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BACHELOR'S DEGREE THESIS

Inequalities and Applications

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According to the famous romanian mathematician Tiberiu Popoviciu: "Algebra is a science of the equalities while Analysis is one of the inequalities." Prof. Cobzaş is also the person who made me understand that in the social domain the situation is quite similar: "Considering that all the attempts of building a society based on equality have failed lamentably, we have to console ourselves with the idea of a society based on inequalities. We do not know which is the opinion of the algebraists concerning this problem. Do they still hope in a society based on equality?"(see in the book [32]).

Under the above circumstances, we can just try to solve the inequalities that we are confronting with, hoping that they lead us to the results we desire. We wish that this work will be of some help in this sense.

Contents

Introduction

The inequalities appear in most of the domains of Mathematics, but it was just in 30's when the first monograph dedicated to them was published. This first book called "Inequalities" written by Hardy, Littlewood and Polya at Cambridge University Press in 1934 represents the first effort to systemize a rapidly expanding domain. There are several editions of this book, that made it to be actual even today. During the time, the growing interest for the inequalities led to the apparition of several books in the area. In this sense, we mention [34],[43], [41] or [30].

The aim of this work is to present some inequalities along with applications and recent results related to them, as follows:

- In the first chapter we expose some of the most commonly used inequalities. These inequalities often appear especially in analysis so they worth to be known.
- In the second chapter we present shortly some of the means of two variables and some recent results related to their inequalities.
- The last part of the present work contains some personal results related to inequalities. Some of the results have been published in different Mathematical Journals.

We use Halmos' "tombstone" symbol $\Box$ for the end of the proof and "iff" for "if and only if". When the proof is only sketched, we end by $\clubsuit$.

Chapter 1

Remarkable Inequalities

In this chapter we shall present some classical inequalities. Most of the results given here are contained in [34], [30], [33],[32] or [36] and represent basic tools for anybody who has to deal with some problems related to inequalities.

1.1 The $AM - GM - HM$ inequalities

Some of the most common inequalities that often appear in the first chapters of any book on inequalities are the inequalities between the Harmonic, Geometric and Arithmetic means.

1.1.1 Two variable version

In the case of two numbers, these means are known since the antiquity. A surviving fragment of the work of Archytas of Tarentum (ca. 350 BC) states, "There are three means in music: one is the arithmetic, the second is the geometric, and the third is the subcontrary, which they call harmonic." The term "harmonic mean" was also used by Aristotle. The above mentioned means also have geometrical interpretations, as it will be shown in Chapter 2. For more information about the Greek Means, see [57] or [35].
An inequality between these means states that for any two positive numbers a, b the following inequality holds:

$$\frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \sqrt{ab} \leq \frac{a+b}{2}. \tag{1.1.1}$$

Two geometric proofs of this inequality will be given in Chapter 2.

1.1.2 Extended version for n variables

An extended version of the previous inequality states that, considering the positive numbers $a_1, \ldots, a_n$, the following inequality holds:

$$\frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}. \tag{1.1.2}$$

There are several proofs of this inequality, as can be seen in [33]. These proofs are using diverse techniques related to the mathematical induction or to other different tricks.

1.1.3 Pondered version

The pondered version of the $AM - GM - HM$ inequality states that

$$\frac{1}{\frac{p_1}{a_1} + \frac{p_2}{a_2} + \cdots + \frac{p_n}{a_n}} \leq a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} \leq p_1 a_1 + p_2 a_2 + \cdots + p_n a_n \tag{1.1.3}$$

where $p_1 + p_2 + \cdots + p_n = 1$, $p_1, p_2, \ldots, p_n$, $a_1, a_2, \ldots, a_n > 0$, $n \geq 2$.
As it will be shown in the next chapter, the proof of this inequality can be done in a few rows, by the application of the Jensen Inequality (that will be presented in Chapter 1.5) for an appropriate convex function.

1.2 The power means

The power means are direct generalizations of the means given in the previous section.

1.2.1 Two variable version

For two positive numbers a, b the power mean of order s of a and b is defined by:

$$M_s(a,b) := \begin{cases} \left(\dfrac{a^s + b^s}{2} \right)^{\frac{1}{s}} & \text{if } s \neq 0 \\ (ab)^{\frac{1}{2}} & \text{if } s = 0. \end{cases} \tag{1.2.1}$$

1.2.2 Extended version for n variables

The power mean can be defined not only for two numbers, but for any finite set of nonnegative real numbers. Given $a_1, \ldots, a_n \in [0, \infty[$ and $s \in \mathbb{R}$, the power mean $M_s(a_1, \ldots, a_n)$ of $a_1, \ldots, a_n$ is defined by

$$M_s(a_1, \ldots, a_n) = \begin{cases} \left(\dfrac{a_1^s + \cdots + a_n^s}{n} \right)^{\frac{1}{s}} & \text{if } s \neq 0 \\ \\ (a_1 \cdots a_n)^{\frac{1}{n}} & \text{if } s = 0. \end{cases} \tag{1.2.2}$$

As one can see easily, the power mean of order r is a generalization for all the $AM - GM - HM$ means. Clearly

$$AM(a_1, \dots, a_n) = M_1(a_1, \dots, a_n)$$

$$GM(a_1, \dots, a_n) = M_0(a_1, \dots, a_n)$$

$$HM(a_1, \dots, a_n) = M_{-1}(a_1, \dots, a_n).$$

1.2.3 Monotonicity

It is well known (see, for instance, [34],[43], [41] or [36]), that for fixed $a_1, \dots, a_n$, the function $s \in \mathbb{R} \mapsto M_s(a_1, \dots, a_n) \in \mathbb{R}$ is nondecreasing. Moreover, if $r < s$, then $M_r(a_1, \dots, a_n) < M_s(a_1, \dots, a_n)$, unless $a_1 = \dots = a_n$.
This clearly implies that (1.1.2) is a consequence of the previous statement, for $(r, s) = (0, 1)$ and $(r, s) = (-1, 0)$, respectively.

1.2.4 Pondered version

The pondered mean of order s is defined by

$$M_s(a,p) = \begin{cases} \left(p_1 a_1^s + p_2 a_2^s + \dots + p_n a_n^s\right)^{\frac{1}{s}}, & \text{if } s \neq 0 \\ {a_1}^{p_1} \cdots {a_n}^{p_n}, & \text{if } s = 0 \\ \min(a_1, a_2, \dots, a_n), & \text{if } s = -\infty \\ \max(a_1, a_2, \dots, a_n), & \text{if } s = -\infty \end{cases} \tag{1.2.3}$$

where $p_1 + p_2 + \dots + p_n = 1$, $p = (p_1, p_2, \dots, p_n)$ and $a = (a_1, a_2, \dots, a_n)$ are n-tuples of positive numbers, $n \geq 2$.

Remark. For $s = -1$ we get the Harmonic Pondered Mean, for $s = 0$ we get the Geometric Pondered Mean, for $s = 1$ we get the Arithmetic Pondered Mean, for $s = 2$ we get the Quadratic Pondered Mean.
For the pondered means, the same monotonicity property holds (see Chapter 2).

1.3 Cauchy-Buniakowsky-Schwartz and related inequalities

1.3.1 The abstract inequality

A remarkable inequality is the Cauchy-Buniakowsky-Schwartz inequality. For more information about the name and the history of this inequality, see [28, p.139].
Let $(E, \langle \cdot, \cdot \rangle)$ be a prehilbertian space. The scalar product provides the space E with a norm N, defined by

$$N(x) = \sqrt{|\langle x, x \rangle|}. \tag{1.3.1}$$

Some elementary computations show that the following inequality holds

$$|\langle x, y\rangle| \leq N(x)N(y). \tag{1.3.2}$$

This is called the Cauchy-Schwartz inequality.

1.3.2 Complex version

Considering the Euclidean space $\mathbb{K}^n$ with the canonical scalar product

$$\langle a, b\rangle = \sum_{i=1}^{n} a_i \bar{b}_i,$$

we get the following version of (1.3.2).

Theorem 1.3.1 *Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be complex numbers. Then the next inequality holds*

$$|a_1b_1 + a_2b_2 + \cdots + a_nb_n|^2 \leq (|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2)(|b_1|^2 + |b_2|^2 + \cdots + |b_n|^2). \tag{1.3.3}$$

This inequality also has several direct proofs. (see [34],[33])

1.3.3 Integral version

Considering the space $C([a, b], \mathbb{R})$ of the continuous real-valued functions on the interval $[a, b]$, equipped with the scalar product

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx,$$

we obtain the following proposition.

Theorem 1.3.2 *Let $f, g \in C([a, b], \mathbb{R})$. Then the next inequality holds*

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right). \tag{1.3.4}$$

1.3.4 Applications and refinements

These well known inequalities have a large number of generalizations and refinements.
Application 1. A direct consequence of (1.3.3) is the Minkovsky Inequality.
For the nonnegative numbers $a_i, b_i, i = \overline{1, n}$, the following inequality holds:

$$\sqrt{\sum_{i=1}^{n} (a_i + b_i)^2} \leq \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}. \tag{1.3.5}$$

A refinement of the Minkovsky Inequality Let $a_i, b_i \geq 0$, $i = \overline{1,n}$ and $\alpha \in [0,1]$. Then the following inequality holds:
(1.3.6)
$$\sqrt{\sum_{i=1}^{n}(a_i+b_i)^2} \leq \sqrt{\sum_{i=1}^{n}(\alpha a_i+(1-\alpha)b_i)^2}+\sqrt{\sum_{i=1}^{n}((1-\alpha)a_i+\alpha b_i)^2} \leq \sqrt{\sum_{i=1}^{n}a_i^2}+\sqrt{\sum_{i=1}^{n}b_i^2}.$$

Proof: Using the Minkowsky inequality, we get

$$\sqrt{\sum_{i=1}^{n}(a_i+b_i)^2} = \sqrt{\sum_{i=1}^{n}((\alpha a_i+(1-\alpha)b_i)+((1-\alpha)a_i+\alpha b_i))^2} \leq$$

$$\leq \sqrt{\sum_{i=1}^{n}(\alpha a_i+(1-\alpha)b_i)^2} + \sqrt{\sum_{i=1}^{n}((1-\alpha)a_i+\alpha b_i)^2} \leq$$

$$\leq \left(\sqrt{\sum_{i=1}^{n}(\alpha a_i)^2} + \sqrt{\sum_{i=1}^{n}((1-\alpha)b_i)^2}\right) + \left(\sqrt{\sum_{i=1}^{n}((1-\alpha)a_i)^2} + \sqrt{\sum_{i=1}^{n}(\alpha b_i)^2}\right) =$$

$$= \left(\alpha\sqrt{\sum_{i=1}^{n}a_i^2} + (1-\alpha)\sqrt{\sum_{i=1}^{n}b_i^2}\right) + \left((1-\alpha)\sqrt{\sum_{i=1}^{n}a_i^2} + \alpha\sqrt{\sum_{i=1}^{n}b_i^2}\right) =$$

$$= \sqrt{\sum_{i=1}^{n}a_i^2} + \sqrt{\sum_{i=1}^{n}b_i^2}.$$

$\square$

A refinement of the Cauchy-Schwartz inequality. The following result is due to Callebaut (see [31]). For the numbers $0 < x < y < 1$, $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \geq 0$ and $n \geq 0$, the next inequality holds

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n}a_i^{1+x}b_i^{1-x}\right)\left(\sum_{i=1}^{n}a_i^{1-x}b_i^{1+x}\right) \leq$$

$$\leq \left(\sum_{i=1}^{n}a_i^{1+y}b_i^{1-y}\right)\left(\sum_{i=1}^{n}a_i^{1-y}b_i^{1+y}\right) \leq \left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right). \tag{1.3.7}$$

Other results that refine or generalize results related to the Cauchy-Buniakowsky-Schwartz inequality, can be found in [23] or [41].

1.4 Chebyshev's Inequality

This is another commonly used inequality. The main property that the Chebyshev type inequalities are exploiting is the monotonicity.

1.4.1 Real version

The version for real numbers of Chebyshev's Inequality states that

$$n(x_1y_1 + x_2y_2 + \cdots + x_ny_n) \geq (x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n), \tag{1.4.1}$$

where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ are n-tuples of real numbers that have the same monotonicity, with n natural number.

Remark. If one of the n-tuples is increasing and the other one is decreasing, then the reversed inequality holds.
The equality holds iff $x_1 = x_2 = \cdots = x_n$ or $y_1 = y_2 = \cdots = y_n$. Some generalizations of this result can be found in [32].

1.4.2 Integral version

The pondered form of the Chebyshev inequality in the integral form is

$$\int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx \geq \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx, \tag{1.4.2}$$

where p, f, g are continuous on $[a, b]$, p is nonnegative and f, g have the same monotonicity (i.e., f, g are monotone and $(f(x) - f(y))\,(g(x) - g(y)) \geq 0$, $\forall x, y \in [a, b]$).
The Chebyshev inequality has many applications, as one can see in [33] and in the following chapters.

1.5 Inequalities for convex functions

The convexity and its generalizations are very important tools in the study of the inequalities. In this chapter we study some of the most common inequalities that hold in conditions of convexity. For a detailed treatment of the convexity, together with valuable historical data, see for instance [46].

1.5.1 Definitions

Let I be an interval and $f : I \to \mathbb{R}$. We say that the function f is *convex* if

$$f(p_1x_1 + p_2x_2) \leq p_1f(x_1) + p_2f(x_2), \tag{1.5.1}$$

for any $x_1, x_2 \in I$ and $p_1, p_2 \geq 0$, $p_1 + p_2 = 1$.
If the inequality (1.5.1) holds strictly, we say that f is *strictly convex*. We say that f is *(strictly) concave* iff $-f$ is (strictly) convex.

The convexity also has a geometrical interpretation as follows.
Geometric definition: A function f is *convex* iff for any two points situated on the graph of the function, the part of the graph between them is situated under (or on) the line segment joining these points.
A sufficient simple condition for the convexity is the following.

Theorem 1.5.1 *If f is twice differentiable in the interval I and $f'' \geq 0 (\leq 0)$ on I, then f is convex (concave).*

1.5.2 Higher order convexity

Another way to introduce the convexity is by divided differences. As one can see in [45, Chapter XV] and [46], the one who initiated this kind of generalization was T. Popoviciu in 1934 (see [50]). A function $f : [a, b] \to \mathbb{R}$ is said to be *n-convex* ($n \in \mathbb{N}^*$) if for every choice of $n + 1$ distinct points in the interval $[a, b]$ such that $x_0 < x_1 < \cdots < x_n$, we have that the n-th order divided difference satisfies

$$f[x_0, \ldots, x_n] \geq 0.$$

Following the notation from [46], the divided differences are given by

$$\begin{aligned} f[x_0, x_1] &= \frac{f(x_0) - f(x_1)}{x_0 - x_1} \\ f[x_0, x_1, x_2] &= \frac{f[x_0, x_1] - f[x_1, x_2]}{x_0 - x_2} \\ &\vdots \\ f[x_0, \ldots, x_n] &= \frac{f[x_0, \ldots, x_{n-1}] - f[x_1, \ldots, x_n]}{x_0 - x_n}. \end{aligned}$$

By this definition we get that the 1-convex functions are the nondecreasing functions, while the 2-convex functions are the classical convex functions.
The following sufficient condition for the n-convexity, given by T. Popoviciu [50], extends Theorem 1.5.1 for the n-convex functions.

Theorem 1.5.2 *If f is n times differentiable with $f^{(n)} \geq 0$, then f is n-convex.*

For more detailed results about this topic, see [50], [46] or [45, Chapter XV].

1.5.3 Jensen's Inequality

The Jensen Inequality was discovered at the end of the XIX-th century and states that for any convex function

$$f(p_1x_1 + p_2x_2 + \cdots + p_nx_n) \leq p_1f(x_1) + p_2f(x_2) + \cdots + p_nf(x_n) \tag{1.5.2}$$

for any $x_1, x_2, \ldots, x_n \in I$ and $p_1 + p_2 + \cdots + p_n = 1$, $p_1, p_2, \ldots, p_n \geq 0$, $n \geq 2$. For f strictly convex, the equality holds iff $x_1 = x_2 = \cdots = x_n$.

