 f\left(ta+\left(1-t\right)b\right)dt=\int_0^1 f\left(tb+\left(1-t\right)a\right)dt=\frac{1}{b-a}\int_a^b f\left(x\right)dx$$

and the inequality (5.174) is obtained. ∎

Another result of this type which holds for differentiable functions is embodied in the following theorem [**69**].

THEOREM 194. *Let $f:[0,\infty)\to\mathbb{R}$ be a $m-$convex function with $m\in(0,1]$. If $0\leq a<b<\infty$ and f is differentiable on $(0,\infty)$, then one has the inequality:*

$$\begin{aligned}(5.175)\qquad \frac{f\left(mb\right)}{m}-\frac{b-a}{2}f'\left(mb\right) &\leq \frac{1}{b-a}\int_a^b f\left(x\right)dx\\ &\leq \frac{\left(b-ma\right)f\left(b\right)-\left(a-mb\right)f\left(a\right)}{2\left(b-a\right)}.\end{aligned}$$

PROOF. Using Lemma 22, we have for all $x,y\geq 0$ with $x\geq my$ that

$$(5.176)\qquad \left(x-my\right)f'\left(x\right)\geq f\left(x\right)-mf\left(y\right).$$

Choosing in the above inequality $x=mb$ and $a\leq y\leq b$, then $x\geq my$ and

$$\left(mb-my\right)f'\left(mb\right)\geq f\left(mb\right)-mf\left(y\right).$$

Integrating over y on $[a,b]$, we get

$$m\frac{\left(b-a\right)^2}{2}f'\left(mb\right)\geq\left(b-a\right)f\left(mb\right)-m\int_a^b f\left(y\right)dy,$$

thus proving the first inequality in (5.175).
Putting in (5.176) $y=a$, we have

$$\left(x-ma\right)f'\left(x\right)\geq f\left(x\right)-mf\left(a\right),\ x\geq ma.$$

Integrating over x on $[a,b]$, we obtain the second inequality in (5.175). ∎

REMARK 93. *The second inequality from* (5.175) *is also valid for* $m=0$. *That is, if* $f:[0,\infty)\to\mathbb{R}$ *is a differentiable starshaped function, then for all* $0\leq a<b<\infty$ *one has:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{bf(b)-af(a)}{2(b-a)},$$

which also holds from Corollary 64.

We will now point out another result of Hermite-Hadamard type [**36**].

THEOREM 195. *Let* $f:[0,\infty)\to\mathbb{R}$ *be a* $m-$*convex function with* $m\in(0,1]$ *and* $0\leq a<b$. *If* $f\in L_1[a,b]$, *then one has the inequalities*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b\frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx\leq\frac{m+1}{4}\left[\frac{f(a)+f(b)}{2}+m\cdot\frac{f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)}{2}\right].\tag{5.177}$$

PROOF. By the $m-$convexity of f we have that

$$f\left(\frac{x+y}{2}\right)\leq\frac{1}{2}\left[f(x)+mf\left(\frac{x}{m}\right)\right]$$

for all $x,y\in[0,\infty)$.

If we choose $x=ta+(1-t)b,\ y=(1-t)a+tb$, we deduce

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2}\left[f(ta+(1-t)b)+mf\left(t\cdot\frac{a}{m}+(1-t)\cdot\frac{b}{m}\right)\right]$$

for all $t\in[0,1]$.

Integrating over $t\in[0,1]$ we get

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2}\left[\int_0^1 f(ta+(1-t)b)\,dt+m\int_0^1 f\left(t\cdot\frac{a}{m}+(1-t)\cdot\frac{b}{m}\right)dt\right].\tag{5.178}$$

Taking into account that

$$\int_0^1 f(ta+(1-t)b)\,dt=\frac{1}{b-a}\int_a^b f(x)\,dx,$$

and

$$\int_0^1 f\left(t\cdot\frac{a}{m}+(1-t)\cdot\frac{b}{m}\right)dt=\frac{m}{b-a}\int_{\frac{a}{m}}^{\frac{b}{m}}f(x)\,dx=\frac{1}{b-a}\int_a^b f\left(\frac{x}{m}\right)dx,$$

we deduce from (5.178) the first part of (5.177).

By the $m-$convexity of f we also have

$$\frac{1}{2}\left[f(ta+(1-t)b)+mf\left(t\cdot\frac{a}{m}+(1-t)\cdot\frac{b}{m}\right)\right]\leq\frac{1}{2}\left[tf(a)+(1-t)f(b)+mtf\left(\frac{a}{m}\right)+m^2(1-t)f\left(\frac{b}{m}\right)\right]\tag{5.179}$$

for all $t\in[0,1]$.

Integrating the inequality (5.179) over t on $[0,1]$, we deduce

$$\begin{aligned}(5.180)\qquad & \frac{1}{b-a}\int_a^b \frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx \\ \leq\ & \frac{1}{2}\left[\frac{f(a)+mf(b)}{2}+\frac{mf\left(\frac{a}{m}\right)+m^2 f\left(\frac{b}{m}\right)}{2}\right].\end{aligned}$$

By a similar argument we can state:

$$\begin{aligned}(5.181)\qquad & \frac{1}{b-a}\int_a^b \frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx \\ \leq\ & \frac{1}{4}\left[\frac{f(a)+f(b)+m(f(a)+f(b))}{2}\right. \\ & \left. + m\cdot\frac{f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)+m\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)}{2}\right] \\ =\ & \frac{m+1}{4}\left[\frac{f(a)+f(b)}{2}+m\cdot\frac{f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)}{2}\right]\end{aligned}$$

and the proof is completed. ∎

REMARK 94. *For $m=1$, we can drop the assumption $f\in L_1[a,b]$ and* (5.177) *exactly becomes the Hermite-Hadamard inequality.*

The following result also holds [**36**].

THEOREM 196. *Let $f:[0,\infty)\to\mathbb{R}$ be a $m-$convex function with $m\in(0,1]$. If $f\in L_1[am,b]$ where $0\leq a<b$, then one has the inequality:*

$$\begin{aligned}(5.182)\qquad & \frac{1}{m+1}\cdot\left[\frac{1}{mb-a}\int_a^{mb} f(x)\,dx+\frac{1}{b-ma}\int_{ma}^b f(x)\,dx\right] \\ \leq\ & \frac{f(a)+f(b)}{2}.\end{aligned}$$

PROOF. By the $m-$convexity of f we can write:

$$\begin{aligned} f(ta+m(1-t)b) &\leq tf(a)+m(1-t)f(b), \\ f((1-t)a+mtb) &\leq (1-t)f(a)+mtf(b), \\ f(tb+(1-t)ma) &\leq tf(b)+m(1-t)f(a)\end{aligned}$$

and

$$f((1-t)b+tma)\leq(1-t)f(b)+mtf(a)$$

for all $t\in[0,1]$ and a,b as above.

If we add the above inequalities we get

$$\begin{aligned} & f(ta+m(1-t)b)+f((1-t)a+mtb) \\ & +f(tb+(1-t)ma)+f((1-t)b+tma) \\ \leq\ & f(a)+f(b)+m(f(a)+f(b))=(m+1)(f(a)+f(b)).\end{aligned}$$

Integrating over $t\in[0,1]$, we obtain

$$\begin{aligned}(5.183)\qquad &\int_0^1 f(ta+m(1-t)b)\,dt+\int_0^1 f((1-t)a+mtb)\,dt\\ &+\int_0^1 f(tb+m(1-t)a)\,dt+\int_0^1 f((1-t)b+mta)\,dt\\ \le\ &(m+1)(f(a)+f(b)).\end{aligned}$$

As it is easy to see that

$$\int_0^1 f(ta+m(1-t)b)\,dt=\int_0^1 f((1-t)a+mtb)\,dt=\frac{1}{mb-a}\int_a^{mb} f(x)\,dx$$

and

$$\int_0^1 f(tb+m(1-t)a)\,dt=\int_0^1 f((1-t)b+mta)\,dt=\frac{1}{b-ma}\int_{ma}^{b} f(x)\,dx,$$

from (5.183) we deduce the desired result, namely, the inequality (5.182). ∎

In the paper [**53**], V. G. Miheşan introduced the following class of mappings.

Definition 11. *The function $f:[0,b]\to\mathbb{R}$ is said to be $(\alpha,m)-$convex, where $(\alpha,m)\in[0,1]^2$, if for every $x,y\in[0,b]$ and $t\in[0,1]$ we have*

$$f(tx+m(1-t)y)\le t^{\alpha}f(x)+m(1-t^{\alpha})f(y). \tag{5.184}$$

Note that for $(\alpha,m)\in\{(0,0),(\alpha,0),(1,0),(1,m),(1,1),(\alpha,1)\}$ one obtains the following classes of functions: increasing, $\alpha-$starshaped, starshaped, $m-$convex, convex and $\alpha-$convex.

Denote by $K_m^{\alpha}(b)$ the set of the $(\alpha,m)-$convex functions on $[0,b]$ for which $f(0)\le 0$. Then the following result holds [**109**].

Theorem 197. *The mapping f belongs to $K_m^{\alpha}(b)$ if and only if*

$$p_{a,m}^{\alpha}(x):=\frac{f(x)-mf(a)}{(x-ma)^{\alpha}}$$

is increasing on $(ma,b]$.

Proof. Let $f\in K_m^{\alpha}(b)$ and let $x,y\in(ma,b]$ with $ma<x<y$. Then there exists some $t\in(0,1)$ such that $x=ty+m(1-t)a$. We have:

$$\begin{aligned}p_{a,m}^{\alpha}(x)&=\frac{f(x)-mf(a)}{(x-ma)^{\alpha}}\\ &=\frac{f(ty+m(1-t)a)-mf(a)}{(ty+m(1-t)a-ma)^{\alpha}}\\ &\le\frac{t^{\alpha}f(y)+m(1-t^{\alpha})f(a)-mf(a)}{t^{\alpha}(y-ma)^{\alpha}}\\ &=\frac{f(y)-mf(a)}{(y-ma)^{\alpha}}\\ &=p_{a,m}^{\alpha}(y).\end{aligned}$$

Reciprocally, if $p_{a,m}^{\alpha}$ is increasing, for $ma<x<y$ (a arbitrary in $(0,b]$), we have $p_{a,m}^{\alpha}(x)\le p_{a,m}^{\alpha}(y)$. That is,

$$\frac{f(x)-mf(a)}{(x-ma)^{\alpha}}\le\frac{f(y)-mf(a)}{(y-ma)^{\alpha}}.$$

Hence

$$f(x) \leq \left(\frac{x-ma}{y-ma}\right)^{\alpha} f(y) + m\left(1-\left(\frac{x-ma}{y-ma}\right)^{\alpha}\right) f(a).$$

Denote $t = \frac{(x-ma)}{(y-ma)} \in (0,1)$, then we obtain

$$f(ty + m(1-t)a) \leq t^{\alpha} f(y) + m(1-t)^{\alpha} f(a). \tag{5.185}$$

Consequently, for all $a, y \in (0,b]$ with $a < y$ and $t \in (0,1)$ we have the inequality (5.185).

The other cases go likewise and we shall omit the details. ∎

Corollary 65. *If f is differentiable on $(0,b)$, then $f \in K_m^{\alpha}(b)$ if and only if we have*

$$f'(x) \geq \frac{\alpha(f(x) - mf(a))}{(x-ma)}, \quad \text{for } x > ma. \tag{5.186}$$

Proof. By Theorem 197, $f \in K_m^{\alpha}(b)$ iff $\left(p_{a,m}^{\alpha}(x)\right)' \geq 0$ for all $x \in (ma,b]$, which is obviously equivalent with (5.186). ∎

Remark 95. *Similar results of Hermite-Hadamard type can be stated for this class of mappings, but we omit the details* [**27**].

13. Inequalities for Convex-Dominated Functions

In [**54**], S. S. Dragomir and N. M. Ionescu introduced the following class of functions.

Definition 12. *Let $g : I \to \mathbb{R}$ be a given convex function on the interval I from $\mathbb{R}$. The real function $f : I \to \mathbb{R}$ is called $g-$convex dominated on I if the following condition is satisfied:*

$$\begin{aligned} & |\lambda f(x) + (1-\lambda) f(y) - f(\lambda x + (1-\lambda) y)| \\ \leq \quad & \lambda g(x) + (1-\lambda) g(y) - g(\lambda x + (1-\lambda) y) \end{aligned} \tag{5.187}$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

The next simple characterisation of convex-dominated functions holds [**54**]:

Lemma 23. *Let g be a convex function on I and $f : I \to \mathbb{R}$. The following statements are equivalent:*

(i) *f is $g-$convex dominated on I;*
(ii) *The mappings $g - f$ and $g + f$ are convex on I;*
(iii) *There exist two convex mappings h, k defined on I such that*

$$f = \frac{1}{2}(h-k) \ \text{ and } g = \frac{1}{2}(h+k).$$

Proof. "$(i) \Longleftrightarrow (ii)$". The condition (5.187) is equivalent with

$$\begin{aligned} & g(\lambda x + (1-\lambda) y) - \lambda g(x) - (1-\lambda) g(y) \\ \leq \quad & \lambda f(x) + (1-\lambda) f(y) - f(\lambda x + (1-\lambda) y) \\ \leq \quad & \lambda g(x) + (1-\lambda) g(y) - g(\lambda x + (1-\lambda) y) \end{aligned}$$

for all $x, y \in I$ and $\lambda \in [0,1]$, or additionally, with

$$\begin{aligned} & \lambda(f(x) + g(x)) + (1-\lambda)(f(x) + g(x)) \\ \geq \quad & f(\lambda x + (1-\lambda) y) + g(\lambda x + (1-\lambda) y) \end{aligned}$$

and

$$\begin{aligned} & \lambda(g(x)-f(x))+(1-\lambda)(g(x)-f(x)) \\ \geq\ & g(\lambda x+(1-\lambda) y)-f(\lambda x+(1-\lambda) y) \end{aligned}$$

for all $x, y \in I$ and $\lambda \in[0,1]$.

The equivalence "$(ii) \Longleftrightarrow (iii)$" is obvious and we shall omit the details. ∎

The following inequality of Hermite-Hadamard type for functions that are convex-dominated holds [**63**].

Theorem 198. *Let $g: I \rightarrow \mathbb{R}$ be a convex mapping on I and $f: I \rightarrow \mathbb{R}$ a $g-$convex-dominated mapping. Then, for all $a, b \in I$ with $a<b$, one has the inequalities:*

$$\begin{aligned} & \left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a} \int_a^b f(x) d x\right| \\ \leq\ & \frac{1}{b-a} \int_a^b g(x) d x-g\left(\frac{a+b}{2}\right) \end{aligned} \tag{5.188}$$

and

$$\begin{aligned} & \left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a} \int_a^b f(x) d x\right| \\ \leq\ & \frac{g(a)+g(b)}{2}-\frac{1}{b-a} \int_a^b g(x) d x . \end{aligned} \tag{5.189}$$

Proof. We shall give two proofs. The fact that f is integrable follows by Lemma 23 (iii).

(1) As the mapping f is $g-$convex-dominated, we have that

$$\left|\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right| \leq \frac{g(x)+g(y)}{2}-g\left(\frac{x+y}{2}\right)$$

for all $x \in[a, b]$.

Choose $x=t a+(1-t) b,\ y=(1-t) a+t b,\ t \in[0,1]$. Then we get

$$\begin{aligned} & \left|\frac{f(t a+(1-t) b)+f((1-t) a+t b)}{2}-f\left(\frac{a+b}{2}\right)\right| \\ \leq\ & \frac{g(t a+(1-t) b)+g((1-t) a+t b)}{2}-g\left(\frac{a+b}{2}\right) \end{aligned}$$

for all $t \in[0,1]$.

Integrating over t on $[0,1]$ we deduce that

$$\begin{aligned} & \left|\frac{\int_0^1 f(t a+(1-t) b) d t+\int_0^1 f((1-t) a+t b) d t}{2}-f\left(\frac{a+b}{2}\right)\right| \\ \leq\ & \frac{\int_0^1 g(t a+(1-t) b) d t+\int_0^1 g((1-t) a+t b) d t}{2}-g\left(\frac{a+b}{2}\right) \end{aligned}$$

and the inequality (5.188) is proved.
For the second inequality we observe that

$$\begin{aligned} & |tf(a)+(1-t)f(b)-f(ta+(1-t)b)| \\ \leq\ & tg(a)+(1-t)g(b)-g(ta+(1-t)b) \end{aligned}$$

for all $t\in[0,1]$.
Integrating this inequality over $t\in[0,1]$, we obtain

$$\begin{aligned} & \left|f(a)\int_0^1 tdt+f(b)\int_0^1(1-t)\,dt-\int_0^1 f(ta+(1-t)b)\,dt\right| \\ \leq\ & g(a)\int_0^1 tdt+g(b)\int_0^1(1-t)\,dt-\int_0^1 g(ta+(1-t)b)\,dt, \end{aligned}$$

which is equivalent with (5.189).

(2) Since f is $g-$convex-dominated, then by Lemma 23, it follows that $f+g$ and $g-f$ are convex on $[a,b]$. Then we have, by the classical Hermite-Hadamard inequality:

$$\begin{aligned} (f+g)\left(\frac{a+b}{2}\right) & \leq \frac{1}{b-a}\int_a^b (f+g)(x)\,dx \\ & \leq \frac{(f+g)(a)+(f+g)(b)}{2} \end{aligned}$$

and

$$\begin{aligned} (g-f)\left(\frac{a+b}{2}\right) & \leq \frac{1}{b-a}\int_a^b (g-f)(x)\,dx \\ & \leq \frac{(g-f)(a)+(g-f)(b)}{2}, \end{aligned}$$

which are equivalent with (5.188) and (5.189) respectively.

∎

The following corollaries are interesting as they also contain some examples of convex-dominated functions [**63**].

Corollary 66. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a twice differentiable mapping with the property that $|f''(x)|\leq Mx^p$ $(M>0)$ where $p\in\mathbb{R}\setminus\{-3,-2,-1\}$ and $x\in[a,b]$. Then we have the inequalities:*

$$\begin{aligned} & \left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq\ & \frac{M}{(p+1)(p+2)}\left[[L_{p+2}(a,b)]^{p+2}-[A(a,b)]^{p+2}\right], \end{aligned} \tag{5.190}$$

where L_p is the generalised logarithmic mean, i.e.,

$$L_p(x,y)=\left[\frac{y^{p+1}-x^{p+1}}{(p+1)(y-x)}\right]^{\frac{1}{p}} \quad \textit{for } x\neq y$$

and $A(x,y)=\frac{x+y}{2}$ *is the usual arithmetic mean.*
We also have:

$$\begin{aligned}(5.191)\qquad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\leq \frac{M}{(p+1)(p+2)}\left[A\left(a^{p+2},b^{p+2}\right)-\left[L_{p+2}(a,b)\right]^{p+2}\right].\end{aligned}$$

Proof. Define the mapping $g:[a,b]\to\mathbb{R}$, $g(x)=\frac{Mx^{p+2}}{(p+1)(p+2)}$,
$p\in\mathbb{R}\backslash\{-3,-2,-1\}$. Then $g''(x)=Mx^p$, i.e., the mapping g is convex on $[a,b]$. Moreover, since $|f''(x)|\leq Mx^p$, $x\in[a,b]$, then f is $g-$convex-dominated on $[a,b]$ as $f''(x)+Mx^p\geq 0$, $Mx^p-f''(x)\geq 0$, $x\in[a,b]$ (see Lemma 23).
Applying the above theorem, we deduce the inequalities (5.190) and (5.191). ∎

Remark 96. *Let f be a twice differentiable mapping on $[a,b]$ and assume that $M:=\sup_{x\in[a,b]}|f''(x)|<\infty$. Then we have the inequalities:*

$$(5.192)\qquad \left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{M}{24}(b-a)^2$$

and

$$(5.193)\qquad \left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{M}{12}(b-a)^2.$$

The proof is obvious by the above theorem [**63**].

