Ostrowski Type Inequalities and Applications in Numerical Integration

Edited By:

Sever S. Dragomir

and

Themistocles M. Rassias

(S.S. Dragomir) School and Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City, MC 8001, Victoria, Australia

E-mail address, S.S. Dragomir: sever@matilda.vu.edu.au

URL: http://rgmia.vu.edu.au/SSDragomirWeb.html

(T.M. Rassias) Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece

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Abstract. Ostrowski type inequalities for univariate and multivariate real functions and their natural applications for numerical quadratures are presented.

Contents

Preface

It was noted in the preface of the book "*Inequalities Involving Functions and Their Integrals and Derivatives",* Kluwer Academic Publishers, 1991, by D.S. Mitrinović, J.E. Pečarić and A.M. Fink; since the writing of the classical book by Hardy, Littlewood and Polya (1934), the subject of differential and integral inequalities has grown by about 800%. Ten years on, we can confidently assert that this growth will increase even more significantly. Twenty pages of Chapter XV in the above mentioned book are devoted to integral inequalities involving functions with bounded derivatives, or, Ostrowski type inequalities. This is now itself a special domain of the Theory of Inequalities with many powerful results and a large number of applications in Numerical Integration, Probability Theory and Statistics, Information Theory and Integral Operator Theory.

The main aim of this present book, jointly written by the members of the Victoria University node of RGMIA (*Research Group in Mathematical Inequalities and Applications*, `http://rgmia.vu.edu.au`), is to present a selected number of results on Ostrowski type inequalities. Results for univariate and multivariate real functions and their natural applications in the error analysis of numerical quadrature for both simple and multiple integrals as well as for the Riemann-Stieltjes integral are given.

In Chapter 1, authored by S.S. Dragomir and T.M. Rassias, generalisations of the Ostrowski integral inequality for mappings of bounded variation and for absolutely continuous functions via kernels with $n-$branches including applications for general quadrature formulae, are given.

Chapter 2, authored by A. Sofo, builds on the work in Chapter 1. He investigates generalisations of integral inequalities for n-times differentiable mappings. With the aid of the modern theory of inequalities and by use of a general Peano kernel, explicit bounds for interior point rules are obtained. Firstly, he develops integral equalities which are then used to obtain inequalities for n-times differentiable mappings on the Lebesgue spaces $L_\infty[a, b]$, $L_p\,[a, b]$, $1 < p < \infty$ and $L_1[a, b]$. Secondly, some particular inequalities are obtained which include explicit bounds for perturbed trapezoid, midpoint, Simpson's, Newton-Cotes, left and right rectangle rules. Finally, inequalities are also applied to various composite quadrature rules and the analysis allows the determination of the partition required for the accuracy of the result to be within a prescribed error tolerance.

In Chapter 3, authored by P. Cerone and S.S. Dragomir, a unified treatment of three point quadrature rules is presented in which the classical rules of mid-point, trapezoidal and Simpson type are recaptured as particular cases. Riemann integrals are approximated for the derivative of the integrand belonging to a variety of norms. The Grüss inequality and a number of variants are also presented which provide a variety of inequalities that are suitable for numerical implementation. Mappings that are of bounded total variation, Lipschitzian and monotonic are also investigated with relation to Riemann-Stieltjes integrals. Explicit *a priori* bounds are provided allowing the determination of the partition required to achieve a prescribed error tolerance.

It is demonstrated that with the above classes of functions, the average of a midpoint and trapezoidal type rule produces the best bounds.

In Chapter 4, authored by P. Cerone, product branches of Peano kernels are used to obtain results suitable for numerical integration. In particular, identities and inequalities are obtained involving evaluations at an interior and at the end points. It is shown how previous work and rules in numerical integration are recaptured as particular instances of the current development. Explicit *a priori* bounds are provided allowing the determination of the partition required for achieving a prescribed error tolerance. In the main, Ostrowski-Grüss type inequalities are used to obtain bounds on the rules in terms of a variety of norms.

In Chapter 5, authored by N.S. Barnett, P. Cerone and S.S. Dragomir, new results for Ostrowski type inequalities for double and multiple integrals and their applications for cubature formulae are presented. This work is then continued in Chapter 6, authored by G. Hanna, where an Ostrowski type inequality in two dimensions for double integrals on a rectangle region is developed. The resulting integral inequalities are evaluated for the class of functions with bounded first derivative. They are employed to approximate the double integral by one dimensional integrals and function evaluations using different types of norms. If the one-dimensional integrals are not known, they themselves can be approximated by using a suitable rule, to produce a cubature rule consisting only of sampling points.

In addition, some generalisations of an Ostrowski type inequality in two dimensions for n - time differentiable mappings are given. The result is an integral inequality with bounded n - time derivatives. This is employed to approximate double integrals using one dimensional integrals and function evaluations at the boundary and interior points.

In Chapter 7, authored by John Roumeliotis, weighted quadrature rules are investigated. The results are valid for general weight functions. The robustness of the bounds is explored for specific weight functions and for a variety of integrands. A comparison of the current development is made with traditional quadrature rules and it is demonstrated that the current development has some advantages. In particular, this method allows the nodes and weights of an n point rule to be easily obtained, which may be preferential if the region of integration varies. Other explicit error bounds may be obtained in advance, thus making it possible to determine the weight dependent partition required to achieve a certain error tolerance.

In the last chapter, S.S. Dragomir presents recent results in approximating the Riemann-Stieltjes integral by the use of Trapezoid type, Ostrowski type and Grüss type inequalities. Applications for certain classes of weighted integrals are also given.

This book is intended for use in the fields of integral inequalities, approximation theory, applied mathematics, probability theory and statistics and numerical analysis.

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CHAPTER 1

Generalisations of Ostrowski Inequality and Applications

by

S.S. DRAGOMIR and T.M. RASSIAS

ABSTRACT Generalizations of Ostrowski integral inequality for mappings of bounded variation and for absolutely continuous functions via kernels with $n-$branches plus applications for general quadrature formulae are given.

1.1. Introduction

The following result is known in the literature as Ostrowski's inequality (see for example [**22**, p. 468]).

THEOREM 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)|\leq M$ for all $t\in(a,b)$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)M \tag{1.1}$$

for all $x\in[a,b]$.
The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

A simple proof of this fact can be done by using the identity:

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{b-a}\int_a^b p(x,t)\,f'(t)\,dt,\quad x\in[a,b], \tag{1.2}$$

where

$$p(x,t):=\begin{cases} t-a & \text{if} \quad a\leq t\leq x \\ t-b & \text{if} \quad x<t\leq b \end{cases}$$

which also holds for absolutely continuous functions $f:[a,b]\to\mathbb{R}$.

The following Ostrowski type result for absolutely continuous functions holds (see [**17**], [**20**] and [**18**]).

Theorem 1.2. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $x\in[a,b]$, we have:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.3}$$

$$\leq\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left\|f'\right\|_\infty & \text{if } f'\in L_\infty[a,b];\\[2ex] \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{x-a}{b-a}\right)^{p+1}+\left(\frac{b-x}{b-a}\right)^{p+1}\right]^{\frac{1}{p}}(b-a)^{\frac{1}{p}}\left\|f'\right\|_q & \text{if } f'\in L_q[a,b],\\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1;\\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left\|f'\right\|_1; & \end{cases}$$

where $\|\cdot\|_r$ $(r\in[1,\infty])$ are the usual Lebesgue norms on $L_r[a,b]$, i.e.,

$$\|g\|_\infty := ess\sup_{t\in[a,b]}|g(t)|$$

and

$$\|g\|_r := \left(\int_a^b |g(t)|^r\,dt\right)^{\frac{1}{r}},\ r\in[1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented in Theorem 1.1.