A remarkable particular case is obtained for $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$. The inequality becomes:

$$f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \le \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \tag{1.5.3}$$

where f is convex on the interval I, $x_1, x_2, \ldots, x_n \in I$, $n \ge 2$.
The integral form of Jensen's Inequality has the following form.

Theorem 1.5.3 *If the function f is convex on the interval I, $u : [a, b] \longrightarrow I$, then*

$$f\left(\int_a^b p(x)u(x)dx\right) \le \int_a^b p(x)f(u(x))dx, \tag{1.5.4}$$

where p is nonnegative and continuous on $[a, b]$ with $\int_a^b p(x)dx = 1$.

1.5.4 Karamata's Inequality

This is a strong inequality based on convexity [34]. We present here the pondered form.

Theorem 1.5.4 *If f is a convex function on the interval I, then*

$$p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) \ge p_1 f(y_1) + p_2 f(y_2) + \cdots + p_n f(y_n), \tag{1.5.5}$$

for every $x_1 \ge x_2 \ge \cdots \ge x_n$, $y_1 \ge y_2 \ge \cdots \ge y_n$ are from I and

$$p_1 x_1 \ge p_1 y_1$$

$$p_1 x_1 + p_2 x_2 \ge p_1 y_1 + p_2 y_2$$

$$\cdots\cdots\cdots$$

$$p_1 x_1 + p_2 x_2 + \cdots + p_{n-1} x_{n-1} \ge p_1 y_1 + p_2 y_2 + \cdots + p_{n-1} y_{n-1}$$

$$p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = p_1 y_1 + p_2 y_2 + \cdots + p_n y_n$$

with $p_1, p_2, \ldots, p_n > 0$ and $n \ge 2$.

If f is strictly convex, the equality holds iff $(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$.

Remark. In the literature, the Karamata inequality is often referred as the inequality of Hardy-Littlewood-Polya.

1.5.5 Tiberiu Popoviciu's Inequality

The inequality of T. Popoviciu is another result obtained for the convex functions. It states that

$$f(x)+f(y)+f(z)+3f\left(\frac{x+y+z}{3}\right) \ge 2f\left(\frac{x+y}{2}\right)+2f\left(\frac{y+z}{2}\right)+2f\left(\frac{z+x}{2}\right), \tag{1.5.6}$$

where f is convex on the interval I, and $x, y, z \in I$. If f is strictly convex, then the equality holds in (1.5.6) iff $x = y = z$. This result along with some generalization first appeared in ([51],1965).
The first proof of (1.5.6) using the Karamata inequality appears in [58]. Some generalizations and applications of this result can be found in [59],[33].
A pondered version of (1.5.6) is due to Al. Lupaş and states that

$$pf(x) + qf(y) + rf(z) + (p+q+r)f\left(\frac{px+qy+rz}{p+q+r}\right) \geq$$

$$\geq (p+q)f\left(\frac{px+qy}{p+q}\right) + (q+r)f\left(\frac{qy+rz}{q+r}\right) + (r+p)f\left(\frac{rz+px}{r+p}\right), \tag{1.5.7}$$

where f is convex on I interval, $p, q, r > 0$ and $x, y, z \in I$. For a proof of this result, see [34, pp.294].

1.5.6 Niculescu's Inequality

In this paragraph we present shortly another result about the convex functions, obtained by C. Niculescu(1998). This is given by the following

Theorem 1.5.5 *If $f : [a, b] \longrightarrow \mathbb{R}$ is a convex function and $c, d \in [a, b]$, then*

$$\frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right) \geq \frac{f(c) + f(d)}{2} - f\left(\frac{c+d}{2}\right). \tag{1.5.8}$$

A proof using Karamata's inequality and some applications can be found in [24]. We present two of them.

Applications.

A.1

If $a, b \in \mathbb{R}$ and $c, d \in [a, b]$, then

$$|a| + |b| - |a+b| \geq |c| + |d| - |c+d|.$$

Proof: In the theorem we take $f(x) = x$. □

A.2

If $a, b \in \mathbb{R}_+^*$ and $c, d \in [a, b]$, then

$$\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \geq \sqrt{\frac{c}{d}} + \sqrt{\frac{d}{c}}.$$

Proof: In the theorem we take $f(x) = -\ln x$. □

1.5.7 The Hadamard Inequality

This is another famous inequality that we have to mention.

Theorem 1.5.6 *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.5.9}$$

We give in the next chapter some applications. Several generalizations of Hadamard's inequality can be found in [33].

1.6 The Bernoulli, Hölder and Minkovski Inequalities

This class of inequalities is very important, having a wide range of applications in the Functional Analysis and in the Theory of Probability.

1.6.1 Bernoulli's Inequality

A well-known inequality is due to Bernoulli and can be sated as follows

Theorem 1.6.1 *If $x > 0$, then $x^\alpha - \alpha x \leq 1 - \alpha$, $0 < \alpha < 1$;*
$x^\alpha - \alpha x \geq 1 - \alpha$, $\alpha < 0$ or $\alpha > 1$, with equality iff $x = 1$.

Even the proof of this inequality makes no great effort, the inequality has incredibly many applications (see for instance [60]).

1.6.2 Hölder's inequality

As it is mentioned in [43], one of the most important inequalities of analysis is Hölder's inequality.

Theorem 1.6.2 *Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be two positive n-tuples and p, q two nonzero numbers such that*

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{1.6.1}$$

Then

(**a**) *if p, q are positive,we have*

$$\sum_{i=1}^n a_i b_i \leq \left(\sum_{i=1}^n a_i^p\right)^{1/p} \left(\sum_{i=1}^n b_i^q\right)^{1/q}, \tag{1.6.2}$$

(**b**) *if either p or q is negative, then the opposite inequality is valid.*
In both cases (**a**) *and* (**b**) *equality holds iff a^p and b^q are proportional.*

Proof:

(a) The given inequality can be written as:

$$\sum_{i=1}^{n}\left(\frac{a_i^p}{\sum_{i=1}^n a_i^p}\right)^{1/p}\left(\frac{b_i^q}{\sum_{i=1}^n b_i^q}\right)^{1/q} \leq 1.$$

Due to the AG inequality, the left hand side is not greater than:

$$\sum_{i=1}^{n}\left(\frac{1}{p}\cdot\frac{a_i^p}{\sum_{i=1}^n a_i^p}+\frac{1}{q}\cdot\frac{b_i^q}{\sum_{i=1}^n b_i^q}\right)=\frac{1}{p}+\frac{1}{q}=1,$$

with equality iff $a_i^p = b_j^q$ for $i, j = \overline{1,n}$.

(b) Suppose that $p < 0$. Consider $P = -p/q$, $Q = 1/q$. Then $P > 0$ and $Q > 0$ and satisfy $\frac{1}{P} + \frac{1}{Q} = 1$. Let the n-tuples u and v be defined by $u = a^{-q}$, $v = a^q b^q$. Due to (a),we have

$$\sum_{i=1}^{n} u_i v_i \leq \left(\sum_{i=1}^{n} u_i^P\right)^{1/P}\left(\sum_{i=1}^{n} v_i^Q\right)^{1/Q},$$

which is in fact the reverse of the inequality in the theorem.

Remark. Inequality (1.6.2) is due to the German mathematician Hölder first appeared in [37](1889).

The integral version of Hölder's inequality can be formulated as follows.

Theorem 1.6.3 *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on $[a, b]$ and if $|f|^p$ and $|g|^q$ are integrable functions on $[a, b]$ then*

$$\left(\int_a^b |f(x)g(x)|dx\right) \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p}\left(\int_a^b |g(x)|^q dx\right)^{1/q}.$$

Equality holds iff $A|f(x)|^p = B|g(x)|^q$ almost everywhere, where A, B are constants.

Remark. Considering $p = q = \frac{1}{2}$, in this inequality we get the Cauchy-Schwartz inequality (1.3.4), in the both versions we have mentioned in the Chapter 1.3.

1.6.3 Minkovsky's Inequality

Another basic inequality in Analysis is Minkovsky's inequality.

Theorem 1.6.4 *Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be two positive n-tuples and $p > 1$. Then the next inequality holds*

$$\left(\sum_{i=1}^{n}(a_i+b_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} b_i^p\right)^{1/p}, \tag{1.6.3}$$

If $p < 1$ $(p \neq 0)$, we have the reverse inequality. In both cases equality holds iff a and b are proportional.

Proof: The next identity holds:

$$\sum_{i=1}^{n}(a_i+b_i)^p=\sum_{i=1}^{n}a_i(a_i+b_i)^{p-1}+\sum_{i=1}^{n}b_i(a_i+b_i)^{p-1}. \tag{1.6.4}$$

Let q be defined by $1/p+1/q=1$, i.e. $q=p/(p-1)$. If $p>1$, then Hölder's inequality implies

$$\sum_{i=1}^{n}(a_i+b_i)^p\le\left(\sum_{i=1}^{n}a_i^p\right)^{1/p}\left(\sum_{i=1}^{n}(a_i+b_i)^p\right)^{p-1/p}+\left(\sum_{i=1}^{n}b_i^p\right)^{1/p}\left(\sum_{i=1}^{n}(a_i+b_i)^p\right)^{p-1/p}.$$

From here the required inequality follows easily. Due to the equality conditions in Hölder's inequality, we get that equality holds iff a and b are proportional. If $p<1$, q becomes negative, and the result follows by the same way by application of corresponding case of Hölder's inequality.

1.7 Reversed inequalities

In this chapter we present some inequalities that give some reverse information about some classical inequalities.
The inequality we are going to present now is due to Polya and Szegö and states that

Theorem 1.7.1 *If $0<a<A$, $0<b<B$, $a_1,a_2,\ldots,a_n\in[a,A]$, $b_1,b_2,\ldots,b_n\in[b,B]$ and $n\ge 2$ then the inequality*

$$\frac{\left(a_1^2+a_2^2+\cdots+a_n^2\right)\left(b_1^2+b_2^2+\cdots+b_n^2\right)}{\left(a_1b_1+a_2b_2+\cdots+a_nb_n\right)^2}\le\frac{1}{4}\left(\sqrt{\frac{AB}{ab}}+\sqrt{\frac{ab}{AB}}\right)^2 \tag{1.7.1}$$

holds.

Proof: An idea for the proof is to use the inequality

$$\left(x-\frac{a_i}{b_i}\sqrt{\frac{AB}{ab}}\right)\left(x-\frac{a_i}{b_i}\sqrt{\frac{ab}{AB}}\right)\le 0,$$

Clearly, for $i=\overline{1,n}$ there are x satisfying the above relation. In this case, we have:

$$\left(b_ix-a_i\sqrt{\frac{AB}{ab}}\right)\left(b_ix-a_i\sqrt{\frac{ab}{AB}}\right)\le 0,$$

By summing we get

$$\left(\sum_{i=1}^{n}b_i^2\right)x^2-\left(\sum_{i=1}^{n}a_ib_i\right)\left(\sqrt{\frac{AB}{ab}}+\sqrt{\frac{ab}{AB}}\right)x+\left(\sum_{i=1}^{n}a_i^2\right)\le 0.$$

Because this is a function of degree 2 which is negative, the discriminant $\triangle$ must be positive. This shows that (1.7.1) holds. □
A special case of (1.7.1) is given by a result due to P. Schweitzer.

Theorem 1.7.2 *If* $0 < m \leq a_i \leq M$ $(i = 1, 2, \ldots, n)$, *then*

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \frac{1}{a_i}\right) \leq \frac{n^2(m+M)^2}{4mM}. \tag{1.7.2}$$

Proof: Indeed, considering $A = B = M$, $a = b = m$ and $a_i \to \sqrt{a_i}$, $b_i \to \sqrt{\frac{1}{a_i}}$ in (1.7.1), we get (1.7.2). □

A solution based on the Jensen inequality can be consulted in [25].

We end this section with the pondered version of (1.7.2), named the inequality of Kantorovici.

Theorem 1.7.3 *If* $0 < m \leq a_i \leq M$, $p_1, p_2, \ldots, p_n > 0$ $(i = 1, 2, \ldots, n)$, *then*

$$\left(\sum_{i=1}^{n} p_i a_i\right)\left(\sum_{i=1}^{n} \frac{p_i}{a_i}\right) \leq \frac{n^2(m+M)^2}{4mM}(p_1 + p_2 + \cdots + p_n)^2. \tag{1.7.3}$$

Remark The subject of the first part of Chapter 3 is also a problem related to a reversed inequality.

1.8 Reciprocal Inequalities

In this chapter we study properties of functions induced by some inequalities that these functions are satisfying.

1.8.1 A sufficient condition for monotony

A result that gives a sufficient condition that a function to be monotone is given by the next proposition.

Theorem 1.8.1 *The function* f *is increasing on* $D \subset \mathbb{R}$ *iff for any* $n \geq 2$, *for any* $x_1, x_2, \ldots, x_n \in D$ *and for any* $p_1, p_2, \ldots, p_n > 0$,

$$\left(\sum_{i=1}^{n} p_i\right)\left(\sum_{i=1}^{n} p_i x_i f(x_i)\right) \geq \left(\sum_{i=1}^{n} p_i x_i\right)\left(\sum_{i=1}^{n} p_i f(x_i)\right), \tag{1.8.1}$$

If f *is strictly increasing, then the equality holds iff* $x_1 = x_2 = \cdots = x_n$.

Proof: If f is increasing, then the n-tuples $(x_1, x_2, \ldots, x_n)$, $(f(x_1), f(x_2), \ldots, f(x_n))$, are ordered in the same way, and by applying the Chebyshev Inequality, we get the result. Reciprocally, considering the case when $n = 2$ we obtain that

$$(p_1 + p_2)(p_1 x_1 f(x_1) + p_2 x_2 f(x_2)) \geq (p_1 x_1 + p_2 x_2)(p_1 f(x_1) + p_2 f(x_2)).$$

This is equivalent to

$$p_1 p_2 (x_1 - x_2)(f(x_1) - f(x_2)) \geq 0.$$

Because $x_1, x_2 \in D$ are arbitrary, the condition is equivalent to f increasing. The equality holds iff one of the conditions $x_1 = x_2 = \cdots = x_n$, and $f(x_1) = f(x_2) = \cdots = f(x_n)$. In the case when f strictly increasing, they are equivalent.

1.8.2 A sufficient condition for convexity

The Theorem that we present here represents a reciprocal of Theorem 1.5.1. With more details, it can be found in [46, Chapter 1, Corollary 1.3.10].

Theorem 1.8.2 *Suppose $f : I \to \mathbb{R}$ is a twice differentiable function. Then:*

(i) f is convex iff $f'' \geq 0$.

(ii) f is strictly convex iff $f'' \geq 0$ and the set where f'' vanishes does not contains intervals of positive length.

Chapter 2

Applications

In this chapter we present some results related to the inequalities mentioned in the first part of our work.

2.1 Applications of classical inequalities

2.1.1 Proof of certain inequalities by using the Bernoulli inequality

The Bernoulli Inequality has many applications. For instance, in the article [60], it is showed that many inequalities can be proved by using the inequality of Bernoulli (see Chapter 1.6.1).
We present here a proof for the inequality between the power means, based on the Bernoulli inequality.
We have mentioned in the first chapter that the following inequality holds. Now we shall give the proof.

