# Semi-Inner Products and Applications

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Abstract. Semi-Inner Products, that can be naturally defined in general Banach spaces over the real or complex number field, play an important role in describing the geometric properties of these spaces.

In the first chapter of the book, a short introduction to the main properties of the duality mapping that will be used in the next chapters is given. Chapter 2 is devoted to the semi-inner products in the sense of Lumer-Giles while the 3rd chapter is concerning with the main properties of the superior and inferior semi-inner products. In the next chapter the main properties of Milicics semi-inner product and the properties of normed spaces of $(G)-$type are presented. The next two chapters investigate the geometric properties of $(Q)$, $(SQ)$ and $2k-$inner product spaces introduced by the author, while Chapter 7 is entirely devoted to the study of different mappings that can naturally be associated to the norm derivatives in general normed spaces and, in particular, in inner product spaces. Chapters 8 and 9 investigate different orthogonalities that may be introduced in normed spaces and their intimate relationship with semi-inner products. In Chapter 11, orthogonal decomposition theorems in general normed spaces are provided, while in the next chapter the problem of approximating continuous linear functionals in general normed spaces and characterizations of reflexivity in this context are given. A deeper insight on this problem is then considered in Chapter 13, where some classes of continuous functionals are introduced and a density result based on the famous Bishop-Phelps theorem is obtained. In Chapter 14, the class of smooth normed spaces of $(BD)$-type and their application for non-linear operators is presented. In the next chapter the continuous sublinear functionals defined in Reflexive Banach spaces is investigated, while Chapter 16 deals with convex functions defined in more general spaces endowed with subinner products. The monograph concludes by considering the representation problem of linear forms defined on modules endowed with general semi-subinner products.

# Contents

## 1. Preface

Semi-Inner Products, that can be naturally defined in general Banach spaces over the real or complex number field, play an important role in describing the geometric properties of these spaces.

In the last forty years a large number of authors including: G. Lumer, P.S. Phillips, J.R. Giles, J.R. James, B.W. Glickfeld, E. Torrance, G. Godini, I. Singer, T. Precupanu, I. Rosca, T. Husain, B.D. Malviya, D.O. Koehler, P.M. Milicic, B. Nath, R.A. Tapia, A. Torgasev, S.M. Khaleelulla, N.J. Kalton, G.V. Wood, S. Gudder, S. Strawther, P.L. Papini, G.D. Faulkner, J.A. Canavati, J.L. Abreu, S.S. Dragomir, D.K. Sen, C. Benitez, G. Marino, P. Pietramala, M.A. Noor, J.J. Koliha, M. Crasmareanu and others, have used them as a powerful tool in investigating various properties such as; reflexivity, strict convexity and smoothness of Banach spaces as well as the possibility to represent the continuous linear functionals or to bound sub-linear functionals or convex functions defined on these spaces. Characterizations of different types of orthogonality or other geometric properties in normed linear spaces were also provided by the use of different semi-inner products as will be shown further in this book.

In the first chapter of the book, a short introduction to the main properties of the duality mapping that will be used in the next chapters is given. Chapter 2 is devoted to the semi-inner products in the sense of Lumer-Giles while the third chapter is concerned with the main properties of the superior and inferior semi-inner products. In the next chapter the main properties of Milicic?s semi-inner product and the properties of normed spaces of $(G)-$type are presented. The next two chapters investigate the geometric properties of $(Q)$, $(SQ)$ and $2k-$inner product spaces introduced by the author, while Chapter 7 is entirely devoted to the study of different mappings that can naturally be associated to the norm derivatives in general normed spaces and, in particular, in inner product spaces. Chapters 8 and 9 investigate different orthogonalities that may be introduced in normed spaces and their intimate relationship with semi-inner products. In Chapter 11, orthogonal decomposition theorems in general normed spaces are provided, while in the next chapter the problem of approximating continuous linear functionals in general normed spaces and characterizations of reflexivity in this context are given. A deeper insight on this problem is then considered in Chapter 13, where some classes of continuous functionals are introduced and a density result based on the famous Bishop-Phelps theorem is obtained. In Chapter 14, the class of smooth normed spaces of $(BD)-$type and their application for non-linear operators is presented. In the next chapter the continuous sublinear functionals defined in Reflexive Banach spaces are investigated, while Chapter 16 deals with convex functions defined in more general spaces endowed with subinner products. The monograph concludes by considering the representation problem of linear forms defined on modules endowed with general semi-subinner products.

The bibliography at the end of each chapter contains only a list of the papers cited in the chapter. The interested reader may find more information on the subject by consulting the list of papers provided at the end of the work.

The book is intended for use by both researchers and postgraduate students interested in Functional Analysis. It also provides helpful tools to mathematicians using Functional Analysis in other domains such as: Linear and Non-linear Operator Theory, Optimisation Theory, Game Theory or other related fields.

The author,

January 2003,
Melbourne

CHAPTER 1

# The Normalized Duality Mapping

## 1. Definition and Some Fundamental Properties

In what follows, we recall some of the main properties for the normalized duality mapping which will be used in the sequel. For more information and details concerning this concept we recommend the monograph by Ioana Ciorănescu [**2**] where further references are given.

Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field which will be denoted by $\mathbb{K}$.

Definition 1. *The mapping $J : X \to 2^{X^*}$, where $X^*$ is the dual space of $X$, given by:*

$$J(x) := \{x^* \in X^* | \langle x^*, x\rangle = \|x^*\| \|x\|, \|x^*\| = \|x\|\}, \quad x \in X$$

*will be called the normalised duality mapping of normed linear space $X$.*

Definition 2. *A mapping $\tilde{J} : X \to X^*$ will be called a section of normalised duality mapping if $\tilde{J}(x) \in J(x)$ for all $x$ in $X$.*

The next proposition contains some fundamental properties of these multifunctions (see for example [**1**], [**2**] or [**3**]):

Proposition 1. *Let $(X, \|\cdot\|)$ be a normed space. Then the following statements are true:*

a) *For each $x \in X$ the set $J(x)$ is convex and nonempty in $X^*$;*
b) *$J$ is monotonic in the following sense:*

$$\operatorname{Re}\langle x^* - y^*, x - y\rangle \geq 0$$

*for every $x, y \in X$ and $x^* \in J(x)$, $y^* \in J(y)$.*

c) *$J$ is antihomogeneous, i.e.,*

$$J(\lambda x) = \bar{\lambda} J(x)$$

*for all $x \in X$ and every scalar $\lambda \in \mathbb{K}$.*

Proof. The proof is as follows.

a) If $x = 0$, then obviously $J(0) = \{0\}$. Let $x \in X$, $x \neq 0$. Consider the subspace $S_p(x) := \{\lambda x | \lambda \in \mathbb{K}\}$ and define the functional $g : S_p(x) \to \mathbb{K}$, $g(u) = \lambda \|x\|^2$ where $u \in S_p(x)$,

$u = \lambda u$ $(\lambda \in \mathbb{K})$. It is clear that $g$ is a bounded linear functional on $S_p(x)$ and $\|y\| = \|x\|$. By a well-known corollary of the Hahn-Banach theorem, there is a functional $x^* \in X^*$ which extends the mapping $g$ to $X$ and is such that

$$\|x^*\| = \|y\| = \|x\|.$$

Since

$$\langle x, x \rangle = g(x) = g(1 \cdot x) = \|x\|^2 = \|x^*\| \|x\|,$$

it follows that $x^* \in J(x)$ which shows that $J(x)$ is nonempty. Now, we will show that $J(x)$ is convex in $X^*$.
Suppose $x \neq 0$ and let $x_1^*,\ x_2^* \in J(x)$. Then one has

$$\begin{aligned} \langle t x_2^* + (1-t) x_1^*, x \rangle &= t \langle x_2^*, x \rangle + (1-t) \langle x_1^*, x \rangle \\ &= t \|x\|^2 + (1-t) \|x\|^2 = \|x\|^2 \end{aligned}$$

for all $t \in [0, 1]$.

On the other hand,

$$\begin{aligned} 0 < \|x\| &= \left\langle t x_2^* + (1-t) x_1^*, \frac{x}{\|x\|} \right\rangle \\ &= \left| \left\langle t x_2^* + (1-t) x_1^*, \frac{x}{\|x\|} \right\rangle \right| \\ &\leq \sup_{y \in X \setminus \{0\}} \left| \left\langle t x_2^* + (1-t) x_1^*, \frac{y}{\|y\|} \right\rangle \right| \\ &= \|t x_2^* + (1-t) x_1^*\| \end{aligned}$$

which shows that

$$\|x\| \leq \|t x_2^* + (1-t) x_1^*\|$$

for all $t \in [0, 1]$.
However, $\|x_2^*\| = \|x_1^*\| = \|x\|$, hence

$$\|t x_2^* + (1-t) x_1^*\| \leq t \|x_2^*\| + (1-t) \|x_1^*\| = \|x\|$$

for all $t \in [0, 1]$, which gives that

$$\|t x_2^* + (1-t) x_1^*\| \leq \|x\|$$

and consequently $t x_2^* + (1-t) x_1^* \in J(x)$ for all $t \in [0, 1]$, i.e., $J(x)$ is a convex subset of $X^*$.

b) Let $x, y \in X$ and $x^* \in J(x)$, $y^* \in J(y)$. Then we have:

$$\begin{aligned}
&\operatorname{Re}\langle x^* - y^*, x - y\rangle \\
&= \langle x^*, x\rangle + \langle y^*, y\rangle - \operatorname{Re}\langle x^*, y\rangle - \operatorname{Re}\langle y^*, x\rangle \\
&\geq \|x^*\|\,\|x\| + \|y^*\|\,\|y\| - \|x^*\|\,\|y\| - \|y^*\|\,\|x\| \\
&= \|x\|^2 + \|y\|^2 - 2\,\|x\|\,\|y\| = (\|x\| - \|y\|)^2 \geq 0,
\end{aligned}$$

which proves the assertion.

c) If $\lambda = 0$, the statement is true.
Suppose that $\lambda \neq 0$ and $x^* \in J(\lambda x)$, i.e.,

$$\langle x^*, \lambda x\rangle = \|x^*\|\,\|\lambda x\| \text{ and } \|x^*\| = \|\lambda x\|$$

which yields that

$$\left\langle \frac{1}{\bar{\lambda}} x^*, x \right\rangle = \left\| \frac{1}{\bar{\lambda}} x^* \right\| \|x\| \quad \text{and} \quad \left\| \frac{1}{\bar{\lambda}} x^* \right\| = \|x\|,$$

i.e., $\frac{1}{\bar{\lambda}} x^* \in J(x)$ and then $x^* \in \bar{\lambda} J(x)$ which gives the inclusion $J(\lambda x) \subseteq \bar{\lambda} J(x)$.

The reverse inclusion goes likewise and we omit the details. ∎

Now we will give a characterization of surjectivity of the dual mapping in terms of continuous linear functionals (see for example [**1**], [**2**] or [**3**]).

Proposition 2. *Let $(X, \|\cdot\|)$ be a normed linear space. Then the following statements are equivalent:*

(i) *Every continuous linear functional on $X$ achieves its maximum on the unit sphere, i.e.,*

$$(\forall)\, x^* \in X^*, \quad (\exists)\, x \in X,\ \|x\| = 1 \quad \textit{such that} \quad \langle x^*, x\rangle = \|x^*\|.$$

(ii) *The normalised duality mapping is surjective, i.e.,*

$$(\forall)\, x^* \in X^*, \quad (\exists)\, y \in X \quad \textit{such that} \quad x^* \in J(y).$$

Proof. "(i) $\Longrightarrow$(ii)". Let $x^* \in X^*$. Then there exists an element $x \in X$, $\|x\| = 1$ and $\langle x^*, x\rangle = \|x^*\|$. Let $y = \|x^*\|\, x$. We will show that $x^* \in J(y)$.

Indeed, we have:

$$\langle x^*, y\rangle = \langle x^*, \|x^*\|\, x\rangle = \|x^*\|\,\langle x^*, x\rangle = \|x^*\|^2 = \|y\|^2$$

and

$$\|x^*\| = \|y\|$$

i.e., $x^* \in J(y)$.

"(ii) $\Longrightarrow$(i)". Let $x^* \in X^*$. Then there exists an element $y \in X$ such that $x^* \in J(y)$. We will show that $x = \frac{y}{\|y\|}$ achieves its maximum of $x^*$ on the unit sphere. Indeed, we have:

$$\langle x^*, x\rangle = \left\langle x^*, \frac{y}{\|y\|}\right\rangle = \frac{1}{\|y\|}\langle x^*, y\rangle = \frac{1}{\|y\|}\|x^*\|\|y\| = \|x^*\|,$$

and the implication is proved. ∎

## 2. Characterisations of Some Classes of Normed Spaces

In this section we point out some characterisations of smooth or reflexive normed linear spaces in terms of normalised duality mapping. A characterisation of strictly convex normed spaces is also given.

We start with the following definition (see for example [**1**], [**2**] or [**3**]).

Definition 3. *A normed linear space $(X, \|\cdot\|)$ is said to be smooth in the point $x \neq 0$ if there is a unique continuous linear functional $x^*$ such that:*

$$\langle x^*, x\rangle = \|x\| \quad \textit{and} \quad \|x^*\| = 1.$$

The following characterisation theorem holds (see for example [**3**]).

Theorem 1. *Let $(X, \|\cdot\|)$ be as above and $x_0 \in X$ with $\|x_0\| = 1$. Then the following statements are equivalent:*

(i) *$X$ is smooth in $x_0$;*
(ii) *$J(x_0)$ contains a unique element in $X^*$;*
(iii) *Every section $\tilde{J}$ of normalised duality mapping $J$ has the property:*

$$(\forall)\, x_n \in X, \quad \|x_n\| = 1, \quad x_n \overset{\|\cdot\|}{\longrightarrow} x \Longrightarrow \tilde{J}(x_n) \to \tilde{J}(x_0)$$

*in the weak topology $\sigma(X^*, X)$ of $X^*$;*
(iv) *The norm $\|\cdot\|$ is Gâteaux differentiable in $x_0$.*

Proof. "(i) $\Longrightarrow$(ii)". Let us assume that there exists $x_1^*, x_2^* \in J(x_0)$ with $x_1^* \neq x_2^*$. Then we have:

$$\langle x_1^*, x_0\rangle = \|x_1^*\|\|x_0\| = \|x_1^*\|^2 \quad \text{and} \quad \|x_1^*\| = \|x_0\| = 1$$

and

$$\langle x_2^*, x_0\rangle = \|x_2^*\|\|x_0\| = \|x_2^*\|^2 \quad \text{and} \quad \|x_2^*\| = \|x_0\| = 1,$$

which contradicts the smoothness of $X$ at the point $x_0$.

"(ii) $\Longrightarrow$(iii)". Now, let us assume that (iii) is not true. Then there exists a section $\tilde{J}$ of the normalised duality mapping $J$ and a sequence $(x_n)_{n\in\mathbb{N}}$, $\|x_n\| = 1$, $x_n \overset{\|\cdot\|}{\longrightarrow} x_0$ and with the property that $\tilde{J}(x_n) \not\to \tilde{J}(x_0)$ in $\sigma(X^*, X)$. Thus, one has a neighbourhood $U$ in

$\sigma(X^*, X)$ of $J(x_0)$ so that in the exterior of $U$ there are an infinite number of terms of the sequence $\tilde{J}(x_n)$. Let us denote these terms by $\left(\tilde{J}(x_p)\right)_{p\in\mathbb{N}}$. Since the unit ball of the dual space $X^*$ is $\sigma(X^*, X)$ – compact (c.f. the theorem of Alaoglu) then by $\left(\tilde{J}(x_p)\right)_{p\in\mathbb{N}}$ we can extract a subsequence $\left(\tilde{J}(x_q)\right)_{q\in\mathbb{N}}$ which converges to a functional $x^*$ in $\sigma(X^*, X)$ and $\|x^*\| \leq 1$ (we also used the fact that $\left\|\tilde{J}(x_q)\right\| = \|x_q\| = 1$). Since we have

$$\begin{aligned}|x^*(x_0) - 1| &= \left|x^*(x_0) - \left\langle \tilde{J}(x_q), x_q \right\rangle\right| \\ &\leq \left|x^*(x_0) - \left\langle \tilde{J}(x_q), x_q \right\rangle\right| + \left|\left\langle \tilde{J}(x_q), x_0 \right\rangle - \left\langle \tilde{J}(x_q), x_q \right\rangle\right| \\ &\leq \left|x^*(x_0) - \left\langle \tilde{J}(x_q), x_q \right\rangle\right| + \|x_q - x_0\|, \quad q \in \mathbb{N},\end{aligned}$$

hence, by passing at limit over $q$, $q \to \infty$, we get $x^*(x_0) = 1$. However, we know that $\|x^*\| \leq 1$, and thus $\|x^*\| = 1$.

Consequently, the continuous linear functional $x^*$ has the properties:

$$\|x^*\| = 1 = \|x_0\| \quad \text{and} \quad \langle x^*, x_0 \rangle = 1 = \|x_0\|^2,$$

which implies that $x^* = \tilde{J}(x_0) = J(x_0)$ (because $J(x_0)$ contains a unique element). In conclusion $\tilde{J}(x_q)$ converges to $\tilde{J}(x_0)$ in $\sigma(X^*, X)$, which contradicts the choice of the neighbourhood $U$.

"(iii) $\Longrightarrow$(iv)". Let $\tilde{J}$ be a section of the normalised duality mapping $J$. Then for all $t \neq 0$ and $h \neq 0$, $h \in X$, we have:

$$\begin{aligned}\|x_0 + th\| - \|x_0\| &= \frac{1}{\|x_0\|}\left[\|x_0\| \|x_0 + th\| - \|x_0\|^2\right] \\ &\geq \frac{1}{\|x_0\|}\left[\operatorname{Re}\left\langle \tilde{J}x_0, x_0 + th \right\rangle - \|x_0\|^2\right] \\ &= \frac{t}{\|x_0\|} \operatorname{Re}\left\langle \tilde{J}x_0, h \right\rangle,\end{aligned}$$

which implies that

$$\frac{1}{\|x_0\|} \operatorname{Re}\left\langle \tilde{J}x_0, h \right\rangle \leq \frac{1}{t}\left(\|x_0 + th\| - \|x_0\|\right) \tag{1.1}$$

for all $t > 0$.

On the other hand, one has:

$$\begin{aligned}
&\frac{1}{t}\left(\|x_0+th\|-\|x_0\|\right)\\
&=\frac{1}{t\,\|x_0+th\|}\left(\|x_0+th\|^2-\|x_0\|\,\|x_0+th\|\right)\\
&=\frac{1}{t\,\|x_0+th\|}\left(\left\langle \tilde{J}\left(x_0+th\right),x_0+th\right\rangle-\|x_0\|\,\|x_0+th\|\right)\\
&=\frac{1}{t\,\|x_0+th\|}\Big(\operatorname{Re}\left\langle \tilde{J}\left(x_0+th\right),x_0\right\rangle\\
&\qquad+t\operatorname{Re}\left\langle J\left(x_0+th\right),h\right\rangle-\|x_0\|\,\|x_0+th\|\Big)\\
&\le \operatorname{Re}\left\langle J\left(\frac{x_0+th}{\|x_0+th\|}\right),h\right\rangle
\end{aligned}$$

because

$$\operatorname{Re}\left\langle \tilde{J}\left(x_0+th\right),x_0\right\rangle\le\|x_0\|\,\|x_0+th\|$$

and then

$$\frac{1}{t}\left(\|x_0+th\|-\|x_0\|\right)\le\operatorname{Re}\left\langle \tilde{J}\left(\frac{x_0+th}{\|x_0+th\|}\right),h\right\rangle. \tag{1.2}$$

Using the inequalities (1.1) and (1.2) we summarize

$$\begin{aligned}
\operatorname{Re}\left\langle \tilde{J}\left(\frac{x_0}{\|x_0\|}\right),h\right\rangle &\le \frac{1}{t}\left(\|x_0+th\|-\|x_0\|\right)\\
&\le \operatorname{Re}\left\langle \tilde{J}\left(\frac{x_0+th}{\|x_0+th\|}\right),h\right\rangle
\end{aligned} \tag{1.3}$$

for all $t>0$ and $h\in X$, $h\neq 0$.

It is well known that for every normed space $(X,\|\cdot\|)$, the mapping $X\ni x\longmapsto\|x\|\in\mathbb{R}$ is Gâteaux differentiable at the right on $X\diagdown\{0\}$, i.e., there exists the limit

$$\left(\vee_+\|\cdot\|\right)(x)\cdot h:=\lim_{t\to 0+}\frac{\|x+th\|-\|x\|}{t},\quad(\forall)\,x\in X\diagdown\{0\}\,,\ (\forall)\,h\in X.$$

It is also known that the norm $\|\cdot\|$ is Gâteaux differentiable on $X\diagdown\{0\}$ if and only if

$$\left(\vee_+\|\cdot\|\right)(x)\cdot h=-\left(\vee_+\|\cdot\|\right)(x)\cdot(-h)$$

for all $h\in X$.

By the relation (1.3) and taking into account the fact that:

$$\lim_{t\to 0^+}\operatorname{Re}\left\langle \tilde{J}\left(\frac{x_0+th}{\|x_0+th\|}\right),h\right\rangle=\operatorname{Re}\left\langle \tilde{J}\left(\frac{x_0}{\|x_0\|}\right),h\right\rangle,\ (\forall)\,h\in X$$

(this follows by (iii)), we deduce that

$$(\vee_+ \|\cdot\|)(x_0) \cdot h = \left\langle \tilde{J}\left(\frac{x}{\|x_0\|}\right), h \right\rangle, \ (\forall)\, h \in X.$$

On the other hand, we can see that:

$$\begin{aligned} -(\vee_+ \|\cdot\|)(x_0) \cdot (-h) &= -\left\langle \tilde{J}\left(\frac{x_0}{\|x_0\|}\right), -h \right\rangle \\ &= (\vee_+ \|\cdot\|)(x_0) \cdot h, \ (\forall)\, h \in X, \end{aligned}$$

which shows that the norm $\|\cdot\|$ is Gâteaux differentiable in $x_0$ and

$$(\vee_+ \|\cdot\|)(x_0) \cdot h = \left\langle \tilde{J}\left(\frac{x_0}{\|x_0\|}\right), h \right\rangle, \ (\forall)\, h \in X.$$

"(iv) $\Longrightarrow$(ii)". By the inequality (1.1) we have for all $t > 0$ and for every $\tilde{J}$ a section of the normalised duality mapping

$$\frac{1}{\|x_0\|}\left\langle \tilde{J}x_0, h \right\rangle \le \frac{1}{t}\left(\|x_0 + th\| - \|x_0\|\right), \ (\forall)\, h \in X, \tag{1.4}$$

which implies for all $s < 0$:

$$\frac{1}{s}\left(\|x_0 + sh\| - \|x_0\|\right) \le \frac{1}{\|x_0\|}\operatorname{Re}\left\langle \tilde{J}x_0, h \right\rangle, \ (\forall)\, h \in X. \tag{1.5}$$

Since the norm $\|\cdot\|$ is assumed Gâteaux differentiable in $x_0$, then (1.4) yields that:

$$\begin{aligned} (\vee \|\cdot\|)(x_0) \cdot h &= (\vee_+ \|\cdot\|)(x_0) \cdot h = \lim_{t \to 0+} \frac{1}{t}\left(\|x_0 + th\| - \|x_0\|\right) \\ &\ge \frac{1}{\|x_0\|}\operatorname{Re}\left\langle \tilde{J}x_0, h \right\rangle, \ (\forall)\, h \in X, \end{aligned}$$

and the relation (1.5) shows that:

$$\begin{aligned} (\vee \|\cdot\|)(x_0) \cdot h &= (\vee_- \|\cdot\|)(x_0) \cdot h = \lim_{s \to 0^-} \frac{1}{s}\left(\|x_0 + sh\| - \|x_0\|\right) \\ &\le \frac{1}{\|x_0\|}\operatorname{Re}\left\langle \tilde{J}x_0, h \right\rangle, \ (\forall)\, h \in X. \end{aligned}$$

Consequently, we get:

$$(\vee \|\cdot\|)(x_0) \cdot h = \frac{1}{\|x_0\|}\operatorname{Re}\left\langle \tilde{J}x_0, h \right\rangle, \ (\forall)\, h \in X. \tag{1.6}$$

Now, if we suppose that $J(x_0)$ contains two distinct functionals $x_{0,1}^*$ and $x_{0,2}^*$ and $\tilde{J}_1$, $\tilde{J}_2$ are two sections of normalised duality mapping $J$ such that;

$$\tilde{J}_1(x_0) = x_{0,1}^* \ \text{ and } \ \tilde{J}_2(x_0) = x_{0,2}^*$$

then the relation (1.6) written for $\tilde{J}_1$ and $\tilde{J}_2$ gives

$$\operatorname{Re}\left\langle \tilde{J}_1(x_0), h\right\rangle = \operatorname{Re}\left\langle \tilde{J}_2(x_0), h\right\rangle, \ (\forall)\, h \in X.$$

On the other hand:

$$\begin{aligned}\operatorname{Im}\left\langle \tilde{J}_1(x_0), h\right\rangle &= -\operatorname{Re}\left\langle \tilde{J}_1(x_0), ih\right\rangle \\ &= -\operatorname{Re}\left\langle \tilde{J}_2(x_0), ih\right\rangle = \operatorname{Im}\left\langle \tilde{J}_2(x_0), h\right\rangle, \ (\forall)\, h \in X,\end{aligned}$$

which gives:

$$\left\langle \tilde{J}_1(x_0), h\right\rangle = \left\langle \tilde{J}_2(x_0), h\right\rangle, \ (\forall)\, h \in X,$$

i.e.,

$$\tilde{J}_1(x_0) = \tilde{J}_2(x_0) \quad \text{and} \quad x_{0,1}^* = x_{0,2}^*$$

which produces a contradiction; and the implication is thus proved.

"(ii) $\Longrightarrow$(i)". Let us assume that $X$ is not smooth in $x_0$. Then there exists $x^*$, $y^* \in X^*$, $x^* \neq y^*$, $\|x^*\| = \|y^*\| = 1$ and $\langle x^*, x_0\rangle = \|x_0\| = \langle y^*, x_0\rangle$. If we put

$$x_1^* = \|x_0\|\, x^* \quad \text{and} \quad y_1^* = \|x_0\|\, y^*,$$

then

$$x_1^* \in J(x_0), \quad y_1^* \in J(x_0), \quad \text{and} \quad x_1^* \neq y_1^*,$$

which contradicts the fact that $J(x_0)$ contains a unique element.

The proof of the theorem is thus completed. ∎

The following corollary is a natural consequence by the above considerations.

Corollary 1. *Let $(X, \|\cdot\|)$ be a normed linear space. Then the following statements are equivalent:*

(i) *$X$ is smooth;*
(ii) *The normalised duality mapping is univocal;*
(iii) *Every section $\tilde{J}$ of the normalised duality mapping $J$ is continuous from $X$ endowed with the norm topology at $X^*$ with the weak topology $\sigma(X^*, X)$;*
(iv) *The norm $\|\cdot\|$ is Gâteaux differentiable on $X \diagdown \{0\}$.*

Now, let $(X, \|\cdot\|)$ be a normed space, $X^*$ its dual, $X^{**}$ the bidual of $X$. For a fixed element $x \in X$, we define the mapping $F_x : X^* \to \mathbb{K}$,

$$F_x(f) = f(x), \quad f \in X^*.$$

It is obvious that $F_x$ is a linear functional on $X^*$. Moreover, since

$$|F_x(f)| = |f(x)| \leq \|f\|\,\|x\|, \quad (\forall)\, f \in X^*,$$

it follows that $F_x$ is also continuous on $X^*$. In addition, we have

$$\|F_x\| \leq \|x\| .$$

In this way, we can establish a mapping $X \ni x \xrightarrow{\Phi} F_x \in X^{**}$ which satisfies the inequality

$$\|\Phi(x)\| \leq \|x\| , \quad x \in X.$$

On the other hand, it is clear that $\Phi$ is a linear operator, and by the above inequality, also a bounded operator on $X$. Now, by a well known consequence of the Hahn-Banach theorem, there exists a functional $f_x \in X^*$ such that $f_x(x) = \|x\|$ and $\|f_x\| = 1$. Consequently, we have:

$$\|x\| = f_x(x) = (\Phi(x))(f_x) \leq \|\Phi(x)\| \|f(x)\| = \|\Phi(x)\| ,$$

which shows that $\Phi$ is an isometry on $X$ with values in $X^{**}$.

DEFINITION 4. *Let $(X, \|\cdot\|)$ be a normed space. Then it will be said to be reflexive if the mapping $\Phi$ defined as above is an isomorphism of normed linear spaces or, equivalently, $\Phi$ is surjective.*

The following characterisation of reflexivity in terms of the normalised duality mapping holds (see for example [**3**]).

THEOREM 2. *Let $(X, \|\cdot\|)$ be a Banach space. Then the following statements are equivalent:*

(i) *$X$ is reflexive;*
(ii) *The normalised duality mapping $J$ is surjective.*

PROOF. We use the following result due to R.C. James (see [**4**] or [**5**]) which states:

THEOREM 3. *Let $(X, \|\cdot\|)$ be a Banach space. Then $X$ is reflexive if and only if for every $x^*$ a continuous linear functional there exists an element $x \in X$ such that:*

$$\langle x^*, x \rangle = \|x^*\| \|x\| .$$

*This element is said to be a maximal element for $x^*$.*

Now, if we assume that $X$ is reflexive, then for every $x^* \in X \diagdown \{0\}$ there exists, by James' result, an element $u$ $\left(u = \frac{x}{\|x\|}, x \neq 0\right)$ in which the functional $x^*$ achieves its norm and, by Proposition 2, we obtain that $J$ is surjective.

The converse of this implication goes likewise and we omit the details. ∎

Finally, we recall the concept of strictly convex normed spaces and we give a result containing a characterisation of this class of spaces in terms of normalised duality mapping.

DEFINITION 5. *A normed linear space* $(X, \|\cdot\|)$ *will be called strictly convex if for every* $x, y$ *from* $X$, $x \neq y$ *and* $\|x\| = \|y\| = 1$ *we have:*

$$\|\lambda x + (1-\lambda) y\| < 1$$

*for all* $\lambda \in (0,1)$.

Now we can state the following result.

THEOREM 4. *Let* $(X, \|\cdot\|)$ *be a normed linear space. Then the following statements are equivalent.*

(i) $X$ *is strictly convex;*
(ii) *The duality mapping* $J$ *is strictly monotonic;*
(iii) *The duality mapping is injective, i.e.,*

$$J(x) \cap J(y) = \emptyset \quad \text{for } x \neq y.$$

The proof follows by the following well known results due to M.G. Klein (see for example [**6**]) which states:

THEOREM 5. *A normed linear space* $(X, \|\cdot\|)$ *is strictly convex iff every continuous linear functional on it has at most one maximal element having the same norm one.*

We omit the details.

For other classical characterisations of reflexive or strictly convex normed linear spaces we refer the reader to [**7**] where further references are given.

## 3. Other Properties of Normalised Duality Mappings

We start with the following theorem which improves the equivalence "(i)$\Longleftrightarrow$(iii)" of Theorem 1 and also gives another characterisation of smoothness as follows.

THEOREM 6. *Let* $(X, \|\cdot\|)$ *be a normed linear space and* $x_0 \in X \setminus \{0\}$. *Then the following statements are equivalent:*

(i) $X$ *is smooth in* $x_0$*;*
(ii) *For every* $\tilde{J}$ *a section of the normalised duality mapping* $J$ *we have*

$$\lim_{t \to 0} \operatorname{Re} \left\langle \tilde{J}(x_0 + ty), y \right\rangle = \operatorname{Re} \left\langle \tilde{J} x_0, y \right\rangle \tag{1.7}$$

*for all* $y \in X$*;*

(iii) *For every* $J$ *as above, we have:*

$$\lim_{t \to 0} \operatorname{Re} \left\langle \frac{\tilde{J}(x_0 + ty) - \tilde{J} x_0}{t}, x_0 \right\rangle = \operatorname{Re} \left\langle \tilde{J} x_0, y \right\rangle \tag{1.8}$$

*for all* $y \in X$.

Proof. As in the proof of Theorem 1 (see the relation (1.3)), we have the double inequality:

$$\operatorname{Re}\left\langle \frac{\tilde{J}x_0}{\|x_0\|}, y\right\rangle \le \frac{\|x_0+ty\|-\|x_0\|}{t} \le \operatorname{Re}\left\langle \frac{\tilde{J}(x_0+ty)}{\|x_0+ty\|}, y\right\rangle \tag{1.9}$$

for all $y \in X$ and $t > 0$ ($\tilde{J}$ is a section of a duality mapping).

On the other hand, by the first inequality in (1.9), we have

$$\operatorname{Re}\left\langle \frac{\tilde{J}(x_0+ty)}{\|x_0+ty\|}, y\right\rangle \le \frac{\|x_0+2ty\|-\|x_0+ty\|}{t} \tag{1.10}$$

for all $y \in X$ and $t > 0$. By the inequalities (1.9) and (1.10) we have:

$$\begin{aligned}
&\lim_{t\to 0^+} \frac{\|x_0+ty\|-\|x_0\|}{t} \\
&\le \lim_{t\to 0^+} \operatorname{Re}\left\langle \frac{\tilde{J}(x_0+ty)}{\|x_0+ty\|}, y\right\rangle \le \lim_{t\to 0^+} \frac{\|x_0+2ty\|-\|x_0+ty\|}{t} \\
&= 2\lim_{t\to 0^+} \frac{\|x_0+2ty\|-\|x_0\|}{2t} - \lim_{t\to 0^+} \frac{\|x_0+ty\|-\|x_0\|}{t} \\
&= 2\lim_{s\to 0^+} \frac{\|x_0+sy\|-\|x_0\|}{s} - \lim_{t\to 0^+} \frac{\|x_0+ty\|-\|x_0\|}{t} \\
&= \lim_{t\to 0^+} \frac{\|x_0+ty\|-\|x_0\|}{t},
\end{aligned}$$

for all $y \in X$, which shows that in every normed space we have the equality:

$$\lim_{t\to 0^+} \frac{\|x_0+ty\|-\|x_0\|}{t} = \lim_{t\to 0^+} \operatorname{Re}\left\langle \frac{\tilde{J}(x_0+ty)}{\|x_0+ty\|}, y\right\rangle \tag{1.11}$$

for all $y \in X$.

"(i) $\Longrightarrow$ (ii)". If $X$ is smooth in $x_0$, then by (1.11) we have:

$$\|x_0\| \left(\vee \|\cdot\|\right)(x_0)\, y = \lim_{t\to 0^+} \operatorname{Re}\left\langle \tilde{J}(x_0+ty), y\right\rangle$$

for all $y \in X$.

On the other hand, one has

$$\lim_{t\to 0^+} \operatorname{Re}\left\langle \tilde{J}(x_0+t(-y)), (-y)\right\rangle = \|x_0\| \left(\vee \|\cdot\|\right)(x_0)(-y)$$

and then

$$\begin{aligned}
\lim_{t\to 0^+} \operatorname{Re}\left\langle \tilde{J}(x_0-ty), y\right\rangle &= -\|x_0\| \left(\vee \|\cdot\|\right)(x_0)(-y) \\
&= \|x_0\| \left(\vee \|\cdot\|\right)(x_0)\, y.
\end{aligned}$$

However,

$$\lim_{t\to0^+}\operatorname{Re}\left\langle\tilde{J}\left(x_0-ty\right),y\right\rangle=\lim_{s\to0^-}\operatorname{Re}\left\langle\tilde{J}\left(x_0+sy\right),y\right\rangle.$$

Consequently, the limit $\lim\limits_{t\to0}\operatorname{Re}\left\langle\tilde{J}\left(x_0+ty\right),y\right\rangle$ exists and

$$\lim_{t\to0}\operatorname{Re}\left\langle\tilde{J}\left(x_0+ty\right),y\right\rangle=\|x_0\|\left(\triangledown\|\cdot\|\right)\left(x_0\right)\cdot y \tag{1.12}$$

for all $y\in X$.

Now, by the inequality (1.9) we also have:

$$\|x_0\|\frac{\left(\|x_0+sy\|-\|x_0\|\right)}{s}\le\operatorname{Re}\left\langle\tilde{J}\left(x_0\right),y\right\rangle\le\|x_0\|\frac{\left(\|x_0+ty\|-\|x_0\|\right)}{t}$$

for all $t>0$, $s<0$ and $y\in X$ is smooth in $x_0$ we obtain

$$\|x_0\|\left(\triangledown\|\cdot\|\right)\left(x_0\right)\cdot y=\operatorname{Re}\left\langle\tilde{J}\left(x_0\right),y\right\rangle \tag{1.13}$$

for all $y\in X$ and then (1.12) and (1.13) show the relation (1.7), and the implication is thus proven.

"(ii) $\Longrightarrow$ (i)". By the inequality (1.9) we deduce:

$$\begin{aligned}\operatorname{Re}\left\langle\frac{\tilde{J}\left(x_0\right)}{\|x_0\|},y\right\rangle&\le\lim_{t\to0^+}\frac{\|x_0+ty\|-\|x_0\|}{t}=\left(\triangledown_+\|\cdot\|\right)\left(x_0\right)\cdot y\\&\le\lim_{t\to0^+}\operatorname{Re}\frac{\left\langle\tilde{J}\left(x_0+ty\right),y\right\rangle}{\|x_0+ty\|}=\frac{\operatorname{Re}\left\langle\tilde{J}x_0,y\right\rangle}{\|x_0\|}\end{aligned}$$

then

$$\left(\triangledown_+\|\cdot\|\right)\left(x_0\right)\cdot y=\frac{\operatorname{Re}\left\langle\tilde{J}\left(x_0\right),y\right\rangle}{\|x_0\|}\quad\text{for all } y\in X.$$

On the other hand, one has:

$$\begin{aligned}\left(\triangledown_-\|\cdot\|\right)\left(x_0\right)\cdot y&=-\left(\triangledown_+\|\cdot\|\right)\left(x_0\right)\cdot\left(-y\right)=-\frac{\operatorname{Re}\left\langle\tilde{J}x_0,-y\right\rangle}{\|x_0\|}\\&=\frac{\operatorname{Re}\left\langle\tilde{J}x_0,y\right\rangle}{\|x_0\|}=\left(\triangledown_+\|\cdot\|\right)\left(x_0\right)\cdot y\end{aligned}$$

for $y\in X$, which shows that the norm $\|\cdot\|$ is Gâteaux differentiable in $x_0$, i.e., $X$ is smooth in $x_0$.

"(i) $\Longrightarrow$ (iii)". Firstly, let us observe that the following equality holds

$$\begin{aligned}&\frac{\|x_0+ty\|^2-\|x_0\|^2}{t}\\&=\frac{\operatorname{Re}\left\langle \tilde{J}(x_0+ty),x_0+ty\right\rangle-\left\langle \tilde{J}x_0,x_0\right\rangle}{t}\\&=\operatorname{Re}\left\langle \frac{\tilde{J}(x_0+ty)-\tilde{J}x_0}{t},x_0\right\rangle+\operatorname{Re}\left\langle \tilde{J}(x_0+ty),y\right\rangle\end{aligned}$$

for all $y\in X$ and $t\neq 0$, where $\tilde{J}$ is a section of a duality mapping. Now, assume that $X$ is smooth in $x_0$. Then by the above equality we have:

$$\begin{aligned}&\lim_{t\to 0}\operatorname{Re}\left\langle \frac{\tilde{J}(x_0+ty)-\tilde{J}x_0}{t},x_0\right\rangle\\&=\lim_{t\to 0}\frac{\|x_0+ty\|^2-\|x_0\|^2}{t}-\operatorname{Re}\left\langle \tilde{J}(x_0+ty),y\right\rangle\\&=2\|x_0\|(\vee\|\cdot\|)(x_0)\cdot y-\operatorname{Re}\langle Jx_0,y\rangle\\&=\operatorname{Re}\left\langle \tilde{J}x_0,y\right\rangle\end{aligned}$$

(we also used the statement (ii)), and thus (1.8) holds.

Conversely, if (1.8) holds, then, by the use of identity (1.11), we deduce:

$$\begin{aligned}2\|x_0\|(\vee_+\|\cdot\|)(x_0)\cdot y&=\lim_{t\to 0^+}\frac{\|x_0+ty\|^2-\|x_0\|^2}{t}\\&=\operatorname{Re}\left\langle \tilde{J}x_0,y\right\rangle+\lim_{t\to 0^+}\operatorname{Re}\left\langle \tilde{J}(x_0+ty),y\right\rangle\\&=\operatorname{Re}\left\langle \tilde{J}x_0,y\right\rangle+\|x_0\|(\vee_+\|\cdot\|)(x_0)\cdot y\end{aligned}$$

for all $y\in X$. Consequently,

$$\|x_0\|(\vee_+\|\cdot\|)(x_0)\cdot y=\operatorname{Re}\langle Jx_0,y\rangle$$

for all $y\in X$.

Since

$$\begin{aligned}\|x_0\|(\vee_+\|\cdot\|)(x_0)\cdot(-y)=\operatorname{Re}\left\langle \tilde{J}x_0,(-y)\right\rangle&=-\operatorname{Re}\left\langle \tilde{J}x_0,y\right\rangle\\&=\|x_0\|(\vee_+\|\cdot\|)(x_0)\cdot y\end{aligned}$$

for all $y\in X$, it follows that $\|\cdot\|$ is Gâteaux differentiable in $x_0$ and the proof is completed. ∎

COROLLARY 2. *Let $X$ be a normed linear space. Then the following statements are equivalent:*

(i) *$X$ is smooth;*

(ii) *For a section $\tilde{J}$ of normalised duality mapping $J$, we have*

$$\lim_{t\to 0} \operatorname{Re}\left\langle \tilde{J}\left(x+ty\right), y\right\rangle = \operatorname{Re}\left\langle \tilde{J}x, y\right\rangle$$

*for all $x, y \in X$;*

(iii) *For a section $\tilde{J}$ of duality mapping we have:*

$$\lim_{t\to 0} \operatorname{Re}\left\langle \frac{\tilde{J}\left(x+ty\right)-\tilde{J}x}{t}, x\right\rangle = \operatorname{Re}\left\langle \tilde{J}x, y\right\rangle$$

*for all $x, y \in X$.*

The proof is clearly embodied in the above theorem and we omit the details.

REMARK 1. *The equivalence "(i) $\Longleftrightarrow$ (ii)" is similar in a sense with the result of J.R. Giles [**8**] which holds for semi-inner products. On the other hand, equivalence "(i) $\Longleftrightarrow$ (ii)" of Theorem 6 improves the equivalence "(i) $\Longleftrightarrow$ (iii)" of Theorem 1.*

The following result is due to M. Golomb and R.A. Tapia [**9**] (see also [**3**, p. 283]).

THEOREM 7. *Let $X$ be a real (complex) Banach space on which the normalised duality mapping is univocal. Then $J$ is linear (antilinear) iff $X$ is an inner product space.*

PROOF. If $X$ is a Hilbert space then by Riesz's representation theorem it follows that $J$ is a linear operator on $X$ with values in $X^*$.

Conversely, if $J$ is linear, then one has:

$$\begin{aligned}\|x \pm y\|^2 &= \langle J\left(x \pm y\right), x \pm y\rangle = \langle Jx \pm Jy, x \pm y\rangle \\ &= \langle Jx, x\rangle + \langle Jy, y\rangle \pm \langle Jx, y\rangle \pm \langle Jy, x\rangle\end{aligned}$$

for all $x, y \in X$.

Consequently,

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

which shows, by the well known result of von Neumann and Jordan, that $X$ is an inner product space. ∎

Now, we list some other properties of the normalised duality mapping.

THEOREM 8. *Let $(X, \|\cdot\|)$ be a normed linear space. Then the following assertions are true:*

(i) *If $X^*$ is smooth (strictly convex) then $X$ is strictly convex (smooth);*
(ii) *If $X$ is reflexive, then in the above statement we have an equivalence.*

For the proof of this fact, we refer the reader to [**3**, p. 50].
Another result is embodied in the following theorem.

THEOREM 9. *Let $(X, \|\cdot\|)$ be a normed linear space. Then one has:*
(i) *If $X^*$ is strictly convex, then the normalised duality mapping is univocal and continuous to $X$ endowed with the norm topology at $X^*$ with the weak topology $\sigma(X, X^*)$.*
(ii) *If $X$ and $X^*$ are strictly convex and $X$ is reflexive, then $J$ is strictly monotonic and bijective.*

For a proof of these facts, see [**3**, pp. 283-284].

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CHAPTER 2

# Semi-Inner Products in the Sense of Lumer-Giles

## 1. Definition and Fundamental Properties

In what follows, we assume that $X$ is a linear space over the real or complex number field $\mathbb{K}$.

The following concept was introduced in 1961 by G. Lumer [**1**] but the main properties of it were discovered by J.R. Giles [**2**], P.L. Papini [**3**], P.M. Miličić [**4**] – [**13**], I. Roşca [**14**], B. Nath [**15**] and others.

In this introductory section we give the definition of this concept and point out the main facts which are derived directly from the definition.

Definition 6. *The mapping* $[\cdot,\cdot] : X \times X \to \mathbb{K}$ *will be called the semi-inner product in the sense of Lumer-Giles or* $L.-G.-s.i.p.$*, for short, if the following properties are satisfied:*

(i) $[x+y,z] = [x,z] + [y,z]$ *for all* $x, y \in X$*;*
(ii) $[\lambda x, y] = \lambda [x,y]$ *for all* $x, y \in X$ *and* $\lambda$ *a scalar in* $\mathbb{K}$*;*
(iii) $[x,x] \geq 0$ *for all* $x \in X$ *and* $[x,x] = 0$ *implies that* $x = 0$*;*
(iv) $|[x,y]|^2 \leq [x,x]\,[y,y]$ *for all* $x, y \in X$*;*
(v) $[x, \lambda y] = \bar{\lambda} [x,y]$ *for all* $x, y \in X$ *and* $\lambda$ *a scalar in* $\mathbb{K}$*.*

Now, we will state and prove the first result.

Proposition 3. *Let* $X$ *be a linear space and* $[\cdot,\cdot]$ *a* $L.-G.-s.i.p.$ *on* $X$*. Then the following statements are true:*

(i) *The mapping* $X \ni x \xrightarrow{\|\cdot\|} [x,x]^{\frac{1}{2}} \in \mathbb{R}_+$ *is a norm on* $X$*;*
(ii) *For every* $y \in X$ *the functional* $X \ni x \xrightarrow{f_y} [x,y] \in \mathbb{K}$ *is a continuous linear functional on* $X$ *endowed with the norm generated by* $L.-G.-s.i.p.$ *Moreover, one has* $\|f_y\| = \|y\|$*.*

Proof. The proof is as follows.

(i) We will verify the properties of the norm. Let $x \in X$. Then $\|x\| = [x,x]^{\frac{1}{2}} \geq 0$ and if $\|x\| = 0$, then $[x,x] = 0$, which implies that $x = 0$.
If $x \in X$ and $\lambda \in \mathbb{K}$, then one has

$$\|\lambda x\| = [\lambda x, \lambda x]^{\frac{1}{2}} = \left[\lambda \cdot \bar{\lambda}\right]^{\frac{1}{2}} [x,x]^{\frac{1}{2}} = |\lambda|\,\|x\|.$$

Finally, for every $x, y \in X$, we deduce:

$$\begin{aligned}
\|x+y\|^2 &= [x+y, x+y] = |[x, x+y] + [y, x+y]| \\
&\leq |[x, x+y]| + |[y, x+y]| \\
&\leq \|x\| \|x+y\| + \|y\| \|x+y\|
\end{aligned}$$

from where we get:

$$\|x+y\| \leq \|x\| + \|y\|$$

for all $x, y \in X$.

(ii) The fact that $f_y$ is linear follows by (i) and (ii) of Definition 6. Now, using Schwartz's inequality (iv) we get;

$$|f_y(x)| \leq \|x\| \|y\| \quad \text{for all } x \text{ in } X,$$

which implies that $f_y$ is bounded and

$$\|f_y\| \leq \|y\|.$$

On the other hand, we have;

$$\|f_y\| \geq \frac{|f_y(y)|}{\|y\|} = \frac{\|y\|^2}{\|y\|} = \|y\|$$

and then $\|f_y\| = \|y\|$.

∎

The following theorem due to I. Roşca [**14**] establishes a natural connection between the normalised duality mapping and the semi-inner products in the sense of Lumer-Giles.

THEOREM 10. *Let $(X, \|\cdot\|)$ be a normed space. Then every L.$-$G.$-$s.i.p. which generates the norm $\|\cdot\|$ is of the form*

$$[x, y] = \left\langle \tilde{J}(y), x \right\rangle \quad \text{for all } x, y \text{ in } X,$$

*where $\tilde{J}$ is a section of the normalised duality mapping.*

PROOF. Let $\tilde{J}$ be a section of the normalised duality mapping $J$. Define the functional;

$$[\cdot, \cdot] : X \times X \to \mathbb{K}, \quad [x, y] = \left\langle \tilde{J}(y), x \right\rangle.$$

Then:

$$\begin{aligned}
[\alpha x + \beta y, z] &= \left\langle \tilde{J}z, \alpha x + \beta y \right\rangle = \alpha \left\langle \tilde{J}z, x \right\rangle + \beta \left\langle \tilde{J}z, y \right\rangle \\
&= \alpha [x, z] + \beta [y, z]
\end{aligned}$$

for every $\alpha$, $\beta \in \mathbb{K}$ and $x, y, z \in X$. We also have:

$$[x, \alpha y] = \left\langle \tilde{J}(\alpha y), x \right\rangle = \left\langle \bar{\alpha} \tilde{J}(y), x \right\rangle = \bar{\alpha} \left\langle \tilde{J}(y), x \right\rangle = \bar{\alpha}[x, y]$$

for all $x, y \in X$ and a scalar $\alpha$ in $\mathbb{K}$.

Now, let us observe that one has

$$[x, x] = \left\langle \tilde{J}x, x \right\rangle = \left\| \tilde{J}x \right\| \|x\| = \|x\|^2 \geq 0 \quad \text{for all } x \in X$$

and

$$[x, x] = 0, \text{ i.e, } \|x\| = 0 \text{ implies } x = 0.$$

Finally, by the properties of bounded linear functionals, we have:

$$\begin{aligned} |[x, y]|^2 &= \left| \left\langle \tilde{J}(y), x \right\rangle \right|^2 \leq \left\| \tilde{J}(y) \right\|^2 \|x\|^2 = \|y\|^2 \|x\|^2 \\ &= [x, x][y, y] \end{aligned}$$

for all $x, y \in X$, and then the mapping $[\cdot, \cdot]$ is a $L. - G.-$s.i.p. which generates the norm $\|\cdot\|$ of $X$.

Conversely, let $[\cdot, \cdot]$ be a $L. - G.-$s.i.p. which generates the norm $\|\cdot\|$ of $X$. Define $\tilde{J} : X \to X^*$ such that the functional $\tilde{J}(y)$ $(y \in X)$ is given by:

$$\left\langle \tilde{J}(y), x \right\rangle := [x, y] \quad \text{for all } x \in X.$$

Then

$$\left\langle \tilde{J}x, x \right\rangle = [x, x] = \|x\|^2, \quad x \in X$$

and

$$\left\| \tilde{J}(y) \right\| = \|x\|, \quad y \in X \quad \text{(see Proposition 3).}$$

Consequently,

$$\left\langle \tilde{J}(x), x \right\rangle = \left\| \tilde{J}x \right\| \|x\| \text{ and } \left\| \tilde{J}x \right\| = \|x\|$$

for all $x \in X$, i.e., $\tilde{J}$ is a section of the normalised duality mapping. ∎

## 2. Characterisation of Some Classes of Normed Spaces

We will start with the next proposition which is a natural consequence of Roşca's result.

Proposition 4. *Let $(X, \|\cdot\|)$ be a normed linear space. Then the following statements are equivalent:*

(i) *$X$ is smooth;*

(ii) *There exists a unique $L. - G.-$s.i.p. which generates the norm $\|\cdot\|$.*

PROOF. "(i) $\Longrightarrow$ (ii)". If $X$ is smooth, then $J$ is univocal (see Corollary 1) and there is a unique section of $J$, and then, by Theorem 10, a unique $L.-G.-$s.i.p. which generates the norm $\|\cdot\|$.

"(ii) $\Longrightarrow$ (i)". If there exists a unique $L.-G.-$s.i.p. which generates the norm $\|\cdot\|$, then $J$ is univocal and by the same corollary it follows that $X$ is smooth. ∎

Before we can state a remarkable result due to J.R. Giles [**2**] that contains a classical characterisation of smooth normed spaces, we need the following definition.

DEFINITION 7. *A $L.-G.-$s.i.p. $[\cdot,\cdot]$ defined on the linear space $X$ is said to be continuous* [**2**], *if for every $x, y \in X$ one has the equality:*

$$\lim_{t\to 0} \operatorname{Re}[y, x+ty] = \operatorname{Re}[y,x]. \tag{2.1}$$

Now we can state and give a partially new proof of this established result (compare with [**2**]).

THEOREM 11. *Let $(X, \|\cdot\|)$ be a normed linear space and $[\cdot,\cdot]$ a $L.-G.-$s.i.p. which generates the norm $\|\cdot\|$. Then $[\cdot,\cdot]$ is continuous if and only if the space $X$ is smooth.*

PROOF. Let us suppose that $[\cdot,\cdot]$ is continuous. Using the properties of $L.-G.-$s.i.p.s we can easily obtain (see also [**16**, p. 387]):

$$\frac{\operatorname{Re}[y,x]}{\|x\|} \le \frac{\|x+ty\| - \|x\|}{t} \le \frac{\operatorname{Re}[y, x+ty]}{\|x+ty\|} \tag{2.2}$$

for every $x, y \in X$, $x \neq 0$ and $t > 0$. Passing at limit after $t$, $t \to 0^+$, we have

$$(\nabla_+ \|\cdot\|)(x)\cdot y = \frac{\operatorname{Re}[y,x]}{\|x\|}$$

for all $x, y \in X$, $x \neq 0$.

On the other hand, one has:

$$\begin{aligned}
(\nabla_- \|\cdot\|)(x)\cdot y &= -(\nabla_+ \|\cdot\|)(x)\cdot(-y) \\
&= -\frac{\operatorname{Re}[-y,x]}{\|x\|} = \frac{\operatorname{Re}[y,x]}{\|x\|} = (\nabla_+ \|\cdot\|)(x)\cdot y
\end{aligned}$$

which shows that the norm $\|\cdot\|$ is Gâteaux differentiable, i.e., $X$ is smooth.

Conversely, let us assume that the norm $\|\cdot\|$ is Gâteaux differentiable on $X \setminus \{0\}$. Then, by the inequalities (2.2) we can write:

$$\frac{\|x+ty\| - \|x\|}{t} \le \frac{\operatorname{Re}[y, x+ty]}{\|x+ty\|} \le \frac{\|x+2ty\| - \|x+ty\|}{t}$$

for all $t>0$, i.e.,

$$\begin{aligned}\frac{1}{t}\left(\|x+ty\|-\|x\|\right)\|x+ty\| &\leq \operatorname{Re}[y,x+ty]\\ &\leq \frac{1}{t}\left(\|x+2ty\|-\|x+ty\|\right)\|x+ty\|\end{aligned}$$

for all $t>0$ and $x,y\in X$. Taking $t\to 0^{+}$, we obtain

$$\lim_{t\to 0^{+}}\operatorname{Re}[y,x+ty]=\|x\|\left(\vee_{+}\|\cdot\|\right)(x)\cdot y$$

because a simple computation shows that:

$$\lim_{t\to 0^{+}}\frac{\|x+2ty\|-\|x+ty\|}{t}=\left(\vee_{+}\|\cdot\|\right)(x)\cdot y.$$

On the other hand, we have:

$$\lim_{t\to 0^{+}}\operatorname{Re}[-y,x+t(-y)]=\|x\|\left(\vee_{+}\|\cdot\|\right)(x)\cdot(-y)$$

and then

$$\begin{aligned}\lim_{t\to 0^{+}}\operatorname{Re}[y,x-ty] &= -\|x\|\left(\vee_{+}\|\cdot\|\right)(x)\cdot(-y)\\ &= \|x\|\left(\vee_{-}\|\cdot\|\right)(x)\cdot y\end{aligned}$$

but

$$\lim_{t\to 0^{+}}\operatorname{Re}[y,x-ty]=\lim_{s\to 0^{-}}\operatorname{Re}[y,x+sy]$$

and in conclusion, we derive:

$$\lim_{t\to 0^{+}}\operatorname{Re}[y,x+ty]=\|x\|\left(\vee_{-}\|\cdot\|\right)(x)\cdot y$$

for all $x,y\in X$, $x\neq 0$.

Since the norm $\|\cdot\|$ is Gâteaux differentiable on $X\diagdown\{0\}$, the above considerations yield that

$$\lim_{t\to 0}\operatorname{Re}[y,x+ty]=\|x\|\left(\vee\|\cdot\|\right)(x)\cdot y \tag{2.3}$$

for all $x,y\in X$, $x\neq 0$.

Now, by the inequalities in (2.2), we also have:

$$\|x\|\frac{\left(\|x+sy\|-\|x\|\right)}{s}\leq \operatorname{Re}[y,x]\leq \|x\|\frac{\left(\|x+ty\|-\|x\|\right)}{t}$$

where $s<0$ and $t<0$. Passing at limit after $s\to 0^{-}$, and $t\to 0^{+}$, we deduce:

$$\operatorname{Re}[y,x]=\|x\|\left(\vee\|\cdot\|\right)(x)\cdot y \tag{2.4}$$

for all $x,y\in X$, $x\neq 0$. Taking into account the equalities (2.3) and (2.4), we obtain the continuity of $[\cdot,\cdot]$ in the sense of Definition 7. ∎

Further on, we will state a result due to Nath [**15**] containing a characterisation of strictly convex spaces in terms of semi-inner product in Lumer-Giles' sense.

THEOREM 12. *Let* $(X, \|\cdot\|)$ *be a normed linear space and* $[\cdot,\cdot]$ *a* $L.-G.-s.i.p.$ *which generates its norm. Then the following statements are equivalent:*

(i) $X$ *is strictly convex;*
(ii) *For every* $x, y \in X$, $x, y \neq 0$ *so that* $[x, y] = \|x\| \|y\|$ *there exists a positive number* $\lambda$ *with* $x = \lambda y$.

PROOF. "(i) $\Longrightarrow$ (ii)". Assume that $(X, \|\cdot\|)$ is a strictly convex space and $x, y$ belong to $X$, $x, y \neq 0$ such that $[x, y] = \|x\| \|y\|$. Using Theorem 10, there exists a section of normalised duality mapping so that

$$\left\langle \tilde{J}(y), x \right\rangle = \|x\| \|y\|.$$

From whence we get

$$\left\langle \tilde{J}(y), \frac{x}{\|x\|} \right\rangle = \|y\| = \left\| \tilde{J}(y) \right\| \quad \text{and} \quad \left\langle \tilde{J}(y), \frac{y}{\|y\|} \right\rangle = \|y\| = \left\| \tilde{J}(y) \right\|.$$

Since $X$ is strictly convex, every continuous linear functional achieves its norm on at most one point which means that

$$\frac{x}{\|x\|} = \frac{y}{\|y\|}$$

and putting $\lambda = \frac{\|x\|}{\|y\|}$, we deduce that $x = \lambda y$.

"(ii) $\Longrightarrow$ (i)". Now, we will show that the condition "(ii)" implies the property:

$$(\forall)\, x, y \in X \diagdown \{0\} \quad \text{and} \quad \|x + y\| = \|x\| + \|y\| \Longrightarrow x = \lambda y,$$

with $\lambda > 0$, which is equivalent with the strict convexity of $X$.

If $\|x + y\| = \|x\| + \|y\|$ with $x, y \in X \diagdown \{0\}$, then :

$$\operatorname{Re}[x, x + y] = \|x\| \|x + y\| \quad \text{or} \quad \operatorname{Re}[y, x + y] = \|y\| \|x + y\|. \tag{2.5}$$

Indeed, by Schwartz's inequality, we have

$$\operatorname{Re}[x, x + y] \leq \|x\| \|x + y\| \quad \text{and} \quad \operatorname{Re}[y, x + y] \leq \|y\| \|x + y\|.$$

Let us assume that both inequalities are strict, then, by addition, we get:

$$\operatorname{Re}[x, x + y] + \operatorname{Re}[y, x + y] < (\|x\| + \|y\|) \|x + y\|$$

and since the left membership is $\|x + y\|^2$, we deduce that

$$\|x + y\| < \|x\| + \|y\|$$

which contradicts the initial assumption. Consequently, at least one of the equalities embodied in (2.5) is valid.

Suppose that

$$\operatorname{Re}[x, x+y] = \|x\| \|x+y\|$$

then by (ii) we get

$$x = t(x+y) \quad \text{with} \quad t \neq 1$$

from where results $x = \lambda y$ with $\lambda = \frac{t}{1-t} > 0$.

The theorem is thus proved. ∎

## 3. Other Properties of $L.-G.$–s.i.p.s

Firstly, we will give the following slight improvement of Giles' theorem.

THEOREM 13. *Let $(X, \|\cdot\|)$ be a normed space and $[\cdot,\cdot]$ a $L.-G.$–s.i.p. which generates its norm. Then the following statements are equivalent:*

(i) *$X$ is smooth;*
(ii) *The following limit exists:*

$$\lim_{t\to 0} \operatorname{Re}[y, x+ty]$$

*for all $x, y \in X$.*

PROOF. We need only prove the implication "(ii) $\Longrightarrow$ (i)". In the proof of Theorem 12, we have pointed out that:

$$\lim_{t\to 0^+} \operatorname{Re}[y, x+ty] = \|x\| \left(\triangledown_+ \|\cdot\|\right)(x) \cdot y$$

and

$$\lim_{t\to 0^-} \operatorname{Re}[y, x+ty] = \|x\| \left(\triangledown_- \|\cdot\|\right)(x) \cdot y$$

for all $x, y \in X$, $x \neq 0$, where $X$ is an arbitrary normed linear space.

Now, if the limit $\lim\limits_{t\to 0} \operatorname{Re}[y, x+ty]$ exists it follows that

$$\left(\triangledown_- \|\cdot\|\right)(x) \cdot y = \left(\triangledown_+ \|\cdot\|\right)(x) \cdot y$$

for all $x, y \in X$, $x \neq 0$, which shows that the space $X$ is smooth. ∎

Another result of this type is embodied in the following theorem.

THEOREM 14. *Let $(X, \|\cdot\|)$ be a normed space and $[\cdot,\cdot]$ a $L.-G.$–s.i.p. which generates the norm $\|\cdot\|$. Then the following statements are equivalent:*

(i) *$X$ is smooth;*

(ii) *The following limit exists:*

$$\lim_{t\to 0}\frac{\operatorname{Re}[x,x+ty]-\|x\|^2}{t} \tag{2.6}$$

*for all $x,y\in X$.*

*Moreover, if (i) or (ii) hold, then one has the equality*

$$\lim_{t\to 0}\frac{\operatorname{Re}[x,x+ty]-\|x\|^2}{t}=\operatorname{Re}[y,x] \tag{2.7}$$

*for all $x,y\in X$.*

PROOF. Firstly, let us observe that

$$\frac{\|x+ty\|^2-\|x\|^2}{t}=\frac{\operatorname{Re}[x,x+ty]-\|x\|^2}{t}+\operatorname{Re}[y,x+ty] \tag{2.8}$$

for every $x,y\in X$ and $t\in\mathbb{R}\diagdown\{0\}$.

On the other hand, in every normed space one has:

$$\lim_{t\to 0^+}\operatorname{Re}[y,x+ty]=\|x\|\left(\nabla_+\|\cdot\|\right)(x)\cdot y$$

and

$$\lim_{t\to 0^-}\operatorname{Re}[y,x+ty]=\|x\|\left(\nabla_-\|\cdot\|\right)(x)\cdot y$$

for all $x,y\in X$ (see the proof of Theorem 11).

We also note that:

$$\lim_{t\to 0^+}\frac{\|x+ty\|^2-\|x\|^2}{t}=2\|x\|\left(\nabla_+\|\cdot\|\right)(x)\cdot y$$

and

$$\lim_{t\to 0^-}\frac{\|x+ty\|^2-\|x\|^2}{t}=2\|x\|\left(\nabla_-\|\cdot\|\right)(x)\cdot y$$

for all $x,y\in X$ and $x\neq 0$.

"(i) $\Longrightarrow$ (ii)". If $X$ is smooth, then $\lim_{t\to 0}\frac{\|x+ty\|^2-\|x\|^2}{t}$ and $\lim_{t\to 0}\operatorname{Re}[y,x+ty]$ exist, which implies that, by (2.8), the limit (2.6) also exists for all $x,y\in X$.

"(ii) $\Longrightarrow$ (i)". By the equality (2.8) and by the above remarks, we deduce that

$$\lim_{t\to 0^+}\frac{\operatorname{Re}[x,x+ty]-\|x\|^2}{t}=\|x\|\left(\nabla_+\|\cdot\|\right)(x)\cdot y$$

and

$$\lim_{t\to 0^-}\frac{\operatorname{Re}[x,x+ty]-\|x\|^2}{t}=\|x\|\left(\nabla_-\|\cdot\|\right)(x)\cdot y$$

for all $x, y \in X$, $x \neq 0$. Since the limit (2.6) exists, it follows that $(\vee_{+}\|\cdot\|)(x)\cdot y = (\vee_{-}\|\cdot\|)(x)\cdot y$ for all $x, y \in X$, $x \neq 0$, which shows that $X$ is smooth.