Corollary 67. *Let $f:[a,b]\to(0,\infty)$ be a twice differentiable function such that $|f''(x)|\leq\frac{M}{x^3}$ $(M>0)$, $x\in(a,b)$, $0<a<b$. Then one has the inequalities:*

$$(5.194)\qquad \left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{M}{2}\cdot\left[\frac{A(a,b)-L(a,b)}{A(a,b)\,L(a,b)}\right]$$

and

$$(5.195)\qquad \left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{M}{2}\cdot\left[\frac{L(a,b)-H(a,b)}{H(a,b)\,L(a,b)}\right]$$

respectively.

Proof. Consider the mapping $g:[a,b]\to\mathbb{R}$, $g(x)=\frac{M}{2x}$. Then $g''(x)=\frac{M}{x^3}$ and, as $|f''(x)|\leq\frac{M}{x^3}$, it follows that f is $g-$convex-dominated.
Now, if we apply Theorem 198, we can easily deduce the inequalities (5.194) and (5.195). ∎

Corollary 68. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a twice differentiable function such that $|f''(x)|\leq\frac{M}{x^2}$ for all $x\in(a,b)$. Then one has the inequalities:*

$$(5.196)\qquad \exp\left[\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\right]\leq\left[\frac{A(a,b)}{I(a,b)}\right]^M$$

and

$$(5.197)\qquad \exp\left[\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\right]\leq\left[\frac{I(a,b)}{G(a,b)}\right]^M.$$

PROOF. Consider the mapping $g:[a,b]\to\mathbb{R}$, $g(x)=-M\ln x$. Then $g''(x)=\frac{M}{x^2}$ and, since $|f''(x)|\le\frac{M}{x^2}$, it follows that f is $g-$convex-dominated on $[a,b]$. Applying Theorem 198, we can write that

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le M\left[\ln\frac{a+b}{2}-\frac{\int_a^b\ln x dx}{b-a}\right]$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le M\left[\frac{\int_a^b\ln x dx}{b-a}-\frac{\ln a+\ln b}{2}\right].$$

A simple calculation shows us that

$$\int_a^b\ln x dx=I(a,b),$$

and then the above two inequalities yield that (5.196) and (5.197) hold true. ∎

Finally, the following corollary also holds [**63**]:

COROLLARY 69. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a twice differentiable mapping such that $|f''(x)|\le\frac{M}{x}$ for all $x\in(a,b)$. Then one has the inequalities:*

$$\exp\left[\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\right]\le\frac{\left[\left(\frac{b^{\frac{b^2}{2}}}{a^{\frac{a^2}{2}}}\right)e^{-\frac{3}{4}(b^2-a^2)}\right]^{\frac{M}{b-a}}}{\left[\left(\frac{a+b}{2}\right)^{\frac{a+b}{2}}e^{-\frac{a+b}{2}}\right]^M}\tag{5.198}$$

and

$$\exp\left[\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\right]\le\frac{\left[\left(a^ab^b\right)^{\frac{1}{2}}e^{-\frac{a+b}{2}}\right]^M}{\left[\left(\frac{b^{\frac{b^2}{2}}}{a^{\frac{a^2}{2}}}\right)e^{-\frac{3}{4}(b^2-a^2)}\right]^{\frac{M}{b-a}}}\tag{5.199}$$

respectively.

PROOF. Consider the mapping $g(x)=M\ln x-Mx$. Then $g'(x)=M\ln x$ and $g''(x)=\frac{M}{x}$. As $|f''(x)|\le\frac{M}{x}$, it follows that f is $g-$convex-dominated and, by Theorem 198, we can state that

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{1}{b-a}\int_a^b g(x)\,dx-g\left(\frac{a+b}{2}\right)$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(x)\,dx.$$

As

$$\int_a^b(x\ln x-x)\,dx=\ln\left[\left(\frac{b^{\frac{b^2}{2}}}{a^{\frac{a^2}{2}}}\right)e^{-\frac{3}{4}(b^2-a^2)}\right]^{\frac{M}{b-a}},$$

and

$$
\begin{aligned}
g\left(\frac{a+b}{2}\right) &= \ln\left[\left(\frac{a+b}{2}\right)^{\frac{a+b}{2}} e^{-\frac{a+b}{2}}\right]^{M}, \\
\frac{1}{b-a}\int_a^b g(x)\,dx &= \ln\left[\left(\frac{b^{\frac{b^2}{2}}}{a^{\frac{a^2}{2}}}\right) e^{-\frac{3}{4}\left(b^2-a^2\right)}\right]^{\frac{M}{b-a}}
\end{aligned}
$$

and

$$
\frac{g(a)+g(b)}{2} = \ln\left[\left(a^a b^b\right)^{\frac{1}{2}} e^{-\frac{a+b}{2}}\right]^{M},
$$

then we get the inequalities:

$$
\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \ln\left\{\frac{\left[\left(\frac{b^{\frac{b^2}{2}}}{a^{\frac{a^2}{2}}}\right) e^{-\frac{3}{4}\left(b^2-a^2\right)}\right]^{\frac{M}{b-a}}}{\left[\left(\frac{a+b}{2}\right)^{\frac{a+b}{2}} e^{-\frac{a+b}{2}}\right]^{M}}\right\}
$$

and

$$
\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \ln\left\{\frac{\left[\left(a^a b^b\right)^{\frac{1}{2}} e^{-\frac{a+b}{2}}\right]^{M}}{\left[\left(\frac{b^{\frac{b^2}{2}}}{a^{\frac{a^2}{2}}}\right) e^{-\frac{3}{4}\left(b^2-a^2\right)}\right]^{\frac{M}{b-a}}}\right\},
$$

from where results the desired results (5.198) and (5.199). ∎

Now, for a mapping $f : [a,b] \to \mathbb{R}$ with $f \in L_1[a,b]$, we can define the mapping (see also Section 5 of Chapter 3):

$$
H_f(t) := \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx.
$$

The following theorem contains some results of this type for convex-dominated functions [**63**].

THEOREM 199. *Let $g : [a,b] \to \mathbb{R}$ be a convex mapping on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ a $g-$convex-dominated mapping on $[a,b]$. Then:*

(*i*) *H_f is H_g-convex dominated on $[0,1]$;*

(*ii*) *One has the inequalities*

$$
0 \leq \left|H_f(t_2) - H_f(t_1)\right| \leq H_g(t_2) - H_g(t_1) \tag{5.200}
$$

for all $0 \leq t_1 < t_2 \leq 1$;

(*iii*) *One has the inequalities*

$$
0 \leq \left|f\left(\frac{a+b}{2}\right) - H_f(t)\right| \leq H_g(t) - g\left(\frac{a+b}{2}\right) \tag{5.201}
$$

and

$$
0 \leq \left|\frac{1}{b-a}\int_a^b f(x)\,dx - H_f(t)\right| \leq \frac{1}{b-a}\int_a^b g(x)\,dx - H_g(t) \tag{5.202}
$$

for all $t \in [0,1]$.

PROOF. (i) Since f is $g-$convex dominated on $[a,b]$, it follows (see Lemma 23) that $g-f$ and $g+f$ are convex on $[a,b]$. Now, using Theorem 71 (i), we get that $H_{(g-f)}$ and $H_{(g+f)}$ are convex on $[0,1]$. By the linearity of the mapping $f \mapsto H_f$, one gets that $H_{(g-f)} = H_g - H_f$ and $H_{(g+f)} = H_g + H_f$, and, as H_g is convex, then by the same lemma we deduce that H_f is H_g-dominated on $[0,1]$.

(ii) By Theorem 71, (iii), we can state that $H_{(g-f)}$ and $H_{(g+f)}$ are monotonically nondecreasing on $[0,1]$ and thus we have

$$H_g(t_1) - H_f(t_1) = H_{(g-f)}(t_1) \le H_{(g-f)}(t_2) = H_g(t_2) - H_f(t_2)$$

and

$$H_g(t_1) + H_f(t_1) = H_{(g+f)}(t_1) \le H_{(g+f)}(t_2) = H_g(t_2) + H_f(t_2)$$

from where we obtain

$$H_f(t_2) - H_f(t_1) \le H_g(t_2) - H_g(t_1)$$

and

$$H_f(t_2) - H_f(t_1) \ge H_g(t_1) - H_g(t_2),$$

which are equivalent with (5.200).

(iii) The inequalities (5.201) and (5.202) follow by the statement (ii) of Theorem 71 with a similar argument. We shall omit the details.

∎

Now, for a given integrable mapping $f : [a,b] \to \mathbb{R}$ we can also consider the mapping (see Section 5 of Chapter 3) $F_f : [0,1] \to \mathbb{R}$

$$F_f(t) := \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx + (1-t)y)\, dxdy.$$

By the use of Theorem 74, we can state the following result [**63**].

THEOREM 200. *Let $g : [a,b] \to \mathbb{R}$ be a convex mapping and $f : [a,b] \to \mathbb{R}$ be a g-convex dominated function on $[a,b]$. Then:*

(i) *F_f is F_g-convex dominated on $[0,1]$;*

(ii) *We have the inequalities:*

$$0 \le |F_f(t_2) - F_f(t_1)| \le F_g(t_2) - F_g(t_1) \quad \textit{for } \frac{1}{2} \le t_1 < t_2 \le 1$$

and

$$0 \le |F_f(t_2) - F_f(t_1)| \le F_g(t_1) - F_g(t_2) \quad \textit{for } 0 \le t_1 < t_2 \le \frac{1}{2}.$$

(iii) *One has the inequalities:*

$$0 \le \left| \frac{1}{b-a} \int_a^b f(x)\, dx - F_f(t) \right| \le \frac{1}{b-a} \int_a^b g(x)\, dx - F_g(t),$$

and

$$\begin{aligned} 0 &\le \left| \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dxdy - F_f(t) \right| \\ &\le F_g(t) - \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dxdy \end{aligned}$$

and

$$0 \leq |F_f(t) - H_f(t)| \leq F_g(t) - H_g(t)$$

for all $t \in [0,1]$.

The argument follows by Theorem 74 in a similar fashion to that of the proof of the previous theorem and we shall omit the details.

14. $H. - H.$ Inequality for Lipschitzian Mappings

14.1. $H. - H.$ **Type Inequality.** We will start with the following theorem containing two inequalities of $H. - H.$ type for Lipschitzian mappings [**51**].

THEOREM 201. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be an* M*-Lipschitzian mapping on* I *and* $a, b \in I$ *with* $a < b$. *Then we have the inequalities:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{M}{4}(b-a), \tag{5.203}$$

and

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{M}{3}(b-a). \tag{5.204}$$

PROOF. Let $t \in [0,1]$. Then we have, for all $a, b \in I$, that

$$\begin{aligned} & |tf(a) + (1-t)f(b) - f(ta + (1-t)b)| \\ = \; & |t(f(a) - f(ta + (1-t)b) + (1-t)(f(b) - f(ta + (1-t)b)| \\ \leq \; & t|f(a) - f(ta + (1-t)b)| + (1-t)|f(b) - f(ta + (1-t)b)| \\ \leq \; & tM|a - (ta + (1-t)b)| + (1-t)M|b - (ta + (1-t)b)| \\ = \; & 2t(1-t)M|b-a|. \end{aligned} \tag{5.205}$$

If we choose $t = \frac{1}{2}$, we have also

$$\left| \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \right| \leq \frac{M}{2}|b-a|. \tag{5.206}$$

If we put $ta+(1-t)b$ instead of a and $(1-t)a+tb$ instead of b in (5.206), respectively, then we have

$$\left| \frac{f(ta + (1-t)b) + f((1-t)a + tb)}{2} - f\left(\frac{a+b}{2}\right) \right| \leq \frac{M|2t-1|}{2}|b-a| \tag{5.207}$$

for all $t \in [0,1]$. If we integrate the inequality (5.207) on $[0,1]$, we have

$$\begin{aligned} & \left| \frac{1}{2}\left[\int_0^1 f(ta + (1-t)b)dt + \int_0^1 f((1-t)a + tb)dt\right] - f\left(\frac{a+b}{2}\right) \right| \\ \leq \; & \frac{M|b-a|}{2}\int_0^1 |2t-1|dt. \end{aligned}$$

Thus, from

$$\int_0^1 f(ta + (1-t)b)dt = \int_0^1 f((1-t)a + tb)dt = \frac{1}{b-a}\int_a^b f(x)dx$$

and

$$\int_0^1 |2t-1|dt = \frac{1}{2},$$

we obtain the inequality (5.203).

Note that, by the inequality (5.205), we have

$$|tf(a)+(1-t)f(b)-f(ta+(1-t)b)| \leq 2t(1-t)M(b-a)$$

for all $t \in [0,1]$ and $a,b \in I$ with $a<b$. Integrating on $[0,1]$, we have

$$\begin{aligned} & \left| f(a)\int_0^1 tdt + f(b)\int_0^1 (1-t)dt - \int_0^1 f(ta+(1-t)b)dt \right| \\ \leq\ & 2M(b-a)\int_0^1 t(1-t)dt. \end{aligned}$$

Hence, from

$$\int_0^1 tdt = \int_0^1 (1-t)dt = \frac{1}{2}, \quad \int_0^1 t(1-t)dt = \frac{1}{6},$$

we have

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{M}{3}(b-a)$$

and so we have the inequality (5.204). This completes the proof. ∎

The following corollary is important in applications:

COROLLARY 70. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex mapping on I, $a,b \in I$ with $a<b$ and $M := \sup_{t\in[a,b]} |f'(t)| < \infty$. Then we have the following complements of $H.-H.$ inequalities:*

$$0 \leq \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \leq \frac{M}{4}(b-a) \tag{5.208}$$

and

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{M}{3}(b-a). \tag{5.209}$$

PROOF. The proof is obvious by Lagrange's theorem, i.e., we recall that for any $x,y \in (a,b)$ there exists a c between them so that

$$|f(x)-f(y)| = |x-y|\,|f'(c)| \leq M\,|x-y|\,,$$

and Theorem 201. We shall omit the details. ∎

The following corollaries for elementary inequalities hold:

COROLLARY 71. (1) *Let $p \geq 1$ and $a,b \in \mathbb{R}$ with $0 \leq a < b$. Then we have the inequalities:*

$$0 \leq L_p^p(a,b) - A^p(a,b) \leq \frac{pb^{p-1}}{4}(b-a)$$

and

$$0 \leq A(a^p,b^p) - L_p^p(a,b) \leq \frac{pb^{p-1}}{3}(b-a).$$

(2) *Let $a,b \in \mathbb{R}$ with $0<a<b$. Then we have the inequalities:*

$$0 \leq L^{-1}(a,b) - A^{-1}(a,b) \leq \frac{1}{4a^2}(b-a)$$

and

$$0 \leq H^{-1}(a,b) - L^{-1}(a,b) \leq \frac{1}{3a^2}(b-a).$$

(3) *Let $a, b \in \mathbb{R}$ with $a < b$. Then we have the inequalities*

$$0 \leq \frac{\exp(b) - \exp(a)}{b-a} - \exp\left(\frac{a+b}{2}\right) \leq \frac{\exp(b)}{4}(b-a)$$

and

$$0 \leq \frac{\exp(a) + \exp(b)}{2} - \frac{\exp(b) - \exp(a)}{b-a} \leq \frac{\exp(b)}{3}(b-a).$$

(4) *Let $a, b \in \mathbb{R}$ with $0 < a < b$. Then we have the inequalities*

$$1 \leq \frac{A(a,b)}{L(a,b)} \leq \exp\left(\frac{1}{4a}(b-a)\right)$$

and

$$1 \leq \frac{L(a,b)}{G(a,b)} \leq \exp\left(\frac{1}{3a}(b-a)\right).$$

PROOF. (1) The proof follows by Corollary 70 applied for the convex mapping $f(x) = x^p$ on $[a, b]$.

(2) The proof follows by Corollary 70 applied for the convex mapping $f(x) = \frac{1}{x}$ on $[a, b]$.

(3) The proof is obvious by Corollary 70 applied for the convex mapping $f(x) = \exp(x)$ on $\mathbb{R}$.

(4) The proof follows by Corollary 70 applied for the convex mapping $f(x) = -\ln x$ on $[a, b]$.

This completes the proof. ∎

Now, we shall point out some other inequalities of the types in Corollary 71, but these hold for the mappings which are not convex on $[a, b]$.

COROLLARY 72. (1) *Let $a, b \in \mathbb{R}$ with $a < b$ and $k \in \mathbb{N}$. Then we have the inequalities:*

$$\left|\left(\frac{a+b}{2}\right)^{2k+1} - \frac{b^{2k+2} - a^{2k+2}}{(2k+2)(b-a)}\right| \leq \frac{(2k+1)\max\{a^{2k}, b^{2k}\}}{4}(b-a)$$

and

$$\left|\frac{a^{2k+1} + b^{2k+1}}{2} - \frac{b^{2k+2} - a^{2k+2}}{(2k+2)(b-a)}\right| \leq \frac{(2k+1)\max\{a^{2k}, b^{2k}\}}{3}(b-a).$$

(2) *Let $a, b \in \mathbb{R}$ with $a < b$. Then we have the inequalities:*

$$\left|\cos\left(\frac{a+b}{2}\right) - \frac{\sin b - \sin a}{b-a}\right| \leq \frac{b-a}{4}$$

and

$$\left|\frac{\cos a + \cos b}{2} - \frac{\sin b - \sin a}{b-a}\right| \leq \frac{b-a}{3}.$$

PROOF. (1) The proof follows by Theorem 201 applied for the mapping $f(x) = x^{2k+1}$ on $[a, b]$.

(2) The proof is obvious by Theorem 201 applied for the mapping $f(x) = \cos x$ on $[a, b]$. This completes the proof.

∎

14.2. The Mapping H. For an M-Lipschitzian function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, we can define a mapping $H : [0,1] \to \mathbb{R}$ by

$$H(t) = \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx$$

for all $t \in [0,1]$ and we shall give some properties of the mapping H, [**51**].