Theorem 2.1.1 *If* $a_i > 0$, $p_i > 0$, $i = 1, 2, \ldots, n$ *and* $r \leq s$. *Let*

$$M_r(a,p) = \left(\frac{\sum_{i=1}^{n} p_i a_i^r}{\sum_{i=1}^{n} p_i} \right)^{1/r}, M_s(a,p) = \left(\frac{\sum_{i=1}^{n} p_i a_i^s}{\sum_{i=1}^{n} p_i} \right)^{1/s}.$$

Then it holds the inequality

$$M_r(a,p) \leq M_s(a,p). \tag{2.1.1}$$

Proof: We give the proof when $r > 0$. Consider in the Bernoulli inequality (1.6.1), $x = \frac{a_i^r}{M_r^r(a,p)}$ and take $\alpha = \frac{s}{r} > 1$, one obtains

$$\frac{a_i^s}{M_r^s(a,p)} - \frac{s}{r} \frac{\sum_{i=1}^{n} p_i}{\sum_{i=1}^{n} p_i a_i^r} \geq 1 - \frac{s}{r}.$$

It follows

$$\frac{p_i a_i^s}{M_r^s(a,p)} - \frac{s}{r}\frac{p_i \sum_{i=1}^n p_i}{\sum_{i=1}^n p_i a_i^r} \geq \left(1 - \frac{s}{r}\right) p_i,$$

hence

$$\frac{\sum_{i=1}^n p_i a_i^s}{M_r^s(a,p)} - \frac{s}{r}\sum_{i=1}^n p_i \geq \left(1 - \frac{s}{r}\right)\sum_{i=1}^n p_i.$$

This is equivalent to

$$\frac{\sum_{i=1}^n p_i a_i^s}{M_r^s(a,p)} \geq \sum_{i=1}^n p_i,$$

which ends the proof. □

2.1.2 The pondered AM-GM inequality

The inequality (1.1.3) can be treated as a consequence of the previous inequality (2.1.1), since the pondered geometric mean can be defined as $M_0(a,p) = \lim_{s\to 0} M_s(a,p)$.
We present another proof, based on the Jensen inequality. Using the convexity of the function $\exp(x)$ we apply the Jensen inequality for $x_i = \ln a_i$ and for the positive numbers $p_1, \ldots, p_n$ with $p_1 + \cdots + p_n = 1$. We get

$$\exp\left(\sum_{i=1}^n p_i \ln a_i\right) \leq \sum_{i=1}^n p_i \exp(\ln a_i).$$

The left hand side member is equal to $a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n}$ while the right hand side one is equal to $p_1 a_1 + p_2 a_2 + \cdots + p_n a_n$, which yield (1.1.3).

2.1.3 An inequality for convex functions

In the paper [38], W. Janous has obtained some results for monotone and for convex functions, presented in the following two theorems.

Theorem 2.1.2 *Let $I \in (0,\infty)$ be an interval and $f : I \to [0,\infty)$ be an increasing function. Then for all $x_1, x_2, \ldots, x_n \in I (n \geq 2)$, the inequality*

$$\sum_{i=1}^n \frac{x_i f(x_i)}{s - x_i} \geq \frac{1}{n-1}\sum_{i=1}^n f(x_i), \tag{2.1.2}$$

holds, where $s = x_1 + x_2 + \cdots + x_n$. Furthermore, the equality occurs for all increasing functions $f : I \to (0,\infty)$ iff the fixed n numbers are all equal.

The solution is based on an application of the Chebyshev inequality and of the inequality between the arithmetic and harmonic mean.
In the case when we admit that the function also has the property of convexity, it was obtained the following result.

Theorem 2.1.3 *Let $I \in (0,\infty)$ be an interval and $f : I \to [0,\infty)$ be an increasing and convex function. Then for all $x_1, x_2, \ldots, x_n \in I (n \geq 2)$, the inequality*

$$\sum_{i=1}^{n} \frac{x_i f(x_i)}{s - x_i} \geq \frac{n}{n-1} f\left(\frac{s}{n}\right), \tag{2.1.3}$$

holds, where $s = x_1 + x_2 + \cdots + x_n$.

One can obtain the last inequality by applying the Jensen inequality to the right hand side member of (2.1.2)

2.2 Some Means in two variables and inequalities

We present briefly some means in two variables along with some of their inequalities. For a detailed treatment and for some more information about the means contained here, see [29](1987) or [30](1988). The paper [40] also can be of help.

2.2.1 Means in two variables

Let a, b be two positive numbers such that $0 < a < b$. The next expressions are defining for a, b the following means:

- $A(a,b) = \frac{a+b}{2}$ - the arithmetic mean
- $G(a,b) = (ab)^{\frac{1}{2}}$ - the geometric mean
- $H(a,b) = \frac{2ab}{a+b}$ - the harmonic mean
- $Q(a,b) = \sqrt{\frac{a^2+b^2}{2}}$ - the quadratic mean
- $I(a,b) = \frac{1}{e} \cdot (\frac{b^b}{a^a})^{\frac{1}{b-a}}$ - the identric mean [56](1975)
- $L(a,b) = \frac{b-a}{\ln b - \ln a}$ - the logarithmic mean [49](1951)
- $S(a,b) = \frac{a-b}{2arcsin\frac{a-b}{a+b}}$ - the Seiffert mean (1995)
- $B(a,b) = \frac{a-b}{2arctg\frac{a-b}{a+b}}$ - the Bencze mean [26](1980)
- $A_p(a,b) = \left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}$ - the power mean of order p

Some more general means are defined by

⋄ Lehmer's mean of order $p \in \mathbb{R}$.

$$L_p(a,b) = \frac{a^p + b^p}{a^{p-1} + b^{p-1}} \tag{2.2.1}$$

◇ Gini's mean, defined for real $r \neq s$

$$G_{r,s}(a,b) = \left(\frac{a^r + b^r}{a^s + b^s}\right)^{\frac{1}{r-s}} \tag{2.2.2}$$

◇ Stolarsky's mean for real $p \neq 0, 1$

$$S_p(a,b) = \left(\frac{a^p - b^p}{p(a-b)}\right)^{\frac{1}{p-1}} \tag{2.2.3}$$

In the first section of the next chapter it is stated a more general definition of the Stolarsky mean for two variables.

Remarks

1) It is easily seen that $L_0 = H$, $L_{\frac{1}{2}} = G$ and $L_1 = A$.

2) One can prove that the AM-GM-HM are the only power means that are Lehmer means.

3) The Stolarsky means have the property that

$S_{-1}(a,b) = G(a,b)$ the geometric mean.

$S_0(a,b) = \lim_{p\to 0} S_p(a,b) = \frac{b-a}{\ln b - \ln a} = L(a,b)$ the logarithmic mean.

$S_1(a,b) = \lim_{p\to 1} S_p(a,b) = \frac{1}{e} \cdot (\frac{b^b}{a^a})^{\frac{1}{b-a}} = I(a,b)$ the identric mean.

$S_2(a,b) = A(a,b)$ the arithmetic mean.

4) $M_p(a,b) = G_{r,0}(a,b)$.

2.2.2 Some geometrical proofs of the AM-GM-HM inequality

We give two geometrical proofs. Following [55, Chapter 12], we use the next picture to show a geometrical interpretation of the AGH means.(see Figure 2.1)

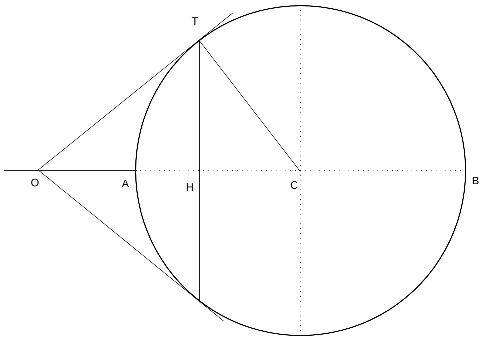

Figure 2.1:

Solution. Consider two points situated on the Ox axes, and two points A, B with the positive coordinates, $0 < a < b$. Construct then the circle of diameter A, B and let C be its center. Then consider T, P the points where the tangents from the origin intersect the circle. The line $[TP]$ intersects the Ox axes in H. From the rectangular triangles OTH, OTC we get that $m(OH) \leq m(OT) \leq m(OC)$. An easy computation shows that $m(OH) = H(a,b)$, $m(OT) = G(a,b)$ and $m(OC) = A(a,b)$, so (1.1.1) follows.

Second solution. Consider a trapezoid having the parallel lines of lengths $0 < a < b$. Then the following parallel lines with the bases of the trapezoid have the lengths:

- $A(a,b)$ - the median of the trapezoid.
- $G(a,b)$ - the line that splits the trapezoid into two similar trapezoids
- $H(a,b)$ - the line containing the intersection of the diagonals.

By the construction from Figure 2.2, we clearly have that $H(a,b) \leq G(a,b) \leq A(a,b)$. (due to the fact that the lines are closer and closer to the smallest basis)

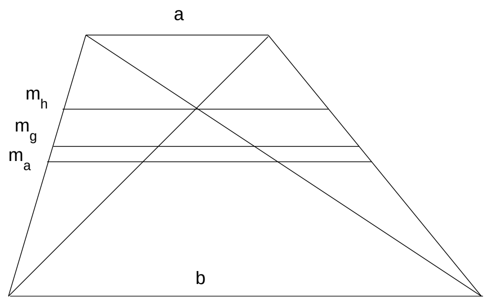

Figure 2.2:

Remark A figure containing representations of more means using the trapezoid can be found in [40, Figure 3.3, pp.26].

2.2.3 Other inequalities between means

1. Inequalities for the Lehmer means

Theorem 2.2.1 *If $p, q \in \mathbb{R}$ such that $p < q$ then $L_p(a,b) < L_q(a,b)$.*

Proof: We prove that
$$\frac{a^p + b^p}{a^{p-1} + b^{p-1}} < \frac{a^q + b^q}{a^{q-1} + b^{q-1}}.$$
By the substitution $t = b/a$ the inequality becomes
$$\frac{1 + t^p}{1 + t^{p-1}} < \frac{1 + t^q}{1 + t^{q-1}}.$$
By simple computations the last expression is equivalent to $t^{p-1}(t-1)(t^{q-p} - 1) > 0$ that is clear. $\square$

Remark Based on this result, one can give another proof of (1.1.1).

2. Inequalities for the logarithmic and identric means
In the paper [53], we find that

$$G < L < I < A. \tag{2.2.4}$$

Due to the remarks we have specified in Chapter 2.2.1, this inequality is just a consequence of the monotony of the Stolarsky mean.
There are also other proofs of (2.2.4). For instance, as is it shown in [22], the inequalities $G < L < A$ and $G < I < A$, can be obtained from Hadamard's inequality, and from some of its refinements.

3. Some more mixed inequalities
The following inequalities can be found in [54]

$$I > \frac{2A+G}{3} > \sqrt[3]{G^2A} > \sqrt{AG}.$$

Some improvements for the inequality $L > \sqrt[3]{G^2A}$ (Leach, Scholander) are given in [52]. By using a refinement of Hadamard's inequality, it is proved that

$$L < M_{1/3}.$$

(The inequality of Tung-Po-Liu, see [22]).

4. Inequalities for Bencze mean
For the Bencze mean, the following inequalities are mentioned in [26].

Theorem 2.2.2 *With the above notations, the following inequalities hold*

$$A < B < Q,$$
$$\frac{2}{\pi} \le \frac{B}{Q} \le 1.$$

Proof: We present a short proof of the second statement.
We apply Jordan's inequality, namely $\frac{2}{\pi} \le \frac{\sin t}{t} \le 1$, for $0 \le t \le \frac{\pi}{2}$. Consider $t = \operatorname{arctg} \frac{a-b}{a+b}$. Using that $\sin t = \frac{\operatorname{tg} t}{\sqrt{1+\operatorname{tg}^2 t}}$ we have that

$$\frac{\sin t}{t} = \frac{\frac{a-b}{a+b}}{\sqrt{1+\left(\frac{a-b}{a+b}\right)^2}} \cdot \frac{1}{\operatorname{arctg} \frac{a-b}{a+b}},$$

which is exactly $\frac{B}{Q}$. This ends the proof. □

5. Inequalities for Seiffert mean We indicate just that

$$\frac{2}{\pi} \le \frac{M}{A} \le 1.$$

For a proof of this inequality one can use the same argument, applied to $t = \arcsin \frac{a-b}{a+b}$. In [27] there are some more inequalities related to Seiffert's mean.

Chapter 3

Some personal results

In this chapter we present some results about inequalities that have been obtained by the author. Some of them are published or submitted to different mathematical journals. We give the results with some proofs or indications.

3.1 A best approximation for the difference of expressions related to the power means

In this section we give the solution to a problem of optimum, also providing the optimal configuration, and the asymptotical behavior as well.

The Problem. Let n be a positive integer, let $p > q$ and $0 < a < b$.
We prove that the maximum of

$$\frac{a_1^p + \cdots + a_n^p}{n} - \left(\frac{a_1^q + \cdots + a_n^q}{n}\right)^{\frac{p}{q}}$$

when $a_1, \ldots, a_n \in [a, b]$ is attained if and only if k of the variables $a_1, \ldots, a_n$ are equal to a and $n - k$ are equal to b, where k is either

$$\left[\frac{b^q - D_{p,q}^q(a,b)}{b^q - a^q} \cdot n\right]$$

or

$$\left[\frac{b^q - D_{p,q}^q(a,b)}{b^q - a^q} \cdot n\right] + 1,$$

and $D_{p,q}(a, b)$ denotes the Stolarsky mean of a and b. Moreover, if p and q are fixed, then

$$\lim_{b \searrow a} \lim_{n \to \infty} \frac{k}{n} = \frac{1}{2}.$$

3.1.1 Introduction and main results

Given the positive real numbers a and b and the real numbers p and q, the Stolarsky mean (or difference mean) $D_{p,q}(a,b)$ of a and b is defined by (see, for instance, [56] or [39]).

$$D_{p,q}(a,b) := \begin{cases} \left(\dfrac{q(a^p-b^p)}{p(a^q-b^q)}\right)^{\frac{1}{p-q}} & \text{if } pq(p-q)(b-a)\neq 0, \\ \left(\dfrac{a^p-b^p}{p(\ln a-\ln b)}\right)^{\frac{1}{p}} & \text{if } p(a-b)\neq 0, q=0, \\ \left(\dfrac{q(\ln a-\ln b)}{(a^q-b^q)}\right)^{-\frac{1}{q}} & \text{if } q(a-b)\neq 0, p=0, \\ exp\left(-\frac{1}{p}+\frac{a^p\ln a-b^p\ln b}{a^p-b^p}\right) & \text{if } q(a-b)\neq 0, p=q, \\ (ab)^{\frac{1}{2}} & \text{if } a-b\neq 0, p=q=0, \\ a & \text{if } a-b=0. \end{cases}$$

Note that $D_{2p,p}(a,b)$ is the power mean of order p of a and b:

$$D_{2p,p}(a,b) = M_p(a,b) := \begin{cases} \left(\dfrac{a^p+b^p}{2}\right)^{\frac{1}{p}} & \text{if } p\neq 0 \\ (ab)^{\frac{1}{2}} & \text{if } p=0. \end{cases}$$

The power mean can be defined not only for two numbers, but for any finite set of nonnegative real numbers. Given $a_1,\dots,a_n \in [0,\infty[$, and $p\in\mathbb{R}$, the power mean $M_p(a_1,\dots,a_n)$ of $a_1,\dots,a_n$ is defined by

$$M_p(a_1,\dots,a_n) = \begin{cases} \left(\dfrac{a_1^p+\cdots+a_n^p}{n}\right)^{\frac{1}{p}} & \text{if } p\neq 0 \\ (a_1\cdots a_n)^{\frac{1}{n}} & \text{if } p=0 \end{cases}$$

It is well known (see for instance [34],[43], [41] or [36]), that for fixed $a_1,\dots,a_n$, the function $p\in\mathbb{R}\mapsto M_p(a_1,\dots,a_n)\in\mathbb{R}$ is nondecreasing. That is

$$\left(\frac{a_1^p+\cdots+a_n^p}{n}\right)^{\frac{1}{p}} \geq \left(\frac{a_1^q+\cdots+a_n^q}{n}\right)^{\frac{1}{q}}.$$

Moreover, if $q<p$, then $M_q(a_1,\dots,a_n) < M_p(a_1,\dots,a_n)$, unless $a_1=\cdots=a_n$.
This result implies that for every $p>q$ one has

$$\frac{a_1^p+\cdots+a_n^p}{n} - \left(\frac{a_1^q+\cdots+a_n^q}{n}\right)^{\frac{p}{q}} \geq 0,$$

with equality if and only if $a_1=\cdots=a_n$.
Therefore, for fixed p and q such that $p>q$, the function $f:[0,\infty]^n\to\mathbb{R}$ defined by

$$f(a_1,\dots,a_n) = \frac{a_1^p+\cdots+a_n^p}{n} - \left(\frac{a_1^q+\cdots+a_n^q}{n}\right)^{\frac{p}{q}}, \tag{3.1.1}$$

satisfies $f(a_1,\dots,a_n)\geq 0$ for all $a_1,\dots,a_n\in[0,\infty)$.
Having in mind that the minimum of f over $[0,\infty)^n$ is 0 and it is attained when $a_1=\cdots=a_n$, it is natural to ask when is attained the maximum of f. Since

$$\sup_{a_1,\dots,a_n\in[0,\infty[} f(a_1,\dots,a_n)=\infty,$$

this question is relevant only when all the variables $a_1,\dots,a_n$ of f are restricted to a compact interval $[a,b]\subseteq[0,\infty[$. The answer is given in the next theorem.