The proof of the relation (2.7) follows by that identity (2.8) and by the fact that in smooth normed spaces the $L.-G.-$s.i.p. is continuous.

We will omit the details. ∎

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CHAPTER 3

# The Superior and Inferior Semi-Inner Products

## 1. Definition and Some Fundamental Properties

Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. The mapping $f : X \to \mathbb{R}$, $f(x) = \frac{1}{2}\|x\|^2$ is obviously convex and then there exists the following limits:

$$\begin{aligned}(x,y)_i &= \lim_{t\to 0^-}\frac{\|y+tx\|^2-\|y\|^2}{2t};\\(x,y)_s &= \lim_{t\to 0^+}\frac{\|y+tx\|^2-\|y\|^2}{2t};\end{aligned}$$

for every two elements in $X$. The mapping $(\cdot,\cdot)_s$ $((\cdot,\cdot)_i)$ will be called *the superior semi-inner product (the inferior semi-inner product)* associated to the norm $\|\cdot\|$. These mapping were considered by P.M. Miličić [**1**] – [**3**], R.A. Tapia [**4**], N. Pavel [**5**], G. Dincă [**6**] and others who pointed out their main properties.

We will start with the following properties which may easily be derived from the definitions of $(\cdot,\cdot)_i$ and $(\cdot,\cdot)_s$.

Proposition 5. *Let $(X, \|\cdot\|)$ be a normed linear space. Then the following statements are true.*

(i) $(x,x)_p = \|x\|^2$ *for all* $x \in X$;
(ii) $(ix,x)_p = (x,ix)_p = 0$ *for all* $x \in X$;
(iii) $(\lambda x,y)_p = \lambda (x,y)_p$ *for all nonnegative scalar* $\lambda$ *and* $x,y \in X$;
(iv) $(x,\lambda y)_p = \lambda (x,y)_p$ *for all nonnegative scalar* $\lambda$ *and* $x,y \in X$;
(v) $(\lambda x,y)_p = \lambda (x,y)_q$ *if* $\lambda < 0$ *and* $x,y \in X$;
(vi) $(x,\lambda y)_p = \lambda (x,y)_q$ *if* $\lambda < 0$ *and* $x,y \in X$;
(vii) $(ix,y)_p = -(x,iy)_p = 0$ *for all* $x \in X$;

*where* $p,q \in \{s,i\}$ *and* $p \neq q$.

Proof. The proof is as follows.

(i) One has:

$$(x,x)_p = \lim_{t\to\pm 0}\frac{\|x+tx\|^2-\|x\|^2}{2t}$$
$$= \|x\|^2 \lim_{t\to\pm 0}\frac{|1+t|-1}{t} = \|x\|^2,$$

for all $x \in X$, which proves the assertion.

(ii) It is clear that;

$$(ix,x)_p = (x,ix)_p = \lim_{t\to\pm 0}\frac{\|ix+tx\|^2-\|ix\|^2}{2t}$$
$$= \|x\|^2 \lim_{t\to\pm 0}\frac{|i+t|^2-1}{t}$$
$$= \|x\|^2 \lim_{t\to\pm 0}\frac{\sqrt{1+t^2}-1}{2t} = 0,$$

for all $x \in X$.

(iii), (v) For every $x \in X$, we have:

$$(\lambda x,y)_p = \lim_{t\to\pm 0}\frac{\|y+\lambda tx\|^2-\|y\|^2}{2t}.$$

Denoting $u=\lambda t$, we have

$$(\lambda x,y)_p = \begin{cases} \lambda \lim\limits_{u\to\pm 0}\frac{\|y+ux\|^2-\|y\|^2}{2u}, & \lambda \geq 0 \\ \\ \lambda \lim\limits_{u\to\mp 0}\frac{\|y+ux\|^2-\|y\|^2}{2u}, & \lambda < 0 \end{cases}$$
$$= \begin{cases} \lambda (x,y)_p, & \lambda \geq 0 \\ \\ \lambda (x,y)_p, & \lambda < 0 \end{cases}.$$

The proofs of the statements (iv) and (vi) go likewise and we omit the details.

(vii) We have:

$$(ix,y)_p = \lim_{t\to\pm 0}\frac{\|y+itx\|^2-\|y\|^2}{2t} = \lim_{t\to\pm 0}\frac{\|iy-tx\|^2-\|iy\|^2}{2t}$$
$$= (x,-iy)_p = -(x,iy)_q$$

for all $x,y \in X$, and the assertion is proven.

∎

Corollary 3. *With the above assumptions, we have*

$$(\alpha x,\beta y)_p = \alpha\beta\,(x,y)_p,$$

*for all* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha\beta \geq 0$ *and* $x, y$ *are two elements in* $X$*.*

COROLLARY 4. *We also have:*

$$(-x, y)_p = (x, -y)_p = -(x, y)_q,$$

*where* $x, y \in X$*;* $p, q \in \{s, i\}$ *and* $p \neq q$*.*

The next proposition contains some other properties of $(\cdot, \cdot)_i$ and $(\cdot, \cdot)_s$.

PROPOSITION 6. *Let* $(X, \|\cdot\|)$ *be a normed space. Then one has:*

(i) *The following inequality is valid*

$$\frac{\|x + ty\|^2 - \|x\|^2}{2t} \geq (y, x)_s \geq (y, x)_i \geq \frac{\|x + sy\|^2 - \|x\|^2}{2s}$$

*for all* $x, y \in X$ *and* $t > 0$*,* $s < 0$*;*

(ii) $\left|(x, y)_p\right| \leq \|x\| \|y\|$ *for all* $x, y \in X$*;*

(iii) *The mapping* $(\cdot, \cdot)_s$ $((\cdot, \cdot)_i)$ *is sub(super)-additive in the first variable, i.e., for* $x_1$*,* $x_2$ *and* $y$ *in* $X$*:*

$$(x_1 + x_2, y)_{s(i)} \leq (\geq)\, (x_1, y)_{s(i)} + (x_2, y)_{s(i)} \tag{3.1}$$

*holds.*

PROOF. The proof is as follows.

(i) Let us consider the mapping $g : [0, \infty) \to \mathbb{R}$, $g(t) := \frac{1}{2}\|x + ty\|^2$ for two $x, y$ fixed in $X$. It is clear that $g$ is convex on $[0, \infty)$ and then:

$$\frac{g(t) - g(0)}{t - 0} \geq g'_+(0), \text{ for } t > 0$$

which means that

$$\frac{\|x + ty\|^2 - \|x\|^2}{2t} \geq \lim_{t \to 0+} \frac{\|x + ty\|^2 - \|x\|^2}{2t} = (y, x)_s.$$

The second inequality follows by the fact that

$$g'_+(0) \geq g'_-(0),$$

if $g$ is any convex mapping of a real variable.

The last fact is also obvious.

(ii) Let $x, y \in X$. Then

$$\begin{aligned}
\left|(x,y)_p\right| &= \left|\lim_{t\to\pm 0}\frac{\|y+tx\|^2-\|y\|^2}{2t}\right| \\
&= \left|\lim_{t\to\pm 0}\frac{\|y+tx\|+\|y\|}{2t}\right|\left|\lim_{t\to\pm 0}\frac{\|y+tx\|-\|y\|}{2t}\right| \\
&\le \|y\|\lim_{t\to\pm 0}\frac{\|y+tx-y\|}{|t|} = \|y\|\,\|x\|,
\end{aligned}$$

and the statement is proved.

(iii) By the usual properties of the norm, one has:

$$\begin{aligned}
&(x_1+x_2,y)_{s(i)} \\
&= \frac{1}{2}\|2y\|\lim_{t\to\pm 0}\frac{\|y+t(x_1+x_2)\|-\|2y\|}{t} \\
&= \|y\|\lim_{t\to\pm 0}\frac{\|y+tx_1+y+tx_2\|-2\|y\|}{t} \\
&\le (\ge)\ \|y\|\lim_{t\to\pm 0}\frac{\|y+tx_1\|+\|y+tx_2\|-2\|y\|}{t} \\
&= \|y\|\lim_{t\to\pm 0}\frac{\|y+tx_1\|-\|y\|}{t}+\|y\|\lim_{t\to\pm 0}\frac{\|y+tx_2\|-\|y\|}{t} \\
&= (x_1,y)_{s(i)}+(x_2,y)_{s(i)}
\end{aligned}$$

for every $x_1$, $x_2$ and $y$ in $X$.

∎

## 2. The Connection Between $(\cdot,\cdot)_{s(i)}$ and the Duality Mapping

In this section we will point out the natural connection that exists between the semi-inner products $(\cdot,\cdot)_s$ and $(\cdot,\cdot)_i$ and the normalised duality mapping $J$ in every normed linear space $X$.

The following lemma is important in itself as well (see for example [**5**]).

LEMMA 1. *Let $x, y$ be two given elements in $X$. Then there exists the real functionals $w_1$, $w_2 \in J(x)$ such that:*

$$(y,x)_s = w_1(y) \quad \textit{and} \quad (y,x)_i = w_2(y).$$

PROOF. Let $\Psi = \{\alpha x+\beta y \mid \alpha,\beta\in\mathbb{R}\} \subset X$. Consider the linear functional:

$$f:\Psi\to\mathbb{R},\ \ f(\alpha x+\beta y) = \alpha\|x\|^2+\beta(y,x)_s.$$

Then one has $f(x) = \|x\|^2$. We will show that

$$f(\alpha x + \beta y) \leq \|\alpha x + \beta y\| \|x\| \tag{3.2}$$

for every $\alpha, \beta \in \mathbb{R}$.

Let us denote

$$\lambda_+ := \lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t} \quad \text{and} \quad \lambda_- := \lim_{t \to 0^-} \frac{\|x + ty\| - \|x\|}{t}.$$

Then it is clear that

$$(y, x)_s = \|x\| \lambda_+ \quad \text{and} \quad (y, x)_i = \|x\| \lambda_-.$$

To show the inequality (3.2), it is sufficient to prove the inequality

$$\alpha \|x\| + \beta \lambda_+ \leq \|\alpha x + \beta y\| \quad \text{for all } \alpha, \beta \in \mathbb{R}. \tag{3.3}$$

On the other hand, since the mapping $\mathbb{R} \ni t \rightarrow \|x + ty\|$ is convex, we have (as above):

$$t\lambda_+ \leq \|x + ty\| - \|x\|, \ t \in \mathbb{R}$$

which is equivalent with;

$$\|x\| + t\lambda_+ \leq \|x + ty\| \quad \text{for all } t \in \mathbb{R}.$$

If $\alpha > 0$, then by the above inequality, we get:

$$\alpha \|x\| + \beta \lambda_+ = \alpha \left[ \|x\| + \frac{\beta}{\alpha} \lambda_+ \right] \leq \|\alpha x + \beta y\|.$$

If $\alpha < 0$, then we also have:

$$\begin{aligned} \alpha \|x\| + \beta \lambda_+ &= (-\alpha) \left[ -\|x\| + \frac{\beta}{-\alpha} \lambda_+ \right] \\ &\leq (-\alpha) \left[ -2\|x\| + \left\| x - \frac{\beta}{\alpha} y \right\| \right] \\ &\leq (-\alpha) \left\| -x - \frac{\beta}{\alpha} y \right\| = \|\alpha x + \beta y\|. \end{aligned}$$

If $\alpha = 0$, we get:

$$\beta \lambda_+ \leq \|\beta y\|,$$

which results by $\beta \lambda_+ \leq \|x + \beta y\| - \|x\|$.

Consequently, by (3.2) we can conclude that $f$ is bounded and

$$\|f\| \leq \|x\|.$$

Now, by the Hahn-Banach theorem, there exists a functional $w_1 : X \rightarrow \mathbb{R}$ such that $w_1(x) = f(x) = \|x\|^2$ and $\|w_1\| = \|f\| \leq \|x\|$.

On the other hand,

$$\|w_1\| \geq \frac{|w_1(x)|}{\|x\|} = \frac{|f(x)|}{\|x\|} = \|x\|$$

and then

$$\|w_1\| = \|x\|$$

which shows that $w_1 \in J(x)$.

Since $\langle y, x \rangle_s = f(y) = w_1(y)$, the first part of the lemma is proven.

The second part goes likewise and we omit the details. ∎

Now, we can state and prove the following main result containing a representation of the norm derivatives $(\cdot, \cdot)_{s(i)}$ in terms of duality mappings (see for example [**5**]).

THEOREM 15. *Let $(X, \|\cdot\|)$ be a normed space. Then:*

(i) $(y, x)_s = \sup \{w(y), \ w \in J(x), \ w \text{ is a real functional}\}$;

(ii) $(y, x)_i = \inf \{w(y), \ w \in J(x), \ w \text{ is a real functional}\}$;

*and $x, y$ are vectors in $X$.*

PROOF. The proof is as follows.

(i) Let $y, x \in X$ and $w \in J(x)$, $w$ is a real functional on $X$. Then

$$\frac{1}{2}\|x + ty\|^2 - \frac{1}{2}\|x\|^2 \geq \|x + ty\| (\|x\| - w(x)) \geq tw(y)$$

because the first inequality is equivalent with:

$$\|x + ty\|^2 + 2w(x) \geq \|x\|^2 + \|x + ty\| \|x\|$$

i.e.,

$$\|x + ty\|^2 + \|x\|^2 \geq 2\|x + ty\| \|x\|, \ \text{ for all } t \in \mathbb{R}$$

and the second inequality follows by the fact that:

$$\|x + ty\| \|x\| \geq w(x + ty), \ \text{ for all } t \in \mathbb{R}.$$

Now, for $t > 0$, we deduce

$$\frac{\|x + ty\|^2 - \|x\|^2}{2t} \geq w(y)$$

which gives:

$$(y, x)_s \geq w(y), \ w \in J(x).$$

On the other hand, from the above lemma, there exists a real functional $w_1 \in J(x)$ such that $(y, x)_s = w_1(y)$, which proves the statement.

(ii) The proof is similar to that in the above statements and we will omit the details.

∎

Now, we will give two very important properties of semi-inner products $(\cdot,\cdot)_s$ and $(\cdot,\cdot)_i$ which may be proved with the help of the above results (see for example [**5**]).

THEOREM 16. *Let* $(X, \|\cdot\|)$ *be a normed linear space. Then the following equality:*

$$(\alpha x + y, x)_p = \alpha \|x\|^2 + (y, x)_p\,, \quad p \in \{s, i\}\,, \tag{3.4}$$

*holds, for all* $x, y$ *in* $X$ *and* $\alpha$ *a real number .*

PROOF. Let $\alpha \in \mathbb{R}$ and $x, y \in X$. Then by Theorem 15, we can write

$$\begin{aligned}(\alpha x + y, x)_s &= \sup\{w(\alpha x + y)\,|w \in J(x)\,,\, w \text{ is real}\}\\ &= \sup\{\alpha w(x) + w(y)\,|w \in J(x)\,,\, w \text{ is real}\}\\ &= \alpha\|x\|^2 + \sup\{w(y)\,|w \in J(x)\,,\, w \text{ is real}\}\\ &= \alpha\|x\|^2 + (y, x)_s\end{aligned}$$

which shows the equality (3.4) for $p = s$.

The case $p = i$ goes likewise and we omit the details. ∎

The second property is embodied in the following theorem.

THEOREM 17. *Let* $x, y, z$ *belong to* $X$. *Then one has the inequality:*

$$\left|(y + z, x)_p - (z, x)_p\right| \le \|y\|\,\|x\|\,,$$

*where* $p = s$ *or* $p = i$.

PROOF. We will prove only in the case $p = s$.

Let $w \in J(x)$ be a real functional. Then

$$w(y + z) - w(z) = w(y) \le \|x\|\,\|y\|\,,$$

and then

$$w(y + z) \le \|x\|\,\|y\| + w(z)\,.$$

Taking the supremum after $w \in J(x)$, $w$ is real, we deduce

$$(y + z, x)_s \le \|x\|\,\|y\| + (z, x)_s\,, \quad (\forall)\, x, y, z \in X.$$

Now, taking the infimum after $w \in J(x)$, $w$ is real, we also have:

$$(y + z, x)_i \le \|x\|\,\|y\| + (z, x)_i$$

which is equivalent with

$$(y + z, x)_s \ge -\|x\|\,\|y\| + (z, x)_s\,, \quad (\forall)\, x, y, z \in X.$$

The statement is thus proved. ∎

COROLLARY 5. *The mapping* $(\cdot, x)_p$ *is continuous on* $(X, \|\cdot\|)$ *for every* $x \in X$, $p \in \{s, i\}$.

PROOF. Let $y_n \to y$ in $(X, \|\cdot\|)$. Then one has, by the above theorem, that

$$\left|(y_n, x)_p - (y, x)_p\right| \leq \|y_n - y\| \, \|x\| \to 0 \text{ as } x \to \infty,$$

which shows the assertion. ∎

Now we give a new theorem of representation of semi-inner products $(\cdot, \cdot)_s$ and $(\cdot, \cdot)_i$ in terms of the normalised duality mapping.

THEOREM 18. *Let* $(X, \|\cdot\|)$ *be a real or complex normed space and* $\tilde{J}$ *a section of the normalised duality mapping* $J$. *Then we have the representation:*

$$(y, x)_s = \lim_{t \to 0+} \operatorname{Re} \left\langle \tilde{J}(x + ty), y \right\rangle \tag{3.5}$$

*and*

$$(y, x)_i = \lim_{t \to 0-} \operatorname{Re} \left\langle \tilde{J}(x + ty), y \right\rangle \tag{3.6}$$

*for all* $x, y \in X$.

PROOF. Let $\tilde{J}$ be a section of the normalised dual mapping $J$. Then for all $x \in X \diagdown \{0\}$ and $t \in \mathbb{R}$ one has

$$\begin{aligned} &\|x + ty\| - \|x\| \\ &= \frac{\|x + ty\| \, \|x\| - \|x\|^2}{\|x\|} \geq \frac{\left\langle \tilde{J}x, x + ty \right\rangle - \|x\|^2}{\|x\|} \\ &= \frac{\left\langle \tilde{J}x, x \right\rangle + t \operatorname{Re} \left\langle \tilde{J}x, y \right\rangle - \|x\|^2}{\|x\|} = t \cdot \frac{\operatorname{Re} \left\langle \tilde{J}x, y \right\rangle}{\|x\|} \end{aligned}$$

from where results:

$$\frac{\|x\| \left(\|x + ty\| - \|x\|\right)}{t} \geq \operatorname{Re} \left\langle \tilde{J}x, y \right\rangle, \tag{3.7}$$

for all $x, y \in X$ and $t > 0$.

On the other hand, for $t \neq 0$ and $x + ty \neq 0$, we get

$$\begin{aligned}
&\frac{\|x+ty\|-\|x\|}{t} \\
&= \frac{\|x+ty\|^2-\|x\|\,\|x+ty\|}{t\,\|x+ty\|} = \frac{\left\langle \tilde{J}\,(x+ty)\,,x+ty\right\rangle - \|x\|\,\|x+ty\|}{t\,\|x+ty\|} \\
&= \frac{\operatorname{Re}\left\langle \tilde{J}\,(x+ty)\,,x\right\rangle + t\operatorname{Re}\left\langle \tilde{J}\,(x+ty)\,,y\right\rangle - \|x\|\,\|x+ty\|}{t\,\|x+ty\|} \\
&\leq \frac{\operatorname{Re}\left\langle \tilde{J}\,(x+ty)\,,y\right\rangle}{\|x+ty\|}
\end{aligned}$$

because

$$\operatorname{Re}\left\langle \tilde{J}\,(x+ty)\,,x\right\rangle \leq \|x\|\,\|x+ty\|\,.$$

Consequently, we obtain the inequality:

$$\|x+ty\|\,\frac{(\|x+ty\|-\|x\|)}{t} \leq \operatorname{Re}\left\langle \tilde{J}\,(x+ty)\,,y\right\rangle \tag{3.8}$$

for all $t > 0$ and $x, y \in X$.

If we replace in inequality (3.7) the element $x$ with $x+ty$, we deduce:

$$\|x+ty\|\,\frac{(\|x+2ty\|-\|x+ty\|)}{t} \geq \operatorname{Re}\left\langle \tilde{J}\,(x+ty)\,,y\right\rangle \tag{3.9}$$

for all $t > 0$ and $x, y \in X$.

Now, we observe that the relations (3.8) and (3.9) give:

$$\begin{aligned}
&\|x+ty\|\,\frac{(\|x+ty\|-\|x\|)}{t} \\
&\leq \operatorname{Re}\left\langle \tilde{J}\,(x+ty)\,,y\right\rangle \leq \|x+ty\|\,\frac{(\|x+2ty\|-\|x+ty\|)}{t}
\end{aligned} \tag{3.10}$$

for all $t > 0$ and $x, y \in X$.

Since

$$\lim_{t\to 0+}\|x+ty\|\,\frac{(\|x+ty\|-\|x\|)}{t} = \|x\|\lim_{t\to 0+}\frac{\|x+ty\|-\|x\|}{t} = (y,x)_s$$

and

$$\begin{aligned}
&\lim_{t\to 0+}\|x+ty\|\frac{(\|x+2ty\|-\|x+ty\|)}{t}\\
&= \|x\|\left[2\lim_{t\to 0+}\frac{\|x+2ty\|-\|x\|}{t}-\lim_{t\to 0+}\frac{\|x+ty\|-\|x\|}{t}\right]\\
&= \|x\|\lim_{t\to 0+}\frac{\|x+ty\|-\|x\|}{t}=(y,x)_s
\end{aligned}$$

for $x,y\in X$, then, if we pass at limit after $t$, $t\to 0^+$ in the inequality (3.10), we can conclude that the limit

$$\lim_{t\to 0+}\operatorname{Re}\left\langle \tilde{J}(x+ty),y\right\rangle$$

exists for all $x,y\in X$. Moreover,

$$\lim_{t\to 0+}\operatorname{Re}\left\langle \tilde{J}(x+ty),y\right\rangle=(y,x)_s \text{ for all } x,y\in X.$$

On the other hand, by (3.5), we get

$$\begin{aligned}
&(y,x)_i\\
&= (-y,x)_s=-\lim_{t\to 0+}\operatorname{Re}\left\langle \tilde{J}(x+t(-y)),-y\right\rangle\\
&= \lim_{t\to 0-}\operatorname{Re}\left\langle \tilde{J}(x+(-t)y),y\right\rangle=\lim_{t\to 0-}\operatorname{Re}\left\langle \tilde{J}(x+ty),y\right\rangle
\end{aligned}$$

for all $x,y\in X$, and the theorem is proved. ∎

The following result also holds.

Theorem 19. *Let $(X,\|\cdot\|)$ be a normed space and $\tilde{J}$ a section of the normalised duality mapping. Then we have the representation:*

$$(y,x)_s=\lim_{t\to 0+}\operatorname{Re}\left\langle \frac{\tilde{J}(x+ty)-\tilde{J}x}{t},x\right\rangle \tag{3.11}$$

*and*

$$(y,x)_i=\lim_{t\to 0-}\operatorname{Re}\left\langle \frac{\tilde{J}(x+ty)-\tilde{J}x}{t},x\right\rangle \tag{3.12}$$

*for all $x,y\in X$.*

PROOF. For every $x, y \in X$ and $t \in \mathbb{R}$, $t \neq 0$, we have the equality

$$\begin{aligned}
&\frac{\|x+ty\|^2 - \|x\|^2}{t} \\
&= \frac{\left\langle \tilde{J}(x+ty), x+ty \right\rangle - \left\langle \tilde{J}x, x \right\rangle}{t} \\
&= \frac{\operatorname{Re}\left\langle \tilde{J}(x+ty), x \right\rangle + t \operatorname{Re}\left\langle \tilde{J}(x+ty), y \right\rangle - \left\langle \tilde{J}x, x \right\rangle}{t} \\
&= \operatorname{Re}\left\langle \frac{\tilde{J}(x+ty) - \tilde{J}x}{t}, x \right\rangle + \operatorname{Re}\left\langle \tilde{J}(x+ty), y \right\rangle .
\end{aligned}$$

Since

$$\lim_{t\to 0^+} \frac{\|x+ty\|^2 - \|x\|^2}{t} = 2(y,x)_s$$

and

$$\lim_{t\to 0^+} \operatorname{Re}\left\langle \tilde{J}(x+ty), y \right\rangle = (y,x)_s$$

hence from the above equality we deduce that the limit

$$\lim_{t\to 0^+} \operatorname{Re}\left\langle \frac{\tilde{J}(x+ty) - \tilde{J}x}{t}, x \right\rangle$$

exists and is equal with $(y,x)_s$ for all $x, y \in X$.

The relation (3.11) is proven,

The argument of (3.12) goes likewise and we omit the details. ∎

## 3. Other Properties of $(\cdot,\cdot)_s$ and $(\cdot,\cdot)_i$

The following result contains a connection between the norm derivatives $(\cdot,\cdot)_s$ and $(\cdot,\cdot)_i$ and the semi-inner product in the sense of Lumer-Giles (see for example [**6**]).

THEOREM 20. *Let $(X, \|\cdot\|)$ be a real normed space and $\mathcal{S}_p$ the set of all $L.-G.-s.i.p$ on $X$ which generates the norm $\|\cdot\|$. Then one has the representation:*

$$(y,x)_s = \sup\{[y,x] \mid [\cdot,\cdot] \in \mathcal{S}_p\} \tag{3.13}$$

*and*

$$(y,x)_i = \inf\{[y,x] \mid [\cdot,\cdot] \in \mathcal{S}_p\} \tag{3.14}$$

*for all $x, y \in X$.*

Proof. Let us consider the mapping $f_x : X \to \mathbb{R}$, $f_x(y) = [y, x]$ for all $y$ in $X$. Then $f_x \in X^*$, $\|f_x\| = \|x\|$ and $f_x(x) = \|x\| \|f_x\|$ which shows that $f_x \in J(x)$. Consequently,

$$\sup\{[y,x] \mid [\cdot,\cdot] \in \mathcal{S}_p\} \leq \sup\{w(y) \mid w \in J(x)\} = (y,x)_s$$

(see Theorem 15).

Now, by Lemma 1, there exists a $w_1 \in J(x)$ such that

$$(y,x)_s = w_1(y).$$

Let us consider a section $\tilde{J}$ of $J$ such that $\tilde{J}(x) = w_1$ and define the mapping:

$$[y,z] := \left\langle \tilde{J}(z), y \right\rangle, \quad z, y \in X.$$

Then $[\cdot,\cdot]$ is a $L.-G.-$s.i.p which generates the norm $\|\cdot\|$ and

$$[y,x] = \left\langle \tilde{J}(x), y \right\rangle = w_1(y) = (y,x)_s.$$

Consequently, there exists a $L.-G.-$s.i.p $[\cdot,\cdot]$ in $\mathcal{S}_p$ such that

$$[y,x] = (y,x)_s$$

which shows that the identity (3.13) holds.

To prove the relation (3.14), we observe that

$$\begin{aligned} (y,x)_i &= (-y,x)_s = -\sup\{[-y,x] \mid [\cdot,\cdot] \in \mathcal{S}_p\} \\ &= -\sup\{-[y,x] \mid [\cdot,\cdot] \in \mathcal{S}_p\} \\ &= \inf\{[y,x] \mid [\cdot,\cdot] \in \mathcal{S}_p\}, \end{aligned}$$

which ends the proof. ∎

Corollary 6. *Let $(X, \|\cdot\|)$ be a real normed space and $[\cdot,\cdot]$ a $L.-G.-$s.i.p. which generates the norm $\|\cdot\|$. Then*

$$(y,x)_i \leq [y,x] \leq (y,x)_s,$$

*for all $x, y$ in $X$.*

Another representation of $(\cdot,\cdot)_{s(i)}$ in terms of $L.-G.-$s.i.p. is the following.

Theorem 21. *Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K}$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, and $[\cdot,\cdot]$ a $L.-G.-$s.i.p. which generates its norm. Then*

$$(y,x)_s = \lim_{t \to 0+} \operatorname{Re}[y, x+ty]$$

*and*

$$(y,x)_i = \lim_{t \to 0-} \operatorname{Re}[y, x+ty]$$

*for all $x, y$ in $X$.*

The proof is derived by applying Theorem 18 via Roşca's result of representation and we will omit the details.

Finally, we note that the next result of this type also holds.

THEOREM 22. *For the above assumption, we have:*

$$(y,x)_s = \lim_{t\to 0+} \frac{\operatorname{Re}[y, x+ty] - \|x\|^2}{t}$$

*and*

$$(y,x)_i = \lim_{t\to 0-} \frac{\operatorname{Re}[y, x+ty] - \|x\|^2}{t}$$

*for all $x, y$ in $X$.*

Now we will give a characterisation of smooth normed spaces in terms of the superior (inferior) semi-inner product.

THEOREM 23. *Let $(X, \|\cdot\|)$ be a normed space. Then the following statements are equivalent.*

(i) *The norm is Gâteaux differentiable on $X \diagdown \{0\}$, i.e., the space is smooth;*
(ii) *The semi-inner product $(\cdot,\cdot)_p$ is homogeneous in the second argument;*
(iii) *The semi-inner product $(\cdot,\cdot)_p$ is homogeneous in the first argument;*
(iv) *The semi-inner product $(\cdot,\cdot)_p$ is linear in the first argument,*

*where $p = s$ or $p = i$.*

PROOF. We only prove in the case $p = s$.

"(i) $\Longrightarrow$ (ii)". Since $(\cdot,\cdot)_s$ is positive-homogeneous in the second argument, it is sufficient to show that:

$$(x, -y)_s = -(x,y)_s$$

for all $x, y$ in $X$.

The Gâteaux differentiability of the norm implies that

$$\begin{aligned}(x,-y)_s &= \lim_{t\to 0} \frac{\|(-y)+tx\|^2 - \|-y\|^2}{2t} \\ &= \lim_{t\to 0} \frac{\|y-tx\|^2 - \|y\|^2}{2t} \\ &= \lim_{t\to 0} \frac{\|y+(-t)\,x\|^2 - \|y\|^2}{2t} \\ &= -\lim_{s\to 0} \frac{\|y+sx\|^2 - \|y\|^2}{2s} = -(x,y)_s\end{aligned}$$

and the implication is proved.

"(ii) $\Longrightarrow$ (iii)". We will show that

$$(-x, y)_s = -(x, y)_s \quad \text{for all } x, y \in X.$$

Indeed, since

$$(-x, y)_s = (x, -y)_s = -(x, y)_s$$

for all $x, y \in X$, and the proof of the statement is completed.

"(iii) $\Longrightarrow$ (iv)". Since $(\cdot, \cdot)_s$ is subadditive (see Proposition 6 (iii)) and homogeneous, it is linear in the first argument.

"(iv) $\Longrightarrow$ (i)". Let $x, y \in X$ with $y \neq 0$. Then

$$\begin{aligned}
\lim_{t \to 0+} \frac{\|y + tx\| - \|y\|}{t} = \frac{(x, y)_s}{\|y\|} &= -\frac{(-x, y)_s}{\|y\|} \\
&= -\lim_{t \to 0+} \frac{\|y + (-t)\, x\| - \|y\|}{t} \\
&= \lim_{s \to 0-} \frac{\|y + sx\| - \|y\|}{s}
\end{aligned}$$

i.e., the norm $\|\cdot\|$ is Gâteaux differentiable on $X \diagdown \{0\}$, and the theorem is thus proved. ∎

Finally, we have:

THEOREM 24. *Let $(X, \|\cdot\|)$ be a smooth normed space and $[\cdot, \cdot]$ be the semi-inner product in the sense of Lumer-Giles which generates the norm $\|\cdot\|$. Then*

(i) *$[x, y] = (x, y)_s$, $x, y \in X$; if $X$ is a real space and*

(ii) *$[x, y] = (x, y)_s - i\,(ix, y)_s$, $x, y \in X$; if $X$ is complex.*

PROOF. The proof is as follows.

(i) Since in a smooth normed space there exists a unique $L.-G.-$s.i.p. which generates the norm (see Proposition 4) and $(\cdot, \cdot)_s$ satisfies the conditions of such a semi-inner product, it follows that the equality $[x, y] = (x, y)_s$ for all $x, y \in X$ holds.

(ii) The argument follows as above and we will omit the details.

∎

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CHAPTER 4

# Semi-Inner Products in the Sense of Miličić

## 1. Definition and the Main Properties

In paper [**1**], P.M. Miličić introduced the following concept.

DEFINITION 8. *Let* $(X, \|\cdot\|)$ *be a normed linear space. The mapping* $(\cdot,)_g : X \times X \to \mathbb{R}$ *given by*

$$(x, y)_g := \frac{1}{2}\left[(x, y)_s + (x, y)_i\right], \quad x, y \in X;$$

*is said to be the semi-inner product in the sense of Miličić or* $M-$*semi-inner product, for short.*

It is clear that the above mapping is well-defined for all $x, y \in X$ and the following properties hold (see [**1**] – [**3**]).

PROPOSITION 7. *Let* $(X, \|\cdot\|)$ *be a normed linear space. Then the following statements are true:*

(i) $(x, x)_g = \|x\|^2$ *for all* $x \in X$*;*
(ii) $(ix, x)_g = (x, ix)_g = 0$ *for every* $x \in X$*;*
(iii) $(ix, y)_g = (x, iy)_g = 0$ *for all* $x, y \in X$*;*
(iv) $(ix, iy)_g = (x, y)_g = 0$ *for all* $x, y \in X$*;*
(v) $(\alpha x, \beta y)_g = \alpha\beta (x, y)_g = 0$ *for all* $x, y \in X$ *and* $\alpha\beta \geq 0$, $\alpha, \beta \in \mathbb{R}$*;*
(vi) *The following inequality of Schwartz's type*

$$\left|(x, y)_g\right| \leq \|x\| \|y\|, \quad \text{for all } x, y \in X$$

*holds.*
(vii) *We have*

$$(-x, y)_g = -(x, y)_g, \quad \text{for all } x, y \in X.$$

PROOF. The argument is obvious from Propositions 5 and 6 and we will omit the details. ∎

Another important property which will be used in the sequel is:

PROPOSITION 8. *Let* $(X, \|\cdot\|)$ *be a normed space. Then we have the equality:*

$$(\alpha x + y, x)_g = \alpha \|x\|^2 + (y, x)_g,$$

*for any* $\alpha$ *a real number and* $x, y$ *two vectors in* $X$*.*

Proof. By Theorem 16 one has

$$(\alpha x + y, x)_s = \alpha \|x\|^2 + (y, x)_s$$

and

$$(\alpha x + y, x)_i = \alpha \|x\|^2 + (y, x)_i$$

for all $\alpha \in \mathbb{R}$, $x, y \in X$.

Now, if we add the above equalities, we deduce the desired results. ∎

By Theorem 17, we also can state:

Proposition 9. *Let* $x, y, z$ *belong to* $X$*. Then we have the inequality*

$$\left| (y + z, x)_g - (z, x)_g \right| \le \|y\| \|x\| .$$

Corollary 7. *The mapping* $(\cdot, x)_g$ *is continuous on* $(X, \|\cdot\|)$*, for all* $x \in X$*.*

The following representation theorem of the semi-inner product in Miličić's sense in terms of the normalised duality mapping holds.

Theorem 25. *Let* $(X, \|\cdot\|)$ *be a real or complex normed space and* $\tilde{J}$ *a section of the normalised dual mapping* $J$*. Then we have the representation:*

$$(y, x)_g = \lim_{t \to 0^+} \operatorname{Re} \left\langle \frac{\tilde{J}(x + ty) + \tilde{J}(x - ty)}{2}, y \right\rangle$$

*for all* $x, y \in X$*.*

Proof. By the use of Theorem 18, we have:

$$\begin{aligned}
(y, x)_g &= \frac{1}{2} \left[ (y, x)_s + (y, x)_i \right] = \frac{1}{2} \left[ (y, x)_s - (-y, x)_s \right] \\
&= \frac{1}{2} \left[ \lim_{t \to 0^+} \operatorname{Re} \left\langle \tilde{J}(x + ty), y \right\rangle + \lim_{t \to 0^+} \left\langle \tilde{J}(x - ty), y \right\rangle \right] \\
&= \lim_{t \to 0^+} \operatorname{Re} \left\langle \frac{\tilde{J}(x + ty) + \tilde{J}(x - ty)}{2}, y \right\rangle
\end{aligned}$$

for all $x, y \in X$, and the statement is proved. ∎

Another result of this type which can be proved with the help of Theorem 19 is the following.

THEOREM 26. *Let $(X, \|\cdot\|)$ be a normed space. Then for every $\tilde{J}$ a section of the normalised dual mapping $J$, we have the representation:*

$$(y,x)_g = \lim_{t\to 0^+} \operatorname{Re}\left\langle \frac{\tilde{J}(x+ty) - \tilde{J}(x-ty)}{2t}, x \right\rangle$$

*for all $x, y \in X$.*

In [**6**], G. Godini introduced the *smoothness subspace* of the point $x$, denoted by $G_x$, and given by

$$G_x := \{y \in X | \mathcal{T}_-(x,y) = \mathcal{T}_+(x,y)\},$$

where $\mathcal{T}_\pm$ are the *tangent functionals:*

$$\mathcal{T}_+(x,y) := \lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t}, \quad \mathcal{T}_-(x,y) := \lim_{t\to 0^-} \frac{\|x+ty\| - \|x\|}{t}$$

and $x, y \in X$, $x \neq 0$. We note that:

$$(y,x)_s = \|x\| \mathcal{T}_+(x,y), \quad (y,x)_i = \|x\| \mathcal{T}_-(x,y), \quad x,y \in X,\ x \neq 0$$

and

$$(y,x)_g = \frac{\|x\|}{2} (\mathcal{T}_+(x,y) + \mathcal{T}_-(x,y)), \quad x,y \in X,\ x \neq 0.$$

In the above cited paper [**6**], G. Godini pointed out a representation theorem for the smoothness subspace $G_x$ in terms of the normalised dual mapping for the case of real normed spaces. Recently, P.M. Miličić [**4**, Theorem 2] extended this result to the case of complex normed spaces using another technique of proof. We will present here this result. The reader can find the proof in [**4**].

THEOREM 27. *Let $(X, \|\cdot\|)$ be a real or complex normed linear space and $x$ a fixed element in $X \diagdown \{0\}$. Then we have the representation:*

$$G_x = (\operatorname{Re} J(x))_\perp \oplus S_p(x) \tag{4.1}$$

*where*

$$(\operatorname{Re} J(x))_\perp := \{h \in X | (\forall) f \in J(x),\ \operatorname{Re} f(x) = 0\}.$$

Using this result, we can present the following approximation theorem due to P.M. Miličić (see [**4**, Theorem 3]):

THEOREM 28. *Let $(X, \|\cdot\|)$ be a normed space. Then one has the estimation*

$$(y,x)_s - \|x\| d(y, G_x) \leq (y,x)_g \leq (y,x)_i - \|x\| d(y, G_x) \tag{4.2}$$

*for all $x, y \in X$ and $x \neq 0$, where $d(y, G_x) := \inf\{\|y - z\|, z \in G_x\}$.*

Proof. Let $X_r$ be the real normed space associated to $X$. The set $G_x$ is a linear subspace of the space $X_r$ [**6**].

Now, let $f_1,\ f_2 \in J(x)$ so that:

$$\operatorname{Re} f_1(y) = \|x\| \mathcal{T}_+(x,y),\quad \operatorname{Re} f_2(y) = \|x\| \mathcal{T}_-(x,y).$$

Using the representation (4.1), it follows that $G_x \subset Ker\left[\left(\frac{f_1-f_2}{2}\right)\right]$ where $\operatorname{Re}\left[\left(\frac{f_1-f_2}{2}\right)\right] \in (G_x)_\perp$.

On the other hand, $\operatorname{Re}\left[\left(\frac{f_1-f_2}{2}\right)\right] \in \operatorname{Re} X^*$ and $\left\|\frac{\operatorname{Re}(f_1-f_2)}{2}\right\| \leq 1$. Now, using the relation (8) from [**4**], for every real normed spaces $X$, we have

$$d(y,G_x) = \max\{\varphi(y) \,|\varphi \in (G_x)_\perp, \|\varphi\| \leq 1\}. \tag{4.3}$$

For $\varphi = \operatorname{Re}\left[\left(\frac{f_1-f_2}{2}\right)\right]$ from (4.3) we get

$$d(y,G_x) \geq \operatorname{Re}\left[\left(\frac{f_1-f_2}{2}\right)\right] = \frac{[\mathcal{T}_+(x,y) - \mathcal{T}_-(x,y)]}{2}.$$

Consequently, for all real or complex normed spaces, we have:

$$\mathcal{T}_+(x,y) - \mathcal{T}_-(x,y) \leq 2d(y,G_x),\quad x,y \in X. \tag{4.4}$$

Now, by the definition of the functional $(\cdot,\cdot)_g$, we get

$$\begin{aligned}(y,x)_g &= \|x\| \mathcal{T}_-(x,y) + \frac{\|x\|}{2}\left(\mathcal{T}_+(x,y) - \mathcal{T}_-(x,y)\right)\\ &= \|x\| \mathcal{T}_+(x,y) + \frac{\|x\|}{2}\left(\mathcal{T}_+(x,y) - \mathcal{T}_-(x,y)\right)\end{aligned}$$

and applying (4.4) we deduce (4.2). ∎

The next corollary is interesting (see also [**4**]).

Corollary 8. *In every normed linear space we have the estimation*

$$\begin{aligned}&(y,x)_g + (z,x)_g - \|x\|[d(y,G_x) + d(z,G_x)]\\ \leq\ &(y+z,x)_g\\ \leq\ &(y,x)_g + (z,x)_g + \|x\|[d(y,G_x) + d(z,G_x)],\end{aligned}$$

*where $x,y,z \in X$.*

Proof. We have:

$$\begin{aligned}(y,x)_g + (z,x)_g &\leq \|x\|[\mathcal{T}_-(x,y) + \mathcal{T}_-(x,z) + d(y,G_x) + d(z,G_x)]\\ &\leq \|x\|[\mathcal{T}_-(x,y+z) + d(y,G_x) + d(z,G_x)]\\ &\leq (y+z,x)_g + \|x\|[d(y,G_x) + d(z,G_x)]\end{aligned}$$

because

$$\mathcal{T}_{-}(x, y+z) \geq \mathcal{T}_{-}(x, y) + \mathcal{T}_{-}(x, z)$$

and

$$\|x\| \mathcal{T}_{-}(x, y) \leq (y, x)_g \leq \|x\| \mathcal{T}_{+}(x, y).$$

The second inequality goes likewise and we omit the details. ∎

## 2. Normed Space of $(G)$-Type

It is clear that if $(x, y)_s = (x, y)_i$ for all $x, y \in X$, or equivalently, the space $(X, \|\cdot\|)$ is smooth, the semi-inner product in the sense of Miličić $(\cdot, \cdot)_g$ is linear in the first argument. However, we observe that there also exists non-smooth spaces from which the mapping $(\cdot, \cdot)_g$ is linear in the first variable too.

Indeed, if we consider the space $l^1$, then by [**1**, Example 8.1] we have:

$$(x, y)_s = \|y\| \left( \sum_{y_i \neq 0} \frac{y_i}{|y_i|} x_i + \sum_{y_i = 0} |x_i| \right), \quad x, y \in l^1$$

and

$$(x, y)_i = \|y\| \left( \sum_{y_i \neq 0} \frac{y_i}{|y_i|} x_i - \sum_{y_i = 0} |x_i| \right), \quad x, y \in l^1.$$

Now we observe that

$$(x, y)_g = \|y\| \sum_{i=1}^{\infty} (sgn\ y_i)\, x_i, \quad x, y \in l^1,$$

which shows that $(\cdot, \cdot)_g$ is linear in the first variable.

Similarly, if we consider the space $L^1(0, 1)$, then $(\cdot, \cdot)_g$ will be given by

$$(h, f)_g := \|f\| \int_A \frac{f(t)}{|f(t)|} h(t)\, dt,$$

where $A := \{t \in (0, 1) \,|\, f(t) \neq 0\}$.

These facts give us the possibility to introduce the following concept (see [**5**]).

Definition 9. *Let $(X, \|\cdot\|)$ be a normed space. Then $X$ will be called semi-smooth of $(G)$ – type, or of $(G)$ –type, for short, if the following condition:*

$$(x + y, z)_g = (x, z)_g + (y, z)_g, \text{ for all } x, y, z \in X,$$

*holds.*

The following simple proposition also holds.

Proposition 10. *Let $(X, \|\cdot\|)$ be a real or complex normed space of the $(G)$ – type. Then $(\cdot,\cdot)_g$ is a L. − G.−s.i.p over the real number field.*

The proof is obvious by the properties of the mapping $(\cdot,\cdot)_g$ defined on a normed space of $(G)$ – type. We will omit the details.

The following result for complex spaces is also valid.

Proposition 11. *Let $(X, \|\cdot\|)$ be a complex normed space of the $(G)$ – type. Then the functional*

$$[x, y]_g := (x, y)_g - i\,(ix, y)_g\,, \quad x, y \in X;$$

*satisfies the conditions:*

(i) $[x, y]_g \geq 0$ *and* $[x, x]_g = 0$ *implies* $x = 0$*;*
(ii) $[x + y, z]_g = [x, z]_g + [y, z]_g$ *for all* $x, y, z \in X$*;*
(iii) $[\lambda x, y]_g = \lambda\,[x, y]_g$ *for all* $x, y \in X$ *and* $\lambda$ *a complex number;*
(iv) *One has the inequality:*

$$\left|[x, y]_g\right|^2 \leq 2\,\|x\|^2\,\|y\|^2 - |z_g\,(x, y)|^2\,, \text{ for all } x, y, z \in X,$$

*where*

$$z_g\,(x, y) = \frac{1}{2}\left\{(x, y)_s - (x, y)_i + i\,[(ix, y)_i - (ix, y)_s]\right\}.$$

Proof. The proof is as follows.

(i) We have:

$$[x, x]_g = (x, x)_g - i\,(ix, x)_g = (x, x)_g = \|x\|^2$$

for all $x \in X$ as

$$(ix, x)_g = 0 \text{ for all } x \text{ in } X.$$

(ii) It is obvious by the properties of the functional $(\cdot,\cdot)_g$.
(iii) It is sufficient to show that

$$[ix, y]_g = i\,[x, y]_g$$

for all $x, y \in X$.
Then we have:

$$\begin{aligned} [ix, y]_g &= (ix, y)_g - i\left(i^2 x, y\right)_g = (ix, y)_g + i\,(x, y)_g \\ &= i\left[(x, y)_g - i\,(ix, y)_g\right] = i\,[x, y]_g\,, \end{aligned}$$

where $x, y \in X$; and the statement is proved.

(iv) For $z_1 = \frac{1}{2}\left[(x,y)_s - i\,(ix,y)_s\right],\ z_1 = \frac{1}{2}\left[(x,y)_i - i\,(ix,y)_i\right]$ and put $z = z_1 - z_2$. Then $[x,y]_g = z_1 + z_2$ and from the fact that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$$

it follows that

$$\left|[x,y]_g\right|^2 = |z|^2 = \frac{1}{2}\left[(x,y)_s^2 + (x,y)_i^2 + (ix,y)_s^2 + (ix,y)_i^2\right].$$

This inequality and Schwartz's inequality

$$\left|(x,y)_{s(i)}\right|^2 \le \|x\|^2\,\|y\|^2$$

yield that the statement (iv) is valid.

∎

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CHAPTER 5

# $(Q)$ and $(SQ)$-Inner Product Spaces

## 1. $(Q)$ – Inner Product Spaces

In the paper [**1**] (see also [**2**] and [**3**]) the author introduced the following generalisation of inner products in a real linear space that extends this concept in a different manner than the extensions due to Lumer-Giles, Tapia or Miličić.

Definition 10. *A mapping* $(\cdot,\cdot,\cdot,\cdot)_g : X^4 \to \mathbb{R}$ *will be called a quaternary-inner product, or* $(Q)$ *- inner product, for short, if the following conditions are satisfied:*

(i) $(\alpha x_1 + \beta x_2, x_3, x_4, x_5)_q = \alpha (x_1, x_3, x_4, x_5)_q + \beta (x_2, x_3, x_4, x_5)_q$ *where* $\alpha, \beta \in \mathbb{R}$ *and* $x_i \in X$ $\left(i = \overline{1,5}\right)$;

(ii) $\left(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}\right)_q = (x_1, x_2, x_3, x_4)$ *for any* $\sigma$ *a permutation of the indices* $(1,2,3,4)$ *and* $x_i \in X$, $\left(i = \overline{1,4}\right)$;

(iii) *One has the following Schwartz type inequality*

$$\left|(x_1, x_2, x_3, x_4)_q\right|^4 \le \prod_{i=1}^{4} (x_i, x_i, x_i, x_i)_q$$

*for all* $x_i \in X$, $i = \overline{1,4}$.

Definition 11. *A real linear space* $X$ *endowed with a* $(Q)$ *- inner product* $(\cdot,\cdot,\cdot,\cdot)_q$ *on it will be called a* $(Q)$ *- inner product space.*

Now, by the definition of $(Q)-$ inner product space, we can state the following simple properties:

$$(0, x_2, x_3, x_4) = 0 \text{ for every } x_2, x_3, x_4 \in X$$

and

$$(\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4)_q = \alpha^4 (x_1, x_2, x_3, x_4)_q$$

for any $\alpha \in \mathbb{R}$ and $x_1, x_2, x_3, x_4 \in X$.

Let us now give some examples of $(Q)-$ inner product spaces.

Assume that $(\Omega, \mathcal{A}, \mu)$ is a measure space consisting of a set $\Omega$, a $\sigma-$algebra $\mathcal{A}$ of subsets of $\Omega$, and a countably additive and positive measure $\mu$ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. If $x_1, x_2, x_3, x_4$ are in the real vector space $L^4(\Omega) \equiv L^4(\Omega, \mathcal{A}, \mu)$,

$$(x_1, x_2, x_3, x_4)_q = \int_\Omega x_1(s)\, x_2(s)\, x_3(s)\, x_4(s)\, d\mu(s)$$

then this defines a $(Q)-$ inner product in $L^4(\Omega)$. When $\mu(\Omega) < \infty$, then the above formula defines a $(Q)-$ inner product in space $L^p(\Omega)$ with $p > 4$.

The following proposition is important in the sequel.

Proposition 12. *Let $\left(X; (\cdot,\cdot,\cdot,\cdot)_q\right)$ be a $(Q)-$ inner product space. Then the mapping*

$$\|\cdot\|_q : X \to \mathbb{R},\ \ \|x\|_q = (x,x,x,x)_q^{\frac{1}{4}}$$

*is a norm on $X$.*

Proof. Firstly, we observe that

$$\|x_1 + x_2\|_q^4 = \|x_1\|_q^4 + 4(x_1,x_1,x_1,x_2)_q + 6(x_1,x_1,x_2,x_2)_q + 4(x_1,x_2,x_2,x_2)_q + \|x_2\|_q^4, \tag{5.1}$$

for all $x_1,\, x_2 \in X$.

Using the property (iv) of Definition 10, we have

$$\begin{aligned}(x_1,x_1,x_1,x_2)_q &\le \|x_1\|_q^3 \|x_2\|_q \\ (x_1,x_1,x_2,x_2)_q &\le \|x_1\|_q^2 \|x_2\|_q^2\end{aligned}$$

and

$$(x_1,x_2,x_2,x_2)_q \le \|x_1\|_q \|x_2\|_q^3$$

for all $x_1,\, x_2 \in X$.

Now, taking into account the equality (5.1), we have that

$$\|x_1 + x_2\|_q^4 \le \left(\|x_1\|_q + \|x_2\|_q\right)^4,\ \ x_1, x_2 \in X$$

which produces the triangle inequality:

$$\|x_1 + x_2\|_q \le \|x_1\|_q + \|x_2\|_q,\ \ x_1, x_2 \in X.$$

On the other hand, we have:

$$\|x_1\|_q \ge 0 \ \text{ for all } x_1 \in X$$

and

$$\|x_1\|_q = 0 \ \text{ implies } x_1 = 0$$

and finally, we also have:

$$\|\alpha x_1\|_q = |\alpha|\, \|x_1\|_q,\ \text{ where } \alpha \in \mathbb{R} \text{ and } x_1 \in X.$$

Consequently, $\|\cdot\|_q$ is a norm and the proposition is proved. ∎

The following definition is also natural.

DEFINITION 12. *A real normed (Banach) linear space is said to be a $(Q)-$ normed (Banach) space if its norm is generated by a $(Q)-$ inner product space.*

By the above considerations, we see that the real Banach space $(L^4(\Omega), \|\cdot\|_4)$ where:

$$\|x\|_4 = \left(\int_\Omega |x(s)|^4 \, d\mu(s)\right)^{\frac{1}{4}}$$

is a $Q-$ Banach space.

The following proposition also holds.

PROPOSITION 13. *Every real prehilbertian space is a $Q-$ normed space. The converse is not generally true.*

PROOF. Suppose that $(X, \|\cdot\|)$ is a prehilbertian space and $(\cdot, \cdot)$ denotes the inner product which generates its norm. Let us defined the mapping:

$$\begin{aligned}
&(x_1, x_2, x_3, x_4)_q \\
&\qquad := \frac{1}{3}\left[(x_1, x_2)(x_3, x_4) + (x_1, x_3)(x_2, x_4) + (x_1, x_4)(x_2, x_3)\right],
\end{aligned}$$

where $x_i \in X$ $\left(i = \overline{1,4}\right)$.

It is evident that $(\cdot, \cdot, \cdot, \cdot)_q$ is linear in the first variable and symmetrical. The condition of positivity holds by the same condition as the inner product $(\cdot, \cdot)$. We must therefore only prove the Schwartz inequality.

We have

$$\begin{aligned}
&\left|(x_1, x_2, x_3, x_4)_q\right|^4 \\
&= \left\{\frac{1}{3}\left[(x_1, x_2)(x_3, x_4) + (x_1, x_3)(x_2, x_4) + (x_1, x_4)(x_2, x_3)\right]\right\}^4 \\
&\leq \left\{\frac{1}{3}\left[|(x_1, x_2)||(x_3, x_4)| + |(x_1, x_3)||(x_2, x_4)| + |(x_1, x_4)||(x_2, x_3)|\right]\right\}^4 \\
&\leq (\|x_1\|\|x_2\|\|x_3\|\|x_4\|)^4 \\
&= \prod_{i=1}^4 (x_i, x_i, x_i, x_i)_q
\end{aligned}$$

for all $x_i \in X$ $\left(i = \overline{1,4}\right)$.

To show the last part of the proposition, it is sufficient to choose the $Q-$ Banach space $L^4(\Omega)$ which is not a Hilbert space.

The proof is thus completed. ∎

Now we will point out some natural properties that follow by the definition of $Q-$ inner product.

PROPOSITION 14. *Let $\left(X, \|\cdot\|_q\right)$ be a $Q-$ normed space. Then for all $x_1, x_2 \in X$ we have:*
(5.2)
$$\|x_1+x_2\|_q^4+\|x_1-x_2\|_q^4=2\left(\|x_1\|_q^4+\|x_2\|_q^4\right)+12\left(x_1,x_2,x_2,x_2\right)_q$$

*and*

$$\|x_1+x_2\|_q^4+\|x_1-x_2\|_q^4\le 2\left(\|x_1\|_q^4+\|x_2\|_q^4\right)+12\|x_1\|_q^2\|x_2\|_q^2 . \tag{5.3}$$

PROOF. By the equality (5.1) we can state:

$$\begin{aligned}\|x_1+x_2\|_q^4 &= \|x_1\|_q^4+4\left(x_1,x_1,x_1,x_2\right)_q+6\left(x_1,x_1,x_2,x_2\right)_q \\ &\quad +4\left(x_1,x_2,x_2,x_2\right)_q+\|x_2\|_q^4\end{aligned}$$

and

$$\begin{aligned}\|x_1-x_2\|_q^4 &= \|x_1\|_q^4-4\left(x_1,x_1,x_1,x_2\right)_q+6\left(x_1,x_1,x_2,x_2\right)_q \\ &\quad -4\left(x_1,x_2,x_2,x_2\right)_q+\|x_2\|_q^4\end{aligned}$$

for all $x_1$, $x_2 \in X$.

Adding these equalities, we easily deduce (5.2).

The inequality (5.3) follows by (5.2) and observing that

$$\left(x_1,x_1,x_2,x_2\right)_q\le\|x_1\|_q^2\|x_2\|_q^2,\quad x_1,x_2\in X.$$

The proposition is thus proven. ∎

PROPOSITION 15. *In the above assumption, we also have the representation:*

$$\begin{aligned}&\left(x_1,x_2,x_3,x_4\right)_q \\ = &\ \frac{1}{4^3\cdot 3}\Big[\|x_1+x_2+x_3+x_4\|_q^4+\|x_1+x_2-x_3-x_4\|_q^4 \\ &+\|x_1+x_3-x_2-x_4\|_q^4+\|x_1+x_4-x_2-x_3\|_q^4 \\ &-\|x_1+x_2+x_3-x_4\|_q^4-\|x_1+x_2+x_4-x_3\|_q^4 \\ &-\|x_1+x_3+x_4-x_2\|_q^4-\|x_2+x_3+x_4-x_1\|_q^4\Big]\end{aligned}$$

*for all $x_i \in X \quad \left(i=\overline{1,4}\right)$.*

The proof follows by a simple computation. We will omit the details.

COROLLARY 9. *For every* $x_1,\, x_2 \in X$ *we have*

$$\begin{aligned}&(x_1, x_1, x_2, x_2)_q \\ &= \frac{1}{4^3 \cdot 3}\left[\|x_1 + 3x_2\|_q^4 + 3\|x_1 - x_2\|_q^4 - 3\|x_1 + x_2\|_q^4 - \|x_1 - 3x_2\|_q^4\right].\end{aligned}$$

Further on, we will give two theorems of classification for $Q-$ normed linear spaces.

THEOREM 29. *Every* $Q-$ *normed linear space* $\left(X, \|\cdot\|_q\right)$ *is a uniformly convex space.*

PROOF. Let $0 < \varepsilon < 2$ and $x_1,\, x_2$ be two elements from $X$ such that

$$\|x_1\|_q \leq 1, \quad \|x_2\|_q \leq 1 \text{ and } \|x_1 - x_2\|_q \geq \varepsilon.$$

Then, from the inequality (5.3) we can conclude that

$$\begin{aligned}\|x_1 + x_2\|_q^4 &\leq 2\left(\|x_1\|_q^4 + \|x_2\|_q^4\right) + 12\|x_1\|_q^2\|x_2\|_q^2 \\ &\leq 16 - \varepsilon^4\end{aligned}$$

from where results

$$\left\|\frac{x_1 + x_2}{2}\right\|_q \leq 1 - \left[1 - \left(1 - \frac{\varepsilon^4}{16}\right)^{\frac{1}{4}}\right].$$

Choosing

$$\delta(\varepsilon) := 1 - \left(1 - \frac{\varepsilon^4}{16}\right)^{\frac{1}{4}}$$

we have $\delta(\varepsilon) > 0$, which shows that the space $\left(X, \|\cdot\|_q\right)$ is uniformly convex. ∎

The second result is

THEOREM 30. *Every* $Q-$ *normed linear space* $\left(X, \|\cdot\|_q\right)$ *is uniformly smooth.*

PROOF. Let $t \in \mathbb{R}$ and $x, y \in X$ with $x \neq 0$. Then

$$\begin{aligned}\frac{1}{t}\left(\|x + ty\|_q^4 - \|x\|_q^4\right) &= 4(x, x, x, y)_q + 6(x, x, y, y)_q\, t \\ &\quad + 4(x, y, y, y)_q\, t^2 + \|y\|_q^4\, t^3\end{aligned}$$

from where results, for $\|x\|_q, \|y\|_q < 1$ that:

$$\left|\frac{\|x + ty\|_q^4 - \|x\|_q^4}{t} - 4(x, x, x, y)_q\right| \leq 6|t| + 4t^2 + t^3$$

and then

$$\lim_{t\to 0}\frac{\|x+ty\|_q^4-\|x\|_q^4}{t}=4\,(x,x,x,y)_q$$

uniformly by rapport with $x,y$ on the unit ball

$$\bar{B}\,(1):=\{x\in X|\,\|x\|\le 1\}\,.$$

On the other hand, we have:

$$\begin{aligned}&\frac{1}{t}\left(\|x+ty\|_q-\|x\|_q\right)\\ &\qquad=\frac{\|x+ty\|_q^4-\|x\|_q^4}{t}\cdot\frac{1}{\left(\|x+ty\|^2+\|x\|^2\right)\left(\|x+ty\|+\|x\|\right)}\end{aligned}$$

and since

$$\lim_{t\to 0}\left(\|x+ty\|^2+\|x\|^2\right)=2\,\|x\|^2\ \text{ uniformly for } x,y\in\bar{B}\,(1)$$

and

$$\lim_{t\to 0}\left(\|x+ty\|+\|x\|\right)=2\,\|x\|\ \text{ uniformly for } x,y\in\bar{B}\,(1)$$

we deduce that:

$$\lim_{t\to 0}\frac{\|x+ty\|_q-\|x\|_q}{t}=\frac{(x,x,x,y)_q}{\|x\|_q^3}$$

uniformly by rapport with $x,y$ in $\bar{B}\,(1)$, and then $\|\cdot\|_q$ is uniformly Fréchet differentiable on $X\diagdown\{0\}$ which means that (see [**4**, p. 36]) the space is uniformly smooth. ∎

The following proposition establishes a connection between the $Q-$ inner product and the superior semi-inner product (which is equal with the inferior semi-inner product because the space is smooth) and which will be denoted by $(\cdot,\cdot)$.

Proposition 16. *Let $\left(X,\|\cdot\|_q\right)$ be a $Q-$ normed linear space. Then for every $\alpha\in X$ one has:*

$$(x,y)=\begin{cases}\frac{(x,y,y,y)_q}{\|y\|_q^2} & \text{if }\ y\neq 0\\ \\ 0 & \text{if }\ y=0\end{cases}.$$

Proof. If $y=0$, the equality is obvious.

Suppose that $y \neq 0$. Then we have

$$\begin{aligned}(x,y) &= \lim_{t\to 0}\frac{\|y+tx\|^2-\|y\|^2}{2t}\\ &= \lim_{t\to 0}\frac{\|y+tx\|^4-\|y\|^4}{t}\cdot\lim_{t\to 0}\frac{1}{2\left(\|y+tx\|^2+\|y\|^2\right)}\\ &= \frac{4\,(x,y,y,y)_q}{4\,\|y\|_q^2}=\frac{(x,y,y,y)_q}{\|y\|_q^2}\end{aligned}$$

and the statement is proved. ∎

## 2. $(SQ)-$ Inner Product Spaces

This concept is another natural generalisation of inner products on real or complex linear spaces [**3**].

Definition 13. *Let $X$ be a real or complex linear space. A mapping $(\cdot,\cdot,\cdot,\cdot)_{sq} : X^4 \to \mathbb{K}$ $(\mathbb{K}=\mathbb{C},\ \mathbb{R})$ is said to be a sesqui-quaternary-inner product or $(SQ)-$ inner product, for short, if the following conditions are satisfied:*

(i) $(\alpha x_1+\beta x_2,x_3,x_4,x_5)_{sq}=\alpha\,(x_1,x_3,x_4,x_5)_{sq}+\beta\,(x_2,x_3,x_4,x_5)_{sq}$ *where* $\alpha,\beta\in\mathbb{K}$ *and* $x_i\in X$ $\left(i=\overline{1,5}\right)$;

(ii) $\overline{(x_1,x_2,x_3,x_4)_{sq}}=(x_2,x_1,x_4,x_3)_{sq}$ *for all* $x_i\in X$ $\left(i=\overline{1,4}\right)$;

(iii) $(x_1,x_2,x_3,x_4)_{sq}=(x_3,x_4,x_1,x_2)_{sq}$ *for all* $x_i\in X$ $\left(i=\overline{1,4}\right)$;

(iv) $(x_1,x_1,x_1,x_1)_{sq}>0$ *if* $x_1\in X,\ x_1\neq 0$;

(v) $\left|(x_1,x_2,x_3,x_4)_{sq}\right|^4\le\prod_{i=1}^4(x_i,x_i,x_i,x_i)_{sq}$ *for* $x_i\in X$ $\left(i=\overline{1,4}\right)$.

By the definition of $(SQ)-$ inner product, it is easy to see that $(\cdot,\cdot,\cdot,\cdot)_{sq}$ is linear in the third variable and antilinear in the second and fourth variables and the number $(x,x,y,y)_{sq}$ is real for every $x,y\in X$.