The above inequalities can also be obtained from the Fink result in [**21**] on choosing $n=1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see [**15**])

Theorem 1.3. *Let $f:[a,b]\to\mathbb{R}$ be of $r-H-$Hölder type, i.e.,*

$$|f(x)-f(y)|\leq H|x-y|^r,\ \textit{for all }\ x,y\in[a,b], \tag{1.4}$$

where $r\in(0,1]$ and $H>0$ are fixed. Then for all $x\in[a,b]$ we have the inequality:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.5}$$

$$\leq \frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1}+\left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r.$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r=1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) due to S.S. Dragomir ([**13**], see also [**2**])

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)L. \tag{1.6}$$

Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result due to Dragomir [**11**] may be stated (see also [**14**] or [**2**]).

THEOREM 1.4. *Assume that* $f : [a,b] \to \mathbb{R}$ *is of bounded variation and denote by* $\bigvee_a^b$ *its total variation. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f) \tag{1.7}$$

for all $x \in [a,b]$.
The constant $\frac{1}{2}$ *is the best possible.*

If we assume more about f, i.e., f is monotonically increasing, then the inequality (1.7) may be improved in the following manner [**12**] (see also [**2**]).

THEOREM 1.5. *Let* $f : [a,b] \to \mathbb{R}$ *be monotonic nondecreasing. Then for all* $x \in [a,b]$*, we have the inequality:*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \\ \leq\ & \frac{1}{b-a} \left\{ [2x - (a+b)]\, f(x) + \int_a^b sgn(t-x)\, f(t)\,dt \right\} \\ \leq\ & \frac{1}{b-a} \left\{ (x-a)[f(x) - f(a)] + (b-x)[f(b) - f(x)] \right\} \\ \leq\ & \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] [f(b) - f(a)]. \end{aligned} \tag{1.8}$$

All the inequalities in (1.8) are sharp and the constant $\frac{1}{2}$ *is the best possible.*

For a recent generalisation of this result see [**16**] where further extensions were given.

The main aim of the present chapter is to provide a number of generalisations for kernels with $N-$branches of the above Ostrowski type inequality. Natural applications for quadrature formulae are also given.

1.2. Generalisations for Functions of Bounded Variation

1.2.1. Some Inequalities.

We start with the following theorem [**14**].

THEOREM 1.6. *Let* $I_k : a = x_0 < x_1 < ... < x_{k-1} < x_k = b$ *be a division of the interval* $[a,b]$ *and* α_i $(i = 0, ..., k+1)$ *be "*$k+2$*" points so that* $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, ..., k)$ *and* $\alpha_{k+1} = b$. *If* $f : [a,b] \to \mathbb{R}$ *is of bounded variation on* $[a,b]$*, then* *we have the inequality:*

$$
\begin{aligned}
(1.9) \qquad & \left| \int_a^b f(x)\,dx - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \\
& \le \left[\frac{1}{2}\nu(h) + \max\left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right|, i = 0, ..., k-1 \right\} \right] \bigvee_a^b (f) \\
& \le \nu(h) \bigvee_a^b (f),
\end{aligned}
$$

where $\nu(h) := \max\{h_i | i = 0, ..., k-1\}$, $h_i := x_{i+1} - x_i$ $(i = 0, ..., k-1)$ *and* $\bigvee_a^b (f)$ *is the total variation of* f *on the interval* $[a, b]$.

Proof. Define the kernel $K : [a, b] \to \mathbb{R}$ given by (see also [**14**])

$$
K(t) := \begin{cases} t - \alpha_1, t \in [a, x_1) \\ t - \alpha_2, t \in [x_1, x_2) \\ \dots\dots\dots\dots\dots\dots \\ t - \alpha_{k-1}, t \in [x_{k-2}, x_{k-1}) \\ t - \alpha_k, t \in [x_{k-1}, b]. \end{cases}
$$

Integrating by parts in Riemann-Stieltjes integral, we have successively

$$
\begin{aligned}
\int_a^b K(t)\,df(t) &= \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} K(t)\,df(t) = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} (t - \alpha_{i+1})\,df(t) \\
&= \sum_{i=0}^{k-1} \left[(t - \alpha_{i+1}) f(t) \Big|_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} f(t)\,dt \right] \\
&= \sum_{i=0}^{k-1} \left[(\alpha_{i+1} - x_i) f(x_i) + (x_{i+1} - \alpha_{i+1}) f(x_{i+1}) \right] - \int_a^b f(t)\,dt \\
&= (\alpha_1 - a) f(a) + \sum_{i=1}^{k-1} (\alpha_{i+1} - x_i) f(x_i) \\
&\quad + \sum_{i=0}^{k-2} (x_{i+1} - \alpha_{i+1}) f(x_{i+1}) + (b - \alpha_n) f(b) - \int_a^b f(t)\,dt
\end{aligned}
$$

$$
\begin{aligned}
&= (\alpha_1 - a) f(a) + \sum_{i=1}^{k-1} (\alpha_{i+1} - x_i) f(x_i) + \sum_{i=1}^{k-1} (x_i - \alpha_i) f(x_i) \\
&\quad + (b - \alpha_n) f(b) - \int_a^b f(t)\,dt \\
&= (\alpha_1 - a) f(a) + \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i) f(x_i) + (b - \alpha_n) f(b) - \int_a^b f(t)\,dt \\
&= \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_{i+1}) - \int_a^b f(t)\,dt,
\end{aligned}
$$

and then we have the integral equality which is of interest in itself too:

$$\int_a^b f(t)\,dt = \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) - \int_a^b K(t)\,df(t). \tag{1.10}$$

Using the modulus' properties, we have

$$\begin{aligned}\left|\int_a^b K(t)\,df(t)\right| &= \left|\sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} K(t)\,df(t)\right| \\ &\le \sum_{i=0}^{k-1} \left|\int_{x_i}^{x_{i+1}} K(t)\,df(t)\right| := T.\end{aligned}$$

However,

$$\begin{aligned}\left|\int_{x_i}^{x_{i+1}} (t-\alpha_{i+1}) f(t)\,dt\right| &\le \sup_{t\in[x_i,x_{i+1}]} |t-\alpha_{i+1}| \bigvee_{x_i}^{x_{i+1}}(f) \\ &= \max\{\alpha_{i+1}-x_i, x_{i+1}-\alpha_{i+1}\} \bigvee_{x_i}^{x_{i+1}}(f) \\ &= \left[\frac{1}{2}(x_{i+1}-x_i) + \left|\alpha_{i+1} - \frac{x_i+x_{i+1}}{2}\right|\right] \bigvee_{x_i}^{x_{i+1}}(f)dt.\end{aligned}$$

Then

$$\begin{aligned}T &\le \sum_{i=0}^{k-1}\left[\frac{1}{2}h_i + \left|\alpha_{i+1} - \frac{x_i+x_{i+1}}{2}\right|\right] \bigvee_{x_i}^{x_{i+1}}(f) \\ &\le \max_{i=0,\dots,k-1}\left[\frac{1}{2}h_i + \left|\alpha_{i+1} - \frac{x_i+x_{i+1}}{2}\right|\right] \sum_{i=0}^{k-1}\bigvee_{x_i}^{x_{i+1}}(f) \\ &\le \left[\frac{1}{2}\nu(h) + \max\left\{\left|\alpha_{i+1} - \frac{x_i+x_{i+1}}{2}\right|, i=0,\dots,k-1\right\}\right] \bigvee_a^b(f) =: V.\end{aligned}$$