Theorem 3.1.1 *Given the positive integer n, the real numbers $p>q>0$, $0<a<b$, and the function $f:[a,b]^n\to\mathbb{R}$, defined by (3.1.1), the following assertions are true:*
1°. The function f attains its maximum if and only if
$a_1=\cdots=a_k=a$ and $a_{k+1}=\cdots=a_n=b$, where k is either

$$\left[\frac{b^q-D^q_{p,q}(a,b)}{b^q-a^q}\cdot n\right]$$

or

$$\left[\frac{b^q-D^q_{p,q}(a,b)}{b^q-a^q}\cdot n\right]+1.$$

2°. If n,p and q are held fixed, then it holds that

$$\lim_{b\searrow a}\lim_{n\to\infty}\frac{k}{n}=\frac{1}{2}.$$

The previous result can be extended.

Theorem 3.1.2 *Given the positive integer n, the real numbers $|p|>|q|>0$, $0<a<b$, and the function $f:[a,b]^n\to\mathbb{R}$, defined by (3.1.1), the following assertions are true:*
1°. The function f attains its maximum if and only if
$a_1=\cdots=a_k=a$ and $a_{k+1}=\cdots=a_n=b$, where k is either

$$\left[\frac{b^q-D^q_{p,q}(a,b)}{b^q-a^q}\cdot n\right]$$

or

$$\left[\frac{b^q-D^q_{p,q}(a,b)}{b^q-a^q}\cdot n\right]+1.$$

2°. If n,p and q are held fixed, then it holds that

$$\lim_{b\searrow a}\lim_{n\to\infty}\frac{k}{n}=\frac{1}{2}.$$

As an application of Theorem (3.1.1), we solve the following problem.
Given the positive integer n, determine the smallest value of α such that

$$\frac{a_1^2+\cdots+a_n^2}{n}-\left(\frac{a_1+\cdots+a_n}{n}\right)^2\leq\alpha\max_{1\leq i\leq j\leq n}(a_i-a_j)^2, \tag{3.1.2}$$

holds true for all positive real numbers $a_1,\dots,a_n$.(see [34], 70-72)

Theorem 3.1.3 *Given the positive integer n, the smallest value of α such that* (3.1.2) *holds true for all positive real numbers $a_1, \ldots, a_n$ is*

$$\alpha = \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right].$$

3.1.2 Proofs

Proof of Theorem 3.1.1

1° Since f is continuous on the compact interval $[a,b]^n$, there is a point $(\overline{a}_1, \ldots, \overline{a}_n) \in [a,b]^n$ at which f attains its maximum. If $(\overline{a}_1, \ldots, \overline{a}_n)$ is an interior point of $[a,b]^n$, then $\frac{\partial f}{\partial a_i}(\overline{a}_1, \ldots, \overline{a}_n) = 0$ for all $i = 1, \ldots, n$.
Therefore

$$p \cdot \frac{\overline{a}_i^{p-1}}{n} - \frac{p}{q} \cdot \frac{q\overline{a}_i^{q-1}}{n}\left(\frac{\overline{a}_1^q + \cdots + \overline{a}_n^q}{n}\right)^{\frac{p}{q}-1} = 0,$$

whence

$$\overline{a}_i = \left(\frac{\overline{a}_1^q + \cdots + \overline{a}_n^q}{n}\right)^{\frac{1}{q}},$$

for all $i = 1, \ldots, n$. But, if $\overline{a}_1 = \cdots = \overline{a}_n$, then $f(\overline{a}_1, \ldots, \overline{a}_n) = 0$ and f cannot attain its maximum at $(\overline{a}_1, \ldots, \overline{a}_n)$. Consequently, $(\overline{a}_1, \ldots, \overline{a}_n)$ lies on the boundary of $[a,b]^n$. Taking into account that f is symmetric in its variables, and that $f(\underbrace{a, \ldots, a}_{n}) = f(\underbrace{b, \ldots, b}_{n}) = 0$, it follows that there exist $k \in \{1, \ldots, n-1\}$ and $l \in \{k+1, \ldots, n\}$ such that

$$\overline{a}_1 = \cdots = \overline{a}_k = a \qquad \text{and} \qquad \overline{a}_{k+1} = \cdots = \overline{a}_l = b.$$

If $l < n$ then $\overline{a}_{l+1}, \ldots, \overline{a}_n \in (a,b)$. We consider the function $g_l : (a,b)^{n-l} \to \mathbb{R}$, defined by

$$g_l(a_{l+1}, \ldots, a_n) = f(\underbrace{a, \ldots, a}_{k}, \underbrace{b, \ldots, b}_{l-k}, a_{l+1}, \ldots, a_n).$$

Note that g_l attains its maximum at $(\overline{a}_{l+1}, \ldots, \overline{a}_n)$, which is an interior point of $[a,b]^{n-l}$. By virtue of the Fermat theorem, we deduce that for all $i \in \{l+1, \ldots, n\}$, one has

$$\frac{\partial g_l}{\partial a_i}(\overline{a}_{l+1}, \ldots, \overline{a}_n) = 0,$$

for all $i = l+1, \ldots, n$, that is

$$p \cdot \frac{\overline{a}_i^{p-1}}{n} - \frac{p}{q} \cdot \frac{q\overline{a}_i^{q-1}}{n}\left(\frac{\overline{a}_1^q + \cdots + \overline{a}_n^q}{n}\right)^{\frac{p}{q}-1} = 0,$$

hence

$$\overline{a}_i = \left(\frac{\overline{a}_1^q + \cdots + \overline{a}_n^q}{n}\right)^{\frac{1}{q}} = c,$$

where c satisfies

$$c^q = \frac{ka^q + (l-k)b^q + (n-l)c^q}{n}.$$

A simple computation shows that

$$c^q = \frac{ka^q + (l-k)b^q}{l}.$$

We have

$$g_l(\underbrace{c, \ldots, c}_{n-l}) = \frac{ka^p + (l-k)b^p + (n-l)c^p}{n} - c^p$$

$$= \frac{k(a^p - b^p) + l\left[b^p - \left(b^q - \frac{k}{l}(b^q - a^q)\right)^{\frac{p}{q}}\right]}{n} = M_k.$$

Consider now the function $h : [k+1, n] \to \mathbb{R}$, defined by

$$h(x) = x\left[b^p - \left(b^q - \frac{k}{x}(b^q - a^q)\right)^{\frac{p}{q}}\right].$$

We claim that h is increasing. Indeed, one has

$$h'(x) = \left[b^p - \left(b^q - \frac{k}{x}(b^q - a^q)\right)^{\frac{p}{q}}\right]$$

$$-x \cdot \frac{p}{q}\left(b^q - \frac{k}{x}(b^q - a^q)\right)^{\frac{p}{q}-1} \frac{k}{x^2}(b^q - a^q)$$

$$= b^p - \left[b^q - \frac{k}{x}(b^q - a^q)\right]^{\frac{p}{q}} - \frac{p}{q} \cdot \frac{k}{x}(b^q - a^q)\left[b^q - \frac{k}{x}(b^q - a^q)\right]^{\frac{p}{q}-1}.$$

Let $\alpha = b^q - a^q$, $\quad \eta = \dfrac{k}{x} < 1$, $\quad$ and let

$$\varphi(\eta) := b^p - (b^q - \alpha\eta)^{\frac{p}{q}} - \frac{p}{q}\alpha\eta(b^q - \alpha\eta)^{\frac{p}{q}-1}.$$

Since

$$a^q < b^q - \alpha\eta = b^q - \frac{k}{x}(b^q - a^q) < b^q,$$

it follows that $h'(x) > 0$. Therefore h is increasing as claimed. Finally, we get

$$\max g_l = \frac{k(a^p - b^p) + h(l)}{n} \le \frac{k(a^p - b^p) + h(n)}{n}$$

$$= \frac{ka^p + (n-k)b^p}{n} - \left[\frac{ka^q + (n-k)b^q}{n}\right]^{\frac{p}{q}} = M_k.$$

Our problem is now reduced to the one of finding the $k \in [0, \ldots, n]$ for which M_k attains its maximum, where

$$M_k = \frac{a^p - b^p}{n}k + b^p - \left(\frac{a^q - b^q}{n}k + b^q\right)^{\frac{p}{q}}.$$

To do this, we consider the function $g : [0, n] \to \mathbb{R}$, defined by

$$g(x) = \frac{a^p - b^p}{n}x + b^p - \left(\frac{a^q - b^q}{n}x + b^q\right)^{\frac{p}{q}}.$$

It is clear that our function satisfies
$g(k) := M_k$, for $k \in [0, \ldots, n]$.
We find first the extremal points of g which lie in the interior of the interval $[0, n]$.
In these points, due to the Theorem of Fermat we have that

$$g'(x) = \frac{a^p - b^p}{n} - \frac{p}{q} \cdot \frac{a^q - b^q}{n}\left(\frac{a^q - b^q}{n}x + b^q\right)^{\frac{p}{q}-1} = 0,$$

that is

$$\frac{q\,(a^p - b^p)}{p\,(a^q - b^q)} = \left(\frac{a^q - b^q}{n}x + b^q\right)^{\frac{p}{q}-1},$$

hence, as we have seen in the definition of the Stolarsky mean that we are using in our case

$$D_{p,q}^{p-q}(a, b) = \left[\frac{a^q - b^q}{n}x + b^q\right]^{\frac{p-q}{q}},$$

and from here,

$$x^* = \frac{b^q - D_{p,q}^q(a, b)}{b^q - a^q} \cdot n,$$

is the only extremal point contained in the interior of $[0, n]$.
Taking into account that the second derivative of g is :

$$g''(x) = -\frac{p}{q} \cdot (\frac{p}{q} - 1) \cdot \left(\frac{a^q - b^q}{n}\right)^2 \cdot \left(\frac{a^q - b^q}{n}x + b^q\right)^{\frac{p}{q}-2} < 0,$$

we get that the extremal point x^* we have just found, is a point of maximum for g.
This relation also tells us that the function g' is decreasing on the interval $(0, n)$. Because $g'(x^*) = 0$, we get then that $g'(y) > 0$ for $y \in (0, x^*)$, and also that $g'(y) < 0$ for $y \in (x^*, n)$.
Finally this means that g is increasing on $(0, x^*)$ and decreasing on (x^*, n).
We conclude that:

$$g(1) < g(2) < \cdots < g([x^*]),$$

and

$$g(n) < g(n-1) < \cdots < g([x^*]+1).$$

From here we get that in order to obtain the maximum for M_k, k has to take one of the values $[x^*]$ and $[x^*]+1$, where

$$x^* = \frac{b^q - D^q_{p,q}(a,b)}{b^q - a^q} \cdot n.$$

Because in our case

$$pq(p-q)(b-a) \neq 0,$$

the Stolarsky mean has the property that $a < D^q_{p,q}(a,b) < b$, so we clearly have that $0 < x < n$.

2° We use the fact that for a positive number α, $\lim_{n\to\infty} \frac{[\alpha\cdot n]}{n} = \alpha$.

Let

$$\ell = \lim_{b\searrow a} \lim_{n\to\infty} \frac{k}{n} = \lim_{b\searrow a} \frac{b^q - \left[\dfrac{q(b^p - a^p)}{p(b^q - a^q)}\right]^{\frac{q}{p-q}}}{b^q - a^q}.$$

Using l'Hospital's rule we get

$$\ell = \lim_{b\searrow a} \frac{qb^{q-1} - \dfrac{q}{p-q}\left[\dfrac{q(b^p - a^p)}{p(b^q - a^q)}\right]^{\frac{q}{p-q}-1} \cdot \dfrac{q}{p} \cdot \bar{\ell}}{qb^{q-1}}.$$

But

$$\lim_{b\searrow a} \frac{b^p - a^p}{b^q - a^q} = \frac{p}{q} \cdot a^{p-q},$$

so,

$$\ell = \lim_{b\searrow a} \left\{1 - \frac{q}{(p-q)p} a^{2q-p} \cdot \bar{\ell}\right\},$$

where

$$\begin{aligned}
\bar{\ell} &= \lim_{b\searrow a} \frac{pb^{p-1}(b^q - a^q) - qb^{q-1}(b^p - a^p)}{b^{q-1}(b^q - a^q)^2} \\
&= \lim_{b\searrow a} \frac{(p-q)b^p - pb^{p-q}a^q + qa^p}{(b^q - a^q)^2}.
\end{aligned}$$

Using l'Hospital's rule we get

$$\begin{aligned}
\bar{\ell} &= \lim_{b\searrow a} \frac{p(p-q)b^{p-1} - p(p-q)b^{p-q-1}a^q}{2qb^{q-1}(b^q - a^q)} \\
&= \lim_{b\searrow a} \frac{p(p-q)b^{p-q} - p(p-q)b^{p-2q}a^q}{2q(b^q - a^q)} \\
&= \lim_{b\searrow a} \frac{p(p-q)(p-q)b^{p-q-1} - p(p-q)(p-2q)b^{p-2q-1}a^q}{2q^2b^{q-1}}
\end{aligned}$$

$$= \frac{p}{2q^2}(p-q)qa^{p-2q} = \frac{1}{2}(p-q)\frac{p}{q}.$$

Finally,

$$\ell = 1 - \frac{q}{(p-q)p} \cdot \frac{1}{2}(p-q)\frac{p}{q} = \frac{1}{2}.$$

In conclusion, $\lim\limits_{b \searrow a} \frac{k}{\eta} = \frac{1}{2}$, for any $p > q$. $\square$

Proof of Theorem 3.1.2

One can see that replacing negative values for p and q, the proof gives the required result. $\square$

Proof of Theorem 3.1.3

Considering $p = 2$, $q = 1$ in Theorem 1, we can see that:

$$D_{2,1}(a,b) = \frac{1}{2} \cdot \frac{b^2 - a^2}{b - a} = \frac{1}{2}(b+a),$$

and it follows that

$$\frac{k}{n} = \frac{b - \frac{1}{2}(b+a)}{b-a} = \frac{1}{2}.$$

From here, we get immediately the best constant α for which:

$$\frac{a_1^2 + \cdots + a_n^2}{n} - \left(\frac{a_1 + \cdots + a_n}{n}\right)^2 \leq \alpha \max_{1 \leq i \leq j \leq n} (a_i - a_j)^2.$$

Following the steps mentioned before, the function gets the maximum for

$$a_1 = \cdots = a_k = a,$$

$$a_{k+1} = \cdots = a_n = b,$$

where $k = \left[\frac{n}{2}\right]$ or $k = \left[\frac{n+1}{2}\right]$.

We have that

$$\frac{a_1^2 + \cdots + a_n^2}{n} - \left(\frac{a_1 + \cdots + a_n}{n}\right)^2 \leq \frac{(b-a)^2}{n^2}(nk - k^2).$$

So the best constant α will be

$$\alpha = \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right].$$

$\square$

Observation: The results of this section are contained in the paper [4].

3.2 The extension of a cyclic inequality to the symmetric form

In this section we extend a well known inequality to a symmetric form. We also give a generalization of another known result.
Let n be a natural number such that $n \geq 2$, and let $a_1, \dots a_n$ be positive numbers. Considering the notations

$$S_{i_1 \dots i_k} = a_{i_1} + \cdots + a_{i_k},$$

$$S = a_1 + \cdots + a_n,$$

we prove certain inequalities connected to conjugate sums of the form:

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{S_{i_1 \dots i_k}}{S - S_{i_1 \dots i_k}}$$

Then provided that $1 \leq k \leq n-1$ we give certain lower estimates for expressions of the above form, that extend some cyclic inequalities of Mitrinovic and others.
We also give certain inequalities that are more or less direct applications of the previous mentioned results.