Let us now give some examples of $(SQ)-$ inner product spaces, i.e., linear spaces endowed with $(SQ)-$ inner products.

a) Every $Q-$ inner product space is a $(SQ)-$ inner product space;

b) Let $(\cdot,\cdot)_p : X\times X\to\mathbb{K}$ be an inner product space over the real or complex number field $\mathbb{K}$. Then the mapping $(\cdot,\cdot,\cdot,\cdot)_{sq} : X^4\to\mathbb{K}$ given by

$$(x_1,x_2,x_3,x_4)_{sq}:=(x_1,x_2)_p\,(x_3,x_4)_p\,,\quad x_i\in X,\ i=\overline{1,4}$$

is an $(SQ)-$ inner product.

c) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. If $x_1, x_2, x_3, x_4$ are vectors in the real or complex linear space $L^4(\Omega)$ and

$$(x_1, x_2, x_3, x_4) := \int_\Omega x_1(s)\overline{x_2(s)}x_3(s)\overline{x_4(s)}d\mu(s) \tag{5.4}$$

then this defines a $(SQ)-$ inner product on $L^4(\Omega)$. If $\mu(\Omega) < \infty$, then the formula (5.4) on the relation:

$$(x_1, x_2, x_3, x_4)'_{sq} := \int_\Omega x_1(s)\overline{x_2(s)\,d\mu(s)}\int_\Omega x_3(s)\overline{x_4(s)}d\mu(s)$$

for every $x_1, x_2, x_3, x_4$ in $L^p(\Omega)$ define a $(SQ)-$ inner product on $L^p(\Omega)$ with $p > 4$.

The following proposition will be important later.

Proposition 17. *Let $\left(X, (\cdot,\cdot,\cdot,\cdot)_{sq}\right)$ be a $(SQ)-$ inner product space. Then the mapping $\|\cdot\|_{sq} : X \to \mathbb{R}$ given by*

$$\|x\|_{sq} := \left[(x,x,x,x)_{sq}\right]^{\frac{1}{4}}, \quad x \in X$$

*is a norm on $X$.*

Proof. Let us observe that for every $x, y \in X$ one has the identity

$$\begin{aligned} &\|x+y\|_{sq}^4 \\ &= \|x\|_{sq}^4 + 4\operatorname{Re}(x,x,x,y)_{sq} + 2(x,x,y,y)_{sq} + 2\operatorname{Re}(x,y,x,y)_{sq} \\ &\qquad + 2\operatorname{Re}(x,y,y,x)_{sq} + 4\operatorname{Re}(x,y,y,y)_{sq} + \|y\|_{sq}^4. \end{aligned} \tag{5.5}$$

By the use of this equality and by Schwartz's inequality (v) of Definition 13, we observe that:

$$\begin{aligned} \|x+y\|_{sq}^4 &\le \|x\|_{sq}^4 + 4\|x\|_{sq}^3\|y\|_{sq} + 6\|x\|_{sq}^2\|y\|_{sq}^2 \\ &\quad +4\|x\|_{sq}\|y\|_{sq}^3 + \|y\|_{sq}^4 \\ &= \left(\|x\|_{sq} + \|y\|_{sq}\right)^4 \end{aligned}$$

for all $x, y \in X$, which shows that the triangle inequality

$$\|x+y\|_{sq} \le \|x\|_{sq} + \|y\|_{sq}, \quad x, y \in X$$

holds.

The proofs of the other properties of the norm are obvious and the proposition is thus proved. ∎

Now, it is natural to introduce the following definition.

DEFINITION 14. *A real or complex normed (Banach) space is said to be a $(SQ)-$ normed ($(SQ)-$ Banach) space if its norm is generated by a $(SQ)-$ inner product.*

It is obvious that $\left(L^4(\Omega), \|\cdot\|_4\right)$ where:

$$\|x\|_4 = \left(\int_\Omega |x(s)|^4 \, d\mu(s)\right)^{\frac{1}{4}}, \quad x \in L^4(\Omega)$$

is an $(SQ)-$ Banach space.

The following proposition also holds.

PROPOSITION 18. *Every inner product space over the real or complex number field may be regarded as a $(SQ)-$ normed linear space. The converse is not generally true.*

PROOF. Let $(\cdot,\cdot)_p : X \times X \to \mathbb{K}$ be the inner product which generates the norm of $X$. We may define the mapping:

$$(\cdot,\cdot,\cdot,\cdot)_{sq} : X^4 \to \mathbb{K}, \ (x_1,x_2,x_3,x_4)_{sq} := (x_1,x_2)_{sq}(x_3,x_4)_{sq},$$

where $x_i \in X \ \left(i = \overline{1,4}\right)$.

The fact that $(\cdot,\cdot,\cdot,\cdot)_{sq}$ defined above satisfies the axioms of a $(SQ)-$ inner product is obvious and we will omit the details.

For the converse, it is sufficient to choose the $(SQ)-$ Banach space $\left(L^4(\Omega), \|\cdot\|_4\right)$ which is not a Hilbert space. ∎

The following proposition will be used later as well.

PROPOSITION 19. *Let $\left(X, \|\cdot\|_{sq}\right)$ be a $(SQ)-$ normed space. Then*

$$\begin{aligned} \|x+y\|_{sq}^4 + \|x-y\|_{sq}^4 &= 2\left(\|x\|_{sq}^4 + \|y\|_{sq}^4\right) + 4(x,x,y,y)_{sq} \\ &\quad + 4\,\mathrm{Re}\,(x,y,x,y)_{sq} + 4\,\mathrm{Re}\,(x,y,y,x)_{sq} \end{aligned} \tag{5.6}$$

*and*

$$\|x+y\|_{sq}^4 + \|x-y\|_{sq}^4 \le 2\left(\|x\|_{sq}^4 + \|y\|_{sq}^4\right) + 12\,\|x\|_{sq}^2\,\|y\|_{sq}^2 \tag{5.7}$$

*for all $x,y \in X$.*

PROOF. By the identity (5.5) we have

$$\begin{aligned} &\|x+y\|_{sq}^4 \\ &= \|x\|_{sq}^4 + 4\,\mathrm{Re}\,(x,x,x,y)_{sq} + 2(x,x,y,y)_{sq} + 2\,\mathrm{Re}\,(x,y,x,y)_{sq} \\ &\qquad + 2\,\mathrm{Re}\,(x,y,y,x)_{sq} + 4\,\mathrm{Re}\,(x,y,y,y)_{sq} + \|y\|_{sq}^4 . \end{aligned}$$

and

$$\begin{aligned} & \|x-y\|_{sq}^4 \\ = & \|x\|_{sq}^4 - 4\operatorname{Re}(x,x,x,y)_{sq} + 2(x,x,y,y)_{sq} + 2\operatorname{Re}(x,y,x,y)_{sq} \\ & +2\operatorname{Re}(x,y,y,x)_{sq} - 4\operatorname{Re}(x,y,y,y)_{sq} + \|y\|_{sq}^4, \end{aligned}$$

which, by addition, give exactly the desired equality (5.6).

The inequality (5.7) follows by the above equality and by Schwartz's inequality from (v), Definition 13. ∎

Now, we can give the following two main results concerning the classification of $(SQ)-$ normed linear spaces in the class of normed spaces.

THEOREM 31. *Every $(SQ)-$ normed space $\left(X, \|\cdot\|_{sq}\right)$ is a uniformly convex space.*

PROOF. Let $0<\varepsilon<2$ and assume that $x_1$, $x_2$ are two elements in $X$ such that

$$\|x_1\|_{sq} \le 1, \quad \|x_2\|_{sq} \le 1 \quad \text{and} \quad \|x_1 - x_2\|_{sq} \ge \varepsilon.$$

Then, by the inequality (5.7) we deduce that:

$$\begin{aligned} \|x_1+x_2\|_{sq}^4 & \le 2\|x_1\|_{sq}^4 + 2\|x_2\|_{sq}^4 + 12\|x_1\|_{sq}^2\|x_2\|_{sq}^2 - \|x_1-x_2\|_{sq}^4 \\ & \le 16-\varepsilon^4. \end{aligned}$$

Putting $\delta(\varepsilon) := 1 - \left(1 - \frac{\varepsilon^4}{16}\right)^{\frac{1}{4}} > 0$ the last inequality shows that

$$\left\|\frac{x_1+x_2}{2}\right\|_{sq} \le 1 - \delta(\varepsilon)$$

i.e., the normed space $\left(X, \|\cdot\|_{sq}\right)$ is uniformly convex. ∎

The second result is embodied in the following theorem.

THEOREM 32. *Every $(SQ)-$ normed linear space is a uniformly smooth space.*

PROOF. Let $t \in \mathbb{R}$ and $x, y \in X$ with $x \neq 0$. Then:

$$\begin{aligned} \frac{\|x+y\|_{sq}^4 - \|x\|_{sq}^4}{t} & = 4\operatorname{Re}(x,x,x,y)_{sq} + \Big[2(x,x,y,y)_{sq} + 2\operatorname{Re}(x,y,x,y)_{sq} \\ & \quad +2\operatorname{Re}(x,y,y,x)_{sq}\Big]t + 4\operatorname{Re}(x,y,y,y)_{sq}t^2 + \|y\|_{sq}^4 t^3, \end{aligned}$$

which implies:

$$\left|\frac{\|x+y\|_{sq}^4-\|x\|_{sq}^4}{t}-4\operatorname{Re}(x,x,x,y)_{sq}\right|\le 6|t|+4t^2+|t|^3$$

for all $x,y$ with $\|x\|_{sq}, \|y\|_{sq}<1$, and consequently

$$\lim_{t\to 0}\frac{\|x+y\|_{sq}^4-\|x\|_{sq}^4}{t}=4\operatorname{Re}(x,x,x,y)_{sq}$$

for all $x,y$ in the unit ball

$$\bar{B}(1):=\left\{x\in X|\ \|x\|_{sq}\le 1\right\}.$$

Now, the argument is similar to that embodied in the proof of Theorem 30 and we will omit the details. ∎

Finally, we will establish the connection between the superior (inferior) semi-inner product and the $(SQ)-$ inner product. Namely, we have the following proposition.

Proposition 20. *Let $\left(X,\|\cdot\|_{sq}\right)$ be a $(SQ)-$ normed space and $(\cdot,\cdot)$ the superior (inferior) semi-inner product. Then*

$$(x,y)=\begin{cases}\frac{\operatorname{Re}(x,y,y,,y)_{sq}}{\|y\|_{sq}^2} & \text{if } \ y\neq 0\\ \\ 0 & \text{if } \ y=0\end{cases}.$$

The proof is similar to that embodied in the proof of Proposition 16 and we will omit the details.

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CHAPTER 6

# $2k$-Inner Products on Real Linear Spaces

## 1. Introduction

In the last decade, the author gave (see [**4**] – [**9**]) an extension of the usual notion of inner product, namely the quaternary inner product, or, for short, the $Q$-inner product. Some of the properties of an inner product and of the associated norm, such as:

(i) uniform convexity,
(ii) Gâteaux differentiability,
(iii) equivalence of Birkhoff orthogonality with the inner product orthogonality,
(iv) the Riesz form of linear continuous functionals

were reobtained in this new framework.

The present chapter, following the recent paper [**3**], is devoted to a generalization of both the classical inner product and the $Q$-inner product.

In the first section we introduce the concept of $2k$-inner products and prove the properties (i)-(ii) above. Also, it is proved that a $2k$-inner product space is a smooth space of (BD)-type, and two remarkable identities, equivalent with the parallelogram identity, are given. The following two sections deal with the properties (iii) and (iv) and some results related to projections are obtained.

## 2. Main Properties of $2k$-Inner Products

Let $X$ be a real linear space and $k \neq 0$ a natural number. As usual, we shall denote $X^{2k} = \underbrace{X \times \ldots \times X}_{2k \ times}$. We introduce the following new concept [**3**]:

Definition 15. *A mapping $(\cdot, \ldots, \cdot) : X^{2k} \to \mathbb{R}$ is said to be a $2k$-inner product if:*

(i) $(\alpha_1 x_1 + \alpha_2 x_2, x_3, \ldots, x_{2k+1}) = \alpha_1 (x_1, x_3, \ldots, x_{2k+1}) + \alpha_2 (x_2, x_3, \ldots, x_{2k+1}), \quad \alpha_1, \alpha_2 \in \mathbb{R}$;
(ii) $\left(x_{\sigma(1)}, \ldots, x_{\sigma(2k)}\right) = (x_1, \ldots, x_{2k}), \quad \sigma \in S_{2k}$, *where $S_{2k}$ denotes the set of all permutations of the indices $\{1, \ldots, 2k\}$;*

(iii) $(x, \ldots, x) > 0$ *if* $x \neq 0$;

(iv) *Cauchy-Buniakowski-Schwarz's inequality (CBS for short)*

$$|(x_1, \ldots, x_{2k})|^{2k} \leq \prod_{i=1}^{2k} (x_i, \ldots, x_i)$$

*with equality if and only if* $x_1, \ldots, x_{2k}$ *are linearly dependent.*

The pair $(X, (\cdot, \ldots, \cdot))$ is called $2k$*-inner product space* [**3**]. Let us remark that our notion is different from the $n$-inner product of Misiak ([**10**]).

For $k = 1$ we have the usual notion of inner product and for $k = 2$ we obtain the notion of $Q$*-inner product* from [**4**]-[**8**]. Also, it follows that

$$(0, x_2, \ldots, x_{2k}) = 0 \text{ and } \quad (\alpha x_1, \ldots, \alpha x_{2k}) = \alpha^{2k} (x_1, \ldots, x_{2k}).$$

Example 1. I) $X = \mathbb{R}^n$, $(x_1, \ldots, x_{2k}) = \sum_{i=1}^{n} \left( \prod_{j=1}^{2k} x_j^i \right)$ *if* $x_j = \left( x_j^1, \ldots, x_j^n \right)$

II) *Let* $(\Omega, \mathcal{A}, \mu)$ *be a measure space consisting of a set* $\Omega$*, a* $\sigma$*-algebra* $\mathcal{A}$ *of subsets of* $\Omega$*, and a countably additive and positive measure* $\mu$ *on* $\mathcal{A}$ *with* $\mu(\Omega) < \infty$*. Then on* $X = L^{2k}(\Omega, \mathcal{A}, \mu)$ *we have the* $2k$*-inner product*

$$(x_1, \ldots, x_{2k}) = \int_{\Omega} \prod_{i=1}^{2k} x_i(t)\, d\mu(t).$$

A remarkable class of $2k$-inner products is provided by [**3**]:

Proposition 21. *An usual inner product* $(\cdot, \cdot)$ *on* $X$ *gives rise to a* $2k$*-inner product on* $X$ *for every* $k$*.*

Proof. By induction after $k$. Let us suppose that the given inner product yields the $2k$-inner product $(\cdot, \ldots, \cdot)_{2k}$. Then:

$$\begin{aligned} &(x_1, \ldots, x_{2k+2})_{2k+2} \\ &:= \frac{1}{2k+1} [(x_1, x_2)(x_3, \ldots, x_{2k+2})_{2k} + (x_1, x_3)(x_2, x_4, \ldots, x_{2k+2})_{2k} \\ &\qquad + \ldots + (x_1, x_{2k+2})(x_3, \ldots, x_{2k+1})_{2k}] \end{aligned}$$

is a $(2k+2)$-inner product. ∎

In the following we call *simple* the above type of $2k$-inner products.

EXAMPLE 2. (i) *For $k=2$ ([**6**, p. 76], [**8**, p. 20]) we have the following 4-inner product:*

$$(x_1,x_2,x_3,x_4)_4 = \frac{1}{3}\left[(x_1,x_2)(x_3,x_4)+(x_1,x_3)(x_2,x_4)+(x_1,x_4)(x_2,x_3)\right].$$

(ii) *For $k=3$ we have the 6-inner product*

$$\begin{aligned}(x_1,\dots,x_6)_6 &= \frac{1}{15}\{(x_1,x_2)\left[(x_3,x_4)(x_5,x_6)+(x_3,x_5)(x_4,x_6)+(x_3,x_6)(x_4,x_5)\right]\\ &+(x_1,x_3)\left[(x_2,x_4)(x_5,x_6)+(x_2,x_5)(x_4,x_6)+(x_2,x_6)(x_4,x_5)\right]\\ &+(x_1,x_4)\left[(x_2,x_3)(x_5,x_6)+(x_2,x_5)(x_3,x_6)+(x_2,x_6)(x_3,x_5)\right]\\ &+(x_1,x_5)\left[(x_2,x_3)(x_4,x_6)+(x_2,x_4)(x_3,x_6)+(x_2,x_6)(x_3,x_4)\right]\\ &+(x_1,x_6)\left[(x_2,x_3)(x_4,x_5)+(x_2,x_4)(x_3,x_5)+(x_2,x_5)(x_3,x_4)\right]\}.\end{aligned}$$

(iii) *In the general case we have $(2k-1)!!=1\cdot 3\cdot\cdots\cdot(2k-1)$ terms. So, for $k=4$ we have $7!!=3\cdot 5\cdot 7=105$ terms.*

The previous proposition leads to the definition of orthogonal basis. Let us suppose that $X$ has dimension $n$ and let $B=\{e_i\}_{1\le i\le n}$ be a basis for $X$. For $k=1$ as usual $B$ is said to be *orthogonal* if $(e_i,e_j)=\delta_{ij}$ and for $k>1$ we define recurrently using the relation from the proof of Proposition 21. For example, $B$ is orthogonal for a $Q$-inner product if:

$$(e_{i_1},e_{i_2},e_{i_3},e_{i_4})=\frac{1}{3}\left(\delta_{i_1i_2}\delta_{i_3i_4}+\delta_{i_1i_3}\delta_{i_2i_4}+\delta_{i_1i_4}\delta_{i_2i_3}\right).$$

Then, for $i\neq j$, we have $(e_i,e_i,e_j,e_j)=\frac{1}{3}$ and $(e_i,e_i,e_i,e_j)=0$.

A first property is [**3**]:

PROPOSITION 22. *If $(\cdot,\dots,\cdot)$ is a $2k$-inner product then $\|\cdot\|_{2k}:X\to\mathbb{R}_+$, $\|x\|_{2k}=(x,\dots,x)^{\frac{1}{2k}}$ is a norm on $X$ for which the following generalization of parallelogram identity holds:*

$$\|x+y\|_{2k}^{2k}+\|x-y\|_{2k}^{2k}=2\sum_{i=0}^{k}\binom{2k}{2(k-i)}\Big(\underbrace{x,\dots,x}_{2i\ \text{times}},\underbrace{y,\dots,y}_{2(k-i)\ \text{times}}\Big).$$

PROOF. By definition of the $2k-$norm, we get

$$\|x+y\|_{2k}^{2k}=\sum_{i=0}^{2k}\binom{2k}{i}\left(\underbrace{x,\dots,x}_{i\ \text{times}},\underbrace{y,\dots,y}_{2k-i\ \text{times}}\right).$$

However,

$$\Big(\underbrace{x,\ldots,x}_{i\text{ times}},\underbrace{y,\ldots,y}_{2k-i\text{ times}}\Big) \leq \|x\|_{2k}^{i}\|y\|_{2k}^{2k-i}$$

and then

$$\|x+y\|_{2k}^{2k} \leq \sum_{i=0}^{2k}\binom{2k}{i}\|x\|_{2k}^{i}\|y\|_{2k}^{2k-i} = \left(\|x\|_{2k}+\|y\|_{2k}\right)^{2k}$$

which gives the triangle inequality. The relations:

$$\|x\|_{2k} \geq 0, \|x\|_{2k} = 0 \Leftrightarrow x = 0$$

and $\|\lambda x\|_{2k} = \mid \lambda \mid \|x\|_{2k}$, $\lambda$ a real number, immediately follow. The parallelogram identity is obvious. ∎

Remark 2. (i) *For Example 1 part I, we have*

$$\|x\|_{2k} = \left(\sum_{i=1}^{n}\left(x^{i}\right)^{2k}\right)^{\frac{1}{2k}}$$

*if* $x = \left(x^{i}\right)_{1\leq i\leq n}$.

(ii) *CBS has the form*

$$\mid (x_1,\ldots,x_{2k}) \mid \leq \prod_{i=1}^{2k}\|x_i\|_{2k}.$$

(iii) *If* $(\cdot,\ldots,\cdot)_{2k}$ *is a simple* $2k$*-inner product with the inner product* $(\cdot,\cdot)$ *as generator then* $\|\cdot\|_{2k}$ *is exactly the norm* $\|\cdot\|$ *of* $(\cdot,\cdot)$. *Also, we have*

$$(x,\ldots,x,y)_{2k} = \|x\|_{2k}^{2(k-1)}(x,y),$$

*a relation important for orthogonality theory, see Remark 3 part (ii) of Section 3.*

The previous result leads to [**3**]:

Definition 16. *A real normed space is said to be a* $2k$*-normed space if its norm is defined by a* $2k$*-inner product.*

An important property of $2k-$normed spaces is provided by [**3**]:

Theorem 33. *A* $2k$*-normed space is uniformly convex.*

Proof. Let $0 < \varepsilon < 2$ and $x, y \in X$ with $\|x\|_{2k} \leq 1$, $\|y\|_{2k} \leq 1$ and $\|x-y\|_{2k} \geq \varepsilon$. Applying the parallelogram identity and the CBS inequality, we have that

$$\begin{aligned}\|x+y\|_{2k}^{2k} &\leq 2\sum_{i=0}^{k}\binom{2k}{2(k-i)}\|x\|_{2k}^{2i}\|y\|_{2k}^{2(k-i)}-\|x-y\|_{2k}^{2k}\\ &\leq 2^{2k}-\varepsilon^{2k}=2^{2k}\left[1-\left(\frac{\varepsilon}{2}\right)^{2k}\right]\end{aligned}$$

and then

$$\left\|\frac{x+y}{2}\right\|\leq 1-\left[1-\left(1-\left(\frac{\epsilon}{2}\right)^{2k}\right)^{\frac{1}{2k}}\right].$$

Putting

$$\delta(\varepsilon)=1-\left(1-\left(\frac{\varepsilon}{2}\right)^{2k}\right)^{\frac{1}{2k}}$$

we have $\delta(\varepsilon)>0$, which gives the desired result. ∎

Another remarkable result of this section is [**3**]:

THEOREM 34. *The norm of a $2k$-normed space is Gâteaux differentiable with:*

$$\tau(x,y):=(\|\cdot\|_{2k}')(x)(y)=\frac{(x,\dots,x,y)}{\|x\|_{2k}^{2k-1}},\quad x\neq 0.$$

PROOF. Let $x,y\in X$, $x\neq 0$ and $t\neq 0$ a real number. Since

$$\frac{1}{t}\left(\|x+ty\|_{2k}^{2k}-\|x\|_{2k}^{2k}\right)=\frac{1}{t}\sum_{i=0}^{2k-1}\binom{2k}{i}\Big(\underbrace{x,\dots,x}_{i\text{ times}},\underbrace{ty,\dots,ty}_{2k-i\text{ times}}\Big),$$

we have

$$\lim_{t\to 0}\frac{1}{t}\left(\|x+y\|_{2k}^{2k}-\|x\|_{2k}^{2k}\right)=2k(x,\dots,x,y).$$

Also, from:

$$\begin{aligned}&\frac{1}{t}\left(\|x+ty\|_{2k}-\|x\|_{2k}\right)\\ &\qquad=\frac{1}{t}\cdot\frac{\|x+ty\|_{2k}^{2k}-\|x\|_{2k}^{2k}}{\left(\|x+ty\|_{2k}^{k}+\|x\|_{2k}^{k}\right)\sum_{i=1}^{k}\|x+ty\|_{2k}^{k-i}\|x\|_{2k}^{i-1}}\end{aligned}$$

we get:

$$\lim_{t\to 0}\frac{1}{t}\left(\|x+ty\|_{2k}-\|x\|_{2k}\right)=\frac{2k(x,\dots,x,y)}{2\|x\|_{2k}^{k}k\|x\|_{2k}^{k-1}},$$

which is the required relation. ∎

Let us recall, following [**9**], the following notions:

DEFINITION 17. (i) *On a normed linear space* $(X, \|\cdot\|)$ *the semi-inner-product* $(\cdot,\cdot)_T : X \times X \to \mathbb{R}$,

$$(x,y)_T := \lim_{t\downarrow 0} \frac{1}{2t}\left(\|y+tx\|^2 - \|y\|^2\right)$$

*is called semi-inner-product in the Tapia sense.*

(ii) *A smooth normed space is called of (D)-type* [**9**] *if there exists:*

$$(x,y)_T^{'} := \lim_{t\to 0} \frac{1}{t}\left[(x,y+tx)_T - (x,y)_T\right]$$

*and a space of (D)-type is called of (BD)-type if there exists a real number* $k$ *so that* $(x,y)_T^{'} \leq k^2\|y\|^2$. *The least number* $k$ *is called the boundedness modulus (for details, see Chapter 14).*

The following result is known.

PROPOSITION 23. *(*[**9**, p. 1]*) A normed linear space is smooth if and only if* $(\cdot,\cdot)_T$ *is linear in the first variable.*

A straightforward computation for the $2k-$normed spaces gives [**3**]:

PROPOSITION 24. *A* $2k$*-normed space is smooth since*

$$(x,y)_T = \frac{(y,\ldots,y,x)}{\|y\|_{2k}^{2(k-1)}}.$$

*Also, a* $2k$*-normed space is of (BD)-type with boundedness modulus* $1$ *because* $(x,y)_T^{'} = \|y\|_{2k}^2$.

We finish this section with two identities in a $2k$-inner product space. A simple calculation gives the equivalences:

$$a^2 + c^2 = 2b^2 \Longleftrightarrow \frac{1}{b+c} + \frac{1}{a+b} = \frac{2}{a+c},$$

$$a^2 + c^2 = 2b^2 \Longleftrightarrow \frac{a}{b+c} + \frac{c}{a+b} = \frac{2b}{a+c}.$$

Using the above parallelogram identity, let

$$\begin{aligned} a &= \|x+y\|_{2k}^k, \quad c = \|x-y\|_{2k}^k \text{ and} \\ b &= \left(\sum_{i=0}^{k}\binom{2k}{2(k-i)}\left(\underbrace{x,\ldots,x}_{2i\ \text{times}},\underbrace{y,\ldots,y}_{2(k-i)\ \text{times}}\right)\right)^{\frac{1}{2}} \end{aligned}$$

to obtain:

$$\frac{1}{\|x-y\|_{2k}^{k}+\left(\sum_{i=0}^{k}\binom{2k}{2(k-i)}\left(\underbrace{x,\dots,x}_{2i\text{ times}},\underbrace{y,\dots,y}_{2(k-i)\text{ times}}\right)\right)^{\frac{1}{2}}}$$
$$+\frac{1}{\|x+y\|_{2k}^{k}+\left(\sum_{i=0}^{k}\binom{2k}{2(k-i)}\left(\underbrace{x,\dots,x}_{2i\text{ times}},\underbrace{y,\dots,y}_{2(k-i)\text{ times}}\right)\right)^{\frac{1}{2}}}$$
$$=\frac{2}{\|x+y\|_{2k}^{k}+\|x-y\|_{2k}^{k}}$$

and

$$\frac{\|x+y\|_{2k}^{k}}{\|x-y\|_{2k}^{k}+\left(\sum_{i=0}^{k}\binom{2k}{2(k-i)}\left(\underbrace{x,\dots,x}_{2i\text{ times}},\underbrace{y,\dots,y}_{2(k-i)\text{ times}}\right)\right)^{\frac{1}{2}}}$$
$$+\frac{\|x-y\|_{2k}^{k}}{\|x+y\|_{2k}^{k}+\left(\sum_{i=0}^{k}\binom{2k}{2(k-i)}\left(\underbrace{x,\dots,x}_{2i\text{ times}},\underbrace{y,\dots,y}_{2(k-i)\text{ times}}\right)\right)^{\frac{1}{2}}}$$
$$=\frac{2\left(\sum_{i=0}^{k}\binom{2k}{2(k-i)}\left(\underbrace{x,\dots,x}_{2i\text{ times}},\underbrace{y,\dots,y}_{2(k-i)\text{ times}}\right)\right)^{\frac{1}{2}}}{\|x+y\|_{2k}^{k}+\|x-y\|_{2k}^{k}}.$$

## 3. $2k$-Orthogonality

We shall begin with:

DEFINITION 18. *If $x,y\in(X,(\cdot,\dots,\cdot))$ then $x$ is said to be $2k$-orthogonal to $y$ if $(x,\dots,x,y)=0$ and we denote this fact by $x\perp_{2k}y$.*

REMARK 3. (i) *Obviously, $x\perp_{2k}x\Rightarrow x=0$.*

(ii) *From Remark 2 part (iii), it follows that for a simple $2k$-inner product generated by $(\cdot,\cdot)$ we have $x\perp_{2k}y\Leftrightarrow x\perp_{2}y$.*

Let us recall that on a normed space $(X,\|\cdot\|)$, $x$ is called *Birkhoff orthogonal to $y$* if $\|x+\lambda y\|\geq\|x\|$ for all real $\lambda$ and denote this fact by $x \perp_B y$. The following characterization of Birkhoff orthogonality is due by R. C. James:

PROPOSITION 25. *([**11**, p. 92]) $x \perp_B y \Leftrightarrow \tau_-(x,y) \le 0 \le \tau_+(x,y)$ where:*

$$\tau_-(x,y) := \lim_{t \downarrow 0} \frac{1}{t}\left(\|x+ty\| - \|x\|\right),$$
$$\tau_+(x,y) := \lim_{t \uparrow 0} \frac{1}{t}\left(\|x+ty\| - \|x\|\right).$$

The following lemma is useful [**3**]:

LEMMA 2. *If $(X,(\cdot,\ldots,\cdot))$ is a $2k$-inner product space then the $2k$-orthogonality is equivalent with Birkhoff orthogonality.*

PROOF. If $x \perp_B y$ then applying Proposition 25 it results that

$$0 \le \tau_-(x,y) \le 0 \le \tau_+(x,y)$$

which implies

$$\tau(x,y) = \tau_-(x,y) = \tau_+(x,y) = 0$$

and then $x \perp_{2k} y$. Conversely, if $x \perp_{2k} y$ and $x \neq 0$ then

$$\tau_-(x,y) = \tau_+(x,y) = \frac{(x,\ldots,x,y)}{\|x\|_{2k}^{2k-1}} = 0$$

and applying Proposition 25 we have the conclusion. ∎

This result has an important consequence. Thus, applying Ex. 24 from [**2**, V. 66] it results that $x \perp_{2k} y$ is equivalent with $y \perp_{2k} x$ if and only if $\|\cdot\|_{2k}$ is generated by an usual inner product. For example, this is the case of simple $2k$-inner products, see Remark 2 part (iii) or Remark 3 part (ii).

DEFINITION 19. *Given a subset $Y \subset (X,(\cdot,\ldots,\cdot))$, the set $Y^{\perp_{2k}} = \{z \in X; z \perp_{2k} y \text{ for all } y \in Y\}$ is called the $2k$-orthogonal complement of $Y$.*

Remark that $Y \cap Y^{\perp_{2k}} = \{0\}$ and if $\lambda \in \mathbb{R}$ and $z \in Y^{\perp_{2k}}$ then $\lambda z \in Y^{\perp_{2k}}$ showing that $Y^{\perp_{2k}}$ is a linear subspace. However, from Proposition 24 $X$ is smooth and applying Ex. 26 from [**2**, V. 66] it results that $Y^{\perp_{2k}}$ is a linear subspace.

The following orthogonal decomposition theorem holds [**3**].

THEOREM 35. *Let $Y$ be a closed linear subspace in a complete $2k$-inner product space $(X,(\cdot,\ldots,\cdot))$. Then, for $x \in X$ there exists a unique $y \in Y$ and $z \in Y^{\perp_{2k}}$ such that $x = y + z$.*

PROOF. *Existence.* From uniform convexity it follows that $X$ is reflexive ( [**11**, p. 368]), and thus there exists a projection of $x$ on $Y$, i.e., an element $y \in Y$ such that

$$\|x-y\|_{2k} \leq \|x-y^{'}\|_{2k}$$

for all $y' \in Y$. Denoting $z = x - y$ we have the required relation.

Now, we prove that $z \in Y^{\perp_{2k}}$. For $y' \in Y$ we have

$$\|z+\lambda y'\|_{2k} = \|x-(y-\lambda y')\|_{2k} \geq \|x-y\|_{2k} = \|z\|_{2k}$$

for all real $\lambda$ and then $z \perp_B y'$. Applying Lemma 2 we obtain $z \in Y^{\perp_{2k}}$.

*Unicity.* The above $y$ is in $P_Y(x)$, where $P_Y(x)$ denotes the set of best approximation elements in $Y$ referring to $x$. Since $X$ is uniformly convex it results that $X$ is strictly convex and then $P_Y(x)$ contains a unique element ( [**11**, p. 110]). ∎

In the following we obtain some results in the spirit of [**10**], which appear as a counterpart of the above results.

Let $a \in X \backslash \{0\}$ and denote by $X(a)$ the linear subspace generated by $a$. Let us consider the mapping

$$pr_a : X \to X, \ \ pr_a(x) := \frac{(a,\ldots,a,x)}{||a||_{2k}^{2k}} a.$$

It follows that [**3**]:

PROPOSITION 26. (i) *$pr_a$ is independent of the choice of $a$ in $X(a)$ i.e. for $\lambda \in \mathbb{R}$ we have $pr_{\lambda a} = pr_a$.*

(ii) *$pr_a$ is a projection onto $X(a)$.*

(iii) *For arbitrary $x \in X$, $a$ is $2k$-orthogonal to $x - pr_a x$ and*

$$\|pr_a(x)\|_{2k} \leq \|x\|_{2k}.$$

PROOF. The proof is as follows.

(i) We observe that

$$pr_{\lambda a}(x) = \frac{(\lambda a,\ldots,\lambda a,x)}{\|\lambda a\|_{2k}^{2k}} \lambda a = \frac{\lambda^{2k}(a,\ldots,a,x)}{\lambda^{2k}\|a\|_{2k}^{2k}} a = pr_a(x).$$

(ii) We note that $pr_a$ is onto because $pr_a(a) = a$. Obviously, $pr_a$ is linear and:

$$\begin{aligned} pr_a(pr_a(x)) &= \frac{(a,\ldots,a,pr_a(x))}{\|a\|_{2k}^{2k}} a \\ &= \frac{(a,\ldots,a)(a,\ldots,a,x)}{\|a\|_{2k}^{4k}} a \\ &= pr_a(x). \end{aligned}$$

(iii) We remark that

$$\begin{aligned}(a,\dots,a,x-pr_a(x)) &= (a,\dots,a,x)-(a,\dots,a,pr_a(x))\\ &= (a,\dots,a,x)-\frac{(a,\dots,a)(a,\dots,a,x)}{\|a\|_{2k}^{2k}}=0\end{aligned}$$

and

$$\begin{aligned}\|pr_a(x)\|_{2k} &= \frac{|(a,\dots,a,x)|\|a\|_{2k}}{\|a\|_{2k}^{2k}}=\frac{|(a,\dots,a,x)|}{\|a\|_{2k}^{2k-1}}\\ &\le \frac{\|a\|_{2k}^{2k-1}\|x\|_{2k}}{\|a\|_{2k}^{2k-1}}=\|x\|_{2k},\end{aligned}$$

and the proposition is proved.

∎

## 4. The Riesz Property

Let us denote by $X^*$ the usual dual of $X$, that is, the space of linear continuous functionals $f: X\to\mathbb{R}$. Fix an element $y\in X$ and consider the functional $f: X\to\mathbb{R}$, $f(x):=(x,y,\dots,y)$. It follows that $f\in X^*$ with

$$|f(x)|\le\|x\|_{2k}\|y\|_{2k}^{2k-1}\text{ for all }x\in X,$$

hence

$$\|f\|\le\|y\|_{2k}^{2k-1}.$$

Also,

$$\|f\|\|y\|_{2k}\ge f(y)=\|y\|_{2k}^{2k},$$

so that

$$\|f\|=\|y\|_{2k}^{2k-1}.$$

Conversely, we shall show that any $f\in X^*$ has the above form if $X$ is complete, obtaining the following generalization of the Riesz representation theorem [**3**]:

THEOREM 36. *If $(X,(\cdot,\dots,\cdot))$ is a complete $2k$-inner product space and $f\in X^*$ then there exists an element $y\in X$ such that $f(x)=(x,y,\dots,y)$ for all $x\in X$ and $\|f\|=\|y\|_{2k}^{2k-1}$.*

PROOF. If $f=0$ then $y=0$. If $f\neq0$ let $x_0\in X$ with $f(x_0)\neq0$. Applying the Proposition 35 for $x_0$ and $Y=Ker(f)$ which is a closed linear subspace of $X$, there is a unique $y_0\in Ker(f)$ and a unique $z_0\in Ker(f)^{\perp_{2k}}$ such that $x_0=y_0+z_0$. It results that $z_0\notin Ker(f)$.

Let $\lambda\in\mathbb{R}$ with

$$\lambda^{2k-1}=\frac{f(x_0)}{\|z_0\|_{2k}^{2k}}$$

and $y = \lambda z_0$. Because $f(x) z_0 - f(z_0) x \in Ker(f)$ for all $x \in X$ we have

$$z_0 \perp_{2k} (f(x) z_0 - f(z_0) x),$$

that is,

$$(f(x) z_0 - f(z_0) x, z_0, \ldots, z_0) = 0$$

which implies

$$\begin{aligned} f(x) &= \frac{f(z_0)}{\|z_0\|_{2k}^{2k}} (x, z_0, \ldots, z_0) = \lambda^{2k-1} (x, z_0, \ldots, z_0) \\ &= (x, \lambda z_0, \ldots, \lambda z_0) = (x, y, \ldots, y) \end{aligned}$$

for all $x \in X$. ∎

Finally, we shall prove the theorem of unicity for the representation element [**3**].

THEOREM 37. *Let $(X, (\cdot, \ldots, \cdot))$ be a complete $2k$-inner product space and $f \in X^* \setminus \{0\}$. Then there exists an unique $u \in X$ with $\|u\|_{2k} = 1$ such that $f(x) = \|f\| (x, u, \ldots, u)$ for all $x \in X$.*

PROOF. *Existence.* As above, there exists a $z_0 \in \ Ker(f)^{\perp_{2k}} \setminus \{0\}$ such that

$$f(x) = \frac{f(z_0)}{\|z_0\|_{2k}} \left( x, \frac{z_0}{\|z_0\|_{2k}}, \ldots, \frac{z_0}{\|z_0\|_{2k}} \right)$$

for all $x \in X$ and

$$\|f\| = \frac{f(z_0)}{\|z_0\|_{2k}}.$$

With

$$\lambda = \left( \frac{f(z_0)}{|f(z_0)|} \right)^{1/2k-1}$$

we get

$$\begin{aligned} f(x) &= \|f\| \frac{f(z_0)}{|f(z_0)|} \left( x, \frac{z_0}{\|z_0\|_{2k}}, \ldots, \frac{z_0}{\|z_0\|_{2k}} \right) \\ &= \|f\| \lambda^{2k-1} \left( x, \frac{z_0}{\|z_0\|_{2k}}, \ldots, \frac{z_0}{\|z_0\|_{2k}} \right) = \|f\| (x, u, \ldots, u), \end{aligned}$$

where $u = \frac{\lambda z_0}{\|z_0\|_{2k}}$. Obviously $\|u\|_{2k} = 1$.

*Unicity.* We have $f(u) = \|f\|$. Since $(X, (\cdot, \cdot))$ is strictly convex and $u$ satisfy the last relations, by the Krein theorem ([**11**, p. 110]), it follows that $u$ is unique. ∎

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CHAPTER 7

# Mappings Associated with the Norm Derivatives

## 1. Introduction

In this chapter we introduce some natural mappings associated to the semi-inner products $(\cdot,\cdot)_i$ and $(\cdot,\cdot)_s$ and study their main properties both in the general setting of normed linear spaces and in the case of inner product spaces.

## 2. Some Mappings Associated with the Norm Derivatives

Let $(X,\|\cdot\|)$ be a real normed linear space and $x,y$ two fixed elements in $X$. We can defined the following mappings:

$$\begin{aligned}
n_{x,y} &: \mathbb{R}\to\mathbb{R},\;\; n_{x,y}(t)=\|x+ty\|,\\
\delta_{x,y} &: \mathbb{R}\to\mathbb{R},\;\; \delta_{x,y}(t)=2\|x+ty\|-\|x+2ty\|,\\
v_{x,y} &: \mathbb{R}\diagdown\{0\}\to\mathbb{R},\;\; v_{x,y}(t)=\frac{\|x+ty\|-\|x\|}{t},\\
\gamma_{x,y} &: \mathbb{R}\diagdown\{0\}\to\mathbb{R},\;\; \gamma_{x,y}(t)=\frac{\|x+2ty\|-\|x+ty\|}{t}.
\end{aligned}$$

Using the semi-inner products $(\cdot,\cdot)_i$ and $(\cdot,\cdot)_s$ and assuming that $x,y$ are linearly independent, we can also consider the mappings:

$$\Phi^p_{x,y}:\mathbb{R}\to\mathbb{R},\;\; \Phi^p_{x,y}(t)=\frac{(y,x+ty)_p}{\|x+ty\|}$$

and

$$\Psi^p_{x,y}:\mathbb{R}\to\mathbb{R},\;\; \Psi^p_{x,y}(t)=\frac{(x,x+ty)_p}{\|x+ty\|},$$

where $p\in\{s,i\}$.

There are some natural connections between the previous mappings. We shall incorporate them in the following proposition:

Proposition 27. *If $x,y$ are two linearly independent vectors in the normed linear space $X$, then we have that:*

(i) *The following equalities for the mappings* $\gamma, \delta$ *hold:*

$$\gamma_{x,y}(t) = \delta_{y,\frac{x}{2}}\left(\frac{1}{t}\right) \quad \textit{for} \quad t>0; \tag{7.1}$$

$$\gamma_{x,y}(u) = -\delta_{y,\frac{x}{2}}\left(\frac{1}{u}\right) \quad \textit{for} \quad u<0; \tag{7.2}$$

$$\delta_{x,y}(t) = \gamma_{2y,x}\left(\frac{1}{t}\right) \quad \textit{for} \quad t>0 \tag{7.3}$$

*and*

$$\delta_{x,y}(u) = -\gamma_{2y,x}\left(\frac{1}{u}\right) \quad \textit{for} \quad u<0. \tag{7.4}$$

(ii) *The following equalities for the mappings* $\Phi^p, \Psi^p$ *hold:*

$$\Phi^p_{x,y}\left(\frac{1}{t}\right) = \Psi^p_{y,x}(t) \quad \textit{for} \quad t>0 \tag{7.5}$$

*and*

$$\Phi^p_{x,y}\left(\frac{1}{u}\right) = \Psi^q_{y,x}(u) \quad \textit{for} \quad u<0. \tag{7.6}$$

(iii) *The following equalities for the mappings* $\Phi^p, \Psi^p$, *and* $n$ *hold:*

$$\Psi^p_{x,y}(t) = n_{x,y}(t) - t\Phi^q_{x,y}(t) \quad \textit{for} \quad t>0 \tag{7.7}$$

*and*

$$\Psi^p_{x,y}(u) = n_{x,y}(u) - u\Phi^p_{x,y}(u) \quad \textit{for} \quad u<0, \tag{7.8}$$

*where* $p,q \in \{s,i\}$ *and* $p \neq q$.

Proof. The proof is as follows:

(i) For $\alpha > 0$, we have that

$$\begin{aligned} \gamma_{x,y}\left(\frac{1}{\alpha}\right) &= \|2y+\alpha x\| - \|y+\alpha x\| = 2\left\|y+\alpha\frac{x}{2}\right\| - \left\|y+2\alpha\frac{x}{2}\right\| \\ &= \delta_{y,\frac{x}{2}}(\alpha) \end{aligned}$$

from where results (7.1).

We observe that (7.3) follows by (7.1).

For $\beta < 0$ we have that:

$$\begin{aligned} \gamma_{x,y}\left(\frac{1}{\beta}\right) &= \beta\left(\left\|\frac{2y+\beta x}{\beta}\right\| - \left\|\frac{y+\beta x}{\beta}\right\|\right) \\ &= \beta\left\|\frac{2y+\beta x}{-(-\beta)}\right\| - \beta\left\|\frac{y+\beta x}{-(-\beta)}\right\| = \|y+\beta x\| - \|2y+\beta x\| \\ &= -\delta_{y,\frac{x}{2}}(\beta) \end{aligned}$$

from where results (7.2) and by (7.2), we obtain (7.4).

(ii) We have for $t>0$ that:

$$\Phi^p_{x,y}\left(\frac{1}{t}\right)=\frac{\left(y,x+\frac{1}{t}y\right)_p}{\left\|x+\frac{1}{t}y\right\|}=\frac{(y,y+tx)_p}{\|y+tx\|}=\Psi^p_{y,x}(t).$$

If $u<0$, then

$$\begin{aligned}\Phi^p_{x,y}\left(\frac{1}{u}\right) &= \frac{\left(y,x+\frac{1}{u}y\right)_p}{\left\|x+\frac{1}{u}y\right\|}=\frac{(-u)\,(y,-(y+ux))_p}{(-u)\,\|y+ux\|}\\ &= \frac{(y,y+ux)_q}{\|y+ux\|}=\Psi^q_{y,x}(u)\end{aligned}$$

and the statement is proved.

(iii) If $t>0$, then

$$\begin{aligned}\Phi^p_{x,y}(t) &= \frac{(x+ty-ty,x+ty)_p}{\|x+ty\|}=\frac{\|x+ty\|^2+t\,(-y,x+ty)_p}{\|x+ty\|}\\ &= n_{x,y}(t)-t\frac{(y,x+ty)_q}{\|x+ty\|}=n_{x,y}(t)-t\Phi^q_{x,y}(t)\end{aligned}$$

and the identity (7.7) is proved.

If $u<0$, then:

$$\begin{aligned}\Phi^p_{x,y}(u) &= \frac{\|x+uy\|^2+u\,(y,x+uy)_p}{\|x+uy\|}\\ &= n_{x,y}(u)-u\Phi^p_{x,y}(u)\end{aligned}$$

and the proposition is thus proved. ∎

For the sake of completeness, we shall point out here some properties of the mappings $u$ and $v$ as well.

Proposition 28. *Let $x,y$ be fixed in $X$. We have*

(i) *$n_{x,y}$ is continuous convex on $\mathbb{R}$;*

(ii) *$n_{x,y}$ has lateral derivatives in each point on $\mathbb{R}$;*

(iii) *If $x,y$ are linearly independent, then*

$$\frac{d^+n_{x,y}(t)}{dt}=\Phi^s_{x,y}(t)\,,\ t\in\mathbb{R} \tag{7.9}$$

*and*

$$\frac{d^-n_{x,y}(t)}{dt}=\Phi^i_{x,y}(t)\,,\ t\in\mathbb{R}. \tag{7.10}$$

PROOF. (i), (ii). Are obvious.

(iii). Let $t \in \mathbb{R}$. Then

$$\begin{aligned}
\frac{d^{+}u_{x,y}(t)}{dt} &= \lim_{\substack{\alpha \to t \\ \alpha > t}} \left( \frac{\|x+\alpha y\| - \|x+ty\|}{\alpha - t} \right) \\
&= \lim_{\substack{\beta \to 0 \\ \beta > 0}} \left( \frac{\|x+ty+\beta y\| - \|x+ty\|}{\beta} \right) \\
&= \lim_{\beta \to 0+} \frac{\|x+ty+\beta y\|^{2} - \|x+ty\|^{2}}{2\beta} \\
&\qquad \times \lim_{\beta \to 0+} \frac{1}{\|x+ty+\beta y\| + \|x+ty\|} \\
&= \frac{(y, x+ty)_{s}}{\|x+ty\|} = \Phi^{s}_{x,y}(t),
\end{aligned}$$

and the relation (7.9) is proved.

The equality (7.10) goes likewise and we shall omit the details. ∎

For the mapping $v_{x,y}$ we have the following properties [**3**].

THEOREM 38. *Let $x, y$ be fixed in $X$. Then:*

(i) $v_{x,y}$ *is monotonic decreasing on* $\mathbb{R} \setminus \{0\}$*;*

(ii) $v_{x,y}$ *is bounded and*

$$|v_{x,y}(t)| \leq \|y\| \quad \textit{for all } t \in \mathbb{R} \setminus \{0\}; \tag{7.11}$$

(iii) *We have the inequalities:*

$$\Phi^{s}_{x,y}(u) \leq v_{x,y}(u) \leq \frac{(y,x)_{i}}{\|x\|} \quad \textit{for all } u < 0 \tag{7.12}$$

*and*

$$\Phi^{i}_{x,y}(t) \geq v_{x,y}(t) \geq \frac{(y,x)_{s}}{\|x\|} \quad \textit{for all } t > 0; \tag{7.13}$$

*assuming that $x, y$ are linearly independent.*

(iv) *We have the limits:*

$$\lim_{u \to -\infty} v_{x,y}(u) = -\|y\| \quad \textit{and} \quad \lim_{u \to \infty} v_{x,y}(u) = \|y\| \tag{7.14}$$

*and*

$$\lim_{u \to 0-} v_{x,y}(u) = \frac{(y,x)_{i}}{\|x\|} \quad \textit{and} \quad \lim_{u \to 0+} v_{x,y}(u) = \frac{(y,x)_{s}}{\|x\|}; \tag{7.15}$$

*assuming that $x \neq 0$;*

(v) *$v_{x,y}$ is laterally derivable in each point $t_0 \in \mathbb{R} \diagdown \{0\}$ and if $x, y$ are linearly independent we have that*

$$\frac{d^+ v_{x,y}(t)}{dt} = \frac{1}{t^2}\left[t\Psi^s_{x,y}(t) - u_{x,y}(t) + \|x\|\right] \tag{7.16}$$

*and*

$$\frac{d^- v_{x,y}(t)}{dt} = \frac{1}{t^2}\left[t\Psi^i_{x,y}(t) - u_{x,y}(t) + \|x\|\right] \tag{7.17}$$

*for all $t \in \mathbb{R} \diagdown \{0\}$.*

Proof. The proof is as follows.

(i) The mapping $n_{x,y}$ being convex, we have that

$$v_{x,y}(t_2) = \frac{n_{x,y}(t_2) - n_{x,y}(0)}{t_2 - 0} \geq \frac{n_{x,y}(t_1) - n_{x,y}(0)}{t_1} = v_{x,y}(t_1)$$

for all $t_2 > t_1$, $t_1, t_2 \in \mathbb{R}$.

(ii) By the continuity of the norm, we have that

$$|\|x + ty\| - \|x\|| \leq \|x + ty - x\| = |t| \|x\|, \quad t \in \mathbb{R}$$

from where results the inequality (7.11).

(iii) Let $u < 0$. Then by Schwartz's inequality we have that

$$(x, x + uy)_s \leq \|x\| \|x + ty\|.$$

By the properties of semi-inner product $(\cdot, \cdot)_s$, we can state that

$$\begin{aligned}(x, x + uy)_s &= (x + uy - uy, x + uy)_s \\ &= \|x + uy\|^2 + (-uy, x + uy)_s \\ &= \|x + uy\|^2 - u(y, x + uy)_s\end{aligned}$$

and thus, by the previous inequality, we can state that

$$\|x + uy\|^2 - u(y, x + uy)_s \leq \|x\| \|x + ty\|$$

for all $u < 0$, from where we get

$$v_{x,y}(u) = \frac{\|x + uy\| - \|x\|}{u} \geq \frac{(y, x + uy)_s}{\|x + uy\|} = \Phi^s_{x,y}(u)$$

and the first inequality in (7.12) is proved.

By Schwartz's inequality, we can also state

$$\|x\| \|x + uy\| \geq (x + uy, x)_s$$

for all $u < 0$.

A simple calculation shows us that

$$\begin{aligned}(x+uy,x)_s &= \|x\|^2+(uy,x)_s=\|x\|^2-u(-y,x)_s\\ &= \|x\|^2+(uy,x)_i\end{aligned}$$

for all $u<0$, and thus, the above inequality gives us

$$\|x\|\,\|x+uy\|-\|x\|^2\geq u\,(y,x)_i\,,\quad u<0$$

from where we obtain

$$v_{x,y}(u)=\frac{\|x+uy\|-\|x\|}{u}\leq\frac{(y,x)_i}{\|x\|}$$

and the second part of (7.12) is also proved.

The inequality (7.13) goes likewise and we shall omit the details.

(iv) We have

$$\begin{aligned}\lim_{t\to\infty}v_{x,y}(t) &= \lim_{\alpha\to 0+}v_{x,y}\left(\frac{1}{\alpha}\right)=\lim_{\alpha\to 0+}\frac{\left\|x+\frac{1}{\alpha}y\right\|-\|u\|}{\frac{1}{\alpha}}\\ &= \lim_{\alpha\to 0+}\left(\|y+\alpha x\|-\alpha\|x\|\right)=\|y\|\,.\end{aligned}$$

The second limit in (7.14) goes likewise and we shall omit the details.

Now, let us observe that

$$\begin{aligned}\lim_{t\to 0+}v_{x,y}(t) &= \lim_{t\to 0+}\frac{\|x+ty\|^2-\|x\|^2}{2t}\cdot\lim_{t\to 0+}\frac{2}{\|x+ty\|+\|x\|}\\ &= \frac{(y,x)_s}{\|x\|}\end{aligned}$$

for all $x\in X\diagdown\{0\}$.

The second limit in (7.15) is similar and we shall omit the details.

(v) The fact that $v_{x,y}$ is laterally derivable in each point $t\in\mathbb{R}\diagdown\{0\}$ is obvious. Let us compute the lateral derivatives. We have

$$\begin{aligned}\frac{d^+v_{x,y}(t)}{dt} &= \frac{d^+}{dt}\left(\frac{u_{x,y}(t)-\|x\|}{t}\right)\\ &= \frac{1}{t^2}\left[\frac{d^+u_{x,y}(t)}{dt}\cdot t-\left(u_{x,y}(t)-\|x\|\right)\right]\\ &= \frac{1}{t^2}\left[t\Psi^s_{x,y}(t)-u_{x,y}(t)+\|x\|\right],\end{aligned}$$

and the relation (7.16) is obtained.

The identity (7.17) goes likewise and we shall omit the details.

∎

Remark 4. *In the case of general normed linear spaces the graph of the mapping $n_{x,y}$ for fixed linearly independent vectors $x, y$ is incorporated in Figure 1.*

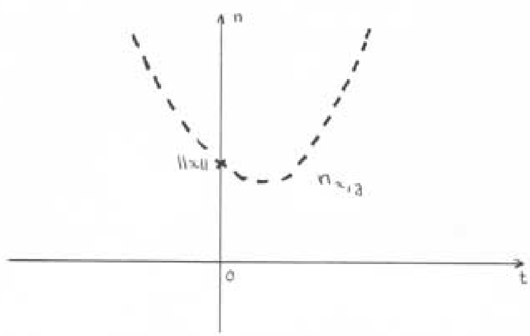


Figure 1.

Remark 5. *In the case of general normed linear spaces, the graph of the mapping $v_{x,y}$ for fixed linearly independent vectors $x, y$ is incorporated in Figure 2.*

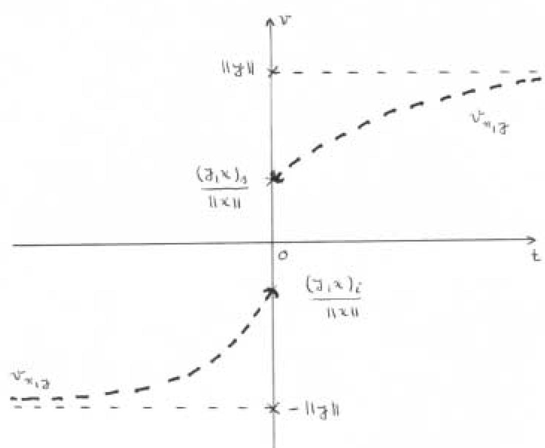


Figure 2.

*Note that if the space* $(X, \|\cdot\|)$ *is smooth in* $x$, *then* $(y,x)_s = (y,x)_i$. *The line* $v = \|y\|$ *in Figure 2 is the asymptotic of* $v$ *at* $t = +\infty$ *and* $v = -\|y\|$ *is the asymptotic for* $t = -\infty$.

## 3. Properties of the Mapping $\delta_{x,y}$

The following theorem contains the main properties of the mapping $\delta_{x,y}$ in the general case of normed linear spaces [**5**].

THEOREM 39. *Let* $(X, \|\cdot\|)$ *be a real normed linear space and* $x, y$ *two fixed vectors in* $X$. *We have:*

(i) $\delta_{x,y}$ *is bounded and*

$$|\delta_{x,y}(t)| \leq \|x\| \quad \text{for all } t \in \mathbb{R}; \tag{7.18}$$

(ii) *If* $x, y$ *are linearly independent, then we have the inequalities:*

$$\delta_{x,y}(t) \leq \Psi^i_{x,y}(t) \leq \Psi^s_{x,y}(t) \leq \|x\| \tag{7.19}$$

*and*

$$\begin{aligned} \delta_{x,y}(t) &\geq \Psi^s_{x,2y}(t) \geq \Psi^i_{x,2y}(t) \geq \|x+2ty\| - 2|t|\|y\| \\ &\geq \begin{cases} \dfrac{(x,y)_s}{\|y\|} & \text{if } t \geq 0, \\ -\dfrac{(x,y)_i}{\|y\|} & \text{if } t < 0. \end{cases} \end{aligned} \tag{7.20}$$

(iii) *The mapping* $\delta_{x,y}$ *is continuous on* $\mathbb{R}$ *and we have the limits:*

$$\lim_{t\to+\infty} \delta_{x,y}(t) = \frac{(x,y)_s}{\|y\|}, \quad \lim_{t\to-\infty} \delta_{x,y}(t) = \frac{-(x,y)_i}{\|y\|}, \tag{7.21}$$

*where* $x, y$ *are linearly independent;*

(iv) *The mapping* $\delta_{x,y}$ *is laterally derivable in each point and if* $x, y$ *are linearly independent, then we have*

$$\frac{d^+\delta_{x,y}(t)}{dt} = 2\left(\Phi^s_{x,y}(t) - \Phi^s_{x,y}(2t)\right) \tag{7.22}$$

*and*

$$\frac{d^-\delta_{x,y}(t)}{dt} = 2\left(\Phi^i_{x,y}(t) - \Phi^i_{x,y}(2t)\right) \tag{7.23}$$

*for all* $t \in \mathbb{R}$;

(v) *The mapping* $\delta_{x,y}$ *is monotonic nondecreasing on* $(-\infty, 0]$ *and monotonic nonincreasing on* $(0, \infty)$.

PROOF. The proof is as follows:

(i) By the continuity property of the norm, we have

$$|\delta_{x,y}(t)| = |\|2x+2ty\| - \|x+2ty\|| \leq \|2x+2ty-x-2ty\| = \|x\|$$

for all $t \in \mathbb{R}$, and the inequality (7.18) is obtained.

(ii) Using Schwartz's inequality and the properties of norm derivatives $(\cdot,\cdot)_p$, we have that

$$\begin{aligned} & \|x+2ty\| \|2x+2ty\| \\ \geq\ & (x+2ty, 2x+2ty)_s = (2x+2ty-x, 2x+2ty)_s \\ =\ & \|2x+2ty\|^2 - (x, 2x+2ty)_i \end{aligned}$$

from where we get

$$\|x+2ty\| - \|2x+2ty\| \geq -\frac{(x,2x+2ty)_i}{\|2x+2ty\|}$$

which is equivalent with

$$2\|x+ty\| - \|x+2ty\| \leq \frac{(x,x+ty)_i}{\|x+ty\|}$$

for all $t \in \mathbb{R}$, and the first inequality in (7.19) is proved. The second inequality is obvious.

The third inequality follows by Schwartz's inequality:

$$(x, x+ty)_s \leq \|x+ty\| \|x\|, \quad t \in \mathbb{R}.$$

To prove the first inequality in (7.20), we also use Schwartz's inequality:

$$\begin{aligned} & \|2x+2ty\| \|x+2ty\| \\ \geq\ & (2x+2ty, x+2ty)_s = (x+x+2ty, x+2ty)_s \\ =\ & \|x+2ty\|^2 + (x, x+2ty)_s \end{aligned}$$

from where we get

$$2\|x+ty\| - \|x+2ty\| \geq \frac{(x,x+2ty)_s}{\|x+2ty\|} = \Psi^s_{x,2y}(t)$$

for all $t \in \mathbb{R}$, and the first inequality in (7.20) is proved. The second inequality is obvious.

By Schwartz's inequality, we also have

$$\begin{aligned} & \|x+2ty\| \|2ty\| \\ \geq\ & (2ty, x+2ty)_s = (x+2ty-x, x+2ty)_s \\ =\ & \|x+2ty\|^2 - (x, x+2ty)_i \end{aligned}$$

from where we get

$$\frac{(x, x+2ty)_i}{\|x+2ty\|} \geq \|x+2ty\| - 2|t|\|y\|$$

for all $t \in \mathbb{R}$ and the third inequality in (7.20) is proved.

Now, suppose that $t \geq 0$. Then

$$\|x+2ty\| - 2|t|\|y\| = \|x+2ty\| - 2t\|y\|.$$

By Schwartz's inequality, we have that

$$\|x+2ty\|\|y\| \geq (x+2ty, y)_s = (x,y)_s + 2t\|y\|^2,$$

from where we get

$$\|x+2ty\| - 2t\|y\| \geq \frac{(x,y)_s}{\|y\|}.$$

If $t < 0$, let $t = -u$ with $u > 0$. Then

$$\|x+2ty\| - 2|t|\|y\| = \|x-2uy\| - 2u\|y\|.$$

By Schwartz's inequality, we also have that

$$\|x-2uy\|\|y\| \geq (x-2uy, y)_s = (x,-y)_s + 2u\|y\|^2$$

from where we get

$$\|x-2uy\| - 2u\|y\| \geq \frac{(x,-y)_s}{\|y\|} = -\frac{(x,y)_i}{\|y\|} \geq -\frac{(x,y)_s}{\|y\|},$$

and the last inequality in (7.20) is also proved.

(iii) The continuity of $\delta_{x,y}$ on $\mathbb{R}$ is obvious.

By the inequalities (7.19) and (7.20) we have that

$$\Psi^s_{x,y}(t) \geq \delta_{x,y}(t) \geq \Psi^s_{x,2y}(t) = \Psi^s_{x,y}(2t).$$

As

$$\lim_{t\to+\infty} \Psi^s_{x,y}(t) = \frac{(x,y)_s}{\|y\|} \quad \text{(see Section 6)}$$

we get the first limit in (7.21). The second limit goes likewise and we shall omit the details.

(iv) The fact that $\delta_{x,y}$ is laterally derivable in each point of $\mathbb{R}$ follows by the same property of the norm $\|\cdot\|$.

We now have

$$\begin{aligned}\frac{d^+\delta_{x,y}(t)}{dt} &= 2\frac{d^+u_{x,y}(t)}{dt} - \frac{d^+u_{x,y}(2t)}{dt} \\ &= 2\left(\Phi^s_{x,y}(t) - \Phi^s_{x,y}(2t)\right)\end{aligned}$$

and similarly,

$$\frac{d^{-}\delta_{x,y}(t)}{dt}=2\left(\Phi_{x,y}^{i}(t)-\Phi_{x,y}^{i}(2t)\right)$$

for all $t\in\mathbb{R}$.

(v) We know that the mappings $\Phi_{x,y}^{p}$, $p\in\{s,i\}$ are monotonic nondecreasing on $\mathbb{R}$ (see Section 5).

If $t<0$, then $2t<t$ and then $\Phi_{x,y}^{p}(t)\geq\Phi_{x,y}^{p}(2t)$ from where we get that

$$\frac{d^{\pm}\delta_{x,y}(t)}{dt}\geq 0 \ \text{ for } \ t\in(-\infty,0).$$

If $t\geq 0$, then $2t\geq t$ and then $\Phi_{x,y}^{p}(2t)\geq\Phi_{x,y}^{p}(t)$ from where we get that

$$\frac{d^{\pm}\delta_{x,y}(t)}{dt}\leq 0 \ \text{ for } \ t\in(-\infty,0).$$

In conclusion, the mapping $\delta_{x,y}$ is monotonic nondecreasing on $(-\infty,0)$ and nonincreasing on $[0,\infty)$.

The theorem is thus proved. ∎

Remark 6. *In what follows, we shall show the approximative graph of the mapping $\delta_{x,y}$ in the general case of normed spaces.*

a) *If we assume that $(x,y)_s\geq 0$, we have the following graph incorporated in Figure 3. We are not sure about the convexity*

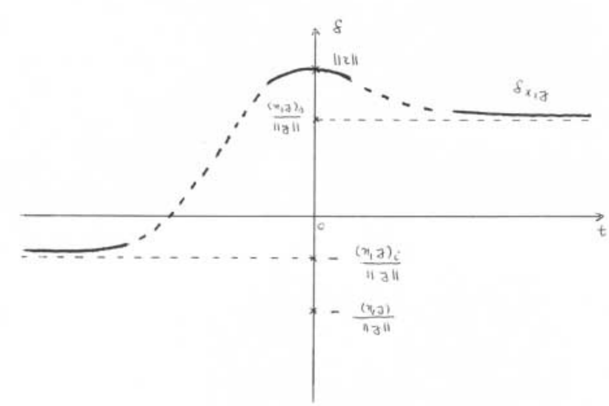

Figure 3.