Now, as

$$\left|\alpha_{i+1} - \frac{x_i+x_{i+1}}{2}\right| \le \frac{1}{2}h_i,$$

then

$$\max\left\{\left|\alpha_{i+1} - \frac{x_i+x_{i+1}}{2}\right|, i=0,\dots,k-1\right\} \le \frac{1}{2}\nu(h)$$

and, consequently,

$$V \le \nu(h) \bigvee_a^b(f).$$

The theorem is completely proved. ∎

Now, if we assume that the points of the division I_k are given, then the best inequality we can obtain from Theorem 1.6 is embodied in the following corollary:

Corollary 1.7. Let f and I_k be as above. Then we have the inequality:

$$\begin{aligned} &\left|\int_a^b f(x)\,dx - \frac{1}{2}\left[(x_1-a)f(a) \right.\right. \\ &\left.\left. + \sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})f(x_i) + (b-x_{k-1})f(b)\right]\right| \\ \leq\ & \frac{1}{2}\nu(h)\bigvee_a^b(f). \end{aligned} \tag{1.11}$$

Proof. We choose in Theorem 1.6,

$$\begin{aligned} \alpha_0 &= a, \alpha_1 = \frac{a+x_1}{2}, \alpha_2 = \frac{x_1+x_2}{2}, ..., \\ \alpha_{k-1} &= \frac{x_{k-2}+x_{k-1}}{2}, \alpha_k = \frac{x_{k-1}+x_k}{2} \text{ and } \alpha_{k+1} = b. \end{aligned}$$

In this case we get

$$\begin{aligned} & \sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i) \\ =\ & (\alpha_1-\alpha_0)f(a) + (\alpha_2-\alpha_1)f(x_1) + \cdots \\ & + (\alpha_k-\alpha_{k-1})f(x_{k-1}) + (b-\alpha_k)f(b) \\ =\ & \left(\frac{a+x_1}{2}-a\right)f(a) + \left(\frac{x_1+x_2}{2}-\frac{a+x_1}{2}\right)f(x_1) \\ & + \cdots + \left(\frac{x_{k-1}+b}{2}-\frac{x_{k-2}+x_{k-1}}{2}\right)f(x_{k-1}) + \left(b-\frac{x_{k-1}+b}{2}\right)f(b) \\ =\ & \frac{1}{2}\left[(x_1-a)f(a) + \sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})f(x_i) + (b-x_{k-1})f(b)\right]. \end{aligned}$$

Now, applying the inequality (1.9), we get (1.11). ∎

The following corollary for equidistant partitioning also holds.

Corollary 1.8. Let

$$I_k : x_i := a + (b-a)\frac{i}{k}\,(i = 0, ..., k)$$

be an equidistant partitioning of $[a, b]$. If f is as above, then we have the inequality:

$$\begin{aligned} &\left|\int_a^b f(x)\,dx - \left[\frac{1}{k}\cdot\frac{f(a)+f(b)}{2}(b-a) + \frac{(b-a)}{k}\sum_{i=1}^{k-1} f\left[\frac{(k-i)a+ib}{k}\right]\right]\right| \\ \leq\ & \frac{1}{2k}(b-a)\bigvee_a^b(f). \end{aligned} \tag{1.12}$$

1.2.2. A General Quadrature Formula. Let $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ be a sequence of division of $[a,b]$ and consider the sequence of numerical integration formulae

$$I_n(f,\Delta_n,w_n) := \sum_{j=0}^{n} w_j^{(n)} f\left(x_j^{(n)}\right)$$

where $w_j^{(n)}$ $(j = 0,...,n)$ are the quadrature weights and $\sum_{j=0}^{n} w_j^{(n)} = b - a$.

The following theorem provides a sufficient condition for the weights $w_j^{(n)}$ so that $I_n(f,\Delta_n,w_n)$ approximates the integral $\int_a^b f(x)\,dx$ (see also [**14**]).

THEOREM 1.9. *Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. If the quadrature weights $w_j^{(n)}$ satisfy the condition*

$$x_i^{(n)} - a \le \sum_{j=0}^{i} w_j^{(n)} \le x_{i+1}^{(n)} - a \text{ for all } i = 0,...,n-1, \tag{1.13}$$

then we have the estimate

$$\begin{aligned} &\left| I_n(f,\Delta_n,w_n) - \int_a^b f(x)\,dx \right| \\ \le\; &\left[\frac{1}{2}\nu\left(h^{(n)}\right) + \max\left\{ \left| a + \sum_{j=0}^{i} w_j^{(n)} - \frac{x_i^{(n)} + x_{i+1}^{(n)}}{2} \right|, i = 0,...,n-1 \right\} \right] \bigvee_a^b (f) \\ \le\; &\nu\left(h^{(n)}\right) \bigvee_a^b (f), \end{aligned} \tag{1.14}$$

where $\nu\left(h^{(n)}\right) := \max\left\{ h_i^{(n)} | i = 0,...,n-1 \right\}$ and $h_i^{(n)} := x_{i+1}^{(n)} - x_i^{(n)}$.
In particular,

$$\lim_{\nu\left(h^{(n)}\right)\to 0} I_n(f,\Delta_n,w_n) = \int_a^b f(x)\,dx \tag{1.15}$$

uniformly by rapport of the w_n.

PROOF. Define the sequence of real numbers

$$\alpha_{i+1}^{(n)} := a + \sum_{j=0}^{i} w_j^{(n)}, i = 0,...,n.$$

Note that

$$\alpha_{n+1}^{(n)} = a + \sum_{j=0}^{n} w_j^{(n)} = a + b - a = b,$$

and observe also that $\alpha_{i+1}^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)}\right]$.

Define $\alpha_0^{(n)} := a$ and compute

$$\alpha_1^{(n)} - \alpha_0^{(n)} = a,$$

$$\alpha_{i+1}^{(n)} - \alpha_i^{(n)} = a + \sum_{j=0}^{i} w_j^{(n)} - a - \sum_{j=0}^{i-1} w_j^{(n)} = w_i^{(n)} \; (i = 1, ..., n-1),$$

$$\alpha_{n+1}^{(n)} - \alpha_n^{(n)} = b - \left(a + \sum_{j=0}^{n-1} w_j^{(n)} \right) = w_n^{(n)}.$$

Then

$$\sum_{i=0}^{n} \left(\alpha_{i+1}^{(n)} - \alpha_i^{(n)} \right) f\left(x_i^{(n)} \right) = \sum_{i=0}^{n} w_i^{(n)} f\left(x_i^{(n)} \right) = I_n (f, \Delta_n, w_n).$$

Applying the inequality (1.9), we get the estimate (1.14).