3.2.1 Introduction

Consider the natural numbers $1 \leq k \leq n-1$, and the positive numbers $a_1, \dots, a_n$.
We first prove the inequality

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{S_{i_1 \dots i_k}}{S - S_{i_1 \dots i_k}} \leq \frac{k^2}{(n-k)^2} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{S - S_{i_1 \dots i_k}}{S_{i_1 \dots i_k}}, \tag{3.2.1}$$

in the case $k \leq \left[\dfrac{n}{2}\right]$.
Then we present a result which states that if $\mathcal{I} = \{(i_1, \dots, i_k) |\ 1 \leq i_1 < \cdots < i_k \leq n\}$, the following inequality holds:

$$\sum_{I \in \mathcal{I}} \frac{S_I}{S - S_I} \geq \frac{k}{n-k} \binom{n}{k}. \tag{3.2.2}$$

This result extends some cyclic inequalities to their symmetric form, as follows.
For $k = 1$ and $n = 3$ we obtain the result of Nesbit (see [44] and [34]),

$$\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} \geq \frac{3}{2} \tag{3.2.3}$$

For $k = 1$, we obtain the result of Peixoto [47],

$$\frac{a_1}{S - a_1} + \cdots + \frac{a_n}{S - a_n} \geq \frac{n}{n-1} \tag{3.2.4}$$

For arbitrary natural numbers n, k, with $1 \leq k \leq n-1$, we get the result of Mitrinovic [42],

$$\frac{a_1 + a_2 + \cdots + a_k}{a_{k+1} + \cdots + a_n} + \frac{a_2 + a_3 + \cdots + a_{k+1}}{a_{k+2} + \cdots + a_n + a_1} + \cdots + \frac{a_n + a_1 + \cdots + a_{k-1}}{a_k + \cdots + a_{n-1}} \geq$$

$$\geq \frac{nk}{n-k}. \tag{3.2.5}$$

It is easy to see that in (3.2.3) − (3.2.5) we had a cyclic summation.
By considering $n = 3$ and $k = 1$ in Theorem 3.2.3, we obtain the following result of J. Nesbitt(see [34], *pp*.87):

$$\frac{a_1 + a_2}{a_3} + \frac{a_2 + a_3}{a_1} + \frac{a_3 + a_1}{a_2} \geq \frac{a_1}{a_2 + a_3} + \frac{a_2}{a_3 + a_1} + \frac{a_3}{a_1 + a_2} + \frac{9}{2}. \tag{3.2.6}$$

3.2.2 Main results

In this section we are going to present in detail the results mentioned in the previous section.

Theorem 3.2.1 *Let n and k be natural numbers, such that $n \geq 2$ and $k \leq \left[\frac{n}{2}\right]$.*
Then considering the positive numbers $a_1, \ldots, a_n$ the following inequality holds:

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{S_{i_1 \ldots i_k}}{S - S_{i_1 \ldots i_k}} \leq \frac{k^2}{(n-k)^2} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{S - S_{i_1 \ldots i_k}}{S_{i_1 \ldots i_k}}, \tag{3.2.7}$$

where

$$S_{i_1 \ldots i_k} = a_{i_1} + \cdots + a_{i_k},$$
$$S = a_1 + \cdots + a_n.$$

Theorem 3.2.2 *Let n and k be natural numbers, such that $n \geq 2$ and $1 \leq k \leq n-1$. Then, considering the positive numbers $a_1, \ldots, a_n$ and $\mathcal{I} = \{(i_1, \ldots, i_k) | \ 1 \leq i_1 < \cdots < i_k \leq n\}$, the next inequality holds:*

$$\sum_{I \in \mathcal{I}} \frac{S_I}{S - S_I} \geq \frac{k}{n-k} \binom{n}{k}. \tag{3.2.8}$$

We have considered that $S_I = a_{i_1} + \cdots + a_{i_k}$, for $I = (i_1, \ldots, i_k)$.

As applications of the previous Theorems, we have the following results.

Theorem 3.2.3 *Let n and k be natural numbers, such that $n \geq 2$ and $k \leq \left[\frac{n}{2}\right]$.*
Then considering the positive numbers $a_1, \ldots, a_n$ the next inequality holds:

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{S - S_{i_1 \ldots i_k}}{S_{i_1 \ldots i_k}} - \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{S_{i_1 \ldots i_k}}{S - S_{i_1 \ldots i_k}} \geq \frac{(n-2k) \cdot n}{(n-k) \cdot k} \binom{n}{k}. \tag{3.2.9}$$

Theorem 3.2.4 *Let n and k be natural numbers, such that $1 \le k \le n-1$, and $a_1, \ldots, a_n$ positive numbers. Then the next inequality holds:*

$$\sum_{I \in \mathcal{I}} \frac{S_I}{S - S_I} + \sum_{I \in \mathcal{I}} \frac{S - S_I}{S_I} \ge \frac{(n-k)^2 + k^2}{k(n-k)} \binom{n}{k}. \tag{3.2.10}$$

3.2.3 Proofs

Proof of Theorem 3.2.1**.** Denote $I = \{i_1, \ldots, i_k\}$ and $\mathcal{I}$ the set of all k-tuples I ($1 \le i_1 < \cdots < i_k \le n$). The inequality in this result is equivalent to:

$$\sum_{I \in \mathcal{I}} \frac{S_I}{S - S_I} \le \frac{k^2}{(n-k)^2} \sum_{I \in \mathcal{I}} \frac{S - S_I}{S_I}. \tag{3.2.11}$$

Denote by

$$E = \sum_{I \in \mathcal{I}} \frac{S - S_I}{S_I} = \sum_{I \in \mathcal{I}} \frac{\sum\limits_{j \notin I} a_j}{S_I} \tag{3.2.12}$$

and $|\{j \in \{1, \ldots, n\} |\ j \notin I\}| = n - k \ge k$.
We write the sum $\sum\limits_{j \notin I} a_j$ as a sum of terms of the kind S_I.
We write the sum of the first $n-k$ terms (the other ones being symmetrical):

$$a_1 + \cdots + a_{n-k} = \frac{(a_1 + \cdots + a_k) + \cdots + (a_{n-2k+1} + \cdots + a_{n-k})}{\alpha}. \tag{3.2.13}$$

It is now cleat that in the right member a_1 appears for $\binom{n-k-1}{k-1}$ times, so

$$\alpha = \binom{n-k-1}{k-1}.$$

It is now easy to see that

$$\sum_{j \notin I} a_j = \frac{\sum\limits_{J \in \mathcal{I}} S_J}{\binom{n-k-1}{k-1}}, \tag{3.2.14}$$

where $J = \{j_1, \ldots, j_k\}$, with $I \cap J = \emptyset$.
With our notations (3.2.14) is equivalent to

$$S - S_I = \sum_{\substack{J \in \mathcal{I} \\ J \cap I = \emptyset}} \frac{S_J}{\binom{n-k-1}{k-1}}.$$

We obtain

$$E = \sum_{I\in\mathcal{I}} \frac{1}{\binom{n-k-1}{k-1}} \sum_{\substack{J\in\mathcal{J}\\ J\cap I=\emptyset}} \frac{S_J}{S_I},$$

so

$$E = \frac{1}{\binom{n-k-1}{k-1}} \sum_{J\in\mathcal{I}} S_J \sum_{\substack{I\in\mathcal{I}\\ I\cap J=\emptyset}} \frac{1}{S_I}.$$

We can interchange I and J now and we obtain:

$$E = \frac{1}{\binom{n-k-1}{k-1}} \cdot \sum_{I\in\mathcal{I}} S_I \sum_{\substack{J\in\mathcal{I}\\ I\cap J=\emptyset}} \frac{1}{S_J}.$$

Denote

$$E_I = \frac{S_I}{\binom{n-k-1}{k-1}} \cdot \sum_{\substack{J\in\mathcal{I}\\ I\cap\mathcal{J}=\emptyset}} \frac{1}{S_J}.$$

We will prove the following relation:

$$\frac{S_I}{S-S_I} \le \frac{\beta}{\binom{n-k-1}{k-1}} \cdot S_I \cdot \sum_{\substack{J\in\mathcal{I}\\ I\cap J=\emptyset}} \frac{1}{S_J} = \beta \cdot E_I. \tag{3.2.15}$$

It is easy to see that summing (3.2.15) after $I \in \mathcal{I}$ we get (3.2.11) and β will be determined later. We have that (3.2.15) is equivalent to

$$\frac{1}{S-S_I} \le \frac{\beta}{\binom{n-k-1}{k-1}} \sum_{\substack{J\in\mathcal{I}\\ I\cap J=\emptyset}} \frac{1}{S_J}, \tag{3.2.16}$$

hence we have obtained that:

$$1 \le \frac{\beta}{\binom{n-k-1}{k-1}^2} \left(\sum_{\substack{J\in\mathcal{I}\\ I\cap J=\emptyset}} S_J \right) \left(\sum_{\substack{J\in\mathcal{I}\\ I\cap J=\emptyset}} \frac{1}{S_J} \right).$$

But each of the sums in the right hand side has exactly $\binom{n-k}{k}$ terms and by Cauchy's inequality we obtain that:

$$\binom{n-k}{k}^2 \le \left(\sum_{\substack{J\in\mathcal{I}\\ J\cap I=\emptyset}} S_J \right) \left(\sum_{\substack{J\in\mathcal{I}\\ J\cap I=\emptyset}} \frac{1}{S_J} \right).$$

Finally we get the required β which is:

$$\beta := \frac{\binom{n-k-1}{k-1}^2}{\binom{n-k}{k}^2} = \left[\frac{(n-k-1)!}{(k-1)!(n-2k)!} \cdot \frac{k!(n-2k)!}{(n-k)!}\right]^2 = \left[\frac{k}{n-k}\right]^2.$$

Hence in view of (3.2.15) we have obtained that that:

$$\frac{S_I}{S-S_I} \leq \left(\frac{k}{n-k}\right)^2 E_I,$$

and by summing we have (3.2.11). □

Proof of Theorem 3.2.2**.** By the inequality of Cauchy we have that:

$$\left(\sum_{I\in\mathcal{I}} \frac{S_I}{S-S_I}\right)\left(\sum_{I\in\mathcal{I}} S_I(S-S_I)\right) \geq \left(\sum_{I\in\mathcal{I}} S_I\right)^2. \tag{3.2.17}$$

In order to prove (3.2.8) it is enough to show that

$$\left(\sum_{I\in\mathcal{I}} S_I\right)^2 \geq \frac{k}{n-k}\binom{n}{k}\sum_{I\in\mathcal{I}} S_I(S-S_I) \tag{3.2.18}$$

Indeed we obtain (3.2.8) by making the product of (3.2.17) and (3.2.18).
Let us now prove (3.2.18). We begin with the next lemma.

Lemma. $\sum S_I = \binom{n-1}{k-1} S.$

Proof: We have to find the multiplicity of a_1 in $\sum\limits_{I\in\mathcal{I}} S_I$. If a_1 appears on the first position, the other $k-1$ position from I may be chosen in $\binom{n-1}{k-1}$ ways and because the sum is symmetric, the conclusion follows. Lemma is proved.

Because

$$\sum S_I \cdot S = \binom{n-1}{k-1} S^2,$$

the inequality (3.2.18) becomes

$$\binom{n-1}{k-1}^2 \cdot S^2 \geq \frac{k}{n-k}\binom{n}{k}\left[\binom{n-1}{k-1}S^2 - \sum_{I\in\mathcal{I}} S_I^2\right], \tag{3.2.19}$$

which is

$$\frac{k}{n-k}\binom{n}{k}\left(\sum_{I\in\mathcal{I}} S_I^2\right) \geq \binom{n-1}{k-1} S^2 \left[\frac{k}{n-k}\binom{n}{k} - \binom{n-1}{k-1}\right]. \tag{3.2.20}$$

The identity

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1},$$

shows that (3.2.20) is equivalent to

$$\frac{k}{n-k}\binom{n}{k}\left(\sum_{I\in\mathcal{I}} S_I^2\right) \geq \binom{n-1}{k-1} S^2 \left[\frac{k}{n-k}\binom{n-1}{k-1}\right],$$

which implies that

$$\left(\sum_{I\in\mathcal{I}} S_I^2\right)\binom{n}{k} \geq \binom{n-1}{k-1}^2 S^2.$$

Because the sums have $\binom{n}{k}$ terms, by the inequality of Cauchy and the Lemma we have that

$$\left(\sum_{I\in\mathcal{I}} S_I^2\right)\binom{n}{k} \geq \left(\sum_{I\in\mathcal{I}} S_I\right)^2.$$

This shows that (3.2.18) holds.
Note that the equality holds if and only if $S_I = S_J$ for $I, J \in \mathcal{I}$, which gives that $a_1 = \cdots = a_n$. □

Remark. In [3] is given a shorter proof for this Theorem which uses Jensen's inequality for some convex function.

Proof of Theorem 3.2.3. Using Theorem 3.1.1, we get that our sum is in fact greater or equal to

$$\left(\frac{(n-k)^2}{k^2} - 1\right) \sum_{1\leq i_1<\cdots<i_k\leq n} \frac{S_{i_1\ldots i_k}}{S - S_{i_1\ldots i_k}}.$$

By the previous Theorem, this is greater than

$$\frac{(n-2k)\cdot n}{k^2}\cdot\frac{k}{n-k}\cdot\binom{n}{k}.$$

□

Proof of Theorem 3.2.4. Using Theorem 3.2.1 and Theorem 3.2.2 for $I = \{(i_1,\ldots,i_k)|\ 1 \leq i_1 < \cdots < i_k \leq n\}$ and $J = \{(j_1,\ldots,j_{n-k})|\ 1 \leq j_1 < \cdots < j_{n-k} \leq n\}$ one obtains

$$\sum_{I\in\mathcal{I}} \frac{S - S_I}{S_I} = \sum_{J\in\mathcal{I}} \frac{S_J}{S - S_J} \geq \frac{n-k}{k}\binom{n}{k}.$$

It is clear that

$$\sum_{I\in\mathcal{I}} \frac{S_I}{S - S_I} + \sum_{I\in\mathcal{I}} \frac{S - S_I}{S_I} \geq \left(\frac{n-k}{k} + \frac{k}{n-k}\right)\cdot\binom{n}{k},$$

and this is exactly the required inequality. □

Observation: The above results are contained in the paper [5], submitted to J.Ineq.Appl.Math.

3.3 Notes and answers to some questions published in the Octogon Mathematical Magazine

In this chapter we expose some of the results that the author has obtained during his collaboration with the Octogon Mathematical Magazine that appears in Brasov, Romania, in English. We have presented presented just the results related to the inequalities.
Most of the results have been published in the same journal.
The results are original generalizations of inequalities, or answers to the Open Questions rubric of Octogon, abbreviated OQ. in the following.

3.3.1 About an Inequality

In this paragraph we present two inequalities obtained by the author. The proof is not short enough po put it here. We recommend the reader to consult [1].
We have that for $a_i \geq 0$, $i = \overline{1,m}$ and $m, n \geq 2$ natural numbers with $n_j \in \mathbb{N}$, $j = \overline{1,m}$, the next inequality holds:

$$\frac{\sum_{n_1+\cdots+n_m=n} a_1^{n_1}\cdots a_m^{n_m}}{\binom{n+m-1}{m-1}} \geq \left(\frac{\sum_{i=1}^m a_i}{m}\right)^n. \tag{3.3.1}$$

To prove this result, we use in fact the stronger inequality

$$\frac{S_{m,n+1}}{\binom{n+m}{m-1}} \geq \frac{S_{m,n}}{\binom{n+m-1}{m-1}} \cdot \frac{\sum_{i=1}^m a_i}{m}, \tag{3.3.2}$$

where $S_{m,n} = \sum_{n_1+\cdots+n_m=n} a_1^{n_1}\cdots a_m^{n_m}$.
The proof of the above result needs some steps of induction as well as a good skill in applying Chebyshev's inequality.

Remark. The standard notations can be found in [41, Chapter 2.16], but here we kept the notations from [1].

3.3.2 A New Class of Non-similarities Between the Triangle and the Tethraedron

In this chapter we refer to the paper [2], where it is proved that in space we can extend just partially the result given by the next theorem.

Theorem 3.3.1 *Given a triangle, the monotonicity of the lengths of the segments say $a < b < c$, determines the inverse monotonicity for the heights, medians and bisectrices. (i.e. $h_a > h_b > hc$, $m_a > m_b > m_c$, $l_a > l_b > l_c$)*

I have proved that in space the monotonicity of the surfaces of the faces of a tetrahedron implies just the inverse monotony of the heights.