*of the mapping $\delta_{x,y}$. We know that*

$$-\frac{(x,y)_i}{\|y\|}\geq-\frac{(x,y)_s}{\|y\|}$$

*but* $-(x,y)_i$ *do not always have to be negative.*

b) *If* $(x,y)_s \leq 0$*, then we have the following graph incorprated in Figure 4. We are not sure about the convexity of the mapping*

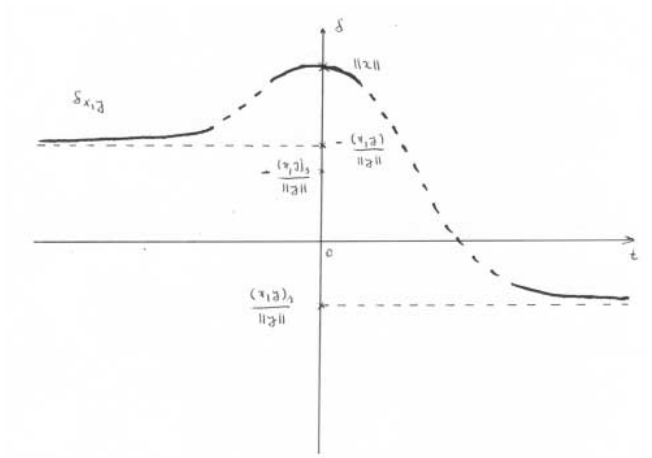

FIGURE 4.

$\delta_{x,y}$*. We know that*

$$-\frac{(x,y)_i}{\|y\|} \geq -\frac{(x,y)_s}{\|y\|}$$

*but* $-(x,y)_i$ *do not have to be positive in each case.*

## 4. Properties of the Mapping $\gamma_{x,y}$

It is natural to consider the following mapping

$$\gamma_{x,y}(t) := \frac{\|x+2ty\| - \|x+ty\|}{t}, \quad t \in \mathbb{R} \setminus \{0\},$$

where $x, y$ are two fixed elements in $X$.

The main properties of this mapping are embodied in the following theorem [**2**].

THEOREM 40. *Let* $(X, \|\cdot\|)$ *be a real normed linear space and* $x, y$ *two fixed vectors in* $X$*. We have:*

(i) *The mapping* $\gamma_{x,y}$ *is bounded on* $\mathbb{R} \setminus \{0\}$ *and*

$$\left|\gamma_{x,y}(t)\right| \leq \|y\| \quad \text{for all } t \in \mathbb{R} \setminus \{0\}; \tag{7.24}$$

(ii) *If* $x, y$ *are linearly independent, then we have the inequalities:*

$$-\|y\| \leq \gamma_{x,y}(u) \leq \Phi^i_{x,y}(u) \quad \text{for all} \quad u < 0 \tag{7.25}$$

*and*

$$\|y\| \geq \gamma_{x,y}(t) \geq \Phi^s_{x,y}(t) \quad \text{for all } t > 0; \tag{7.26}$$

(iii) *The mapping $\gamma_{x,y}$ is continuous on $\mathbb{R} \diagdown \{0\}$ and we have the limits;*

$$\lim_{u \to 0-} \gamma_{x,y}(u) = \frac{(y,x)_i}{\|y\|}, \quad \lim_{u \to 0+} \gamma_{x,y}(t) = \frac{(y,x)_s}{\|y\|} \tag{7.27}$$

*and*

$$\lim_{u \to -\infty} \gamma_{x,y}(u) = -\|y\|, \quad \lim_{t \to +\infty} \gamma_{x,y}(t) = \|y\| \tag{7.28}$$

*if $x, y$ are linearly independent;*

(iv) *We have the inequalities:*

$$\gamma_{x,y}(t) \leq \Phi^i_{\frac{x}{2},y}(t) \leq \Phi^s_{\frac{x}{2},y}(t) \leq \|y\| \quad \text{for all } t > 0 \tag{7.29}$$

*and*

$$\gamma_{x,y}(u) \geq \Phi^s_{\frac{x}{2},y}(u) \leq \Phi^i_{\frac{x}{2},y}(u) \geq -\|y\| \quad \text{for all } u < 0 \tag{7.30}$$

*if $x, y$ are linearly independent;*

(v) *The mapping $\gamma_{x,y}$ has one sided derivatives at each point of $\mathbb{R} \diagdown \{0\}$ and, if $x, y$ are linearly independent, then*

$$\frac{d^- \gamma_{x,y}(t)}{dt} = \begin{cases} \frac{1}{t^2}\left[\Psi^i_{x,y}(t) - \Psi^i_{x,y}(2t)\right] & \text{if } t < 0 \\ \frac{1}{t^2}\left[\Psi^s_{x,y}(t) - \Psi^s_{x,y}(2t)\right] & \text{if } t > 0 \end{cases} \tag{7.31}$$

*and*

$$\frac{d^+ \gamma_{x,y}(t)}{dt} = \begin{cases} \frac{1}{t^2}\left[\Psi^s_{x,y}(t) - \Psi^s_{x,y}(2t)\right] & \text{if } t < 0 \\ \frac{1}{t^2}\left[\Psi^i_{x,y}(t) - \Psi^i_{x,y}(2t)\right] & \text{if } t > 0; \end{cases} \tag{7.32}$$

(vi) *The mapping $\gamma_{x,y}$ is monotonic nondecreasing on $\mathbb{R} \diagdown \{0\}$.*

Proof. The proof is as follows.

(i) By the continuity property of the norm, we have that

$$|\|x + 2ty\| - \|x + ty\|| \leq \|x + 2ty - x - ty\| = |t| \|y\|, \quad t \in \mathbb{R}$$

from where results the inequality (7.24).

(ii) By Schwartz's inequality and by the properties of the norm derivatives $(\cdot,\cdot)_p$, we have that:

$$\begin{aligned} \|x + 2uy\| \|x + uy\| &\geq (x + 2uy, x + uy)_s \\ &= (x + uy + uy, x + uy)_s \\ &= \|x + uy\|^2 + u(y, x + uy)_i \end{aligned}$$

from where we get

$$(\|x+2uy\| - \|x+uy\|)\,\|x+uy\| \geq u\,(y, x+uy)_i$$

for all $u < 0$, which implies

$$\frac{\|x+2uy\| - \|x+uy\|}{u} \leq \frac{u\,(y, x+uy)_i}{\|x+uy\|}$$

and the second inequality in (7.25) is proved.

The second inequality in (7.26) goes likewise and we shall omit the details.

(iii) The continuity of the mapping $\gamma_{x,y}$ on $\mathbb{R}\diagdown\{0\}$ is obvious.

We have:

$$\begin{aligned}
\lim_{u\to 0-} \gamma_{x,y}(u) &= \lim_{u\to 0-} \frac{\|x+2uy\| - \|x\| - (\|x+uy\| - \|x\|)}{u} \\
&= 2\lim_{u\to 0-} \frac{\|x+2uy\| - \|x\|}{u} - \lim_{u\to 0-} \frac{\|x+uy\| - \|x\|}{u} \\
&= 2\frac{(y,x)_i}{\|x\|} - \frac{(y,x)_i}{\|x\|} = \frac{(y,x)_i}{\|x\|}
\end{aligned}$$

and the first limit in (7.27) is obtained.

The second limit goes likewise and we shall omit the details.

We have

$$\begin{aligned}
\lim_{u\to -\infty} \gamma_{x,y}(u) &= \lim_{u\to -\infty} \frac{\|x+2uy\| - \|x+uy\|}{u} \\
&= \lim_{u\to -\infty} (-u)\,\frac{\left[\left\|-\frac{1}{u}x - 2y\right\| - \left\|-\frac{1}{u}x - y\right\|\right]}{u} \\
&= -\lim_{u\to -\infty} \left(\left\|2y + \frac{1}{u}x\right\| - \left\|y + \frac{1}{u}x\right\|\right) \\
&= -\lim_{\alpha\to 0^-} (\|2y+\alpha x\| - \|y+\alpha x\|) \\
&= -2\|y\| + \|y\| = -\|y\|
\end{aligned}$$

and the first limit in (7.28) is obtained.

The second limit goes likewise and we shall omit the details.

(iv) We shall prove the inequality (7.30).

By Schwartz's inequality, we have that

$$\begin{aligned}
\|x+2uy\|\,\|x+uy\| &\geq (x+uy, x+2uy)_s \\
&= (x+2uy-uy, x+2uy)_s \\
&= \|x+2uy\|^2 - u\,(y, x+2uy)_s\,,
\end{aligned}$$

which is equivalent with

$$u\,(y, x+2uy)_s \geq \|x+2uy\|\,(\|x+2uy\| - \|x+uy\|)$$

that is,

$$\frac{(y,x+2uy)_s}{\|x+2uy\|} \leq \frac{\|x+2uy\| - \|x+uy\|}{u} \quad \text{for all } u<0.$$

However,

$$\Phi^s_{\frac{x}{2},y}(u) = \Phi^s_{\frac{x}{2},y}(2u) = \frac{(y,x+2uy)_s}{\|x+2uy\|}$$

and the first inequality in (7.30) is proved.

The second and third inequalities are obvious, and the statement is proved.

The inequality (7.29) goes likewise and we shall omit the details.

(v) The fact that the mapping $\gamma_{x,y}$ has one sided derivatives at each point of $\mathbb{R}\diagdown\{0\}$ is obvious.

We have

$$\begin{aligned}
\frac{d^+\gamma_{x,y}(t)}{dt} &= \frac{d^+}{dt}\left(\frac{u_{x,y}(2t)-u_{x,y}(t)}{t}\right) \\
&= \frac{1}{t^2}\left[\frac{d^+}{dt}\left(u_{x,y}(2t)-u_{x,y}(t)\right)t - \left(u_{x,y}(2t)-u_{x,y}(t)\right)\right] \\
&= \frac{1}{t^2}\left[\left(2\frac{d^+u_{x,y}(2t)}{dt} - \frac{d^+u_{x,y}(t)}{dt}\right)t - u_{x,y}(2t)+u_{x,y}(t)\right] \\
&= \frac{1}{t^2}\left[\left(2\Phi^s_{x,y}(2t)-\Phi^s_{x,y}(t)\right)t - u_{x,y}(2t)+u_{x,y}(t)\right] \\
&= \frac{1}{t^2}\left[2t\Phi^s_{x,y}(2t)-u_{x,y}(2t)-\left(t\Phi^s_{x,y}(t)-u_{x,y}(t)\right)\right]
\end{aligned}$$

for all $t\in\mathbb{R}$.

If $t>0$, we have that (see Proposition 27)

$$n_{x,y}(2t)-2t\Phi^s_{x,y}(2t) = \Psi^i_{x,y}(2t)$$

and

$$n_{x,y}(t)-t\Phi^s_{x,y}(t) = \Psi^i_{x,y}(t).$$

If $t<0$, we know that

$$n_{x,y}(2t)-2t\Phi^s_{x,y}(2t) = \Psi^s_{x,y}(2t)$$

and

$$n_{x,y}(t)-t\Phi^s_{x,y}(t) = \Psi^s_{x,y}(t).$$

Thus,

$$\frac{d^+\gamma_{x,y}(t)}{dt} = \begin{cases} \frac{1}{t^2}\left[\Psi^i_{x,y}(t)-\Psi^i_{x,y}(2t)\right] & \text{if} \quad t>0 \\ \\ \frac{1}{t^2}\left[\Psi^s_{x,y}(t)-\Psi^s_{x,y}(2t)\right] & \text{if} \quad t<0 \end{cases}$$

The equality (7.32) goes likewise, and we shall omit the details.

(vi) We know that the mapping $\Psi_{x,y}^{p}$, $p \in \{s, i\}$ are nondecreasing on $(-\infty, 0]$ and nonincreasing on $(0, +\infty)$.

If $t < 0$, then $2t < t$ and then $\Psi_{x,y}^{i}(t) > \Psi_{x,y}^{i}(2t)$ which gives us that

$$\frac{d^{+}\gamma_{x,y}(t)}{dt} \geq 0 \text{ if } t < 0.$$

If $t > 0$, then $2t > t$ and $\Psi_{x,y}^{s}(t) > \Psi_{x,y}^{s}(2t)$ which gives us that

$$\frac{d^{+}\gamma_{x,y}(t)}{dt} \geq 0 \text{ if } t > 0.$$

In conclusion, the mapping $\gamma_{x,y}$ is monotonic nondecreasing on $\mathbb{R} \diagdown \{0\}$. ∎

REMARK 7. *In the general case of normed linear spaces, the graph of $\gamma_{x,y}$ is as follows (see Figure 5). We are not sure about the convexity*

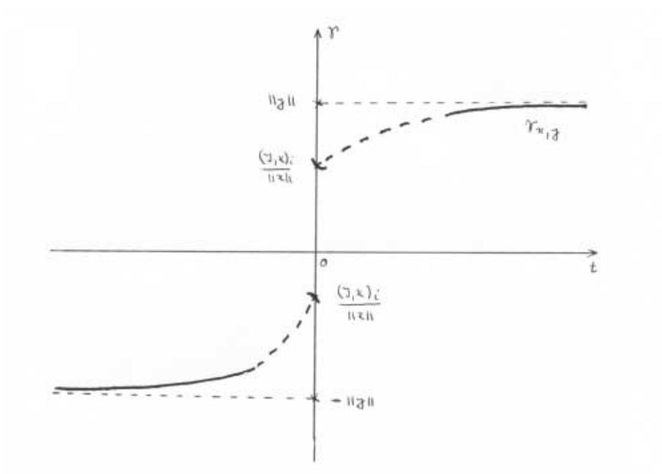

FIGURE 5.

*of $\gamma_{x,y}$.*

## 5. Properties of $\Phi_{x,y}^{p}$ Mappings

For two linearly independent vectors in $X$, $x, y$, we consider the mapping

$$\Phi_{x,y}^{p}(t) := \frac{(y, x+ty)_{p}}{\|x+ty\|}, \quad p = s \text{ or } p = i;$$

which is well defined for all $t \in \mathbb{R}$.

The main properties of these mappings are embodied in the following theorem [**1**].

Theorem 41. *Let $(X, \|\cdot\|)$ be a real normed linear space and $x, y$ two linearly independent vectors in $X$. Then*

(i) *The mapping $\Phi^p_{x,y}$ is bounded on $\mathbb{R}$ and*

$$\left|\Phi^p_{x,y}(t)\right| \leq \|y\| \quad \text{for all } t \in \mathbb{R}; \tag{7.33}$$

(ii) *We have the inequalities*

$$\gamma_{x,y}(u) \leq \Phi^i_{x,y}(u) \leq \Phi^s_{x,y}(u) \leq v_{x,y}(u) \quad \text{for all } u < 0 \tag{7.34}$$

*and*

$$\gamma_{x,y}(t) \geq \Phi^i_{x,y}(t) \geq \Phi^i_{x,y}(t) \geq v_{x,y}(t) \quad \text{for all } t > 0; \tag{7.35}$$

(iii) *The mappings $\Phi^p_{x,y}$ are monotonic nondecreasing on $\mathbb{R}$;*

(iv) *We have the limits:*

$$\lim_{u \to -\infty} \Phi^p_{x,y}(u) = -\|y\| \quad \text{and} \quad \lim_{t \to +\infty} \Phi^p_{x,y}(t) = \|y\| \tag{7.36}$$

*and*

$$\lim_{t \to 0^+} \Phi^p_{x,y}(t) = \frac{(y,x)_s}{\|x\|}, \quad \lim_{u \to 0^-} \Phi^p_{x,y}(u) = \frac{(y,x)_i}{\|x\|}; \tag{7.37}$$

(v) *The mapping $\Phi^s_{x,y}$ is right continuous in every point of $\mathbb{R}$ and $\Phi^i_{x,y}$ is left continuous.*

Proof. The proof is as follows.

(i) Follows by the Schwartz inequality.

(ii) The first inequalities in (7.34) and (7.35) were proved in Theorem 40.

The last inequalities in (7.34) and (7.35) were proved in Theorem 38.

(iii) Suppose that $p \in \{s, i\}$ and $t_2 > t_1$. Then, by Schwartz's inequality, we have that

$$\|x + t_2 y\| \|x + t_1 y\| \geq (x + t_2 y, x + t_1 y)_p$$

for all $x, y \in X$.

Using the properties of the norm derivatives, we get that

$$\begin{aligned} (x + t_2 y, x + t_1 y)_p &= ((t_2 - t_1) y + x + t_1 y, x + t_1 y)_p \\ &= \|x + t_1 y\|^2 + (t_2 - t_1)(y, x + t_1 y)_p \end{aligned}$$

and thus, by the above inequality, we deduce

$$\|x + t_2 y\| \|x + t_1 y\| \geq \|x + t_1 y\|^2 + (t_2 - t_1)(y, x + t_1 y)_p,$$

from where we get

$$\Phi^p_{x,y}(t_1) = \frac{(y, x + t_1 y)_p}{\|x + t_1 y\|} \le \frac{\|x + t_2 y\| - \|x + t_1 y\|}{t_2 - t_1}.$$

Now, let us put $t := t_2 - t_1 > 0$. Then by (7.35) we have that:

$$\begin{aligned} &\frac{\|x + t_2 y\| - \|x + t_1 y\|}{t_2 - t_1} \\ = &\frac{\|x + t_1 y + ty\| - \|x + t_1 y\|}{t_2 - t_1} \le \Phi^p_{x+t_1 y,y}(t) \\ = &\frac{(y, x + t_1 y + ty)_p}{\|x + t_1 y + ty\|} = \frac{(y, x + t_2 y)_p}{\|x + t_2 y\|} = \Phi^p_{x,y}(t_2) \end{aligned}$$

and the statement is proved.

(iv) We know from Theorem 38 that

$$\lim_{u \to -\infty} \gamma_{x,y}(u) = -\|y\| \quad \text{and} \quad \lim_{t \to +\infty} \gamma^p_{x,y}(t) = \|y\|$$

and from Theorem 40 that

$$\lim_{u \to -\infty} \gamma_{x,y}(u) = -\|y\| \quad \text{and} \quad \lim_{t \to +\infty} \gamma_{x,y}(t) = \|y\|.$$

Using the inequalities (7.34) and (7.35) we deduce the desired limits (7.36).

The proof of limits (7.37) go likewise, and we shall omit the details.

(v) Let $t_0 \in \mathbb{R}$. Then we have

$$\begin{aligned} \lim_{\alpha \to t_0+} \Phi^p_{x,y}(\alpha) &= \lim_{t \to 0^+} \Phi^p_{x,y}(t_0 + t) = \lim_{t \to 0^+} \frac{(y, x + t_0 y + ty)}{\|x + t_0 y + ty\|} \\ &= \lim_{t \to 0^+} \Phi^p_{x+t_0 y,y}(t) = \frac{(y, x + t_0 y)}{\|x + t_0 y\|} = \Phi^p_{x,y}(t_0) \end{aligned}$$

and the right continuity is proved.

The proof of left continuity goes likewise and we shall omit the details.

∎

Remark 8. *In the general case of normed linear spaces, the graphs of $\Phi^i_{x,y}$ and $\Phi^s_{x,y}$ are incorporated in Figure 6. We are not sure about the convexity of the mappings $\Phi^p_{x,y}$.*

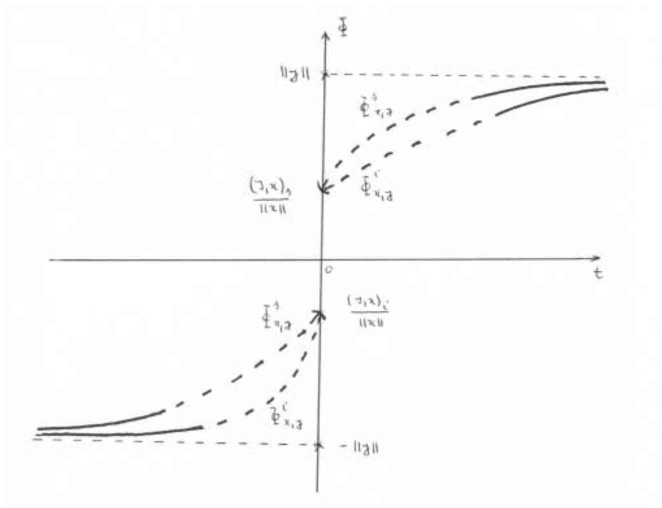


FIGURE 6.

## 6. Properties of the Mappings $\Psi^p_{x,y}$

Let $x, y$ be two fixed linearly independent vectors in the normed linear space $(X, \|\cdot\|)$. Consider the mapping

$$\Phi^p_{x,y}(t) := \frac{(x, x + ty)_p}{\|x + ty\|}, \quad p \in \{s, i\}$$

which is well defined for all $t \in \mathbb{R}$.

The main properties of these mappings are embodied in the following theorem [**4**].

THEOREM 42. *With the above assumptions, we have:*

(i) *The mapping $\Psi^p_{x,y}$ is bounded on $\mathbb{R}$ and we have the inequality*

$$\left|\Psi^p_{x,y}(t)\right| \leq \|x\| \quad \textit{for all } t \in \mathbb{R}; \tag{7.38}$$

(ii) *We have the inequalities:*

$$\delta_{x,t}(t) \leq \Psi^i_{x,y}(t) \leq \Psi^s_{x,y}(t) \leq \|x\| \tag{7.39}$$

*and*

$$\begin{aligned}
\delta_{x,t}(t) &\geq \Psi^s_{x,y}(t) \geq \Psi^i_{x,y}(t) \\
&\geq \|x - 2ty\| - 2|t|\|y\| \\
&\geq \begin{cases} \frac{(x,y)_s}{\|y\|} & \textit{if} \quad t \geq 0 \\ \\ \frac{-(x,y)_i}{\|y\|} & \textit{if} \quad t < 0 \end{cases}
\end{aligned}$$

*for all* $t \in \mathbb{R}$*;*

(iii) $\Psi^p_{x,y}$ *is continuous in 0 and we have the limits:*

$$\lim_{t\to\infty} \Psi^p_{x,y}(t) = \frac{(x,y)_s}{\|y\|} \tag{7.40}$$

*and*

$$\lim_{u\to-\infty} \Psi^p_{x,y}(u) = \frac{-(x,y)_i}{\|y\|} \tag{7.41}$$

(iv) $\Psi^p_{x,y}$ *is monotonic nondecreasing on* $(-\infty,0]$ *and nonincreasing on* $[0,\infty)$.

Proof. The proof is as follows.

(i) Goes by the Schwartz inequality.
(ii) Were proved in Theorem 39.
(iii) We know that

$$\lim_{t\to 0} \delta_{x,t}(t) = \|x\|.$$

Then by the inequality (7.39) the limits $\lim\limits_{t\to 0} \Psi^p_{x,y}(t)$ exist and are equal to $\|x\|$.

We have

$$\begin{aligned}\lim_{t\to+\infty} \Psi^p_{x,y}(t) &= \lim_{t\to+\infty} \frac{(x,x+ty)_p}{\|x+ty\|} = \lim_{t\to+\infty} \frac{t\left(x,\frac{1}{t}x+y\right)_p}{t\left\|y+\frac{1}{t}x\right\|} \\ &= \lim_{\alpha\to 0+} \frac{(x,y+\alpha x)_p}{\|y+\alpha x\|} = \lim_{\alpha\to 0+} \Phi^p_{y,x}(\alpha).\end{aligned}$$

By Theorem 42, we have that:

$$\lim_{\alpha\to 0+} \Phi^p_{y,x}(\alpha) = \frac{(x,y)_s}{\|y\|}$$

and the limit (7.40) is obtained.

On the other hand, we also have

$$\begin{aligned}\lim_{u\to-\infty} \Psi^p_{x,y}(u) &= \lim_{u\to-\infty} \frac{(x,x+uy)_p}{\|x+uy\|} = \lim_{u\to-\infty} \frac{(-u)\left(x,-y-\frac{1}{u}x\right)_p}{(-u)\left\|-y-\frac{1}{u}x\right\|} \\ &= \lim_{u\to-\infty} \frac{-\left(x,y+\frac{1}{u}x\right)_q}{\left\|y+\frac{1}{u}x\right\|} = -\lim_{\beta\to 0-} \frac{(x,y+\beta x)_q}{\|y+\beta x\|} \\ &= \lim_{\beta\to 0-} \Phi^q_{y,x}(\beta) = \frac{(x,y)_i}{\|y\|}.\end{aligned}$$

By Theorem 42, we have that

$$\lim_{\beta \to 0-} \Phi_{y,x}^{q}(\beta) = \frac{(x,y)_i}{\|y\|},$$

and the limit (7.41) is obtained.

(iv) Let $-\infty < t_1 < t_2 \le 0$. By Proposition 27, we have:

$$\begin{aligned} &\frac{\Psi_{x,y}^{p}(t_2) - \Psi_{x,y}^{p}(t_1)}{t_2 - t_1} \\ &= \frac{n_{x,y}(t_2) - t_2 \Phi_{x,y}^{p}(t_2) - n_{x,y}(t_1) + t_1 \Phi_{x,y}^{p}(t_1)}{t_2 - t_1} \\ &= \frac{n_{x,y}(t_2) - n_{x,y}(t_1)}{t_2 - t_1} - \frac{t_2 \Phi_{x,y}^{p}(t_2) - t_1 \Phi_{x,y}^{p}(t_1)}{t_2 - t_1}. \end{aligned}$$

In Theorem 42, we proved among others that the following inequality holds

$$\Phi_{x,y}^{p}(t_1) \le \frac{n_{x,y}(t_2) - n_{x,y}(t_1)}{t_2 - t_1} \le \Phi_{x,y}^{p}(t_2) \tag{7.42}$$

where $t_1,\, t_2 \in \mathbb{R}$ and $t_1 < t_2$.

Using (7.42), we have that

$$\begin{aligned} &\frac{\Psi_{x,y}^{p}(t_2) - \Psi_{x,y}^{p}(t_1)}{t_2 - t_1} \\ \ge\ & \Phi_{x,y}^{p}(t_1) - \frac{t_2 \Phi_{x,y}^{p}(t_2) - t_1 \Phi_{x,y}^{p}(t_1)}{t_2 - t_1} \\ =\ & \frac{(t_2 - t_1)\Phi_{x,y}^{p}(t_1) - t_2 \Phi_{x,y}^{p}(t_2) + t_1 \Phi_{x,y}^{p}(t_1)}{t_2 - t_1} \\ =\ & \frac{t_2 \left(\Phi_{x,y}^{p}(t_1) - \Phi_{x,y}^{p}(t_2)\right) + t_1 \left(\Phi_{x,y}^{p}(t_1) - \Phi_{x,y}^{p}(t_2)\right)}{t_2 - t_1} \\ =\ & \frac{\left(\Phi_{x,y}^{p}(t_1) - \Phi_{x,y}^{p}(t_2)\right)(t_1 + t_2)}{t_2 - t_1}. \end{aligned}$$

However,

$$\Phi_{x,y}^{p}(t_1) \le \Phi_{x,y}^{p}(t_2),\ t_1 + t_2 \le 0 \ \text{ and } \ t_2 > t_1$$

then

$$\frac{\Psi_{x,y}^{p}(t_2) - \Psi_{x,y}^{p}(t_1)}{t_2 - t_1} \ge 0$$

which shows the monotonicity of $\Psi_{x,y}^{p}$ on $(-\infty, 0]$.

Let $+\infty > t_2 > t_1 \geq 0$. Then, by Proposition 27, we have

$$\begin{aligned}
& \frac{\Psi^p_{x,y}(t_2) - \Psi^p_{x,y}(t_1)}{t_2 - t_1} \\
= \ & \frac{n_{x,y}(t_2) - n_{x,y}(t_1)}{t_2 - t_1} - \frac{t_2 \Phi^q_{x,y}(t_2) - t_1 \Phi^q_{x,y}(t_1)}{t_2 - t_1} \\
\leq \ & \Phi^q_{x,y}(t_2) - \frac{t_2 \Phi^q_{x,y}(t_2) - t_1 \Phi^q_{x,y}(t_1)}{t_2 - t_1} \\
= \ & \frac{(t_2 - t_1)\Phi^q_{x,y}(t_2) - t_2 \Phi^q_{x,y}(t_2) + t_1 \Phi^q_{x,y}(t_1)}{t_2 - t_1} = 0
\end{aligned}$$

and the monotonicity of $\Psi^p_{x,y}$ on $[0, \infty)$ is proved. ∎

REMARK 9. *In the general case of normed linear spaces, the graph of the mapping $\Psi^p_{x,y}$ are incorporated in Figure 7 and Figure 8.*

a) *If $(x, y)_s \geq 0$, then we have the graph below (see Figure 7). We know that*

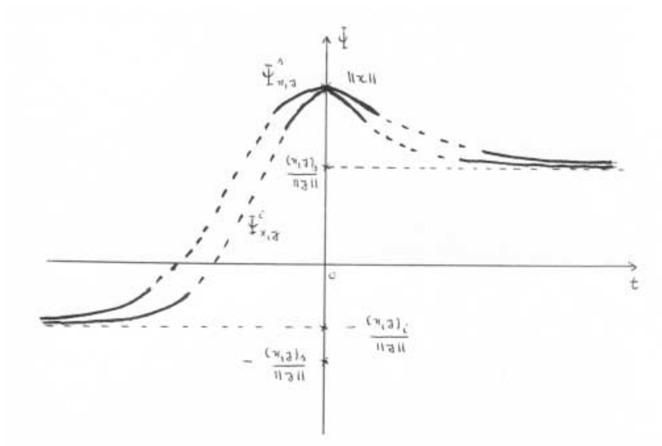

FIGURE 7.

$$\frac{-(x,y)_i}{\|y\|} \geq \frac{-(x,y)_s}{\|y\|}$$

*but $-(x,y)_i$ do not always have to be negative.*
*We are not sure about the convexity of the mappings $\Psi^p_{x,y}$, $p = s$ or $p = i$.*

b) *If $(x, y)_s \leq 0$, then we have the following graph (see Figure 8). We know that*

$$\frac{-(x,y)_i}{\|y\|} \geq \frac{-(x,y)_s}{\|y\|}$$

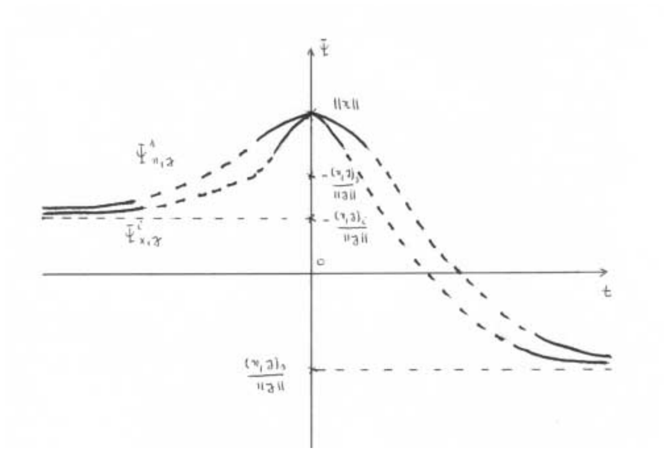

FIGURE 8.

*but* $-(x,y)_i$ *do not always have to be positive.*

## 7. The Case of Inner Products

In this section we will investigate the properties of the mappings $n_{x,y}$, $\delta_{x,y}$, $v_{x,y}$, $\gamma_{x,y}$, $\Phi^p_{x,y}$ and $\Psi^p_{x,y}$ in the particular case of inner product spaces.

The following proposition holds [**3**].

THEOREM 43. *If* $(X;(\cdot,\cdot))$ *is a real linear inner product space, then the mapping* $v_{x,y}$ *is convex on* $(0,\infty)$*, where* $x,y$ *are fixed linearly independent vectors in* $X$*.*

PROOF. If $(X;(\cdot,\cdot))$ is an inner product space, then $v_{x,y}$ is derivable on $\mathbb{R}\diagdown\{0\}$ and

$$\frac{dv_{x,y}(t)}{dt}=\frac{1}{t^2}\left[t\frac{(y,x)+t\|y\|^2}{u_{x,y}(t)}-n_{x,y}(t)+\|x\|\right].$$

The second derivative of $v_{x,y}$ also exists and

$$\frac{d^2v_{x,y}(t)}{dt^2}=\frac{I}{t^4n^2_{x,y}(t)},$$

where

$$
\begin{aligned}
I &= \frac{d}{dt}\left(t\left(y,x\right)+t^{2}\left\|y\right\|^{2}-n_{x,y}^{2}\left(t\right)+\left\|x\right\|n_{x,y}\left(t\right)\right)t^{2}n_{x,y}\left(t\right)\\
&\quad-\left(t\left(y,x\right)+t^{2}\left\|y\right\|^{2}-n_{x,y}^{2}\left(t\right)+\left\|x\right\|n_{x,y}\left(t\right)\right)\frac{d}{dt}\left(t^{2}n_{x,y}\left(t\right)\right)\\
&= \left(\left(y,x\right)+2t\left\|y\right\|^{2}-2u_{x,y}\left(t\right)n_{x,y}'\left(t\right)+\left\|x\right\|n_{x,y}'\left(t\right)\right)t^{2}n_{x,y}\left(t\right)\\
&\quad-\left(t\left(y,x\right)+t^{2}\left\|y\right\|^{2}-n_{x,y}^{2}\left(t\right)+\left\|x\right\|n_{x,y}\left(t\right)\right)\left(2tn_{x,y}\left(t\right)+t^{2}n_{x,y}'\left(t\right)\right)\\
&= -t^{2}\left(y,x\right)n_{x,y}\left(t\right)-t^{2}n_{x,y}^{2}\left(t\right)n_{x,y}'\left(t\right)+2tn_{x,y}^{3}\left(t\right)-2t\left\|x\right\|n_{x,y}^{2}\left(t\right)\\
&\quad-t^{3}n_{x,y}'\left(t\right)-t^{4}\left\|y\right\|^{2}n_{x,y}'\left(t\right)\\
&= -t^{2}\left(y,x\right)n_{x,y}\left(t\right)+2tn_{x,y}^{3}\left(t\right)-2t\left\|x\right\|n_{x,y}^{2}\left(t\right)\\
&\quad-t^{2}n_{x,y}^{2}\left(t\right)\frac{\left(y,x\right)+t\left\|y\right\|^{2}}{n_{x,y}\left(t\right)}-t^{3}\left(y,x\right)\frac{\left(\left(y,x\right)+t\left\|y\right\|^{2}\right)}{n_{x,y}\left(t\right)}\\
&\quad-t^{4}\left\|y\right\|^{2}\frac{\left(y,x\right)+t\left\|y\right\|^{2}}{n_{x,y}\left(t\right)}=\frac{J}{n_{x,y}\left(t\right)},
\end{aligned}
$$

where

$$
\begin{aligned}
J &= -t^{2}\left(y,x\right)n_{x,y}^{2}\left(t\right)+2tn_{x,y}^{4}\left(t\right)-2t\left\|x\right\|n_{x,y}^{3}\left(t\right)\\
&\quad-t^{2}n_{x,y}^{2}\left(t\right)\left(y,x\right)-t^{3}n_{x,y}^{2}\left(t\right)\left\|y\right\|^{2}-t^{3}\left(y,x\right)^{2}-t^{4}\left\|y\right\|^{2}\left(y,x\right)\\
&\quad-t^{4}\left\|y\right\|^{2}\left(y,x\right)-t^{5}\left\|y\right\|^{4}.
\end{aligned}
$$

However,

$$
\begin{aligned}
&-t^{5}\left\|y\right\|^{4}-t^{4}\left\|y\right\|^{2}\left(y,x\right)-t^{4}\left\|y\right\|^{2}\left(y,x\right)-t^{3}\left(y,x\right)^{2}\\
&\qquad\qquad=-t^{3}\left(t\left\|y\right\|^{2}+\left(y,x\right)\right)^{2}=-t^{3}\left(y,x+ty\right)^{2}
\end{aligned}
$$

and

$$
\begin{aligned}
&2tn_{x,y}^{4}\left(t\right)-2t\left\|x\right\|n_{x,y}^{3}\left(t\right)-2t^{2}\left(y,x\right)n_{x,y}^{2}\left(t\right)-t^{3}n_{x,y}^{2}\left(t\right)\left\|y\right\|^{2}\\
&=tn_{x,y}^{2}\left(t\right)\left(2n_{x,y}^{2}\left(t\right)+2\left\|x\right\|n_{x,y}\left(t\right)-2t\left(y,x\right)-t^{2}\left\|y\right\|^{2}\right)\\
&=tn_{x,y}^{2}\left(t\right)\left(n_{x,y}^{2}\left(t\right)+\left\|x+ty\right\|^{2}-2\left\|x\right\|n_{x,y}\left(t\right)-2t\left(y,x\right)-t^{2}\left\|y\right\|^{2}\right)\\
&=tn_{x,y}^{2}\left(t\right)\left(n_{x,y}^{2}\left(t\right)+\left\|x\right\|^{2}+2t\left(y,x\right)+t^{2}\left\|y\right\|^{2}\right.\\
&\qquad\qquad\left.-2\left\|x\right\|n_{x,y}\left(t\right)-2t\left(y,x\right)-t^{2}\left\|y\right\|^{2}\right)\\
&=tn_{x,y}^{2}\left(t\right)\left(n_{x,y}\left(t\right)-\left\|x\right\|\right)^{2}.
\end{aligned}
$$

In conclusion, we obtain

$$
\frac{d^{2}v_{x,y}\left(t\right)}{dt^{2}}=t\cdot\frac{u_{x,y}^{2}\left(t\right)\left(n_{x,y}\left(t\right)-\left\|x\right\|\right)^{2}-t^{2}\left(y,x+ty\right)^{2}}{t^{4}n_{x,y}^{3}\left(t\right)},\quad t\in\mathbb{R}\diagdown\left\{0\right\}.
$$

Using known inequalities

$$\frac{(y, x+ty)}{\|x+ty\|} \leq \frac{\|x+ty\| - \|x\|}{t} \quad \text{if} \quad t < 0$$

and

$$\frac{\|x+ty\| - \|x\|}{t} \leq \frac{(y, x+ty)}{\|x+ty\|} \quad \text{if} \quad t > 0.$$

We have for all $t \in \mathbb{R} \setminus \{0\}$ that

$$\left| \frac{\|x+ty\| - \|x\|}{t} \right| \leq \frac{|(y, x+ty)|}{\|x+ty\|}$$

from where results

$$n_{x,y}^2 (t) \left(n_{x,y} (t) - \|x\|\right)^2 \leq t^2 (y, x+ty)^2$$

which shows us that

$$\frac{d^2 v_{x,y} (t)}{dt^2} \geq 0 \quad \text{if} \quad t < 0$$

and

$$\frac{d^2 v_{x,y} (t)}{dt^2} \leq 0 \quad \text{if} \quad t > 0$$

and the proposition is proved. ∎

Remark 10. *If we assume for the mapping $n_{x,y}$ that $(X; (\cdot, \cdot))$ is an inner product space, then we can provide more information (see Figure 9). Here the mapping $n_{x,y}$ is strictly convex, has a unique minimum in*

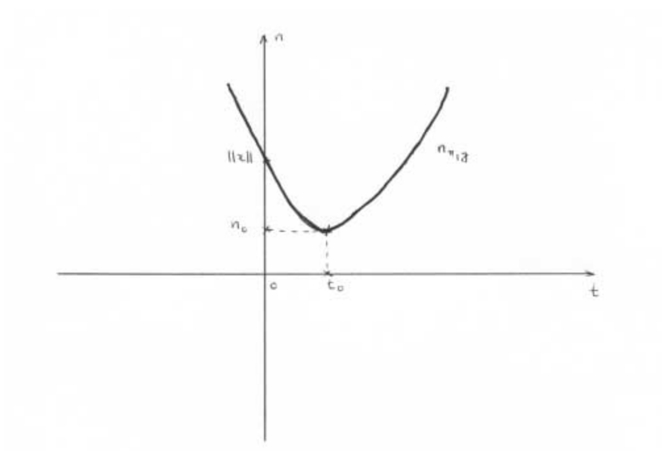


Figure 9.

$t_0 = -\frac{(y,x)}{\|y\|^2}$ *and*

$$n_0 := n_{x,y}(t_0) = \frac{\left(\|x\|^2 \|y\|^2 - (x,y)^2\right)^{\frac{1}{2}}}{\|y\|}.$$

*Indeed,*

$$\frac{dn_{x,y}(t)}{dt} = \Phi_{x,y}(t) = \frac{(y, x+ty)}{\|x+ty\|}$$

*and*

$$\frac{dn_{x,y}(t)}{dt} = 0 \quad \text{iff} \quad t = t_0$$

*and*

$$n_{x,y}(t_0) = \left\| x - \frac{(y,x)_s}{\|y\|^2} \right\| = \frac{\left(\|x\|^2 \|y\|^2 - (x,y)^2\right)^{\frac{1}{2}}}{\|y\|}.$$

REMARK 11. *If we assume that $(X;(\cdot,\cdot))$ is an inner product space, then $v_{x,y}$ is strictly convex and monotonic increasing on $(-\infty,0)$ and strictly concave and monotonic increasing on $(0,\infty)$.*
*The line $v = \|y\|$ is an asymptote at $t = \infty$ and the line $v = -\|y\|$ is an asymptote at $t = -\infty$ (see Figure 10).*

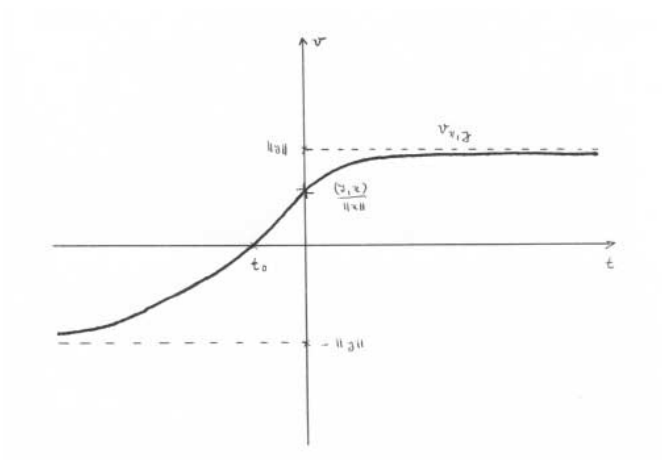

FIGURE 10.

*Note that $v_{x,y}(t) = 0$ iff $\|x+ty\| = \|x\|$, $t \neq 0$, i.e.,*

$$\|x\|^2 + 2t(y,x) = t^2 \|y\|^2 = \|x\|^2$$

*from where we get*

$$t_0 = -2\frac{(y,x)}{\|y\|^2}$$

*is the point where the graph of $v_{x,y}$ intersects the $t$ axis.*

We shall now investigate the function $\delta_{x,y}$ in the case of inner products [**5**].

THEOREM 44. *Let $(X;(\cdot,\cdot))$ be an inner product space over the real number field $\mathbb{R}$. The mapping $\delta_{x,y}$ is twice differentiable on $\mathbb{R}$ and*

$$\frac{d^2\delta_{x,y}(t)}{dt^2} = \frac{2\left(\|x\|^2\|y\|^2-(x,y)^2\right)\left(n_{x,y}^3(2t)-2n_{x,y}^3(t)\right)}{n_{x,y}^3(2t)\,n_{x,y}^3(t)},\quad t\in\mathbb{R}, \tag{7.43}$$

*where $x,y$ are linearly independent.*

*Moreover, $\delta_{x,y}$ is convex on $(-\infty,t_1]\cup[t_2,+\infty)$ and concave on $(t_1,t_2)$, where*

$$t_1 := \frac{\left(2\sqrt[3]{4}-4\right)(x,y)-\sqrt{\Delta_{x,y}}}{2\left(4-\sqrt[3]{4}\right)}, \tag{7.44}$$

$$t_2 := \frac{\left(2\sqrt[3]{4}-4\right)(x,y)+\sqrt{\Delta_{x,y}}}{2\left(4-\sqrt[3]{4}\right)} \tag{7.45}$$

*and*

$$\Delta_{x,y} := \left(4-2\sqrt[3]{4}\right)^2(x,y)^2+4\left(4-\sqrt[3]{4}\right)\left(\sqrt[3]{4}-1\right)\|x\|^2\|y\|^2>0.$$

PROOF. It is obvious, by the above proposition, that

$$\frac{d\delta_{x,y}(t)}{dt} = 2\left(\Phi_{x,y}(t)-\Phi_{x,y}(2t)\right),$$

where

$$\Phi_{x,y}(t) = \frac{(y,x)+t\|y\|^2}{\|x+ty\|}.$$

As $\Phi_{x,y}$ is differentiable on $\mathbb{R}$ and

$$\frac{d\Phi_{x,y}(t)}{dt} = \frac{\|x\|^2\|y\|^2-(x,y)^2}{u_{x,y}^3(t)},\quad t\in\mathbb{R},$$

and we get that

$$\begin{aligned}\frac{d^2\delta_{x,y}(t)}{dt^2} &= 2\left(\frac{d\Phi_{x,y}(t)}{dt}-2\frac{d\Phi_{x,y}(2t)}{dt}\right)\\ &= 2\left(\|x\|^2\|y\|^2-(x,y)^2\right)\left(\frac{1}{u_{x,y}^3(t)}-\frac{2}{u_{x,y}^3(2t)}\right)\end{aligned}$$

and the relation (7.43) is obtained.

Note that the equation

$$\frac{d^2\delta_{x,y}(t)}{dt^2} = 0$$

is equivalent with

$$\|x + 2ty\|^2 = \sqrt[3]{4}\,\|x + ty\|^2$$

i.e.,

$$\left(4 - \sqrt[3]{4}\right)\|y\|^2 t^2 + \left(4 - 2\sqrt[3]{4}\right)(x, y)\,t + \left(1 - \sqrt[3]{4}\right)\|x\|^2 = 0.$$

The solutions of this equation on $t_1$, $t_2$ are given by (7.44) and (7.45).

Note that $t_1 < 0 < t_2$.

In addition, we should observe that

$$\frac{d^2\delta_{x,y}(t)}{dt^2} \geq 0 \ \text{ if } \ t \in (-\infty, t_1] \cup [t_2, \infty)$$

and

$$\frac{d^2\delta_{x,y}(t)}{dt^2} \leq 0 \ \text{ if } \ t \in (t_1, t_2)$$

and the convexity of $\delta_{x,y}$ is thus proved.

In the particular case of inner product spaces, we have

$$\delta_{x,y}(t) = 0 \ \text{ iff } \ 4\|x + ty\|^2 = \|x + 2ty\|^2$$

that is,

$$4(x, y)\,t = -3\|x\|^2.$$

In this case, we are certain about the convexity of $\delta_{x,y}$.

The graph of $\delta_{x,y}$ is the following one:

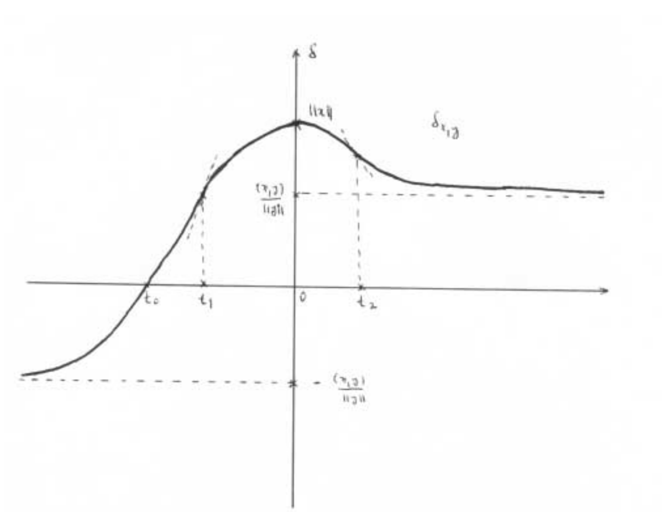


FIGURE 11.

if $(x, y) > 0$ (see Figure 11).

If $(x, y) < 0$, then we have Figure 12, and if $(x, y) = 0$, i.e., the

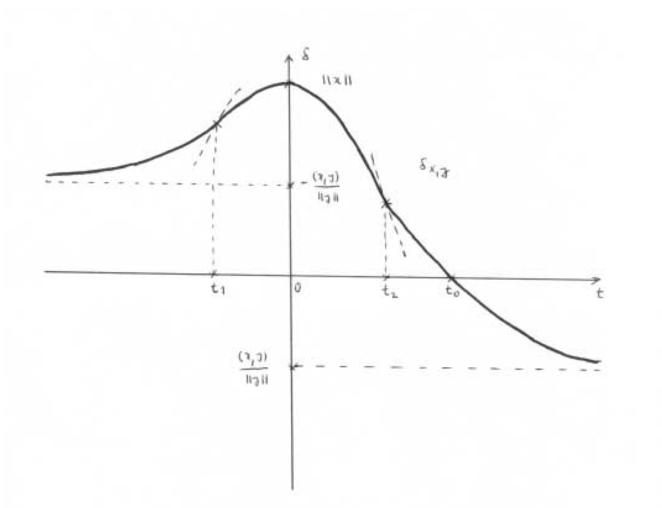

FIGURE 12.

vectors $x, y$ are orthogonal, we have Figure 13 where $t_0 = -\frac{3\|x\|^2}{4(x,y)}$ and

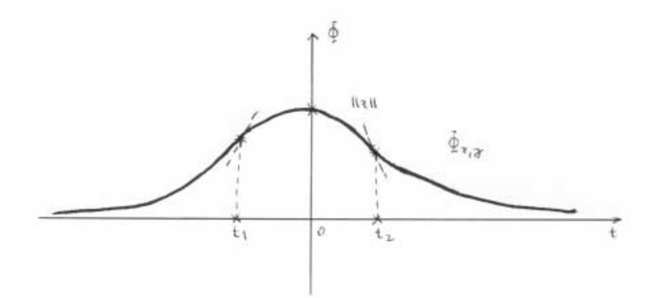

FIGURE 13.

$t_1$, $t_2$ are as above. Here $t_2 = -t_1$. ∎

Now we point out some results for the mapping $\gamma_{x,y}$ [**2**].

PROPOSITION 29. *Let $(X; (\cdot, \cdot))$ be an inner product space. The mapping $\gamma_{x,y}$, where $x, y$ are two linearly independent vectors in $X$, is twice differentiable on $\mathbb{R} \setminus \{0\}$ and*

$$\frac{d^2 \gamma_{x,y}(t)}{dt^2} = \frac{K_{x,y}(2t) - K_{x,y}(t)}{t^3}, \quad t \in \mathbb{R} \setminus \{0\}, \tag{7.46}$$

*where*

$$K_{x,y}(t) = \frac{n_{x,y}^2(t)\left(n_{x,y}(t) - \|x\|\right)^2 - t^2 (y, x+ty)^2}{n_{x,y}^3(t)}.$$

PROOF. We have

$$\begin{aligned}\gamma_{x,y}(t) &= \frac{\|x+2ty\| - \|x+ty\|}{t} \\ &= \frac{\|x+2ty\| - \|x\| - (\|x+ty\| - \|x\|)}{t} \\ &= 2\frac{\|x+2ty\| - \|x\|}{2t} - \frac{\|x+ty\| - \|x\|}{t} \\ &= 2v_{x,y}(2t) - v_{x,y}(t).\end{aligned}$$

Then we obtain:

$$\frac{d\gamma_{x,y}(t)}{dt} = 4\frac{dv_{x,y}(2t)}{dt} - \frac{dv_{x,y}(t)}{dt}$$

and

$$\frac{d^2\gamma_{x,y}(t)}{dt^2} = 8\frac{d^2 v_{x,y}(2t)}{dt^2} - \frac{d^2 v_{x,y}(t)}{dt^2}.$$

We know (see the proof of Theorem 43) that

$$\frac{d^2 v_{x,y}(t)}{dt^2} = t\frac{n_{x,y}^2(t)\left(n_{x,y}(t) - \|x\|\right)^2 - t^2 (y, x+ty)^2}{t^4 n_{x,y}^3(t)}.$$

Then

$$\begin{aligned}\frac{d^2\gamma_{x,y}(t)}{dt^2} &= 8\cdot\frac{2tn_{x,y}^2(2t)\left(n_{x,y}(2t) - \|x\|\right)^2 - (2t)^2 (y, x+2ty)^2}{(2t)^4 n_{x,y}^3(2t)} \\ &\quad -t\cdot\frac{n_{x,y}^2(t)\left(n_{x,y}(t) - \|x\|\right)^2 - t^2 (y, x+ty)^2}{t^4 n_{x,y}^3(t)}\end{aligned}$$

and the identity (7.46) is proved. ∎

PROPOSITION 30. *With the above assumptions, the mapping $K_{x,y}$ is differentiable on $\mathbb{R}\diagdown\{0\}$ and*

$$\frac{dK_{x,y}(t)}{dt} = 3t^2\frac{(y, x+ty)\left[(y, x+ty)^2 - \|y\|^2 n_{x,y}^2(t)\right]}{n_{x,y}^5(t)} \tag{7.47}$$

*for all $t \in \mathbb{R}$. Moreover, $K_{x,y}$ is monotonic increasing on $\left(-\infty, -\frac{(x,y)}{\|y\|^2}\right)$, and decreasing on $\left(-\frac{(x,y)}{\|y\|^2}, +\infty\right)$.*

PROOF. We have

$$\begin{aligned}
&\frac{dK_{x,y}(t)}{dt} \\
&= \frac{1}{n_{x,y}^{6}(t)}\left[\left(\frac{d}{dt}\left(n_{x,y}^{2}(t)\left(n_{x,y}(t)-\|x\|\right)^{2}\right)-t^{2}(y,x+ty)^{2}\right)n_{x,y}^{3}(2t)\right. \\
&\qquad\left. -\left(n_{x,y}^{2}(t)\left(n_{x,y}(t)-\|x\|\right)^{2}-t^{2}(y,x+ty)^{2}\right)\frac{dn_{x,y}^{3}(t)}{dt}\right].
\end{aligned}$$

However,

$$\begin{aligned}
&\frac{d}{dt}\left(n_{x,y}^{2}(t)\left(n_{x,y}(t)-\|x\|\right)^{2}\right) \\
&= 2n_{x,y}(t)\,n_{x,y}'(t)\left(n_{x,y}(t)-\|x\|\right)^{2}+2n_{x,y}^{2}(t)\left(n_{x,y}(t)-\|x\|\right)n_{x,y}'(t) \\
&= 2n_{x,y}(t)\left(n_{x,y}(t)-\|x\|\right)\left[n_{x,y}'(t)\left(n_{x,y}(t)-\|x\|\right)+n_{x,y}(t)\,n_{x,y}'(t)\right] \\
&= 2n_{x,y}(t)\left(n_{x,y}(t)-\|x\|\right)\left(2n_{x,y}'(t)\,n_{x,y}(t)-n_{x,y}'(t)\|x\|\right) \\
&= 2n_{x,y}(t)\,n_{x,y}'(t)\left(n_{x,y}(t)-\|x\|\right)\left(2n_{x,y}(t)-\|x\|\right) \\
&= 2n_{x,y}(t)\frac{(y,x+ty)}{n_{x,y}(t)}\left(n_{x,y}(t)-\|x\|\right)\left(2n_{x,y}(t)-\|x\|\right) \\
&= 2(y,x+ty)\left(n_{x,y}(t)-\|x\|\right)\left(2n_{x,y}(t)-\|x\|\right).
\end{aligned}$$

We also have

$$\begin{aligned}
\frac{d\left(t^{2}(y,x+ty)^{2}\right)}{dt^{2}} &= 2t(y,x+ty)^{2}+2t^{2}(y,x+ty)\|y\|^{2} \\
&= 2t(y,x+ty)\left[(y,x+ty)+t\|y\|^{2}\right] \\
&= 2t(y,x+ty)\left[2t\|y\|^{2}+(x,y)\right].
\end{aligned}$$

We have:

$$\begin{aligned}
A_{x,y} &:= 2(y,x+ty)\left(n_{x,y}(t)-\|x\|\right)\left(2n_{x,y}(t)-\|x\|\right) \\
&\quad -2t(y,x+ty)\left[2t\|y\|^{2}+(x,y)\right] \\
&= 2(y,x+ty)\left[\left(n_{x,y}(t)-\|x\|\right)\left(2n_{x,y}(t)-\|x\|\right)-2t\|y\|^{2}-t(x,y)\right] \\
&= 2(y,x+ty)\left(2n_{x,y}^{2}(t)-3n_{x,y}(t)\|x\|+\|x\|^{2}-2t^{2}\|y\|^{2}-t(x,y)\right) \\
&= 2(y,x+ty)\left[2\left(\|x\|^{2}+2(x,y)t+t^{2}\|y\|^{2}\right)\right. \\
&\quad \left. -3n_{x,y}(t)\|x\|+\|x\|^{2}-2t^{2}\|y\|^{2}-t(x,y)\right] \\
&= 6(y,x+ty)\left[(x,x+ty)-n_{x,y}(t)\|x\|\right].
\end{aligned}$$

Consequently,

$$\frac{dK_{x,y}(t)}{dt}=\frac{B_{x,y}}{n_{x,y}^{6}(t)},$$

where

$$
\begin{aligned}
B_{x,y} &:= 6\left(y, x+ty\right)\left[\left(x, x+ty\right) - n_{x,y}\left(t\right)\|x\|\right] n_{x,y}^{3}\left(t\right) \\
&\quad - \left[n_{x,y}^{2}\left(t\right)\left(n_{x,y}\left(t\right) - \|x\|\right)^{2} - t^{2}\left(y, x+ty\right)^{2}\right] \times 3n_{x,y}^{2}\left(t\right) n_{x,y}'\left(t\right) \\
&= 6\left(y, x+ty\right)\left[\left(x, x+ty\right) - n_{x,y}\left(t\right)\|x\|\right] n_{x,y}^{3}\left(t\right) \\
&\quad - 3\left(y, x+ty\right) n_{x,y}\left[n_{x,y}^{2}\left(t\right)\left(n_{x,y}\left(t\right) - \|x\|\right)^{2} - t^{2}\left(y, x+ty\right)^{2}\right] \\
&= 3n_{x,y}\left(t\right)\left(y, x+ty\right)\left[2n_{x,y}^{2}\left(t\right)\left(\left(x, x+ty\right) - n_{x,y}\left(t\right)\|x\|\right)\right. \\
&\quad \left. - n_{x,y}^{2}\left(t\right)\left(n_{x,y}\left(t\right) - \|x\|\right)^{2} - t^{2}\left(y, x+ty\right)^{2}\right] \\
&= 3n_{x,y}\left(t\right)\left(y, x+ty\right)\left\{n_{x,y}^{2}\left(t\right)\left[2\left(x, x+ty\right) - 2n_{x,y}\left(t\right)\|x\|\right.\right. \\
&\quad \left.\left. +\ 2n_{x,y}\left(t\right)\|x\| - \|x\|^{2}\right] - t^{2}\left(y, x+ty\right)^{2}\right\} \\
&= 3n_{x,y}\left(t\right)\left(y, x+ty\right)\left[n_{x,y}^{2}\left(t\right)\left(2\|x\|^{2} + 2t\left(x,y\right) - 2n_{x,y}\left(t\right)\|x\|^{2}\right.\right. \\
&\quad \left.\left. - n_{x,y}^{2}\left(t\right) +\ 2n_{x,y}\left(t\right)\|x\| - \|x\|^{2}\right) + t^{2}\left(y, x+ty\right)^{2}\right] \\
&= 3n_{x,y}\left(t\right)\left(y, x+ty\right)\left[n_{x,y}^{2}\left(t\right)\left(\|x\|^{2} + 2t\left(x,y\right) - n_{x,y}^{2}\left(t\right)\right)\right. \\
&\quad \left. + t^{2}\ \left(y, x+ty\right)^{2}\right] \\
&= 3n_{x,y}\left(t\right)\left(y, x+ty\right)\left[n_{x,y}^{2}\left(t\right)\left(\|x\|^{2} + 2t\left(x,y\right) - \|x\|^{2}\right.\right. \\
&\quad \left.\left. -2t\left(x,y\right) - t^{2}\|y\|^{2}\right) + t^{2}\left(y, x+ty\right)^{2}\right] \\
&= 3n_{x,y}\left(t\right)\left(y, x+ty\right)\left[t^{2}\left(y, x+ty\right)^{2} - t^{2}\|y\|^{2} n_{x,y}^{2}\left(t\right)\right] \\
&= 3t^{2} n_{x,y}\left(t\right)\left(y, x+ty\right)\left[\left(y, x+ty\right)^{2} - \|y\|^{2} n_{x,y}^{2}\left(t\right)\right]
\end{aligned}
$$

and the equality (7.47) is obtained.

Note that

$$\|y\|^{2} n_{x,y}^{2}\left(t\right) \geq \left(y, x+ty\right)^{2}$$

with equality iff $y$ and $x$ are linearly dependent.

Also,

$$\left(ty + x, y\right) = 0 \quad \text{iff} \quad t = -\frac{\left(x,y\right)}{\|y\|^{2}},$$

then $\frac{dK_{x,y}(t)}{dt} \geq 0$ for $t \in \left(-\infty, -\frac{(x,y)}{\|y\|^{2}}\right]$, and $\frac{dK_{x,y}(t)}{dt} \leq 0$ for $t \in \left[-\frac{(x,y)}{\|y\|^{2}}, +\infty\right)$, which shows that $K_{x,y}$ is monotonic increasing on $\left(-\infty, -\frac{(x,y)}{\|y\|^{2}}\right)$, and decreasing on $\left(-\frac{(x,y)}{\|y\|^{2}}, +\infty\right)$. ∎

We are now able to give the following partial result on the convexity of $\gamma_{x,y}$ in the particular case of inner product spaces [**2**].

PROPOSITION 31. *If $x, y$ are orthogonal, then the mapping $\gamma_{x,y}$ is strictly convex on $(-\infty, 0)$ and strictly concave on $(0, +\infty)$.*

PROOF. If $x \perp y$, the mapping $K_{x,y}$ is strictly increasing on $(-\infty, 0)$ and strictly decreasing on $(0, +\infty)$.

If $t < 0$, then $2t < t$ and $K_{x,y}(2t) < K_{x,y}(t)$, which give us that

$$\frac{1}{t}\left[K_{x,y}(2t) - K_{x,y}(t)\right] < 0$$

i.e.,

$$\frac{d^2\gamma_{x,y}(t)}{dt^2} < 0 \quad \text{for} \quad t \in (0, +\infty),$$

which proves the strict concavity of $\gamma_{x,y}$ on $(0, +\infty)$. ∎

REMARK 12. *The convexity of $\gamma_{x,y}$ in the general case of fixed linearly independent vectors $x, y$ is still open.*

The following result for the function $\Phi_{x,y}$ holds [**1**].

THEOREM 45. *Let $(X; (\cdot, \cdot))$ be an inner product space over the real number field $\mathbb{R}$. If $x, y$ are linearly independent, then we have*

$$\frac{d\Phi_{x,y}(t)}{dt} = \frac{\|y\|^2 \|x\|^2 - (x, y)^2}{n_{x,y}^3(t)}$$

*and*

$$\frac{d^2\Phi_{x,y}(t)}{dt^2} = \frac{-3(y, x + ty)\left(\|y\|^2 \|x\|^2 - (x, y)^2\right)}{n_{x,y}^5(t)}.$$

*Moreover, $\Phi_{x,y}$ is convex on $\left(-\infty, -\frac{(x,y)}{\|y\|^2}\right]$ and concave on $\left(-\frac{(x,y)}{\|y\|^2}, \infty\right)$.*

PROOF. We have successively,

$$\begin{aligned}
&\frac{d\Phi_{x,y}(t)}{dt} \\
&= \frac{\|y\|^2 n_{x,y}(t) - (y, x + ty)\, n'_{x,y}(t)}{n_{x,y}^2(t)} = \frac{\|y\|^2 n_{x,y}(t) - (y, x + ty)\frac{(y, x+ty)}{n_{x,y}(t)}}{n_{x,y}^2(t)} \\
&= \frac{\|y\|^2 n_{x,y}^2(t) - (y, x + ty)^2}{n_{x,y}^3(t)} \\
&= \frac{\|y\|^2\left(\|x\|^2 + 2t(y, x) + t^2\|y\|^2\right) - \left((y, x)^2 + 2t(y, x)\|y\|^2 + t^2\|y\|^4\right)}{n_{x,y}^3(t)} \\
&= \frac{\|y\|^2 \|x\|^2 - (x, y)^2}{n_{x,y}^3(t)}.
\end{aligned}$$

We also have:

$$\begin{aligned}\frac{d^2\Phi_{x,y}(t)}{dt^2} &= \frac{-3n_{x,y}^2(t)\, n'_{x,y}(t)\left(\|y\|^2\|x\|^2-(y,x)^2\right)}{n_{x,y}^6(t)}\\ &= \frac{-3n_{x,y}^2(t)\,\frac{(y,x+ty)}{n_{x,y}(t)}\left(\|y\|^2\|x\|^2-(y,x)^2\right)}{n_{x,y}^6(t)}\\ &= \frac{-3(y,x+ty)\left(\|y\|^2\|x\|^2-(y,x)^2\right)}{n_{x,y}^5(t)}.\end{aligned}$$

It is clear now that $\frac{d^2\Phi_{x,y}(t)}{dt^2} \geq 0$ if $t \in \left(-\infty, -\frac{(x,y)}{\|y\|^2}\right)$ and $\frac{d^2\Phi_{x,y}(t)}{dt^2} \leq 0$ if $t \in \left(-\frac{(x,y)}{\|y\|^2}, \infty\right)$. ∎

REMARK 13. *In the particular case of inner product spaces, we have the following graph for $\Phi_{x,y}$ (see Figure 14). Note that $t_0 = -\frac{(y,x)}{\|y\|^2}$ is*

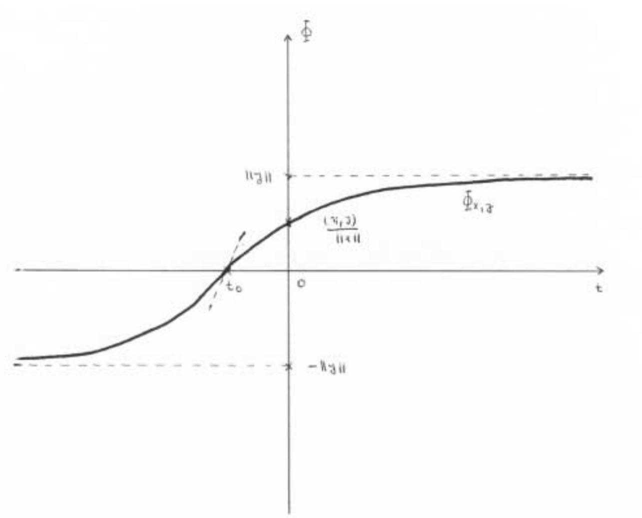


FIGURE 14.