The uniform convergence by rapport of quadrature weights $w_j^{(n)}$ is obvious by the last inequality. ∎

Now, consider the equidistant partitioning of $[a, b]$ given by

$$E_n : x_i^{(n)} := a + \frac{i}{n} (b - a) \quad (i = 0, ..., n)$$

and define the sequence of numerical quadrature formulae

$$I_n (f, w_n) := \sum_{i=0}^{n} w_i^{(n)} f \left[a + \frac{i}{n} (b - a) \right].$$

The following corollary which can be more useful in practice holds:

Corollary 1.10. Let f be as above. If the quadrature weight $w_j^{(n)}$ satisfy the condition:

$$\frac{i}{n} \le \frac{1}{b-a} \sum_{j=0}^{i} w_j^{(n)} \le \frac{i+1}{n}, i = 0, ..., n-1; \tag{1.16}$$

then we have:

$$\begin{aligned} & \left| I_n (f, w_n) - \int_a^b f(x)\, dx \right| \\ \le \; & \left[\frac{b-a}{2n} + \max \left\{ \left| a + \sum_{j=0}^{i} w_j^{(n)} - \frac{2i+1}{2} \cdot \frac{(b-a)}{n} \right|, i = 0, ..., n-1 \right\} \right] \bigvee_a^b (f) \\ \le \; & \frac{(b-a)}{n} \bigvee_a^b (f). \end{aligned} \tag{1.17}$$

In particular, we have the limit

$$\lim_{n \to \infty} I_n (f, w_n) = \int_a^b f(x)\, dx,$$

uniformly by rapport of w_n.

1.2.3. Particular Inequalities. The following proposition holds [**14**] (see also [**4**]).

PROPOSITION 1.11. Let $f:[a,b]\rightarrow\mathbb{R}$ be a function of bounded variation on $[a,b]$. Then we have the inequality:

$$\left|\int_a^b f(x)\,dx-[(\alpha-a)f(a)+(b-\alpha)f(b)]\right| \leq \left[\frac{1}{2}(b-a)+\left|\alpha-\frac{a+b}{2}\right|\right]\bigvee_a^b(f) \tag{1.18}$$

for all $\alpha\in[a,b]$.

The proof follows by Theorem 1.6 choosing $x_0=a, x_1=b, \alpha_0=a, \alpha_1=\alpha\in[a,b]$ and $\alpha_2=b$.

REMARK 1.1. a) If in (1.18) we put $\alpha=b$, then we get the "*left rectangle inequality*"

$$\left|\int_a^b f(x)\,dx-(b-a)f(a)\right|\leq(b-a)\bigvee_a^b(f); \tag{1.19}$$

b) If $\alpha=a$, then by (1.18) we get the "*right rectangle inequality*"

$$\left|\int_a^b f(x)\,dx-(b-a)f(b)\right|\leq(b-a)\bigvee_a^b(f); \tag{1.20}$$

c) It is easy to see that the best inequality we can get from (1.18) is for $\alpha=\frac{a+b}{2}$ obtaining the "*trapezoid inequality*" (see also [**4**])

$$\left|\int_a^b f(x)\,dx-\frac{f(a)+f(b)}{2}(b-a)\right|\leq\frac{1}{2}(b-a)\bigvee_a^b(f). \tag{1.21}$$

Another proposition with many interesting particular cases is the following one [**14**] (see also [**4**]):

PROPOSITION 1.12. Let f be as above and $a\leq x_1\leq b$, $a\leq\alpha_1\leq x_1\leq\alpha_2\leq b$. Then we have

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-[(\alpha_1-a)f(a)+(\alpha_2-\alpha_1)f(x_1)+(b-\alpha_2)f(b)]\right| \\
&\leq \frac{1}{2}\left[\frac{1}{2}(b-a)+\left|x_1-\frac{a+b}{2}\right|+\left|\alpha_1-\frac{a+x_1}{2}\right|\right. \\
&\quad\left.+\left|\alpha_2-\frac{x_1+b}{2}\right|+\left|\left|\alpha_1-\frac{a+x_1}{2}\right|-\left|\alpha_2-\frac{x_1+b}{2}\right|\right|\right]\bigvee_a^b(f) \\
&\leq \left[\frac{(b-a)}{2}+\left|x_1-\frac{a+b}{2}\right|\right]\bigvee_a^b(f)\leq(b-a)\bigvee_a^b(f).
\end{aligned} \tag{1.22}$$

Proof. Consider the division $a = x_0 \le x_1 \le x_2 \le b$ and the numbers $\alpha_0 = a, \alpha_1 \in [a, x_1], \alpha_2 \in [x_1, b]$ and $\alpha_3 = b$. Now, applying Theorem 1.6, we get

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx - [(\alpha_1 - a) f(a) + (\alpha_2 - \alpha_1) f(x_1) + (b - \alpha_2) f(b)]\right| \\
\le\ & \frac{1}{2}\left[\max\{x_1 - a, b - x_1\} + \max\left\{\left|\alpha_1 - \frac{a + x_1}{2}\right|, \left|\alpha_2 - \frac{x_1 + b}{2}\right|\right\}\right] \bigvee_a^b (f) \\
=\ & \left[\frac{1}{4}(b - a) + \frac{1}{2}\left|x_1 - \frac{a+b}{2}\right| + \frac{1}{2}\left|\alpha_1 - \frac{a + x_1}{2}\right|\right. \\
& \left. + \frac{1}{2}\left|\alpha_2 - \frac{x_1 + b}{2}\right| + \frac{1}{2}\left|\left|\alpha_1 - \frac{a + x_1}{2}\right| - \left|\alpha_2 - \frac{x_1 + b}{2}\right|\right|\right] \bigvee_a^b (f)
\end{aligned}$$

and the first inequality in (1.22) is proved.

Now, let observe that

$$\left|\alpha_1 - \frac{a + x_1}{2}\right| \le \frac{x_1 - a}{2}, \left|\alpha_2 - \frac{x_1 + b}{2}\right| \le \frac{b - x_1}{2}.$$

Consequently,

$$\max\left\{\left|\alpha_1 - \frac{a + x_1}{2}\right|, \left|\alpha_2 - \frac{x_1 + b}{2}\right|\right\} \le \frac{1}{2}\max\{x_1 - a, b - x_1\}$$

and the second inequality in (1.22) is proved.

The last inequality is obvious. ∎

Remark 1.2. a) If we choose above $\alpha_1 = a, \alpha_2 = b$, then we get the following Ostrowski type inequality obtained by Dragomir in the recent paper [**11**]:

$$\left|\int_a^b f(x)\,dx - (b - a) f(x_1)\right| \le \left[\frac{1}{2}(b - a) + \left|x_1 - \frac{a+b}{2}\right|\right] \bigvee_a^b (f) \tag{1.23}$$

for all $x_1 \in [a, b]$.

We note that the best inequality we can get in (1.23) is for $x_1 = \frac{a+b}{2}$ obtaining the "*midpoint inequality*" (see also [**2**]).

$$\left|\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)(b - a)\right| \le \frac{1}{2}(b - a) \bigvee_a^b (f). \tag{1.24}$$

b) If we choose in (1.22) $\alpha_1 = \frac{5a+b}{6}, \alpha_2 = \frac{a+5b}{6}$ and $x_1 \in \left[\frac{5a+b}{6}, \frac{a+5b}{6}\right]$, then we get

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx - \frac{b-a}{3}\left[\frac{f(a) + f(b)}{2} + 2f(x_1)\right]\right| \\
\le\ & \frac{1}{2}\left[\frac{1}{2}\cdot(b - a) + \left|x_1 - \frac{a+b}{2}\right|\right. \\
& \left. + \max\left\{\left|x_1 - \frac{2a+b}{3}\right|, \left|\frac{a+2b}{3} - x_1\right|\right\}\right] \bigvee_a^b (f).
\end{aligned} \tag{1.25}$$

In particular, if we choose in (1.25), $x_1 = \frac{a+b}{2}$, then we get the following "*Simpson's inequality*" [**9**]

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{1}{3}(b-a)\bigvee_a^b (f). \tag{1.26}$$

1.2.4. Particular Quadrature Formulae. Let us consider the partitioning of the interval $[a,b]$ given by $\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ and put $h_i := x_{i+1} - x_i$ $(i = 0, ..., n-1)$ and $\nu(h) := \max\{h_i | i = 0, ..., n-1\}$.