THEOREM 202. *Let a mapping $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be M-Lipschitzian on I and $a, b \in I$ with $a < b$. Then*

(1) *The mapping H is $\frac{M}{4}(b-a)$-Lipschitzian on $[0,1]$.*
(2) *We have the inequalities:*

$$\left| H(t) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{M(1-t)}{4}(b-a), \tag{5.210}$$

$$\left| f\left(\frac{a+b}{2}\right) - H(t) \right| \leq \frac{Mt}{4}(b-a), \tag{5.211}$$

and

$$\left| H(t) - t\frac{1}{b-a} \int_a^b f(x) dx - (1-t) f\left(\frac{a+b}{2}\right) \right| \leq \frac{t(1-t)M}{2}(b-a) \tag{5.212}$$

for all $t \in [0,1]$.

PROOF. (1) Let $t_1, t_2 \in [0,1]$. Then we have

$$\begin{aligned}
&|H(t_2) - H(t_1)| \\
= \ & \frac{1}{b-a} \left| \int_a^b f\left(t_2 + (1-t_2)\frac{a+b}{2}\right) dx \right. \\
&\left. - \int_a^b f\left(t_1 x + (1-t_1)\frac{a+b}{2}\right) dx \right| \\
\leq \ & \frac{1}{b-a} \int_a^b \left| f\left(t_2 x + (1-t_2)\frac{a+b}{2}\right) - f\left(t_1 x + (1-t_1)\frac{a+b}{2}\right) \right| dx \\
\leq \ & \frac{M}{b-a} \int_a^b \left| t_2 x + (1-t_2)\frac{a+b}{2} - t_1 x - (1-t_1)\frac{a+b}{2} \right| dx \\
= \ & \frac{M|t_2 - t_1|}{b-a} \int_a^b \left| x - \frac{a+b}{2} \right| dx \\
= \ & \frac{M(b-a)}{4} |t_2 - t_1|,
\end{aligned}$$

i.e., for all $t_1, t_2 \in [0,1]$,

$$|H(t_2) - H(t_1)| \leq \frac{M(b-a)}{4} |t_2 - t_1|, \tag{5.213}$$

which yields that the mapping H is $\frac{M(b-a)}{4}$-Lipschitzian on $[0,1]$.

(2) The inequalities (5.210) and (5.211) follow from (5.213) by choosing $t_1 = 0$, $t_2 = t$ and $t_1 = 1$, $t_2 = t$, respectively.
Inequality (5.212) follows by adding t times (5.210) and $(1-t)$ times (5.211). This completes the proof.

∎

Another result which is connected in a sense with the inequality (5.204) is also given in the following (cf. [**51**]).

THEOREM 203. *With the above assumptions, we have the inequality:*

$$\left|\frac{f\left(tb+(1-t)\frac{a+b}{2}\right)+f\left(ta+(1-t)\frac{a+b}{2}\right)}{2}-H(t)\right| \leq \frac{Mt}{3}(b-a) \tag{5.214}$$

for all $t \in [0,1]$.

PROOF. If we denote $u = tb+(1-t)\frac{a+b}{2}$ and $v = ta+(1-t)\frac{a+b}{2}$, then we have

$$H(t) = \frac{1}{u-v}\int_v^u f(z)dz.$$

Now, using the inequality (5.204) applied for u and v, we have

$$\left|\frac{f(u)+f(v)}{2}-\frac{1}{u-v}\int_v^u f(z)dz\right| \leq \frac{M}{3}(u-v),$$

from which we have the inequality (5.214). This completes the proof. ∎

Theorems 202 and 203 imply the following theorem which is important in applications for convex functions [**51**]:

THEOREM 204. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable convex mapping on* I, $a, b \in I$ *with* $a < b$ *and* $M = \sup_{x\in[a,b]} |f'(x)| < \infty$. *Then we have the inequalities:*

$$0 \leq \frac{1}{b-a}\int_a^b f(x)dx - H(t) \leq \frac{M(1-t)}{4}(b-a), \tag{5.215}$$

$$0 \leq H(t) - f\left(\frac{a+b}{2}\right) \leq \frac{Mt}{4}(b-a), \tag{5.216}$$

$$0 \leq \frac{f\left(tb+(1-t)\frac{a+b}{2}\right)+f\left((ta+(1-t)\frac{a+b}{2}\right)}{2} - H(t) \leq \frac{Mt}{3}(b-a), \tag{5.217}$$

for all $t \in [0,1]$.

14.3. The Mapping F. For an M-Lipschitzian function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ we can define a mapping $F : [0,1] \to \mathbb{R}$ by

$$F(t) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)dxdy,$$

and give some properties of the mapping F as follows [**51**].

THEOREM 205. *Let a mapping* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be* M*-Lipschitzian on* I *and* $a, b \in I$ *with* $a < b$. *Then*

(1) *The mapping* F *is symmetrical, i.e.,* $F(t) = F(1-t)$ *for all* $t \in [0,1]$.
(2) *The mapping* F *is* $\frac{M(b-a)}{3}$*-Lipschitzian on* $[0,1]$.

(3) *We have the inequalities:*

$$\left|F(t)-\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy\right| \le \frac{M|2t-1|}{6}(b-a), \tag{5.218}$$

$$\left|F(t)-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{Mt}{3}(b-a), \tag{5.219}$$

and

$$|F(t)-H(t)| \le \frac{M(1-t)}{4}(b-a) \tag{5.220}$$

for all $t\in[0,1]$.

Proof. (1) It is obvious by the definition of the mapping F.

(2) Let $t_1,t_2\in[0,1]$. Then we have

$$\begin{aligned} & |F(t_2)-F(t_1)| \\ = & \frac{1}{(b-a)^2}\left|\int_a^b\int_a^b [f(t_2x+(1-t_2)y)-f(t_1x+(1-t_1)y)]dxdy\right| \\ \le & \frac{1}{(b-a)^2}\int_a^b\int_a^b |f(t_2x+(1-t_2)y)-f(t_1x+(1-t_1)y)|dxdy \\ \le & \frac{M|t_2-t_1|}{(b-a)^2}\int_a^b\int_a^b |x-y|dxdy. \end{aligned} \tag{5.221}$$

Now, note that

$$\int_a^b\int_a^b |x-y|\,dxdy = \frac{(b-a)^3}{3}. \tag{5.222}$$

Therefore, from (5.221) and (5.222), it follows that

$$|F(t_2)-F(t_1)| \le \frac{M|t_2-t_1|}{3}(b-a) \tag{5.223}$$

for all $t_1,t_2\in[0,1]$ and so the mapping F is $\frac{M(b-a)}{3}$-Lipschitzian on $[0,1]$.

(3) The inequalities (5.218) and (5.219) follow from (5.223) if we choose $t_1=\frac{1}{2}$, $t_2=t$ and $t_1=0$, $t_2=t$, respectively.

Now, we prove the inequality (5.220). Since f is M-Lipschitzian, we can write

$$\begin{aligned} & \left|f(tx+(1-t)y)-f\left(tx+(1-t)\frac{a+b}{2}\right)\right| \\ \le & M\left|tx+(1-t)y-tx-(1-t)\frac{a+b}{2}\right| \\ = & (1-t)M\left|y-\frac{a+b}{2}\right| \end{aligned} \tag{5.224}$$

for all $t \in [0,1]$ and $x, y \in [a,b]$. Integrating the inequality (5.224) on $[a,b] \times [a,b]$, we have

$$\begin{aligned} &\left| \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx + (1-t)y)\, dxdy - \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx \right| \\ \leq\ & (1-t) M \frac{1}{b-a} \int_a^b \left| y - \frac{a+b}{2} \right| dy = \frac{M(1-t)(b-a)}{4} \end{aligned}$$

for all $t \in [0,1]$ and so the inequality (5.220) is proved. This completes the proof.

∎

Theorem 205 implies the following converses of the known results holding for convex functions (see the results listed in Subsection 1).

Corollary 73. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex mapping and $M := \sup_{x \in [a,b]} |f'(x)|$ for $a, b \in I$ with $a < b$. Then we have the inequalities:*

$$0 \leq F(t) - \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dxdy \leq \frac{M|2t-1|}{6}(b-a),$$

$$0 \leq \frac{1}{b-a} \int_a^b f(x)\, dx - F(t) \leq \frac{Mt}{2}(b-a),$$

and

$$0 \leq F(t) - H(t) \leq \frac{M(1-t)}{4}(b-a)$$

for all $t \in [0,1]$.

Remark 97. *Similar results can be obtained if we consider the more general class of $r-H-$Hölder type mappings, i.e.,*

$$|f(x) - f(y)| \leq H |x-y|^r, \quad x, y \in I, \tag{5.225}$$

where $H > 0$ and $r \in (0,1]$, [**26**].

CHAPTER 6

The $H. - H.$ Inequalities for Mappings of Several Variables

1. An Inequality for Convex Functions on the Co-ordinates

1.1. Hermite-Hadamard's Inequality. Let us consider the bidimensional interval $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called *convex on the co-ordinates* if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) := f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) := f(u,v)$ are convex where defined for all $y \in [c,d]$ and $x \in [a,b]$.

Recall that the mapping $f : \Delta \to \mathbb{R}$ is convex in Δ if the following inequality:

$$f(\lambda x + (1-\lambda) z, \lambda y + (1-\lambda) w) \leq \lambda f(x,y) + (1-\lambda) f(z,w) \tag{6.1}$$

holds, for all $(x,y), (z,w) \in \Delta$ and $\lambda \in [0,1]$.

The following lemma holds:

Lemma 24. *Every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates, but the converse is not generally true.*

Proof. Suppose that $f : \Delta \to \mathbb{R}$ is convex in Δ. Consider $f_x : [c,d] \to \mathbb{R}$, $f_x(v) := f(x,v)$. Then for all $\lambda \in [0,1]$ and $v, w \in [c,d]$ one has:

$$\begin{aligned} f_x(\lambda v + (1-\lambda) w) &= f(x, \lambda v + (1-\lambda) w) \\ &= f(\lambda x + (1-\lambda) x, \lambda v + (1-\lambda) w) \\ &\leq \lambda f(x,v) + (1-\lambda) f(x,w) \\ &= \lambda f_x(v) + (1-\lambda) f_x(w) \end{aligned}$$

which shows the convexity of f_x.
The fact that $f_y : [a,b] \to \mathbb{R}$, $f_y(u) := f(u,y)$ is also convex on $[a,b]$ for all $y \in [c,d]$ goes likewise and we shall omit the details.
Now, consider the mapping $f_0 : [0,1]^2 \to [0,\infty)$ given by $f_0(x,y) = xy$. It's obvious that f is convex on the co-ordinates but is not convex on $[0,1]^2$.
Indeed, if $(u,0), (0,w) \in [0,1]^2$ and $\lambda \in [0,1]$ we have:

$$f(\lambda(u,0) + (1-\lambda)(0,w)) = f(\lambda u, (1-\lambda) w) = \lambda(1-\lambda) xw$$

and

$$\lambda f(u,0) + (1-\lambda) f(0,w) = 0.$$

Thus, for all $\lambda \in (0,1)$, $u, w \in (0,1)$, we have

$$f(\lambda(u,0) + (1-\lambda)(0,w)) > \lambda f(u,0) + (1-\lambda) f(0,w)$$

which shows that f is not convex on $[0,1]^2$. ∎

The following inequalities of Hadamard type hold [**31**].

THEOREM 206. *Suppose that $f:\Delta=[a,b]\times[c,d]\rightarrow\mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities:*

$$
\begin{aligned}
(6.2)\qquad & f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy \\
\leq\ & \frac{1}{4}\left[\frac{1}{b-a}\int_c^d f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right. \\
& \left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}
$$

The above inequalities are sharp.

PROOF. Since $f:\Delta\rightarrow\mathbb{R}$ is convex on the co-ordinates it follows that the mapping $g_x:[c,d]\rightarrow\mathbb{R}$, $g_x(y)=f(x,y)$ is convex on $[c,d]$ for all $x\in[a,b]$. Then by Hadamard's inequality (1.1) one has:

$$g_x\left(\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d g_x(y)\,dy\leq\frac{g_x(c)+g_x(d)}{2},\ x\in[a,b].$$

That is,

$$f\left(x,\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d f(x,y)\,dy\leq\frac{f(x,c)+f(x,d)}{2},\ x\in[a,b].$$

Integrating this inequality on $[a,b]$, we have:

$$
\begin{aligned}
(6.3)\qquad & \frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy \\
\leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right].
\end{aligned}
$$

By a similar argument applied for the mapping $g_y:[a,b]\rightarrow\mathbb{R}$, $g_y(x):=f(x,y)$ we get

$$
\begin{aligned}
(6.4)\qquad & \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy \\
\leq\ & \frac{1}{2}\left[\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right].
\end{aligned}
$$

Summing the inequalities (6.3) and (6.4) , we get the second and the third inequality in (6.2) .
By Hadamard's inequality we also have:

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx$$

and

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy,$$

which give, by addition, the first inequality in (6.2) .
Finally, by the same inequality we can also state:

$$\frac{1}{b-a}\int_a^b f(x,c)\,dx\leq\frac{f(a,c)+f(b,c)}{2},$$

$$\frac{1}{b-a}\int_a^b f(x,d)\,dx\leq\frac{f(a,d)+f(b,d)}{2},$$

$$\frac{1}{d-c}\int_c^d f(a,y)\,dy\leq\frac{f(a,c)+f(a,d)}{2},$$

and

$$\frac{1}{d-c}\int_c^d f(b,y)\,dy\leq\frac{f(b,c)+f(d,b)}{2},$$

which give, by addition, the last inequality in (6.2) .
If in (6.2) we choose $f(x)=xy$, then (6.2) becomes an equality, which shows that (6.2) are sharp. ∎

1.2. Some Mappings Associated to $H.-H.$ Inequality. Now, for a mapping $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ as above, we can define the mapping $H:[0,1]^2\to\mathbb{R}$,

$$\begin{aligned}&H(t,s)\\ :&=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)dxdy.\end{aligned}$$

The properties of this mapping are embodied in the following theorem [**31**].

THEOREM 207. *Suppose that $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ is convex on the co-ordinates on $\Delta:=[a,b]\times[c,d]$. Then:*

(*i*) *The mapping H is convex on the co-ordinates on $[0,1]^2$;*
(*ii*) *We have the bounds:*

$$\sup_{(t,s)\in[0,1]^2}H(t,s)=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy=H(0,0);$$

$$\inf_{(t,s)\in[0,1]^2}H(t,s)=f\left(\frac{a+b}{2},\frac{c+d}{2}\right)=H(1,1);$$

(*iii*) *The mapping H is monotonic nondecreasing on the co-ordinates.*

PROOF. (i) Fix $s\in[0,1]$. Then for all $\alpha,\beta\geq 0$ with $\alpha+\beta=1$ and $t_1,t_2\in[0,1]$, we have:

$$\begin{aligned}
&H(\alpha t_1+\beta t_2,s)=\frac{1}{(b-a)(d-c)}\\
&\times\int_a^b\int_c^d f\left((\alpha t_1+\beta t_2)x+[1-(\alpha t_1+\beta t_2)]\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)dxdy\\
=\ &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(\alpha\left(t_1x+(1-t_1)\frac{a+b}{2}\right)\right.\\
&\left.+\beta\left(t_2x+(1-t_2)\frac{a+b}{2}\right),sy+(1-s)\frac{c+d}{2}\right)dxdy\\
\leq\ &\alpha\cdot\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(t_1x+(1-t_1)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)dxdy\\
&+\beta\cdot\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(t_2x+(1-t_2)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)dxdy\\
=\ &\alpha H(t_1,s)+\beta H(t_2,s).
\end{aligned}$$

If $t\in[0,1]$ is fixed, then for all $s_1,s_2\in[0,1]$ and $\alpha,\beta\geq 0$ with $\alpha+\beta=1$, we also have:

$$H(t,\alpha s_1+\beta s_2)\leq\alpha H(t,s_1)+\beta H(t,s_2)$$

and the statement is proved.

(ii) Since f is convex on the co-ordinates we have, by Jensen's inequality for integrals, that:

$$\begin{aligned}
&H(t,s)\\
=\ &\frac{1}{b-a}\int_a^b\left[\frac{1}{d-c}\int_c^d f\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)dy\right]dx\\
\geq\ &\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2},\frac{1}{d-c}\int_c^d\left[sy+(1-s)\frac{c+d}{2}\right]dy\right)dx\\
=\ &\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2},\frac{c+d}{2}\right)dx\\
\geq\ &f\left(\frac{1}{b-a}\int_a^b\left[tx+(1-t)\frac{a+b}{2}\right],\frac{c+d}{2}\right)dx\\
=\ &f\left(\frac{a+b}{2},\frac{c+d}{2}\right).
\end{aligned}$$

By the convexity of H on the co-ordinates we have:

$$\begin{aligned}
&H(t,s)\\
\leq\ &s\cdot\frac{1}{b-a}\int_a^b\left[\frac{1}{d-c}\int_c^d f\left(tx+(1-t)\frac{a+b}{2},y\right)dy\right.\\
&\left.+(1-s)\cdot\frac{1}{d-c}\int_c^d f\left(tx+(1-t)\frac{a+b}{2},\frac{c+d}{2}\right)dy\right]dx
\end{aligned}$$

$$\begin{aligned}
&\leq s\cdot\frac{1}{d-c}\int_c^d\left[t\cdot\frac{1}{b-a}\int_a^b f(x,y)\,dxdy\right.\\
&\quad\left.+(1-t)\cdot\frac{1}{b-a}\int_a^b f\left(\frac{a+b}{2},y\right)dx\right]dy\\
&\quad+(1-s)\cdot\frac{1}{d-c}\int_c^d\left[t\cdot\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\right.\\
&\quad\left.+(1-t)\cdot f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]dy\\
&= st\cdot\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy+s(1-t)\cdot\frac{1}{d-c}\int_a^b f\left(\frac{a+b}{2},y\right)dy\\
&\quad+(1-s)\,t\cdot\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+(1-s)(1-t)\cdot f\left(\frac{a+b}{2},\frac{c+d}{2}\right).
\end{aligned}$$

By Hadamard's inequality we also have:

$$f\left(\frac{a+b}{2},y\right)\leq\frac{1}{b-a}\int_a^b f(x,y)\,dx,\ \ y\in[c,d]$$

and

$$f\left(x,\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d f(x,y)\,dy,\ \ x\in[a,b]\,.$$

Thus, by integration, we get that:

$$\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy$$

and

$$\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy.$$

Using the above inequality, we deduce that

$$\begin{aligned}
&H(t,s)\\
&\leq [st+s(1-t)+(1-s)\,t+(1-s)(1-t)]\,\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy\\
&= \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy,\ \ (s,t)\in[0,1]^2
\end{aligned}$$

and the second bound in (ii) is proved.