3.3.3 Answer to OQ.603

This proof was published in [18] and gives a solution to an open problem proposed by M. Bencze in [33, OQ.603, Vol. 9, No.1, April 2001]. This is an application of the inequality of Chebyshev for monotone sequences of numbers.
We have to find all positive integers n such that the following inequality

$$\prod_{1\leq k\leq n} \frac{a_1^k+\cdots+a_m^k}{m} \leqslant \frac{1}{m}\cdot\left(a_1^{\frac{n(n+1)}{2}}+\cdots+a_m^{\frac{n(n+1)}{2}}\right), \tag{3.3.3}$$

holds, where $a_1,\ldots,a_m \geqslant 0$.
We will prove that (3.3.3) holds for every natural n, and that even the more general inequality

$$\prod_{1\leq k\leq n} \frac{a_1^{x_k}+\cdots+a_m^{x_k}}{m} \leqslant \frac{1}{m}\cdot\left(a_1^S+\cdots+a_m^S\right), \tag{3.3.4}$$

holds, where $a_1,\ldots,a_m \geqslant 0$ and $S = x_1+\cdots+x_m$, with $x_1,\ldots,x_n$ nonnegative numbers.
Proof of (3.3.4)
Let x, y be two nonnegative numbers.
Then for $a_1 \leqslant \cdots \leqslant a_m$, we get the following two inequalities:

$${a_1}^x \leqslant \cdots \leqslant {a_m}^x$$

$${a_1}^y \leqslant \cdots \leqslant {a_m}^y.$$

From here, by Chebyshev's inequality (see Chapter 1.4) we get that

$$\frac{{a_1}^x+\cdots+{a_m}^x}{m}\cdot\frac{{a_1}^y+\cdots+{a_m}^y}{m} \leqslant \frac{{a_1}^{x+y}+\cdots+{a_m}^{x+y}}{m}. \tag{3.3.5}$$

Now, the following inequalities can be obtained easily from (3.3.5):

$$\frac{a_1^{x_1}+\cdots+a_m^{x_1}}{m}\cdot\frac{a_1^{x_2}+\cdots+a_m^{x_2}}{m} \leqslant \frac{a_1^{x_1+x_2}+\cdots+a_m^{x_1+x_2}}{m}$$

$$\frac{{a_1}^{x_1+x_2}+\cdots+{a_m}^{x_1+x_2}}{m}\cdot\frac{{a_1}^{x_3}+\cdots+{a_m}^{x_3}}{m} \leqslant \frac{{a_1}^{x_1+x_2+x_3}+\cdots+{a_m}^{x_1+x_2+x_3}}{m}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{{a_1}^{S-x_n}+\cdots+{a_m}^{S-x_n}}{m}\cdot\frac{{a_1}^{x_n}+\cdots+{a_m}^{x_n}}{m} \leqslant \frac{{a_1}^{S}+\cdots+{a_m}^{S}}{m}.$$

Multiplying them, we clearly get (3.3.4).

Remark. Considering $x_1 = 1,\ldots,x_n = n$ in (3.3.4), we obtain $S = \frac{n(n+1)}{2}$, and from here we get that (3.3.3) holds as a particular case of (3.3.4). $\square$

3.3.4 Some inequalities that hold only for sequences having a cardinal smaller that a certain number

In this paragraph we present some results related to $OQ.1759$, $OQ.1760$ and $OQ.1761$. The tool that we are using, is the property of the continuous functions, that being positive in a point, implies being positive in a neighborhood.

a) Open Question 1759

In [33, OQ.1759, Vol. 13, No.1, April 2005, pp.873], M. Bencze has proposed the problem of proving that the next inequality holds:

$$\prod_{1\le i<j\le n}\left(a_i^2+a_ia_j+a_j^2\right)\geqslant\left(\sum_{1\le i<j\le n}a_ia_j\right)^3, \tag{3.3.6}$$

where $a_k\in\mathbb{R}$, $k=\overline{1,n}$.

Because of reasons of symmetry, (3.3.6) has to look like:

$$\prod_{1\le i<j\le n}\left(a_i^2+a_ia_j+a_j^2\right)\geqslant\left(\sum_{1\le i<j\le n}a_ia_j\right)^{\frac{n(n-1)}{2}}\cdot\left(\frac{6}{n(n-1)}\right)^{\frac{n(n-1)}{2}} \tag{3.3.7}$$

where $a_k\in\mathbb{R}$, $k=\overline{1,n}$.

We will prove that even in the more restrictive assumptions $a_1,\dots,a_n>0$, the inequality (3.3.7) does not hold for $n\geqslant 4$.

Consider the function $f:(\mathbb{R}_+)^n\longrightarrow\mathbb{R}$, defined as:

$$f(a_1,\dots,a_n)=\prod_{1\le i<j\le n}\left(a_i^2+a_ia_j+a_j^2\right)-\left(\sum_{1\le i<j\le n}a_ia_j\right)^{\frac{n(n-1)}{2}}\cdot\left(\frac{6}{n(n-1)}\right)^{\frac{n(n-1)}{2}}.$$

For the point $x=(a_1,a_2,0,\dots,0)$, with $a_1,a_2>0$ we have clearly that

$$f(x)=-\left(a_1a_2\right)^{\frac{n(n-1)}{2}}\cdot\left(\frac{6}{n(n-1)}\right)^{\frac{n(n-1)}{2}}<0.$$

Clearly f is continuous on its domain, so there is a neighborhood of x, where the function takes only negative values. In that neighborhood there are points having only positive coordinates $a_1,\dots,a_n$, and so, for these numbers we get that $f(a_1,\dots,a_n)<0$.

This proves that the inequality (3.3.7) does not hold.

Remark 1. Considering for instance the numbers $a_1=\cdots=a_3=1$, $a_4=2$, we get that $f(a_1,\dots,a_4)>0$. This thing implies that the reverse inequality also cannot hold, in this case for $n=4$.

Remark 2. In [21], the inequality (3.3.7) is solved for the case $n=3$, but in a more restricted domain, because the authors have considered that a_1,a_2 and a_3 represent the length of the sides of a triangle. It is not known if the inequality holds in general in the case $n = 3$.

b) Open Question 1760

In [33, OQ.1760, Vol. 13, No.1, April 2005, pp.873], M. Bencze has proposed the following problem:

$$\left(\frac{3}{4}\right)^n \dot{\prod}_{1\le i<j\le n} (a_i + a_j)^3 \geqslant \left(\sum_{1\le i<j\le n} a_i a_j\right)^3, \tag{3.3.8}$$

where $a_k \in \mathbb{R}$, $k = \overline{1,n}$.

Because of reasons of symmetry, (3.3.8) has to look like:

$$\prod_{1\le i<j\le n} (a_i + a_j)^2 \geqslant \left(\sum_{1\le i<j\le n} a_i a_j\right)^{\frac{n(n-1)}{2}} \cdot \left(\frac{8}{n(n-1)}\right)^{\frac{n(n-1)}{2}}, \tag{3.3.9}$$

where $a_k \in \mathbb{R}$, $k = \overline{1,n}$.

We will prove first that even under the more restrictive assumptions $a_1, \ldots, a_n > 0$, the inequality (3.3.9) does not hold for $n \geqslant 3$.

Suppose $n \geq 4$.

Consider the function $f : (\mathbb{R}_+)^n \longrightarrow \mathbb{R}$, defined as:

$$f(a_1, \ldots, a_n) = \prod_{1\le i<j\le n} (a_i + a_j)^2 - \left(\frac{8}{n(n-1)} \sum_{1\le i<j\le n} a_i a_j\right)^{\frac{n(n-1)}{2}}.$$

For the point $x = (a_1, a_2, 0, \ldots, 0)$, with $a_1, a_2 > 0$ we have

$$f(x) = -(a_1 a_2)^{\frac{n(n-1)}{2}} \cdot \left(\frac{8}{n(n-1)}\right)^{\frac{n(n-1)}{2}} < 0.$$

Clearly f is continuous on its domain, so there is a neighborhood of x, where the function takes only negative values. In that neighborhood there are points having only positive coordinates $a_1, \ldots, a_n$, and so, for these numbers we get that $f(a_1, \ldots, a_n) < 0$.

This proves that the inequality (3.3.9) does not hold.

Consider now the case when $n = 3$.

The inequality becomes

$$(a_1 + a_2)^2 \cdot (a_2 + a_3)^2 \cdot (a_3 + a_1)^2 \geq (a_1 a_2 + a_2 a_3 + a_3 a_1)^3 \cdot \frac{64}{27}. \tag{3.3.10}$$

By the identities:

$$(a_1 + a_2)(a_1 + a_3) = a_1^2 + a_1 a_2 + a_2 a_3 + a_3 a_1,$$
$$(a_2 + a_3)(a_2 + a_1) = a_2^2 + a_1 a_2 + a_2 a_3 + a_3 a_1,$$

$$(a_3 + a_1)(a_3 + a_2) = a_3^2 + a_1 a_2 + a_2 a_3 + a_3 a_1,$$

and making the substitution $s = a_1 a_2 + a_2 a_3 + a_3 a_1$, the inequality becomes

$$(a_1^2 + s)(a_2^2 + s)(a_3^2 + s) \geq \frac{64}{27} s^3.$$

In the case of positive numbers this inequality seems to hold (to me), but I have no proof of this.

Remark. Considering again the inequality (3.3.10) for $n = 3$, for the left member we have a superior estimation we have by applying the ln and the Jensen inequality for the concave function ln :

$$\frac{\ln(a_1^2 + s) + \ln(a_2^2 + s) + \ln(a_3^2 + s)}{3} \leq \ln\left(\frac{a_1^2 + a_2^2 + a_3^2}{3} + s\right). \tag{3.3.11}$$

Because we clearly have that $a_1^2 + a_2^2 + a_3^2 \geq s$, we obtain some information about the inequality(3.3.10), that has to be a refinement of

$$\frac{a_1^2 + a_2^2 + a_3^2}{3} + s \geq \frac{4}{3} s,$$

or equivalently, to

$$a_1^2 + a_2^2 + a_3^2 \geq a_1 a_2 + a_2 a_3 + a_3 a_1.$$

c) Open Question 1761

In [33, OQ.1761, Vol. 13, No.1, April 2005, pp.873], M. Bencze has proposed the problem of proving the following inequality:

$$\left(\frac{k}{k+1}\right)^n \dot{\prod}_{1 \leq i_1 < \cdots < i_k \leq n} (a_{i_1} + \cdots + a_{i_k})^{k-1} \geqslant \left(\sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1} \ldots a_{i_k}\right)^k, \tag{3.3.12}$$

where $a_k \in \mathbb{R}$, $k = \overline{1, n}$ and $k \geqslant 3$.

Because of reasons of symmetry, (3.3.12) has to look like:

$$\prod_{1 \leq i_1 < \cdots < i_k \leq n} \left(\frac{a_{i_1} + \cdots + a_{i_k}}{k}\right)^{k-1} \geqslant \left(\frac{\sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1} \ldots a_{i_k}}{\binom{n}{k}}\right)^{\binom{n}{k} \cdot \frac{k-1}{k}}, \tag{3.3.13}$$

where $a_k \in \mathbb{R}$, $k = \overline{1, n}$.

We will prove that for some k, the inequality (3.3.13) does not hold even under the more restrictive assumptions $a_1, \ldots, a_n > 0$, and that the equality holds for some particular choices of the $k's$.

Case 1. $k = n$

Then (3.3.13) is equivalent to $\frac{a_1 + \cdots + a_n}{n} \geqslant (a_1 \cdots a_n)^{\frac{1}{n}}$, which is the AM-GM inequality. In this case, when the numbers $a_1, \ldots, a_n$ are nonnegative, the GM is well defined for any n, so (3.3.13) holds.

Case 2. $k \leqslant [\frac{n}{2}]$ (where $[x]$ denotes the integer part of x)
In this case we choose the function $f : (\mathbb{R}_+)^n \longrightarrow \mathbb{R}$, defined by the expression

$$\prod_{1\le i_1<\cdots<i_k\le n}\left(\frac{a_{i_1}+\cdots+a_{i_k}}{k}\right)^{k-1}-\left(\frac{\sum_{1\le i_1<\cdots<i_k\le n}a_{i_1}\ldots a_{i_k}}{\binom{n}{k}}\right)^{\binom{n}{k}\cdot\frac{k-1}{k}}.$$

Considering the point x with the coordinates $a_1 = \cdots = a_k = 0$ and $a_{k+1} = \cdots = a_n = a > 0$ then $f(x) < 0$ and because f is continuous on its domain, there is a neighborhood of x where the function takes only negative values. In that neighborhood there are points having only positive coordinates $a_1, \ldots, a_n$, and so, for these numbers, we get that $f(a_1, \ldots, a_n) < 0$.
The conclusion is that (3.3.13) cannot hold for such k.
At present there is no proof was provided for the other cases.

3.3.5 A way of obtaining new inequalities between different means

In this chapter we obtain some new inequalities between means that are defined by the two variable versions of the logarithmic mean, geometric mean and arithmetic mean. We give some versions of n-ary logarithmic mean maintaining some inequalities.
This is because the logarithmic mean does not quite lend itself for a natural generalization, the mean failing some axiom for the n-ary means. We refer the interested reader to the work [48].
We first give a solution to the next problem of M. Bencze [33, OQ.407, Vol. 9, No.1, April 2001]: "If $x_k > 0$, $(k = 1, 2, \ldots, n)$ then holds the following inequality:

$$\sqrt[n]{\prod_{k=1}^{n}x_k}<$$

$$<\frac{1}{n}\left(\frac{x_2-x_1}{\ln x_2-\ln x_1}+\frac{x_3-x_2}{\ln x_3-\ln x_2}+\cdots+\frac{x_1-x_n}{\ln x_1-\ln x_n}\right)<$$

$$<\frac{1}{n}\sum_{i=1}^{n}x_k.\text{''} \tag{3.3.14}$$

We will show that both of the inequalities hold, considering the existence condition for the expressions above, that are: $x_i \neq x_j$, for $i \neq j$, and $i, j = 1, \ldots, n$. We also define the logarithmic mean for n numbers in some ways.

Proof of the inequalities (3.3.14**)**
In the paper [53], we find that

$$G < L < I < A, \tag{3.3.15}$$

where for distinct positive numbers a, b we have:

$A(a,b) = \frac{a+b}{2}$, the arithmetic mean

$G(a,b) = (ab)^{\frac{1}{2}}$, the geometric mean

$I(a,b) = \frac{1}{e} \cdot (\frac{b^b}{a^a})^{\frac{1}{b-a}}$, the identric mean and

$L(a,b) = \frac{b-a}{\ln b - \ln a}$, the logarithmic mean, as defined before in the previous chapters.

Using (3.3.15) we get that:

$$G(x_i, x_{i+1}) < L(x_i, x_{i+1}) < A(x_i, x_{i+1}),$$

for $i = 1, \ldots, n$, and $x_{n+1} = x_1$. Summing these expressions we get that

$$\sum_{i=1}^{n} G(x_i, x_{i+1}) < \sum_{i=1}^{n} L(x_i, x_{i+1}), \tag{3.3.16}$$

$$\sum_{i=1}^{n} L(x_i, x_{i+1}) < \sum_{i=1}^{n} A(x_i, x_{i+1}). \tag{3.3.17}$$

Clearly multiplying in (3.3.17) by $\frac{1}{n}$, we get the inequality from the right hand side of (3.3.14).

In (3.3.16) we multiply by $\frac{1}{n}$. Then applying the $AM - GM$ inequality for the set of positive numbers $G(x_i, x_{i+1})$, we have that

$$\sqrt[n]{\prod_{k=1}^{n} x_k} < \frac{1}{n} \left(\sum_{i=1}^{n} G\left(x_i, x_{i+1}\right) \right). \tag{3.3.18}$$

From (3.3.16) and (3.3.18) it follows that the left hand side of the inequality (3.3.14) holds.

This ends the proof. $\square$

Further results.