*the point where $\Phi_{x,y}(t)$ is zero and also the point where $\Phi_{x,y}$ changes its convexity.*

The following result for the mapping $\Psi_{x,y}$ holds [**4**].

THEOREM 46. *Let $(X;(\cdot,\cdot))$ be an inner product space over the real number field $\mathbb{R}$ and $x, y$ two given linearly independent vectors in $X$. The mapping $\Psi_{x,y} : \mathbb{R} \to \mathbb{R}$,*

$$\Psi_{x,y}(t) = \frac{\|x\|^2 + t(x,y)}{\|x+ty\|}, \quad t \in \mathbb{R}$$

*is twice differentiable on* $\mathbb{R}$,

$$\frac{d\Psi_{x,y}(t)}{dt} = t \cdot \frac{(x,y)^2 - \|x\|^2 \|y\|^2}{\|x+ty\|^3}, \quad t \in \mathbb{R} \tag{7.48}$$

*and*

$$\frac{d^2\Psi_{x,y}(t)}{dt^2} = t \cdot \frac{\|x\|^2 \|y\|^2 - (x,y)^2}{\|x+ty\|^5} \left(2t^2 \|y\|^2 + t(x,y) - \|x\|^2\right). \tag{7.49}$$

*Moreover, the mapping* $\Psi_{x,y}$ *is convex on* $(-\infty, t_1] \cup [t_2, +\infty)$ *and concave on* $(t_1, t_2)$ *where*

$$t_1 = \frac{-(x,y) - \sqrt{\Delta_{x,y}}}{4\|y\|^2}, \quad t_2 = \frac{-(x,y) + \sqrt{\Delta_{x,y}}}{4\|y\|^2}$$

*and* $\Delta_{x,y} := 8\|x\|^2 \|y\|^2 + (x,y)^2 > 0$.

Proof. We have

$$\begin{aligned}
& \frac{d\Psi_{x,y}(t)}{dt} \\
= \; & \frac{1}{n_{x,y}^2(t)} \left[\frac{d}{dt}\left(\|x\|^2 + t(x,y)\right) n_{x,y}(t) - \left(\|x\|^2 + t(x,y)\right) \frac{dn_{x,y}(t)}{dt}\right] \\
= \; & \frac{1}{n_{x,y}^2(t)} \left[(x,y)\, n_{x,y}(t) - \left(\|x\|^2 + t(x,y)\right) \frac{\left((x,y) + t\|y\|^2\right)}{n_{x,y}(t)}\right] \\
= \; & \frac{1}{n_{x,y}^3(t)} \left[(x,y)\, n_{x,y}^2(t) - \left(\|x\|^2 + t(x,y)\right)\left((x,y) + t\|y\|^2\right)\right] \\
= \; & \frac{1}{n_{x,y}^3(t)} \left[(x,y)\left(\|x\|^2 + 2t(x,y) + t^2\|y\|^2\right)\right. \\
& \left. - \left(\|x\|^2 + t(x,y)\right)\left((x,y) + t\|y\|^2\right)\right] \\
= \; & t \cdot \frac{(x,y)^2 - \|x\|^2 \|y\|^2}{n_{x,y}^3(t)}
\end{aligned}$$

and the relation (7.48) is proved.

We have

$$\begin{aligned}
\frac{d^2\Psi_{x,y}(t)}{dt^2} &= \frac{\left((x,y)^2 - \|x\|^2\|y\|^2\right)}{n_{x,y}^6(t)}\left[n_{x,y}^3(t) - 3tn_{x,y}^2(t)\,n_{x,y}'(t)\right] \\
&= \frac{\left((x,y)^2 - \|x\|^2\|y\|^2\right)}{n_{x,y}^4(t)}\left[n_{x,y}(t) - 3t\frac{(y,x)+t\|y\|^2}{n_{x,y}(t)}\right] \\
&= \frac{\left((x,y)^2 - \|x\|^2\|y\|^2\right)}{n_{x,y}^5(t)}\left(n_{x,y}^2(t) - 3t(y,x) + 3t\|y\|^2\right) \\
&= \left((x,y)^2 - \|x\|^2\|y\|^2\right) \\
&\quad\times\frac{\left(\|x\|^2 + 2t(y,x) + t^2\|y\|^2 - 3t(y,x) + 3t\|y\|^2\right)}{n_{x,y}^5(t)} \\
&= \frac{(x,y)^2 - \|x\|^2\|y\|^2}{n_{x,y}^5(t)}\left(2t^2\|y\|^2 + t(y,x) - \|x\|^2\right).
\end{aligned}$$

Consider the equation

$$2t^2\|y\|^2 + t(y,x) - \|x\|^2 = 0, \quad t \in \mathbb{R}.$$

This equation has two distinct solutions $t_1$, $t_2$ given by

$$t_{1,2} = \frac{-(y,x) \pm \sqrt{\Delta_{x,y}}}{4\|y\|^2},$$

where $\Delta_{x,y} := 8\|x\|^2\|y\|^2 + (x,y)^2 > 0$.

Now, it is clear that $\frac{d^2\Psi_{x,y}(t)}{dt^2} \geq 0$ if $t \in (-\infty, t_1] \cup [t_2, +\infty)$ and

$$\frac{d^2\Psi_{x,y}(t)}{dt^2} < 0 \quad \text{if} \quad t \in (t_1, t_2).$$

The theorem is thus proved. ∎

In the case of inner product spaces, we have

$$\Psi_{x,y}(t) = 0 \quad \text{iff} \quad \|x\|^2 + t(x,y) = 0.$$

In this case, we are certain about the convexity of $\Psi_{x,y}$.

The graph of $\Psi_{x,y}$ is the following one

a) If $(x,y) > 0$, then the plot of $\Psi_{x,y}$ is incorporated in Figure 15

b) If $(x,y) < 0$, then the plot of $\Psi_{x,y}$ is incorporated in Figure 16

c) If $(x,y) = 0$, i.e., the vectors $x, y$ are orthogonal, then the plot of $\Psi_{x,y}$ is incorporated in Figure 17, where $t_1$, $t_2$ are as above and $t_0 = -\frac{\|x\|^2}{(x,y)}$.

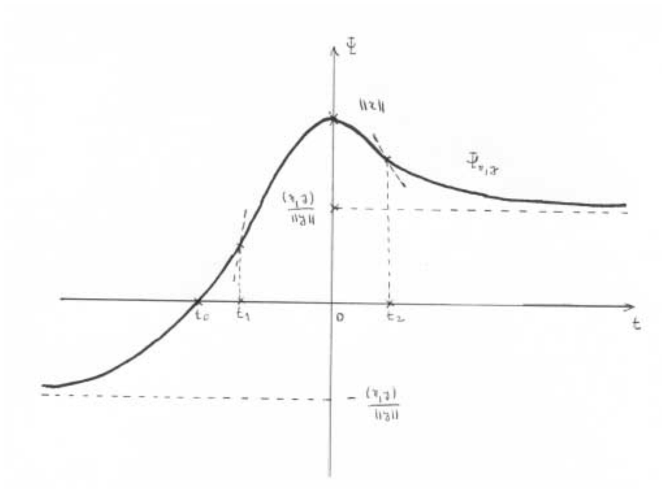

FIGURE 15.

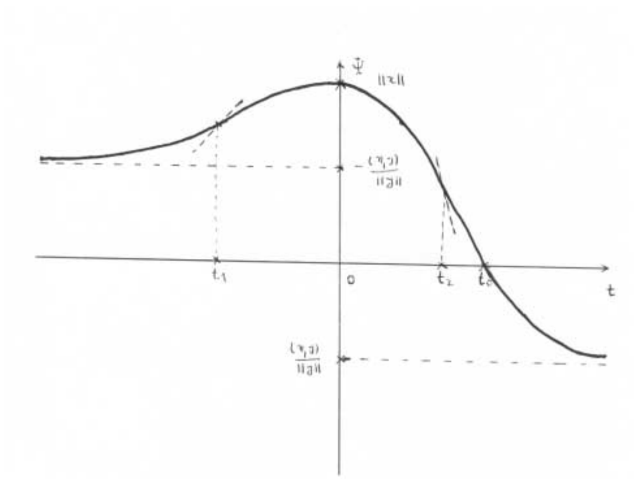

FIGURE 16.

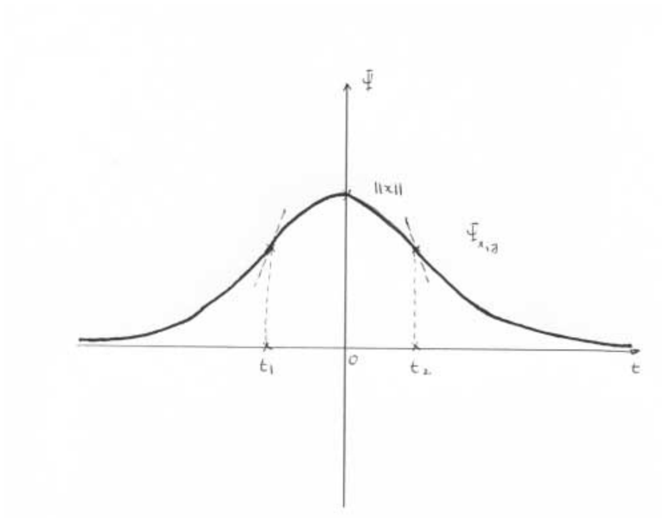


FIGURE 17.

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CHAPTER 8

# Orthogonality in the Sense of Birkhoff-James

## 1. Definition and Preliminary Results

In 1935, G. Birkhoff [**1**] introduced the following concept that is a natural generalisation of the usual orthogonality which holds in inner product spaces over the real number field.

Definition 20. *Let $(X, \|\cdot\|)$ be a real or complex normed linear space and $x, y$ be two given elements in $X$. We will say that $x$ is Birkhoff-orthogonal over $y$ and denote this $x \perp y\,(B)$ iff:*

$$\|x\| \leq \|x + ty\| \text{ for all } t \in \mathbb{R}.$$

It is clear that if $(X; (\cdot, \cdot))$ is an inner product space then the usual orthogonality introduced by the inner product, i.e., $x \perp y$ iff $(x, y) = 0$ is equivalent with Birkhoff's orthogonality.

In 1947, R.C. James [**2**] extended this concept of orthogonality for the case of complex normed spaces. Namely, we have:

Definition 21. *Let $(X, \|\cdot\|)$ be a complex normed space and $x, y$ two vectors in $X$. We will say that $x$ is James-orthogonal over $y$ and we will denote this by $x \perp y$ $(J)$, iff*

$$\|x\| \leq \|x + \lambda y\| \text{ for all } \lambda \in \mathbb{C}.$$

Now, we note here that if $(X; (\cdot, \cdot))$ is a complex prehilbertian space, then the usual orthogonality is equivalent with James' orthogonality.

Remark 14. *It is obvious that $x \perp y\,(B\,[J])$ implies that $x \perp (\alpha y)\,(B\,[J])$ for every scalar $\alpha \in \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $x \perp x\,(B\,[J])$ implies $x = 0$ and $x \perp y\,(B\,[J])$, $x \perp z\,(B\,[J])$ do not imply $x \perp (y + z)\,(B\,[J])$ and also $x \perp y\,(B\,[J])$ is not connected with $y \perp x\,(B\,[J])$.*

The following theorem holds (see [**3**, p. 25]).

Theorem 47. *Let $(X, \|\cdot\|)$ be a real or complex normed space, $f : X \to \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) a bounded linear functional on $X$ and $x \in X$, $x \neq 0$. Then the following statements are equivalent:*

(i) *$x \perp Ker\,(f)$ in the sense of Birkhoff or James;*

(ii) *$x$ is a maximal element for $f$, i.e.,*

$$|f\,(x)| = \|f\| \, \|x\| \,. \tag{8.1}$$

PROOF. Let us assume that $x \perp Ker(f)$, i.e., $x \perp y$ for all $y \in Ker(f)$. Suppose also that:

$$|f(x)| = p\|x\|. \tag{8.2}$$

Now, for all $y$ in $Ker(f)$ we have:

$$|f(x+y)| = |f(x)| = p\|x\| \leq p\|x+y\|.$$

Since $x \notin Ker(f)$, we have:

$$X = \{\lambda(x+y) \,|\lambda \in \mathbb{K},\, y \in Ker(f)\}$$

and then

$$\|f\| = \sup \frac{\|f(\lambda(x+y))\|}{\|\lambda(x+y)\|} \leq p.$$

On the other hand, we have $|f(x)| = p\|x\|$ and then $\|f\| = p$, which gives by (8.2) the desired relation (8.1).

Conversely, if we assume that $x$ is a maximal element for the functional $f$, then for every $\lambda \in \mathbb{K}$, we have:

$$\|x\| = \frac{|f(x)|}{\|f\|} = \frac{|f(x+\lambda y)|}{\|f\|} \leq \frac{\|f\|\,\|x+\lambda y\|}{\|f\|} = \|x+\lambda y\|$$

which shows that $x \perp y\,(B[J])$ wherever $y \in Ker(f)$, i.e., $x \perp Ker(f)\,(B[J])$ and the theorem is proved. ∎

The following two corollaries are obvious by the above theorem (see also [**3**, p. 25]).

COROLLARY 10. *Let $x$ be a nonzero vector in Banach space $X$. Then $x$ is Birkhoff (James) orthogonal over a hyperplane containing the null element.*

COROLLARY 11. *Let $x, y \in X$ with $x \neq 0$. Then there exists a $\alpha \in \mathbb{R}$ such that $x \perp (\alpha x + y)\,(B[J])$.*

REMARK 15. *We observe that $x \perp (\alpha x + y)\,(B[J])$, $\alpha \in \mathbb{K}$ if and only if there exists a functional $f \in X^*$, $\|f\| = 1$ such that $f(x) = \|x\|$ and $\alpha = -\frac{f(y)}{f(x)}$.*

REMARK 16. *If $x \perp (\alpha x + y)$, then $|\alpha| \leq \frac{\|y\|}{\|x\|}$.*

In the following section, we will give some characterisations of smoothness and strict convexity in terms of Birkhoff orthogonality.

## 2. Characterisation of Some Classes of Normed Spaces

We will start with the following theorem which contains two characterisation of smooth normed spaces in terms of Birkhoff's orthogonality (see for example [**3**, p. 26]).

THEOREM 48. *Let $(X, \|\cdot\|)$ be a normed space. Then the following statements are equivalent:*

(i) *$X$ is smooth;*
(ii) *the Birkhoff orthogonality is unique in the right hand side, i.e., for every $x \in X \setminus \{0\}$ and $y \in X$ there exists a unique scalar $\alpha$ such that $x \perp (\alpha x + y)\,(B)$.*
(iii) *the Birkhoff orthogonality is additive at right, i.e., for every $x, y, z \in X$ with $x \perp y\,(B)$ and $x \perp z\,(B)$, we also have $x \perp (y + z)\,(B)$.*

PROOF. "(i) $\Longrightarrow$ (ii)". Let us assume that $X$ is smooth and let $x \in X$, $x \neq 0$. Then there exists a unique functional $f \in X^*$ with $\|f\| = 1$ and such that $f(x) = \|x\|$. Using Remark 16, the scalar $\alpha = -\frac{f(y)}{f(x)}$ is unique with the property that $x \perp (\alpha x + y)\,(B)$.

"(ii) $\Longrightarrow$ (i)". Suppose that the orthogonality in the sense of Birkhoff is unique at the right hand and let $x \in X \setminus \{0\}$. Take $f \in X^*$, $\|f\| = 1$ with the property that $f(x) = \|x\|$. If $y \in X$, then by the unicity of "$\perp$", $x \perp \alpha x + y$ with the unique scalar $\alpha = -\frac{f(y)}{f(x)}$, and then $f(y) = -\alpha f(x)$, which implies that the element $x$ has a unique *support functional*, i.e., a bounded linear functional $g \in X^*$ with $\|g\| = 1$ and $g(x) = 1$, i.e., $X$ is smooth.

"(i) $\Longrightarrow$ (iii)". Now, assume that $X$ is smooth and consider the *support mapping*, i.e., the mapping $X \setminus \{0\} \ni x \longmapsto f_x \in X \setminus \{0\}$ given by

a) $\|x\| = 1$ implies $\|f_x\| = 1 = f_x(x)$;
b) $\lambda \geq 0$ implies $f_{\lambda x} = \lambda f_x$.

Let $x \in X \setminus \{0\}$ and assume that $x \perp y\,(B)$ and $x \perp z\,(B)$ where $y, z \in X$. Then by the unicity at right the unique scalar with $x \perp (\alpha x + y)\,(B)$ is $\alpha = 0$ and the unique $\beta$ with $x \perp (\beta x + y)\,(B)$ is also $\beta = 0$. In both cases (see Remark 15) we have $f_x(y) = f_x(z) = 0$. Then $f_x(y + z) = f_x(y) + f_x(z) = 0$ and by Theorem 47 we deduce that $x \perp (y + z)\,(B)$.

"(iii) $\Longrightarrow$ (ii)". Let us assume that Birkhoff orthogonality is additive at right and let $x \in X \setminus \{0\}$ such that $x \perp (\alpha x + y)$ and $x \perp (\beta x + y)$. Then $x \perp -(\beta x + y)$ and by the additivity at right we have:

$$x \perp [(\alpha x + y) - (\beta x + y)]$$

i.e., $x \perp (\alpha - \beta)\, x$ and then

$$\|x\| \leq \|x + \lambda (\alpha - \beta)\, x\| = |1 + \lambda (\alpha - \beta)|\, \|x\|$$

for all $\lambda \in \mathbb{K}$, which implies that $\alpha = \beta$.

The theorem is thus proved. ∎

The second result is embodied in the following theorem (see also [**3**, p. 27]).

THEOREM 49. *Let $(X, \|\cdot\|)$ be a real normed space. Then the following statements are equivalent:*

(i) *$X$ is strictly convex;*
(ii) *the Birkhoff orthogonality is unique at left, i.e., for every $x, y \in X$ with $x \neq 0$, there exists a unique $\alpha$ such that $(\alpha x + y) \perp x\,(B)$.*

PROOF. We will firstly prove the following lemma which guarantees the existence of a scalar $\alpha$ such that $(\alpha x + y) \perp x\,(B)$.

LEMMA 3. *Let $x, y \in X$. Then there exists a real number $\alpha$ such that $(\alpha x + y) \perp x\,(B)$. Moreover, this scalar $\alpha$ is the real number which achieves the minimum of the following real functionals:*

$$\mathbb{R} \ni k \longmapsto \|kx + y\| \in \mathbb{R}_+.$$

*In addition, if $(ax + y) \perp x\,(B)$ and $(bx + y) \perp x\,(B)$, then for all $\alpha$ between $a$ and $b$ we also have $(\alpha x + y) \perp x\,(B)$.*

PROOF. Let us consider the mapping $n : \mathbb{R} \to \mathbb{K}$, $n(t) = \|tx + y\|$. This mapping is clearly convex on $\mathbb{R}$ and then $n$ achieves its minimum for a certain $\alpha \in \mathbb{R}$. Moreover, the set of points in which $n$ achieves its minimum is an interval.
Now, let us observe that $(\alpha x + y) \perp x\,(B)$ if and only if

$$\|\alpha x + y\| \leq \|\alpha x + y + \lambda x\| = \|(\alpha + \lambda)\, x + y\|$$

for every $\lambda \in \mathbb{R}$ which is equivalent with

$$\|\alpha x + y\| \leq \|kx + y\| \quad \text{for all } k \in \mathbb{R},$$

i.e., $\alpha$ is the point in which the mapping $n$ achieves its minimum.
The lemma is thus proven. ∎

Let us now prove the theorem.

"(ii) $\Longrightarrow$ (i)". It is obvious.

"(i) $\Longrightarrow$ (ii)". Let us assume that $X$ is not strictly convex and there exist $x, y \in X$, $x \in y$, $\|x\| = \|y\| = 1$ such that $\lambda x + (1-\lambda) y \in S(X) := \{z \in X | \|z\| = 1\}$ for all $\lambda \in [0,1]$.

Denote $u = x + y$ and $v = x - y$. We will show that:

$$\|u\| \leq \|u + \mu v\| \quad \text{for all } \mu \in \mathbb{R}.$$

It is sufficient to prove the above inequality for $\mu > 0$.

Let us observe that:

$$\|u+v\| = 2\|x\| = 2, \quad \|u-v\| = 2\|y\| = 2,$$

and

$$\|u\| = \|x+y\| = 2.$$

Consider $0 \leq \mu \leq 1$. Then

$$\begin{aligned} \|u + \mu v\| &= \|x + y + \mu x - \mu y\| = \|(1+\mu) x + (1-\mu) y\| \\ &= 2 = \|u\| \end{aligned}$$

because

$$0 \leq 1 - \mu \leq 1 + \mu \leq 2 \;\text{ and }\; 1 + \mu + (1-\mu) = 2.$$

Consequently, for $0 \leq \mu \leq 1$ we get $\|u + \mu v\| = \|u\|$.

If $\mu > 1$, then $\mu - 1 > 0$ and thus

$$\begin{aligned} \|u + \mu v\| &= \|x + y + \mu x - \mu y\| = \|(1+\mu) x + (1-\mu) y\| \\ &\geq |\mu| \left\| \left(1 + \frac{1}{\mu}\right) x - \left(\frac{\mu - 1}{\mu}\right) y \right\| \\ &\geq |\mu| \left[ \left\| \left(1 + \frac{1}{\mu}\right) x \right\| - \left\| \left(\frac{\mu - 1}{\mu}\right) y \right\| \right] \\ &= |\mu| \left[ \left(1 + \frac{1}{\mu}\right) \|x\| - \left(\frac{\mu - 1}{\mu}\right) \|y\| \right] = 2 = \|u\|. \end{aligned}$$

In conclusion, $\|u\| \leq \|u + \mu v\|$ for every $\mu \in \mathbb{R}$ and the equality holds in the above inequality for $|\mu| \leq 1$.

Now, we will show that for $|\mu| \leq 1$ we have $(u + \mu v) \perp v\,(B)$ $(v \neq 0)$, which contradicts the unicity at left of Birkhoff orthogonality.

Indeed, we have:

$$\|u + \mu v\| = \|u\| \leq \|u + (\lambda + \mu) v\| = \|u + \mu v + \lambda v\|$$

for all $\mu$ with $|\mu| \leq 1$, i.e., $(u + \mu v) \perp v$, and the statement is proved. ∎

## 3. Birkhoff's Orthogonality and the Semi-inner Products

Let $(X, \|\cdot\|)$ be a real normed space. Then the following characterisation of Birkhoff's orthogonality in terms of semi-inner products $(\cdot,\cdot)_s$ and $(\cdot,\cdot)_i$ holds.

THEOREM 50. *Let $(X, \|\cdot\|)$ be as above. Then the following statements are equivalent:*

(i) $x \perp z\,(B)$;
(ii) $(z,x)_i \le 0 \le (z,x)_s$

*where $x, z \in X$.*

PROOF. "(i) $\Longrightarrow$ (ii)". Let us assume that $x \perp z\,(B)$, i.e., $\|x+tz\| \ge \|x\|$, for all $t \in \mathbb{R}$. Then we have:

$$\frac{\|x+tz\|^2 - \|x\|^2}{2t} \ge 0 \quad \text{and} \quad \frac{\|x+sz\|^2 - \|x\|^2}{2s} \le 0$$

for all $t > 0$ and $s < 0$, which implies that $(z,x)_s \ge 0 \ge (z,x)_i$.

"(ii) $\Longrightarrow$ (i)". Now, let us observe that for all $t \in \mathbb{R}$, we have:

$$(tz + x, x)_s \le \|x+tz\| \cdot \|x\| .$$

On the other hand, we have

$$(x+tz, x)_s = t\,(z,x)_s + \|x\|^2 , \quad t \ge 0,$$

which implies:

$$t\,(z,x)_s \le (\|x+tz\| - \|x\|)\,\|x\|$$

for all $t \in \mathbb{R}_+$. Since $(z,x)_s \ge 0$, then

$$\|x+tz\| - \|x\| \ge 0 \quad \text{for all } t \ge 0.$$

Now, as $(z,x)_i \le 0$, we get $-(z,x)_i = (-z,x)_s \ge 0$ which shows that $\|x + s(-z)\| - \|x\| \ge 0$ for all $s \ge 0$. Consequently, $\|x+tz\| \ge \|x\|$, for all $t \le 0$ and thus $x \perp z\,(B)$.

The theorem is thus proved. ∎

The following corollary is due to R.C. James [**2**].

COROLLARY 12. *Let $(X, \|\cdot\|)$ be a normed space over the real number field and $\alpha$ a given real number. Then the following statements are equivalent:*

(i) $x \perp (\alpha x + y)\,(B)$;
(ii) *we have the estimation*

$$(y,x)_i \le -\alpha \|x\|^2 \le (y,x)_s ,$$

*where $x, y$ belong to $X$.*

PROOF. Let $z := \alpha x + y$. Then $x \perp z\,(B)$ iff $(z,x)_i \le 0 \le (z,x)_s$. However, $(z,x)_p = (\alpha x + y, x)_p = \alpha \|x\|^2 + (y,x)_p$ where $p \in \{s,i\}$, and the corollary is thus proved. ∎

The following theorem gives us the opportunity to approximate the bounded linear functionals defined on a real normed space with the help of semi-inner products $(\cdot,\cdot)_s$ and $(\cdot,\cdot)_i$ (see also [**4**] and [**5**]).

THEOREM 51. *Let $(X, \|\cdot\|)$ be a real normed space, $f : X \to \mathbb{R}$ a nonzero bounded linear functional on $X$ and $w \in X$, $w \neq 0$. Then the following statements are equivalent:*

(i) $w \perp Ker\,(f)\,(B)$;

(ii) *we have the estimation:*

$$\left(x, \frac{f(w)}{\|w\|^2} w\right)_i \le f(x) \le \left(x, \frac{f(w)}{\|w\|^2} w\right)_s \tag{8.3}$$

*for all $x \in X$.*

PROOF. "(i) $\Longrightarrow$ (ii)". Let us assume that $w \perp Ker\,(f)$. Then, by Theorem 50, we have

$$(y,x)_i \le 0 \le (y,x)_s,$$

for all $y \in Ker\,(f)$.

Now, let $x \in X$. Then the element $y = f(x)\,w - f(w)\,x$ belongs to $Ker\,(f)$ because $f(y) = f(f(x)\,w - f(w)\,x, w) = f(x)\,f(w) - f(w)\,f(x) = 0$. Consequently, one has:

$$(f(x)\,w - f(w)\,x, w)_i \le 0 \le (f(x)\,w - f(w)\,x, w)_s \tag{8.4}$$

for all $x \in X$.

Using the properties of semi-inner products $(\cdot,\cdot)_s$ and $(\cdot,\cdot)_i$, we derive

$$(f(x)\,w - f(w)\,x, w)_i = f(x)\,\|w\|^2 - (x, f(w)\,w)_s$$

and

$$(f(x)\,w - f(w)\,x, w)_s = f(x)\,\|w\|^2 - (x, f(w)\,w)_i$$

for all $x \in X$. By the double inequality (8.4), we deduce

$$(x, f(w)\,w)_i \le f(x)\,\|w\|^2 \le (x, f(w)\,w)_s$$

for all $x \in X$, which is equivalent with (8.3).

"(ii) $\Longrightarrow$ (i)". Firstly, we observe that $f(w) \neq 0$, because $f(w) = 0$ easily implies that $f(x) = 0$ for all $x \in X$, which is false.

By (8.3), it follows that

$$(x, f(w)\,w)_i \le 0 \le (x, f(w)\,w)_s$$

i.e., $f(w)w \perp Ker(f)(B)$. However $f(w) \neq 0$ and then $w \perp Ker(f)$ which completes the proof. ∎

The following corollary is obvious.

COROLLARY 13. *Let $(X, \|\cdot\|)$ be a real normed space, $f \in X^* \diagdown \{0\}$ and $w \in X \diagdown \{0\}$. Then the following statements are equivalent:*

(i) $w \perp Ker(f)(B)$;
(ii) $|f(w)| = \|f\| \|w\|$;
(iii) *We have the estimation (8.3).*

Now, we can state the following general result, which contains the Birkhoff orthogonality of an element over a closed linear subspace in a normed space.

Namely, we have the following theorem.

THEOREM 52. *Let $(X, \|\cdot\|)$ be a real normed space, $G$ a closed linear subspace in $X$ and $x_0 \in X \diagdown \{G\}$. Then the following statements are equivalent.*

(i) $x_0 \perp G(B)$;
(ii) *Wherever $f \in (G \oplus S_p(x_0))^*$ with $Ker(f) = G$, we have the estimation:*

$$\left(x, \frac{f(x_0)}{\|x_0\|^2} x_0\right)_i \leq f(x) \leq \left(x, \frac{f(x_0)}{\|x_0\|^2} x_0\right)_s$$

*for all $x \in G \oplus S_p(x_0)$.*

The proof is obvious.

Now, we will give some characterisations of Birkhoff-James' orthogonality in terms of quadratic functionals.

The first result is embodied in the following theorem.

THEOREM 53. *Let $(X, \|\cdot\|)$ be a real normed space, $f : X \to \mathbb{R}$ a nonzero continuous linear functional and $w \in X \diagdown \{0\}$. Then the following statements are equivalent:*

(i) *We have the estimation:*

$$(x, w)_i \leq f(x) \leq (x, w)_s \tag{8.5}$$

*for all $x \in X$;*
(ii) *$w$ minimizes the quadratic functional $F_f : X \to \mathbb{R}$,*

$$F_f(u) := \|u\|^2 - 2f(u).$$

PROOF. "(i) $\Longrightarrow$ (ii)". If $w$ satisfies (8.5), we have $f(w) = \|w\|^2$. Now, for all $u \in X$ we can obtain

$$\begin{aligned} F_f(u) - F_f(w) &= \|u\|^2 - 2f(u) - \|w\|^2 + 2f(w) \\ &= \|u\|^2 - 2f(u) + \|w\|^2 \\ &\geq \|u\|^2 - 2\|u\|\|w\| + \|w\|^2 \\ &= (\|u\| - \|w\|)^2 \geq 0 \end{aligned}$$

because

$$f(u) \leq (u, w)_s \quad \text{for all } u \in X$$

and

$$-(u, w)_s \geq -\|u\|\|w\| \quad \text{for all } u \in X.$$

In conclusion,

$$F_f(u) \geq F_f(w) \quad \text{for all } u \in X,$$

i.e., $w$ minimizes the functional $F_f$.

"(ii) $\Longrightarrow$ (i)". If $w$ minimizes the functional $F_f$, then for all $u \in X$ and $\lambda \in \mathbb{R}$, we have:

$$F_f(w + \lambda u) - F_f(w) \geq 0.$$

On the other hand, we have:

$$\begin{aligned} F_f(w + \lambda u) - F_f(w) &= \|w + \lambda u\|^2 - 2f(w + \lambda u) - \|w\|^2 + 2f(w) \\ &= \|w + \lambda u\|^2 - \|w\|^2 - 2\lambda f(u) \end{aligned}$$

and thus

$$2\lambda f(u) \leq \|w + \lambda u\|^2 - \|w\|^2 \tag{8.6}$$

for all $u \in X$ and $\lambda \in \mathbb{R}$.

Suppose that $\lambda > 0$. Then from (8.6), we have:

$$f(u) \leq \frac{\|w + \lambda u\|^2 - \|w\|^2}{2\lambda}$$

which gives, by passing at limit after $\lambda$, $\lambda \to 0+$,

$$f(u) \leq (u, w)_s$$

for all $u \in X$.

Now, if we replace $u$ by $-u$, we get

$$f(u) \geq -(-u, w)_s = (u, w)_i$$

for all $u \in X$, which completes the proof. ∎

The second result is the following.

THEOREM 54. *Let* $(X, \|\cdot\|)$ *be a real normed space,* $f : X \to \mathbb{R}$ *a nonzero bounded linear functional and* $w \in X \diagdown \{0\}$. *Then the following statements are equivalent:*

(i) $w \perp Ker\,(f)\,(B)$;
(ii) *The element* $u_0 := \frac{f(w)}{\|w\|^2} w$ *minimizes the quadratic functional:*

$$F_f : X \to \mathbb{R}, \quad F_f\,(u) := \|u\|^2 - 2f\,(u)\,.$$

PROOF. By Theorem 51, we have that $w \perp Ker\,(f)\,(B)$ iff one has the estimation:

$$\left(x, \frac{f\,(w)}{\|w\|^2} w\right)_i \leq f\,(x) \leq \left(x, \frac{f\,(w)}{\|w\|^2} w\right)_s$$

for all $x \in X$.

Now, the above estimation is equivalent, by Theorem 53, to the fact that the vector $u_0 = \frac{f(w)}{\|w\|^2} w$ minimizes the quadratic functional $F_f$.

The proof is thus completed. ∎

COROLLARY 14. *Let* $(X, \|\cdot\|)$ *be a real normed space,* $G$ *a closed linear subspace in* $X$ *and* $x_0 \in X \diagdown \{G\}$. *Then the following statements are equivalent.*

(i) $x_0 \perp G\,(B)$;
(ii) *Wherever* $f \in (G \oplus S_p\,(x_0))^*$ *with* $Ker\,(f) = G$, *the element* $u_0 = \frac{f(x_0)}{\|x_0\|^2} x_0$ *minimizes the quadratic functional* $F_{x_0,f}\,(u) : G \oplus S_p\,(x_0) \to \mathbb{R}$ *given by*

$$F_{x_0,f}\,(u) = \|u\|^2 - 2f\,(u)\,.$$

The proof is obvious from the above theorem for $X_{x_0} := G \oplus S_p\,(x_0)$. We omit the details.

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CHAPTER 9

# Orthogonality Associated to the Semi-Inner Product

## 1. Orthogonality in the Sense of Giles

Let $(X, \|\cdot\|)$ be a normed linear space and $[\cdot,\cdot]$ a $L.-G.-$s.i.p which generates the norm $\|\cdot\|$. In [**1**], J. R. Giles introduced the following concept.

Definition 22. *An element $x \in X$ is said to be Giles-orthogonal over the element $y \in X$ relative to $L.-G.-s.i.p$ $[\cdot,\cdot]$ or $G-$orthogonal, for short, if the condition*

$$[y, x] = 0$$

*holds. We denote this by $x \perp y\,(G)$.*

It is obvious that $x \perp x\,(G)$ implies that $x = 0$, $x \perp y\,(G)$ and $\alpha \in \mathbb{K}$ imply that $(\alpha x) \perp y\,(G)$ and $x \perp (\alpha y)\,(G)$ and $x \perp y\,(G)$, $x \perp z\,(G)$ imply the right additivity, i.e., $x \perp (y+z)\,(G)$. The argument of these facts follows by the properties of semi-inner product in the sense of Lumer-Giles.

Now, if $E$ is a linear subspace in normed linear space $X$, then by $E^{\perp}(G)$ we will denote the orthogonal complement in Giles' sense associated to $E$. It is easy to see that it satisfies $E \cap E^{\perp}(G) = \{0\}$, $\alpha \in \mathbb{K}$ and $x \in E^{\perp}(G)$ imply that $\alpha x \in E^{\perp}(G)$ and generally $E^{\perp}(G)$ is not a linear subspace in $X$.

The following theorem contains a result concerning the $(G)-$ orthogonality of an element over a hyperplane defined by a bounded linear functional on a normed linear space (see also [**2**] or [**3**]).

Theorem 55. *Let $(X, \|\cdot\|)$ be a normed linear space and $[\cdot,\cdot]$ a $L.-G.-s.i.p$ which generates the norm $\|\cdot\|$. If $f : X \to \mathbb{K}$ is a bounded linear functional on $X$, $f \neq 0$, and $w \in X \diagdown \{0\}$, then the following statements are equivalent:*

(i) $w \perp Ker\,(f)\,(G)$;

(ii) *We have the representation:*

$$f(x) = \left[x, \frac{\overline{f(w)}}{\|w\|^2} w\right]. \tag{9.1}$$

*Moreover, if (i) or (ii) holds, then:*

$$\|f\| = \frac{|f(w)|}{\|w\|}. \tag{9.2}$$

PROOF. "(i) $\Longrightarrow$ (ii)" Let us assume that $w \perp Ker(f)(G)$. Then we have $[y, w] = 0$ for all $y \in Ker(f)$.

Let $x \in X$ and $y = f(x)w - f(w)x$. It is obvious (see Theorem 51) that $y \in Ker(f)$ and then:

$$[f(x)w - f(w)x, w] = 0 \quad \text{for all } x \in X, \tag{9.3}$$

which is equivalent with

$$f(x) = \left[x, \frac{\overline{f(w)}}{\|w\|^2} w\right], \quad x \in X,$$

and the implication is proven.

"(ii) $\Longrightarrow$ (i)" If the representation (9.1) holds, then clearly, $f(w) \neq 0$, which gives:

$$[x, w] = 0 \quad \text{for all } x \in Ker(f)$$

i.e., $w \perp Ker(f)(G)$, and the implication is proven.

The relation (9.2) follows by Proposition 3 and we shall omit the details.

The proof of the theorem is thus completed. ∎

By the use of the above result, we can also state the following theorem.

THEOREM 56. *Let $(X, \|\cdot\|)$ be a normed linear space, $E$ a closed linear subspace in $X$ and $x_0 \in X \setminus E$. Then the following statements are equivalent:*

(i) $x_0 \perp E(G)$;
(ii) *For every $f \in (E \oplus S_p(x_0))^*$ with $E = Ker(f)$, we have the representation:*

$$f(x) = \left[x, \frac{\overline{f(x_0)}}{\|x_0\|^2} x_0\right]$$

*for all $x$ in $E \oplus S_p(x_0)$.*

*In addition, if (i) or (ii) holds, then one has:*

$$\|f\|_{G \oplus S_p(x_0)} = \frac{|f(x_0)|}{\|x_0\|}.$$

The proof is obvious from Theorem 55 applied for the normed space $X_{x_0} := E \oplus S_p(x_0)$.

Let us now establish the connection between Birkhoff-James' and Giles' orthogonality.

Proposition 32. *Let $(X, \|\cdot\|)$ be a real or complex normed space and $[\cdot,\cdot]$ a $L.-G.-$s.i.p which generates the norm $\|\cdot\|$. If $x, y \in X$ and $x \perp y\,(G)$ then $x \perp y\,(B\,[J])$. The converse is generally not true.*

Proof. Let us assume that $x \perp y\,(G)$, i.e., $[y,x] = 0$. Then

$$\|x\|^2 = [x,x] = \operatorname{Re}[x + \lambda y, x] \le \|x\|\,\|x + \lambda y\|$$

for all $\lambda \in \mathbb{K}$, i.e., $\|x\| \le \|x + \lambda y\|$ for all $\lambda \in \mathbb{K}$ which is equivalent with $x \perp y\,(B\,[J])$.

For the converse, let us consider the space $l^1(\mathbb{C})$. It is known that

$$[y,x] = \|x\| \sum_{x_k \ne 0} \frac{\overline{x_k} y_k}{|x_k|}, \quad x, y \in l^1(\mathbb{C})$$

is a $L. - G.-$s.i.p on $l^1(\mathbb{C})$.

Consider the vectors

$$x = (i, 1, 0, \ldots, 0) \quad \text{and} \quad y = (2, i, 0, \ldots, 0).$$

We obtain

$$\|x\| = 2 \quad \text{and} \quad \|x + \lambda y\| = \left(1 + 4\lambda^2\right)^{\frac{1}{2}} + \left(1 + \lambda^2\right)^{\frac{1}{2}}, \quad \lambda \in \mathbb{R}$$

and then

$$\|x + \lambda y\| \ge \|x\| \quad \text{for all} \quad \lambda \in \mathbb{R},$$

because a simple calculation shows that

$$\left(1 + 4\lambda^2\right)^{\frac{1}{2}} + \left(1 + \lambda^2\right)^{\frac{1}{2}} \ge 2 \quad \text{for all} \quad \lambda \in \mathbb{R}.$$

On the other hand, it is obvious that $[y,x] = -2i \ne 0$ and the proof is completed. ∎

We note that the following result also holds.

Proposition 33. *Let $(X, \|\cdot\|)$ be a real or complex normed space, $E$ its linear subspace and $x \in X \setminus E$. If $x \perp E\,(B\,[J])$, then there exists at least one $L. - G.-$s.i.p which generates the norm $\|\cdot\|$ and for which we have $x \perp E\,(G)$.*

Proof. Let us consider the subspace $E_1 := S_p(x_0) \oplus E$ and $g_1 \in E_1$. Then $g_1 = \lambda x + g$ and this decomposition is unique ($\lambda \in \mathbb{K}$, $g \in E$). Define the functional

$$f_0 : E_1 \to \mathbb{K}, \;\; f_0(g_1) = \lambda \|x\|^2.$$

Then $f_0$ is well-defined and $f_0$ is linear on $E_1$. We also have $f_0(x) = \|x\|^2$ and $f_0(g) = 0$ for all $g \in G$.

Now, for all $g_1 \in E_1$ with: $g_1 = \lambda x + g$ and $\lambda \neq 0$, one has

$$\frac{|f_0(g_1)|}{\|g_1\|} = \frac{\lambda \|x\|^2}{\|\lambda x + g\|} = \frac{\|x\|^2}{\left\|x + \frac{1}{\lambda} g\right\|} \leq \|x\|,$$

because $x \perp E(B[J])$, which shows that $\|f_0\|_{E_1} = \|x\|$. On the other hand, one has:

$$\|f_0\|_{E_1} \geq \frac{|f_0(x)|}{\|x\|} = \frac{\|x\|^2}{\|x\|} = \|x\|$$

which shows that $\|f_0\|_{E_1} = \|x\|$.

By the Hahn-Banach theorem, there exists a functional $f : X \to \mathbb{K}$ such that:

$$f_{E_1} = f_0 \quad \text{and} \quad \|f\| = \|f_0\|_{E_1} = \|x\|$$

and then

$$f(x) = f_0(x) = \|x\|^2 \quad \text{and} \quad \|f\| = \|x\|, \quad \text{i.e., } f \in J(x)$$

($J$ is the normalised duality mapping).

Now, let $\tilde{J}$ be a section of the duality mapping such that $\tilde{J}(x) = f$, then the $L.-G.-$s.i.p which can be generated by $\tilde{J}$ is given by $[y, z] := \left\langle \tilde{J}(z), y \right\rangle$, $z, x \in X$.

It is easy to see that $[y, x] = \left\langle \tilde{J}(x), y \right\rangle = f(y) = 0$ for every $y \in E$ and then $x \perp E(G)$ relatively at $L. - G.-$s.i.p defined above.

The proposition is thus proved. ∎

By the use of the above proposition, we can state the following characterisation of Birkhoff-James orthogonality in terms of $G-$ orthogonality.

Theorem 57. *Let $(X, \|\cdot\|)$ be a normed space, $E$ its linear subspace and $x_0 \in X \setminus E$. Then the following statements are equivalent:*

(i) $x_0 \perp E(B[J])$;
(ii) *There exists a $L.-G.-s.i.p$ $[\cdot, \cdot]$ which generates the norm $\|\cdot\|$ and for which $x_0 \perp E(G)$.*

Finally, using Theorems 56 and 57, we can state the following theorem of representation for the continuous linear functionals.

Theorem 58. *Let $(X, \|\cdot\|)$ be a normed space, $E$ a closed linear subspace in $X$ and $x_0 \in X \setminus E$. Then the following statements are equivalent:*

(i) $x_0 \perp E(B[J])$;

(ii) *There exists a* $L.-G.-s.i.p$ *on* $X$ *which generates the norm* $\|\cdot\|$ *and is such that for all* $f \in (E \oplus S_p(x_0))^*$ *with* $E = Ker(f)$ *one has the representation:*

$$f(x) = \left[x, \frac{f(x_0)}{\|x_0\|^2} x_0\right]$$

*for all* $x \in E \oplus S_p(x_0)$*.*

## 2. Orthogonality in the Sense of Miličić

Let $(X, \|\cdot\|)$ be a real or complex normed space and $(\cdot,\cdot,)_g$ the semi-inner product in the sense of Miličić associated to the norm $\|\cdot\|$, i.e., the mapping $(\cdot,\cdot,)_g : X \times X \to \mathbb{R}$ given by

$$(x, y)_g := \frac{1}{2}\left[(x, y)_s + (x, y)_i\right] \quad \text{for all } x, y \in X,$$

where $(\cdot,\cdot,)_i$ and $(\cdot,\cdot,)_s$ are given as:

$$(x, y)_i = \lim_{t \to 0^-} \frac{\|y + tx\|^2 - \|y\|^2}{2t}$$

and

$$(x, y)_s = \lim_{t \to 0^+} \frac{\|y + tx\|^2 - \|y\|^2}{2t}$$

where $x, y \in X$.

In 1987, P.M. Miličić [**4**] introduced the following concept of orthogonality associated to the semi-inner product $(\cdot,\cdot,)_g$ on a real or complex normed space.

Definition 23. *Let* $x, y$ *be two vectors in* $X$*. The vector* $x$ *is said to be* $g-$*orthogonal over the vector* $y$ *iff* $(y, x)_g = 0$*. We denote this by* $x \perp y\,(g)$*.*

In the case when the space $X$ is complex, we can also introduce the concept of complex $g-$orthogonality [**4**, Definition 2]:

Definition 24. *Let* $x, y \in X$*,* $X$ *is here a complex normed space. Then* $x$ *is said to be complex-g-orthogonal over* $y$ *of* $cg-$*orthogonal, for short, if* $(y, x)_g = (iy, x)_g = 0$*. We denote this by* $x \perp y\,(cg)$*.*

Remark 17. *([**4**]). If in* $X$ *we can introduce an inner product* $(\cdot,\cdot)$ *then* $(y, x)_g = i\,(iy, x)_g = (y, x)$ *in the case of complex cases.*

If the normed space $X$ has the $(G)-$property, i.e., the functional $(\cdot,\cdot)_g$ is linear in the first variable (see Section 2 of Chapter 4), then it is easy to see that $[x, y] = (x, y)_g$ is a $L.-G.-$s.i.p in the real case and $(y, x)_g = (y, x)$ in the case of real spaces and $[x, y] = (x, y)_g = i\,(ix, y)_g$ is also a $L.-G.-$s.i.p in the complex case.

Consequently, in the case of normed spaces of $(G)-$type, we have the equivalence:

(i) $x \perp y\,(G)$ iff $x \perp y\,(g)$ if $X$ is real

and

(ii) $x \perp y\,(G)$ iff $x \perp y\,(cg)$ if $X$ is complex.

Now, we will point out the connection between Birkhoff-James', Giles' and Miličić's orthogonality in the case of general normed spaces.

The first result is embodied in the following proposition [**4**].

Proposition 34. *Let $(X, \|\cdot\|)$ be a complex normed space. Then $x \perp y\,(cg)$ implies $x \perp y\,(g)$. The converse is not generally true.*

Proof. The implication is obvious by the definition of the involved orthogonalities.

For the converse, let us consider the complex space $l^1(\mathbb{C})$ endowed with the usual norm $\|x\| = \sum_{i=1}^{\infty} |x_i| < \infty$. It is well known that (see [**4**])

$$(y, x)_g = \|x\| \sum_{x_k \neq 0} \frac{\operatorname{Re}(\overline{x_k}, y_k)}{|x_k|},$$

and

$$(y, x)_g - i\,(iy, x)_g = \|x\| \sum_{x_k \neq 0} \frac{\overline{x_k}, y_k}{|x_k|}.$$

Now, if we put

$$x = (i, 1, 0, \dots) \quad \text{and} \quad y = (2, i, 0, 0, \dots)$$

we obtain

$$(y, x)_g = 0 \quad \text{and} \quad (iy, x)_g = 2i$$

which completes the proof. ∎

Proposition 35. *Let $(X, \|\cdot\|)$ be a real or complex normed space. Then $x \perp y\,(g)$ implies $x \perp y\,(B)$. The converse is generally not true.*

Proof. ([**4**]) Let us assume that $x \perp y\,(g)$, i.e., $(y, x)_g = 0$. Then for all $\lambda \in \mathbb{R}$ we have:

$$(x + \lambda y, x)_g = \|x\|^2 + (\lambda y, x)_g = \|x\|^2 + \lambda\,(y, x)_g = \|x\|^2.$$

On the other hand we have:

$$(x + \lambda y, x)_g \leq \|x\|\,\|x + \lambda y\|$$

from where results

$$\|x + \lambda y\| \geq \|x\| \quad \text{for all} \quad \lambda \in \mathbb{R}$$

which shows that $x \perp y\,(B)$.

For the converse, we choose $x, y \in l^1(\mathbb{C})$ with

$$x = (1, 0, 0, \dots) \quad \text{and} \quad y = (1, 5i, 0, \dots)$$

Then we have:

$$\|x + \lambda y\| = |1 + \lambda| + 5\,|\lambda| \geq 1 = \|x\| \quad \text{for all} \quad \lambda \in \mathbb{R}$$

i.e., $x \perp y\,(B)$. However, a simple calculation shows that

$$(y, x)_g = 1$$

and the proof is completed. ∎

The following proposition established the connection between James' orthogonality and $cg-$orthogonality in a complex normed space.

Proposition 36. *Let $(X, \|\cdot\|)$ be a complex normed space. If $x \perp y\,(cg)$ then $x \perp y\,(J)$. The converse is not generally true.*

Proof. Let $x \perp y\,(cg)$, i.e., $(y, x)_g = (iy, x)_g = 0$. Consider the functional $f_y : X \to \mathbb{C}$ given by

$$f_x(z) := (z, x)_i - i\,(iz, x)_s\,, \ \ z \in X.$$

This functional is linear on $X$ (see Proposition 11, (ii), (iii)) and bounded (the same proposition (iv)). In the paper [**4**, Theorem 1], P.M. Miličić proved that $f_x$ also belongs to $J(x)$ and then $\|f_x\| = \|x\|$. Thus, we can state:

$$|f_x(x + \lambda y)| = |f_x(x) + \lambda f_x(y)| = |f_x(x)| = \|x\|^2$$

for all $\lambda \in \mathbb{C}$.

On the other hand, one has:

$$|f_x(x + \lambda y)| \leq \|x\|\,\|x + \lambda y\| \text{ for all } \lambda \in \mathbb{C},$$

which shows that $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in \mathbb{C}$, i.e., $x \perp y\,(J)$.

For the converse, we may choose:

$$x = (1, 0, 0, \dots) \quad \text{and} \quad y = (1, 5i, 0, \dots) \in l^1(\mathbb{C})\,.$$

Then one has:

$$\|x + \lambda y\| = |1 + \lambda| + 5\,|\lambda i| = |1 + \lambda| + 5\,|\lambda| \geq 1 = \|x\|$$

for all $\lambda \in \mathbb{C}$, which is equivalent with

$$x \perp y\,(J)\,.$$

Now, we observe that $(y, x)_g = 1$, which completes the proof. ∎

Next, we present some other characterisations of $(g)-$orthogonality in terms of bounded linear functionals which will improve some known results obtained by P.M. Miličić in [**4**].

THEOREM 59. *Let $(X, \|\cdot\|)$ be a real normed space and $f : X \to \mathbb{R}$ a bounded linear functional on $X$, $f \neq 0$ and $w$ an element from $X \setminus \{0\}$. Then the following statements are equivalent:*

(i) $w \perp Ker(f)(g)$;
(ii) *we have the representation:*

$$f(x) = \left(x, \frac{f(w)}{\|w\|^2} w\right)_g \tag{9.4}$$

*for all $x \in X$.*

*In addition, if (i) or (ii) holds, then $\|f\| = \frac{|f(w)|}{\|w\|}$.*

PROOF. "(i) $\Longrightarrow$ (ii)". Let us assume that $w \perp Ker(f)(g)$, i.e., $(y, w)_g = 0$ for all $y \in Ker(f)$. Let $x \in X$ and put $y = f(x) w - f(w) x$. Then $y \in Ker(f)$ and

$$(f(x) w - f(w) x, w)_g = 0 \text{ for all } x \in X.$$

Using the properties of $(\cdot, \cdot)_g$, we have:

$$f(x) \|w\|^2 - f(w)(x, w)_g = 0 \text{ for all } x \in X$$

which gives:

$$f(x) = \frac{f(w)}{\|w\|^2}(x, w)_g = \left(x, \frac{f(w)}{\|w\|^2} w\right)_g$$

for all $x \in X$.

"(ii) $\Longrightarrow$ (i)". It is clear that $f(w) \neq 0$ because $f(w) = 0$ implies the fact $f \equiv 0$, which is a contradiction.

Since $f(x) = \frac{f(w)}{\|w\|^2}(x, w)_g$ for all $x \in X$, which implies that $(x, w)_g = 0$ for all $x$ in $Ker(f)$, which is equivalent with $w \perp Ker(f)(g)$.

Now, let us prove the last part of theorem.

By the representation (9.4), we have:

$$|f(x)| = \frac{|f(w)|}{\|w\|^2}\left|(x, w)_g\right| \leq \frac{|f(w)|}{\|w\|}\|x\|$$

for all $x$ in $X$, which gives:

$$\|f\| \leq \frac{|f(w)|}{\|w\|}.$$

On the other hand, we have $\frac{|f(w)|}{\|w\|} \leq \|f\|$, which gives the desired result. ∎

As a corollary of this theorem, we can state the following result.

Theorem 60. *Let $(X, \|\cdot\|)$ be a real normed space, $G$ a closed linear subspace in $X$ and $x_0 \in X \diagdown G$. Then the following statements are equivalent:*

(i) $x_0 \perp G\,(g)$;

(ii) *For every $f \in (G \oplus S_p(x_0))^*$ such that $Ker\,(f) = G$, we have the representation:*

$$f(x) = \left(x, \frac{f(x_0)}{\|x_0\|^2} x_0\right)_g$$

*for all $x \in G \oplus S_p(x_0)$.*

*Moreover, if (i) or (ii) holds, then one has:*

$$\|f\|_{G \oplus S_p(x_0)} = \frac{|f(x_0)|}{\|x_0\|}.$$

The proof is obvious from the above theorem for the space $X_{x_0} := G \oplus S_p(x_0)$.

## 3. The Superior and Inferior Orthogonality

Let $(X, \|\cdot\|)$ be a real or complex normed space and let $(\cdot,\cdot)_s$, $(\cdot,\cdot)_i$ be the semi-inner products associated with this normed space. The following definition is natural to be considered.

Definition 25. *Let $(X, \|\cdot\|)$ be a normed space and let $x, y$ be two fixed elements in $X$. Then $x$ is said to be superior-orthogonal (inferior orthogonal) or $(s)-$orthogonal ($(i)-$orthogonal) over $y$, for short, iff*

$$(y, x)_s = 0 \quad ((y, x)_i = 0).$$

We denote this by $x \perp y\ (s\,[i])$. If $x \perp y\,(s)$ and $x \perp y\,(i)$, then we will write this as $x \perp y\,(s, i)$.

Remark 18. *Since the mapping $\mathbb{R} \ni t \longmapsto \|y + tx\|^2 \in \mathbb{R}$ is a convex mapping, then one has:*

$$(x, y)_i = \lim_{t \to 0^-} \frac{\|y + tx\|^2 - \|y\|^2}{2t} \le \lim_{t \to 0^+} \frac{\|y + tx\|^2 - \|y\|^2}{2t} = (x, y)_s$$

*for all $x, y$ in $X$.*

The following proposition is obvious by the definition of $(s)$ and $(i)$ orthogonality.

Proposition 37. *Let $x, y \in X$. Then the following statements are true:*

(i) $x \perp x\ [s\,(i)] \Longrightarrow x = 0$;

(ii) $x \perp y\ [s\,(i)] \Longrightarrow (-x) \perp y\ [i\,(s)] \Longleftrightarrow x \perp (-y)\ [i\,(s)]$;

(iii) $x \perp y\ [s\,(i)] \Longleftrightarrow (-x) \perp (-y)\ [s\,(i)]$;

(iv) $x \perp y \ [s(i)] \Longleftrightarrow \alpha x \perp \beta y \ [s(i)]$ *with* $\alpha\beta > 0$.

Now, we are able to establish the connection between these orthogonalities and those introduced by Miličić and Birkhoff.

Proposition 38. *Let $x, y$ be two vectors in $X$. Then $x \perp y \ [s(i)]$ implies that $x \perp y\,(B)$. The converse generally does not hold.*

Proof. Let us assume that $x \perp y\,(s)$, i.e., $(y,x)_s = 0$. Since $(y,x)_i \leq (y,x)_s$, we can write $(y,x)_i \leq 0 \leq (y,x)_s$ which implies that (see Theorem 50) $x \perp y\,(B)$.

For the converse, let us consider the space $l^1(\mathbb{C})$ in which we know that:

$$(x,y)_{s(i)} = \|y\| \left( \sum_{y_i \neq 0} \frac{\operatorname{Re}(\overline{y_i} x_i)}{|y_i|} \pm \sum_{y_i = 0} |x_i| \right).$$

If we choose the vectors

$$x = (i, 1, 1, 0, \dots) \quad \text{and} \quad y = (1, i, 1, 1, 0, \dots) \in l^1(\mathbb{C})$$

we have:

$$\|x\| = 3 \quad \text{and} \quad \|x + \lambda y\| = 2\left(1 + \lambda^2\right)^{\frac{1}{2}} + |1 + \lambda| + |\lambda|, \quad \lambda \in \mathbb{R}.$$

A simple calculation shows that

$$\|x + \lambda y\| \geq \|x\| \quad \text{for all } \lambda \in \mathbb{R}$$

which means that $x \perp y\,(B)$.

On the other hand, it is easy to see that

$$(x,y)_s = (x,y)_i = 1$$

which proves the assertion. ∎

Another result which establishes the connection between $(g)$ −orthogonality and $(s,i)$ −orthogonality is the following one.

Proposition 39. *Let $x, y$ be two elements from $X$. If $x \perp y\,(s,i)$, then also $x \perp y\,(g)$. The converse is not generally true.*

Proof. If $x \perp y\,(s,i)$, then $(y,x)_i = (y,x)_s = 0$, which gives $(y,x)_s = 0$, i.e., $x \perp y\,(g)$.

For the converse, let us consider in $l^1(\mathbb{C})$, the vectors:

$$\begin{aligned} x &= (i, 1, 1, 0, \dots) \\ y &= (1, i, 0, \dots). \end{aligned}$$

Then we have:

$$(x,y)_g = \frac{1}{2}\|y\| \sum_{y_i \neq 0} \frac{\operatorname{Re}(\overline{y_i} x_i)}{|y_i|} = 0$$

and

$$(x, y)_s = 1, \quad (x, y)_i = -1.$$

The proposition is thus proven. ∎

REMARK 19. *We will show that* $(s)-$*orthogonality or* $(i)-$*orthogonality does not imply the orthogonality in the sense of Miličić.*
*Indeed, if we choose* $x = (-1, -1, 2, 0, \dots)$ *and* $y = (1, 1, 0, \dots)$ *in* $l^1(\mathbb{C})$, *we get:*

$$(x, y)_g = -4 \quad \text{and} \quad (x, y)_s = 0$$

*which shows that* $y \perp x\,(i)$ *but* $y \not\perp x\,(g)$.

Now, we will state and prove a result which gives a characterisation of $(s)\,[(i)]-$ orthogonality.

THEOREM 61. *Let* $(X, \|\cdot\|)$ *be a real normed space and* $f : X \to \mathbb{R}$ *a bounded linear functional on* $X$, $f \neq 0$. *If* $w \in X \diagdown \{0\}$, *then the following statements are equivalent:*

(i) $w \perp Ker\,(f)\,(s)$;
(ii) $w \perp Ker\,(f)\,(i)$;
(iii) *We have the representation*

$$f(x) = \left(x, \frac{f(w)}{\|w\|^2} w\right)_s$$

*for all* $x$ *in* $X$;
(iv) *We have the representation*

$$f(x) = \left(x, \frac{f(w)}{\|w\|^2} w\right)_i$$

*for all* $x$ *in* $X$;

*Moreover, each of the above statements implies the following statements which are also equivalent:*

(v) *We have* $(x, w)_s = (x, w)_i$ *for every* $x \in X$;
(vi) *The norm* $\|\cdot\|$ *is Gâteaux differentiable in* $w$;
(vii) $w$ *is a point of smoothness for the space* $x$.

PROOF. The equivalences "(v) $\Longleftrightarrow$ (vi) $\Longleftrightarrow$ (vii)" follow by Theorem 1 and Theorem 23.

"(iii) $\Longrightarrow$ (iv)". Suppose that $f(x) = \left(x, \frac{f(w)}{\|w\|^2} w\right)_s$ for all $x$ in $X$. Then we have:

$$f(x) = -f(-x) = -\left(-x, \frac{f(w)}{\|w\|^2} w\right)_s = \left(x, \frac{f(w)}{\|w\|^2} w\right)_i$$

which proves the implication.

"(iv) $\Longrightarrow$ (iii)". Is similar.

"(iii) $\Longrightarrow$ (i)". Is obvious.

"(i) $\Longrightarrow$ (iii)". Let us assume that $w \perp Ker\,(f)\,(s)$. Then for all $x \in X$ we have $y = f\,(x)\,w - f\,(w)\,x \in Ker\,(f)$ and thus

$$(f\,(x)\,w - f\,(w)\,x, w)_s = 0, \quad x \in X.$$

A simple calculation shows that

$$f\,(x) = \left(x, \frac{f\,(w)\,w}{\|w\|^2}\right)_i; \quad x \in X$$

and since "(iv) $\Longleftrightarrow$ (iii)", the implication is also proved.

"(i) $\Longleftrightarrow$ (ii)". It is obvious by the definition.

"(iii) $\Longrightarrow$ (v)". Let us assume that $f\,(w) > 0$. Then we have:

$$(x, w)_i = \frac{f\,(x)\,\|w\|^2}{f\,(w)}, \; x \in X,$$

which shows that $(x, w)_s = (x, w)_i$ for all $x$ in $X$.

The case when $f\,(w) < 0$ follows likewise and we will omit the details. ∎

Remark 20. *If any one of the statements (i), (ii), (iii) or (iv) from above is valid, we also have that:*

$$\|f\| = \frac{|f\,(w)|}{\|w\|}.$$

As a consequence of the above theorem, we can also state:

Theorem 62. *Let $(X, \|\cdot\|)$ be a real normed space and $G$ its closed linear subspace. Suppose $x_0 \in X \diagdown G$. Then the following assertions are equivalent:*

(i) *$x_0 \perp G\,(s)$;*
(ii) *$x_0 \perp G\,(i)$;*
(iii) *For all $f \in (G \oplus S_p\,(x_0))^*$ with $Ker\,(f) = G$, we have the representation*

$$f\,(x) = \left(x, \frac{f\,(x_0)}{\|x_0\|^2} x_0\right)_s$$

*for all $x \in G \oplus S_p\,(x_0)$;*

(iv) *For all $f \in (G \oplus S_p\,(x_0))^*$ with $Ker\,(f) = G$, we have the representation*

$$f\,(x) = \left(x, \frac{f\,(x_0)}{\|x_0\|^2} x_0\right)_i$$

*for all $x \in G \oplus S_p\,(x_0)$.*

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CHAPTER 10

# Characterisations of Certain Classes of Spaces

## 1. The Case of Giles Orthogonality

Let $X$ be a complex normed space and $[\cdot,\cdot]$ a $L.-G.-$s.i.p. which generates the norm of $X$. We will denote by $S$ the unit sphere, i.e., $S := \{x \in X | \, \|x\| = 1\}$ and by $B$, the unit ball given by $B := \{x \in X | \, \|x\| \leq 1\}$.

Definition 26. *We will say that the point $x \in S$ is an extremal point of the ball $B$ if $x_1, x_2 \in B$ and $x = \frac{1}{2}(x_1 + x_2)$ implies that $x = x_1 = x_2$.*

It is known that if every point $x$ of $S$ is an extremal point for the unit ball $B$, then $(X, \|\cdot\|)$ is a strictly convex space.

Definition 27. *We will say that the point $x \in S$ is a point of smoothness of the unit ball $B$ if there exists a unique functional $f \in X^*$ such that $f(x) = \|f\| = 1$.*

It is also clear that if every point of the sphere $S$ is a point a smoothness, then the space $(X, \|\cdot\|)$ is smooth.

The following characterization of extremal points in terms of the semi-inner product in Lumer-Giles' sense holds (see [**1**]).

Theorem 63. *Let $(X, \|\cdot\|)$ be a complex normed space and $x \in S$. Then the following statements are equivalent:*

(i) *$x$ is an extremal point of $B$;*
(ii) *$\|x \pm y\| \leq 1$ implies that $y = 0$;*
(iii) *$\mathrm{Re}\,[y, x \pm y] = 0$ implies that $y = 0$;*
(iv) *If $y \in S$ and $y \neq x$, then $\mathrm{Re}\,[y, x] < 1$.*

We need the following lemmas which are also interesting in themselves [**1**].