(iii) Firstly, we will show that

$$H(t,s)\geq H(0,s)\ \text{ for all }\ (t,s)\in[0,1]^2\,. \tag{6.5}$$

By Hadamard's inequality we have:

$$\begin{aligned}
&H(t,s)\\
&\geq \frac{1}{d-c}\int_c^d f\left(\frac{1}{b-a}\int_a^b\left[tx+(1-t)\,\frac{a+b}{2}\right]dx,sy+(1-s)\,\frac{c+d}{2}\right)dy\\
&= \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},sy+(1-s)\,\frac{c+d}{2}\right)dy=H(0,s)
\end{aligned}$$

for all $(t,s)\in[0,1]^2$.
Now let $0\le t_1<t_2\le 1$. By the convexity of the mapping $H(\cdot,s)$ for all $s\in[0,1]$ we have

$$\frac{H(t_2,s)-H(t_1,s)}{t_2-t_1}\ge\frac{H(t_1,s)-H(0,s)}{t_1}\ge 0.$$

For the last inequality we use (6.5).

∎

The following theorem also holds.

THEOREM 208. *Suppose that $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is convex on Δ. Then*

(*i*) *The mapping H is convex on Δ;*
(*ii*) *Define the mapping $h:[0,1]\to\mathbb{R}$, $h(t)=H(t,t)$. Then h is convex, monotonic nondecreasing on $[0,1]$ and one has the bounds:*

$$\sup_{t\in[0,1]}h(t)=h(1)=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy$$

and

$$\inf_{t\in[0,1]}h(t)=h(0)=f\left(\frac{a+b}{2},\frac{c+d}{2}\right).$$

PROOF. (*i*) Let $(t_1,s_1),(t_2,s_2)\in[0,1]^2$ and $\alpha,\beta\ge 0$ with $\alpha+\beta=1$. Since $f:\Delta\to\mathbb{R}$ is convex on Δ we have:

$$H(\alpha(t_1,s_1)+\beta(t_2,s_2))$$

$$\begin{aligned}
&= H(\alpha t_1+\beta t_2,\alpha s_1+\beta s_2)\\
&= \frac{1}{(b-a)(d-c)}\\
&\quad\times\int_a^b\int_c^d f\left(\alpha\left(t_1x+(1-t_1)\frac{a+b}{2},s_1y+(1-s_1)\frac{c+d}{2}\right)\right.\\
&\quad\left.+\beta\left(t_2x+(1-t_2)\frac{a+b}{2},s_2y+(1-s_2)\frac{c+d}{2}\right)\right)dxdy\\
&\le \alpha\cdot\frac{1}{(b-a)(d-c)}\\
&\quad\times\int_a^b\int_c^d f\left(t_1x+(1-t_1)\frac{a+b}{2},s_1y+(1-s_1)\frac{c+d}{2}\right)dxdy\\
&\quad+\beta\cdot\frac{1}{(b-a)(d-c)}\\
&\quad\times\int_a^b\int_c^d f\left(t_2x+(1-t_2)\frac{a+b}{2},s_2y+(1-s_2)\frac{c+d}{2}\right)dxdy\\
&= \alpha H(t_1,s_1)+\beta H(t_2,s_2),
\end{aligned}$$

which shows that H is convex on $[0,1]^2$.

(ii) Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then

$$\begin{aligned} h(\alpha t_1 + \beta t_2) &= H(\alpha t_1 + \beta t_2, \alpha t_1 + \beta t_2) \\ &= H(\alpha(t_1, t_1) + \beta(t_2, t_2)) \\ &\leq \alpha H(t_1, t_1) + \beta H(t_2, t_2) \\ &= \alpha h(t_1) + \beta h(t_2) \end{aligned}$$

which shows the convexity of h on $[0,1]$.

We have, by the above theorem, that

$$h(t) = H(t,t) \geq H(0,0) = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right), \ t \in [0,1]$$

and

$$h(t) = H(t,t) \leq H(1,1) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dxdy, \ t \in [0,1]$$

which prove the required bounds.

Now, let $0 \leq t_1 < t_2 \leq 1$. Then, by the convexity of h we have that

$$\frac{h(t_2) - h(t_1)}{t_2 - t_1} \geq \frac{h(t_1) - h(0)}{t_1} \geq 0,$$

and the theorem is proved.

∎

Next, we shall consider the following mapping which is closely connected with Hadamard's inequality: $H : [0,1]^2 \to [0, \infty)$ given by

$$\begin{aligned} &\tilde{H}(t,s) \\ :&= \frac{1}{(b-a)^2 (d-c)^2} \int_a^b \int_a^b \int_c^d \int_c^d f(tx + (1-t)y, sz + (1-s)u)\, dxdydzdu. \end{aligned}$$

The next theorem contains the main properties of this mapping.

THEOREM 209. *Suppose that $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then:*

(i) *We have the equalities:*

$$\tilde{H}\left(t + \frac{1}{2}, s\right) = \tilde{H}\left(\frac{1}{2} - t, s\right) \ \text{for all } t \in \left[0, \frac{1}{2}\right], \ s \in [0,1];$$

$$\tilde{H}\left(t, s + \frac{1}{2}\right) = \tilde{H}\left(t, \frac{1}{2} - s\right) \ \text{for all } t \in [0,1], \ s \in \left[0, \frac{1}{2}\right];$$

$$\tilde{H}(1-t, s) = \tilde{H}(t,s) \ \text{and } \tilde{H}(t, 1-s) = \tilde{H}(t,s) \ \text{for all } (t,s) \in \Delta;$$

(ii) *$\tilde{H}$ is convex on the co-ordinates;*

(iii) *We have the bounds*

$$\begin{aligned} &\inf_{(t,s)\in[0,1]^2} \tilde{H}(t,s) = \tilde{H}\left(\frac{1}{2}, \frac{1}{2}\right) \\ = \ &\frac{1}{(b-a)^2 (d-c)^2} \int_a^b \int_a^b \int_c^d \int_c^d f\left(\frac{x+y}{2}, \frac{z+u}{2}\right) dxdydzdu \end{aligned}$$

and

$$\sup_{(t,s)\in[0,1]^2} \tilde{H}(t,s) = \tilde{H}(0,0) = \tilde{H}(1,1) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,z)\,dxdz.$$

(iv) *The mapping $\tilde{H}(\cdot,s)$ is monotonic nonincreasing on $\left[0,\frac{1}{2}\right)$ and nondecreasing on $\left[\frac{1}{2},1\right]$ for all $s\in[0,1]$. A similar property has the mapping $\tilde{H}(t,\cdot)$ for all $t\in[0,1]$.*

(v) *We have the inequality*

$$\tilde{H}(t,s) \geq \max\{H(t,s), H(1-t,s), H(t,1-s), H(1-t,1-s)\} \tag{6.6}$$

for all $(t,s)\in[0,1]^2$.

Proof. $(i),(ii)$ are obvious.
(iii) By the convexity of f in the first variable, we get that

$$\begin{aligned}&\frac{1}{2}\left[f(tx+(1-t)y, sz+(1-s)u) + f((1-t)x+ty, sz+(1-s)u)\right]\\ \geq\ & f\left(\frac{x+y}{2}, sz+(1-s)u\right)\end{aligned}$$

for all $(x,y)\in[a,b]^2$, $(z,u)\in[c,d]^2$ and $(t,s)\in[0,1]^2$.
Integrating on $[a,b]^2$, we get

$$\begin{aligned}&\frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y, sz+(1-s)u)\,dxdy\\ \geq\ & \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}, sz+(1-s)u\right)dxdy.\end{aligned}$$

Similarly,

$$\begin{aligned}&\frac{1}{(d-c)^2}\int_c^d\int_c^d f\left(\frac{x+y}{2}, sz+(1-s)u\right)dzdu\\ \geq\ & \frac{1}{(d-c)^2}\int_c^d\int_c^d f\left(\frac{x+y}{2}, \frac{z+u}{2}\right)dzdu.\end{aligned}$$

Now, integrating this inequality on $[a,b]^2$ and taking into account the above inequality we deduce:

$$\tilde{H}(t,s) \geq \frac{1}{(b-a)^2(d-c)^2}\int_a^b\int_a^b\int_c^d\int_c^d f\left(\frac{x+y}{2},\frac{z+u}{2}\right)dxdydzdu$$

for $(t,s)\in[0,1]$. The first bound in (iii) is therefore proved.
The second bound goes likewise and we shall omit the details.

(iv) The monotonicity of $\tilde{H}(\cdot,s)$ follows by a similar argument as in the proof of Theorem 207, (iii) and we shall omit the details.

(v) By Jensen's inequality we have successively for all $(t,s)\in[0,1]^2$ that

$$\begin{aligned}
&\tilde{H}(t,s)\\
\geq\ & \frac{1}{(b-a)(d-c)^2}\\
&\times\int_a^b\int_c^d\int_c^d f\left(\frac{1}{b-a}\int_a^b[tx+(1-t)y]\,dy, sz+(1-s)u\right)dxdzdu\\
=\ & \frac{1}{(b-a)(d-c)^2}\int_a^b\int_c^d\int_c^d f\left(tx+(1-t)\frac{a+b}{2}, sz+(1-s)u\right)dxdzdu\\
\geq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(tx+(1-t)\frac{a+b}{2}, sz+(1-s)\frac{c+d}{2}\right)dxdz\\
=\ & H(t,s).
\end{aligned}$$

In addition, as

$$\tilde{H}(t,s)=\tilde{H}(1-t,s)=\tilde{H}(t,1-s)=\tilde{H}(1-t,1-s)\ \text{ for all }\ (t,s)\in[0,1]^2,$$

then by the above inequality we deduce (6.6).
The theorem is thus proved. ∎

Finally, we can also state the following theorem which can be proved in a similar fashion to Theorem 208 and we will omit the details.

THEOREM 210. *Suppose that $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ is convex on Δ. Then we have:*

(i) *The mapping $\tilde{H}$ is convex on Δ.*

(ii) *Define the mapping $\tilde{h}:[0,1]\to\mathbb{R}$, $\tilde{h}(t):=\tilde{H}(t,t)$. Then $\tilde{h}$ is convex, monotonic nonincreasing on $\left[0,\frac{1}{2}\right]$ and nondecreasing on $\left[\frac{1}{2},1\right]$ and one has the bounds:*

$$\sup_{t,\in[0,1]}\tilde{h}(t)=\tilde{h}(1)=\tilde{h}(0)=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy.$$

and

$$\begin{aligned}
&\inf_{t\in[0,1]}\tilde{h}(t)=\tilde{h}\left(\frac{1}{2}\right)\\
=\ & \frac{1}{(b-a)^2(d-c)^2}\int_a^b\int_a^b\int_c^d\int_c^d f\left(\frac{x+y}{2},\frac{z+u}{2}\right)dxdydzdu.
\end{aligned}$$

(iii) *One has the inequality:*

$$\tilde{h}(t)\geq\max\{h(t),h(1-t)\}\ \textit{for all}\ t\in[0,1].$$

2. A $H.-H.$ Inequality on the Disk

Let us consider a point $C=(a,b)\in\mathbb{R}^2$ and the disk $D(C,R)$ centered at the point C and having the radius $R>0$. The following inequality of Hadamard's type holds [**33**].

THEOREM 211. *If the mapping $f:D(C,R)\to\mathbb{R}$ is convex on $D(C,R)$, then one has the inequality:*

$$f(C)\leq\frac{1}{\pi R^2}\iint_{D(C,R)}f(x,y)\,dxdy\leq\frac{1}{2\pi R}\int_{\mathfrak{S}(C,R)}f(\gamma)\,dl(\gamma)\tag{6.7}$$

where $\mathfrak{S}(C,R)$ is the circle centered at the point C and having the radius R. The above inequalities are sharp.

PROOF. Consider the transformation of the plane $\mathbb{R}^2$ in itself given by:

$$h:\mathbb{R}^2\to\mathbb{R}^2,\ h=(h_1,h_2)\ \text{ and } h_1(x,y)=-x+2a,\ h_2(x,y)=-y+2b.$$

Then $h(D(C,R))=D(C,R)$ and since

$$\frac{\partial(h_1,h_2)}{\partial(x,y)}=\begin{vmatrix}-1 & 0\\ 0 & -1\end{vmatrix}=1,$$

we have the change of variable:

$$\begin{aligned}&\iint_{D(C,R)} f(x,y)\,dxdy\\ =&\iint_{D(C,R)} f(h_1(x,y),h_2(x,y))\left|\frac{\partial(h_1,h_2)}{\partial(x,y)}\right|dxdy\\ =&\iint_{D(C,R)} f(-x+2a,-y+2b)\,dxdy.\end{aligned}$$

Now, by the convexity of f on $D(C,R)$ we also have:

$$\frac{1}{2}\left[f(x,y)+f(-x+2a,-y+2b)\right]\geq f(a,b)$$

which gives, by integration on the disk $D(C,R)$, that:

$$\begin{aligned}&\frac{1}{2}\left[\iint_{D(C,R)} f(x,y)\,dxdy+\iint_{D(C,R)} f(-x+2a,-y+2b)\,dxdy\right]\\ \geq\ & f(a,b)\iint_{D(C,R)} dxdy=\pi R^2 f(a,b).\end{aligned}\tag{6.8}$$

In addition, as

$$\iint_{D(C,R)} f(x,y)\,dxdy=\iint_{D(C,R)} f(-x+2a,-y+2b)\,dxdy,$$

then by the inequality (6.8) we obtain the first part of (6.7).
Now, consider the transformation

$$g=(g_1,g_2):\ [0,R]\times[0,2\pi]\to D(C,R)$$

given by

$$g:\begin{cases} g_1(r,\theta)=r\cos\theta+a\\ g_2(r,\theta)=r\sin\theta+b\end{cases},\ r\in[0,R],\theta\in[0,2\pi].$$

Then we have

$$\frac{\partial(g_1,g_2)}{\partial(r,\theta)}=\begin{vmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{vmatrix}=r.$$

Thus, we have the change of variable

$$\begin{aligned}\iint_{D(C,R)} f(x,y)\,dxdy &= \int_0^R\int_0^{2\pi} f(g_1(r,\theta),g_2(r,\theta))\left|\frac{\partial(g_1,g_2)}{\partial(r,\theta)}\right|drd\theta\\ &=\int_0^R\int_0^{2\pi} f(r\cos\theta+a,r\sin\theta+b)\,rdrd\theta.\end{aligned}$$

Note that, by the convexity of f on $D(C,R)$, we have:

$$\begin{aligned}
& f(r\cos\theta + a, r\sin\theta + b) \\
= \; & f\left(\frac{r}{R}(R\cos\theta + a, R\sin\theta + b) + \left(1-\frac{r}{R}\right)(a,b)\right) \\
\leq \; & \frac{r}{R} f(R\cos\theta + a, R\sin\theta + b) + \left(1-\frac{r}{R}\right) f(a,b),
\end{aligned}$$

which yields that

$$\begin{aligned}
& f(r\cos\theta + a, r\sin\theta + b)\, r \\
\leq \; & \frac{r^2}{R} f(R\cos\theta + a, R\sin\theta + b) + r\left(1-\frac{r}{R}\right) f(a,b)
\end{aligned}$$

for all $(r,\theta) \in [0,R] \times [0,2\pi]$.
Integrating on $[0,R] \times [0,2\pi]$ we get

$$\begin{aligned}
& \iint_{D(C,R)} f(x,y)\,dxdy \\
\leq \; & \int_0^R \frac{r^2}{R} dr \int_0^{2\pi} f(R\cos\theta + a, R\sin\theta + b)\,d\theta \\
& + f(a,b) \int_0^{2\pi} d\theta \int_0^R r\left(1-\frac{r}{R}\right) dr \\
= \; & \frac{R^2}{3} \int_0^{2\pi} f(R\cos\theta + a, R\sin\theta + b)\,d\theta + \frac{\pi R^2}{3} f(a,b).
\end{aligned} \tag{6.9}$$

Now, consider the curve $\gamma : [0,2\pi] \to \mathbb{R}^2$ given by:

$$\gamma : \begin{cases} x(\theta) := R\cos\theta + a \\ y(\theta) := R\sin\theta + b \end{cases}, \; \theta \in [0,2\pi].$$

Then $\operatorname{Im}(\gamma) = \gamma([0,2\pi]) = \mathfrak{S}(C,R)$ and we write (integrating with respect to arc length):

$$\begin{aligned}
\int_{\mathfrak{S}(C,R)} f(\gamma)\,dl(\gamma) &= \int_0^{2\pi} f(x(\theta), y(\theta)) \left([\dot{x}(\theta)]^2 + [\dot{y}(\theta)]^2\right)^{\frac{1}{2}} d\theta \\
&= R \int_0^{2\pi} f(R\cos\theta + a, R\sin\theta + b)\,d\theta.
\end{aligned}$$

By the inequality (6.9) we obtain

$$\iint_{D(C,R)} f(x,y)\,dxdy \leq \frac{R}{3} \int_{\mathfrak{S}(C,R)} f(\gamma)\,dl(\gamma) + \frac{\pi R^2}{3} f(a,b)$$

which gives the following inequality which is interesting in itself

$$\frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\,dxdy \leq \frac{2}{3} \cdot \frac{1}{2\pi R} \int_{\mathfrak{S}(C,R)} f(\gamma)\,dl(\gamma) + \frac{1}{3} f(a,b). \tag{6.10}$$

As we proved that

$$f(C) \leq \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\,dxdy,$$

then by the inequality (6.10) we deduce the inequality:

$$f(C) \leq \frac{1}{2\pi R} \int_{\mathfrak{S}(C,R)} f(\gamma)\,dl(\gamma). \tag{6.11}$$

Finally, by (6.11) and (6.10) we have

$$\begin{aligned}\frac{1}{\pi R^2}\iint_{D(C,R)} f(x,y)\,dxdy &\leq \frac{2}{3}\cdot\frac{1}{2\pi R}\int_{\mathfrak{S}(C,R)} f(\gamma)\,dl(\gamma)+\frac{1}{3}f(C)\\ &\leq \frac{1}{2\pi R}\int_{\mathfrak{S}(C,R)} f(\gamma)\,dl(\gamma)\end{aligned}$$

and the second part of (6.7) is proved.
Now, consider the map $f_0 : D(C,R) \to \mathbb{R}$, $f_0(x,y) = 1$. Thus

$$\begin{aligned}1 &= f_0(\lambda(x,y)+(1-\lambda)(u,z))\\ &= \lambda f_0(x,y)+(1-\lambda)f_0(u,z) = 1.\end{aligned}$$

Therefore f_0 is convex on $D(C,R) \to \mathbb{R}$. We also have

$$f_0(C) = 1,\ \frac{1}{\pi R^2}\iint_{D(C,R)} f_0(x,y)\,dxdy = 1 \text{ and } \frac{1}{2\pi R}\int_{\mathfrak{S}(C,R)} f_0(\gamma)\,dl(\gamma) = 1$$

which shows us the inequalities (6.7) are sharp. ∎

2.1. Some Mappings Connected to Hadamard's Inequality on the Disk. As above, assume that the mapping $f : D(C,R) \to \mathbb{R}$ is a convex mapping on the disk centered at the point $C = (a,b) \in \mathbb{R}^2$ and having the radius $R > 0$. Consider the mapping $H : [0,1] \to \mathbb{R}$ associated with the function f and given by

$$H(t) := \frac{1}{\pi R^2}\iint_{D(C,R)} f(t(x,y)+(1-t)C)\,dxdy,$$

which is well-defined for all $t \in [0,1]$.

The following theorem contains the main properties of this mapping [**33**].