The following theorem holds [**14**]:

THEOREM 1.13. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$ and $k \geq 1$. Then we have the composite quadrature formula*

$$\int_a^b f(x)\,dx = A_k(\Delta_n, f) + R_k(\Delta_n, f), \tag{1.27}$$

where

$$A_k(\Delta_n, f) := \frac{1}{k}\left[T(\Delta_n, f) + \sum_{i=0}^{n}\sum_{j=1}^{k-1} f\left[\frac{(k-j)x_i + jx_{i+1}}{k}\right] h_i\right] \tag{1.28}$$

and

$$T(\Delta_n, f) := \frac{1}{2}\sum_{i=0}^{n-1}\left[f(x_i) + f(x_{i+1})\right] h_i \tag{1.29}$$

is the trapezoid quadrature formula.

The remainder $R_k(\Delta_n, f)$ satisfies the estimate

$$|R_k(\Delta_n, f)| \leq \frac{1}{2k}\nu(h)\bigvee_a^b (f). \tag{1.30}$$

PROOF. Applying Corollary 1.8 on the intervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ we get

$$\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \left[\frac{1}{k}\frac{f(x_i)+f(x_{i+1})}{2}h_i + \frac{h_i}{k}\sum_{j=1}^{k} f\left[\frac{(k-j)x_i + jx_{i+1}}{k}\right]\right]\right| \leq \frac{1}{2k}h_i \bigvee_{x_i}^{x_{i+1}} (f).$$

Now, using the generalized triangle inequality, we get:

$$
\begin{aligned}
& |R_k(\Delta_n, f)| \\
\leq\ & \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x)\,dx - \left[\frac{1}{k} \cdot \frac{f(x_i)+f(x_{i+1})}{2} h_i + \frac{h_i}{k} \sum_{j=1}^{k-1} f\left[\frac{(k-j)\,x_i + jx_{i+1}}{k} \right] \right] \right| \\
\leq\ & \frac{1}{2k} \sum_{i=0}^{n-1} h_i \bigvee_{x_i}^{x_{i+1}} (f) \leq \frac{\nu(h)}{2k} \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (f) = \frac{\nu(h)}{2k} \bigvee_a^b (f)
\end{aligned}
$$

and the theorem is proved. ∎

The following corollaries hold:

Corollary 1.14. Let f be as above. Then we have the formula:

$$\int_a^b f(x)\,dx = \frac{1}{2}\left[T_n(\Delta_n, f) + M_n(\Delta_n, f)\right] + R_2(\Delta_n, f) \tag{1.31}$$

where $M_n(\Delta_n, f)$ is the midpoint quadrature formula,

$$M_n(\Delta_n, f) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i$$

and the remainder $R_2(\Delta_n, f)$ satisfies the inequality:

$$|R_2(\Delta_n, f)| \leq \frac{1}{4}\nu(h) \bigvee_a^b (f). \tag{1.32}$$

Corollary 1.15. Under the above assumptions we have

$$
\begin{aligned}
\int_a^b f(x)\,dx \quad = \quad & \frac{1}{3}\left[T_n(\Delta_n, f) + \sum_{i=0}^{n-1} f\left(\frac{2x_i + x_{i+1}}{3}\right) h_i \right. \\
& \left. + \sum_{i=0}^{n-1} f\left(\frac{x_i + 2x_{i+1}}{3}\right) h_i \right] + R_3(\Delta_n, f).
\end{aligned} \tag{1.33}
$$

The remainder $R_3(\Delta_n, f)$ satisfies the bound:

$$|R_3(\Delta_n, f)| \leq \frac{1}{6}\nu(h) \bigvee_a^b (f). \tag{1.34}$$

The following theorem holds [**14**] (see also [**4**]):

Theorem 1.16. *Let f and Δ_n be as above and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$. Then we have the quadrature formula:*

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \left[(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})\right] + R(\xi, \Delta_n, f). \tag{1.35}$$

The remainder $R(\xi,\Delta_n,f)$ *satisfies the estimation:*

$$
\begin{aligned}
(1.36)\qquad & \left|R(\xi,\Delta_n,f)\right| \\
&\leq \left[\frac{1}{2}\nu(h)+\max\left\{\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|,i=0,...,n-1\right\}\right]\bigvee_a^b(f) \\
&\leq \nu(h)\bigvee_a^b(f),
\end{aligned}
$$

for all ξ_i *as above.*

Proof. Apply Proposition 1.11 on the interval $[x_i,x_{i+1}]$ $(i=0,...,n-1)$ to get

$$
\begin{aligned}
&\left|\int_{x_i}^{x_{i+1}} f(x)\,dx-\left[(\xi_i-x_i)f(x_i)+(x_{i+1}-\xi_i)f(x_{i+1})\right]\right| \\
&\leq \left[\frac{1}{2}h_i+\max\left\{\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right\}\right]\bigvee_{x_i}^{x_{i+1}}(f).
\end{aligned}
$$

Summing over i from 0 to $n-1$, using the generalized triangle inequality and the properties of the maximum mapping, we get (1.36). ∎

Corollary 1.17. Let f and Δ_n be as above. Then we have

1) the "*left rectangle rule*"

$$\int_a^b f(x)\,dx=\sum_{i=0}^{n-1} f(x_i)\,h_i+R_l(\Delta_n,f);\tag{1.37}$$

2) the "*right rectangle rule*"

$$\int_a^b f(x)\,dx=\sum_{i=0}^{n-1} f(x_{i+1})\,h_i+R_r(\Delta_n,f);\tag{1.38}$$

3) the "*trapezoid rule*"

$$\int_a^b f(x)\,dx=T(\Delta_n,f)+R_T(\Delta_n,f)\tag{1.39}$$

where

$$\left|R_l(\Delta_n,f)\right|\left|R_r(\Delta_n,f)\right|\leq\nu(h)\bigvee_a^b(f)$$

and

$$\left|R_T(\Delta_n,f)\right|\leq\frac{1}{2}\nu(h)\bigvee_a^b(f).$$

The following theorem also holds [**14**].