Lemma 4. *Let $(X, \|\cdot\|)$ be as above, $x \in S$ and $y \in X$ such that $\|x \pm y\| \leq 1$. Then $x \perp y\,(G)$ and for every $t \in [-1, 1]$ one has $\|x + ty\| = 1$.*

Proof. Using the properties of $L.-G.-$s.i.p., we can state that:

$$[\lambda x + \mu y, z] = \lambda\,[x, z] + \mu\,[y, z]$$

for all $\lambda, \mu \in \mathbb{C}$ and $x, y, z \in X$; and one has the inequality

$$|[x, y]| \leq \|x\| \cdot \|y\|$$

for all $x, y \in X$.

Now, if $x \in S$, and $\|x \pm y\| \leq 1$, we get

$$|1 \pm [y, x]| = |[x \pm y, x]| \leq \|x \pm y\| \, \|x\| \leq 1$$

which shows that $[y, x] = 0$ and for every $\lambda \in \mathbb{C}$ we have

$$1 = |[x + \lambda y, x]| \leq \|x + \lambda y\| .$$

For $\lambda = \pm 1$, we obtain $1 \leq \|x \pm y\| \leq 1$, i.e., $\|x \pm y\| = 1$. Now let $t \in [-1, 1]$. Then

$$\begin{aligned} [x \pm ty, x] &= 1 \leq \|x \pm y\| = \|x - ty + tx \pm ty\| \\ &= \|(1 - t)\, x + t\,(x \pm y)\| \leq 1 - t + t = 1 \end{aligned}$$

which gives $\|x \pm ty\| = 1$, and the lemma is thus proven. ∎

LEMMA 5. *Let $(X, \|\cdot\|)$ be as above, and $[\cdot, \cdot]$ a L.−G.−s.i.p. which generates the norm $\|\cdot\|$. Then:*

$$(y, x)_i \leq \operatorname{Re}[y, x] \leq (y, x)_s$$

*for all $x, y$ in $X$.*

PROOF. Let us consider the mapping $f_x : X \to \mathbb{R}$, $f_x(y) = \operatorname{Re}[y, x]$. Then it is obvious that $x \perp Ker\,(f_x)\,(G)$ and by the use of Proposition 32, it follows that $x \perp Ker\,(f_x)\,(B)$. Now, Theorem 51 yields that:

$$\left(y, \frac{f_x(x)}{\|x\|^2} x\right)_i \leq f_x(y) \leq \left(y, \frac{f_x(x)}{\|x\|^2} x\right)_s$$

for all $y \in X$, i.e.,

$$(y, x)_i \leq \operatorname{Re}[y, x] \leq (y, x)_s$$

for all $y \in X$, and the proof is completed. ∎

PROOF. (of the Theorem) "(i) $\Longleftrightarrow$ (ii)". Let $\|x \pm y\| \leq 1$. Put $x + y = u$ and $x - y = v$. Then $x = \frac{1}{2}(u + v)$ and $u, v \in B$. By (i), we have $x = y$; i.e., $y = 0$.

Now, let $x = \frac{1}{2}(x_1 + x_2)$ with $x_1$, $x_2 \in B$. Let us put $y = \frac{1}{2}(x_1 - x_2)$. Then $x + y = x_1$ and $x - y = x_2$ and thus $\|x \pm y\| \leq 1$. By (ii), we get $y = 0$, i.e., $x_1 = x_2 = x$ and the statement is proved.

"(ii) $\Longleftrightarrow$ (iii)". Let us assume that $\operatorname{Re}[y, x \pm y] = 0$. Then

$$\begin{aligned} \|x \pm y\|^2 &= \operatorname{Re}[x \pm y, x \pm y] = \operatorname{Re}[x, x \pm y] \pm \operatorname{Re}[y, x \pm y] \\ &\leq \|x \pm y\| , \end{aligned}$$

thus $\|x \pm y\| \leq 1$.

Now, by Lemma 4, for all $t \in [-1,1]$ we have $\|x \pm ty\| = 1$. Using Lemma 5, we can write:

$$\lim_{\lambda \to 0-} \frac{\|x + (t+\lambda)\, y\| - 1}{\lambda} \le \operatorname{Re}[y, x + ty] \le \lim_{\lambda \to 0+} \frac{\|x + (t+\lambda)\, y\| - 1}{\lambda}.$$

Since $\left\|x + \left(\frac{1}{2} + \lambda\right) y\right\| = 1$ for every $\lambda \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, the above limits are zero for $t = \frac{1}{2}$ and $t = -\frac{1}{2}$. Consequently,

$$\operatorname{Re}\left[y, x \pm \frac{1}{2} y\right] = 0$$

and then

$$\operatorname{Re}\left[\frac{1}{2} y, x \pm \frac{1}{2} y\right] = 0,$$

which implies

$$\frac{1}{2} y = 0, \text{ i.e., } y = 0.$$

"(ii) $\Longleftrightarrow$ (iv)". Let $y \neq x$ with $y \in S$ and $\operatorname{Re}[y, x] = 1$. Then

$$1 + 1 = \operatorname{Re}[x + y, x] = |[x + y, x]| \le \|x + y\| \le 1 + 1$$

i.e., $\|x + y\| = 2$. If $x + y = u$ and $x - y = v$, then $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$ and $\left\|\frac{u}{2}\right\| = 1$. Consequently, $\left\|\frac{u}{2} \pm \frac{v}{2}\right\| \le 1$ and by (ii), $\frac{v}{2} = 0$, i.e., $y = x$ which produces a contradiction.

Now, let $y \neq 0$ and $\|x \pm y\| \le 1$. By virtue of Lemma 4, we have $\operatorname{Re}[y, x] = 0$ and $\|x \pm y\| = 2$, and, by (iv) (for $y \neq 0$) we get

$$\operatorname{Re}[x - y, x] = 1 - \operatorname{Re}[y, x] < 1, \text{ i.e., } \operatorname{Re}[y, x] < 0,$$

which contradicts the relation $\operatorname{Re}[y, x] = 0$.

The theorem is thus proved. ∎

COROLLARY 15. *Let $(X, \|\cdot\|)$ be a complex normed space. The following statements are equivalent:*

(i) *The space $(X, \|\cdot\|)$ is strictly convex;*
(ii) *For all $x \in S$ and $\|x \pm y\| \le 1$ implies that $y = 0$;*
(iii) *For all $x \in S$ and $\operatorname{Re}[y, x \pm y] = 0$ implies that $y = 0$;*
(iv) *For every $x, y \in S$ and $y \neq x$ implies that $\operatorname{Re}[y, x] < 1$.*

REMARK 21. *Using the property (iv) of Corollary 15, we can easily see that the spaces $l^1(\mathbb{C})$ and $\mathfrak{C}[a, b]$ are not strictly convex spaces.*

a) *For the space $l^1(\mathbb{C})$ it is known that the functional*

$$[x, y] = \|y\| \sum_{y_k \neq 0} \frac{x_k \overline{y_k}}{|y_k|}$$

*is a L. − G.−s.i.p. on $l^1(\mathbb{C})$. The vectors $x = (i, 0, \dots)$ and $y = \left(\frac{i}{2}, \frac{i}{2}, 0, \dots\right)$ belong to $S$, $y \neq x$ and* $\operatorname{Re}[y, x] = 1$.

b) *By a function $y \in \mathfrak{C}[a,b]$, let us denote by the real number $t$ in $[a,b]$ for which*

$$\|y\| = \max_{t\in[a,b]} |y(t)| = y(t_y).$$

*Then the functional $[x,y] := x(t_x)\,y(t_y)$ is a $L.-G.-s.i.p.$ on $\mathfrak{C}[a,b]$ which generates the norm of $\mathfrak{C}[a,b]$. For $x(t) = 1$ and $y(t) = (b-a)^{-1}(t-a)$ we have $\|x\| = \|y\| = 1$ and $x \neq y$. On the other hand, we have:*

$$\operatorname{Re}[y,x] = [y,x] = x(b)\,y(b) = 1,$$

*which shows that $\mathfrak{C}[a,b]$ is not strictly convex.*

Now, we will give a characterisation of smooth normed spaces in terms of Birkhoff's and Giles' orthogonality.

THEOREM 64. *Let $(X, \|\cdot\|)$ be a complex normed space. The following statements are equivalent:*

(i) *$X$ is smooth;*
(ii) *We have $y \perp x\,(B)$ if and only if $\operatorname{Re}[x,y] = 0$ where $x, y$ are vectors in $X$.*

PROOF. Firstly, we observe that $\operatorname{Re}[y,x] = 0$ implies that $y \perp x\,(B)$. Indeed, for every $\lambda \in \mathbb{R}$ and $x, y \in X$ we have:

$$\begin{aligned} \|y\|^2 &= \|y\|^2 + \lambda \operatorname{Re}[x,y] = \operatorname{Re}[y + \lambda x, y] \\ &\le |[y+\lambda x, y]| \le \|y\|\,\|y+\lambda x\| \end{aligned}$$

i.e.,

$$\|y\| \le \|y + \lambda x\| \text{ for all } \lambda \in \mathbb{R},$$

which means that $y \perp x\,(B)$.

On the other hand, if $y \perp x\,(B)$, we have:

$$\frac{\|y+\lambda x\| - \|y\|}{\lambda} \le 0 \le \frac{\|y+tx\| - \|y\|}{t} \tag{10.1}$$

for all $\lambda < 0$ and $t > 0$.

Finally, let us observe, by Lemma 5, we also have the estimation:

$$\lim_{t\to 0-} \frac{\|y+tx\| - \|y\|}{t} \le \frac{\operatorname{Re}[x,y]}{\|y\|} \le \lim_{t\to 0+} \frac{\|y+tx\| - \|y\|}{t}. \tag{10.2}$$

Now, let us assume that $X$ is smooth. Then

$$\lim_{t\to 0-} \frac{\|y+tx\| - \|y\|}{t} = \lim_{t\to 0+} \frac{\|y+tx\| - \|y\|}{t}$$

and by (10.1) and (10.2), we can state:

$$\operatorname{Re}[x,y] = 0.$$

Let us assume that (ii) holds. If $[\cdot,\cdot]_1$ is another $L.-G.-$s.i.p. on $X$ which generates the norm $\|\cdot\|$, then it is clear that $\operatorname{Re}[x,y]$ and $\operatorname{Re}[x,y]_1$ are semi inner products in the sense of Lumer-Giles on the real space $X$. Put $f_y(x) := \operatorname{Re}[x,y]$ and $g_y(x) := \operatorname{Re}[x,y]_1$, $x \in X$. Then $f_y$ and $g_y$ are two bounded linear functionals on the real space $X$. We have $g_y(x) = 0$ implies $\operatorname{Re}[x,y]_1 = 0$ and then $y \perp x\,(B)$.

By (ii), it follows that $\operatorname{Re}[y,x] = 0$, i.e., $f_y(x) = 0$. Consequently,

$$Ker\,(g_y) \subseteq Ker\,(f_y)$$

and similarly

$$Ker\,(g_y) \supseteq Ker\,(f_y),$$

which means that

$$Ker\,(g_y) = Ker\,(f_y).$$

Since $g_y(y) = f_y(y) = \|y\|^2$, it follows that $f_y = g_y$, i.e.,

$$\operatorname{Re}[x,y] = \operatorname{Re}[x,y]_1, \quad \text{for all } x,y \in X.$$

On the other hand

$$\begin{aligned}[x,y] &= \operatorname{Re}[x,y] - i\operatorname{Re}[ix,y] = \operatorname{Re}[x,y]_1 - i\operatorname{Re}[ix,y]_1 \\ &= [x,y]_1, \quad \text{for all } x,y \in X,\end{aligned}$$

thus there exists a unique $L.-G.-$s.i.p. on $X$ which generates the norm $\|\cdot\|$. Using Proposition 4, we conclude that $(X,\|\cdot\|)$ is smooth. ∎

Theorem 65. *Let $(X,\|\cdot\|)$ be a real (complex) normed space. The the following statements are equivalent:*

(i) *$X$ is smooth;*
(ii) *we have $y \perp x\,(B[J])$ if and only if $y \perp x\,(G)$, where $x,y$ are vectors from $X$.*

Proof. If the space is real, the proof is contained in the above theorem.

"(i) $\Longrightarrow$ (ii)". Suppose that $X$ is complex and $y \perp x\,(J)$, i.e.,

$$\|y+\lambda x\| \geq \|y\|$$

for all $\lambda \in \mathbb{C}$.

If $\lambda \in \mathbb{R}$, then we have $\|x+\lambda x\| \geq \|x\|$, which implies, by (i) that $\operatorname{Re}[x,y] = 0$.

On the other hand, we have:

$$\|y+itx\| \geq \|y\| \quad \text{for all } t \in \mathbb{R}$$

which implies that $\operatorname{Re}[ix,y] = 0$.

Since $[x, y] = \operatorname{Re}[x, y] - i \operatorname{Re}[ix, y]$, we obtain that $[x, y] = 0$, i.e., $y \perp x\,(G)$.

As $x \perp y\,(G)$ implies $x \perp y\,(J)$ in every complex normed space, the implication "(i) $\Longrightarrow$ (ii)" is proven.

"(ii) $\Longrightarrow$ (i)". Let us assume that $[\cdot, \cdot]$ and $[\cdot, \cdot]_1$ are two $L.-G.-$s.i.p.s which generate the norm $\|\cdot\|$. The functionals $f_y(x) := [x, y]$ and $g_y(x) = [x, y]_1$ are linear and bounded on $X$. As above, we can state: $Ker\,(f_y) = Ker\,(g_y)$ and since $f_y(y) = g_y(y) = \|y\|^2$, it follows that $f_y = g_y$ and consequently, $[\cdot, \cdot] = [\cdot, \cdot]_1$, which shows that $X$ is smooth. ∎

The following lemma is interesting as well.

Lemma 6. *Let $(X, \|\cdot\|)$ be a normed space and $[\cdot, \cdot]_1$, $[\cdot, \cdot]_2$ are two $L.-G.-$s.i.p.s on $X$ which generate the norm $\|\cdot\|$. Then the following statements are equivalent:*

(i) *$x \perp_1 y\,(G)$ implies $x \perp_2 y\,(G)$ where $x, y$ are vectors in $X$;*
(ii) *$[z, w]_1 = [z, w]_2$ for all $z, w$ in $X$.*

Proof. "(ii) $\Longrightarrow$ (i)". It is obvious.

"(i) $\Longrightarrow$ (ii)". Let $w \in X$, $w \neq 0$ and suppose that $x$ is in $X$. Then $[z, w]_1 w - \|w\|^2 z \perp_1 w\,(G)$ because we have:

$$\begin{aligned} \left[[z, w]_1 w - \|w\|^2 z, w\right]_1 &= [z, w]_1 [w, w]_1 - \|w\|^2 [z, w]_1 \\ &= \|w\|^2 [z, w] - \|w\|^2 [z, w]_1 = 0. \end{aligned}$$

Now, by (i) it follows that $[z, w]_1 w - \|w\|^2 z \perp_2 w\,(G)$, which means that:

$$\left[[z, w]_1 w - \|w\|^2 z, w\right]_2 = 0,$$

i.e.,

$$0 = [z, w]_1 [w, w]_1 - \|w\|^2 [z, w]_2 = \|w\|^2 \left([z, w]_1 - [z, w]_2\right).$$

Since $\|w\| \neq 0$, one gets

$$[z, w]_1 = [z, w]_2 \quad \text{for all } z \in X.$$

If $w = 0$, then also

$$[z, 0]_1 = [z, 0]_2 \quad \text{for all } z \in X,$$

which means that $[\cdot, \cdot]_1 = [\cdot, \cdot]_2$. ∎

Now we can state the following theorem.

Theorem 66. *In a normed space the orthogonalities induced by two semi-inner products in the sense of Lumer-Giles are either incomparable or coincidental.*

The proof is evident by the above lemma and we will omit the details.

COROLLARY 16. *Let* $(X, \|\cdot\|)$ *be a real (complex) normed space. Then the following statements are equivalent.*

(i) $X$ *is smooth;*
(ii) *There exists a* $L.-G.-$*s.i.p. for which*

$$x \perp y\,(B\,[J]) \quad \textit{implies} \quad x \perp y\,(G)\,,$$

*where* $x, y$ *are vectors in* $X$.

## 2. The Case of Miličić Orthogonality

Now, we will give some results regarding the characterisation of smooth and strictly convex normed spaces in terms of $g-$orthogonality. We will follow the paper [**2**] of Miličić.

The first result is embodied in the following theorem.

THEOREM 67. *Let* $(X, \|\cdot\|)$ *be a real or complex normed space. Then the following statements are equivalent.*

(i) $X$ *is smooth;*
(ii) $x \perp y\,(B)$ *iff* $x \perp y\,(g)$, *where* $x, y$ *are vectors in* $X$.

PROOF. Firstly, let us assume that the space $X$ is real. If $X$ is smooth, then there exists a unique $L.-G.-$s.i.p. which generates the norm $\|\cdot\|$ and this s.i.p. is given by

$$[x,y] = (x,y)_s = (x,y)_i = (x,y)_g\,, \quad x,y \in X. \tag{10.3}$$

Now using the implication "(i) $\Longrightarrow$ (ii)" of Theorem 65 (the real case), we obtain that the orthogonality in Birkhoff's sense is equivalent with that of Giles, and, by the above equality, with that of Miličić.

Let us suppose that "(ii)" holds. Fix $f$ in $J(x)$ ($J$ is the normalized duality mapping). Then $x \perp Ker\,(f)\,(B)$ (see for example Corollary 13) and by "(ii)" we deduce that $x \perp Ker\,(f)\,(g)$. This shows that $Ker\,(f) \subseteq Ker\,(\cdot, x)_g$.

On the other hand, if there exists $y \in Ker\,(\cdot, x)_g$ and $y \notin Ker\,(f)$, then we have $y = \lambda x + h$ with $\lambda \in \mathbb{R}$ and $h \in Ker\,(f)$. Consequently, by the use of the properties of $(\cdot,\cdot)_g$, we have:

$$0 = (y,x)_g = (\lambda x + h, x) = \lambda \|x\|^2 + (h,x)_g = \lambda \|x\|^2$$

from where results $\lambda = 0$ and $y = h \in Ker\,(f)$, which produces a contradiction. In conclusion, we have $Ker\,(\cdot,x)_g \subseteq Ker\,(f)$, i.e., $Ker\,(\cdot,x)_g = Ker\,(f)$. Since $(x,x)_g = \|x\|^2 = f(x)$, we deduce that $(\cdot,x)_g = f$.

Since the formula (10.3) defines a $L.-G.-$s.i.p. for which $\mathrm{Re}\,[x,y] = 0 \Longleftrightarrow x \perp y\,(B)$, in virtue of Theorem 65, we can conclude that $X$ is smooth.

Now, suppose that $X$ is complex and let $X_{\mathbb{R}}$ be the restriction of $X$ over the real number field $\mathbb{R}$. $X_{\mathbb{R}}$ is a normed space. If $X$ is smooth, then

$$\mathcal{T}_{-}(x,y) = \mathcal{T}_{+}(x,y) \quad \text{for every} \quad x,y \in X;\ x \neq 0$$

where

$$\mathcal{T}_{-(+)}(x,y) := \lim_{t \to -(+)0} \frac{\|x+ty\| - \|x\|}{t},\ x,y \in X;\ x \neq 0$$

which clearly implies that

$$\mathcal{T}_{-}(x,y) = \mathcal{T}_{+}(x,y) \quad \text{for every} \quad x,y \in X_{\mathbb{R}},$$

i.e., $X_{\mathbb{R}}$ is also a smooth normed space in which we have the condition "(ii)".

Conversely, if "(ii)" holds, then $\mathcal{T}_{-}(x,y) = \mathcal{T}_{+}(x,y)$ for all $x,y \in X_{\mathbb{R}}$, which implies that the same relation holds for all $x,y$ in $X$, i.e., $X$ is smooth. ∎

The second result contains a characterisation of strictly convex spaces in terms of $g-$ orthogonality.

THEOREM 68. *Let $(X,\|\cdot\|)$ be a normed space. Then the following statements are equivalent:*

(i) *$X$ is strictly convex;*
(ii) *If $x \perp (x-y)\,(g)$ and $x,y \in S(X)$, then $x=y$, where $S(X)$ is the unit sphere $\{x \in X |\, \|x\| = 1\}$.*

PROOF. Let us assume that $X$ is strictly convex and $x,y \in S(X)$ with $x \perp (x-y)\,(g)$. Then we have

$$0 = (x-y,x)_g = \|x\|^2 - (y,x)_g = 1 - (y,x)_g\,;\ \text{i.e.,}\ (y,x)_g = 1$$

and

$$(x+y,x)_g = \|x\|^2 + (y,x)_g = 2.$$

On the other hand, we have:

$$2 = \left|(x+y,x)_g\right| \leq \|x\|\,\|x+y\| \leq 2$$

and then $\|x+y\| = 2$ and by the strict convexity of $X$ we deduce that $x=y$.

For the converse, we need the following lemma due the Guder and Strawther (see for example [**2**]).

LEMMA 7. *A normed space is strictly convex iff $x \neq y$ implies that $J(x) \cap J(y) = \emptyset$, where $J$ is the normalised duality mapping.*

The proof of this lemma is obvious.

Let us assume that (ii) holds, but $X$ is not strictly convex. Then, by Lemma 7, there exists $x, y \in X$ with $x \neq y$ and $J(x) \cap J(y) \neq \emptyset$. Let $\varphi \in J(x) \cap J(y)$. Then we have

$$\varphi(x) = \|\varphi\| \|x\|, \quad \varphi(y) = \|\varphi\| \|y\| \quad \text{and} \quad \|x\| = \|\varphi\| = \|y\|.$$

Put

$$x_0 := x / \|x\|, \; y_0 := y / \|y\| \quad \text{and} \quad \varphi_0 := \varphi \|\varphi\|.$$

It is obvious that $\varphi_0 \in J(x_0) \cap J(y_0)$. Then

$$\varphi_0(x_0 + y_0) = 2 \leq \|\varphi_0\| \|x_0 + y_0\| = \|x_0 + y_0\| \leq 2,$$

which gives $\|x_0 + y_0\| = 2$. Let us put $u = \frac{x_0 + y_0}{2}$, $v = \frac{x_0 - y_0}{2}$. Then $\|u\| = 1$, $\|v\| \leq 1$ and $\|u \pm v\| = 1$.

Now we will state two lemmas that are also important in themselves.

LEMMA 8. *If $x, y \in X$, $\|x \pm y\| \leq \|x\|$ $(x \neq 0)$ then for all $f \in J(x)$ we have $f(y) = 0$.*

PROOF. Let $f \in J(X)$. Then

$$|f(x \pm y)| = |f(x) \pm f(y)| = \left| \|x\|^2 \pm f(y) \right| \leq \|x\| \|x \pm y\| \leq \|x\|^2.$$

It is easy to see that the system of inequalities

$$\begin{cases} \left| \|x\|^2 + z \right| \leq \|x\|^2, \\ \\ \left| \|x\|^2 - z \right| \leq \|x\|, \end{cases} \quad z \in \mathbb{C}$$

has the unique solution $z = 0$, i.e., $f(y) = 0$. ∎

LEMMA 9. *The vector $x \in X$ is normal on the hyperplane $H$, i.e., $d(x, H) = \|x\|$ iff there exists a bounded linear functional $f$ with $f^{-1}(\{0\}) = H$ and such that $f(x) = \|f\| \|x\|$.*

Now, using Lemma 8, we can state that $\operatorname{Re} f(v) = 0$ for every $f \in J(v)$. By Lemma 9 we deduce that there exists $f \in J(v)$ such that $\operatorname{Re} f(v) = (v, u)_g$. Consequently, $(v, u)_g = 0$ from where we have $\left(\frac{x_0 - y_0}{2}, u\right)_g = 0$ or $(u - y_0, u)_g = 0$ which is equivalent to $u \perp (u - y_0, u)(g)$ $(u, y_0 \in S(X))$.

By condition "(ii)", it follows that $u = y_0$, thus $x_0 = y_0$ and also $x = y$, which is a contradiction.

Consequently, $X$ is strictly convex and the theorem is proved. ∎

# Bibliography


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CHAPTER 11

# Orthogonal Decomposition Theorems

## 1. The Case of General Normed Linear Spaces

Let $(X, \|\cdot\|)$ be a real or complex normed linear space and $E$ a nonempty subset of $X$. By $E^{\perp(B)}$ we will denote the *orthogonal complement of $E$ in Birkhoff's sense,* i.e.

$$E^{\perp(B)} := \{y \in X | y \perp x\,(B) \text{ for each } x \in E\}. \tag{11.1}$$

It is obvious that $0 \in E^{\perp(B)}$ and $E \cap E^{\perp(B)} \subseteq \{0\}$. However, $E^{\perp(B)}$ is not generally a linear subspace of $X$.

The notation $X = E + E^{\perp(B)}$ $\left(X = E \oplus E^{\perp(B)}\right)$ will be understood as: for any $x \in X$, there exists a (unique) $x' \in E$ and a (unique) $x'' \in E^{\perp(B)}$ such that $x = x' + x''$.

The following result holds [**2**].

Theorem 69. *Let $(X, \|\cdot\|)$ be a reflexive Banach space. Then for any $E$ a closed linear subspace of $X$, we have the orthogonal decomposition:*

$$X = E + E^{\perp(B)}. \tag{11.2}$$

Proof. Let $E$ be a closed linear subspace of $X$ with $E \neq X$ and $x \in X$. If $x \in E$, then $x = x + 0$ with $0 \in E^{\perp(B)}$.

If $x \notin E$, since $E$ is reflexive, then there exists a best approximant in $E$, i.e., there exists an element $x' \in E$ such that $\|x - x'\| = d(x, E)$.

Let $\lambda \in \mathbb{R}$ and $y \in E$. Denote $x'' := x - x'$. Then

$$\|x'' + \lambda y\| = \|x - x' + \lambda y\| = \|x - (x' - \lambda y)\| \geq \|x - x'\| = \|x''\|$$

for any $\lambda \in \mathbb{R}$ and $y \in E$ (since, obviously, $x' - \lambda y \in E$). Thus, $x'' \in E^{\perp(B)}$, i.e., $x = x' + x''$ where $x' \in E$ and $x'' \in E^{\perp(B)}$, and the proof is completed. ∎

The following theorem provides a decomposition in a direct sum of the space $X$ [**2**].

Theorem 70. *Let $(X, \|\cdot\|)$ be a strictly convex reflexive Banach space. Then for any $E$ a closed linear subspace in $X$, we have the* *decomposition:*

$$X = E \oplus E^{\perp(B)}. \tag{11.3}$$

Proof. We need to prove only the unicity of the decomposition with elements from $E$ and $E^{\perp(B)}$.

Assume that there exists an element $x \in X$, so that

$$\begin{aligned} x &= x' + x'' \ \text{ with } \ x' \in E \text{ and } x'' \in E^{\perp(B)}, \\ y &= y' + y'' \ \text{ with } \ y' \in E \text{ and } y'' \in E^{\perp(B)}. \end{aligned}$$

Then, as above, $x - x' \perp E\,(B)$, $x - y' \perp E\,(B)$.

We utilise the following well known lemma characterising the best approximants in normed linear spaces (see for example [**5**, p. 85]).

Lemma 10. *Let $(X, \|\cdot\|)$ be a normed linear space and $E$ its non-dense linear subpace. If $x_0 \in X \backslash E$ and $g_0 \in E$, then $g_0 \in P_E(x_0)$ (where $P_E(x_0) := \left\{ g_0 \in E | \, \|g_0 - x_0\| = \inf_{g \in G} \|g - x_0\| \right\}$) if and only if $x_0 - g_0 \perp E\,(B)$.*

We deduce that $x', y' \in P_E(x)$, which contradicts the strict convexity of $X$ (see for example [**5**, p. 102]). ∎

## 2. The Case of Smooth Normed Linear Spaces

In what follows, we will apply the general results obtained above for the particular case of smooth normed linear spaces.

Let $E$ be a nonempty subset on the normed linear space $(X, \|\cdot\|)$ and $[\cdot,\cdot]$ a L.-G. s.i.p. generating the norm $\|\cdot\|$. The set $E^{\perp(G)}$ defined by

$$E^{\perp(G)} := \{ y \in X | y \perp x\,(G) \ \text{ for each } \ x \in E \} \tag{11.4}$$

will be called the *orthogonal complement in Giles' sense,* or, the Giles' $(G)-$orthogonal complement for short.

We observe that $0 \in E^{\perp(G)}$, $E \cap E^{\perp(G)} \subseteq \{0\}$ and $x \in E^{\perp(G)}$, $\alpha \in \mathbb{K}$ imply $\alpha x \in E^{\perp(G)}$, but, in general, $E^{\perp(G)}$ is not a linear subspace of $X$.

The following result holds (see also [**2**]).

Theorem 71. *Let $(X, \|\cdot\|)$ be a (strictly convex) smooth reflexive Banach space. Then for any $E$ a closed linear subspace in $X$, we have the orthogonal decomposition*

$$X = E + E^{\perp(G)} \quad \left( X = E \oplus E^{\perp(G)} \right). \tag{11.5}$$

Proof. Since, in the case of smooth normed linear spaces, the Birkhoff orthogonality is equivalent to Giles' orthogonality, the proof follows by the above two Theorems 69 and 70. ∎

The following concept was introduced in [**1**].

Definition 28. *Assume that* $(X, \|\cdot\|)$ *is a smooth normed linear space. It is said to be of* $(N)-$*type if the L.-G.-s.i.p.* $[\cdot,\cdot]$ *that generates the norm satisfies the condition:*

$$|[x, y+z]| \le |[x, y]| + |[x, z]| \quad \text{for any } x, y, z \in X. \tag{11.6}$$

Remark 22. *It is obvious that any inner product space is a smooth normed space of* $(N)-$*type. It is an open problem whether the property* $(N)$ *is characteristic for inner product spaces. We may prove the following fact (see for example* [**1**, Theorem 2.3]*).*

Theorem 72. *Let* $(X, \|\cdot\|)$ *be a smooth normed linear space of* $(N)-$*type. If* $E$ *is a closed linear subspace in* $X$, *then* $E^{\perp(G)}$ *is also a closed linear subspace in* $X$.

Proof. Assume that $x, y \in E^{\perp(G)}$. Then for any $x \in E$ we have

$$|[e, x+y]| \le |[e, x]| + |[e, y]| = 0$$

implying $x + y \in E^{\perp(G)}$.

Since $\alpha \in \mathbb{K}$, $x \in E^{\perp(G)}$ obviously imply $\alpha x \in E^{\perp(G)}$, we deduce that $E$ is a linear subspace in $X$.

Consider now the functional $p_e : X \to \mathbb{R}$, $p_e(x) = |[e, x]|$ where $x \in X$, $e \neq 0$.

Let $x_n \to x$ in $X$. Then

$$\begin{aligned} ||[e, x_n]| - |[e, x]|| &= |p_e(x_n) - p_e(x)| \le p_e(x_n - x) \\ &= |[e, x_n - x]| \le \|e\| \|x_n - x\|, \end{aligned}$$

showing that $|[e, x_n]| \to |[e, x]|$.

Now, if $y_n \in E^{\perp(G)}$ and $y_n \to y$, then for any $e \in E$ we have

$$0 = |[e, y_n]| = \lim_{n\to\infty} |[e, y_n]| = \left|\left[e, \lim_{n\to\infty} y_n\right]\right| = |[e, y]|$$

showing that $y \in E^{\perp(G)}$. Thus, $E^{\perp(G)}$ is closed and the theorem is proved. ∎

The following result holds [**1**].

Theorem 73. *Let* $(X, \|\cdot\|)$ *be a smooth reflexive Banach space with the* $(N)-$*property. Then for any* $E$ *a closed linear subspace in* $X$, *we have* $X = E \oplus E^{\perp}$ *as a linear topological direct sum.*

Proof. We need to only prove the unicity of the representation.

Let $x \in X$ and

$$\begin{aligned} x &= x' + x'' \text{ with } x' \in E \text{ and } x'' \in E^{\perp(G)}, \\ y &= y' + y'' \text{ with } y' \in E \text{ and } y'' \in E^{\perp(G)}, \end{aligned}$$

be two representations of the vector $x$ with elements from $E$ and $E^{\perp(G)}$. Then obviously,

$$x' - y' = x'' - y''$$

and since $x' - y' \in E$, $x'' - y'' \in E^{\perp}$ and $E \cap E^{\perp(G)} = \{0\}$, we obtain $x' = y'$ and $x'' = y''$. ∎

The following corollary is natural to be stated [**1**].

COROLLARY 17. *Let $(X, \|\cdot\|)$ be a smooth reflexive Banach space with the $(N)-$property. Then $X$ is topological-linear isomorphic to a Hilbert space.*

PROOF. Follows by the above theorem and by the well known Lindenstrauss-Tzafriri theorem:

THEOREM 74. *Let $(X, \|\cdot\|)$ be a Banach space. If for any $E$ a closed linear subspace in $X$ there exists a closed linear subspace $F$ such that $X = E \oplus F$ as a linear-topological direct sum, then $(X, \|\cdot\|)$ is topological-linear isomorphic to a Hilbert space.*

∎

### 3. The Case of $(Q)-$Banach and $(SQ)-$Banach Spaces

Let $E$ be a nonempty set in the $(Q)\,[(SQ)]-$normed linear space $X$. The set

$$E^{\perp(Q)[(SQ)]} := \{y \in X | y \perp x\,(Q)\,[(SQ)] \text{ for any } x \in E\}$$

is called the $Q-$orthonormal $[(SQ)-$orthonormal$]$ complement of $E$ in $X$.

It is obvious that $0 \in E^{\perp(Q)[(SQ)]}$, $E \cap E^{\perp(Q)[(SQ)]} \subseteq \{0\}$ and neither $E^{\perp(Q)}$ nor $E^{\perp S(Q)}$ are linear subspaces of $X$.

The following result holds [**3**].

THEOREM 75. *Let $\left(X, \|\cdot\|_q\right)$ be a $Q-$Banach space. Then for any $E$ a closed linear subspace in $X$, we have the decomposition*

$$X = E \oplus E^{\perp(Q)}. \tag{11.7}$$

PROOF. We know that $\left(X, \|\cdot\|_q\right)$ is a reflexive and strictly convex Banach space, being a uniformly convex Banach space (see for example Theorem 31). It is also a smooth space, being uniformly smooth (cf. Theorem 32). Since the $Q-$orthogonality is equivalent to $(G)-$orthogonality (this follows from Proposition 16, for example), the result may be obtained via Theorem 70. ∎

Analogously, the following result holds [**4**].

THEOREM 76. *Let* $\left(X, \|\cdot\|_{sq}\right)$ *be a* $(SQ)$ −*Banach space over the real or complex number field. Then for any* $E$ *a closed linear subspace in* $X$, *we have the decomposition*

$$X = E \oplus E^{\perp(SQ)}. \tag{11.8}$$

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CHAPTER 12

# Approximation of Continuous Linear Functionals

## 1. Introduction

Let $(X, \|\cdot\|)$ be a real normed space and consider the norm derivatives (see [**2**] or [**8**]):

$$(x, y)_{i(s)} := \lim_{t \to 0-(+)} \frac{\|y + tx\|^2 - \|y\|^2}{2t} \text{ for all } x, y \text{ in } X.$$

For the sake of completeness we list some usual properties of these mappings that will be used in the sequel [**2**]:

(i) $(x, x)_p = \|x\|^2$ for all $x$ in $X$;
(ii) $(-x, y)_s = (x, -y)_s = -(x, y)_i$ if $x, y$ are in $X$;
(iii) $(\alpha x, \beta y)_p = \alpha\beta\,(x, y)_p$ for all $x, y$ in $X$ and $\alpha\beta \geq 0$;
(iv) $(\alpha x + y, x)_p = \alpha\,(x, x)_p + (y, x)_p$ if $x, y$ belong to $X$ and $\alpha$ is in $\mathbb{R}$;
(v) the element $x$ in $X$ is Birkhoff orthogonal over $y$ in $X$, i.e., $\|x + ty\| \geq \|x\|$ for all $t$ in $\mathbb{R}$ iff $(y, x)_i \leq 0 \leq (y, x)_s$;
(vi) $(x + y, z)_p \leq \|x\| \|z\| + (y, z)_p$ for all $x, y, z$ in $X$;
(vii) the space $(X, \|\cdot\|)$ is smooth iff $(y, x)_i = (y, x)_s$ for all $x, y$ in $X$ or iff $(\cdot, \cdot)_p$ is linear in the first variable;

where $p = s$ or $p = i$.

For other properties of $(\cdot, \cdot)_p$ in connection to best approximation elements or continuous linear functionals, see [**2**] where further references are given.

## 2. A Characterisation of Reflexivity

To recall some well-known theorems of reflexivity due to R.C. James, we need the following concept: the nonzero element $u \in X$ is a *maximal element* for the functional $f \in X^*$ if $f(u) = \|f\| \|u\|$, [**9**, p. 35].

Theorem 77. [**6**] *Let $X$ be a Banach space. $X$ is reflexive iff every nonzero continuous linear functional on $E$ has at least one maximal element in $X$.*

Another famous result of R.C. James is the following.

THEOREM 78. [**7**] *Let $X$ be a Banach space. Then $X$ is reflexive iff for every closed and homogeneous hyperplane $H$ in $X$ (i.e., $H$ contains the null element) there exists a point $u \in X \setminus \{0\}$ such that $u \perp_B H$.*

The following characterisation of reflexivity in terms of norm derivatives also holds.

THEOREM 79. [**3**] *Let $X$ be a Banach space. $X$ is reflexive if and only if for every continuous linear functional $f$ on $X$ there exists an element $u$ in $X$ such that the following inequality holds*

$$(x, u)_i \leq f(x) \leq (x, u)_s \quad \text{for all } x \text{ in } X \tag{12.1}$$

*and* $\|f\| = \|u\|$.

PROOF. Let $H$ be a closed and homogeneous hyperplane in $X$ and $f : X \to \mathbb{R}$ be a continuous linear functional on $X$ such that $H = Ker(f)$. Then from (12.1) it follows that $u \perp_B H$ and by Theorem 78 we conclude that $X$ is reflexive.

Now, assume that $X$ is reflexive and let $f$ be a nonzero continuous linear functional on it. Since $Ker(f)$ is a closed and homogeneous hyperplane in $X$, then there exists, by Theorem 78, a nonzero element $w_0$ in $X$ such that:

$$(x, w_0)_i \leq 0 \leq (x, w_0)_s \quad \text{for all } x \in Ker(f). \tag{12.2}$$

Since $f(x) w_0 - f(w_0) x \in Ker(f)$ for all $x$ in $X$, from (12.2) we derive that:

$$(f(x) w_0 - f(w_0) x, w_0)_i \leq 0 \leq (f(x) w_0 - f(w_0) x, w_0)_s \tag{12.3}$$

for all $x$ in $X$.

On the other hand, by the use of norm derivative properties, we have

$$(f(x) w_0 - f(w_0) x, w_0)_p = f(x) \|w_0\|^2 - (x, f(w_0) w_0)_q, \quad x \in X,$$

where $p \neq q$, $p, q \in \{i, s\}$.

We conclude, by (12.3), that

$$\left(x, \frac{f(w_0) w_0}{\|w_0\|^2}\right)_i \leq f(x) \leq \left(x, \frac{f(w_0) w_0}{\|w_0\|^2}\right)_s, \quad x \in X,$$

from where results

$$(x, u)_i \leq f(x) \leq (x, u)_s \quad \text{for all } x \text{ in } X$$

where $u := \frac{f(w_0) w_0}{\|w_0\|^2}$.

To prove the fact that $\|f\| = \|u\|$, we observe that

$$\begin{aligned} -\|x\|\,\|u\| &\leq -(x,-u)_s = (x,u)_i \leq f(x) \\ &\leq (x,u)_s \leq \|x\|\,\|u\|, \quad x \in X, \end{aligned}$$

and

$$\|f\| \geq \frac{f(u)}{\|u\|} \geq \frac{(u,u)_i}{\|u\|} = \|u\|.$$

The theorem is thus proved. ∎

Remark 23. *If $u$ is an "interpolation" element satisfying the relation (12.1) then $u$ is a maximal element for the functional $f$.*
*Indeed, we have $f(u) = \|u\|^2$ and since $\|u\| = \|f\|$ we obtain $f(u) = \|f\|\,\|u\|$.*

Remark 24. *The above theorem is a natural generalization of Riesz's representation theorem which works in Hilbert spaces via a result of R.A. Tapia [**10**] for smooth spaces which is embodied in the following corollary.*

Corollary 18. [**3**] *Let $X$ be a real Banach space. Then the following statements are equivalent:*

(i) *$X$ is reflexive and smooth;*
(ii) *for every continuous linear functional $f : X \to \mathbb{R}$ there exists an element $u$ in $X$ such that:*

$$f(x) = (x,u)_s \quad \textit{for all } x \in X$$

*and $\|f\| = \|u\|$.*

In what follows, we shall point out other approximations of continuous linear functionals on real normed spaces in terms of norm derivatives.

## 3. Approximation of Continuous Linear Functionals

Let $f \in X^*$ with $\|f\| = 1$ and let $k \geq 0$. Define [**1**, p. 1]:

$$K(f,k) := \{x \in X \mid \|x\| \leq kf(x)\};$$

$K(f,k)$ is a closed convex cone. If $k > 1$, then the interior of $K(f,k)$ is nonempty.

Theorem 80. *([**3**]) Let $X$ be a real normed space, $\varepsilon \in (0,1)$, $f \in X^*$ with $\|f\| = 1$ and $u \in X$, $\|u\| = 1$ such that the norm derivative $(\cdot,u)_p$ ($p = s$ or $p = i$) is linear on $X$. If $k > 1 + 2/\varepsilon$ and $(x,u)_p \geq 0$ on $K(f,k)$ then we have the estimation:*

$$\left| f(x) - (x,u)_p \right| \leq \varepsilon \|x\| \quad \textit{for all } x \in X.$$

PROOF. The proof follows from Lemma 3 of [**1**, p. 3] for the continuous linear functional $g : X \to \mathbb{R}$, $g(x) := (x, u)_p$ and we shall omit the details. ∎

The following approximation theorem for the continuous linear functionals on a general normed linear space also holds [**3**].

THEOREM 81. *Let $f : X \to \mathbb{R}$ be a continuous linear functional such that for any $\delta \in (0, 1)$ there exists a nonzero element $x_{f,\delta}$ in $X$ with the property:*

$$\text{(A)} \qquad (x, x_{f,\delta})_i \leq \delta \|x\| \|x_{f,\delta}\| \quad \textit{for all } x \in Ker(f).$$

*Then for each $\varepsilon > 0$ there exists a nonzero element $u_{f,\varepsilon}$ in $X$ such that the following estimation holds:*

$$-\varepsilon \|x\| + (x, u_{f,\varepsilon})_i \leq f(x) \leq (x, u_{f,\varepsilon})_s + \varepsilon \|x\| \tag{12.4}$$

*for all $x$ in $X$.*

PROOF. Since $f$ is nonzero, it follows that $Ker(f)$ is closed in $X$ and $Ker(f) \neq X$.

Let $\varepsilon > 0$ and put $\delta(\varepsilon) := \frac{\varepsilon}{2\|f\|}$. If $\delta(\varepsilon) \geq 1$, then there exists an element $x_{f,\delta(\varepsilon)}$ in $X \backslash Ker(f)$ such that

$$\left(y, x_{f,\delta(\varepsilon)}\right)_i \leq \delta(\varepsilon) \|y\| \left\|x_{f,\delta(\varepsilon)}\right\| \quad \text{for all } x \in Ker(f). \tag{12.5}$$

If $0 < \delta(\varepsilon) < 1$, and since the functional $f$ has the (A)-property, then there exists an element $x_{f,\delta(\varepsilon)}$ in $X \backslash Ker(f)$ (the fact that $x_{f,\delta(\varepsilon)}$ is not in $Ker(f)$ follows from (A)) such that (12.5) is valid as well.

Put in all cases, $z_{f,\varepsilon} := \frac{x_{f,\delta(\varepsilon)}}{\left\|x_{f,\delta(\varepsilon)}\right\|}$. Then for all $x$ in $X$ we have $y := f(x) z_{f,\varepsilon} - f(z_{f,\varepsilon}) x$ belongs to $Ker(f)$ which implies, by (12.5), that:

$$\begin{aligned}(f(x) z_{f,\varepsilon} - f(z_{f,\varepsilon}) x, z_{f,\varepsilon})_i &\leq \delta(\varepsilon) \|f(x) z_{f,\varepsilon} - f(z_{f,\varepsilon}) x\| \\ &\leq 2\delta(\varepsilon) \|f\| \|x\| \leq \varepsilon \|x\|\end{aligned}$$

for all $x$ in $X$.

On the other hand, as above, we have:

$$(f(x) z_{f,\varepsilon} - f(z_{f,\varepsilon}) x, z_{f,\varepsilon})_i = f(x) - (x, f(z_{f,\varepsilon}) z_{f,\varepsilon})_s$$

for all $x$ in $X$ and denoting $u_{f,\varepsilon} := f(z_{f,\varepsilon}) \neq 0$, we obtain:

$$f(x) \leq (x, u_{f,\varepsilon})_s + \varepsilon \|x\| \quad \text{for all } x \text{ in } X.$$

Now, if we replace $x$ by $-x$ in the above estimation, we derive

$$f(x) \geq (x, u_{f,\varepsilon})_i - \varepsilon \|x\| \quad \text{for all } x \text{ in } X$$

and the proof is finished. ∎

Corollary 19. *([**3**]) Let $X$ be a smooth normed space over the real number field and denote $[x,y]=(x,y)_i=(x,y)_s$, $x,y\in X$. If $f\in X^*$ is a nonzero functional such that for any $\delta\in(0,1)$ there exists an element $x_{f,\delta}\in X\setminus\{0\}$ with the property*

$$\text{(A')}\qquad |[x,x_{f,\delta}]|\le\delta\|x\|\|x_{f,\delta}\|\quad\textit{for all } x\in\ker(f),$$

*then for any $\varepsilon>0$ there is an element $u_{f,\varepsilon}\in X\setminus\{0\}$ such that*

$$|f(x)-[x,u_{f,\varepsilon}]|\le\varepsilon\|x\|\quad\textit{for all } x \textit{ in } X. \tag{12.6}$$

The proof is obvious from the above theorem and by the fact that $[\cdot,\cdot]$ is linear in the first variable.

To give the main result of our paper, we need the famous theorem of Bishop-Phelps which says [**1**, p. 3]:

Theorem 82. *Let $C$ be a closed bounded convex set in the Banach space $X$, then the collection of linear functionals that achieve their maximum on $C$ is dense in $X^*$.*

Now, we can state and prove our main result (see [**3**]).

Theorem 83. *Let $X$ be a real Banach space. Then for every continuous linear functional $f:X\to\mathbb{R}$ and for any $\varepsilon>0$ there exists an element $u_{f,\varepsilon}$ in $X$ such that the estimation (12.4) holds.*

Proof. By the use of Bishop-Phelps' theorem for $C=\bar{B}(0,1)$, it follows that the collection of linear functionals which achieve their norm on the unit closed ball is dense in $X^*$, i.e., for every $f\in X^*$ and $\varepsilon>0$ there exists a continuous linear functional $f_\varepsilon$ on $X$ which achieve their norm on $\bar{B}(0,1)$ and such that

$$|f(x)-f_\varepsilon(x)|\le\varepsilon\|x\|\quad\text{for all } x \text{ in } X. \tag{12.7}$$

Suppose $f_\varepsilon\neq0$ and $f_\varepsilon(v_{f,\varepsilon})=\|f_\varepsilon\|$ with $v_{f,\varepsilon}\in\bar{B}(0,1)$. Then

$$0<\|v_{f,\varepsilon}\|\le1=\frac{f_\varepsilon(v_{f,\varepsilon})}{\|f_\varepsilon\|}=\frac{f_\varepsilon(v_{f,\varepsilon}+\lambda y)}{\|f_\varepsilon\|}\le\|v_{f,\varepsilon}+\lambda y\|$$

for all $\lambda\in\mathbb{R}$ and $y\in Ker(f_\varepsilon)$, i.e., $v_{f,\varepsilon}\perp_B Ker(f_\varepsilon)$.

By a similar argument as in Theorem 79, we get:

$$\left(x,\frac{f_\varepsilon(v_{f,\varepsilon})\,v_{f,\varepsilon}}{\|v_{f,\varepsilon}\|^2}\right)_i\le f_\varepsilon(x)\le\left(x,\frac{f_\varepsilon(v_{f,\varepsilon})\,v_{f,\varepsilon}}{\|v_{f,\varepsilon}\|^2}\right)_s$$

for all $x\in X$. Denoting $u_{f,\varepsilon}:=\frac{f_\varepsilon(v_{f,\varepsilon})v_{f,\varepsilon}}{\|v_{f,\varepsilon}\|^2}$, we obtain

$$(x,u_{f,\varepsilon})_i\le f_\varepsilon(x)\le(x,u_{f,\varepsilon})_s\quad\text{for all } x \text{ in } X. \tag{12.8}$$

If $f_\varepsilon=0$, then (12.4) holds with $u_{f,\varepsilon}=0$.

Now, we observe that the relations (12.7) and (12.8) give the desired evaluation and the proof is completed. ∎

Corollary 20. *([**3**])Let $X$ be a smooth Banach space. Then for every $f \in X^*$ and for any $\varepsilon > 0$ there exists an element $u_{f,\varepsilon}$ in $X$ such that:*

$$|f(x) - [x, u_{f,\varepsilon}]| \leq \varepsilon \|x\| \quad \textit{for all } x \textit{ in } X,$$

*where $[\cdot, \cdot]$ is as above.*

## 4. A Characterization of Reflexivity in Terms of Convex Functions

The following characterisation of reflexivity holds (see [**5**]).

Theorem 84. *Let $X$ be a real Banach space. The following statements are equivalent.*

(i) *$X$ is reflexive;*
(ii) *For every $F : X \to \mathbb{R}$ a continuous convex mapping on $X$ and for any $x_0 \in X$ there exists an element $u_{F,x_0} \in X$ such that the estimation*

$$F(x) \geq F(x_0) + (x - x_0, u_{F,x_0})_i \tag{12.9}$$

*holds for all $x$ in $X$.*

Proof. "(i) $\Longrightarrow$ (ii)". Since $F$ is continuous convex on $X$, $F$ is subdifferentiable on $X$, i.e., for every $x_0 \in X$ there exists a functional $f_{x_0} \in X^*$ such that

$$F(x) - F(x_0) \geq f_{x_0}(x - x_0) \quad \text{for all } x \text{ in } X, \tag{12.10}$$

$X$ being reflexive, then, by James' theorem, there is an element $w_{F,x_0} \in X \backslash \{0\}$ such that $w_{F,x_0} \perp Ker(f_{x_0})$. Since

$$f_{x_0}(x) w_{F,x_0} - f(w_{F,x_0}) x \in Ker(f_{x_0}) \quad \text{for all} \quad x \in X$$

by the property (vi), we get that

$$\begin{aligned} &(f_{x_0}(x) w_{F,x_0} - f_{x_0}(w_{F,x_0}) x, w_{F,x_0})_i \\ &\qquad \leq 0 \leq (f_{x_0}(x) w_{F,x_0} - f_{x_0}(w_{F,x_0}) x, w_{F,x_0})_s \end{aligned}$$

for all $x$ in $X$, which are equivalent, by the above properties of $(\cdot, \cdot)_p$ with

$$(x, u_{F,x_0})_i \leq f_{x_0}(x) \leq (x, u_{F,x_0})_s \text{ for all } x \text{ in } X,$$

where

$$u_{F,x_0} := \frac{f_{x_0}(w_{F,x_0}) w_{F,x_0}}{\|w_{F,x_0}\|^2}.$$

Now, by (12.10), we obtain the estimation (12.9).

"(ii) $\Longrightarrow$ (i)". Let $H$ be as in James' theorem and $f \in X^* \setminus \{0\}$ with $H = Ker\,(f)$. Then, by (ii), for $F = f$ and $x_0 = 0$, there exists an element $u_f \in X$ such that

$$f\,(x) \geq (x, u_f)_i \quad \text{for all } x \text{ in } X.$$

Substituting $x$ with $(-x)$ we also have

$$f\,(x) \leq (x, u_f)_s \quad \text{for all } x \text{ in } X.$$

Now, we observe that $u_f \neq 0$ (because $f \neq 0$) and then

$$(x, u_f)_i \leq 0 \leq (x, u_f)_s \quad \text{for all } x \text{ in } H,$$

i.e., $u_f \perp H$ and by James' theorem we deduce that $X$ is reflexive. ∎

Corollary 21. *Let $X$ be a real Banach space. Then $X$ is reflexive iff for every $p : X \to \mathbb{R}$ a continuous sublinear functional on $X$ there is an element $u_p$ in $X$ such that*

$$p\,(x) \geq (x, u_p)_i \quad \textit{for all } x \textit{ in } X.$$

Corollary 22. [**3**] *Let $X$ be a real Banach space. Then $X$ is reflexive iff for every $f \in X^*$ there is an element $u_f$ in $X$ such that*

$$(x, u_f)_i \leq f\,(x) \leq (x, u_f)_s \quad \textit{for all } x \textit{ in } X.$$

Corollary 23. [**3**] *Let $X$ be a real Banach space. Then $X$ is smooth and reflexive iff for all $f \in X^*$ there is an element $u_f \in X$ such that*

$$f\,(x) = (x, u_f)_p \quad \textit{for all } x \textit{ in } X,$$

*where $p = s$ or $p = i$.*

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CHAPTER 13

# Some Classes of Continuous Linear Functionals

## 1. The Case of Semi-Inner Products

The following local approximation of continuous linear functionals on incomplete normed linear spaces in terms of semi-inner-products holds (see [**2**]).

THEOREM 85. *Let $X$ be a normed linear space and $[\cdot,\cdot]$ be a s.i.p. on it which generates the norm. Then for all $f \in X^{*}\backslash\{0\}$ and for any $\varepsilon > 0$ there exists a nonzero element $u_{f,\varepsilon}$ in $X$ and a positive number $r_{f,\varepsilon}$ such that:*

$$\left|f(x) - [x, u_{f,\varepsilon}]\right| \leq \varepsilon \text{ for all } x \in \bar{B}(0, r_{f,\varepsilon}), \tag{13.1}$$

*where $\bar{B}(0, r_{f,\varepsilon})$ is the closed ball $\{x \in X \,|\, \|x\| \leq r_{f,\varepsilon}\}$.*

PROOF. Let $f \in X^{*}\backslash\{0\}$ and $\varepsilon > 0$. Then there exists an element $y_{f,\varepsilon} \in X\backslash\{0\}$ such that $\|y_{f,\varepsilon}\| = \varepsilon$ and $y_{f,\varepsilon}$ is not in $Ker(f)$.
We obtain:

$$\left|[y, y_{f,\varepsilon}]\right| \leq \|y\| \|y_{f,\varepsilon}\| = \varepsilon \|y\|$$

for all $y \in Ker(f)$.
Let us put $y := f(x) y_{f,\varepsilon} - f(y_{f,\varepsilon}) x$ where $x \in X$. Then $y \in Ker(f)$ and:

$$\left|[f(x) y_{f,\varepsilon} - f(y_{f,\varepsilon}) x, y_{f,\varepsilon}]\right| \leq \varepsilon \|f(x) y_{f,\varepsilon} - f(y_{f,\varepsilon}) x\| \leq 2\varepsilon^2 \|f\| \|x\|,$$

for all $x$ in $X$.

On the other hand, we have:

$$[f(x) y_{f,\varepsilon} - f(y_{f,\varepsilon}) x, y_{f,\varepsilon}] = f(x) \|y_{f,\varepsilon}\|^2 - \left[x, \overline{f(y_{f,\varepsilon})} y_{f,\varepsilon}\right]$$

for all $x \in X$, which gives:

$$\left|f(x)\varepsilon^2 - [x, f(y_{f,\varepsilon}) y_{f,\varepsilon}]\right| \leq 2\varepsilon^2 \|f\| \|x\|, \ x \in X,$$

from where results:

$$\left|f(x) - \left[x, \left(\frac{\overline{f(y_{f,\varepsilon})}}{\varepsilon^2}\right) y_{f,\varepsilon}\right]\right| \leq \frac{2\varepsilon \|f\| \|x\|}{\varepsilon}$$

for $x \in X$.
Putting

$$u_{f,\varepsilon} := \left( \frac{\overline{f\left(y_{f,\varepsilon}\right)}}{\varepsilon^2} \right) y_{f,\varepsilon} \neq 0$$

and

$$r_{f,\varepsilon} := \frac{\varepsilon}{2\left\|f\right\|} > 0$$

the theorem is proved. ∎

Another result which improves in one sense the above theorem is the following ([**2**]):

THEOREM 86. *Let $X$ be a normed linear space and $[\cdot,\cdot]$ a s.i.p. on it which generates the norm. If $f$ is a nonzero continuous linear functional on $X$ such that for every $\delta > 0$ there exists an element $x_{f,\delta}$ in $X \setminus \{0\}$ with the property:*

$$|[x, x_{f,\delta}]| \leq \delta \left\|x\right\| \; \left\|x_{f,\delta}\right\| \text{ for all } x \text{ in } Ker\left(f\right), \tag{13.2}$$

*then for every $\varepsilon > 0$ there exists an element $u_{f,\varepsilon}$ in $X$ such that the following estimation holds:*

$$\left|f\left(x\right) - [x, u_{f,\varepsilon}]\right| \leq \varepsilon \left\|x\right\| \text{ for all } x \text{ in } X. \tag{13.3}$$

PROOF. Let $\varepsilon > 0$ and put $\delta\left(\varepsilon\right) := \frac{\varepsilon}{2\|f\|} > 0$. Then there exists an element $y_{f,\delta(\varepsilon)}$ in $X \setminus \{0\}$ such that:

$$\left|\left[y, y_{f,\delta(\varepsilon)}\right]\right| \leq \delta\left(\varepsilon\right)\left\|y\right\| \left\|y_{f,\delta(\varepsilon)}\right\|$$

for all $y \in Ker\left(f\right)$.
Put

$$z_{f,\varepsilon} := \frac{y_{f,\delta(\varepsilon)}}{\left\|y_{f,\delta(\varepsilon)}\right\|}.$$

Then for all $x \in X$ we have

$$y := f\left(x\right) z_{f,\varepsilon} - f\left(z_{f,\varepsilon}\right)\; x \in Ker\left(f\right),$$

which implies:

$$\begin{aligned} \left|\left[f\left(x\right) z_{f,\varepsilon} - f\left(z_{f,\varepsilon}\right) x, z_{f,\varepsilon}\right]\right| &\leq \delta\left(\varepsilon\right)\left\|f\left(x\right) z_{f,\varepsilon} - f\left(z_{f,\varepsilon}\right) x\right\| \\ &\leq 2\delta\left(\varepsilon\right)\left\|f\right\|\left\|x\right\| \\ &\leq \varepsilon\left\|x\right\| \end{aligned}$$

for all $x$ in $X$.
On the other hand, we have:

$$\left[f\left(x\right) z_{f,\varepsilon} - f\left(z_{f,\varepsilon}\right) x, z_{f,\varepsilon}\right] = f\left(x\right) - \left[x, \overline{f\left(z_{f,\varepsilon}\right)} z_{f,\varepsilon}\right]$$

for all $x$ in $X$ and denoting

$$u_{f,\varepsilon} :=, \overline{f\left(z_{f,\varepsilon}\right)} z_{f,\varepsilon}$$

the estimation (13.3) is obtained. ∎

REMARK 25. *The relations* (13.3) *is equivalent to:*

$$|f(x) - [x, u_{f,\varepsilon}]| \leq \varepsilon \tag{13.4}$$

*for all* $x \in \bar{B}(0,1)$. *So let* $f \in X^*$ *with* $\|f\| = 1$ *and let* $k > 0$. *Define:*

$$K(f,k) = \{x \in X \,|\, \|x\| \leq kf(x)\},$$

*then* $K(f,k)$ *is a closed convex cone and if* $k > 1$ *then the interior of* $K(f,k)$ *is nonempty* [**1**, p. 1].

THEOREM 87. *Let* $X, [\cdot,\cdot]$ *be as above,* $\varepsilon \in (0,1)$ *and* $f \in X^*, \|f\| = 1$. *If* $u \in X, \|u\| = 1, k > 1 + 2/\varepsilon$ *and* $[y,u] \geq 0$ *for all* $y \in K(f,k)$, *then the following estimation holds:*

$$|f(x) - [x,u]| \leq \varepsilon \|x\| \text{ for all } x \text{ in } X \tag{13.5}$$

PROOF. Follows by Lemma 3, [**1**, p. 3] choosing $g(x) = [x,u]$ for all $x$ in $X$. ∎

Now, let $[\cdot,\cdot]$ be a given s.i.p on $X$ which generates the norm of $X$. The subset of $X^*$ given by:

$$R(X^*; [\cdot,\cdot]) = \{f \in X^* \,|\, f(x) = [x,u] \text{ for all } x \in X \text{ and } u \in X\},$$

will be called the *Riesz's class* of continuous linear functional associated with s.i.p. $[\cdot,\cdot]$.
The following theorem holds (see [**2**]).

THEOREM 88. *Let* $X$ *be a normed linear space and* $[\cdot,\cdot]$ *be a given s.i.p. which generates its norm. If for every* $f \in X^* \setminus \{0\}$ *and* $\delta > 0$ *there exists an element* $x_{f,\delta}$ *in* $X \setminus \{0\}$ *such that* (13.2) *holds, then the Reisz's class* $R(X^*; [\cdot,\cdot])$ *is dense in* $X^*$.

PROOF. Let $f \in X^* \setminus \{0\}$. Then for any $\varepsilon > 0$, by Theorem 86, there exists an element $u_{f,\varepsilon} \in X$ such that (13.3) holds. Putting

$$f_\varepsilon : X \to K, f_\varepsilon(x) = [x, u_{f,\varepsilon}]$$

we obtain:

$$|f(x) - f_\varepsilon(x)| \leq \varepsilon \|x\| \text{ for all } x \text{ in } X.$$

That is, $\|f - f_\varepsilon\| < \varepsilon$ and the statement is proved. ∎

REMARK 26. *By the use of Bishop-Phelps' Theorem of density we shall prove in the next section a similar result which works in smooth Banach spaces.*

## 2. Some Classes of Functionals in Smooth Normed Spaces

Let $X$ be a smooth normed linear space and $[\cdot,\cdot]$ be the (unique) s.i.p. which generates its norm. We define the following class of continuous linear functionals on $X$ (see [**2**]):

(1) *James' class:* $J(X^*)$ given by:

$$J(X^*) := \left\{f \in X^* \text{ there exists } u \in \bar{B}(0,1) \text{ such that } f(u) = \|f\|\right\};$$

(2) *Riesz's class:* $R(X^*)$ given by:

$$R(X^*) := R(X^*;[\cdot,\cdot])$$

where $R(X^*;[\cdot,\cdot])$ is as above;

(3) $(P)-class$: $P(X^*)$ given by:

$$P(X^*) := \{f \in X^* \ |Ker(f) \text{ is proximal}\}.$$

The following lemma is important in the sequel (see [**2**]).

Lemma 11. *Let $X$ be a smooth normed linear space. Then we have:*

$$R(X^*) = J(X^*) = P(X^*). \tag{13.6}$$

Proof. "$R(X^*) \subseteq J(X^*)$". Let $f \in X^* \setminus \{0\}$ and $v \in X \setminus \{0\}$ such that $f(x) = [x,v]$. Then $f(v) = \|v\|$. Putting $u = \frac{v}{\|v\|}$, we obtain $f(u) = \|f\|$. If $f \equiv 0$ then $v = 0$ also gives $f(v) = \|f\|$ and the inclusion is proven.

"$J(X^*) \subseteq P(X^*)$". Let $f \in X^* \setminus \{0\}$ and $u \in \bar{B}(0,1)$ such that $f(u) = \|f\|$. Then we have

$$\|u\| \le 1 = \frac{f(u)}{\|f\|} = \frac{f(u+\lambda y)}{\|f\|} \le \|u+\lambda y\|$$

for all $\lambda \in K$ and $y \in Ker(f)$. What this means is that $u \perp Ker(f)$ $(B)$ and, by Lemma 2.1 in [**6**], $Ker(f)$ is proximinal.

"$P(X^*) \subseteq R(X^*)$". Suppose that $f \in X^* \setminus \{0\}$ and $Ker(f)$ is proximinal. Then, by Lemma 2.1 in [**6**], there exists a nonzero element $w_0 \in X$ such that $w_0 \perp Ker(f)$ $(B)$. Since $X$ is smooth, it follows that $w_0 \perp Ker(f)$ $(G)$ and then, for all $x \in X$ we have:

$$w_0 \perp f(x) w_0 - f(w_0) x \,(G)$$

because

$$f(x) w_0 - f(w_0) x \in Ker(f)$$

for all $x \in X$. Consequently,

$$[f(x) w_0 - f(w_0) x, w_0] = 0$$

for all $x$ in $X$, which is equivalent to

$$f(x) = \left[x, \frac{\overline{f(w_0)} w_0}{\|w_0\|^2}\right]$$

for all $x$ in $X$. Putting

$$v := \frac{\overline{f(w_0)} w_0}{\|w_0\|^2},$$

we have the representation $f(x) = [x, v]$ for all $x$ in $X$ and the inclusion is proven. ∎

REMARK 27. *It is easy to see that in every normed linear space we have, by a similar argument, the inclusions:*

$$R(X^*; [\cdot,\cdot]) \subseteq J(X^*) \subseteq P(X^*), \tag{13.7}$$

*for all* $[\cdot,\cdot]$ *a s.i.p. generating its norm.*

By the use of the above lemma and Bishop-Phelp's theorem (see Theorem 82), we have the following density result (see [**2**]).