THEOREM 212. *With the above assumption, we have:*

(*i*) *The mapping H is convex on* $[0,1]$;

(*ii*) *One has the bounds:*

$$\inf_{t\in[0,1]} H(t) = H(0) = f(C) \tag{6.12}$$

and

$$\sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{\pi R^2}\iint_{D(C,R)} f(x,y)\,dxdy; \tag{6.13}$$

(*iii*) *The mapping H is monotonic nondecreasing on* $[0,1]$.

PROOF. (*i*) Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha+\beta = 1$. Then we have:

$$\begin{aligned}H(\alpha t_1+\beta t_2) &= \frac{1}{\pi R^2}\iint_{D(C,R)} f(\alpha(t_1(x,y)+(1-t_1)C)\\ &\quad +\beta(t_2(x,y)+(1-t_2)C))\,dxdy\\ &\leq \alpha\cdot\frac{1}{\pi R^2}\iint_{D(C,R)} f(t_1(x,y)+(1-t_1)C)\,dxdy\\ &\quad +\beta\cdot\frac{1}{\pi R^2}\iint_{D(C,R)} f(t_2(x,y)+(1-t_2)C)\,dxdy\\ &= \alpha H(t_1)+\beta H(t_2),\end{aligned}$$

which proves the convexity of H on $[0,1]$.

(ii) We will prove the following identity:

$$H(t)=\frac{1}{\pi t^2R^2}\iint_{D(C,tR)}f(x,y)\,dxdy \tag{6.14}$$

for all $t\in(0,1]$.

Fix t in $(0,1]$ and consider the transformation $g=(\psi,\eta):\mathbb{R}^2\to\mathbb{R}^2$ given by:

$$\begin{cases}\psi(x,y):=tx+(1-t)a\\ \eta(x,y):=ty+(1-t)b\end{cases},(x,y)\in\mathbb{R}^2.$$

Then $g(D(C,R))=D(C,tR)$.

Indeed, for all $(x,y)\in D(C,R)$ we have:

$$(\psi-a)^2+(\eta-b)^2=t^2\left[(x-a)^2+(y-b)^2\right]\le(tR)^2$$

which shows that $(\psi,\eta)\in D(C,tR)$, and conversely, for all $(\psi,\eta)\in D(C,tR)$, it is easy to see that there exists $(x,y)\in D(C,R)$ so that $g(x,y)=(\psi,\eta)$.

We have the change of variable:

$$\begin{aligned}&\iint_{D(C,tR)}f(\psi,\eta)\,d\psi d\eta\\ =\ &\iint_{D(C,R)}f(\psi(x,y),\eta(x,y))\left|\frac{D(\psi,\eta)}{D(x,y)}\right|dxdy\\ =\ &\iint_{D(C,R)}f(t(x,y)+(1-t)(a,b))\,t^2dxdy\end{aligned}$$

since $\left|\frac{D(\psi,\eta)}{D(x,y)}\right|=t^2$, which gives us the equality (6.14).

Now, by the inequality (6.7), we have:

$$\frac{1}{\pi t^2R^2}\iint_{D(C,tR)}f(x,y)\,dxdy\ge f(C)$$

which gives us $H(t)\ge f(C)$ for all $t\in[0,1]$ and since $H(0)=f(C)$, we obtain the bound (6.12).

By the convexity of f on the disk $D(C,R)$ we have:

$$\begin{aligned}H(t)&\le\frac{1}{\pi R^2}\iint_{D(C,R)}[tf(x,y)+(1-t)f(C)]\,dxdy\\ &=\frac{t}{\pi R^2}\iint_{D(C,R)}f(x,y)\,dxdy+(1-t)f(C)\\ &\le\frac{t}{\pi R^2}\iint_{D(C,R)}f(x,y)\,dxdy+\frac{1-t}{\pi R^2}\iint_{D(C,R)}f(x,y)\,dxdy\\ &=\frac{1}{\pi R^2}\iint_{D(C,R)}f(x,y)\,dxdy.\end{aligned}$$

As we have

$$H(1)=\frac{1}{\pi R^2}\iint_{D(C,R)}f(x,y)\,dxdy,$$

then the bound (6.13) holds.

(*iii*) Let $0 \leq t_1 < t_2 \leq 1$. Then, by the convexity of the mapping H we have:

$$\frac{H(t_2) - H(t_1)}{t_2 - t_1} \geq \frac{H(t_1) - H(0)}{t_1} \geq 0$$

as $H(t_1) \geq H(0)$ for all $t_1 \in [0,1]$, which proves the monotonicity of the mapping H in the interval $[0,1]$.

∎

Further on, we shall introduce another mapping connected to Hadamard's inequality

$$h : [0,1] \to \mathbb{R},\ h(t) := \begin{cases} \frac{1}{2\pi R t} \int_{\mathfrak{S}(C,tR)} f(\gamma)\, dl(\gamma(t)), & t \in (0,1] \\ f(C), & t = 0 \end{cases}$$

where $f : D(C,R) \to \mathbb{R}$ is a convex mapping on the disk $D(C,R)$ centered at the point $C = (a,b) \in \mathbb{R}^2$ and having the same radius R.

The main properties of this mapping are embodied in the following theorem [**33**].

THEOREM 213. *With the above assumptions one has:*

(*i*) *The mapping* $h : [0,1] \to \mathbb{R}$ *is convex on* $[0,1]$;

(*ii*) *One has the bounds*

$$\inf_{t\in[0,1]} h(t) = f(0) = f(C) \tag{6.15}$$

and

$$\sup_{t\in[0,1]} h(t) = h(1) = \frac{1}{2\pi R} \int_{\mathfrak{S}(C,R)} f(\gamma)\, dl(\gamma); \tag{6.16}$$

(*iii*) *The mapping* h *is monotonic nondecreasing on* $[0,1]$;

(*iv*) *We have the inequality:*

$$H(t) \leq h(t) \text{ for all } t \in [0,1].$$

PROOF. For a fixed t in $[0,1]$ consider the curve

$$\gamma : \begin{cases} x(\theta) = tR\cos\theta + a \\ y(\theta) = tR\sin\theta + b \end{cases}, \theta \in [0, 2\pi].$$

Then $\operatorname{Im}(\gamma) = \gamma([0,2\pi]) = \mathfrak{S}(C,tR)$ and

$$\begin{aligned} & \frac{1}{2\pi t R} \int_{\mathfrak{S}(C,tR)} f(\gamma)\, dl(\gamma) \\ = & \frac{1}{2\pi t R} \int_0^{2\pi} f(tR\cos\theta + a, tR\sin\theta + b) \sqrt{(\dot{x}(\theta))^2 + (\dot{y}(\theta))^2} d\theta \\ = & \frac{1}{2\pi} \int_0^{2\pi} f(tR\cos\theta + a, tR\sin\theta + b)\, d\theta. \end{aligned}$$

We note that, then

$$\begin{aligned} h(t) &= \frac{1}{2\pi} \int_0^{2\pi} f(tR\cos\theta + a, tR\sin\theta + b)\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(t(R\cos\theta, R\sin\theta) + (a,b))\, d\theta \end{aligned}$$

for all $t \in [0,1]$.

(i) Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then, by the convexity of f we get that

$$\begin{aligned} h(\alpha t_1 + \beta t_2) &= \frac{1}{2\pi}\int_0^{2\pi} f(\alpha[t_1(R\cos\theta, R\sin\theta) + (a,b)] \\ &\quad + \beta[t_2(R\cos\theta, R\sin\theta) + (a,b)])\,d\theta \\ &\leq \alpha\cdot\frac{1}{2\pi}\int_0^{2\pi} f(t_1(R\cos\theta, R\sin\theta) + (a,b))\,d\theta \\ &\quad +\beta\cdot\frac{1}{2\pi}\int_0^{2\pi} f(t_2(R\cos\theta, R\sin\theta) + (a,b))\,d\theta \\ &= \alpha h(t_1) + \beta h(t_2) \end{aligned}$$

which proves the convexity of h on $[0,1]$.

(iv) In the above theorem we showed that:

$$H(t) = \frac{1}{\pi t^2 R^2}\iint_{D(C,tR)} f(x,y)\,dxdy \text{ for all } t \in (0,1].$$

By Hadamard's inequality (6.7) we can state that:

$$\frac{1}{\pi t^2 R^2}\iint_{D(C,tR)} f(x,y)\,dxdy \leq \frac{1}{2\pi t R}\int_{\mathfrak{S}(C,tR)} f(\gamma)\,dl(\gamma)$$

which gives us that

$$H(t) \leq h(t) \text{ for all } t \in (0,1].$$

As it is easy to see that $H(0) = h(0) = f(C)$, then the inequality embodied in (iv) is proved.

(ii) The bound (6.15) follows by the above considerations and we shall omit the details.

By the convexity of f on the disk $D(0,R)$ we have:

$$\begin{aligned} h(t) &= \frac{1}{2\pi}\int_0^{2\pi} f(t[(R\cos\theta, R\sin\theta) + (a,b)] + (1-t)(a,b))\,d\theta \\ &\leq t\cdot\frac{1}{2\pi}\int_0^{2\pi} f(R\cos\theta + a, R\sin\theta + b)\,d\theta + (1-t)f(a,b)\frac{1}{2\pi}\int_0^{2\pi} d\theta \\ &\leq t\cdot\frac{1}{2\pi}\int_0^{2\pi} f(R\cos\theta + a, R\sin\theta + b)\,d\theta \\ &\quad +(1-t)\cdot\frac{1}{2\pi}\int_0^{2\pi} f(R\cos\theta + a, R\sin\theta + b)\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(R\cos\theta + a, R\sin\theta + b)\,d\theta = h(1) \end{aligned}$$

for all $t \in [0,1]$, which proves the bound (6.16).

(iii) Follows by the above considerations as in the Theorem 212. We shall omit the details.

∎

For a convex mapping f defined on the disk $D(C,R)$ we can also consider the mapping:

$$g(t,(x,y)) := \frac{1}{\pi R^2}\iint_{D(C,R)} f(t(x,y)+(1-t)(z,u))\,dzdu$$

which is well-defined for all $t\in[0,1]$ and $(x,y)\in D(C,R)$.

The main properties of the mapping g are enclosed in the following proposition [**33**]:

PROPOSITION 75. *With the above assumptions on the mapping f one has:*

(i) *For all $(x,y)\in D(C,R)$, the map $g(\cdot,(x,y))$ is convex on $[0,1]$;*

(ii) *For all $t\in[0,1]$, the map $g(t,\cdot)$ is convex on $D(C,R)$.*

PROOF. (i) Let $t_1,t_2\in[0,1]$ and $\alpha,\beta\geq 0$ with $\alpha+\beta=1$. By the convexity of f we have:

$$\begin{aligned} &g(\alpha t_1+\beta t_2,(x,y)) \\ =\;& \frac{1}{\pi R^2}\iint_{D(C,R)} f(\alpha[t_1(x,y)+(1-t_1)(z,u)] \\ &+\beta[t_2(x,y)+(1-t_2)(z,u)])\,dzdu \\ \leq\;& \alpha\cdot\frac{1}{\pi R^2}\iint_{D(C,R)} f(t_1(x,y)+(1-t_1)(z,u))\,dzdu \\ &+\beta\cdot\frac{1}{\pi R^2}\iint_{D(C,R)} f(t_2(x,y)+(1-t_2)(z,u))\,dzdu \\ =\;& \alpha g(t_1,(x,y))+\beta g(t_2,(x,y)) \end{aligned}$$

for all $(x,y)\in D(C,R)$, and the statement is proved.

(ii) Let $(x_1,y_1),(x_2,y_2)\in D(C,R)$ and $\alpha,\beta\geq 0$ with $\alpha+\beta=1$. Then

$$\begin{aligned} &g(t,\alpha(x_1,y_1)+\beta(x_2,y_2)) \\ =\;& \frac{1}{\pi R^2}\iint_{D(C,R)} f[\alpha(t(x_1,y_1)+(1-t)(z,u)) \\ &+\beta(t(x_2,y_2)+(1-t)(z,u))]\,dzdu \\ \leq\;& \alpha\cdot\frac{1}{\pi R^2}\iint_{D(C,R)} f(t(x_1,y_1)+(1-t)(z,u))\,dzdu \\ &+\beta\cdot\frac{1}{\pi R^2}\iint_{D(C,R)} f(t(x_2,y_2)+(1-t)(z,u))\,dzdu \\ =\;& \alpha g(t,(x_1,y_1))+\beta g(t,(x_2,y_2)) \end{aligned}$$

for all $t\in[0,1]$, and the statement is proved.

∎

By the use of this mapping we can introduce the following application as well:

$$G:[0,1]\to\mathbb{R},\; G(t) := \frac{1}{\pi R^2}\iint_{D(C,R)} g(t,(x,y))\,dxdy$$

where g is as above.

The main properties of this mapping are embodied in the following theorem [**33**].

THEOREM 214. *With the above assumptions we have:*

(i) *For all* $s \in \left[0, \frac{1}{2}\right]$

$$G\left(s+\frac{1}{2}\right) = G\left(\frac{1}{2}-s\right),$$

and for all $t \in [0,1]$ *one has*

$$G(1-t) = G(t);$$

(ii) *The mapping* G *is convex on the interval* $[0,1]$;

(iii) *One has the bounds:*

$$\begin{aligned} & \inf_{t\in[0,1]} G(t) = G\left(\frac{1}{2}\right) \\ = \; & \frac{1}{(\pi R^2)^2} \iiiint_{D(C,R)\times D(C,R)} f\left(\frac{x+z}{2}, \frac{y+u}{2}\right) dxdydzdu \geq f(C) \end{aligned}$$

and

$$\sup_{t\in[0,1]} G(t) = G(0) = G(1) = \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\, dxdy;$$

(iv) *The mapping* G *is monotonic nonincreasing on* $\left[0, \frac{1}{2}\right]$ *and nondecreasing on* $\left[\frac{1}{2}, 1\right]$;

(v) *We have the inequality:*

$$G(t) \geq \max\{H(t), H(1-t)\} \tag{6.17}$$

for all $t \in [0,1]$.

PROOF. The statements (i) and (ii) are obvious by the properties of the mapping g defined above and we shall omit the details.

(iii) By (i) and (ii) we have:

$$G(t) = \frac{G(t) + G(1-t)}{2} \geq G\left(\frac{1}{2}\right) \text{ for all } t \in [0,1]$$

which proves the first bound in (iii).
Note that the inequality

$$G\left(\frac{1}{2}\right) \geq f(C)$$

follows by (6.17) for $t = \frac{1}{2}$ and taking into account that $H\left(\frac{1}{2}\right) \geq f(C)$.
We also have:

$$\begin{aligned}
& G(t) \\
= & \frac{1}{(\pi R^2)^2} \iint_{D(C,R)} \left(\iint_{D(C,R)} f(t(x,y) + (1-t)(z,u))\, dzdu \right) dxdy \\
\leq & \frac{1}{(\pi R^2)^2} \iint_{D(C,R)} \left[tf(x,y)\pi R^2 + (1-t) \iint_{D(C,R)} f(z,u)\, dzdu \right] dxdy \\
= & \frac{1}{(\pi R^2)^2} \left[t\pi R^2 \iint_{D(C,R)} f(x,y)\, dxdy + (1-t)\pi R^2 \iint_{D(C,R)} f(x,y)\, dxdy \right] \\
= & \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\, dxdy
\end{aligned}$$

for all $t \in [0,1]$, and the second bound in (iii) is also proved.

(iv) The argument goes likewise as in the proof of Theorem 212 (iii) and we shall omit the details.

(v) By Theorem 211 we have that:

$$\begin{aligned}
G(t) &= \frac{1}{\pi R^2} \iint_{D(C,R)} g(t,(x,y))\, dxdy \\
&\geq g(t,(a,b)) = \frac{1}{\pi R^2} \iint_{D(C,R)} f(t(x,y) + (1-t)(a,b))\, dxdy = H(t)
\end{aligned}$$

for all $t \in [0,1]$.
As $G(t) = G(1-t) \geq H(1-t)$, we obtain the desired inequality (6.17).
The theorem is thus proved.

∎

3. A $H. - H.$ Inequality on a Ball

In this section we will point out some inequalities of Hadamard's type for convex functions defined on a ball $\bar{B}(C,R)$ where $C = (a,b,c) \in \mathbb{R}^3$, $R > 0$ and

$$\bar{B}(C,R) := \left\{ (x,y,z) \in \mathbb{R}^3 \middle| (x-a)^2 + (y-b)^2 + (z-c)^2 \leq R^2 \right\}.$$

The following theorem holds [**32**]:

THEOREM 215. *Let $f : \bar{B}(C,R) \to \mathbb{R}$ be a convex mapping on the ball $\bar{B}(C,R)$. Then we have the inequality:*

$$\begin{aligned}
f(a,b,c) &\leq \frac{1}{\nu\left(\bar{B}(C,R)\right)} \iiint_{\bar{B}(C,R)} f(x,y,z)\, dxdydz \\
&\leq \frac{1}{\sigma\left(\bar{B}(C,R)\right)} \iint_{S(C,R)} f(x,y,z)\, ds
\end{aligned} \tag{6.18}$$

where

$$S(C,R) := \left\{ (x,y,z) \in \mathbb{R}^3 \middle| (x-a)^2 + (y-b)^2 + (z-c)^2 = R^2 \right\}$$

and

$$\nu\left(\bar{B}(C,R)\right) = \frac{4\pi R^3}{3}, \ \sigma\left(\bar{B}(C,R)\right) = 4\pi R^2.$$

PROOF. To prove the first inequality in (6.18), let us consider the transformation:

$$T_1 : \mathbb{R}^3 \to \mathbb{R}^3,\ T_1(u,v,w) = (2a-u, 2b-v, 2c-w).$$

It is easy to see that the Jacobian of T_1 is

$$J(T_1) = \det \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1$$

and T_1 is a one-to-one mapping which transforms the ball $\bar{B}(C,R)$ in itself. Then we have the change of variable:

$$\begin{aligned}(6.19)\qquad & \iiint_{\bar{B}(C,R)} f(x,y,z)\,dxdydz \\ = & \iiint_{\bar{B}(C,R)} f(2a-u, 2b-v, 2c-w)\,|J(T_1)|\,dudvdw \\ = & \iiint_{\bar{B}(C,R)} f(2a-x, 2b-y, 2c-z)\,dxdydz.\end{aligned}$$

Now, by the convexity of f on the ball $\bar{B}(C,R)$, we have:

$$\frac{1}{2}\left[f(x,y,z) + f(2a-x, 2b-y, 2c-z)\right] \geq f(a,b,c)$$

for all $(x,y,z) \in \bar{B}(C,R)$.

Integrating this inequality on $\bar{B}(C,R)$ and taking into account that the equality (6.19) holds, we get

$$\begin{aligned} & \iiint_{\bar{B}(C,R)} f(x,y,z)\,dxdydz \\ \geq & \ f(a,b,c) \iiint_{\bar{B}(C,R)} dxdydz = \nu\left(\bar{B}(C,R)\right) f(a,b,c).\end{aligned}$$

That is, the first inequality in (6.18).