THEOREM 1.18. *Let f and Δ_n be as above and $\xi_i \in [x_i, x_{i+1}]$, $x_i \le \alpha_i^{(1)} \le \xi_i \le \alpha_i^{(2)} \le x_{i+1}$, then we have the quadrature formula:*

$$\begin{aligned}(1.40)\qquad &\int_a^b f(x)\,dx \\ = {} & \sum_{i=0}^{n-1}\left(\alpha_i^{(1)} - x_i\right) f(x_i) + \sum_{i=0}^{n-1}\left(\alpha_i^{(2)} - \alpha_i^{(1)}\right) f(\xi_i) \\ & + \sum_{i=0}^{n-1}\left(x_{i+1} - \alpha_i^{(2)}\right) f(x_{i+1}) + R\left(\xi, \alpha^{(1)}, \alpha^{(2)}, \Delta_n, f\right).\end{aligned}$$

The remainder $R\left(\xi, \alpha^{(1)}, \alpha^{(2)}, \Delta_n, f\right)$ satisfies the estimation

$$\begin{aligned}(1.41)\qquad & \left|R\left(\xi, \alpha^{(1)}, \alpha^{(2)}, \Delta_n, f\right)\right| \\ \le {} & \left\{\frac{1}{2}\left[\frac{1}{2}\nu(h) + \max_{i=0,\dots,n-1}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\right. \\ & \left. + \max\left\{\max_{i=0,\dots,n-1}\left|\alpha_i^{(1)} - \frac{x_i + \xi_i}{2}\right|, \max_{i=0,\dots,n-1}\left|\alpha_i^{(2)} - \frac{\xi_i + x_{i+1}}{2}\right|\right\}\right\}\bigvee_a^b(f) \\ \le {} & \left[\frac{1}{2}\nu(h) + \max_{i=0,\dots n-1}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\|f'\|_1 \le \nu(h)\bigvee_a^b(f).\end{aligned}$$

PROOF. Apply Proposition 1.12 on the interval $[x_i, x_{i+1}]$ to obtain

$$\begin{aligned} & \left|\int_{x_i}^{x_{i+1}} f(x)\,dx \right. \\ & \left. - \left[\left(\alpha_i^{(1)} - x_i\right) f(x_i) + \left(\alpha_i^{(2)} - \alpha_i^{(1)}\right) f(\xi_i) + \left(x_{i+1} - \alpha_i^{(2)}\right) f(x_{i+1})\right]\right| \\ \le {} & \frac{1}{2}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right. \\ & \left. + \max\left\{\left|\alpha_i^{(1)} - \frac{x_i + \xi_i}{2}\right|, \left|\alpha_i^{(2)} - \frac{\xi_i + x_{i+1}}{2}\right|\right\}\right]\bigvee_{x_i}^{x_{i+1}}(f).\end{aligned}$$

Summing over i from 0 to $n-1$ and using the properties of modulus and maximum, we get the desired inequality.

We shall omit the details. ∎

The following corollary is the result of Dragomir from the recent paper [**11**].

COROLLARY 1.19. Under the above assumptions, we have the Riemann's quadrature formula:

$$(1.42)\qquad \int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(\xi_i)\,h_i + R_R(\xi, \Delta_n, f).$$

The remainder $R_R(\xi,\Delta_n,f)$ satisfies the bound

$$\begin{aligned} (1.43)\qquad & \left|R_R(\xi,\Delta_n,f)\right| \\ \leq & \left[\frac{1}{2}\nu(h)+\max\left\{\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|, i=0,...,n-1\right\}\right]\bigvee_a^b(f) \\ \leq & \nu(h)\bigvee_a^b(f) \end{aligned}$$

for all $\xi_i\in[x_i,x_{i+1}]$ $(i=0,...,n)$.

Finally, the following corollary which generalizes Simpson's quadrature formula holds

Corollary 1.20. Under the above assumptions and if $\xi_i\in\left[\frac{x_{i+1}+5x_i}{6},\frac{x_i+5x_{i+1}}{6}\right]$ $(i=0,...,n-1)$, then we have the formula:

$$(1.44)\qquad \int_a^b f(x)\,dx=\frac{1}{6}\sum_{i=0}^{n-1}\left[f(x_i)+f(x_{i+1})\right]h_i+\frac{2}{3}\sum_{i=0}^{n-1}f(\xi_i)\,h_i+S(f,\Delta_n,\xi).$$

The remainder $S(f,\Delta_n,\xi)$ satisfies the estimate:

$$\begin{aligned} (1.45)\ & \left|S(f,\Delta_n,\xi)\right| \\ \leq & \left\{\frac{1}{2}\left[\frac{\nu(h)}{2}+\max_{i=0,\dots,n-1}\left\{\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right\}\right.\right. \\ & \left.\left.+\max\left\{\max_{i=0,\dots,n-1}\left|\xi_i-\frac{2x_i+x_{i+1}}{3}\right|,\max_{i=0,\dots,n-1}\left|\frac{x_i+2x_{i+1}}{3}-\xi_i\right|\right\}\right]\right\}\bigvee_a^b(f). \end{aligned}$$

The proof follows by the inequality (1.25) and we omit the details.

Remark 1.3. Now, if we choose in (1.44), $\xi_i=\frac{x_i+x_{i+1}}{2}$, then we get "*Simpson's quadrature formula*" [**9**]

$$\begin{aligned} (1.46)\qquad \int_a^b f(x)\,dx &= \frac{1}{6}\sum_{i=0}^{n-1}\left[f(x_i)+f(x_{i+1})\right]h_i \\ &\quad +\frac{2}{3}\sum_{i=0}^{n-1}f\left(\frac{x_i+x_{i+1}}{2}\right)h_i+S(f,\Delta_n) \end{aligned}$$

where the remainder term $S(f,\Delta_n)$ satisfies the bound:

$$(1.47)\qquad \left|S(f,\Delta_n)\right|\leq\frac{1}{3}\nu(h)\bigvee_a^b(f).$$

1.3. Generalisations for Functions whose Derivatives are in L_∞

1.3.1. Some Inequalities. We start with the following result [**5**].

Theorem 1.21. *Let $I_k: a=x_0<x_1<...<x_{k-1}<x_k=b$ be a division of the interval $[a,b]$, α_i $(i=0,...,k+1)$ be "$k+2$" points so that $\alpha_0=a,\alpha_i\in$*

$[x_{i-1},x_i]$ $(i=1,...,k)$ and $\alpha_{k+1}=b$. If $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$, then we have the inequality:

$$\begin{aligned}
(1.48)\qquad & \left|\int_a^b f(x)\,dx-\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\,f(x_i)\right| \\
&\leq \left[\frac{1}{4}\sum_{i=0}^{k-1}h_i^2+\sum_{i=0}^{k-1}\left(\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right)^2\right]\|f'\|_\infty \\
&\leq \frac{1}{2}\|f'\|_\infty\sum_{i=0}^{k-1}h_i^2\leq\frac{1}{2}(b-a)\|f'\|_\infty\,\nu(h),
\end{aligned}$$

where $h_i:=x_{i+1}-x_i$ $(i=0,...,k-1)$ and $\nu(h):=\max\{h_i\mid i=0,...,k-1\}$. The constant $\frac{1}{4}$ in the first inequality and the constant $\frac{1}{2}$ in the second and third inequality are the best possible.