THEOREM 89. *([**2**]) Let* $X$ *be a smooth Banach space. Then the set* $J(X^*)[P(X^*)(R(X^*))]$ *is dense in* $X^*$.

PROOF. Consider in Bishop-Phelps' theorem $C = \bar{B}(0,1)$. Then the collection of functionals that achieve their maximum on $\bar{B}(0,1)$ is equal to $J(X^*)$, and the statement is proved. ∎

COROLLARY 24. *([**2**]) Let* $X$ *be a smooth Banach space. Then for all continuous linear functionals* $f$ *on it and for any* $\varepsilon > 0$, *there exists an element* $u_{f,\varepsilon}$ *in* $X$ *such that:*

$$|f(x) - [x, u_{f,\varepsilon}]| \leq \varepsilon \|x\| \text{ for all } x \text{ in } X.$$

The proof follows by the fact that $R(X^*)$ is dense in $X^*$, and we shall omit the details.

REMARK 28. *If* $X$ *is a Banach space, then by Remark 27 and by Bishop-Phelps' theorem we have that* $P(X^*)$ *is dense in* $X^*$ *endowed with the strong topology.*

The following characterization of reflexivity in smooth normed linear spaces holds ([**2**]).

THEOREM 90. *Let* $X$ *be a smooth Banach space. Then the following statements are equivalent:*

(1) $X$ *is reflexive;*
(2) $J(X^*)[P(X^*)(R(X^*))] = X^*$;
(3) $J(X^*)[P(X^*)(R(X^*))]$ *is closed in* $X^*$.

The proof follows by James' Theorem, by Lemma 11 and by Theorem 89 and we shall omit the details.

Remark 29. *Let $X$ be a Banach space. Then $X$ is reflexive iff $P(X^*) = X^*$. This fact follows by the inclusion $J(X^*) \subseteq P(X^*)$ which holds in every normed linear space and, of course, by James' Theorem.*

**Consequences**: Let $X$ be a smooth normed space and $f$ be a nonzero continuous linear functional on it. If

$$\bar{B}_{Ker(f)} := \{h \in Ker(f) \,|\, \|h\| \leq 1\}$$

is weakly sequentially compact in $X$, then there exists an element $u_f \in X$ such that:

$$f(x) = [x, u_f], \;\; \|f\| = \|u_f\| \text{ for all } x \text{ in } X.$$

If $G$ is finite-dimensional in $X$, then there exists an element $u_f \in G$ such that:

$$f(x) = [x, u_f] \text{ and } \|f\|_G = \|u_f\| \text{ for all } x \text{ in } G,$$

where

$$\|f\|_G := \sup\{|f(x)|, \|x\| \leq 1, x \in G\}.$$

The proof of the first statement follows by Klee's Theorem (see [**4**] or [**6**, Corollary 3.1]) and by the fact that in smooth normed spaces $P(X^*) \subseteq R(X^*)$.

The second sentence is obvious.

2. Let $X$ be a normed linear space and suppose that $X^*$ endowed with the canonical norm is smooth. If $\phi \in X^* \setminus \{0\}$ satisfies the conditions: $Ker(\phi)$ is $\sigma(X^*, X)$ - closed or $\bar{B}_{Ker(\phi)}$ is compact in $\sigma(X^*, X)$ or $\bar{B}_{Ker(\phi)}$ is weak* sequentially compact in $X^*$, then there exists a functional $f_\phi \in X^*$ such that the following representation holds:

$$\phi(f) = [f, f_\phi]^*, \;\; \|\phi\| = \|f_\phi\| \text{ for all } f \in X^*,$$

where $[\cdot,\cdot]^*$ is the s.i.p. which generates the norm of $X^*$(see also [**?**]).

The proof follows by Phelps' Theorem [**5**], by Klee's Theorem [**4**], and by Lemma 11. We shall omit the details.

## 3. Applications for Nonlinear Operators

Let $X$ be a normed linear space and $A : X \to X^*$ be a nonlinear operator satisfying the following conditions:

(1) $A(\alpha x) = \bar{\alpha} Ax$ for all $\alpha$ in $K$ and $x$ in $X$;

(2) $|\langle Ay, x\rangle|^2 \leq \langle Ax, x\rangle \langle Ay, y\rangle$ for all $x, y$ in $X$.

The following proposition holds.

PROPOSITION 40. *([**2**]) Let $X$ be a normed space and $A$ an operator satisfying the conditions $(i)-(ii)$. If there exists a constant $m>0$ such that:*
*$(iii)$ $\langle Ax,x\rangle \geq m\|x\|^2$ for all $x$ in $X$,*
*then for every $f\in X^*\setminus\{0\}$ and $\varepsilon>0$, there exists an element $u_{f,\varepsilon}\in X\setminus\{0\}$ and a positive number $r_{f,\varepsilon}$ such that:*

$$|\langle f-A(u_{f,\varepsilon}),x\rangle| \leq \varepsilon \text{ if } \|x\| \leq r_{f,\varepsilon}.$$

PROOF. Let us consider the mapping $[\cdot,\cdot]_A : X\times X\to K, [x,y]_A := \langle Ay,x\rangle$. Then by the use of conditions $(i)-(iii)$ it follows that $[\cdot,\cdot]_A$ is a s.i.p. on $X$ generating a norm $\|\cdot\|_A$ which dominates the norm $\|\cdot\|$ of normed space $X$.
Since $f\in X^*\setminus\{0\}$ it follows that $f$ is continuous in $(X,\|\cdot\|_A)$. By the use of Theorem 85, for every $\varepsilon>0$ there exists a nonzero element $u_{f,\varepsilon}$ in $X$ and a positive number $q_{f,\varepsilon}$ such that:

$$\left|f(x)-[x,u_{f,\varepsilon}]_A\right| \leq \varepsilon$$

for all $x$ such that $\|x\|_A \leq q_{f,\varepsilon}$, which implies:

$$|\langle f-A(u_{f,\varepsilon}),x\rangle| \leq \varepsilon \text{ if } \|x\| \leq \frac{1}{m^{\frac{1}{2}}}\|x\|_A \leq r_{f,\varepsilon}.$$

where

$$r_{f,\varepsilon} = \frac{1}{m^{\frac{1}{2}}} q_{f,\varepsilon},$$

and the proof is completed. ∎

Another result is embodied in the following proposition [**2**].

PROPOSITION 41. *Let $X$ be a normed space, $A$ an operator satisfying the conditions $(i)-(iii)$ and there exists a positive number $M$ such that*
*$(iv)$ $M\|x\|^2 \geq \langle Ax,x\rangle$ for all $x$ in $X$.*
*If $f$ is a nonzero continuous linear functional on $X$ such that for every $\delta>0$ there exists an element $x_{f,\delta}$ in $X\setminus\{0\}$ with the property that:*

$$|\langle A(x_{f,\delta}),x\rangle| \leq \delta\|x\|\|x_{f,\delta}\| \tag{13.8}$$

*for all $x$ in $Ker(f)$, then for every $\varepsilon>0$ the equation:*

$$Au = f \tag{$A;f$}$$

*has an $\varepsilon-$solution in $X$. That is, there exists an element $u_{f,\varepsilon}$ in $X$ such that*

$$\|A(u_{f,\varepsilon})-f\| \leq \varepsilon.$$

PROOF. By condition (13.8), for every $\eta > 0$, there exists an element $x_{f,\eta} \in X \setminus \{0\}$ such that:

$$\left|[x, x_{f,\eta}]_A\right| \leq \eta \|x\|_A \|x_{f,\eta}\|_A$$

for all $x$ in $Ker(f)$.
Applying Theorem 86, for every $\varepsilon > 0$ we can find an element $u_{f,\varepsilon}$ in $X$ such that:

$$\left|f(x) - [x, u_{f,\varepsilon}]_A\right| \leq \frac{\varepsilon}{M^{\frac{1}{2}}} \|x\|_A$$

for all $x$ in $X$, which gives:

$$|\langle f - A(u_{f,\varepsilon}), x\rangle| \leq \varepsilon \|x\|$$

for all $x$ in $X$, which implies the desired inequality. Therefore the proof is completed. ∎

The following result is an interesting consequence of Bishop-Phelp's Theorem of density [**2**].

THEOREM 91. *Let $X$ be a Banach space and $A : X \to X^*$ be an operator satisfying the conditions $(i)-(iv)$. If $A$ also has the property*
$(v)$ $\lim_{t\to 0} \operatorname{Re}\langle A(y + tx), x\rangle = \operatorname{Re}\langle Ay, x\rangle$ *for all $x$ in $X$,*
*then the range $R(A)$ of operator $A$ is dense in $X^*$.*

PROOF. By conditions $(i)-(v)$ if follows that $(X, \|\cdot\|_A)$ is a smooth Banach space isomorphic top-linear with $(X, \|\cdot\|)$. Using Corollary 24 of Theorem 89, then for every $f \in X^*$ and $\varepsilon > 0$, there exists an element $u_{f,\varepsilon}$ in $X$ such that:

$$\left|f(x) - [x, u_{f,\varepsilon}]_A\right| \leq \frac{\varepsilon}{M^{\frac{1}{2}}} \|x\|_A$$

for all $x$ in $X$, which implies

$$|\langle f - A(u_{f,\varepsilon}), x\rangle| \leq \varepsilon \|x\|$$

for all $x$ in $X$, and the statement is proven. ∎

Next, we shall consider the following operatorial equation:

$$(A; f) \qquad Au = f,\ u \in X \text{ and } f \text{ is given in } X^*,$$

where $A$ is an operator satisfying the conditions $(i)-(iii)$ and $(v)$.

PROPOSITION 42. *([**2**]) Let $X$ be a normed space and $f \in X^* \setminus \{0\}$ such that $Ker(f)$ is proximal in the normed space $(X, \|\cdot\|_A)$. Then the equation $(A; f)$ has at least one solution.*

PROOF. It is clear that $(X, \|\cdot\|_A)$ is smooth and its norm is generated by s.i.p. $[\cdot,\cdot]_A$. Using Lemma 11, then there exists an element $u$ in $X$ such that

$$\langle f, x\rangle = \langle Au, x\rangle \text{ for all } x \text{ in } X,$$

and the proof is completed. ∎

COROLLARY 25. *([**2**]) Let $X$ be as above, $f \in X^*\backslash\{0\}$ such that $\tilde{B}_{Ker(f)}$ is weakly sequentially compact in $(X, \|\cdot\|_A)$. Then the equation $(A; f)$ has at least one solution.*

The proof follows by Proposition 42 and by Klee's Theorem (see [**4**] or [**6**, Corollary 3.1]).

Finally, we have the following surjectivity theorem [**2**].

THEOREM 92. *Let $X$ be a reflexive Banach space. If the operator $A : X \to X^*$ satisfies the conditions $(i)-(v)$, then $A$ is surjective.*

The proof follows by Theorem 90 and we will omit the details.

## 4. The Case of General Real Spaces

The following approximation theorem for the continuous linear functionals on a normed linear space holds [**3**].

THEOREM 93. *Let $f : X \to \mathbb{R}$ be a continuous linear functional such that for any $\delta \in (0,1)$ there exists a nonzero element $x_{f,\delta}$ in $X$ with the property:*

$$\text{(A)} \qquad (x, x_{f,\delta})_i \leq \delta \|x\| \|x_{f,\delta}\| \text{ for all } x \text{ in } Ker(f).$$

*Then for each $\varepsilon > 0$ there exists a nonzero element $u_{f,\delta}$ in $X$ such that the following estimation:*

$$-\varepsilon \|x\| + (x, u_{f,\varepsilon})_i \leq f(x) \leq (x, u_{f,\varepsilon})_s + \varepsilon \|x\|$$

*holds, for all $x$ in $X$.*

PROOF. Since $f$ is nonzero, it follows that $Ker(f)$ is closed in $X$ and $Ker(f) \neq X$.

Let $\varepsilon > 0$ and put $\delta(\varepsilon) := \frac{\varepsilon}{2\|f\|}$. If $\delta(\varepsilon) \geq 1$, then there exists an element $x_{f,\delta(\varepsilon)}$ in $X\backslash Ker(f)$ such that

$$(13.9) \qquad \left(y, x_{f,\delta(\varepsilon)}\right)_i \leq \delta(\varepsilon) \|y\| \left\|x_{f,\delta(\varepsilon)}\right\| \text{ for all } y \text{ in } Ker(f).$$

If $0 < \delta(\varepsilon) < 1$ and since the functional $f$ has the (A)-property, then there exists an element $x_{f,\delta(\varepsilon)}$ in $X\backslash Ker(f)$ (the fact that $x_{f,\delta(\varepsilon)}$ is not in $Ker(f)$, follows from (A)) such that (13.9) is also valid.

Put all cases $z_{f,\varepsilon} := \frac{x_{f,\delta(\varepsilon)}}{\left\|x_{f,\delta(\varepsilon)}\right\|}$. Then for all $x$ in $X$ we have $y := f(x) \cdot z_{f,\varepsilon} - f(z_{f,\varepsilon}) x$ belongs to $Ker(f)$ which implies, by (13.9), that

$$\begin{aligned}(f(x) \cdot z_{f,\varepsilon} - f(z_{f,\varepsilon}) x, z_{f,\varepsilon})_i &\leq \delta(\varepsilon) \|f(x) z_{f,\varepsilon} - f(z_{f,\varepsilon}) x\| \\ &\leq 2\delta(\varepsilon) \|f\| \|x\| \leq \varepsilon \|x\|,\end{aligned}$$

for all $x$ in $X$.

On the other hand, by the use of norm derivative properties, we have:

$$(f(x) z_{f,\varepsilon} - f(z_{f,\varepsilon}) x, z_{f,\varepsilon})_i = f(x) - (x, f(z_{f,\varepsilon}) z_{f,\varepsilon})_s, \quad x \in X$$

and denoting $u_{f,\varepsilon} := f(z_{f,\varepsilon}) z_{f,\varepsilon} \neq 0$, we obtain

$$f(x) \leq (x, u_{f,\varepsilon})_s + \varepsilon \|x\| \quad \text{for all } x \in X.$$

Now, if we replace $x$ by $-x$ in the above estimation, we have

$$f(x) \geq (x, u_{f,\varepsilon})_i - \varepsilon \|x\| \quad \text{for all } x \in X,$$

and the proof is completed. ∎

COROLLARY 26. *([**3**]) Let $X$ be a smooth normed space over the real number field and denote $[x, y] := (x, y)_i = (x, y)_s$, $x, y \in X$. If $f \in X^*$ is a nonzero functional such that for all $\delta \in (0, 1)$ there exists an element $x_{f,\delta} \in X \setminus \{0\}$ with the property*

(A') $\qquad |[x, x_{f,\delta}]| \leq \delta \|x\| \|x_{f,\delta}\|$ *for all $x$ in $Ker(f)$,*

*then for any $\varepsilon > 0$ there is an element $u_{f,\delta} \in X \setminus \{0\}$ such that*

(13.10) $\qquad |f(x) - [x, u_{f,\varepsilon}]| \leq \varepsilon \|x\|$ *for all $x \in X$.*

The proof is obvious from the above theorem and by the fact that $[\cdot, \cdot]$ is linear in the first variable.

## 5. Some Classes of Continuous Linear Functionals

Let $X$ be a real normed space. We define the following classes of continuous linear functionals on $X$ (see [**3**]):

(1) *James' class*, denoted $J(X^*)$ and given by

$$J(X^*) := \left\{f \in X^* | \text{there is } v \in \bar{B}(0, 1) \text{ so that } f(v) = \|f\|\right\};$$

(2) $(P)-$*class*, denoted by $P(X^*)$ and given by

$$P(X^*) := \{f \in X^* | Ker(f) \text{ is proximinal in } X\};$$

(3) $(I)-$*class*, denoted $I(X^*)$ and given by

$$I(X^*) := \{f \in X^* | \text{there is } u \in X \text{ so that } (x, u)_i \leq f(x) \leq (x, u)_s \text{ for all } x \text{ in } X\}.$$

The following theorem holds [**3**].

THEOREM 94. *Let $X$ be a real normed space. Then one has*

$$I(X^*) = J(X^*) = P(X^*).$$

PROOF. "$I(X^*) \subseteq J(X^*)$". Let $f \in X^*$ and $u \in X$ such that

$$(x,u)_i \le f(x) \le (x,u)_s \text{ for all } x \text{ in } X. \tag{13.11}$$

Then we have

$$-\|x\|\|u\| \le -(x,u)_s = (x,u)_i \le f(x) \le (x,u)_s \le \|x\|\|u\|, \quad x \in X,$$

which implies

$$|f(x)| \le \|x\|\|u\| \text{ for all } x \text{ in } X,$$

and then $\|f\| \le \|u\|$.

On the other hand, we have

$$\|f\| \ge \frac{f(u)}{\|u\|} \ge \frac{(u,u)_i}{\|u\|} = \|u\|,$$

which shows that $\|f\| \ge \|u\|$ and then $\|f\| = \|u\|$.

Since $f(u) = \|u\|^2$, putting $v = \frac{u}{\|u\|}$, we get $f(v) = \|f\|$ and then the inclusion is proven.

"$J(X^*) \subseteq P(X^*)$". Let $f \in X^* \setminus \{0\}$ and $v \in \bar{B}(0,1)$, $v \neq 0$ so that $f(v) = \|f\|$. Then we have:

$$\|v\| \le 1 = \frac{f(v)}{\|f\|} = \frac{f(v+\lambda y)}{\|f\|} \le \|v+\lambda y\|,$$

for all $\lambda \in \mathbb{R}$ and $y \in Ker(f)$, which means that $v \perp Ker(f)$ and by Lemma 2.1 in [**6**] it follows that $Ker(f)$ is proximal.

"$P(X^*) \subseteq J(X^*)$". Let $f \in X^*$. If $f = 0$, then (13.11) holds with $u = 0$. Suppose $f \neq 0$. Since $Ker(f)$ is proximal, then by Lemma 2.1 in [**6**] there is an element $w_0 \in X$, $w_0 \neq 0$ and $w_0 \perp Ker(f)$. Since the element $y := f(x)w_0 - f(w_0)x$ belongs to $Ker(f)$ for all $x$ in $X$, we have, by (vi), that:

$$(f(x)w_0 - f(w_0)x, w_0)_i \le 0 \le (f(x)w_0 - f(w_0)x, w_0)_s, \quad x \in X. \tag{13.12}$$

On the other hand, by the use of norm derivatives properties, we get

$$(f(x)w_0 - f(w_0)x, w_0)_p = f(x)\|w_0\|^2 - (x, f(w_0)w_0)_q, \quad x \in X,$$

where $p,q \in \{i,s\}$ and $p \neq q$.

Consequently, (13.12) yields that:

$$\left(x, \frac{f(w_0)w_0}{\|w_0\|^2}\right)_i \le f(x) \le \left(x, \frac{f(w_0)w_0}{\|w_0\|^2}\right)_s, \quad x \in X,$$

and putting $u := \frac{f(w_0)w_0}{\|w_0\|^2}$, we obtain the desired estimation.

The proof of the theorem is completed. ∎

REMARK 30. *If $X$ is smooth, then the $(I)-$class can be written as:*

$$I(X^*) := \{f \in X^* \text{ there is } u \in X \text{ so that } f(x) = [x, u] \text{ for all } x \text{ in } X\}$$

*and will be called the Riesz's class associated to the smooth normed linear space $X$. We denote this by $R(X^*)$ (see Section 2 of the present chapter).*

COROLLARY 27. *([**3**]) Let $X$ be a smooth normed space over the real number field and $f : X \to \mathbb{R}$ be a continuous linear functional on it. Then the following statements are equivalent:*

(i) *$f$ is Riesz representable, i.e., $f$ belongs to $R(X^*)$;*
(ii) *$f$ achieves its norm on unit closed ball;*
(iii) *$Ker(f)$ is proximal in $X$.*

The proof is obvious from the above theorem and we shall omit the details.

By the use of the previous result and Bishop-Phelp's theorem we can formulate the following density result ([**3**]):

THEOREM 95. *Let $X$ be a real Banach space. Then the class $J(X^*)[I(X^*)(P(X^*))]$ is dense in $X^*$.*

PROOF. Consider in Bishop-Phelp's theorem $C = \bar{B}(0, 1)$. Then the collection of functionals that achieve their maximum on $\bar{B}(0, 1)$ is equal to $J(X^*)[I(X^*)(P(X^*))]$ (see Theorem 94) and the statement is proven. ∎

The following corollary is important because it gives a way to approximate the continuous linear functionals on an arbitrary real Banach space in terms of norm derivatives.

COROLLARY 28. *([**3**]) Let $X$ be a real Banach space and $f : X \to \mathbb{R}$ a continuous linear functional on it. Then for every $\varepsilon > 0$ there exists an element $u_\varepsilon \in X$ such that the following estimation*

$$-\varepsilon \|x\| + (x, u_\varepsilon)_i \le f(x) \le (x, u_\varepsilon)_s + \varepsilon \|x\| \tag{13.13}$$

*holds, for all $x$ in $X$.*

PROOF. Let $\varepsilon > 0$. Then, by Theorem 95, there is a continuous linear functional $f_\varepsilon \in I(X^*)$ such that

$$|f(x) - f_\varepsilon(x)| \le \varepsilon \|x\| \text{ for all } x \text{ in } X. \tag{13.14}$$

However, $f_s$ satisfies the inequalities:

$$(x, u_\varepsilon)_i \le f_\varepsilon(x) \le (x, u_\varepsilon)_s \text{ for all } x \text{ in } X \tag{13.15}$$

with an element $u_s$ in $X$.

Consequently, the relations (13.14) and (13.15) easily give the desired estimation (13.13). ∎

COROLLARY 29. *([**3**]) Let $X$ be a smooth Banach space and $f: X \to \mathbb{R}$ be as above. Then for every $\varepsilon > 0$ there is an element $u_s$ in $X$ such that:*

$$|f(x) - [x, u_\varepsilon]| \leq \varepsilon \|x\|$$

*for all $x$ in $X$.*

The following characterisation of reflexivity in real normed spaces also holds (see [**3**]).

THEOREM 96. *Let $X$ be a real Banach space. Then the following statements are equivalent:*

(i) *$X$ is reflexive;*
(ii) $J(X^*)[I(X^*)(P(X^*))] = X^*$;
(iii) *The set $J(X^*)[I(X^*)(P(X^*))]$ is closed in $X^*$ endowed with the usual norm topology.*

PROOF. It is known, by James' theorem, that $X$ is a reflexive Banach space iff $J(X^*) = X^*$ which is equivalent, by Theorem 95, with $J(X^*)$ is closed in $X^*$.

The second part follows by Theorem 94 and we omit the details. ∎

COROLLARY 30. *Let $X$ be a real Banach space. Then the following statements are equivalent:*

(i) *$X$ is reflexive;*
(ii) *for every $f \in X^*$ there is an element $u_g \in X$ such that the following "interpolation"*

$$(x, u_f)_i \leq f(x) \leq (x, u_f)_s \text{ for all } x \text{ in } X$$

*holds.*

REMARK 31. *If $X$ is smooth, from the above corollary we recapture the result of R.A. Tapia [**7**].*

## 6. Some Applications

The following results are based on the inclusion "$P(X^*) \subseteq I(X^*)$" which was proved in Theorem 94. We will list these consequences.

(1) Let $X$ be a (smooth) real normed space and $f$ be a nonzero continuous linear functional on it. If

$$\bar{B}_{Ker(f)} := \{k \in Ker(f) \mid \|k\| \leq 1\}$$

is weakly sequentially compact in $X$, then there exists an element $u_f \in E$ such that:

$$(x, u_f)_i \leq f(x) \leq (x, u_f)_s \qquad (f(x) = [x, u_f])$$

for all $x$ in $X$ and

$$\|f\| = \|u_f\|.$$

If $G$ is finite-dimensional in $X$, then there exists an element $u_G \in G$ such that:

$$(x, u_G)_i \leq f(x) \leq (x, u_G)_s \qquad (f(x) = [x, u_G])$$

for all $x$ in $G$ and

$$\|f\|_G = \|u_G\|,$$

where $\|f\|_G := \sup\{|f(x)|, \|x\| \leq 1,\ x \in G\}$.
The proof of the first statement follows by Klee's theorem (see [**4**] or [**6**, Corollary 3.1]) and by the above inclusion.

(2) Let $X$ be a real normed space and $X^*$ its normed dual (and a smooth space). If $\Phi \in X^{**} \setminus \{0\}$ satisfies the conditions: $Ker(\Phi)$ is $\sigma(X^*, X)$ $-$closed or $\bar{B}_{Ker(\Phi)}$ is compact in $\sigma(X^*, X)$ or $\bar{B}_{Ker(\Phi)}$ is weak$^*$ sequentially compact in $X^*$, then there exists a functional $f_\Phi \in X^*$ such that the following "interpolation" (representation) holds:

$$(f, f_\Phi)_i^* \leq \Phi(f) \leq (f, f_\Phi)_s^* \qquad (\Phi(f) = [f, f_\Phi]^*)$$

for all $f \in X^*$ and

$$\|\Phi\| = \|f_\Phi\|,$$

where $(\cdot, \cdot)_p^*$ ($p = s$ or $p = i$) are the norm derivatives of the dual norm.

The proof follows by Phelps' theorem [**5**], by Klee's theorem [**4**] and by the inclusion "$P(X^{**}) \subseteq I(X^{**})$". We omit the details.

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CHAPTER 14

# Smooth Normed Spaces of $(BD)-$Type

## 1. Introduction

In what follows, $X$ will be a normed linear space over the real number field $\mathbb{R}$. Consider the mapping

$$(\cdot,\cdot)_s : X\times X\to\mathbb{R},\ \ (x,y)_s := \lim_{t\downarrow 0}\frac{\left(\|y+tx\|^2-\|y\|^2\right)}{2t}$$

which is well defined for all $x,y$ in $X$. This semi-inner product on $X$ is called the superior semi-inner product (see [**1**], [**2**] and [**6**]). For the sake of completeness, we list some usual properties of this semi-inner-product that will be used in the sequel:

(i) $(x,x)_s=\|x\|^2$ for all $x$ in $X$;
(ii) $(\alpha x,\beta y)_s=\alpha\beta\,(x,y)_s$ if $\alpha\beta\geq 0$ and $x,y\in X$;
(iii) $(\alpha x+y,x)_s=\alpha\,\|x\|^2+(y,x)_s$ for all $\alpha\in\mathbb{R}$ and $x,y\in X$;
(iv) $(-x,y)_s=(x,-y)_s$ for $x,y\in X$;
(v) $(x+y,z)_s\leq\|x\|\,\|z\|+(y,z)_s$ for all $x,y,z\in X$;
(vi) $|(x,y)_s|\leq\|x\|\,\|y\|$ if $x,y\in X$;
(vii) $(\cdot,\cdot)_s$ is continuous subadditive in the first variable;
(viii) $X$ is smooth iff $(\cdot,\cdot)_s$ is linear in the first variable or iff $(\cdot,\cdot)_s$ is homogeneous in the second.

We also recall Tapia's theorem of representation for the continuous linear functional on smooth normed spaces (see [**1**] and [**6**]):

Theorem 97. *Let $X$ be a Banach space. Then the following statements are equivalent:*

(i) *$X$ is reflexive and smooth;*
(ii) *for all $f\in X^*$ there exists an element $u_f\in X$ such that*

$$f(x)=(x,u_f)_s\ \ \textit{for all } x\in X\ \ \textit{and}\ \ \|f\|=\|u_f\|.$$

For other properties of the superior semi-inner-product, see [**1**], [**2**] and [**6**], [**5**].

## 2. Smooth Normed Spaces of $(D)-$Type

Following [**3**], start with the following definition:

DEFINITION 29. *The superior semi-inner-product* $(\cdot,\cdot)_s$ *is said to be continuous on* $X$ *if:*

$$\lim_{t\to 0}(y,x+ty)_s=(y,x)_s \quad \text{for all } x,y \text{ in } X. \tag{14.1}$$

The following proposition holds [**3**].

PROPOSITION 43. *Let* $X$ *be a real normed space. Then* $X$ *is smooth if and only if the superior semi-inner-product is continuous.*

PROOF. "$(\Leftarrow)$". By the superior semi-inner-product properties, we have:

$$\frac{(y,x)_s}{\|x\|}\le\frac{\|x+ty\|-\|x\|}{t}\le\frac{(y,x+ty)_s}{\|x+ty\|} \tag{14.2}$$

and

$$\frac{(y,x+sy)_s}{\|x+sy\|}\le(\|x+sy\|-\|x\|)\le\frac{(y,x)_s}{\|x\|} \tag{14.3}$$

for all $x,y$ in $X$, $x\neq 0$ and $t>0$, $s<0$ such that $x+ty$, $x+sy\neq 0$.

Since $(\cdot,\cdot)_s$ is continuous, we have:

$$\lim_{t\downarrow 0}\frac{\|x+ty\|-\|x\|}{t}=\frac{(y,x)_s}{\|x\|}$$

and

$$\lim_{s\uparrow 0}\frac{\|x+sy\|-\|x\|}{s}=\frac{(y,x)_s}{\|x\|}$$

for all $x,y\in X$, $x\neq 0$, i.e., the space $X$ is smooth.

"$(\Longrightarrow)$". By relations (14.2) and (14.3) we have:

$$\begin{aligned}\frac{(y,x)_s\|x+ty\|}{\|x\|}&\le(y,x+ty)_s\\&\le\frac{(\|x+2ty\|-\|x+ty\|)\|x+ty\|}{t}\end{aligned} \tag{14.4}$$

and

$$\begin{aligned}\frac{\|x+sy\|(\|x+2sy\|-\|x+sy\|)}{s}&\le(y,x+sy)_s\\&\le\frac{(y,x)_s\|x+sy\|}{\|x\|}\end{aligned} \tag{14.5}$$

for all $x,y\in X$, $x\neq 0$ and $t>0$, $s<0$.

Since $X$ is smooth, the inequalities (14.4) and (14.5) yield that $\lim_{t\to 0}(y,x+ty)_s=(y,x)_s$ and the proof is completed. We omit the details. ∎

Now, let $X$ be a smooth real normed space and $[\cdot,\cdot]$ be the semi-inner-product generating its norm $\|\cdot\|$. Then $[\cdot,\cdot]$ is said to be derivable on $X$ if the following limit exists

$$[y,x]' := \lim_{t\to 0}\frac{[y,x+ty]-[y,x]}{t}$$

for all $x,y$ in $X$.

Definition 30. *A smooth normed space $(X;\|\cdot\|)$ is said to be of $(D)-$type if its semi-inner-product is derivable on $X$.*

Example 3. *Every inner-product space $(X;[\cdot,\cdot])$ is a smooth normed space of $(D)-$type.*
*Indeed, for every $x,y\in X$ we have:*

$$[y,x]' = \lim_{t\to 0}\frac{[y,x+ty]-[y,x]}{t} = \|y\|^2 .$$

Example 4. *Let $(X;[\cdot,\cdot])$ be an inner-product space over the real number field and $A:\mathcal{D}(A)\subset X\to X$ be an operator on linear subspace $\mathcal{D}(A)$ with the properties:*

(a) $A(\alpha x)=\alpha A(X)$ *for* $\alpha\in\mathbb{R}$ *and* $x\in\mathcal{D}(A)$;
(aa) $[x,Ax]\geq 0$ *for* $x\in\mathcal{D}(A)$ *and* $[x,Ax]=0$ *implies* $x=0$;
(aaa) $|[x,Ay]|^2\leq [x,Ax][y,Ay]$ *for all* $x,y\in\mathcal{D}(A)$;
(av) *the Gâteaux differential*

$$(VA)(x)\cdot y := \lim_{t\to 0}\frac{[A(x+ty)-A(x)]}{t}$$

*exists for all* $x,y\in\mathcal{D}(A)$;

*Then $(\mathcal{D}(A);\|\cdot\|_A)$ where $\|x\|_A := [x,Ax]^{\frac{1}{2}}$ for $x\in\mathcal{D}(A)$ is a smooth normed linear space of $(D)-$type.*

*Indeed, simple calculus gives:*

$$[y,x]_A := \lim_{t\to 0}\frac{\|x+ty\|_A^2-\|x\|_A^2}{2t} = [y,Ax] \;\; \text{for} \quad x\in\mathcal{D}(A),$$

*$[\cdot,\cdot]_A$ is continuous on $X$ and:*

$$[y,x]_A' = [y,(VA)(x)\cdot y] \quad \text{for all } x,y\in\mathcal{D}(A).$$

Example 5. *Let $(\Omega,\mathcal{A},\mu)$ be a measure space consisting of a set $\Omega$, a $\sigma-$algebra $\mathcal{A}$ of subsets of $\Omega$ and a countably additive and positive measure $\mu$ on $\mathcal{A}$ with values $\mathbb{R}\cup\{\infty\}$. If $L_r^p(\Omega)$ is the real Banach space of $p-$integrable functions on $\Omega$ with $p>1$, then it is well-known that (see for example [**7**]):*

$$\lim_{t\to 0}\frac{\|x+ty\|_p-\|x\|_p}{t} = \|x\|_p^{1-p}\int_\Omega |x(s)|^{p-1}(\operatorname{sgn} x(s))\, y(s)\, d\mu(s)$$

*for all* $x, y \in L_r^p(\Omega)$, $x \neq 0$.

*Suppose* $p \geq 2$ *and put* $p = 2k+2$, $k \geq 0$. *Then*

$$[y, Ax]_p := \lim_{t \to 0} \frac{\|x+ty\|_p^2 - \|x\|_p^2}{2t} = \|x\|_p^{-2k} \int_\Omega [x(s)]^{2k+1} y(s)\, d\mu(s)$$

*for all* $x, y \in L_r^p(\Omega)$, $x \neq 0$ *and* $[y, 0]_p = 0$ *if* $y \in L_r^p(\Omega)$.

*Simple calculus gives:*

$$\begin{aligned}[y, Ax]_p' &= \|x\|_p^{-2k} \int_\Omega x^{2k}(s)\, y^2(s)\, d\mu(s) \\ &\qquad - 2k \|x\|_p^{-2k-2} \left( \int_\Omega x^{2k+1}(s)\, y(s)\, d\mu(s) \right)^2\end{aligned}$$

*for all* $x, y \in L_r^p(\Omega)$, $x \neq 0$ *and* $[y, 0]_p = 0$ *if* $y \in L_r^p(\Omega)$.

*Consequently, the real Banach space* $L_r^p(\Omega)$, $p \geq 2$ *is a smooth Banach space of* $(D)-$*type.*

Now we give some fundamental properties of the semi-inner-product derivative on a smooth normed space of $(D)-$type (see [**3**]).

PROPOSITION 44. *If* $X$ *is as above, then the following statements are valid:*

(i) $[y, y]' = \|y\|^2$ *for all* $y \in X$;
(ii) $[y, 0]' = \|y\|^2$ *for all* $y \in X$;
(iii) $[\alpha y, x]' = \alpha^2 [y, x]'$ *for all* $\alpha \in \mathbb{R}$ *and* $x, y \in X$;
(iv) $[y, \alpha x]' = [y, x]'$ *for all* $\alpha \in \mathbb{R} \setminus \{0\}$ *and* $x, y \in X$;
(v) $\|x\|^2 [y, x]' \geq [y, x]^2$ *for all* $x, y \in X$.

PROOF. We only prove the statement (v). The other sentences are obvious from the definition of semi-inner-product derivatives.

(v) By the properties of semi-inner-products, we have:

$$[y, x+ty] - [y, x] \geq [y, x] \frac{\|x+ty\| - \|x\|}{\|x\|}$$

for all $x, y \in X$, $x \neq 0$ and $t \geq 0$; which implies for $t > 0$:

$$\frac{[y, x+ty] - [y, x]}{t} \geq [y, x] \frac{\|x+ty\| - \|x\|}{t\|x\|}.$$

Taking the limit as $t \to 0$, $t > 0$, we derive:

$$[y, x]' \geq \frac{[y, x]^2}{\|x\|^2} \text{ for all } x, y \in X, \ x \neq 0,$$

and the statement is proven. ∎

Another result is embodied in the next proposition [**3**].

PROPOSITION 45. *Let $X$ be a smooth normed space of $(D)$ −type and $x, y$ be two elements in $X$. Then the mapping:*

$$\varphi_{x,y} : \mathbb{R} \to \mathbb{R},\ \varphi_{x,y}(t) = \|x + ty\|^2,$$

*is derivable of two orders on $\mathbb{R}$, the second derivative is nonnegative on $\mathbb{R}$ and*

$$\varphi'_{x,y}(t) = 2[y, x + ty],\ \varphi''_{x,y}(t) = 2[y, x + ty]'$$

*for all $t \in \mathbb{R}$.*

The proof is obvious and we omit the details.

In what follows, we shall give a characterisation of inner-product spaces in the class of smooth normed linear spaces of $(D)$ −type [**3**].

PROPOSITION 46. *Let $X$ be as above. Then the following statements are equivalent:*

(i) *$X$ is an inner product space;*
(ii) *the mapping $\psi_{x,y} : \mathbb{R} \to \mathbb{R}_+$, $\psi_{x,y}(t) = [y, tx]'$ is continuous at $0$ for all $x, y$ in $X$,*
(iii) *for every $x, y \in X$ there exists a sequence $\alpha_n \in \mathbb{R}\setminus\{0\}$, $\alpha_n \to 0$ such that $\lim\limits_{n\to\infty} [y, \alpha_n x]' = [y, 0]'$;*
(iv) *for every $x, y \in X$ we have: $[y, x]' = \|y\|^2$.*

PROOF. "(i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv)". It is obvious.

"(iv) $\Longrightarrow$ (i)". By Taylor's formula for the mapping $\psi_{x,y}$ $(x, y \in X)$ we have:

$$\|x + ty\|^2 = \|x\|^2 + 2[y, x]t + \|y\|^2 t^2 \text{ for all } t \in \mathbb{R},$$

which implies the parallelogram identity:

$$\|x + ty\|^2 + \|x - ty\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) \text{ for all } x, y \in X,$$

i.e., $X$ is an inner-product space. ∎

## 3. Smooth Normed Spaces of $(BD)$ −Type

Let $X$ be a smooth normed linear space of $(D)$ −type. The semi-inner-product has a bounded derivative if there exists a real number $k \geq 1$ such that:

$$[y, x]' \leq k^2 \|y\|^2 \quad \text{for all } x, y \in X. \tag{14.6}$$

The least number $k$ such that (14.6) is valid will be called the boundedness modulus of the derivative $[\cdot, \cdot]'$ and we shall denote this number with $k_0$.

DEFINITION 31. *([**3**]) A smooth normed space of $(D)$ −type is said to be of $(BD)$ −type if its semi-inner-product has a bounded derivative.*

EXAMPLE 6. *Every inner-product space is a smooth normed space of the $(BD)$ −type.*

EXAMPLE 7. *Let $(X;(\cdot,\cdot))$ be an inner-product space and $A:\mathcal{D}(A)\subset X\to X$ be an operator satisfying conditions (a) – (av) from Example 4. Suppose, in addition, that $A$ is $M-$Lipschitzian $(M\geq 1)$, i.e.,*

$$\text{(aM)} \qquad \|Ax-Ay\|\leq M\|x-y\| \text{ for all } x,y\in\mathcal{D}(A).$$

*Then $(\mathcal{D}(A);\|\cdot\|_A)$ is a smooth normed space of $(BD)$ −type.*

*Indeed, from (aM) we derive:*

$$\|(VA)(x)\cdot y\|\leq M\|y\| \text{ for all } x,y\in\mathcal{D}(A),$$

*which implies that:*

$$[y,x]_A'\leq M\|y\|^2 \text{ for all } x,y\in\mathcal{D}(A),$$

*and the assertion is proven.*

EXAMPLE 8. *The real Banach spaces $L_r^p(\Omega)$ for $p\geq 2$ are smooth normed linear spaces of $(BD)$ −type.*

*Indeed, by Hölder's inequality for integrals, we have:*

$$\int_\Omega x^{2k}(s)\,y^2(s)\,d\mu(s)\leq\left(\int_\Omega x^{2k+2}(s)\,d\mu(s)\right)^{\frac{2k}{2k+2}}\left(\int_\Omega y^{2k+2}(s)\,d\mu(s)\right)^{\frac{2}{2k+2}}$$

*and*

$$\left(\int_\Omega x^{2k+1}(s)\,y(s)\,d\mu(s)\right)^2 \leq\left(\int_\Omega x^{2k+2}(s)\,d\mu(s)\right)^{\frac{4k+2}{2k+2}}\left(\int_\Omega y^{2k+2}(s)\,d\mu(s)\right)^{\frac{2}{2k+2}},$$

*where $p=2k+2$, $k\geq 0$.*

*Then we obtain the evaluation:*

$$[y,x]_p'\leq(4k+1)\|y\|_p^2 \text{ for all } x,y\in L_r^p(\Omega),\ x\neq 0,$$

*and the statement is proven.*

The following result gives a characterisation of inner-product spaces in the class of smooth normed linear spaces of $(BD)$ −type [**3**].

PROPOSITION 47. *Let $X$ be as above. Then the following statements are equivalent:*

(i) *$X$ is an inner-product space;*
(ii) *we have $k_0=1$.*

PROOF. "(i) ⇒ (ii)". It is obvious.

"(ii) ⇒ (i)". By Taylor's formula for $\varphi_{x,y}$ $(x, y \in X)$ we obtain:

$$\|x+y\|^2 \leq \|x\|^2 + 2[y,x] + \|y\|^2 \quad \text{for all } x, y \in X,$$

which implies

$$\|x+y\|^2 \leq \|x\|^2 + 2[x,y] + \|y\|^2 \quad \text{for all } x, y \in X.$$

Since $X$ is smooth, we have:

$$\|x+ty\|^2 \leq \|x\|^2 + 2[x,y]\,t + \|y\|^2\,t^2,$$

for all $x, y \in X$ and $t \in \mathbb{R}$. If we assume that $t > 0$, we have:

$$\frac{\|x+ty\|^2 - \|x\|^2}{2t} \leq [x,y] + \frac{t\,\|y\|^2}{2},$$

hence:

$$[y,x] \leq [x,y] \quad \text{for all } x, y \in X$$

and by symmetry, $[y,x] \geq [x,y]$ for all $x, y \in X$, i.e., $X$ is an inner-product space, see [**6**]. ∎

In what follows, we shall introduce two concepts of $\varepsilon-$orthogonality and we shall establish a result of $\varepsilon-$decomposition for smooth normed spaces of $(BD)$ −type.

DEFINITION 32. *([**3**]) Let $X$ be as above and $k_0$ be the boundedness modulus of semi-inner-product derivative. If $\varepsilon \in [0,1)$, then the element $x \in X$ is said to be $\varepsilon - k_0-$orthogonal over $y \in X$ if*

$$|[y,x]| \leq \varepsilon k_0 \|x\| \|y\|, \tag{14.7}$$

*and we denote $x \perp_{\varepsilon k_0} y$.*

REMARK 32. *If $X$ is an inner-product space, then in (14.7) we can put $k_0 = 1$. We denote $x \perp_{\varepsilon} y$.*

If in the previous definition we choose $\varepsilon = 0$, we recapture the usual orthogonality in the semi-inner-product sense or the usual orthogonality in prehilbertian spaces, respectively.

We now present the following generalisation of Birkhoff's orthogonality which works in general normed spaces (see also [**3**]).

DEFINITION 33. *Let $X$ be a normed linear space, $\varepsilon \in [0,1)$ and $x, y \in X$. The element $x$ is said to be $\varepsilon-$Birkhoff orthogonal over $y$ and we denote $x \perp_{\varepsilon B} y$ if:*

$$\|x+ty\| \geq (1-\varepsilon)\|x\| \quad \textit{for all } t \in \mathbb{R}.$$

The following proposition establishes a connection between $\varepsilon - k_0-$orthogonality and $\varepsilon-$Birkhoff orthogonality in smooth normed linear spaces of $(BD)$ −type [**3**].

Proposition 48. *Let $X$, $k_0$ be as above and $x, y \in X$, $\varepsilon \in [0,1)$. Then the following statements are valid:*

(i) *$x \perp_{\varepsilon B} y$ implies $x \perp_{\delta(\varepsilon)k_0} y$ with $\delta(\varepsilon) := [\varepsilon(2-\varepsilon)]^{\frac{1}{2}}$;*
(ii) *$x \perp_{\eta(\varepsilon)B} y$ implies $x \perp_{\varepsilon k_0} y$ with $\eta(\varepsilon) := 1 - (1-\varepsilon^2)^{\frac{1}{2}}$.*

Proof. We shall start with Taylor's formula:

$$\|x+ty\|^2 = \|x\|^2 + 2[y,x]t + (y, x+\xi_t y)' t^2, \quad \text{for } t \in \mathbb{R},$$

where $\xi_t$ is between 0 and $t$.

(i) If $x \perp_{\varepsilon B} y$, then:

$$(1-\varepsilon)^2 \|x\|^2 \leq \|x+ty\|^2 \quad \text{for all } t \in \mathbb{R},$$

which implies:

$$\left(\varepsilon^2 - 2\varepsilon\right) \|x\|^2 \leq 2[y,x]t + k_0^2 \|x\|^2 t^2 \quad \text{for all } t \in \mathbb{R},$$

from where we get:

$$[y,x]^2 \leq k_0^2 \varepsilon (2-\varepsilon) \|x\|^2 \|y\|^2,$$

i.e. $x \perp_{\delta(\varepsilon)k_0} y$ with $\delta(\varepsilon)$ is as above.

(ii) It follows from (i) substituting $\varepsilon$ by $\eta(\varepsilon) \in [0,1)$.

∎

Remark 33. *In the case of inner-product spaces we have:*

(i) *$x \perp_{\varepsilon B} y$ iff $x \perp_{\delta(\varepsilon)} y$;*
(ii) *$x \perp_{\eta(\varepsilon)B} y$ iff $x \perp_{\varepsilon} y$;*

*where $\delta(\varepsilon)$ and $\eta(\varepsilon)$ are as above.*

The proof is obvious and we omit the details.

Now, let $X$ be a normed linear space and $A$ be its nonempty subset. By $A^{\perp_{\varepsilon B}}$ we shall denote the set:

$$\{y \in X | y \perp_{\varepsilon B} x \ \text{ for all } \ x \in A\},$$

where $\varepsilon$ is a given real number in $[0,1)$. This set will be called the Birkhoff orthogonal complement of $A$. It is easy to see that $0 \in A^{\perp_{\varepsilon B}}$ and $A \cap A^{\perp_{\varepsilon B}} \subseteqq \{0\}$ and for all $\varepsilon \in [0,1)$.

The following lemma ([**3**]) is a variant of F. Riesz result (see for example [**8**, p. 84]):

Lemma 12. *Let $X$ be a normed space and $E$ be its closed linear subspace. Suppose $E \neq X$. Then for every $\varepsilon \in (0,1)$, the $\varepsilon-$Birkhoff orthogonal complement of $E$ is nonzero.*

Proof. Let $\bar{y} \in X \backslash E$. Since $E$ is closed, we have $d(\bar{y}, E) = d > 0$. Then there exists $y_\varepsilon \in E$ such that: $0 \leq \|\bar{y} - y_\varepsilon\| \leq \frac{d}{1-\varepsilon}$. Putting $x_\varepsilon := \bar{y} - y_\varepsilon$, we have $x_\varepsilon \neq 0$ and for every $y \in E$ and $\lambda \in \mathbb{R}$:

$$\|x_\varepsilon + \lambda y\| = \|\bar{y} - y_\varepsilon + \lambda y\| = \|\bar{y} - (y_\varepsilon - \lambda y)\| \geq d \geq (1-\varepsilon)\|x_\varepsilon\|,$$

which means that $x_\varepsilon \in E^{\perp_{\varepsilon B}}$ and the lemma is proven. ∎

The following decomposition theorem holds [**3**].

Theorem 98. *Let $X$ be a normed linear space and $E$ be its closed linear subspace. Then for any $\varepsilon \in (0,1)$ the following decomposition*

$$X = E + E^{\perp_{\varepsilon B}}, \tag{14.8}$$

*is valid.*

Proof. Suppose $E \neq X$ and $x \in X$.

If $x \in E$, then $x = x + 0$ with $x \in E$ and $0 \in E^{\perp_{\varepsilon B}}$.

If $x \notin E$, then there exists $y_\varepsilon \in E$ such that $0 < d = d(x, E) = \|x - y_\varepsilon\| \leq \frac{d}{1-\varepsilon}$. Since $x_\varepsilon := x - y_\varepsilon \in E^{\perp_{\varepsilon B}}$ (see the proof of the above lemma) we obtain $x = y_\varepsilon + x_\varepsilon$ and the relation (14.8) is valid. ∎

In what follows, we apply the above results for the particular case of smooth normed spaces of $(BD)$ −type.

Let $X$ be as above and $A$ be a nonempty subset of $X$. Then by $A^{\perp_{\varepsilon k_0}}$ we shall denote the set:

$$\{y \in X | y \perp_{\varepsilon k_0} x \text{ for all } x \in A\}, \quad \varepsilon \in [0,1),$$

which will be called the $\varepsilon - k_0-$orthogonal complement of $A$ in $X$.

Lemma 13. *([**2**]) Let $X$ be a smooth normed linear space of $(BD)$ −type, $E$ be its closed linear subspace and $\varepsilon \in (0,1)$. Assume $E \neq X$. Then the $\varepsilon - k_0-$orthogonal complement of $E$ is nonzero.*

Proof. Let $\varepsilon \in (0,1)$ and $\eta(\varepsilon) := 1 - (1-\varepsilon^2)^{\frac{1}{2}}$. Then $\eta(\varepsilon)$ belongs to $(0,1)$. Applying Lemma 12 for $\eta(\varepsilon)$, then there exists an element $x_\varepsilon \neq 0$ and $x_\varepsilon \in E^{\perp_{\eta(\varepsilon)B}}$. Since $E^{\perp_{\eta(\varepsilon)B}} \subseteqq E^{\perp_{\eta(\varepsilon)k_0}}$ (see Proposition 48) the lemma is thus proven. ∎

Finally, we have [**3**]:

Theorem 99. *Let $X$ be a smooth normed space of $(BD)$ −type, $E$ its closed linear subspace and $\varepsilon \in (0,1)$. Then the following decomposition holds:*

$$X = E + E^{\perp_{\varepsilon k_0}}$$

The proof is obvious from Theorem 98 and Proposition 48 and we omit the details.

## 4. Riesz Class of $X^*$

Let $X$ be a smooth normed linear space over the real number field $\mathbb{R}$. The following subset of dual space $X^*$ :

$$R(X^*) := \{f_y \in X^* | f_y(x) = [x, y]\,;\ x, y \in X\}$$

will be called Riesz's class of $X^*$. We remark that, in general, $R(X^*)$ is not a linear subspace of $X^*$ and by Tapia's theorem or representation, a smooth Banach space $X$ is reflexive iff $R(X^*) = X^*$.

Remark 34. *If $(X; (\cdot,\cdot))$ is an inner-product space, then $R(X^*)$ is a linear subspace in $X^*$ which will be called Riesz's subspace of $X^*$ and will be denoted by $R(X^*)$. The mapping $\Delta : X \to X^*$ given by $\Delta(y) := f_y$ is a linear isometric operator to $X$ onto $R(X^*)$. Putting $(\cdot,\cdot)^* : R(X^*) \times R(X^*) \to \mathbb{R}$, $(f_x, f_y)^* := (x, y)$, then $(\cdot,\cdot)^*$ is an inner-product on $R(X^*)$ which generates the norm induced by dual space $X^*$ in $R(X^*)$ and by these considerations, $R(X^*)$ is isomorphic and isometric to $X$ as inner-product spaces.*

The following proposition holds [**3**].

Proposition 49. *Let $X$ be a smooth normed space of $(BD)-$type, $E$ be its closed linear subspace and $E \neq X$. Then for any $\varepsilon > 0$, there exists a functional $f_\varepsilon \in R(X^*)$ such that*

$$\|f_\varepsilon\| \leq 1 \quad \textit{and} \quad \|f_\varepsilon\|_E \leq \varepsilon, \tag{14.9}$$

*where* $\|f\|_E := \sup\{|f(x)|, \|x\| = 1,\ x \in E\}$.

Proof. If $\varepsilon \geq 1$, the statement is clear.

Let us assume that $\varepsilon \in (0, 1)$. By Lemma 13 there exists a nonzero element $y_\varepsilon$ in $E^{\perp_{\varepsilon k_0}}$, i.e.,

$$|[x, y_\varepsilon]| \leq \varepsilon k_0 \|x\| \|y_\varepsilon\| \text{ for all } x \text{ in } X.$$

Putting $x_\varepsilon := \frac{y_\varepsilon}{k_0\|y_\varepsilon\|}$ we have $\|x_\varepsilon\| = \frac{1}{k_0} \leq 1$ and the functional $f_\varepsilon : X \to \mathbb{R}$, $f_\varepsilon(x) := [x, x_\varepsilon]$ satisfies the relation (14.9). The proof is thus completed. ∎

The next theorem will play an important role in that to follow.

Theorem 100. *Let $X$ be a smooth normed space of $(BD)-$type and $f$ be a nonzero continuous linear functional on it. Then for any $\varepsilon > 0$ there exists a nonzero element $x_\varepsilon$ in $X$ such that:*

$$|f(x) - [x, x_\varepsilon]| \leq \varepsilon \|x\| \text{ for all } x \text{ in } X. \tag{14.10}$$

Proof. Since $f \neq 0$, the linear subspace $E := Ker(f)$ is closed in $X$ and $E \neq X$.

Let $\varepsilon > 0$ and put $\delta(\varepsilon) := \varepsilon / (2\|f\| k_0) > 0$, where $k_0$ is the boundedness modulus of $(\cdot,\cdot)'_T$.

If $\delta(\varepsilon) \geq 1$, then there exists an element $y_\varepsilon \in X \backslash E$ such that

$$|[y, y_\varepsilon]| \leq \delta(\varepsilon) \|y\| \|y_\varepsilon\| \leq \delta(\varepsilon) k_0 \|y\| \|y_\varepsilon\| . \tag{14.11}$$

If $0 < \delta(\varepsilon) < 1$, then by Lemma 13 there exists an element $y_\varepsilon \in X \backslash E$ such that (14.11) is also valid.

Let us put $z_\varepsilon := \frac{y_\varepsilon}{\|y_\varepsilon\|}$. The for all $x \in X$ we have:

$$y := f(x) z_\varepsilon - f(z_\varepsilon) x \in Ker(f),$$

and then:

$$\begin{aligned} |[f(x) z_\varepsilon - f(z_\varepsilon) x, z_\varepsilon]| &\leq k_0 \delta(\varepsilon) \|f(x) z_\varepsilon - f(z_\varepsilon) x\| \\ &\leq 2k_0 \delta(\varepsilon) \|f\| \|x\| \leq \varepsilon \|x\|, \end{aligned}$$

for all $x \in X$.

On the other hand, we have;

$$[f(x) z_\varepsilon - f(z_\varepsilon) x, z_\varepsilon] = f(x) - [x, f(z_\varepsilon) z_\varepsilon]$$

for all $x \in X$ and putting $x_\varepsilon := f(z_\varepsilon) z_\varepsilon$, the relation (14.10) is obtained. ∎

Now, we shall give the main result of this section.

Theorem 101. *Let $X$ be a smooth normed space of $(BD)-$type. Then Riesz's subset $R(X^*)$ of $X^*$ is dense in $X^*$ endowed with the strong topology.*

Proof. Let $f \in X^*$ and $\varepsilon > 0$. Then by Theorem 100 there exists an element $x_\varepsilon \in X$ such that:

$$|f(x) - f_\varepsilon(x)| \leq \varepsilon \|x\| \text{ for all } x \text{ in } X,$$

where $f_\varepsilon(x) := [x, x_\varepsilon]$, $x \in X$. Consequently, $\|f - f_\varepsilon\| \leq \varepsilon$ and the assertion is proved. ∎

Remark 35. *Let $[\cdot,\cdot] : X \times X \to K$ $(K = \mathbb{R}, C)$ be a semi-inner product on normed linear space $X$ (see for example* [**4**]*) which generates its norm. In paper* [**3**] *we introduced the concept of normed linear spaces of $(APP)-$type relative to $[\cdot,\cdot]$, i.e., a normed space such that for every nonzero continuous linear functional $f$ on it and for any $\varepsilon \in (0,1)$ there exists a nonzero element $y_\varepsilon$ in $X$ such that:*

$$|[y, y_\varepsilon]| \leq \varepsilon \|y\| \|y\| \text{ for all } y \in \mathrm{Ker}(f).$$

We also proved that if such a space, then the Lumer subset $L(X^*) := \{f_y \in X^* | f_y(x) := [x, y] \text{ for } x, y \in X\}$, of dual space $X^*$ associated to semi-inner-product $[\cdot, \cdot]$ is dense in $X^*$ endowed with the strong topology.

If $X$ is a smooth real normed space, it is well-known that there exists a unique semi-inner-product which generates the norm and coincides with the superior semi-inner-product (see [**1**] or [**5**]) and then $L(X^*) = R(X^*)$. We also remark that every smooth normed space of $(BD)-$type is a normed linear space of $(APP)-$type relative with the superior semi-inner-product.

Now, we shall give a corollary of Theorem 101.

COROLLARY 31. *Let $X$ be a smooth Banach space of $(BD)-$type. Then the following statements are equivalent:*

(i) *$X$ is reflexive;*
(ii) *$R(X^*)$ is closed in $X^*$;*
(iii) *$R(X^*) = X^*$.*

PROOF. The equivalence "(i) $\Leftrightarrow$ (iii)" follows by Tapia's theorem of representation and the equivalence "(ii) $\Leftrightarrow$ (iii)" is obvious by the above theorem. ∎

The case of prehilbertian spaces is embodied in the next proposition.

PROPOSITION 50. *Let $X$ be an inner-product-space. Then the following statements are equivalent:*

(i) *$X$ is a Hilbert space;*
(ii) *$R(X^*)$ is closed in $X^*$;*
(iii) *$R(X^*) = X^*$.*

The proof follows by Remark 34 and Theorem 101 for inner-product-spaces.

## 5. Applications to Operator Equations

In this section we shall use Theorem 100 to establish some existence results for $\varepsilon-$solutions of the operator equation:

$$(A; y) \qquad Ax = y, \quad x \in \mathcal{D}(A), \quad y \in X,$$

where $A : \mathcal{D}(A) \subset X \to X$ is an operator defined on dense linear subspace $\mathcal{D}(A)$ of Hilbert space $X$ and having the properties (a) – (av) and (aM) from Example 7.

REMARK 36. *Some examples of operators which verify the above conditions are the symmetric strictly positive operators which are densely defined on a real Hilbert space and satisfy condition:*

$$\|Ax\| \leq M\|x\|, \quad M \geq 1 \quad \textit{for all } x \in \mathcal{D}(A).$$

Now, let $\varepsilon > 0$. The element $x_\varepsilon \in \mathcal{D}(A)$ is called an $\varepsilon-$solution for the equation $(A; y)$ if $\|Ax_\varepsilon - y\| \leq \varepsilon$. It is known that (see Example 7) the mapping $\mathcal{D}(A) \ni x \overset{\|\cdot\|_A}{\mapsto} (x, Ax)^{\frac{1}{2}} \in \mathbb{R}^+$ is a norm on $\mathcal{D}(A)$ and $(\mathcal{D}(A), \|\cdot\|_A)$ is a smooth normed space of $(BD)-$type. Then we can also introduce the following concept of approximative solutions.

DEFINITION 34. *Let $\varepsilon > 0$. The element $x_\varepsilon \in \mathcal{D}(A)$ is called an $A - \varepsilon-$solution for the equation $(A; y)$ if:*

$$\sup_{\|x\|_A \leq 1} |(x, y - Ax_\varepsilon)| \leq \varepsilon.$$

The next existence result for $A - \varepsilon-$solutions of the operatorial equation $(A; y)$ holds.

PROPOSITION 51. *Let $X, A$ be as above and $y$ be a nonzero element in $X$ satisfying the assumption:*

$$|(x, y)| \leq \mu (x, Ax)^{\frac{1}{2}} \quad \textit{for all } x \in \mathcal{D}(A) \;\; (\mu > 0); \tag{14.12}$$

*then for every $\varepsilon > 0$ the equation $(A; y)$ has an $A - \varepsilon-$solution.*

PROOF. Let $f_y : \mathcal{D}(A) \to \mathbb{R}$, $f_y(x) := (x, y)$. By condition (14.12) it follows that $f_y$ is continuous in $(\mathcal{D}(A), \|\cdot\|_A)$ and by Theorem 100 there exists an element $x_\varepsilon \in \mathcal{D}(A) \setminus \{0\}$ such that:

$$|f_y(x) - [x, x_\varepsilon]|_A \leq \varepsilon \|x\|_A \quad \text{for all } x \in \mathcal{D}(A),$$

which is equivalent to the existence of an $A-\varepsilon-$solution for the equation $(A; y)$. ∎

COROLLARY 32. *Let $X, A$ be as above and, in addition, there exists a constant $\eta > 0$ such that:*

$$\eta \|x\|^2 \leq (x, Ax) \quad \textit{for all } x \in \mathcal{D}(A). \tag{14.13}$$

*Then for every $y \in X \setminus \{0\}$ and for any $\varepsilon > 0$ the equation $(A; y)$ has an $A - \varepsilon-$solution.*

The proof is obvious by Proposition 51 observing that condition (14.13) implies condition (14.12) for all $y \in X \setminus \{0\}$.

Finally, we have:

PROPOSITION 52. *Let $X, A$ be as in Proposition 51 and, in addition, there exists a constant $\gamma > 0$ such that:*

$$(x, Ax) \leq \gamma \|x\|^2 \quad \textit{for all } x \in \mathcal{D}(A). \tag{14.14}$$

*If $y \in X \setminus \{0\}$ verifies the assumption (14.12), then for any $\varepsilon > 0$ the equation $(A; y)$ has an $\varepsilon-$solution.*

PROOF. By condition (14.14) we have: $\|x\|_A \leq \gamma^{\frac{1}{2}} \|x\|$ for all $x \in \mathcal{D}(A)$. Since the linear functional $f_y$ is continuous in $(\mathcal{D}(A); \|\cdot\|_A)$ then by Theorem 100, for any $\varepsilon \geq 0$ there exists an element $x_\varepsilon \in \mathcal{D}(A) \setminus \{0\}$ such that:

$$|f_y(x) - (x, x_\varepsilon)_A| \leq \left(\frac{\varepsilon}{\gamma^{\frac{1}{2}}}\right) \|x\|_A \quad \text{for all } x \in \mathcal{D}(A),$$

from which results:

$$|(x, y - Ax_\varepsilon)| \leq \|x\| \quad \text{for all } x \in \mathcal{D}(A).$$

Since $\mathcal{D}(A)$ is dense in $X$, the proposition is proven. ∎

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CHAPTER 15

# Continuous Sublinear Functionals

## 1. Introduction

In paper [**1**] the author proved the following "interpolation" theorem for the continuous linear functionals.

Theorem 102. *Let* $(X, \|\cdot\|)$ *be a real reflexive Banach space and* $f$ *be a continuous linear functional on it. Then there exists an element* $u \in X$ *such that*

$$\langle x, u\rangle_i \leq f(x) \leq \langle x, u\rangle_s \quad \textit{for all } x \in X \quad \textit{and} \quad \|f\| = \|u\|.$$

Note that the next decomposition theorem is also valid.

Theorem 103. *Let* $(X, \|\cdot\|)$ *be as above and* $G$ *be its closed linear subspace. If* $G^{\perp}$ *denotes the orthogonal complement of* $G$ *in the sense of Birkhoff, then*

$$X = G + G^{\perp}.$$

For the proof of this fact see for example [**1**] where further consequences and applications are given.

The main aim of this chapter is to extend the above results for continuous sublinear functionals and closed clins in real reflexive Banach spaces. Applications for inequalities as in [**2**] are also given.

## 2. Semi-orthogonality in Reflexive Banach Spaces

A nonempty subset $K$ of a real linear space $X$ is said to be *clin in* $X$ if the following conditions are satisfied:

(i) $x, y \in K$ imply $x + y \in K$
(ii) $x \in K$, $\alpha \geq 0$ imply $\alpha x \in K$.

A real functional $p$ defined on a clin $K$ is said to be *sublinear on* $K$ if

(s) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in \mathbb{K}$
(ss) $p(\alpha x) = \alpha p(x)$ for all $x \in \mathbb{K}$ and $\alpha \geq 0$.

Definition 35. *(*[**3**]*) The element* $x$ *in real normed space* $(X, \|\cdot\|)$ *will be called semi-orthogonal in the sense of Birkhoff over* $y \in X$ *if* $\langle y, x\rangle_i \leq 0$*. We denote* $x \perp_S y$*.*

It is clear that $0 \perp_S y$; $x \perp_S 0$; $x \perp_S x$ implies $x = 0$ and $x \perp_S y$ implies $\alpha x \perp_S \beta y$ if $\alpha\beta \geq 0$. For a nonempty subset $A$ of $X$ we put

$$A^{\perp_S} := \{y \in X | y \perp_S x \text{ for all } x \in A\}.$$

We also remark that $0 \in A^{\perp_S}$, $A \cap A^{\perp_S} \subseteq \{0\}$ and $x \in A^{\perp_S}$, $\alpha \geq 0$ imply $\alpha x \in A^{\perp_S}$.