To prove the second part of the inequality (6.18), let us consider the transformation $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$T_2(r,\psi,\varphi) := (r\cos\psi\cos\varphi + a, r\cos\psi\sin\varphi + b, r\sin\psi + c).$$

It is well known that the Jacobian of T_2 is

$$J(T_2) = r^2\cos\psi$$

and T_2 is a one-to-one mapping defined on the interval of $\mathbb{R}^3$, $[0,R] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [0, 2\pi]$, with values in the ball $\bar{B}(C,R)$ from $\mathbb{R}^3$. Thus we have the change of variable:

$$\begin{aligned} I \ :\ & = \iiint_{\bar{B}(C,R)} f(x,y,z)\,dxdydz \\ = & \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \left[f(r\cos\psi\cos\varphi + a, r\cos\psi\sin\varphi + b, r\sin\psi + c)\right. \\ & \left.\times\, r^2\cos\psi\right] drd\psi d\varphi.\end{aligned}$$

Now, let us observe that for $(r,\psi,\varphi)\in[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$ we have

$$\begin{aligned}&f\left(r\cos\psi\cos\varphi+a,r\cos\psi\sin\varphi+b,r\sin\psi+c\right)\\=\;&f\left[\left(1-\frac{r}{R}\right)(a,b,c)+\frac{r}{R}\left(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c\right)\right].\end{aligned}$$

Using the convexity of f on the ball $\bar{B}(C,R)$ we can state that

$$\begin{aligned}&f\left[\left(1-\frac{r}{R}\right)(a,b,c)+\frac{r}{R}\left(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c\right)\right]\\\le\;&\left(1-\frac{r}{R}\right)f(a,b,c)+\frac{r}{R}f\left(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c\right)\end{aligned}\tag{6.20}$$

for all $(r,\psi,\varphi)\in[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$.
If we multiply this inequality with $r^2\cos\psi\ge0$ for $(r,\psi)\in[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and integrating the obtained inequality on $[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$ we derive:

$$\begin{aligned}I\;\le\;&f(a,b,c)\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}r^2\cos\psi\left(1-\frac{r}{R}\right)drd\psi d\varphi\\&+\frac{1}{R}\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\left[r^3\cos\psi f\left(R\cos\psi\cos\varphi+a,\right.\right.\\&\left.\left.R\cos\psi\sin\varphi+b,R\sin\psi+c\right)\right]drd\psi d\varphi\\=\;&\frac{\pi R^3}{3}f(a,b,c)+J,\end{aligned}\tag{6.21}$$

where

$$J:=\frac{R^3}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\cos\psi f\left(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c\right)d\psi d\varphi.$$

Now, let us compute the surface integral of the first type

$$K:=\iint_{S(C,R)}f(x,y,z)\,dS,$$

where

$$S(C,R):=\left\{(x,y,z)\in\mathbb{R}^3\middle|\,(x-a)^2+(y-b)^2+(z-c)^2=R^2\right\}.$$

If we consider the parametrization of $S(C,R)$ given by:

$$S(C,R):\begin{cases}x=R\cos\psi\cos\varphi+a\\y=R\cos\psi\sin\varphi+b\\z=R\sin\psi+c\end{cases};(\psi,\varphi)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$$

and putting

$$A:=\begin{vmatrix}\frac{\partial y}{\partial\psi}&\frac{\partial z}{\partial\psi}\\\frac{\partial y}{\partial\varphi}&\frac{\partial z}{\partial\varphi}\end{vmatrix}=-R^2\cos^2\psi\cos\varphi,$$

$$B:=\begin{vmatrix}\frac{\partial x}{\partial\psi}&\frac{\partial z}{\partial\psi}\\\frac{\partial x}{\partial\varphi}&\frac{\partial z}{\partial\varphi}\end{vmatrix}=R^2\cos^2\psi\sin\varphi,$$

and

$$C:=\begin{vmatrix}\frac{\partial x}{\partial\psi}&\frac{\partial y}{\partial\psi}\\\frac{\partial x}{\partial\varphi}&\frac{\partial y}{\partial\varphi}\end{vmatrix}=-R^2\sin\psi\cos\psi,$$

we have that

$$A^2+B^2+C^2=R^4\cos^2\psi \text{ for all } (\psi,\varphi)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi].$$

Thus,

$$\begin{aligned}
K &= \iint_{S(C,R)} f(x,y,z)\,dS \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\left[f(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c)\right. \\
&\quad \left.\times\sqrt{A^2+B^2+C^2}\right]d\psi d\varphi \\
&= R^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\cos\psi f(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c)\,d\psi d\varphi.
\end{aligned}$$

Consequently, using the above notations, we define: $J=\frac{R}{4}K$.
Now, using the inequality (6.21) we get

$$I\le\frac{\pi R^3}{3}f(a,b,c)+\frac{R}{4}\iint_{S(C,R)} f(x,y,z)\,dS. \tag{6.22}$$

If we divide this inequality by $\nu\left(\bar{B}(C,R)\right)=\frac{4\pi R^3}{3}$, we get the following inequality which is interesting in itself:

$$\begin{aligned}
&\frac{1}{\nu\left(\bar{B}(C,R)\right)}\iiint_{\bar{B}(C,R)} f(x,y,z)\,dxdydz \\
\le\ &\frac{1}{4}f(a,b,c)+\frac{3}{4}\cdot\frac{1}{\sigma\left(\bar{B}(C,R)\right)}\iint_{S(C,R)} f(x,y,z)\,dS.
\end{aligned} \tag{6.23}$$

Now, taking into account that we proved the inequality

$$f(a,b,c)\le\frac{1}{\bar{\nu}\left(\bar{B}(C,R)\right)}\iint_{\bar{B}(C,R)} f(x,y,z)\,dxdydz,$$

then, from (6.23) we derive

$$\begin{aligned}
&\frac{3}{4}\cdot\frac{1}{\nu\left(\bar{B}(C,R)\right)}\iiint_{\bar{B}(C,R)} f(x,y,z)\,dxdydz \\
\le\ &\frac{3}{4}\cdot\frac{1}{\sigma\left(\bar{B}(C,R)\right)}\iint_{S(C,R)} f(x,y,z)\,dS.
\end{aligned}$$

That is, the second part of the inequality (6.18).
The proof of the theorem is thus completed. ∎

3.1. Some Mappings Connected to $H.-H.$ Inequality. As above, assume that the mapping $f:\bar{B}(C,R)\to\mathbb{R}$ is a convex mapping on the ball $\bar{B}(C,R)$ centered at the point $C=(a,b,c)\in\mathbb{R}^3$ and having the radius $R>0$. Consider the mapping $H:[0,1]\to\mathbb{R}$ associated with the function f and given by:

$$H(t):=\frac{1}{\nu\left(\bar{B}(C,R)\right)}\iiint_{\bar{B}(C,R)} f(t(x,y,z)+(1-t)C)\,dxdydz$$

which is well defined for all $t\in[0,1]$.

The following theorem contains the main properties of this mapping [**32**].

THEOREM 216. *With the above assumption, we have:*

(i) *The mapping H is convex on $[0,1]$;*

(ii) *One has the bounds:*

$$\inf_{t\in[0,1]} H(t) = H(0) = f(C) \tag{6.24}$$

and

$$\sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{\bar{\nu}\left(\bar{B}(C,R)\right)} \iiint_{\bar{B}(C,R)} f(x,y,z)\, dxdydz \tag{6.25}$$

(iii) *The mapping H is monotonic nondecreasing on $[0,1]$.*

Proof. (i) Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha+\beta = 1$. Then we have:

$$\begin{aligned}
& H(\alpha t_1 + \beta t_2) \\
= {} & \frac{1}{\nu\left(\bar{B}(C,R)\right)} \iiint_{\bar{B}(C,R)} f(\alpha[t_1(x,y,z) + (1-t_1)C] \\
& + \beta[t_2(x,y,z) + (1-t_2)C])\, dxdydz \\
\leq {} & \alpha \cdot \frac{1}{\nu\left(\bar{B}(C,R)\right)} \iiint_{\bar{B}(C,R)} f(t_1(x,y,z) + (1-t_1)C)\, dxdydz \\
& + \beta \cdot \frac{1}{\nu\left(\bar{B}(C,R)\right)} \iiint_{\bar{B}(C,R)} f(t_2(x,y,z) + (1-t_2)C)\, dxdydz \\
= {} & \alpha H(t_1) + \beta H(t_2)
\end{aligned}$$

which proves the convexity on $[0,1]$.

(ii) We will prove the following identity:

$$H(t) = \frac{1}{t^3 \nu\left(\bar{B}, C\right)} \iiint_{\bar{B}(C,tR)} f(x,y,z)\, dxdydz \tag{6.26}$$

for all $t \in [0,1]$.

Fix t in $[0,1]$ and consider the mapping $g = (\psi, \eta, \mu) : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\begin{cases} \psi(x,y,z) = tx + (1-t)a \\ \eta(x,y,z) = ty + (1-t)b \\ \mu(x,y,z) = tz + (1-t)c \end{cases}, (x,y,z) \in \mathbb{R}^3.$$

We have:

$$\left| \frac{D(\psi,\eta,\mu)}{D(x,y,z)} \right| = t^3$$

and $g\left(\bar{B}(C,R)\right) = \bar{B}(C,R)$. Indeed

$$(\psi - a)^2 + (\eta - b)^2 + (\mu - c)^2 = t^2\left[(x-a)^2 + (y-b)^2 + (z-c)^2\right] \leq t^2R^2$$

which shows that $(\psi,\eta,\mu) \in \bar{B}(C,R)$, and, conversely, for $(\psi,\eta,\mu) \in \bar{B}(C,tR)$ there exists $(x,y,z) \in \bar{B}(C,R)$ such that $g(x,y,z) = (\psi,\eta,\mu)$.

We have the following change of variable:

$$\begin{aligned}
& \iiint_{\bar{B}(C,tR)} f(\psi,\eta,\mu)\, d\psi d\eta d\mu \\
= {} & \iiint_{\bar{B}(C,R)} f(\psi(x,y,z), \eta(x,y,z), \mu(x,y,z)) \left| \frac{D(\psi,\eta,\mu)}{D(x,y,z)} \right| dxdydz \\
= {} & \iiint_{\bar{B}(C,R)} f(t(x,y,z) + (1-t)C)\, t^3 dxdydz
\end{aligned}$$

and the equality (6.26) is proved.

Now, by the first inequality in (6.18) we get:

$$\frac{1}{\nu\left(\bar{B}\left(C,tR\right)\right)}\iiint_{\bar{B}(C,tR)} f\left(x,y,z\right)dxdydz \geq f\left(C\right)$$

which gives us $H\left(t\right) \geq f\left(C\right)$ for all $t \in [0,1]$. Since $H\left(0\right) = f\left(C\right)$, we obtain the bound (6.24).

By the convexity of f on the ball $\bar{B}\left(C,R\right)$ we have:

$$\begin{aligned}
H\left(t\right) &\leq \frac{1}{\nu\left(\bar{B}\left(C,R\right)\right)}\iiint_{\bar{B}(C,R)} \left[tf\left(x,y,z\right)+\left(1-t\right)f\left(C\right)\right]dxdydz \\
&= \frac{t}{\nu\left(\bar{B}\left(C,R\right)\right)}\iiint_{\bar{B}(C,R)} f\left(x,y,z\right)dxdydz + \left(1-t\right)f\left(C\right) \\
&\leq \frac{t}{\nu\left(\bar{B}\left(C,R\right)\right)}\iiint_{\bar{B}(C,R)} f\left(x,y,z\right)dxdydz \\
&\quad + \frac{1-t}{\nu\left(\bar{B}\left(C,R\right)\right)}\iiint_{\bar{B}(C,R)} f\left(x,y,z\right)dxdydz \\
&= \frac{1}{\nu\left(\bar{B}\left(C,R\right)\right)}\iiint_{\bar{B}(C,R)} f\left(x,y,z\right)dxdydz.
\end{aligned}$$

As we have

$$H\left(1\right) = \frac{1}{\nu\left(\bar{B}\left(C,R\right)\right)}\iiint_{\bar{B}(C,R)} f\left(x,y,z\right)dxdydz,$$

the bound (6.25) holds.

(*iii*) Let $0 \leq t_1 < t_2 \leq 1$. Thus, by the convexity of the mapping H we have

$$\frac{H\left(t_2\right)-H\left(t_1\right)}{t_2-t_1} \geq \frac{H\left(t_1\right)-H\left(0\right)}{t_1} \geq 0$$

as we proved that $H\left(t_1\right) \geq H\left(0\right)$ for all $t_1 \in [0,1]$; and the monotonicity of H is proved.

∎

Further on, we shall introduce another mapping connected to Hadamard's inequality:

$$h:[0,1]\to\mathbb{R},\ h\left(t\right) := \begin{cases} \frac{1}{\sigma\left(\bar{B}(C,tR)\right)}\iint_{S(C,tR)} f\left(x,y,z\right)dS \text{ if } t\in(0,1] \\ f\left(C\right) \text{ if } t=0 \end{cases}$$

where $f:\bar{B}\left(C,R\right)\to\mathbb{R}$ is a convex mapping on the ball $\bar{B}\left(C,R\right)$ centered at the point $C=(a,b,c)$ and having the radius R and $S\left(C,R\right)$ is the sphere:

$$S\left(C,R\right) := \left\{\left(x,y,z\right)\in\mathbb{R}^3 \middle| \left(x-a\right)^2+\left(y-b\right)^2+\left(z-c\right)^2 = R^2\right\}.$$

The main properties of this mapping are embodied in the following theorem [**32**]:

THEOREM 217. *With the above assumptions, one has:*

(*i*) *The mapping* $h:[0,1]\to\mathbb{R}$ *is convex on* $[0,1]$;

(*ii*) *One has the bounds:*

$$\inf_{t\in[0,1]} h\left(t\right) = h\left(0\right) = f\left(C\right) \tag{6.27}$$

and

$$\sup_{t\in[0,1]} h(t) = h(1) = \frac{1}{\sigma\left(\bar{B}(C,R)\right)} \iint_{S(C,R)} f(x,y,z)\,dS; \tag{6.28}$$

(*iii*) *The mapping h is monotonic nondecreasing on $[0,1]$;*

(*iv*) *We have the inequality:*

$$H(t) \le h(t) \ \text{ for all } t\in[0,1].$$

Proof. For a fixed t in $(0,1]$ consider the surface:

$$S(C,tR):\begin{cases} x = tR\cos\psi\cos\varphi + a \\ y = tR\cos\psi\sin\varphi + b \\ z = tR\sin\psi + c \end{cases};(\psi,\varphi)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi].$$

As in the proof of Theorem 215 we get the equality:

$$\begin{aligned} K &= \iint_{S(C,tR)} f(x,y,z)\,dS \\ &= t^2R^2 \\ &\quad\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} \cos\psi f(tR\cos\psi\cos\varphi + a, tR\cos\psi\sin\varphi + b, tR\sin\psi + c)\,d\psi d\varphi. \end{aligned}$$

Thus,

$$\begin{aligned} h(t) &= \frac{1}{4t^2\pi^2R^2}\iint_{S(C,tR)} f(x,y,z)\,dS \\ &= \frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\cos\psi f(t(R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi) + C)\,d\psi d\varphi \end{aligned}$$

for all $t\in(0,1]$.

Using this representation of the mapping h we can prove the following statements:

(*i*) Let $t_1,t_2\in[0,1]$ and $\alpha,\beta\ge 0$ with $\alpha+\beta=1$. Then, by the convexity of f, we get that:

$$h(\alpha t_1 + \beta t_1)$$

$$\begin{aligned} &= \frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} f[\alpha(t_1(R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi) + C) \\ &\quad + \beta(t_2(R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi) + C)]\cos\psi d\psi d\varphi \\ &\le \alpha\cdot\frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} f[t_1(R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi) + C]\cos\psi d\psi d\varphi \\ &\quad +\beta\cdot\frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} f[t_2(R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi) + C]\cos\psi d\psi d\varphi \\ &= \alpha h(t_1) + \beta h(t_1) \end{aligned}$$

which proves the convexity of h.

(*iv*) In the above theorem we proved among others, that

$$H(t) = \frac{1}{\nu\left(\bar{B}(C,tR)\right)}\iiint_{\bar{B}(C,tR)} f(x,y,z)\,dxdydz$$

for all $t \in (0,1]$.

By Hadamard's inequality (6.18) applied for the ball $\bar{B}(C, tR)$ we have:

$$\begin{aligned} & \frac{1}{\nu\left(\bar{B}(C,tR)\right)} \iiint_{\bar{B}(C,tR)} f(x,y,z)\,dxdydz \\ \leq \;& \frac{1}{\sigma\left(\bar{B}(C,tR)\right)} \iint_{S(C,tR)} f(x,y,z)\,dS \end{aligned}$$

from where we get the inequality

$$H(t) \leq h(t) \text{ for all } t \in (0,1].$$

As it is easy to see that $H(0) = h(0) = f(C)$, the statement is thus proved.

(ii) The bound (6.27) follows by the above considerations and we shall omit the details.

By the convexity of f on the ball $\bar{B}(C,R)$ we have:

$$\begin{aligned} & h(t) \\ = \;& \frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} f\left(t\left[(R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi) + C\right]\right. \\ & \left. + (1-t)\,C\right)\cos\psi d\psi d\varphi \\ \leq \;& \frac{t}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} f(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c)\cos\psi d\psi d\varphi \\ & + (1-t)\,f(C)\,\frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\cos\psi d\psi d\varphi \\ = \;& \frac{t}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} f(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c)\cos\psi d\psi d\varphi \\ & + (1-t)\,f(C) \\ = \;& th(1) + (1-t)\,f(C) \leq th(1) + (1-t)\,h(1) = h(1) \end{aligned}$$

as $f(C) \leq h(t)$ for all $t \in [0,1]$. Thus, the bound (6.28) is proved.

(iii) Follows as in the proof of Theorem 216, and we shall omit the details.

∎

4. A $H. - H.$ Inequality for Functions on a Convex Domain

4.1. A General Mapping Associated with the $H. - H.$ Inequality. Let $D \subset \mathbb{R}^m$ be a convex domain and $A : C(D) \to \mathbb{R}$ be a given positive linear functional such that $A(e_0) = 1$, where $e_0(x) = 1,\ x \in D$. If $x = (x_1, \dots, x_m)$ is a point from D we note by $p_i,\ i = 1, 2, \dots, m$ the function defined on D by

$$p_i(x) = x_i, \qquad i = 1, 2, \dots, m$$

and by $a_i,\ i = 1, 2, \dots, m$ the value of the functional A in p_i, i.e.,

$$A(p_i) = a_i, \qquad i = 1, 2, \dots, m.$$

Let f be a convex mapping on D. We consider the mapping $H : [0,1] \to \mathbb{R}$ associated with the function f and given by:

$$H(t) = A(f(tx + (1-t)a))$$

where $a = (a_1, a_2, \ldots, a_m)$ and the functional A acts concerning the variable x, [**79**].