Proof. Define the mapping $K:[a,b]\to\mathbb{R}$ given by (see the proof of Theorem 1.6)

$$K(t):=\begin{cases} t-\alpha_1, & t\in[a,x_1);\\ t-\alpha_2, & t\in[x_1,x_2);\\ & \vdots \\ t-\alpha_{k-1}, & t\in[x_{k-2},x_{k-1});\\ t-\alpha_k, & t\in[x_{k-1},b].\end{cases}$$

Integrating by parts, we have successively:

$$\begin{aligned}
&\int_a^b K(t)\,f'(t)\,dt \\
&= \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}K(t)\,f'(t)\,dt=\sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}(t-\alpha_{i+1})\,f'(t)\,dt \\
&= \sum_{i=0}^{k-1}\left[(t-\alpha_{i+1})\,f(t)\,\Big|_{x_i}^{x_{i+1}}-\int_{x_i}^{x_{i+1}}f(t)\,dt\right] \\
&= \sum_{i=0}^{k-1}\left[(\alpha_{i+1}-x_i)\,f(x_i)+(x_{i+1}-\alpha_{i+1})\,f(x_{i+1})\right]-\int_a^b f(t)\,dt \\
&= (\alpha_1-a)\,f(a)+\sum_{i=1}^{k-1}(\alpha_{i+1}-x_i)\,f(x_i)+\sum_{i=0}^{k-2}(x_{i+1}-\alpha_{i+1})\,f(x_{i+1}) \\
&\quad +(b-\alpha_n)\,f(b)-\int_a^b f(t)\,dt
\end{aligned}$$

$$\begin{aligned}
&= (\alpha_1 - a) f(a) + \sum_{i=1}^{k-1} (\alpha_{i+1} - x_i) f(x_i) + \sum_{i=1}^{k-1} (x_i - \alpha_i) f(x_i) \\
&\quad + (b - \alpha_n) f(b) - \int_a^b f(t)\,dt \\
&= (\alpha_1 - a) f(a) + \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i) f(x_i) + (b - \alpha_n) f(b) - \int_a^b f(t)\,dt \\
&= \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) - \int_a^b f(t)\,dt
\end{aligned}$$

and then we have the integral equality:

$$\int_a^b f(t)\,dt = \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) - \int_a^b K(t) f'(t)\,dt. \tag{1.49}$$

Using the properties of modulus, we have

$$\begin{aligned}
&\left| \int_a^b K(t) f'(t)\,dt \right| \\
&= \left| \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} K(t) f'(t)\,dt \right| \le \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} |K(t)| \, |f'(t)|\,dt \\
&= \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} |t - \alpha_{i+1}| \, |f'(t)|\,dt \le \|f'\|_\infty \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} |t - \alpha_{i+1}|\,dt.
\end{aligned} \tag{1.50}$$

A simple calculation shows that

$$\begin{aligned}
\int_{x_i}^{x_{i+1}} |t - \alpha_{i+1}|\,dt &= \int_{x_i}^{\alpha_{i+1}} (\alpha_{i+1} - t)\,dt + \int_{\alpha_{i+1}}^{x_{i+1}} (t - \alpha_{i+1})\,dt \\
&= \frac{1}{2} \left[(x_{i+1} - \alpha_{i+1})^2 + (\alpha_{i+1} - x_i)^2 \right] \\
&= \frac{1}{4} h_i^2 + \left(\alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right)^2
\end{aligned} \tag{1.51}$$

for all $i = 0, ..., k-1$.

Now, by $(1.49) - (1.51)$, we get the first inequality in (1.48).

Assume that the first inequality in (1.48) holds for a constant $c > 0$, i.e.,

$$\begin{aligned}
&\left| \int_a^b f(x)\,dx - \sum_{i=0}^{k-1} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \\
&\le \left[c \sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1} \left(\alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \|f'\|_\infty .
\end{aligned} \tag{1.52}$$

If we choose $f : [a,b] \to \mathbb{R}$, $f(x) = x$, $\alpha_0 = a$, $\alpha_1 = b$, $x_0 = a$, $x_1 = b$ in (1.52), we obtain

$$\frac{(b-a)^2}{2} \le c(b-a)^2 + \frac{(b-a)^2}{4},$$

from where we get $c \geq \frac{1}{4}$, and the sharpness of the constant $\frac{1}{4}$ is proved.

The last two inequalities as well as the sharpness of the constant $\frac{1}{2}$ are obvious and we omit the details. ∎

Now, if we assume that the points of the division I_k are given, then the best inequality we can get from Theorem 1.21 is embodied in the following corollary:

Corollary 1.22. Let f, I_k be as above. Then we have the inequality

$$\begin{aligned} & \left| \int_a^b f(x)\, dx \right. \\ & \left. - \frac{1}{2}\left[(x_1 - a) f(a) + \sum_{i=1}^{k-1} (x_{i+1} - x_{i-1}) f(x_i) + (b - x_{k-1}) f(b) \right] \right| \\ \leq\ & \frac{1}{4} \|f'\|_\infty \sum_{i=0}^{k-1} h_i^2. \end{aligned} \tag{1.53}$$

The constant $\frac{1}{4}$ is the best possible one.

Proof. Similar to the proof of Corollary 1.7. ∎

The case of equidistant partitioning is important in practice.

Corollary 1.23. Let $I_k : x_i = a + i \cdot \frac{b-a}{k}$ $(i = 0, ..., k)$ be an equidistant partitioning of $[a, b]$. If f is as above, then we have the inequality

$$\begin{aligned} & \left| \int_a^b f(x)\, dx - \left[\frac{1}{k} \cdot \frac{f(a) + f(b)}{2} (b - a) + \frac{(b-a)}{k} \sum_{i=1}^{k-1} f\left[\frac{(k-i)\, a + ib}{k} \right] \right] \right| \\ \leq\ & \frac{1}{4k} (b-a)^2 \|f'\|_\infty . \end{aligned} \tag{1.54}$$

The constant $\frac{1}{4}$ is the best possible one.

Remark 1.4. If $k = 1$, then we have the inequality (see for example [**4**])

$$\left| \int_a^b f(x)\, dx - \frac{f(a) + f(b)}{2} (b - a) \right| \leq \frac{1}{4} (b-a)^2 \|f'\|_\infty . \tag{1.55}$$

Choose $f : [a, b] \to \mathbb{R},\ f(x) = \left| x - \frac{a+b}{2} \right|$, which is L-Lipschitzian with $L = 1$ and

$$f'(x) = \begin{cases} 1 & \text{if} \quad t \in \left[a, \frac{a+b}{2}\right) \\ -1 & \text{if} \quad t \in \left(\frac{a+b}{2}, b\right] \end{cases} .$$

Then $\|f'\|_\infty = 1$ and

$$\int_a^b f(x)\, dx - \frac{f(a) + f(b)}{2} (b - a) = -\frac{(b-a)^2}{4},$$

and the equality is obtained in (1.55), showing that the constant $\frac{1}{4}$ is sharp.

1.3.2. A General Quadrature Formula. Let $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ be a sequence of divisions of $[a,b]$ and consider the sequence of numerical integration formulae

$$I_n(f, \Delta_n, w_n) := \sum_{j=0}^{n} w_j^{(n)} f\left(x_j^{(n)}\right),$$

where $w_j^{(n)}$ $(j = 0, ..., n)$ are the quadrature weights and assume that $\sum_{j=0}^{n} w_j^{(n)} = b - a$.