The following theorem is a natural generalisation of Theorem 103 [**4**].

Theorem 104. *Let $(X, \|\cdot\|)$ be a real reflexive Banach space and $K$ be a closed clin in $X$. Then the following decomposition holds*

$$X = K + K^{\perp_S}. \tag{15.1}$$

Proof. Let $x \in X$. if $x \in X$ then $x = x + 0$ with $x \in K$ and $0 \in K^{\perp_S}$. If $x \notin K$, since $K$ is a closed convex set in reflexive Banach space $X$, then there exists a best approximation element in $K$ referring to $x$, i.e., there exists an $x' \in K$ such that $d(x, K) = \|x - x'\|$.

Let us put $x'' := x - x'$ and consider $\alpha \geq 0$ and $y \in K$. Then we have

$$\|x'' - \alpha y\| = \|x - x' - \alpha y\| = \|x - (x' + \alpha y)\| \geq \|x''\|,$$

because $x', \alpha y \in K$ and $K$ is a clin in $X$. Hence

$$\|x'' - \alpha y\|^2 \geq \|x''\|^2 \quad \text{for all } \alpha \geq 0$$

which implies that

$$\frac{\|x'' - \alpha y\|^2 - \|x''\|^2}{2\alpha} \geq 0 \quad \text{for all } \alpha > 0.$$

Taking the limit as $s \to 0$ $(s > 0)$ we obtain $\langle -y, x\rangle_s \geq 0$, i.e., $\langle y, x\rangle_i \leq 0$ for all $y \in K$ which means that $x'' \in K^{\perp_S}$ and the theorem is proved. ∎

The following result holds [**3**].

Corollary 33. *If $K$ is a closed linear subspace in $X$, then $K^{\perp_S} = K^{\perp}$ where $K^{\perp}$ denotes the orthogonal complement of $K$ in the sense of Birkhoff.*

Proof. It is clear that $K^{\perp} \subseteq K^{\perp_S}$.

Now, let $x \in K^{\perp_S}$. Then $\langle y, x\rangle_i \leq 0$ for all $y \in K$ and since $K$ is a linear subspace, then it follows $\langle y, x\rangle_i \leq 0$, i.e., $\langle y, x\rangle_s \geq 0$, which implies that $x \in K^{\perp}$ and the statement is proved. ∎

Remark 37. *If $X$ is a Hilbert space, we recapture Theorem 2.1 from* [**2**].

The following lemma will be used in the sequel [**3**].

LEMMA 14. *Let $(X, \|\cdot\|)$ be a Banach space and $p : X \to \mathbb{R}$ be a continuous sublinear functional on it. Then the set $K(p) := \{x \in X, p(x) \leq 0\}$ is a closed clin in $X$. In addition, if we assume that there exists $x_0 \in X$ such that $p(x_0) < 0$ then $K(p)$ is proper in $X$, i.e., $K(p)$ is not a linear subspace.*

The argument is similar to that in the proof of the Lemma 3.1 from [**2**] and we omit the details.

THEOREM 105. *([**3**]) Let $(X, \|\cdot\|)$ be a real reflexive Banach space and $p : X \to \mathbb{R}$ be a continuous sublinear functional on it such that $K(p) \neq X$. Then there exists $u \in X$, $\|u\| = 1$ such that*

$$p(x) \geq p(u) \langle x, u \rangle_i \quad \text{for all } x \in K(p). \tag{15.2}$$

PROOF. Since $K(p)$ is closed and $K(p) \neq X$, then there exists an element $w \in K^{\perp_s}(p)$ such that $w \neq 0$. Since $w \notin K(p)$, we have $p(w) > 0$. On the other hand, for all $x \in K(p)$, we have

$$\begin{aligned} p(p(w)x - p(x)w) &\leq p(p(w)x) + p(-p(x)w) \\ &= p(w)p(x) - p(x)p(w) = 0 \end{aligned}$$

and then

$$p(w)x - p(x)w \in K(p) \quad \text{for all } x \in K(p).$$

Since $w \in K^{\perp_s}(p)$ we get

$$\langle p(w)x - p(x)w, w \rangle_i \leq 0 \quad \text{for all } x \in K(p).$$

Using the properties of semi-inner product $\langle \cdot, \cdot \rangle_i$, we deduce $p(w)\langle x, w \rangle_i - p(x)\|w\|^2 \leq 0$ for all $x \in K(p)$ which implies that

$$p(x) \geq \frac{p(w)}{\|w\|} \left\langle x, \frac{w}{\|w\|} \right\rangle_i \quad \text{for all } x \in K(p)$$

from where results (15.2). ∎

REMARK 38. *If $X$ is a Hilbert space we obtain the first part of Theorem 3.2 from* [**2**].

The following two corollaries hold [**3**].

COROLLARY 34. *Let $p : X \to \mathbb{R}$ be a continuous sublinear functional on reflexive Banach space $X$ such that $K(p) \neq X$. Then there exists an element $u \in X$, $\|u\| = 1$ with the property*

$$\inf_{x \neq 0} \frac{p(x)}{\|x\|} \geq -p(u).$$

Proof. It is clear that

$$\inf_{x\neq 0}\frac{p(x)}{\|x\|}=\inf\left\{\left.\frac{p(x)}{\|x\|}\right| x\in K(p)\setminus\{0\}\right\}.$$

By the above theorem there exists an element $x\in X$, $\|u\|=1$ such that: $p(x)\geq p(u)\langle x,u\rangle_i$ for all $x\in K(p)$. However, $\langle x,u\rangle_i\geq -\|x\|\|u\|=-\|x\|$ which implies that $p(x)\geq -p(u)\|x\|$ for all $x\in K(p)$, from where results the desired inequality. ∎

Remark 39. *The above corollary contains Theorem 3.10 from* [**2**] *which works in the case of Hilbert spaces.*

Corollary 35. *Let $p$ be as above. Then there exists an element $u\in X$, $\|u\|=1$ such that the mappings $p_u:X\to\mathbb{R}$, $p_u(x)=p(x)+p(u)\|x\|$ is a positive continuous sublinear functional on $X$.*

## 3. Clins with the $(H)-$Property in Reflexive Spaces

We start with the following definition [**3**].

Definition 36. *Let $(X,\|\cdot\|)$ be a real normed linear space and $K$ be a clin in it. $K$ is said to be with the $H-$property if the set $H(K):=K^{\perp s}\cap(-K)$ also contains nonzero elements.*

Remark 40. *If the clin $K$ has the $(H)-$property, then $K$ is proper in $X$, i.e., $K$ is not a linear subspace in $X$.*

*Indeed, if we suppose that $K$ is a linear subspace and $w\in K^{\perp s}\cap(-K)\setminus\{0\}$ then $w\in -K=K$ and since $K^{\perp s}\cap K=\{0\}$, we obtain a contradiction.*

The following lemma of characterisation holds [**3**].

Lemma 15. *The clin $K$ has the $(H)-$property if and only if there exists a nonzero element $w\in K$ such that $\langle x,w\rangle_s\geq 0$ for all $x\in K$.*

Proof. Let $-w\in K^{\perp s}\cap(-K)$ then $w\in K$ and since $-w\in K^{\perp s}$, we have $\langle x,-w\rangle_i\leq 0$ for all $x\in K$, i.e., $\langle x,w\rangle_s\geq 0$.

Conversely, if $\langle x,w\rangle_s\geq 0$ for all $x\in K$ then $\langle x,-w\rangle_i\leq 0$, i.e., $-w\in K^{\perp s}$ and since $-w\in -K$ we deduce that $K$ has the $(H)-$property. ∎

Example 9. *Let $f:X\to\mathbb{R}$ be a nonzero continuous linear functional on reflexive Banach space $X$ and put $K_+(f):=\{x\in X|f(x)\geq 0\}$, $K_-(f):=\{x\in X|f(x)\leq 0\}$. Then $K_+(f)$ and $K_-(f)$ are clins with the $(H)-$property.*

*Indeed, by Theorem 102, there exists a nonzero element $u\in X$ such that: $\langle x,u\rangle_i\leq f(x)\leq\langle x,u\rangle_s$ for all $x\in X$.*

*Let* $x \in K_+(f)$, *then* $\langle x, u\rangle_s \geq 0$ *and since* $f(u) = \|u\|^2 > 0$ *we obtain that* $u \in K_+(f)$, $u \neq 0$ *and* $\langle x, u\rangle_s \geq 0$, *i.e.,* $K_+(f)$ *has the* $(H)$ *−property.*

*The proof of the fact that* $K_-(f)$ *is also a clin with the* $(H)$ *−property is similar and we omit the details.*

Note that the following theorem is valid [**3**].

THEOREM 106. *Let* $(X, \|\cdot\|)$ *be a reflexive and strictly convex Banach space and* $K$ *be a closed clin in* $X$ *such that* $K^{\perp_S}$ *is also a clin. Then the following statements are equivalent:*

(i) $K$, $K^{\perp_S}$ *are linear subspaces.*
(ii) *The following decomposition holds*

$$X = K \oplus K^{\perp_S}.$$

PROOF. (i) $\Rightarrow$ (ii). If $K$ is a linear subspace, the $K^{\perp_S} = K^{\perp}$ (see Corollary 33). Since $(X, \|\cdot\|)$ is reflexive and strictly convex, it is known that $X = K \oplus K^{\perp}$.

(ii) $\Rightarrow$ (i). Let $u \in K$, $v \in K^{\perp_S}$ and put $x = u + v$. Then by Theorem 104 there exists $m \in K$, $n \in K^{\perp_S}$ such that $-x = m + n$. Hence $0 = (u + m) + (v + n)$ with $u + m \in K$, $v + n \in K^{\perp_S}$ and since the null element has a unique decomposition we obtain $-u = m \in K$, $-v = n \in K^{\perp_S}$, i.e., $K$ and $K^{\perp_S}$ are linear subspaces. ∎

REMARK 41. *The above theorem contains Theorem 2.1 from* [**2**] *which is valid in Hilbert spaces.*

THEOREM 107. *(*[**3**]*) Let* $(X, \|\cdot\|)$ *be a reflexive and strictly convex Banach space and* $K$ *be a proper closed clin in* $X$ *such that* $K^{\perp_S}$ *is also a clin. Then* $K$ *has the* $(H)$ *−property.*

PROOF. Since $K$ is a proper closed clin in $X$, then by the above theorem there exists at least one element $x$ such that

$$x = x' + x'' \qquad x' \in K \qquad x'' \in K^{\perp_S}$$

$$x = x_1 + x_2 \qquad x_1 \in K \qquad x_2 \in K^{\perp_S}$$

and

$$x' \neq x_1 \qquad\qquad x'' \neq x_2.$$

By Theorem 104, there exists $y' \in K$ and $y'' \in K^{\perp_S}$ such that $-x = y' + y''$ and then

$$0 = (x' + y') + (x'' + y'') \qquad x' + y' \in K \qquad x'' + y'' \in K^{\perp_S}$$

$$x = (x_1 + y') + (x_2 + y'') \qquad x_1 + y' \in K \qquad x_2 + y'' \in K^{\perp_S}$$

with $x' + y' \neq x_1 + y'$ and $x'' + y'' \neq x_2 + y''$.

Consequently, there exists $m \in K$, $n \in K^{\perp_S}$ with $m \neq 0$ and $n \neq 0$ such that $0 = m + n$ which implies that $n = -m$ and then the set $K^{\perp_S} \cap (-K)$ also contains nonzero elements. ∎

COROLLARY 36. *([**3**]) Let $(X; (\cdot, \cdot))$ be a Hilbert space. Then every proper closed clin in $X$ has the $(H)$ −property.*

PROOF. Follows from the above theorem and by the fact that for all clin $K$ in $X$, $K^{\perp_S}$ is also a clin in $X$. ∎

We can now improve Theorem 105.

THEOREM 108. *Let $(X, \|\cdot\|)$ be a real reflexive Banach space and $p : X \to \mathbb{R}$ be a continuous sublinear functional on it such that $K(p)$ has the $(H)$ −property. Then there exists an element $u \in X$, $\|u\| = 1$ such that*

$$p(x) \geq \begin{cases} p(u) \langle x, u \rangle_i & \text{for all} \quad x \in K(p) \\ \\ -p(-u) \langle x, u \rangle_i & \text{for all} \quad x \in X \backslash K(p). \end{cases} \tag{15.3}$$

PROOF. Because $K(p)$ has the $(H)$ −property, there exists $w \neq 0$, $w \in K^{\perp_S}(p) \cap (-K(p))$. Since $w \in K^{\perp_S}(p)$, we have $p(w) > 0$. Then by a similar argument to that in the proof of Theorem 105, we have

$$p(x) \geq \frac{p(w)}{\|w\|} \left\langle x, \frac{w}{\|w\|} \right\rangle_i \quad \text{for all } x \in K(p)$$

and putting $u := \frac{w}{\|w\|}$, we obtain the first part of (15.3).

Now, let $x \in X \backslash K(p)$, then $p(x) > 0$ and since $-w \in K(p)$, it follows that $-p(-w) \geq 0$. We obtain: $p(p(x)(-w) - p(-w)x) \leq p(x)p(-w) + (-p(-w))\, p(x) = 0$ which implies that $-p(x)w - p(-w)x \in K(p)$. Since $w \in K^{\perp_S}(p)$, we derive:

$$\langle -p(x)w - p(-w)x, w \rangle_i \leq 0 \quad \text{for all } x \in X \backslash K(p),$$

which implies $-p(x)\|w\|^2 - p(-w)\langle x, w \rangle_i \leq 0$ for all $x \in X \backslash K(p)$, from where results

$$p(x) \geq \frac{-p(-w)}{\|w\|} \left\langle x, \frac{w}{\|w\|} \right\rangle_i \quad \text{for all } x \in X \backslash K(p)$$

and the second part of relation (15.3) is also valid. ∎

REMARK 42. *If $X$ is a Hilbert space we obtain the main result from* [**2**] *(see Theorem 3.2).*

REMARK 43. *If $f$ is a continuous linear functional on $X$ and since $K(f) = K_-(f)$, then by (15.3) we have: $f(x) \geq f(u)\langle x, u \rangle_i$ for all $x \in X$.*

*On the other hand, substituting $x$ by $-x$ we derive that* $-f(x) \geq f(u)\langle -x,u\rangle_i = -f(u)\langle x,u\rangle_s$ *for all $x \in$, which implies*

$$f(x) \leq f(u)\langle x,u\rangle_s \quad \textit{for all } x \in X.$$

*Consequently, Theorem 108 gives a natural generalisation of Theorem 102 for the case of sublinear and continuous functionals which has the $(H)-$property.*

Now, let us consider the set

$$L(p) := \{x \in X | p(x) + p(-x) = 0\}$$

where $p$ is a continuous sublinear functional on Banach space $X$. Then $L(p)$ is a closed linear subspace in $X$. The proof is similar to that of Lemma 3.4 from [**2**] and we shall omit the details.

DEFINITION 37. *([**3**]) A continuous sublinear functional $p$ is said to be of $(C)-$type (see also [**2**]) if the set $N(p) := H(p) \cap L(p)$ also contains nonzero elements.*

It is easy to see that if $p$ is a continuous linear functional then $p$ is on $(C)-$type.

The following result is an extension of Theorem 3.4 in [**2**] which works in Hilbert spaces [**3**].

THEOREM 109. *Let $p$ be a continuous sublinear functional of $(C)-$type on reflexive Banach space $X$. Then there exists an element $v \in X$ such that*

$$p(x) \geq \langle x,v\rangle_i \quad \textit{for all } x \in X.$$

PROOF. Let $w \in N(p)$, $w \neq 0$. Then, as in Theorem 108, we have

$$p(x) \geq \frac{p(w)}{\|w\|^2}\langle x,w\rangle_i \quad \text{for all } x \in K(p),$$
$$p(x) \geq \frac{-p(-w)}{\|w\|^2}\langle x,w\rangle_i \quad \text{for all } x \in X \backslash K(p).$$

Since $p(-w) = -p(w)$, we obtain

$$p(x) \geq \frac{p(w)}{\|w\|^2}\langle x,w\rangle_i \quad \text{for all } x \in X$$

and putting $v := \frac{p(w)}{\|w\|^2}w$, we obtain the desired inequality. ∎

REMARK 44. *If $p$ is linear, then the above theorems also give Theorem 102.*

## 4. Applications

Let $(X, \|\cdot\|)$ be a real reflexive Banach space and $(e_i)_{i=\overline{1,n}}$ be a linearly independent family of vectors in $X$. Consider the following system of inequations $(x \in X)$

$$\text{(S)} \qquad \langle e_1, x\rangle_s \geq 0 \quad \langle e_2, x\rangle_s \geq 0 \quad \ldots \quad \langle e_n, x\rangle_s \geq 0$$

and put $K(e_1, \ldots, e_n) := \{x | x = \sum_{i=1}^n \alpha^i e_i,\ \alpha^i \geq 0\}$ which is a proper closed clin in $X$ generated by $(e_i)_{i=\overline{1,n}}$. The next result holds [**3**].

Proposition 53. *The following statements are equivalent.*

(i) *$K(e_1, \ldots, e_n)$ has the $(H)$ −property in $X$.*
(ii) *The system (S) has a nonzero solution in $K(e_1, \ldots, e_n)$.*

Proof. If $K(e_1, \ldots, e_n)$ has the $(H)$ −property, then there exists $x_0 \in K(e_1, \ldots, e_n) \setminus \{0\}$ (see Lemma 15) such that $\langle x, x_0\rangle_s \geq 0$ for all $x \in K(e_1, \ldots, e_n)$ which implies that (S) has a nonzero solution in $K(e_1, \ldots, e_n)$.

Conversely, if we suppose that (S) has a nonzero solution $x_0$ in $K(e_1, \ldots, e_n)$, then for all $x := \sum_{i=1}^n \alpha^i e_i$, $\alpha^i \geq 0$ $\left(i = \overline{1,n}\right)$ we get

$$\langle x, x_0\rangle_s = \left\langle \sum_{i=1}^n \alpha^i e_i, x_0 \right\rangle_s = \sum_{i=1}^n \alpha^i \langle e_i, x_0\rangle_s \geq 0$$

and by Lemma 15, it follows that $K(e_1, \ldots, e_n)$ has the $(H)$ −property. ∎

Remark 45. *If $(X; (\cdot,\cdot))$ is a Hilbert space, then for all $(e_i)_{i=\overline{1,n}}$ a linearly independent family of vectors, the system (S) has a nonzero solution in $K(e_1, \ldots, e_n)$ (see [**2**]).*

The following results are valid in Hilbert spaces (see [**2**]).

Proposition 54. *Let $(e_i)_{i=\overline{1,n}}$ be a linearly independent family of vectors in $X$ and $G(e_1, \ldots, e_n)$ be the Gram's matrix associated to it. Then the system of linear inequations*

$$G(e_1, \ldots, e_n)\,\bar{x}^t \geq 0, \quad \bar{x} \in \mathbb{R}^n_+$$

*has nonzero solutions.*

Proposition 55. *If $(e_i)_{i=\overline{1,n}}$ is as above and $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $F(\bar{x}, \bar{y}) := \bar{x}G(e_1, \ldots, e_n)\,\bar{y}^t$, then there exists $\bar{y}_0 \geq 0$ in $\mathbb{R}^n$ and $\bar{y}_0 \neq 0$ such that*

$$F(\bar{x}, \bar{y}_0) \geq 0 \ \text{ for all } \ \bar{x} \geq 0.$$

The following result is in connection to well-known theorems of J. von Neuman which are important in Game Theory (see [**4**] or [**5**, p. 107]).

PROPOSITION 56. *Let* $A = \left(a_j^i\right)_{j=\overline{1,n}}^{i=\overline{1,m}}$ *be a matrix with real elements and* $rang\,(A) = m \leq n$. *Then there exists* $\bar{x}_0 \in \mathbb{R}_+^n$, *such that* $A\bar{x}_0^t \geq 0$ *in* $\mathbb{R}^n$.

Finally, we shall give another result in connection to Ville's theorem (see [**4**] or [**5**, p. 130]), which is also important in Game Theory.

PROPOSITION 57. *Let* $A$ *be a symmetric positive definite matrix and* $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $g\,(\bar{x}, \bar{y}) := \bar{x}A\bar{y}^t$. *Then there exists* $\bar{y}_0 \in \mathbb{R}_+^n$, $\bar{y}_0 \neq 0$ *such that:* $g\,(\bar{x}, \bar{y}_0) \geq 0$ *for all* $\bar{x} \geq 0$.

For the proof of these results see [**2**], where further details and consequences are given.

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CHAPTER 16

# Convex Functions in Linear Spaces

## 1. Introduction

In [**4**], the author introduced the following definition which generalizes the concepts of inner product, semi-inner product in the sense of Lumer-Giles [**6**], [**7**] (s.i.p.) and $R-$semi-inner product [**1**].

Definition 38. *Let $E$ be a linear space over the real or complex number field $K$. A mapping $(\cdot,\cdot)_S$ of $E\times E$ into $K$ will be called a subinner product on $E$ if the following conditions (P1 – P3) are satisfied:*

(P1) $(x,x)_S\neq 0$ *if* $x\neq 0$;
(P2) $(\lambda x,y)_S=\lambda (x,y)_S$ *and* $(x,\lambda y)_S=\bar{\lambda} (x,y)_S$ *for all* $\lambda\in K$ *and* $x,y$ *in* $E$;
(P3) $(x+y,z)_S=(x,z)_S+(y,z)_S$ *for all* $x,y,z$ *in* $E$.

In paper [**4**], the author also considered the following concept of orthogonality which generalizes the classical orthogonality in inner product spaces, the orthogonality in the sense of Giles [**6**] and the $R-$orthogonality which was considered in [**1**].

Definition 39. *Let $E$ be a linear space endowed with a subinner product. The element $x\in E$ is said to be orthogonal to $y\in E$ with respect to the subinner product or $S-$orthogonal, for short, if $(y,x)_S=0$. We denote this by $x\perp_{S\sigma} y$.*

The following properties of $S-$orthogonality are obvious from the above definition (see also [**4**]):

(i) $x\perp_{S\sigma} x$ implies $x=0$;
(ii) $x\perp_{S\sigma} y$, $x\perp_{S\sigma} z$ imply $x\perp_{S\sigma} (y+z)$;
(iii) $x\perp_{S\sigma} y$, $\lambda\in K$ imply $x\perp_{S\sigma} (\lambda y)$ and $(\lambda x)\perp_{S\sigma} y$.

Now, let $G$ be a nonempty subset of the linear space $E$. The set given by: $G^{\perp_{S\sigma}}:=\{y\in E|y\perp_{S\sigma} x \text{ for all } x\in G\}$ will be called the *orthogonal complement of $G$ in the sense of subinner product* $(\cdot,\cdot)_S$ or the *$S-$orthogonal complement of $G$*, for short.

The following properties of the $S-$orthogonal complement are obvious by the above definition:

(i) $0 \in G^{\perp_{S\sigma}}$;
(ii) $G \cap G^{\perp_{S\sigma}} = \{0\}$;
(iii) $\alpha G^{\perp_{S\sigma}} \subseteq G^{\perp_{S\sigma}}$ for all $\alpha \in K$.

We will now introduce another concept connected with the subinner product $(\cdot,\cdot)_S$.

Definition 40. *([5]) Let $(E;(\cdot,\cdot)_S)$ be a subinner product space. The element $x \in E$ will be called sub-$S-$orthogonal over the elementary $y \in E$ if*

$$(y,x)_S \leq 0. \tag{S}$$

We will denote this by $x\,S\,y$. It is clear that

(i) if $x \perp_{S\sigma} y$ then $x\,S\,y$.
(ii) $x\,S\,y$, $x\,S\,z$ imply $x\,S\,(y+z)$;
(iii) $0\,S\,x$ and $x\,S\,0$ for all $x \in E$;
(iv) $x\,S\,y$ implies $(\alpha\,x)\ S\ (\beta\,y)$ for all $\alpha,\beta \in \mathbb{R}$ with $\alpha,\beta \geq 0$ and $x,y \in C$.

As above, if $G$ is a nonempty subset in the linear space $E$, then by $G^S$ we will mean the *sub-$S-$orthogonal complement* of $G$ in $E$, i.e.,

$$G^S := \{y \in E | y\,S\,x \text{ for all } x \in G\}.$$

We have:

(i) $G^{\perp_{S\sigma}} \subseteq G^S$
(ii) $\alpha G^S \subset G^S$ for all $\alpha \geq 0$.

## 2. The Estimation of Convex Functions

Suppose that $(E;(\cdot,\cdot)_S)$ is a subinner product space and $F : E \to \mathbb{R}$ is a convex mapping, i.e., a mapping satisfying the condition:

$$F(tx+(1-t)y) \leq tF(x) + (1-t)F(y) \tag{C}$$

for all $t \in [0,1]$ and $x,y \in E$.

Define the set $F^{\leq}(r)$ for a real number $r \in \mathbb{R}$, i.e.,

$$F^{\leq}(r) := \{x \in E | F(x) \leq r\}.$$

It is known that the set $F^{\leq}(r)$ is a convex subset (or the empty set) in the linear space $E$.

The following theorem of estimation for the mapping $F$ in terms of subinner products, holds ([5]).

Theorem 110. *Let $F : E \to \mathbb{R}$ be a convex function on $E$, $r$ a real number such that $F^{\leq}(r) \neq \emptyset$ and $w \in E \backslash F^{\leq}(r)$ such that $(w)_S^2 := (w,w)_S > 0$. Then the following statements are equivalent:*

(i) $w \in \left(F^{\leq}(r)\right)^S$;

(ii) *One has the estimation:*

$$F(x) \geq r + \frac{(F(w) - r)}{(w)_S^2}(x, w)_S \quad \textit{for all} \;\; x \in F^{\leq}(r) \tag{16.1}$$

*or, equivalently, the estimation:*

$$F(x) \geq F(w) + \frac{(F(w) - r)}{(w)_S^2}(x - w, w)_S \quad \textit{for all} \;\; x \in F^{\leq}(r). \tag{16.2}$$

Proof. "(i) $\Longrightarrow$ (ii)". Let $x \in F^{\leq}(r)$, i.e., $r \geq F(x)$, and put $\beta := r - F(x) \geq 0$. Since $w \in F^{\leq}(r)$, we get that $\alpha := F(w) - r > 0$. By the convexity of $F$ we have:

$$\begin{aligned} F\left(\frac{\alpha x + \beta w}{\alpha + \beta}\right) &\leq \frac{\alpha F(x) + \beta F(w)}{\alpha + \beta} \\ &= \frac{(F(w) - r) F(x) - (r - F(x)) F(w)}{F(w) - F(x)} \\ &= \frac{r(F(w) - F(x))}{F(w) - F(x)} = r \end{aligned}$$

as $\alpha + \beta = F(w) - F(x) > 0$. Thus the element

$$u := \frac{\alpha x + \beta w}{\alpha + \beta}$$

belongs to $F^{\leq}(r)$. Now, as $w \in \left(F^{\leq}(r)\right)^S$, and $u \in F^{\leq}(r)$, we have the inequality $(u, w)_S \leq 0$, i.e.,

$$((F(w) - r)x + (r - F(x))w, w)_S \leq 0 \;\; \text{for all} \;\; x \in F^{\leq}(r). \tag{16.3}$$

A simple calculation shows that

$$\begin{aligned} &((F(w) - r)x + (r - F(x))w, w)_S \\ &\qquad = (F(w) - r)(x, w)_S + r(w)_S^2 - F(x)(w)_S^2. \end{aligned}$$

By the inequality (16.3), we get

$$F(x)(w)_S^2 \geq r(w)_S^2 + (F(w) - r)(x, w)_S, \;\; x \in F^{\leq}(r).$$

Since $(w)_S^2 > 0$, by the above inequality we deduce the desired estimation (16.1).

Now, a simple calculation shows that

$$\begin{aligned} r+\frac{(F(w)-r)}{(w)_S^2}(x,w)_S &= r+\frac{(F(w)-r)\left[(x-w,w)_S+(w)_S^2\right]}{(w)_S^2} \\ &= r+\frac{(F(w)-r)(x-w,w)_S}{(w)_S^2}+(F(w)-r) \\ &= F(w)+\frac{(F(w)-r)(x-w,w)_S}{(w)_S^2}, \end{aligned}$$

which proves the estimation (16.2).

"(ii) = (i)". Now, suppose that the estimation (16.1) holds. Thus, for all $x \in F^{\leq}(r)$ we have:

$$0 \geq F(x) - r + \frac{F(w)-r}{(w)_S^2}(x,w)_S .$$

Since $F(w) - r > 0$ because $w \notin F^{\leq}(r)$ and $(w)_S^2 > 0$, we get that

$$(x,w)_S \leq 0 \quad \text{for all} \quad x \in F^{\leq}(r) .$$

i.e., $w \in \left(F^{\leq}(r)\right)^S$ and the theorem is thus proved. ∎

The above theorem had a corollary for the sublinear functional defined on $E$.

Recall that the functional $P : E \to \mathbb{R}$ is said to be *sublinear* on $E$ if

(a) $P(x+y) \leq P(x) + P(y)$ for all $x, y \in E$;
(aa) $P(\alpha, x) = \alpha P(x)$ for all $x \in E$ and $\alpha \geq 0$.

Corollary 37. *([**5**]) Let $P : E \to \mathbb{R}$ be a sublinear functional on $E$ and $w \in E \backslash K(P)$ with $(w)_S^2 > 0$, where $K(P) := \{x \in E | P(x) \leq 0\}$ and $K(P) \neq \{0\}$. The following statements are equivalent:*

(i) $w \in (K(P))^S$;
(ii) *One has the estimation:*

$$P(x) \geq \frac{P(w)}{(w)_S^2}(x,w)_S \quad \textit{for all} \quad x \in K(P) .$$

Proof. Since the mapping $P$ is convex, then we can apply Theorem 110 for $F = P$ and $r = 0$. ∎

The case of linear functionals is embodied in the following proposition ( [**5**]).

Proposition 58. *Let $f : E \to \mathbb{R}$ be a linear functional on $E$ and $w \in E \backslash Ker(f)$. Then the following statements are equivalent:*

(i) $w \perp_S Ker(f)$;

(ii) *One has the representation:*

$$f(x) = \frac{f(w)}{(w)_S^2}(x,w)_S \quad \textit{for all} \quad x \in E. \tag{16.4}$$

PROOF. "(i) $\Longrightarrow$ (ii)". Let us assume that $w \perp_S Ker(f)$. For all $x \in E$ we have that $f(x)w - f(w)x \in Ker(f)$ as $f(f(x)w - f(w)x) = 0$. Thus we have $(f(x)w - f(w)x, w)_S = 0$ for all $x \in E$.

Since

$$(f(x)w - f(w)x, w)_S = f(x)(w)_S^2 - f(w)(x,w)_S \quad \text{for all} \quad x \in E$$

and $(w)_S^2 \neq 0$ because $w \neq 0$, we derive the desired representation (16.4).

"(ii) $\perp$ (i)". Since $w \notin Ker(f)$, we have that $f(w) \neq 0$. Thus, by the representation (16.4), we have

$$0 = \frac{f(w)}{(w)_S^2}(x,w)_S \quad \text{for all} \quad x \in Ker(f),$$

which gives that $w \perp Ker(f)$ and the proposition is proved. ∎

If we wish to obtain a lower bound for the convex mapping $F$ for all $x$ in $E$ we have to assume more on the element $w$ as in the above theorem.

THEOREM 111. *([**5**]) Let $F : E \to \mathbb{R}$ be a convex mapping and there be a $w \in E \backslash F^{\leq}(r)$ for a real number $r \in \mathbb{R}$ such that $(-w) \in F^{\leq}(r)$. If $w \in \left(F^{\leq}(r)\right)^S$, then we have the estimation:*

$$F(x) \geq \begin{cases} r + \dfrac{F(w) - r}{(w)_S^2}(x,w)_S & \textit{for all} \ \ x \in F^{\leq}(r), \\ \\ r + \dfrac{r - F(-w)}{(w)_S^2}(x,w)_S & \textit{for all} \ \ x \in E \backslash F^{\leq}(r). \end{cases} \tag{16.5}$$

PROOF. As $w \in \left(F^{\leq}(r)\right)^S$ and $(-w) \in F^{\leq}(r)$ then $(w,-w)_S < 0$, i.e., $(w)_S^2 > 0$. Now, by the implication (i) $\Longrightarrow$ (ii) of Theorem 110 we get the first of the estimation (16.5) for all $x \in F^{\leq}(r)$.

Now, let $x \in E \backslash F^{\leq}(r)$. Then $F(x) > r$ and thus $\alpha := F(x) - r > 0$. Since $(-w) \in F^{\leq}(r)$ we get that $\beta := r - F(-w) \geq 0$. Let us put:

$$u := \frac{\alpha(-w) + \beta x}{\alpha + \beta} = \frac{(F(x) - r)(-w) - (r - F(-w))x}{F(x) - F(-w)}.$$

By the convexity of $F$ we get that

$$F(u) \le \frac{(F(x) - r) F(-w) - (r - F(-w)) F(x)}{F(x) - F(-w)} = r,$$

i.e., $u \in F^{\le}(r)$. Since $w \in \left(F^{\le}(r)\right)^S$ we obtain that

$$((r - F(-w)) x + (F(x) - r)(-w), w)_S \le 0 \quad \text{for all } x \in E \backslash F^{\le}(r). \tag{16.6}$$

However,

$$((r - F(-w)) x - (F(x) - r) w, w)_S = (r - F(-w))(x, w)_S - (F(x) - r)(w)_S^2.$$

Then from (16.6) we get the second part of the inequality (16.5). ∎

The following corollary holds ([**5**]).

COROLLARY 38. *Let $P : E \to \mathbb{R}$ be a sublinear mapping on $E$ and $w \in E \backslash K(P)$, with $K(P) \neq \{0\}$ such that $(-w) \in K(P)$. If $w \in (K(P))^S$, then we have the estimation:*

$$P(x) \ge \begin{cases} \dfrac{P(w)}{(w)_S^2}(x, w)_S & \text{for all } x \in K(P), \\[2ex] \dfrac{-P(-w)}{(w)_S^2}(x, w)_S & \text{for all } x \in X \backslash K(P). \end{cases}$$

The proof is obvious by the above theorem and we shall omit the details.

By the above results we can also state the following consequence ([**5**]).

CONSEQUENCE 1. *Let $F : E \to \mathbb{R}$ be a convex mapping on $E$, $x_0, w \in E$ such that $F(w) > F(x_0)$ and $(w)_S^2 > 0$. Then the following statements are equivalent:*

(i) $w \in (L(F, x_0))^S$;
(ii) *The following estimation holds:*

$$F(x) \ge F(x_0) + \frac{F(w) - F(x_0)}{(w)_S^2}(x, w)_S \quad \text{for all } x \in L(F, x_0),$$

*where* $L(F; x_0) := \{x \in E | F(x) \le F(x_0)\}$.

*Now, let $F$ be a convex mapping on $E$ and $x_0, w \in E$ such that $F(w) > F(x_0) > F(-w)$.*

*If* $w \in (L(F,x_0))^S$ , *then we have the estimation:*

$$F(x) \geq \begin{cases} F(x_0) + \dfrac{F(w) - F(x_0)}{(w)_S^2}(x,w)_S & \text{for all } x \in L(F,x_0), \\ \\ F(x_0) + \dfrac{F(x_0) - F(-w_0)}{(w)_S^2}(x,w)_S & \text{for all } x \in E \backslash L(F,x_0). \end{cases}$$

The proofs are obvious by Theorems 110 and 111 by choosing $r = F(x_0)$. We shall omit the details.

In what follows, we will apply the above results in the case of smooth normed spaces which obviously contains the case of inner product spaces.

## 3. Applications to Real Normed Linear Spaces

We give the following definition ([**5**]).

Definition 41. *Let* $E$ *be a real normed space and* $[\cdot,\cdot]$ *a s.i.p. which generates its norm. The element* $x \in E$ *is said to be* $G-$*suborthogonal over the element* $y \in E$ *(relative to the s.i.p.* $[\cdot,\cdot]$*) if* $[y,x] \leq 0$. *We denote by* $A^{S(G)} = \{y \in E : [x,y] \leq 0$ *for all* $x \in A\}$ , *where* $A \subseteq E$.

By the use of the results established in the previous section, we can state the following lemmas and corollaries.

Theorem 112. *(*[**5**]*) Let* $F : E \to \mathbb{R}$ *be a convex mapping on the real normed space* $E[\cdot,\cdot]$ *a s.i.p. which generates the norm of* $E$, $r$ *a real number such that* $F^{\leq}(r) \neq \emptyset$ *and* $w \in E \backslash F^{\leq}(r)$ *with* $w \neq 0$. *Then the following statements are equivalent:*

(i) $w \in \left(F^{\leq}(r)\right)^{S(G)}$;
(ii) *One has the estimation*

$$F(x) \geq r + \frac{F(w) - r}{\|w\|^2}[x,w] \quad \text{for all } x \in F^{\leq}(r). \tag{16.7}$$

The case of sublinear functionals is embodied in the following corollary ( [**5**]).

Corollary 39. *Suppose* $E$, $[\cdot,\cdot]$ *are as above and* $p : E \to \mathbb{R}$ *is a sublinear functional on* $E$. *If* $w \in E \backslash K(p)$ *and* $K(p) \neq \{0\}$ , *then the following statements are equivalent:*

(i) $w \in (K(p))^{S(G)}$;
(ii) *One has the estimation:*

$$p(x) \geq p\left(\frac{w}{\|w\|}\right)\left[x, \frac{w}{\|w\|}\right] \quad \text{for all } x \in K(p).$$

If we want to obtain as estimation for all elements $x$ in $E$ we have to assume more about the element $w$.

THEOREM 113. *([**5**]) Let* $E, [\cdot,\cdot]$ *be as above and* $w \in E \backslash F^{\leq}(r)$ *such that* $(-w) \in F^{\leq}(r)$ *. If* $w \in \left(F^{\leq}(r)\right)^{S(G)}$*, then we have the estimation:*

$$F(x) \geq \begin{cases} r + \dfrac{F(w) - r}{\|w\|}\left[x, \dfrac{w}{\|w\|}\right] & \text{for all } x \in F^{\leq}(r), \\[2ex] r + \dfrac{r - F(-w)}{\|w\|}\left[x, \dfrac{w}{\|w\|}\right] & \text{for all } x \in E \backslash F^{\leq}(r). \end{cases}$$

Finally, the following corollary also holds ([**5**]).

COROLLARY 40. *Let* $p : E \to \mathbb{R}$ *be a sublinear mapping on* $E$, $w \in E \backslash K(p)$ $(K(p) \neq \{0\})$ *such that* $(-w) \in K(p)$ *. If* $w \in (K(p))^{S(G)}$, *then we have the estimation:*

$$p(x) \geq \begin{cases} p\left(\dfrac{w}{\|w\|}\right)\left[x, \dfrac{w}{\|w\|}\right] & \text{for all } x \in K(p), \\[2ex] -p\left(\dfrac{-w}{\|w\|}\right)\left[x, \dfrac{w}{\|w\|}\right] & \text{for all } x \in E \backslash K(p). \end{cases}$$

## 4. Applications in Hilbert Spaces

The following theorem of estimation holds ([**5**]).

THEOREM 114. *Let* $(H; \langle\cdot,\cdot\rangle)$ *be a real Hilbert space,* $F : H \to \mathbb{R}$ *a continuous convex mapping on* $H, r \in \mathbb{R}$ *such that* $0 \notin F^{\leq}(r)$ *. Then there exists an element* $w \in H$ *such that* $w \notin F^{\leq}(r)$, $-w \in F^{\leq}(r)$ *and the following estimation holds:*

(16.8)

$$F(x) \geq \begin{cases} r + \dfrac{F(w) - r}{\|w\|}\left\langle x, \dfrac{w}{\|w\|}\right\rangle & \text{for all } x \in F^{\leq}(r), \\[2ex] r + \dfrac{r - F(-w)}{\|w\|}\left\langle x, \dfrac{w}{\|w\|}\right\rangle & \text{for all } x \in H \backslash F^{\leq}(r). \end{cases}$$

PROOF. As $F$ is a continuous function, $F^{\leq}(r)$ is a closed convex set in $H$. Since $0 \notin F^{\leq}(r)$, there exists a unique element $g_0 \in F^{\leq}(r)$ such that $d\left(0, F^{\leq}(r)\right) = d(0, g_0)$, i.e.,

$$\|g_0\| = \inf_{g \in F^{\leq}(r)} \{\|g\|\}.$$

On the other hand, because $F^{\leq}(r)$ is convex, we have that

$$\|g_0\| \leq \|(1-t) g_0 + tg\| \quad \text{for all } g \in F^{\leq}(r) \text{ and } t \in [0,1]$$

which gives us

$$\|g_0\|^2 \le \|g_0 + t(g - g_0)\|^2 = \|g_0\|^2 + 2\langle g_0, g - g_0\rangle t + t^2 \|g - g_0\|^2$$

for all $g \in F^{\le}(r)$ and $t \in [0,1]$, which implies that

$$t\|g - g_0\|^2 + 2\langle g_0, g - g_0\rangle \ge 0 \text{ for all } t \in [0,1].$$

Letting $t \to 0$, $t > 0$, we get $\langle g_0, g - g_0\rangle \ge 0$, i.e., $\langle g_0, g\rangle \ge \|g_0\|^2$ and thus $\langle -g_0, g\rangle \le 0$ for all $g \in F^{\le}(r)$, i.e., the element $w := -g_0$ satisfies the conditions

$$-w \in F^{\le}(r) \quad \text{and} \quad w \in \left(F^{\le}(r)\right)^S.$$

If we assume that $w \in F^{\le}(r)$, then $\langle w, w\rangle = \|w\|^2 \le 0$ which implies that $w = 0$, i.e., $y_0 = 0$, which produces a contradiction. Thus $w \notin F^{\le}(r)$.

Applying Theorem 111 for $w$ as above, we get the equation (16.8). The theorem is thus proved. ∎

Corollary 41. *([**5**]) Let $H, F$ be as above and $x_0 \in H$ such that $F(0) > F(x_0)$. Then there exists $w \in H$ such that $F(w) > F(x_0) \ge F(-w)$ and the following estimation*

(16.9)

$$F(x) \ge \begin{cases} F(x_0) + \dfrac{F(w) - F(x_0)}{\|w\|}\left\langle x, \dfrac{w}{\|w\|}\right\rangle & \text{for all } x \in L(F, x_0), \\ F(x_0) + \dfrac{F(x_0) - F(-w)}{\|w\|}\left\langle x, \dfrac{w}{\|w\|}\right\rangle & \text{for all } x \in H \backslash L(F, x_0). \end{cases}$$

*holds.*

The case of sublinear functionals is embodied in the following.

Theorem 115. *([**5**]) Let $p : H \to \mathbb{R}$ be a continuous sublinear functional such that $K(p)$ is not a linear subspace. Then there exists an element $u \in H$, $\|u\| = 1$ such that*

$$p(x) \ge \begin{cases} p(u)\langle x, u\rangle & \text{for all } x \in K(p), \\ -p(-u)\langle x, u\rangle & \text{for all } x \in H \backslash K(p). \end{cases} \tag{16.10}$$

Proof. As $K(p)$ is not a linear subspace there exists an element $x_0 \in K(p)$ such that $-x_0 \notin K(p)$.

Indeed, if we assume that for all $x \in K(p)$ we have that $-x \in K(p)$ we would deduce that $K(p) = -K(p)$, i.e., $K(p)$ is a linear subspace of $H$, which produces a contradiction. Put $y_0 := -x_0$ with $x_0$ as above. Since $K(p)$ is a closed subset of $H$ and $H$ is a Hilbert space, there exists a unique element $g_0 \in K(p)$ such that $d(y_0, K(p)) = d(y_0, g_0)$, i.e.,

$$\|y_0 - g_0\| \leq \|y_0 - g\| \quad \text{for all} \quad g \in K(p). \tag{16.11}$$

Since $g_0 \in K(p)$, then for all $\alpha \geq 0$ we have that $g_0 + \alpha g_1 \in K(p)$ for all $g_1 \in K(p)$. Thus, by the inequality (16.11) we get that

$$\|y_0 - g_0\| \leq \|y_0 - g_0 - \alpha g_1\|^2 \quad \text{for all} \quad \alpha \geq 0 \text{ and } g_1 \in K(p)$$

and thus

$$\|y_0 - g_0\|^2 \leq \|y_0 - g_0 - \alpha g_1\|^2 = \|y_0 - g_0\| - 2\alpha \langle y_0 - g_0, g_1 \rangle + \alpha^2 \|g_1\|^2$$

from which we get

$$2\alpha \langle y_0 - g_0, g_1 \rangle \leq \alpha^2 \|g_1\|^2 \quad \text{for all} \quad \alpha \geq 0 \text{ and } g_1 \in K(p)$$

which implies that

$$2 \langle y_0 - g_0, g_1 \rangle \leq 2 \|g_1\| \quad \text{for all} \quad \alpha > 0 \text{ and } g_1 \in K(p).$$

Letting $\alpha \to 0$, $\alpha > 0$ we deduce that $\langle y_0 - g_0, g_1 \rangle \leq 0$, i.e., $y_0 - g_0 \in K(p)^S$. Denote by $w := y_0 - g_0$. Thus $w \in (K(p))^S$ and $-w = g_0 - y_0 = g_0 + x_0 \in K(p)$ as $g_0$ and $x_0 \in K(p)$. Now, if we apply Corollary 38 for the element $w$ in the Hilbert space $H$, we derive

$$p(x) \geq \begin{cases} p\left(\frac{w}{\|w\|}\right) \left\langle x, \frac{w}{\|w\|} \right\rangle & \text{for all} \quad x \in K(p), \\ \\ -p\left(\frac{-w}{\|w\|}\right) \left\langle x, \frac{w}{\|w\|} \right\rangle & \text{for all} \quad x \in H \backslash K(p). \end{cases}$$

Choosing $u := \frac{w}{\|w\|}$, we get the equation (16.10). The proof is completed. ∎

Remark 46. *The above theorem was first proved in the paper* [**3**] *using a different argument.*

Remark 47. *If $p = f \neq 0$ is a continuous linear functional, then by (16.10) we get that $f(x) = f(u) \langle x, u \rangle$ for all $x \in H$, i.e., the well known Riesz's representation theorem in Hilbert spaces.*

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CHAPTER 17

# Representation of Linear Forms

## 1. Introduction

Let $I$ be a unitary associative ring and $M$ a left module over $I$. The following concept is a natural generalization of inner product, semi-inner product in the sense of Lumer-Giles [**3**] and Tapia [**4**], or $R-$semi-inner product introduced in [**1**].

Definition 42. *A mapping $(\cdot,\cdot)_S : M\times M\to I$ is called a semi-subinner product on $M$, if the following conditions hold:*

$$(x+y,z)_S=(x,z)_S+(y,z)_S\,,\ \ x,y,z\in M; \tag{S1}$$

$$(\alpha x,y)_S=\alpha\,(x,y)_S\,,\ \alpha\in I,\ x,y\in M. \tag{S2}$$

*In addition, if the relation (S3) is valid too:*

$$(x,x)_S\neq 0 \ \textit{if}\ x\neq 0; \tag{S3}$$

*then $(\cdot,\cdot)_S$ is called a subinner product on $M$.*

We remark that the above definition can be reformulated for a right or bilateral module over a unitary associative ring. We omit the details.

In what follows, by an involution on $I$, we understand a mapping $I\ni r\overset{*}{\longmapsto} r^*\in I$ satisfying the conditions:

$$(r+t)^*=r^*+t^*,\ \ r,t\in I; \tag{I1}$$

$$(rt)^*=t^*r^*,\ \ r,t\in I; \tag{I2}$$

$$(r^*)^*=r,\ \ r\in I; \tag{I3}$$

$$1^*=1,\ \ \text{where 1 is the unit of } I. \tag{I4}$$

Definition 43. *([**2**]) A semi-subinner product or a subinner product on $M$ is said to be $*-$ homogeneous on $I$, if the following condition is valid*

$$(x,\alpha y)_S = \alpha^* (x,y)_S\,,\quad \alpha \in I,\ x,y \in M. \tag{S4}$$

If $I$ is a commutative and unitary ring and $* : I \to I$, $r^* = r$ for all $r \in I$, then $(\cdot,\cdot)_S$ is said to be homogeneous on $I$.

Further, we shall give some examples of semi-subinner products or subinner products on left $I-$modules.

## 2. Examples of Semi-Subinner Products

Let $I$ be a unitary associative ring, $* : I \to I$ an involution on $I$ and $n$ a natural number, $n \geq 1$. Then $I^n$ endowed with the usual operations is a left $I-$module. Then the mapping is given by:

$$(\cdot,\cdot)_n : I^n \times I^n \to I,\ \ (x,y)_n := \sum_{i=1}^{n} x_i y_i, \tag{17.1}$$

respectively

$$(\cdot,\cdot)_n^* : I^n \times I^n \to I,\ \ (x,y)_n^* := \sum_{i=1}^{n} x_i y_i^*, \tag{17.2}$$

where $x = (x_1,\ldots,x_n)\,,\ y = (y_1,\ldots,y_n) \in I^n$, are semi-subinner products on $I^n$.

If we suppose that the ring $I$ is commutative, then $(\cdot,\cdot)_n$ is homogeneous and $(\cdot,\cdot)_n^*$ is $*-$homogeneous on $I$.

Let $M$ be a free $I-$module of the finite type and $E = \{e_i\}_{i=\overline{1,n}}$ a base in $M$. If the elements $x,y \in M$ are given by:

$$x = \sum_{i=1}^{n} \alpha_i \mathbf{e}_i,\ \ y = \sum_{i=1}^{n} \beta_i \mathbf{e}_i;\ \ \alpha_i,\beta_i \in I,\qquad i = \overline{1,n}, \tag{17.3}$$

then we can define the mappings:

$$M \times M \ni (x,y) \to (x,y)_E := \sum_{i=1}^{n} \alpha_i \beta_i \in I \tag{17.4}$$

and

$$M \times M \ni (x,y) \to (x,y)_E^* := \sum_{i=1}^{n} \alpha_i \beta_i^* \in I. \tag{17.5}$$

It is clear that the mappings $(\cdot,\cdot)_E$ and $(\cdot,\cdot)_E^*$ are semi-subinner products on $M$ and if $I$ is commutative, then $(\cdot,\cdot)_E$ is homogeneous on $I$ and $(\cdot,\cdot)_E^*$ is $*-$homogeneous on $I$.

Let $M$ be a left $I-$module, $\psi$ a nonzero linear form on $M$ and $\chi : M \to M$. Then the mappings:

$$(\cdot,\cdot)_{\psi\chi} : M \times M \to I, \ \ (x,y)_{\psi\chi} := \psi(x)\,\chi(y), \ x,y \in M; \tag{17.6}$$

and

$$(\cdot,\cdot)^*_{\psi\chi} : M \times M \to I, \ \ (x,y)^*_{\psi\chi} := \psi(x)\,\chi^*(y), \ x,y \in M; \tag{17.7}$$

are semi-subinner products in $M$.

If $I$ is commutative and $\chi$ is homogeneous on $M$, i.e., $\chi(\alpha x) = \alpha\chi(x)$, $\alpha \in I$, $x \in M$, then $(\cdot,\cdot)_{\psi\chi}$ will be a homogeneous semi-subinner product on $M$ and $(\cdot,\cdot)^*_{\psi\chi}$ will be $*-$homogeneous on $M$.

Let us consider a semi-subinner product on $M$, $(\cdot,\cdot)_S : M\times M \to I$, $L : M \to M$ a linear transformation of $M$ into $M$ and $\chi : M \to M$. Then

$$(\cdot,\cdot)_L : M \times M \to I, \ \ (x,y)_L := (L(x), \chi(y))_S, \ \ x,y \in M, \tag{17.8}$$

is a semi-subinner product on $M$. If we assume that $(\cdot,\cdot)_S$ is $*-$homogeneous on $I$ and $\chi$ is a homogeneous mapping on $M$, then $(\cdot,\cdot)_L$ is $*-$homogeneous as well.

Let $I$ be an (unitary) integrity ring. Then $I$ is a left $I-$module and the mapping given by

$$(\cdot,\cdot)_I : I \times I \to I, \ \ (x,y)_I := xy, \tag{17.9}$$

is a subinner product on $I$.

In addition, if we suppose that $I$ is commutative, then $(\cdot,\cdot)_1$ is a homogeneous subinner product on $I$.

Every inner product or semi-inner product in the sense of Lumer, are homogeneous or antihomogeneous subinner products on real or complex linear spaces.

The following section of the present chapter is devoted to the study of some theorems of representation for the linear forms defined on left $I-$modules in terms of semi-subinner products.

## 3. Representation of Linear Forms

In this section, we point out the following concept which generalizes the orthogonality in the sense of Lumer-Giles or $R-$orthogonality introduced in [**1**].

Definition 44. *([**2**]) Let $M$ be a left $I-$module and $(\cdot,\cdot)_S : M \times M \to I$ a semi-subinner product on $M$. The element $x \in M$ is said to be orthogonal over $y \in M$ in the sense of semi-subinner product or, for short, $S-$orthogonal over $y$, iff $(y,x)_S = 0$. We note that $x \perp_{S\sigma} y$.*

The following properties of $S-$orthogonality are evident by the above definition:

(i) $x \perp_{S\sigma} y$, $x \perp_{S\sigma} z \Longrightarrow x \perp_{S\sigma} (y+z)$;
(ii) $x \perp_{S\sigma} y$, $\alpha \in I \Longrightarrow x \perp_{S\sigma} \alpha y$;
and if $(\cdot,\cdot)_S$ is $*-$homogeneous, then
(iii) $x \perp_{S\sigma} y$, $\alpha \in I \Longrightarrow \alpha x \perp_{S\sigma} y$.

If $E$ is a non-empty set, then

$$E^{\perp_{S\sigma}} := \{y \in M | y \perp_{S\sigma} x,\ x \in E\},$$

is called the orthogonal complement of $E$ in the semi-subinner product sense or, for short, $S-$orthogonal complement of $E$.

If $(\cdot,\cdot)_S$ is homogeneous, then $0 \in E^{\perp_{S\sigma}}$ and if $(\cdot,\cdot)_S$ is a subinner product on $M$, then $E \cap E^{\perp_{S\sigma}} \subseteq \{0\}$.

Now, we can give the first result of representation for the linear form on a left $I-$module endowed with a semi-subinner product [**2**].

THEOREM 116. *Let $M$ be a left $I-$module, $(\cdot,\cdot)_S : M \times M \to I$ a semi-subinner product on $M$, $f \in M^*$ a nonzero linear form on $M$ and $w \in M \setminus \{0\}$.*

*If the following conditions hold:*

(i) $f(x) f(w) = f(w) f(x),\ x \in M$;
(ii) $(w,w)_S$ *is invertible in* $I$;
(iii) $w \in Ker(f)^{\perp_{S\sigma}}$;

*then we have the representation:*

$$f(x) = f(w)(x,w)_S (w,w)_S^{-1} \tag{17.10}$$

*for all $x \in M$.*

PROOF. Let $x \in M$. Then we have

$$f(f(x)w - f(w)x) = f(x)f(w) - f(w)f(x),$$

and by condition (i), one obtains

$$f(x)w - f(w)x \in Ker(f).$$

On the other hand, since $w \perp_S Ker(f)$, we have

$$(f(x)w - f(w)x, w)_S = 0,\ \ x \in M,$$

and by linearity of $(\cdot,\cdot)_S$ it results that:

$$f(x)(w,w)_S = f(w)(x,w)_S,\ \ x \in M.$$

Since $(w,w)_S$ is invertible on $I$, by multiplying with $(w,w)_S^{-1}$, we deduce (17.10).

The theorem is thus proved. ∎

REMARK 48. *If the scalar ring $I$ is commutative, then condition (i) is satisfied and relation (ii) and (iii) implies representation (17.10).*

COROLLARY 42. *([**2**]) Let $M$ be a left $I-$module on a commutative ring, $(\cdot,\cdot)_S$ a $*-$homogeneous semi-subinner product on $M$, $f \in M^*$ a nonzero linear form and $w \in M\setminus\{0\}$. If conditions (ii) and (iii) of the above theorem are satisfied, then there exists $u_f(w) \in M$ such that:*

$$f(x) = (x, u_f(w))_S\,, \quad x \in M. \tag{17.11}$$

*In addition, the representation element $u_f(w)$ is given by*

$$u_f(w) = \left(f(w)(w,w)_S^{-1}\right)^* w. \tag{17.12}$$

PROOF. By the above theorem, we have $f(x) = f(w)(w,w)_S^{-1}(x,w)_S$, $x \in M$. Putting $u_f(w) := \left(f(w)(w,w)_S^{-1}\right)^* w$ and since $(\cdot,\cdot)_S$ is $*-$homogeneous, then representation (17.11) holds. ∎

The following theorem gives a sufficient and necessary condition of representation for the linear forms defined on $I-$modules.

THEOREM 117. *([**2**]) Let $M$ be a left module on integrity ring $I$, $(\cdot,\cdot)_S$ a semi-subinner product on $M$, $f \in M^*$ a nonzero linear form, $w \in M\setminus\{0\}$.*

*If the following conditions hold:*

(i) $f(x)f(w) = f(w)f(x)$, $x \in M$;
(ii) $(w,w)_S$ *is invertible in* $I$;

*then the following sentences are equivalent:*

(iii) $w \in Ker(f)^{\perp_{S_\sigma}}$;
(iv) $f(x) = f(w)(x,w)_S(w,w)_S^{-1}$ *for all* $x \in M$.

PROOF. The implication "(iii) $\Longrightarrow$ (iv)" follows by Theorem 116. "(iv) $\Longrightarrow$ (iii)". By relation (iv) we have

$$0 = f(x) = f(w)(x,w)_S(w,w)_S^{-1}\,, \quad x \in Ker(f)\,. \tag{17.13}$$

Firstly, we remark that $f(w) \neq 0$, since if we suppose that $f(w) = 0$, we have $f(x) = 0$ for all $x \in M$, which produces a contradiction.

By multiplying with $(w,w)_S \neq 0$, we obtain from (17.13)

$$f(w)(x,w)_S = 0 \quad \text{for all} \quad x \in Ker(f)\,.$$

Since $f(w) \neq 0$ and $I$ is an integrity ring, we deduce $(x,w)_S = 0$ for all $x \in Ker(f)$ which implies $w \in Ker(f)^{\perp_{S_\sigma}}$.

The theorem is thus proved. ∎

REMARK 49. *If the ring $I$ is commutative and condition (ii) holds, then relations (iii) and (iv) are equivalent.*

COROLLARY 43. *([**2**]) Let $M$ be a left $I-$module over an integrity and commutative ring, $(\cdot,\cdot)_S$ a $*-$homogeneous semi-subinner product on $M$, $f \in M^*\setminus\{0\}$ and $w \in M\setminus\{0\}$. If condition (ii) holds, then the following assertions are equivalent:*

(iii) $w \in Ker(f)^{\perp_{S_\sigma}}$;

(iv) $f(x) = (x, u_f(w))_S$ *for all* $x \in M$

*where* $u_f(w)$ *is given by*

$$u_f(w) := \left(f(w)(w,w)_S^{-1}\right)^* w. \tag{17.14}$$

The proof follows by Remark 49 and Corollary 42. We omit the details.

## 4. Applications

Let $I$ be an associative unitary ring, $f \in End(I)$, $f \neq 0$, and $w \in I\setminus\{0\}$. If the following conditions hold:

(i) $f(x)f(w) = f(w)f(x)$, $x \in I$;

(ii) $w^2$ is invertible in $I$;

(iii) $xw = 0$ for all $x \in Ker(f)$;

then we have the representation

$$f(x) = f(w)\,xw\left(w^2\right)^{-1}, \quad x \in I. \tag{17.15}$$

If $I$ is commutative, then relations (ii) and (ii) imply

$$f(x) = f(w)\,w\left(w^2\right)^{-1}x, \quad x \in I. \tag{17.16}$$

The proof follows by Theorem 116 for the $I-$module $I$ endowed with semi-subinner product given by:

$$(\cdot,\cdot) : I \times I \to I, \quad (x,y)_S := xy.$$

If $I$ is commutative and $* : I \to I$ is an involution on $I$ and the following conditions hold:

(ii) $ww^*$ is invertible in $I$;

(iii) $xw^* = 0$ for all $x \in Ker(f)$;

then there exists $u_f(w) \in I$ such that:

$$f(x) = xu_f(w)^*, \quad x \in I \tag{17.17}$$

and $u_f(w)$ is given by

$$u_f(w) := \left(f(w)(ww^*)^{-1}\right) w^*. \tag{17.18}$$

The proof follows by Corollary 42. We omit the details.

Let $I$ be an associative unitary ring $\varphi, f \in End(I)\setminus\{0\}$, and $w \in I\setminus\{0\}$. If the following conditions hold:

(i) $f(x) f(w) = f(w) f(x)$, $x \in I$;
(ii) $\varphi(w) w$ is invertible in $I$;
(iii) $\varphi(x) = 0$, $x \in Ker(f)$;

then we have the representation

$$f(x) = f(w) \varphi(x) w (\varphi(w) w)^{-1} \quad \text{for all } x \in I. \tag{17.19}$$

If the ring is commutative, then relation (ii) and (iii) imply:

$$f(x) = f(w) w (\varphi(w) w)^{-1} \varphi(x) \quad \text{for all } x \in I. \tag{17.20}$$

The proof follows by Theorem 116 for the $I-$module $I$ endowed with semi-subinner product $(\cdot,\cdot)_{\varphi} : I \times I \to I$, $(x,y)_{\varphi} = \varphi(x) y$.

If $I$ is commutative and $* : I \to I$ is an involution on $I$ and the following conditions hold:

(ii) $\varphi(w) w^*$ is invertible in $I$;
(iii) $\varphi(x) w^* = 0$, $x \in Ker(f)$;

then we have the representation

$$f(x) = \varphi(x) u_f^*(w) \quad \text{for all } \ x \in I, \tag{17.21}$$

where $u_f(w)$ is given by

$$u_f(w) := f(w) \left(\varphi(w) w^{-1}\right)^* w^*.$$

The proof follows by Corollary 42. We omit the details.

Finally, if we suppose that $I$ is an integrity ring and the following conditions:

(i) $f(x) f(w) = f(w) f(x)$, $x \in I$;
(ii) $\varphi(w) w$ is invertible in $I$;

are true, then the following assertions are equivalent:

(iii) $\varphi(x) = 0$, $x \in Ker(f)$;
(iv) $f(x) = f(w) \varphi(x) w (\varphi(w) w)^{-1}$.

Let $I$ be an associative unitary ring, $I^n$ $(n \geq 1)$, the left $I-$module and $w \in I^n$, $w \neq 0$. If $f \in (I^n)^*$ is a nonzero linear form and the following conditions hold:

(i) $f(x) f(w) = f(w) f(x)$, $x \in I^n$;
(ii) $\sum_{i=1}^n w_i^2$ is invertible in $I$, where $w = (w_1, \dots, w_n)$;
(iii) $\sum_{i=1}^n x_i w_i = 0$, $x = (x_1, \dots, x_n) \in Ker(f)$;

then we have the representation:

$$f(x) = f(w) \left(\sum_{i=1}^n x_i w_i\right) \left(\sum_{i=1}^n w_i^2\right)^{-1}, \ x \in I^n. \tag{17.22}$$

If the ring is commutative, then condition (i) is fulfilled and (ii) and (iii) imply the existence of an element $u_f(w) = \left(u_f^1(w), \ldots, u_f^2(w)\right) \in I^n$ with the property:

$$f(x) = \sum_{i=1}^{n} x_i u_j^i(w), \quad x = (x_1, \ldots, x_n) \in I^n, \tag{17.23}$$

and, in addition, $u_j^i(w)$ $\left(i = \overline{1,n}\right)$ are given by

$$u_j^i(w) = f(w) \left(\sum_{i=1}^{n} w_i^2\right)^{-1} w_i \quad \left(i = \overline{1,n}\right). \tag{17.24}$$

The proof follows by Theorem 116 and by Corollary 42 for the semi-subinner product $(\cdot,\cdot)_n : I^n \times I^n \to I$, $(x,y)_n := \sum_{i=1}^{n} x_i w_i$.

Now, if we suppose that $I$ is an integrity ring and conditions (i) and (ii) are satisfied, then the following sentences are equivalent:

(iii) $\sum_{i=1}^{n} x_i w_i = 0$, $x = (x_1, \ldots, x_n) \in Ker(f)$;

(iv) $f(x) = f(w) \left(\sum_{i=1}^{n} x_i w_i\right) \left(\sum_{i=1}^{n} w_i^2\right)^{-1}$, $x \in I^n$.

In addition, if we suppose that $I$ is commutative and condition (ii) is verified, then (iii) is equivalent with

$$f(x) = \sum_{i=1}^{n} x_i u_j^i(w), \quad x = (x_1, \ldots, x_n) \in I^n \tag{17.25}$$

and $u_j^i(w)$ $\left(i = \overline{1,n}\right)$, are as given by (17.24).

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