THEOREM 218. *With above assumptions, we have:*

(*i*) *The mapping H is convex on* $[0,1]$;
(*ii*) *The bounds of the function H are given by*

$$\inf_{t\in[0,1]} H(t) = H(0) = f(a) \tag{6.29}$$

and $\sup_{t\in[0,1]} H(t) = H(1) = A(f)$;
(*iii*) *The mapping H is monotonic nondecreasing on* $[0,1]$.

PROOF. (*i*) Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then we have

$$\begin{aligned} H(\alpha t_1 + \beta t_2) &= A[f((\alpha t_1 + \beta t_2)x + (1 - (\alpha t_1 + \beta t_2))a)] \\ &= A[f(\alpha(t_1 x + (1-t_1)a) + \beta(t_2 x + (1-t_2)a))] \\ &\leq \alpha A[f(t_1 x + (1-t_1)a)] + BA[f(t_2 x + (1-t_2)a)] \\ &= \alpha H(t_1) + \beta H(t_2) \end{aligned}$$

which proves the convexity of H on $[0,1]$.

(*ii*) Let g be a convex function on D. Then there exist the real numbers $A_1, A_2, \ldots, A_m$ such that

$$g(x) \geq g(a) + (x_1 - a_1)A_1 + (x_2 - a_2)A_2 + \cdots + (x_m - a_m)A_m \tag{6.30}$$

for any $x = (x_1, \ldots, x_m) \in D$.
Using the fact that the functional A is linear and positive, from the inequality (6.30) we obtain the inequality:

$$A(g) \geq g(a). \tag{6.31}$$

Now, for a fixed number $t,\ t \in [0,1]$ the function $g : D \to \mathbb{R}$ defined by

$$g(x) = f(tx + (1-t)a)$$

is a convex function. From the inequality (6.31) we obtain

$$A(f(tx + (1-t)a)) \geq f(ta + (1-t)a) = f(a)$$

or

$$H(t) \geq H(0)$$

for every $t \in [0,1]$, which proves the equality (6.29).
Let $0 \leq t_1 < t_2 \leq 1$. By the convexity of the mapping H we have:

$$\frac{H(t_2) - H(t_1)}{t_2 - t_1} \geq \frac{H(t_1) - H(0)}{t_1} \geq 0.$$

So the function H is a nondecreasing function and $H(t) \leq H(1)$. The theorem is proved.

∎

4.2. A $H. - H.$ **Inequality on a Convex Domain.** Let D be a bounded convex domain from $\mathbb{R}^3$ with a piecewise smooth boundary S.

We note by:

$$\sigma := \int\int_S dS,$$

$$a_1 := \frac{\int\int_S xdS}{\sigma},$$

$$a_2 := \frac{\int\int_S ydS}{\sigma},$$

$$a_3 := \int\int_S zdS,$$

and

$$v := \int\int\int_V f(x,y,z)dxdydz.$$

Let us assume that the surface S is oriented with the aid of the unit normal h directed to the exterior of D,

$$h = (\cos\alpha, \cos\beta, \cos\gamma).$$

The following theorem is a generalization of the Theorem 215, [**79**].

THEOREM 219. *Let f be a convex function on D. With the above assumption we have the following inequalities:*

$$\begin{aligned} & v\int\int_S fds - \sigma\int\int_S [(a_1 - x)\cos\alpha + (a_2 - y)\cos\beta \\ & +(a_3 - z)\cos\gamma] f(x,y,z)dS \\ \geq\ & 4\sigma\int\int\int_D f(x,y,z)dxdydz, \end{aligned} \tag{6.32}$$

and

$$\int\int\int_D f(x,y,z)dxdydz \geq f(x_\sigma, y_\sigma, z_\sigma)v, \tag{6.33}$$

where

$$x_G = \frac{\int\int\int_D xdxdydz}{v},$$

$$y_G = \frac{\int\int\int_D ydxdydz}{v},$$

and

$$z_G = \frac{\int\int\int_D zdxdydz}{v}.$$

PROOF. We can assume that the function f has the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ and these are continuous on D.

For every point $(u,v,w) \in S$ and $(x,y,z) \in D$ the following inequality holds:

$$\begin{aligned} f(u,v,w) \geq\ & f(x,y,z) + \frac{\partial f}{\partial x}(x,y,z)(u-x) \\ & + \frac{\partial f}{\partial y}(x,y,z)(v-y) + \frac{\partial f}{\partial z}(x,y,z)(w-z). \end{aligned} \tag{6.34}$$

From the inequality (6.34) we have

$$\int\int_S f(x,y,z)dS \geq f(x,y,z)\sigma + \frac{\partial f}{\partial x}(x,y,z)(a_1-x)\sigma + \frac{\partial f}{\partial y}(x,y,z)(a_2-y)\sigma + \frac{\partial f}{\partial z}(x,y,z)(a_3-z)\sigma. \tag{6.35}$$

The above inequality leads us to the inequality

$$v\int\int_S f(x,y,z)dS \geq \sigma\int\int\int_D f(x,y,z)dxdydz + \sigma\int\int\int_D\left[\frac{\partial}{\partial x}((a_1-x)f(x,y,z)) + \frac{\partial}{\partial y}((a_2-y)f(x,y,z)) + \frac{\partial}{\partial z}((a_3-z)f(x,y,z))\right]dxdydz + 3\sigma\int\int\int_D f(x,y,z)dxdydz. \tag{6.36}$$

Using the Gauss-Ostrogradsky theorem we obtain the equality:

$$\int\int\int_D\left[\frac{\partial}{\partial x}((a_1-x)f(x,y,z) + \frac{\partial}{\partial y}((a_2-y)f(x,y,z)) + \frac{\partial}{\partial z}((a_3-z)f(z,y,z)\right]dxdydz = \int\int_S[(a_1-x)\cos\alpha + (a_2-y)\cos\beta + (a_3-z)\cos\gamma]f(x,y,z)dS. \tag{6.37}$$

∎

From the relations (6.36) and (6.37) we obtain the inequality (6.31). The inequality (6.33) is the inequality (6.31) for the functional

$$A(f) = \frac{\int\int\int_D f(x,y,z)dxdydz}{\int\int\int_D dxdydz}.$$

REMARK 98. *For $D = \overline{B}(C,R)$ we have*

$$(a_1,a_2,a_3) = C$$

and

$$\cos\alpha = \frac{x-a_1}{R}, \quad \cos\beta = \frac{y-a_2}{R}, \quad \cos\gamma = \frac{z-a_3}{R}.$$

In this case the inequality (6.31) becomes:

$$\sigma\int\int\int_{\overline{B}(C,R)} f(x,y,z)dxdydz \leq v\int\int_{S(C,R)} f(x,y,z)d\sigma.$$

Bibliography

[1] G. ALLASIA, C. GIORDANO, J. PEČARIĆ, Hadamard-type inequalities for $(2r)$-convex functions with applications, *Atti Acad. Sci. Torino-Cl. Sc. Fis.*, **133** (1999), 1-14.

[2] H. ALZER, A note on Hadamard's inequalities, *C.R. Math. Rep. Acad. Sci. Canada,* **11** (1989), 255-258.

[3] H. ALZER, On an integral inequality, *Math. Rev. Anal. Numer. Theor. Approx.,* **18** (1989), 101-103.

[4] E. ARTIN, *The Gamma Function,* Holt, Rinehart and Winston, New York, 1964. English translation of German original, Einführung in die Theorie der Gammafunktion, Teubner, 1931.

[5] A.G. AZPEITIA, Convex functions and the Hadamard inequality, *Rev.-Colombiana-Mat.*, **28**(1) (1994), 7–12.

[6] D. BARBU, S.S. DRAGOMIR and C. BUŞE, A probabilistic argument for the convergence of some sequences associated to Hadamard's inequality, *Studia Univ. Babeş-Bolyai, Math.,* **38** (1) (1993), 29-33.

[7] E. F. BECKENBACH, Convex functions, *Bull. Amer. Math. Soc.,* **54** (1948), 439-460.

[8] I.S. BEREZIN and N.P. ZHIDKOV, *Computing Methods*, Vol. I, Pergamon Press, Oxford-London-Edinburgh-New York-Paris-Frankfurt, 1965.

[9] C. BORELL, Integral inequalities for generalised concave and convex functions, *J. Math. Anal. Appl.,* **43** (1973), 419-440.

[10] H. BOHR and J. MOLLERUP, *Laerebog i mathematisk Analyse,* vol. III, pp. 149-164, Kopenhagen, 1922.

[11] J.L. BRENNER and H. ALZER, Integral inequalities for concave functions with applications to special functions, *Proc. Roy. Soc. Edinburgh Sect. A,* **118** (1991), 173-192.

[12] P.S. BULLEN, D.S. MITRINOVIĆ and P.M. VASIĆ, *Means and their Inequalities*, Reidel, Dordrecht, 1988.

[13] C. BUŞE, S.S. DRAGOMIR and D. BARBU, The convergence of some sequences connected to Hadamard's inequality, *Demostratio Math.,* **29** (1) (1996), 53-59.

[14] B.C. CARLSON, The logarithmic mean, *Amer. Math. Monthly*, **79** (1972), 615-618.

[15] P. CERONE and S.S. DRAGOMIR, Trapezoid type rules from an inequalities point of view, *Handbook of Analytic Computational Methods in Applied Mathematics,* Editor: G. Anastassiou, CRC Press, N.Y., 2000, 65-134. *Res. Rep. Coll,* **2** (1999), No. 6, Article 8, `http://rgmia.vu.edu.au/v2n6.html`

[16] P. CERONE and S.S. DRAGOMIR, Midpoint type rules from an inequalities point of view, *Handbook of Analytic Computational Methods in Applied Mathematics,* Editor: G. Anastassiou, CRC Press, N.Y., 2000, 135-200. *RGMIA Res. Rep. Coll,* **2** (1999), No. 7, Article 1, `http://rgmia.vu.edu.au/v2n7.html`

[17] S.Y. CHUNG, Functional means and harmonic functional means, *Bull. Austral. Math. Soc.*, **7** (1998), 207-220.

[18] L.J. DEDIĆ, C.E.M. PEARCE and J. PEČARIĆ, On generalizations of Ostrowski inequality via some Euler-type identities, *Math. Inequal. Appl.*, **3** (2000), 337-353.

[19] L.J. DEDIĆ, C.E.M. PEARCE and J. PEČARIĆ, Hadamard and Dragomir-Argarwal inequalities, high-order convexity and the Euler Formula, *submitted.*

[20] L.J. DEDIĆ, C.E.M. PEARCE and J. PEČARIĆ, The Euler formulae and convex functions, *Math. Ineq. & Appl.,* **2** (2000), 211-221.

[21] Y. DING, Two classes of means and their applications, *Mathematics in Practice and Theory*, **25** (1995), no. 2, 16–20. (Chinese)

[22] S.S. DRAGOMIR, A mapping in connection to Hadamard's inequalities, *An. Öster. Akad. Wiss. Math.-Natur.,* (Wien), **128**(1991), 17-20. MR 934:26032. ZBL No. 747:26015.

[23] S. S. DRAGOMIR, A refinement of Hadamard's inequality for isotonic linear functionals, *Tamkang J. of Math.* (Taiwan), **24** (1993), 101-106. MR 94a: 26043. 2BL No. 799: 26016.

[24] S. S. DRAGOMIR, Another sequence of mappings associated to Hermite-Hadamard inequalities and applications, *(submitted).*

[25] S.S. DRAGOMIR, Further properties of some mappings associated with Hermite-Hadamard's inequalities, (*submitted*).

[26] S. S. DRAGOMIR, Hermite-Hadamard type inequalities for Hölder continuous functions, (*in preparation).*

[27] S.S. DRAGOMIR, Hermite-Hadamard's inequality for $(\alpha, m)-$convex functions, (*in preparation*).

[28] S. S. DRAGOMIR, Inequalities of Hermite-Hadamard type for functions of Hölder type, (*in preparation).*

[29] S.S. DRAGOMIR, New refinements of Hermite-Hadamard's integral inequality for convex functions and applications (*submitted*).

[30] S.S. DRAGOMIR, On Hadamard's inequalities for convex functions, *Mat. Balkanica,* **6**(1992), 215-222. MR: 934: 26033.

[31] S.S. DRAGOMIR, On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, *Taiwanese J. of Math.,* in press.

[32] S.S. DRAGOMIR, On Hadamard's inequality for the convex mappings defined on a ball in the space and applications, *Math. Ineq. & Appl.*, **3** (2) (2000), 177-187.

[33] S.S. DRAGOMIR, On Hadamard's inequality on a disk, *Journal of Ineq. in Pure & Appl. Math.,* **1** (2000), No. 1, Article 2, `http://jipam.vu.edu.au/`

[34] S. S. DRAGOMIR, On some inequalities for differentiable convex functions and applications, (*submitted*).

[35] S. S. DRAGOMIR, On some integral inequalities for convex functions, *Zb.-Rad. (Kragujevac),* (1996), No. 18, 21–25.

[36] S.S. DRAGOMIR, On some new inequalities of Hermite-Hadamard type for $m-$convex functions, (*submitted).*

[37] S.S. DRAGOMIR, Properties of some sequence of mappings associated to Hermite-Hadamard's inequality and applications, (*submitted*).

[38] S. S. DRAGOMIR, Refinements of Hermite-Hadamard's inequality for convex functions, *(submitted).*

[39] S. S. DRAGOMIR, Refinements of the Hermite-Hadamard integral inequality for log-convex functions, *The Australian Math. Soc. Gazette,* submitted.

[40] S. S. DRAGOMIR, Some inequalities for twice differentiable functions and applications, (*submitted*).

[41] S.S. DRAGOMIR, Some integral inequalities for differentiable convex functions, *Contributions, Macedonian Acad. of Sci. and Arts*, **13**(1) (1992), 13-17.

[42] S. S. DRAGOMIR, Some remarks on Hadamard's inequalities for convex functions, *Extracta Math.*, **9** (2) (1994), 88-94.

[43] S.S. DRAGOMIR, The estimation of some sequences of multiple integral associated with Hermite-Hadamard's inequality, (*submitted*).

[44] S. S. DRAGOMIR, The quasilinearity of Hermite-Hadamard inequality, (*submitted*).

[45] S.S. DRAGOMIR, Two mappings in connection to Hadamard's inequalities, *J. Math. Anal. Appl.,* **167**(1992), 49-56. MR:934:26038, ZBL No. 758:26014.

[46] S. S. DRAGOMIR, Two refinements of Hadamard's inequalities, *Zb.-Rad. (Kragujevac),* (1990), No. 11, 23–26.

[47] S.S. DRAGOMIR and R.P. AGARWAL, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Mat. Lett.* **11**(5) (1998), 91–95.

[48] S.S. DRAGOMIR and R.P. AGARWAL, Two new mappings associated with Hadamard's inequalities for convex functions, *Appl. Math. Lett.,* **11** (1998), No. 3, 33-38.

[49] S.S. DRAGOMIR and C. BUŞE, Refinements of Hadamard's inequality for multiple integrals, *Utilitas Math* (Canada), **47** (1995), 193-195.

[50] S. S. DRAGOMIR, P. CERONE and A. SOFO, Some remarks on the trapezoid rule in numerical integration, *RGMIA Res. Rep. Coll.,* **2** (5) (1999), Article 1.

http://rgmia.vu.edu.au/v2n5.html, *Indian J. of Pure and Appl. Math.,* **31** (5) (2000), 475-494.

[51] S.S. DRAGOMIR, Y.J. CHO and S.S. KIM, Inequalities of Hadamard's type for Lipschitzian mappings and their applications, *J. of Math. Anal. Appl.,* **245** (2) (2000), 489-501.

[52] S.S. DRAGOMIR and S. FITZPATRICK, The Hadamard's inequality for $s-$convex functions in the first sense, *Demonstratio Math.,* **31** (3) (1998), 633-642.

[53] S.S. DRAGOMIR and S. FITZPATRICK, The Hadamard's inequality for $s-$convex functions in the second sense, *Demonstratio Math.*, **32** (4) (1999), 687-696.

[54] S.S. DRAGOMIR and N.M. IONESCU, On some inequalities for convex-dominated functions, A*nal. Num. Theor. Approx.*, **19** (1990), 21-28. MR 936: 26014 ZBL No. 733 : 26010.

[55] S. S. DRAGOMIR and N. M. IONESCU, Some integral inequalities for differentiable convex functions, *Coll. Pap. of the Fac. of Sci. Kragujevac* (Yugoslavia), **13** (1992), 11-16, ZBL No. 770.

[56] S.S. DRAGOMIR and N.M. IONESCU, Some remarks on convex functions, *Anal. Num. Theor. Approx.,* **21** (1992), 31-36.

[57] S.S. DRAGOMIR and D.S. MILOŠEVIC, Some refinements of Hadamard's inequality, *Journal of the Indian Math. Soc.,* (in press).

[58] S.S. DRAGOMIR, D.S. MILOŚEVIĆ and J. SÁNDOR, On some refinements of Hadamard's inequalities and applications, *Univ. Belgrad, Publ. Elek. Fak. Sci. Math.,* **4**(1993), 21-24.

[59] S.S. DRAGOMIR and B. MOND, Integral inequalities of Hadamard's type for $\log-$convex functions, *Demonstratio Math.,* **31** (2) (1998), 354-364.

[60] S.S. DRAGOMIR and B. MOND, On Hadamard's inequality for a class of functions of Godunova and Levin, *Indian J. Math.*, **39** (1997), no. 1, 1–9.

[61] S.S. DRAGOMIR and C.E.M. PEARCE, On a class of inequalities of Hadamard type with symmetric weights, (*submitted*).

[62] S.S. DRAGOMIR and C.E.M PEARCE, Quasi-convex functions and Hadamard's inequality, *Bull. Austral. Math. Soc.,* **57** (1998), 377-385.

[63] S.S. DRAGOMIR, C.E.M PEARCE and J.E. PEČARIĆ, Hadamard's inequality for convex-dominated functions and applications, (*in preparation*).

[64] S. S. DRAGOMIR, C. E. M. PEARCE and J. E. PEČARIĆ, On Jessen's and related inequalities for isotonic sublinear functionals, *Acta Math. Sci.* (Szeged), **61** (1995), 373-382.

[65] S. S. DRAGOMIR and E. PEARCE, A refinement of the second part of the Hadamard inequality, *The 6th Symp. of Math. and Its Appl., Tech. Univ. of Timisoara, Nov. 1995.*

[66] S.S. DRAGOMIR and J.E. PEČARIĆ, Refinements of some inequalities for isotonic functionals, *Anal. Num. Their Approx.,* **18** (1989), 61-65. MR 91g: 26038. ZBL No. 681:26009.

[67] S.S. DRAGOMIR, J.E. PEČARIĆ and L.E. PERSSON, Some inequalities of Hadamard type, *Soochow J. of Math.* (Taiwan), **21** (1995), 335-341.

[68] S.S. DRAGOMIR, J.E. PEČARIĆ and J. SÁNDOR, A note on the Jensen-Hadamard inequality, *Anal. Num. Theor. Approx.,* **19** (1990), 21-28. MR 93b : 260 14.ZBL No. 733 : 26010.