1). The importance of this problem is that we can define a generalization of the logarithmic mean for n numbers, different 2×2, as

$$L\left(x_1, \ldots, x_n\right) = \frac{1}{n} \left(\frac{x_2 - x_1}{\ln x_2 - \ln x_1} + \frac{x_3 - x_2}{\ln x_3 - \ln x_2} + \cdots + \frac{x_1 - x_n}{\ln x_1 - \ln x_n} \right). \tag{3.3.19}$$

2). We can see that the above definition is somehow a logarithmic mean defined by the arithmetical mean of $x_1, \ldots, x_n$. So the expression defined in the previous remark could be better denoted by AL. We can also define

$$GL\left(x_1, \ldots, x_n\right) = \sqrt[n]{\prod_{k=1}^{n} L\left(x_i, x_{i+1}\right)}. \tag{3.3.20}$$

In the same manner as above, we can prove that $G < GL < A$.
3). Now we have the tools to generalize more the results from this section. Consider that we have a n-ary mean that has the property: $G < M < A$.(as binary means) Then we can define the mean $LM = M\left(L\left(x_i, x_{i+1}\right)\right), i = \overline{1, n}$ which has the property $G < LM < A$.
4). Considering the means $P < M < Q$ and $P < N < Q$, defined for positive numbers, such that

$$R\left(R\left(x_i, x_{i+1}\right)\right) = R(x_i), \tag{3.3.21}$$

where $\overline{i = 1, n}$ and $R \in \{P, Q\}$.
We have that the next inequality holds:

$$P < M\left(N\left(x_i, x_{i+1}\right)\right) < Q, \tag{3.3.22}$$

Clearly, the means G and A satisfy (3.3.21), fact that justifies this generalization.

3.3.6 Solutions to other Open Questions

In this chapter we just give the enounces of some problems that have been solved by the author, with some indications concerning the proofs. We also indicate where the solutions can be found.

Answer to Open Question 1189.
In [33, OQ.1189, Vol. 11, No.1, April 2003, pp 384] Themistocles M. Rassias has proposed the next problem:
If p_n is the n-th prime number, find the integer part of the summation:

$$\sum_{n=1}^{1000} \frac{n p_n}{(1 + \sqrt{p_n})^n}.$$

Proof: A complete solution is contained in [13] shows that the answer is 2. ♣

Answer to Open Question 1190.
In [33, OQ.1190, Vol. 11, No.1, April 2003, pp 384] Themistocles M. Rassias has proposed the next problem:
If p_n is the n-th prime number, find the integer part of the summation:

$$\prod_{n=1}^{1000} \frac{n! p_n}{1 + p_n!}.$$

Proof: A complete solution is contained in [14] shows that the answer is 0. ♣

Answer to Open Question 1243.
In [33, OQ.1243, Vol. 11, No.1, April 2003, pp 396.] M. Bencze has proposed the next problem: Let $x_k > 0, (k = 1, 2, \ldots, n)$. Determine all $\alpha \in \mathbb{R}$ for which

$$\prod_{k=1}^{n} \frac{1 + x_k^{\alpha+1}}{1 + x_k^{\alpha}} \geq 1 + \prod_{k=1}^{n} x_k. \tag{3.3.23}$$

Proof: A complete solution to this problem is published in [8].
The answer is that the inequality holds for $\alpha \in (-\infty, -\frac{1}{2}]$ and does not hold for $\alpha \in (-\frac{1}{2}, +\infty)$. In the same time, it can be mentioned that in the case when it holds, the inequality is strict. ♣

Answer to Open Question 1248.
In [33, OQ.1248, Vol. 11, No.1, April 2003, pp 397] M. Bencze has proposed the next problem:
Let n be natural number and $x_1, \dots, x_n$ be distinct positive integers. Prove that:

$$\sum_{k=1}^{n} {x_k}^2 \geq \frac{2n+1}{3} \sum_{k=1}^{n} x_k, \tag{3.3.24}$$

$$\sum_{k=1}^{n} {x_k}^3 \geq \frac{n\,(n+1)}{2} \sum_{k=1}^{n} x_k, \tag{3.3.25}$$

If k, p are positive integers such that $k > p$ then there exist a polynomial $F(k, p, n)$ such that

$$\sum_{i=1}^{n} {x_i}^k \geq F(k, p, n) \sum_{i=1}^{n} {x_i}^p. \tag{3.3.26}$$

Proof: In our paper [19] I have "closed" the first two parts of this problem, and I have reached very close to the solution of the third part. The polynomial seems to be

$$F(k, p, n) = \frac{S_n^k}{S_n^p},$$

where $S_n^i = 1 + 2^i + \cdots + n^i$. ♣

Answer to Open Question 1263.
In [33, OQ.1263, Vol. 11, No.1, April 2003, pp 400] M. Bencze has proposed the next problem:
If $x_i > 0 (i = 1, 2, \dots, n)$ and $\sum_{i=1}^{n} x_i = 1$, then

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{1 - x_{i_1} x_{i_2} \cdots x_{i_k}} \leq \frac{\binom{n}{k}}{1 - \left(\frac{1}{n}\right)^k}. \tag{3.3.27}$$

Proof: A discussion about this problem is presented in [6]. The author has proved that the inequality holds for $k = n$ and does not holds for $k = 1$. ♣

Answer to Open Question 1330.
In [33, OQ.1330, Vol. 11, No.2, October 2003, pp 860] M. Bencze has proposed the next problem:

Let $a \leq x_1 \leq \cdots \leq x_n \leq b$ and $c \leq y_1 \leq \cdots \leq y_n \leq d$ where $c, d \in [a, b]$. Prove that for all convex function $f : [a, b] \to \mathbb{R}$ holds the following inequality:

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k) - f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) \geq \frac{1}{n}\sum_{k=1}^{n} f(y_k) - f\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right). \tag{3.3.28}$$

Proof: In our note [7] we have proved that this inequality does not hold under the given hypothesis. Anyway, the focus of the problem was to generalize the inequality of Niculescu that is obtained for $n = 2$, result that we have presented in Chapter 1.5.4. It seems that the generalization has to have some other form, the condition from above not being strong enough. ♣

Answer to Open Question 1345.
In [33, OQ.1345, Vol. 12, No.1, 2004, pp. 447] M. Bencze has proposed the next problem: Determine all k and n positive integers such that

$$(kn - k + 1)^n + (kn - k + 1)^n + \cdots + (kn)^n < (kn + 1)^n. \tag{3.3.29}$$

Proof: We present just the solution. This is:

$$\{(1, n); n \in \mathbb{N}\} \bigcup \{(2, n); n \in \mathbb{N}, n \geq 3\}.$$

The complete proof is contained in [9]. ♣

Answer to Open Question 1361.
In [33, OQ.1361,Vol. 12, No.1, 2004, pp. 451] M. Bencze has proposed the next problem: If $p_k > 0$, $x_k > 0 (k = 1, 2, \ldots, n)$ and

$$f(\alpha) = \left(\sum_{k=1}^{n} p_k x_k^{\alpha}\right)\left(\sum_{k=1}^{n} \frac{p_k}{x_k^{\alpha}}\right),$$

then $f(\alpha) \geq f(\beta)$ if $\alpha \geq \beta > 0$.
Proof: The solution of this problem is not so difficult and can be found in [12]. ♣

Answer to Open Question 1381.
In [33, OQ.1381,Vol. 12, No.1, 2004, pp. 455] M. Bencze has proposed the next problem: If $\alpha_k \in (0, \pi), p_k > 0 (k = 1, 2, \ldots, n)$ and $\sum_{k=1}^{n} \alpha_k \leq \pi$ then

$$\frac{\sum_{k=1}^{n} p_k \cos \alpha_k}{\sum_{k=1}^{n} p_k} \leq \cos\left(\frac{\sum_{k=1}^{n} p_k \alpha_k}{\sum_{k=1}^{n} p_k}\right). \tag{3.3.30}$$

Proof: In our paper [10] it is proved that such inequality cannot hold even for $k = 2$. ♣

Answer to Open Question 1385.
In [33, OQ.1385,Vol. 12, No.1, 2004, pp. 455] M. Bencze has proposed the next problem:

If $\alpha \geq \beta > 0$ then determine all $x_k > 0 (k = 1, 2, \ldots, n)$ such that

$$\sum_{k=1}^{n} x_k^{\alpha} \geq \left(\sum_{k=1}^{n} x_k^{\alpha-\beta} \right) \sqrt[n]{\prod_{k=1}^{n} x_k^{\beta}}. \tag{3.3.31}$$

Proof: We have proved that the inequality holds for any $x_k > 0$.The proof is an application of the previous mentioned AM-GM and Chebyshev inequalities. It can be found in [11].♣

Answer to Open Question 1489.
In [33, OQ.1489, Vol. 12, No.2, 2004, pp. 1003] M. Bencze has proposed the next problem: If $x_k > 0 (k = 1, 2, \ldots, n)$ then

$$\binom{n}{k} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{x_{i_1} x_{i_2} \cdots x_{i_k}} \geq k^2 \left(\sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{x_{i_1} + x_{i_2} + \cdots + x_{i_k}} \right)^k. \tag{3.3.32}$$

Proof: Considering all the numbers equal to some x, we get that $k \geq \binom{n}{k}$, inequality that holds only for $k \in \{1, n-1\}$. A more detailed argument is contained in [15] ♣

Answer to Open Question 1490.
In [33, OQ.1490, Vol. 12, No.2, 2004, pp. 1003] M. Bencze has proposed the next problem: Find all $x_k > 0 (k = 1, 2, \ldots, n)$ such that for any $p_k > 0 (k = 1, 2, \ldots, n), \sum_{k=1}^{n} = 1$, the following inequality holds:

$$\sum_{k=1}^{n} \frac{p_k}{1 + x_k} \leq \frac{1}{\prod_{k=1}^{n} 1 + x_k^{p_k}}. \tag{3.3.33}$$

Proof: In [16] it is proved that the inequality holds for $x_k \in (0, 1]$, and that the reversed inequality holds for $x_k \in [1, \infty)$. Moreover it is proved that if the numbers are not all in the same hand side of 1, none of the mentioned inequalities holds. ♣

Answer to Open Question 1637.
In [33, OQ.1637, Vol. 12, No.2, 2004, pp. 1034] M. Bencze has proposed the next problem: If $a_i > 0 (i = 1, 2, \ldots, n)$ then

$$k \sum_{i=1}^{n} \frac{1}{(1 + a_i)^k} \geq \frac{1}{1 + a_{i_1} a_{i_2} \cdots a_{i_k}} + \frac{1}{1 + a_{i_2} a_{i_3} \cdots a_{i_{k+1}}} + \cdots + \frac{1}{1 + a_{i_n} a_{i_1} \cdots a_{i_{k-1}}}. \tag{3.3.34}$$

Proof: In the paper [17] it is proved that the inequality holds for $k = 1, 2$ and does not holds for $k \geq 3$. ♣

Some more results of the author can be found in [33].

3.4 On some identities for means in two variables

In this paragraph we present some identities coming from the integral expressions of some means for two variables. The results are directly connected to the main subject of the present work, due to the fact that using these identities we can obtain new inequalities.

3.4.1 Introduction

Let a, b be two positive numbers such that $0 < a < b$. Consider again the means:

$A(a,b) = \frac{a+b}{2}$, the arithmetic mean
$G(a,b) = (ab)^{\frac{1}{2}}$, the geometric mean
$H(a,b) = \frac{2ab}{a+b}$, the harmonic mean
$I(a,b) = \frac{1}{e} \cdot (\frac{b^b}{a^a})^{\frac{1}{b-a}}$, the identric mean, and
$L(a,b) = \frac{b-a}{\ln b - \ln a}$, the logarithmic mean.
$A_t(a,b) = \left(\frac{a^t+b^t}{2}\right)^{\frac{1}{t}}$, for $t > 0$, the power mean of order t for the numbers a and b.

If the reader has read the previous chapters, then this is not the first time when he sees these definitions.
These means are also studied in (see [33],[34]), and also previous, together with some of its properties properties.
We use the integral representations of some of the means.

$$\ln I(a,b) = \frac{1}{b-a} \int_a^b \ln x dx, \tag{3.4.1}$$

$$\frac{1}{L(a,b)} = \frac{1}{b-a} \int_a^b \frac{1}{x} dx, \tag{3.4.2}$$

$$A(a,b) = \frac{1}{b-a} \int_a^b x dx, \tag{3.4.3}$$

$$\frac{1}{G(a,b)^2} = \frac{1}{b-a} \int_a^b \frac{1}{x^2} dx, \tag{3.4.4}$$

$$A_t(a,b) = \left(\frac{t}{b^t - a^t} \int_a^b x^{2t-1} dx\right)^{\frac{1}{t}}, \quad t > 0. \tag{3.4.5}$$

Using the additivity with the interval for the integral, i.e $\int_a^b = \int_a^c + \int_c^b$, for $a < c < b$, (and c properly chosen) in $(3.4.1 - 3.4.5)$ we obtain some identities for means of two variables.

3.4.2 Main results

In this section we obtain some identities connecting the means considered above.
The first theorem gives a relation between the geometrical and the identric mean of two positive numbers.

Theorem 3.4.1 *Consider the positive numbers a, b such that $0 < a < b$. With the definitions mentioned above for the identric and for the logarithmic means, the next identity holds:*

$$I(a,b)^{b-a} = I(a, G(a,b))^{G(a,b)-a} \cdot I(G(a,b), b)^{b-G(a,b)}. \tag{3.4.6}$$

Proof: From $a < \sqrt{ab} < b$ and (3.4.1) we obtain

$$\ln I(a,b) = \frac{1}{b-a}\int_a^b \ln x dx = \frac{1}{b-a}\int_a^{\sqrt{ab}} \ln x dx + \frac{1}{b-a}\int_{\sqrt{ab}}^b \ln x dx,$$

$$\ln I(a,b) = \frac{1}{b-a} \cdot \frac{\sqrt{ab}-a}{\sqrt{ab}-a}\int_a^{\sqrt{ab}} \ln x dx + \frac{1}{b-a} \cdot \frac{b-\sqrt{ab}}{b-\sqrt{ab}}\int_{\sqrt{ab}}^b \ln x dx,$$

so

$$\ln I(a,b) = \frac{\sqrt{ab}-a}{b-a} \cdot \ln I(a, \sqrt{ab}) + \frac{b-\sqrt{ab}}{b-a} \cdot \ln I(\sqrt{ab}, b).$$

This implies

$$(b-a)\ln I(a,b) = (\sqrt{ab}-a)\ln I(a,\sqrt{ab}) + (b-\sqrt{ab})\ln I(b-\sqrt{ab}).$$

Finally this leads to

$$I(a,b)^{b-a} = I(a,\sqrt{ab})^{\sqrt{ab}-a} \cdot I(\sqrt{ab}, b)^{b-\sqrt{ab}}.$$

Clearly this is (3.4.6) $\square$.
The following theorem gives a relation between the geometrical, arithmetical and the identric mean of two positive numbers.

Theorem 3.4.2 *Consider the positive numbers a, b such that $0 < a < b$. Then with the notations mentioned above, we have that the next identity holds:*

$$I(a,b) = G\left(I(a, A(a,b)), I(A(a,b), b)\right). \tag{3.4.7}$$

Proof: From $a < \frac{a+b}{2} < b$ and (3.4.1) we obtain

$$\frac{1}{b-a}\int_a^b \ln x dx = \frac{\frac{a+b}{2}-a}{b-a} \cdot \frac{1}{\frac{a+b}{2}-a}\int_a^{\frac{a+b}{2}} \ln x dx + \frac{b-\frac{a+b}{2}}{b-a} \cdot \frac{1}{b-\frac{a+b}{2}}\int_{\frac{a+b}{2}}^b \ln x dx,$$

so we have

$$(b-a)\ln I(a,b) = \left(\frac{a+b}{2}-a\right)\ln I\left(a, \frac{a+b}{2}\right) + \left(b-\frac{a+b}{2}\right)\ln I\left(b-\frac{a+b}{2}\right),$$

which implies

$$I\left(a,b\right)^{b-a}=I\left(a,\frac{a+b}{2}\right)^{\frac{b-a}{2}}\cdot I\left(\frac{a+b}{2},b\right)^{\frac{b-a}{2}}.$$

Finally this is equivalent to:

$$I(a,b)^2=I\left(a,\frac{a+b}{2}\right)\cdot I\left(\frac{a+b}{2},b\right).$$

By applying the square root in the last expression, we get (3.4.7). □.
For this Theorem, we obtain some nice results for some certain values of a and b.