[69] S.S. DRAGOMIR and G.H. TOADER, Some inequalities for $m-$convex functions, *Studia Univ. Babeş-Bolyai, Math.*, **38** (1) (1993), 21-28.

[70] N. ELEZOVIĆ, C. GIORDANO, J. PEČARIĆ, Convexity and q-gamma function, *Rend. Circ. Mat. Palermo*, **XLVIII** (2) (1999), 285-298.

[71] A. ENGEL, *Problem-Solving Strategies,* Springer-Verlag, 1998.

[72] L. FÉJER, Über die Fourierreihen, II, *Math. Naturwiss, Anz. Ungar. Akad. Wiss.,* **24** (1906), 369-390. (In Hungarian).

[73] A.M. FINK, A best possible Hadamard inequality, *Math. Ineq. & Appl.,* **2** (1998), 223-230.

[74] A.M. FINK, Hadamard's inequality for log-concave functions, *submitted.*

[75] A.M. FINK, Hadamard inequalities for logarithmic concave functions, *Math. Comput. Modeling,* to appear.

[76] A.M. FINK, Toward a theory of best possible inequalities, *Nieuw Archief von Wiskunde,* **12** (1994), 19-29.

[77] A.M. FINK, Two inequalities, *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat.,* **6** (1995), 48-49.

[78] A.M. FINK and M. JODEIT, Jr., On Chebyshev's other inequality; Inequalities in Statistics and Probability, (*Lecture Notes IMS, No. 5),* Inst. Math. Statistics, Haywood, CA (1984), 115-120.

[79] B. GAVREA, On Hadamard's inequality for the convex mappings defined on a convex domain in the space, *Journal of Ineq. in Pure & Appl. Math.,* **1** (2000), No. 1, Article 9, `http://jipam.vu.edu.au/`

[80] P.M. GILL, C.E.M. PEARCE and J. PEČARIĆ, Hadamard's inequality for $r-$convex functions, *J. of Math. Anal. and Appl.,* **215** (1997), 461-470.

[81] E.K. GODUNOVA and V.I. LEVIN, *Neravenstva Dlja Funkcii Širokogo Klassa*, Moskva, 1985 (138-142).

[82] I.S. GRADSHTEYN and I. M. RYZHIK, *Table of Integrals, Series and Products*, Academic Press, 1996.

[83] B. N. GUO and F. QI, Inequalities for generalized weighted mean values of convex function, *submitted.*

[84] G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA, *Inequalities*, 2nd Ed., Cambridge University Press, 1952.

[85] H.P. HEINIG and L. MALIGRANDA, Chebyshev inequality in functions spaces, *Real Analysis Exchange,* **17** (1991/92), 211-247.

[86] A. HORN, On the eigenvalues of a matrix with prescribed singular values, *Proc. Amer. Math. Soc.,* **5** (1954), 4-7.

[87] H. HUDZIK and L. MALIGRANDA, Some remarks on $s-$convex functions, *Aequationes Math., 48* (1994), 100-111.

[88] J.L.W.V. JENSEN, Sur les fonctions convexes et les inégalités entre les voleurs mogernmes, *Acta. Math.,* **30** (1906), 175-193.

[89] K. JICHANG, Some extensions and refinements of Minc-Sathre inequality, *The Math. Gazette*, March 1999, Note 83.17, pp. 123-127.

[90] E. LEACH and M. SHOLANDER, Extended mean values II, *J. Math. Anal. Appl*., **2** (1983), 207-223.

[91] H. KENYON, Note on convex functions, *Amer. Math. Monthly,* **63** (1956), 107.

[92] C.H. KIMBERLING, Some corollaries to an integral inequality, *Amer. Math. Month.,* **81** (1974), 269-270.

[93] V.L. KLEE, Solution of a problem of E.M. Wright on convex functions, *Amer. Math. Monthly*, **63** (1956), 106-107.

[94] V.R. KRYLOV, *Approximate Calculation of Integrals*, The MacMillan Comp., New York, 1962.

[95] JI-CHANG KUANG, *Applied Inequalities*, 2nd edition, Hunan Education Press, Changsha, China, 1993. (Chinese)

[96] S. S. KUTATELADZE and A.M. RUBINOV, Minkowski duality and its applications, *Russian Mathematical Surveys,* **27** (1972), 137-192.

[97] K.-C. LEE and K.-L. TSENG, On a weighted generalisation of Hadamard's inequality for G-convex functions, *Tamsui Oxford Journal of Math. Sci.*, **16**(1) (2000), 91-104.

[98] T.P. LIN, The power mean and the logarithmic mean, *Amer. Math. Monthly,* **81** (1979), 879-883.

[99] M. LOÈVE, *Probability Theory,* van Nostrand, Princeton 3rd ed., 1963.

[100] A. LUPAŞ, The Jensen-Hadamard inequality for convex functions of higher order, *Octogon Math. Mag.*, **5** (1997), no. 2, 8-9.

[101] A. LUPAŞ, A generalisation of Hadamard inequalities for convex functions, ONLINE: (`http://rgmia.vu.edu.au/authors/ALupas.htm`).

[102] A. LUPAŞ, The Jensen-Hadamard inequality for convex functions of higher order, ONLINE: (`http://rgmia.vu.edu.au/authors/ALupas.htm`)

[103] A. LUPAŞ, A generalisation of Hadamard's inequality for convex functions, *Univ. Beograd. Publ. Elek. Fak. Ser. Mat. Fiz.,* No. **544-576,** (1976), 115-121.

[104] D. M. MAKISIMOVIĆ, A short proof of generalized Hadamard's inequalities, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, (1979), No. 634-677 126–128.

[105] L. MALIGRANDA, J.E. PEČARIĆ and L.E. PERSSON, On some inequalities of the Grüss-Barnes and Borell type, *J. Math. Anal. Appl.,* **187** (1994), 306-323.

[106] A.W. MARSHALL and I. OLKIN, *Inequalities: Theory of Majorisation and its Applications,* Academic Press, 1979.

[107] W. MATUSZEWSKA and W. ORLICZ, A note of the theory of $s-$normed spaces of $\psi-$integrable functions, *Studia Math.*, **21** (1961), 107-115.

[108] M. MIGNOTTE and P.H. GLESSER, Landau's inequality via Hadamard's, *J. of Symbolic Computation,* **18** (1994), No. 4, 379-383.

[109] V.G MIHEŞAN, A generalisation of the convexity, *Seminar on Functional Equations, Approx. and Convex., Cluj-Napoca, Romania,* 1993 *(in press)*.

[110] D. MIHET, An extension of the inequality of Huygens, *Revista Matematica din Timisoara,* **1**(2) (1990), 6-7. (Romanian).

[111] D.S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[112] D.S. MITRINOVIĆ and I. LACKOVIĆ, Hermite and convexity, *Aequat. Math.*, **28** (1985), 229–232.

[113] D.S. MITRINOVIĆ and J.E. PEČARIĆ, Note on a class of functions of Godunova and Levin, *C.R. Math. Acad. Sci. Canada*, **12** (1995), 33-36.

[114] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London.

[115] B. MOND and J.E. PEČARIĆ, A companion to Fink's inequality, *Octagon Math. Mag.,* to appear.

[116] B. MOND and J.E. PEČARIĆ, Matrix inequalities for convex functions, *J. Math. Anal. Appl.,* **209**(1) (1997), 147-153.

[117] P. MONTEL, Sur les fonctions convexes et les fonctions sousharmoniques, *Journal de Math.*, **7**(9) (1928), 29-60.

[118] H.L. MONTGOMERY, *Lectures on multiplicative numeber theory,* Springer Verlag, 1971.

[119] J. MUSIELAK, Orlicz Spaces and Modular Spaces, *Lecture Notes in Mathematics*, 1034, Springer-Verlag, 1983.

[120] E. NEUMAN, The weighted logarithmic mean, *J. Math. Anal. Appl.*, **188** (1994), 885-900.

[121] E. NEUMAN, Inequalities involving generalised symmetric means, *J. Math. Anal. Appl.,* **120** (1986), 315-320.

[122] E. NEUMAN and J.E. PEČARIĆ, Inequalities involving multivariate convex functions, *J. Math. Anal. Appl.*, **137** (1989), 514-549.

[123] E. NEUMAN, Inequalities involving multivariate convex functions II, *Proc. Amer. Math. Soc.*, **109** (1990), 965-974.

[124] C.P. NICULESCU, A multiplicative mean value and applications, submitted.

[125] C.P. NICULESCU, A note on the dual Hermite-Hadamard inequality, *The Math. Gazette*, July 2000

[126] C.P. NICULESCU, An extension of Chebyshev's inequality and its connection with Jensen's Inequality, *J. of Inequal. & Appl.*, to appear.

[127] C.P. NICULESCU, Comparative convexity (*Preprint*)

[128] C.P. NICULESCU, Convexity according to the geometric mean, *Math. Ineq. & Appl.*, **3**(2) (2000), 155-167.

[129] W. ORLICZ, A note on modular spaces I, *Bull. Acad. Polon Sci. Ser. Math. Astronom. Phys.*, **9** (1961), 157-162.

[130] B.G. PACHPATTE, On some integral inequalities involving convex functions, *RGMIA Res. Rep. Coll.*, **3**(3) (2000), Article 16.

[131] Z. PÁLES, Inequalities for differences of powers, *J. Math. Anal. Applic.,* **131** (1988), 271-281.

[132] Z. PÁLES, Comparison of two variable homogeneous means, *General Inequalities* (Oberwolfach 1990), Ed. E. Walter, Birkhäuser Verlag, Basel, **6** (1992), 59-70.

[133] D. PALLASCHKE and S. ROLEWICZ, *Foundations of Mathematical Optimization (Convex Analysis without Linearity),* Kluwer Academic Publishers, Dordrecht, 1997.

[134] C.E.M. PEARCE and J. PEČARIĆ, A continuous analogue and an extension of Radós formulae for convex and concave functions, *Bull. Austral. Math. Soc.,* **53** (1996), 229-233.

[135] C.E.M. PEARCE and J. PEČARIĆ, Inequalities for differentiable mappings with applications to special means and quadrature formulæ, to appear in *Appl. Math. Lett.*

[136] C.E.M. PEARCE and J. PEČARIĆ, Some theorems of Jensen type for generalised logarithmic means, *Rev. Roum. Math. Pure Appl.,* **40** (1995), 789-795.

[137] C.E.M. PEARCE, J. PEČARIĆ and V. ŠIMIĆ, Functional Stolarsky means, *Math. Ineq. & Appl.,* **4** (1999), 479-489.

[138] C.E.M. PEARCE, J. PEČARIĆ and V. ŠIMIĆ, Stolarsky means and Hadamard's inequality, *J. Math. Anal. Appl.,* **220** (1998), 99-109.

[139] C.E.M. PEARCE and A.M. RUBINOV, $P-$functions, quasi-convex functions and Hadamard-type inequalities, J. *Math. Anal. Appl.,* **240** (1999), (1), 92-104.

[140] J.E. PEČARIĆ, Remarks on two interpolations of Hadamard's inequalities, *Contributions, Macedonian Acad. of Sci. and Arts, Sect. of Math. and Technical Sciences,* (Scopje)*,* **13**, (1992), 9-12.

[141] J. PEČARIĆ, G. ALLASIA, C. GIORDANO, Convexity and the Gamma function, *Indian J. Math.*, **41** (1) (1999), 79-93.

[142] J. PEČARIĆ and P.R. BEESACK, On Jessen's inequality for convex functions II, *J. Math. Anal. Appl.,* **118** (1986), 125-144.

[143] J. PEČARIĆ and V. ČULJAK, On Hadamard inequalities for logarithmic convex functions, submitted.

[144] J. PEČARIĆ, V. ČULJAK and A.M. FINK, On some inequalities for convex functions of higher order, *submitted.*

[145] J. PEČARIĆ and V. ŠIMIĆ, The Stolarsky-Tobey mean in n variables, *Math. Ineq. & Appl.,* **2** (1999), 325-341.

[146] J. PEČARIĆ and S. S. DRAGOMIR, A generalization of Hadamard's integral inequality for isotonic linear functionals, *Rudovi Mat.* (Sarajevo), **7** (1991), 103-107. MR 924: 26026. 2BL No. 738: 26006.

[147] J. PEČARIC, F. PROSCHAN and Y. L. TONG, *Convex Functions, Partial Orderings and Statistical Applications,* Academic Press, Inc., 1992.

[148] A.O. PITTENGER, Inequalities between arithmetic and logarithmic means, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.,* **678-715** (1980), 15-18.

[149] G. PÓLYA and G. SZEGÖ, *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, Princeton, 1951.

[150] T. POPOVICIU, Sur certaines inégalités qui caractérisent les fonctions convexes, *Analele Stiintifice Univ. "Al. l. Cuza",* Iasi, Sectia Mat., **11** (1965), 155-164.

[151] J. PONSTEIN, Seven types of convexity, *SIAM Review* **9** (1967), 115-119.

[152] F. QI, Generalized weighted mean values with two parameters, *Proceedings of the Royal Society of London Series A*, **454** (1998), no. 1978, 2723-2732.

[153] F. QI, On a two-parameter family of nonhomogeneous mean values, *Tamkang Journal of Mathematics*, **29**(2) (1998), 155-163.

[154] F. QI and S.-L. GUO, Inequalities for the incomplete Gamma functions, *Math. Ineq. & Appl.,* **2**(1) (1999), 47-53.

[155] F. QI and Q.-M. LUO, Refinements and extensions of an inequality, *Mathematics and Informatics Quarterly*, **9**(1) (1999), 23-25.

[156] F. QI, S.-L. XU, and L. DEBNATH, A new proof of monotonicity for extended mean values, *Intern. J. Math. Math. Sci.*, **22**(2) (1999), 415–420.

[157] T. RADO, On convex functions, *Trans. Amer. Math. Soc.,* **37** (1935), 266-285.

[158] A. W. ROBERTS and P. E. VARBERG, *Convex Functions*, Academic Press, 1973.

[159] A. M. RUBINOV and B.M. GLOVER, On generalized quasiconvex conjugation, *Contemporary Mathematics,* **204**, 199-216.

[160] E. SADOWSKA, Hadamard inequality and a refinement of Jensen inequality for set-valued functions, *Results Math.*, **32** (1997), No. 3-4, 332-337.

[161] F. SAIDI and R. YOUNIS, Hadamard and Fejer-type Inequalities, *Archiv der Mathematik.,* to appear.

[162] S. SAKS, Sur un theoreme de M. Montel, *C.R. Acad. Paris,* **187** (1928), 276-277.

[163] J. SÁNDOR, A note on the Jensen-Hadamard inequality, *Anal. Numer. Theor. Approx.*, **19** (1990), No. 1, 29-34.

[164] J. SÁNDOR, An application of the Jensen-Hadamard inequality, *Nieuw-Arch.-Wisk.*, **8** (1990), No. 1, 63-66.

[165] J. SÁNDOR, On the identric and logarithmic means, *Aequationes Math.,* to appear.

[166] J. SÁNDOR, On the Jensen-Hadamard inequality, *Studia Univ. Babes-Bolyai, Math.,* **36** (1991), No. 1, 9-15.

[167] J. SÁNDOR, Some integral inequalities, *El. Math.,* **43** (1988), 177-180.

[168] J. SÁNDOR, Sur la fonction Gamma, *Center Rech. Math. Pures, Neuchâtel, Série I, Fasc.*, **21** (1989), 4-7.
[169] I.J. SCHOENBERG, Cardianl spline interpolation, *Regional Conference Series in Applied Mathematics #12,* SIAM Philadelphia, 1973.
[170] J. SIVALOGANATHAN, Implications of rank one convexity, *Ann. Inst. H. Poincare. Anal. Non-Lineaire*, **5** (1988), no. 2, 99-118.
[171] I. SINGER, *Abstract Convex Analysis*, Wiley- Interscience Publication, N.Y., 1997.
[172] K.B. STOLARSKY, Generalizations of the logarithmic mean, *Math. Mag.*, **48** (1975),87-92.
[173] M.-B. SUN, Inequalities for two-parameter mean of convex function, *Mathematics in Practice and Theory*, **27**(3) (1997), 193-197. (Chinese)
[174] G.J. TEE, Further generalizations of a curiosity that Feynman remembered all his life, *Math. Mag.*, **72**(1) (1999), p. 44.
[175] K. TETTAMANTI, G. SÁRKÁNY, D. KRÂLIK and R. STOMFAI, Über die annäherung logarithmischer funktionen durch algebraische funktionen, *Period. Polytech. Chem. Engrg.*, **14** (1970), 99-111.
[176] G.H. TOADER, Means and convexity, *Studia Univ. Babes Bolyai Math.*, **36** (1991), No. 4, 45-52.
[177] G.H. TOADER, On a generalisation of the convexity, *Mathematica,* **30** (53) (1988), 83-87.
[178] G.H. TOADER, Some generalisations of the convexity*, Proc. Colloq. Approx. Optim*, Cluj-Napoca (Romania), 1984, 329-338.
[179] M.D. TOBEY, A two-parameter homogeneous mean value, *Proc. Amer. Math. Soc.,* **18** (1967), 9-14.
[180] B. UHRIN, Some remarks about the convolution of unimodal functions, *Ann. Probab.,* **12** (1984), 640-645.
[181] P.M. VASIĆ and I.B. LACKOVIĆ, Some complements to the paper: "On an inequality for convex functions", *Univ. Beograd Publ. Elek. Fak., Ser. Mat. Fiz.,* No. **544-576** (1976), 59-62.
[182] P.M. VASIĆ, I.B. LACKOVIĆ and D.M. MAKSIMOVIĆ, Note on convex functions IV: OnHadamard's inequality for weighted arithmetic means, *Univ. Beograd Publ. Elek. Fak., Ser. Mat. Fiz.,* No. **678-715** (1980), 199-205.
[183] C.L. WANG and X.H. WANG, On an extension of Hadamard inequality for convex functions, *Chin. Ann. Math.,* **3** (1982), 567-570.
[184] H. WEYL, Inequalities between two kinds of eigenvalues of a linear transformation, *Proc. Nat. Acad. Sci. USA.,* **35** (1949), 408-411.
[185] E.T. WHITTAKER and G.N. WATSON, *A Course on Modern Analysis,* 4th Edition, Cambridge, 1927.
[186] E.M. WRIGHT, A generalization of Schur's inequality, *Math. Gaz.,* (1956), 217.
[187] E.M WRIGHT, An inequality for convex functions, *Amer. Math. Monthly,* **61** (1954), 620-622.
[188] G.S YANG and M.C. HONG, A note on Hadamard's inequality, *Tamkang J. Math.,* **28** (1) (1997), 33-37.
[189] G.S YANG and K.L. TSENG, On certain integral inequalities related to Hermite-Hadamard inequalities, *J. Math. Anal. Appl.,* **239** (1999), 180-187.
[190] Z.-H. YANG, Inequalities for power mean of convex function, *Mathematics in Practice and Theory*, **20**(1) (1990), 93-96. (Chinese)