Proposition 3.4.1 *If $b=a+2$ we have that:*

$$I(a,a+2)^2=I(a,a+1)\cdot I(a+1,a+2).(a>0)$$

Proposition 3.4.2 *If $b=a+1$ we have*

$$I(a,a+1)^2=I(a,a+\frac{1}{2})\cdot I(a+\frac{1}{2},a+1).(a>0)$$

Proposition 3.4.3 *If $b=(2n-1)a$ we have that:*

$$I(a,(2n-1)a)^2=I(a,na)\cdot I(na,(2n-1)a).(a>0)$$

As a final application we can give the next result.

Proposition 3.4.4 *Consider that $b=a+2^n$. Denote by $I_i(a)=I(a+i-1,a+i)$. Then the next identity holds:*

$$I(a,a+2^n)^{2^{n-1}}=\prod_{1\le i\le 2^n} I_i(a).$$

The proof follows easily by the induction and it is let to the reader as an exercise.
The following theorem gives a relation between the logarithmic, harmonic and the arithmetical mean of two positive numbers.

Theorem 3.4.3 *Considering the positive numbers a,b such that $0<a<b$, the next identity holds:*

$$L(a,b)=H^{-1}\left(L(a,A(a,b)),L(A(a,b),b)\right). \tag{3.4.8}$$

We mention that we have considered that

$$H^{-1}(a,b)=\frac{1}{H(a,b)}.$$

Proof: From $a < \frac{a+b}{2} < b$ and (3.4.1) we obtain

$$\frac{1}{b-a}\int_a^b \frac{1}{x}dx = \frac{\frac{a+b}{2}-a}{b-a}\cdot\frac{1}{\frac{a+b}{2}-a}\int_a^{\frac{a+b}{2}}\frac{1}{x}dx + \frac{b-\frac{a+b}{2}}{b-a}\cdot\frac{1}{b-\frac{a+b}{2}}\int_{\frac{a+b}{2}}^b \frac{1}{x}dx.$$

This means that

$$\frac{1}{L(a,b)} = \frac{1}{2}\cdot\frac{1}{L(a,\frac{a+b}{2})} + \frac{1}{2}\cdot\frac{1}{L(\frac{a+b}{2},b)}.$$

From here follows clearly that

$$L(a,b) = H^{-1}\left(L(a,A(a,b)), L(A(a,b),b)\right).$$

$\square$.

The following theorem gives a relation between the geometrical, harmonic and the arithmetical mean of two positive numbers.

Theorem 3.4.4 *Consider the positive numbers a, b such that $0 < a < b$. Then the next identity holds:*

$$G^2(a,b) = H^{-1}\left(G^2(a,A(a,b)), G^2(A(a,b),b)\right). \tag{3.4.9}$$

The notations are the ones of the previous theorem.

Proof: From $a < \frac{a+b}{2} < b$ and (3.4.4) we obtain that

$$\frac{1}{b-a}\int_a^b \frac{1}{x^2}dx = \frac{\frac{a+b}{2}-a}{b-a}\cdot\frac{1}{\frac{a+b}{2}-a}\int_a^{\frac{a+b}{2}}\frac{1}{x^2}dx + \frac{b-\frac{a+b}{2}}{b-a}\cdot\frac{1}{b-\frac{a+b}{2}}\int_{\frac{a+b}{2}}^b \frac{1}{x^2}dx.$$

This means that

$$\frac{1}{G^2(a,b)} = \frac{1}{2}\cdot\frac{1}{G^2(a,\frac{a+b}{2})} + \frac{1}{2}\cdot\frac{1}{G^2(\frac{a+b}{2},b)}.$$

From here follows clearly that

$$G^2(a,b) = H^{-1}\left(G^2(a,A(a,b)), G^2(A(a,b),b)\right).$$

$\square$.

The following theorem gives a kind of iteration for the power mean of a, b and power $t > 0$.

Theorem 3.4.5 *Consider the positive numbers a, b such that $0 < a < b$. Then the next identity holds:*

$$A_t(a,b) = A_t\left(A_t(a,A_t(a,b)), A_t(A_t(a,b),b)\right). \tag{3.4.10}$$

Observation: For $t = 1$ we get $A_1(a, b) = A(a, b)$. For $t = 2$ we get $A_2(a, b) = \left(\frac{a^2+b^2}{2}\right)^{\frac{1}{2}}$ which is named the quadratic mean, another well known mean. It is well known that for fixed positive numbers a, b and real t, the function $t \longmapsto A_t(a, b)$ is increasing, with equality if an only if $a = b$. (see [34] or [41]).

Proof: Because $a < A_t(a, b) < b$ and the relation (3.4.5), we obtain:

$$[A_t(a,b)]^t = \frac{t}{b^t - a^t} \int_a^b x^{2t-1} dx$$

$$= \frac{[A_t(a,b)]^t - a^t}{b^t - a^t} \cdot \frac{t}{[A_t(a,b)]^t - a^t} \int_a^{A_t(a,b)} x^{2t-1} dx + \frac{b^t - [A_t(a,b)]^t}{b^t - a^t} \cdot \frac{t}{b^t - [A_t(a,b)]^t} \int_{A_t(a,b)}^b x^{2t-1} dx.$$

But this gives that:

$$[A_t(a,b)]^t = \frac{1}{2} \cdot \left[A_t(a, A_t(a,b))^t + A_t(A_t(a,b), b)^t\right].$$

By the definition of $A_t(a, b)$ if follows easily that

$$A_t(a,b) = A_t\left(A_t(a, A_t(a,b)), A_t(A_(a,b), b)\right).$$

This ends the proof. □.

Observation: This chapter represents the paper [20] that was written in collaboration with Z. Starc from Serbia and was accepted to Elem.Math., Baia Mare.

3.5 Inequalities from Monthly

In this chapter we present some solutions to problems from American Mathematical Monthly thave have been obtained by the author during the year 2006. These problems are from the area of the Inequalities. Some of the problems also have received some generalizations.

$[11177, 749]$. *Proposed by Dorin Marghidanu, Colegiul National "A.I.Cuza", Corabia, Romania.* Let $S_k = \sum_{i=1}^k j = \binom{k+1}{2}$. Prove that

$$1 + \sum_{k=2}^{n} \frac{\left(\prod_{j=1}^k j^j\right)^{\frac{1}{S_k}}}{\sum_{i=1}^k j^2} \leq \frac{2n}{n+1}.$$

Solution by Ovidiu Bagdasar, Babes Bolyai University, Cluj Napoca, Romania. By the AM-GM inequality, we have that

$$\left(\prod_{j=1}^{k} j^j\right)^{\frac{1}{S_k}} \leq \frac{1\cdot 1 + 2\cdot 2 + \cdots + k\cdot k}{S_k}.$$

Then

$$\frac{\left(\prod_{j=1}^k j^j\right)^{\frac{1}{S_k}}}{\sum_{i=1}^k j^2} \leq \frac{1}{S_k} = \frac{2}{k} - \frac{2}{k+1}.$$

Finally by summing, we clearly get that

$$1 + \sum_{k=2}^{n} \frac{\left(\prod_{j=1}^k j^j\right)^{\frac{1}{S_k}}}{\sum_{i=1}^k j^2} \leq 1 + \sum_{k=2}^{n}\left(\frac{2}{k} - \frac{2}{k+1}\right) = 2 - \frac{2}{n+1} = \frac{2n}{n+1}.$$

Remark. Because in the AM-GM inequality the equality holds iff all the numbers are equal, we have in fact that for $n \geq 2$, the initial inequality holds strictly.
Generalization. Let $(a_n)_{n\geq 1}$ be an arithmetical progression of general term $a = 0$ and ratio $r > 0$. Let $S_k = \sum_{i=1}^k a_j$. Then the next inequality holds:

$$\frac{1}{r} + \sum_{k=2}^{n} \frac{\left(\prod_{j=1}^k a_j^{a_j}\right)^{\frac{1}{S_k}}}{\sum_{i=1}^k a_j^2} \leq \frac{1}{r}\cdot\frac{2n}{n+1}.$$

The proof goes in the same way. Clearly considering $r = 1$ in the above inequality, we obtain the initial problem. $\square$

[11189.Dec2005] *Proposed by Lajos Csete, Markotabodoge, Hungary.* Let $a_1, \ldots, a_n$ be positive real numbers, let $a_1 = a_{n+1}$, and let p be a real number greater than 1. Prove that:

$$\sum_{k=1}^{n} \frac{a_k^p}{a_k + a_{k+1}} \geq \frac{1}{2}p \sum_{k=1}^{n} a_k - \frac{p-1}{2^{p/(p-1)}} \sum_{k=1}^{n} (a_k + a_{k+1})^{1/(p-1)}.$$

Solution by Ovidiu Bagdasar, Babes Bolyai University, Cluj Napoca, Romania. We apply the pondered version of the AM-GM inequality for two numbers, that states that for $0 \leq \lambda \leq 1$ and positive numbers x, y we have

$$\lambda \cdot x + (1-\lambda) \cdot y \geq x^{\lambda} \cdot y^{1-\lambda}. \tag{3.5.1}$$

Considering $\lambda = \frac{1}{p}$, $x = \frac{a_k^p}{(a_k + a_{k+1})}$,$y = \left(\frac{a_k + a_{k+1}}{2^p}\right)^{1/(p-1)}$, we apply the inequality (3.5.1) obtaining that:

$$\frac{1}{p} \cdot \frac{a_k^p}{a_k + a_{k+1}} + \frac{p-1}{p} \cdot \left(\frac{a_k + a_{k+1}}{2^p}\right)^{1/p-1} \geq$$

$$\geq \frac{a_k}{(a_k + a_{k+1})^{1/p}} \cdot \left(\frac{a_k + a_{k+1}}{2^p}\right)^{1/p} = \frac{a_k}{2}.$$

Clearly, by summing after $k = 1, \ldots, n$ and multiplying by p, we get the desired inequality.

Remark. We could apply the inequality (3.5.1), because $p > 1$ implies that $0 < \lambda = \frac{1}{p} < 1$. One can see that we can obtain many generalizations of the problem even by considering anything positive instead of $a_k + a_{k+1}$, the problem keeping its general aspect. $\square$

[11193, $Dec.$2005] *Proposed by Marian Tetiva, Bârlad, Romania.* Let $a_1, \ldots, a_n$ be positive real numbers. Let $a_1 = a_{n+1}$. Prove that:

$$\sum_{k=1}^{n} \left(\frac{a_k}{a_{k+1}}\right)^{n-1} \geq -n + 2 \left(\sum_{k=1}^{n} a_k\right) \prod_{k=1}^{n} a_k^{-1/n}.$$

Solution by Ovidiu Bagdasar, Babes Bolyai University, Cluj Napoca, Romania. By using the AM-GM inequality we obtain the next inequalities:

$$\left(\frac{a_1}{a_2}\right)^{n-1} \geq \left(\frac{a_1}{a_2}\right)^{n-1},$$

$$\left(\frac{a_1}{a_2}\right)^{n-1} + \left(\frac{a_2}{a_3}\right)^{n-1} \geq 2 \cdot \left(\frac{a_1}{a_3}\right)^{\frac{n-1}{2}},$$

$$\left(\frac{a_1}{a_2}\right)^{n-1} + \left(\frac{a_2}{a_3}\right)^{n-1} + \cdots + \left(\frac{a_{n-1}}{a_n}\right)^{n-1} \geq (n-1) \cdot \left(\frac{a_1}{a_n}\right)^{\frac{n-1}{n-1}}. \tag{3.5.2}$$

Note at this point that in the left hand side we have used $\frac{n(n-1)}{2}$ terms. Denote here by $G = \prod_{k=1}^{n} a_k^{1/n}$.
Now we apply the pondered version of the AM-GM inequality for the members from the right and get:

$$\left(\frac{a_1}{a_2}\right)^{n-1} + 2 \cdot \left(\frac{a_1}{a_3}\right)^{\frac{n-1}{2}} + \cdots + (n-1) \cdot \left(\frac{a_1}{a_n}\right)^{\frac{n-1}{n-1}} \geq$$

$$\geq \frac{n(n-1)}{2} \cdot \sqrt[n(n-1)/2]{\left(\frac{a_1}{a_2}\right)^{n-1} \cdot \left(\frac{a_1}{a_3}\right)^{2\frac{n-1}{2}} \cdots \left(\frac{a_1}{a_n}\right)^{(n-1)\frac{n-1}{n-1}}}$$

$$= \frac{n(n-1)}{2} \sqrt[n(n-1)/2]{\left(\frac{a_1}{a_1}\right)^{n-1} \cdot \left(\frac{a_1}{a_2}\right)^{n-1} \cdot \left(\frac{a_1}{a_3}\right)^{n-1} \cdots \left(\frac{a_1}{a_n}\right)^{n-1}} = \frac{n(n-1)}{2} \left(\frac{a_1}{G}\right)^2.$$

Considering $k = 2, \ldots, n$ we write again the inequalities (3.5.2) (by cyclic permutations). By summing we clearly get that:

$$\frac{n(n-1)}{2} \sum_{k=1}^{n} \left(\frac{a_k}{a_{k+1}}\right)^{n-1} \geq \frac{n(n-1)}{2} \sum_{k=1}^{n} \left(\frac{a_1}{G}\right)^2.$$

This gives

$$\sum_{k=1}^{n} \left(\frac{a_k}{a_{k+1}}\right)^{n-1} \geq \sum_{k=1}^{n} \left(\frac{a_k}{G}\right)^2. \tag{3.5.3}$$

Using the inequality $1 + x^2 \geq 2x$ for $x = \frac{a_k}{G}$ and summing for $k = 1, \ldots, n$ we obtain that

$$\sum_{k=1}^{n} \left(\frac{a_k}{G}\right)^2 \geq -n + 2\left(\sum_{k=1}^{n} a_k\right) \frac{1}{G}. \tag{3.5.4}$$

By (3.5.3) and (3.5.4) we get the desired inequality.

Remark. We give a form for the pondered version of the AM-GM mean. Considering $\lambda_1, \ldots, \lambda_n$ nonnegative numbers such that $\sum_{k=1}^{n} \lambda_k = 1$, and the positive numbers $x_1, \ldots, x_n$ the inequality $\sum_{k=1}^{n} \lambda_k \cdot x_k \geq \prod_{k=1}^{n} x_k^{\lambda_k}$, holds. (This is 1.1.3) In our proof we have used $\lambda_k = \frac{k}{n(n-1)/2}$. □

$[11197, 79, Jan2006]$. *Proposed by Oleg Faynshteyn, Leipzig, Germany* Let x, y and z be positive real numbers satisfying $x^2 + y^2 + z^2 = 1$, and let n be a positive integer. Show that

$$\frac{x}{1 - x^{2n}} + \frac{x}{1 - x^{2n}} + \frac{x}{1 - x^{2n}} \geq \frac{(2n+1)^{1+1/2n}}{2n}.$$

Solution by Ovidiu Bagdasar, Babes Bolyai University, Cluj Napoca, Romania. We prove that

$$\frac{x}{1-x^{2n}} + \frac{x}{1-x^{2n}} + \frac{x}{1-x^{2n}} \geq \frac{(2n+1)^{1+1/2n}}{2n} \cdot \left(x^2+y^2+z^2\right),$$

by showing that

$$\frac{x}{1-x^{2n}} \geq \frac{(2n+1)^{1+1/2n}}{2n} \cdot x^2. \tag{3.5.5}$$

Clearly, by summing the other inequalities for y, z we get the required result.

To prove (3.5.5), we apply the changing of variable: $a = x\sqrt[2n]{2n+1}$. The inequality (3.5.5) becomes: $\frac{1}{1-\frac{a^{2n}}{2n+1}} \geq \frac{(2n+1)a}{2n}$. This is

$$a^{2n+1} - (2n+1)a + 2n \geq 0. \tag{3.5.6}$$

Because $a^{2n+1} - (2n+1)a + 2n = (a-1)[a^{2n} + \cdots + a - 2n] = (a-1)^2[a^{2n-1} + 2a^{2n-2} + \cdots + 2n] \geq 0$, it follows that (3.5.5) holds, and this ends the proof. $\square$

Observation: These solutions are for problems published in the last numbers of American Mathematical Monthly, so they have to appear soon.

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