The following theorem contains a sufficient condition for the weights $w_j^{(n)}$ so that $I_n(f, \Delta_n, w_n)$ approximates the integral $\int_a^b f(x)\,dx$ with an error expressed in terms of $\|f'\|_\infty$ (see also [**5**])

THEOREM 1.24. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a,b]$. If the quadrature weights $w_j^{(n)}$ $(j = 0, ..., n)$ satisfy the condition*

$$x_i^{(n)} - a \le \sum_{j=0}^{i} w_j^{(n)} \le x_{i+1}^{(n)} - a \text{ for all } i = 0, ..., n-1; \tag{1.56}$$

then we have the estimation

$$\begin{aligned}
&\left| I_n(f, \Delta_n, w_n) - \int_a^b f(x)\,dx \right| \\
\le\ &\left[\frac{1}{4} \sum_{i=0}^{n-1} \left[h_i^{(n)}\right]^2 + \sum_{i=0}^{n-1} \left[a + \sum_{j=0}^{i} w_j^{(n)} - \frac{x_i^{(n)} + x_{i+1}^{(n)}}{2} \right]^2 \right] \|f'\|_\infty \\
\le\ &\frac{1}{2} \|f'\|_\infty \sum_{j=0}^{n} [h_i^{(n)}]^2 \le \frac{1}{2} \|f'\|_\infty (b-a)\, \nu\left(h^{(n)}\right),
\end{aligned} \tag{1.57}$$

where $\nu\left(h^{(n)}\right) := \max\{h_i^{(n)} : i = 0, ..., n-1\}$ and $h_i^{(n)} := x_{i+1}^{(n)} - x_i^{(n)}$.
In particular, if $\|f'\|_\infty < \infty$, then

$$\lim_{\nu\left(h^{(n)}\right) \to 0} I_n(f, \Delta_n, w_n) = \int_a^b f(x)\,dx$$

uniformly by the influence of the weights w_n.

PROOF. Similar to the proof of Theorem 1.9 and we omit the details. ∎

The case when the partitioning is equidistant is important in practice. Consider, then, the partitioning

$$E_n : x_i^{(n)} := a + i \cdot \frac{b-a}{n} \ (i = 0, ..., n),$$

and define the sequence of numerical quadrature formulae

$$I_n(f, w_n) := \sum_{i=0}^{n} w_i^{(n)} f\left(a + i \cdot \frac{b-a}{n}\right), \ \sum_{j=0}^{n} w_j^{(n)} = b - a.$$

The following result holds:

Corollary 1.25. Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. If the quadrature weights $w_i^{(n)}$ satisfy the condition:

$$\frac{i}{n} \leq \frac{1}{b-a}\sum_{j=0}^{i} w_j^{(n)} \leq \frac{i+1}{n} \quad (i = 0, ..., n-1);$$

then we have the estimate

$$\begin{aligned} & \left| I_n(f, w_n) - \int_a^b f(x)\,dx \right| \\ \leq \; & \|f'\|_\infty \left[\frac{1}{4n}(b-a)^2 + \sum_{i=0}^{n-1} \left[\sum_{j=0}^{i} w_j^{(n)} - \frac{2i+1}{2} \cdot \frac{b-a}{n} \right]^2 \right] \\ \leq \; & \frac{1}{2n} \|f'\|_\infty (b-a)^2 . \end{aligned}$$

In particular, if $\|f'\|_\infty < \infty$, then

$$\lim_{n \to \infty} I_n(f, w_n) = \int_a^b f(x)\,dx$$

uniformly by the influence of w_n.

1.3.3. Particular Inequalities.

1.3.3. Particular Inequalities. In this sub-section we point out some particular inequalities which generalize some classical results such as: rectangle inequality, trapezoid inequality, Ostrowski's inequality, midpoint inequality, Simpson's inequality and others in terms of the sup-norm of the derivative [**5**] (see also [**4**]).

Proposition 1.26. Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$ and $\alpha \in [a,b]$. Then we have the inequality:

$$\begin{aligned} & \left| \int_a^b f(x)\,dx - [(\alpha - a) f(a) + (b-\alpha) f(b)] \right| \\ \leq \; & \left[\frac{1}{4}(b-a)^2 + \left(\alpha - \frac{a+b}{2} \right)^2 \right] \|f'\|_\infty \\ \leq \; & \frac{1}{2}(b-a)^2 \|f'\|_\infty . \end{aligned} \tag{1.58}$$

The constant $\frac{1}{4}$ is the best possible one.

Proof. Follows from Theorem 1.21 by choosing $x_0 = a, x_1 = b, \alpha_0 = a, \alpha_1 = \alpha \in [a,b]$ and $\alpha_2 = b$. ∎

Remark 1.5. a) If in (1.58) we put $\alpha = b$, then we get the "left rectangle inequality":

$$\left| \int_a^b f(x)\,dx - (b-a) f(a) \right| \leq \frac{1}{2}(b-a)^2 \|f'\|_\infty . \tag{1.59}$$

b) If $\alpha = a$, then by (1.58) we obtain the "right rectangle inequality"

$$\left| \int_a^b f(x)\,dx - (b-a) f(a) \right| \leq \frac{1}{2}(b-a)^2 \|f'\|_\infty . \tag{1.60}$$

c) It is clear that the best estimation we can have in (1.58) is for $\alpha = \frac{a+b}{2}$ getting the "trapezoid inequality" (see also [**2**])

$$\left| \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a) \right| \leq \frac{1}{4}(b-a)^2 \|f'\|_\infty . \tag{1.61}$$

Another particular integral inequality with many applications is the following one [**5**]:

PROPOSITION 1.27. Let $f : [a,b] \to \mathbb{R}$ be an arbitrary absolutely continuous mapping on $[a,b]$ and $a \leq \alpha_1 \leq x_1 \leq \alpha_2 \leq b$. Then we have the inequality:

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - [(\alpha_1 - a) f(a) + (\alpha_2 - \alpha_1) f(x_1) + (b - \alpha_2) f(b)] \right| \quad (1.62) \\ \leq\ & \left[\frac{1}{8}(b-a)^2 + \frac{1}{2}\left(x_1 - \frac{a+b}{2}\right)^2 \right. \\ & \left. + \left(\alpha_1 - \frac{a+x_1}{2}\right)^2 + \left(\alpha_2 - \frac{x_1+b}{2}\right)^2 \right] \|f'\|_\infty . \end{aligned}$$

PROOF. Follows by Theorem 1.21 and we omit the details. ∎

COROLLARY 1.28. Let f be as above and $x_1 \in [a,b]$. Then we have Ostrowski's inequality:

$$\left| \int_a^b f(x)\,dx - (b-a) f(x_1) \right| \leq \left[\frac{1}{4}(b-a)^2 + (x_1 - \frac{a+b}{2})^2 \right] \|f'\|_\infty . \tag{1.63}$$

REMARK 1.6. If we choose $x_1 = \frac{a+b}{2}$ in (1.63), we obtain the "midpoint inequality"

$$\left| \int_a^b f(x)\,dx - (b-a) f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{4}(b-a)^2 \|f'\|_\infty . \tag{1.64}$$

The following corollary generalizing Simpson's inequality holds:

COROLLARY 1.29. Let f be as above and $x_1 \in \left[\frac{5a+b}{6}, \frac{a+5b}{6}\right]$. Then we have the inequality

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f(x_1) \right] \right| \quad (1.65) \\ \leq\ & \left[\frac{5}{36}(b-a)^2 + \left(x_1 - \frac{a+b}{2}\right)^2 \right] \|f'\|_\infty . \end{aligned}$$

PROOF. Follows by Proposition 1.27 using simple computation and we omit the details. ∎

REMARK 1.7. Let us observe that the best estimation we can obtain from (1.65) is that one for which $x_1 = \frac{a+b}{2}$, obtaining the "Simpson's inequality" [**10**]

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{5}{36}(b-a)^2 \|f'\|_\infty . \tag{1.66}$$

The following corollary also holds

COROLLARY 1.30. Let f be as above and $a \leq \alpha_1 \leq \frac{a+b}{2} \leq \alpha_2 \leq b$. Then we have the inequality

$$\begin{aligned} & \left| \int_a^b f(x)\,dx - \left[(\alpha_1 - a)f(a) + (\alpha_2 - \alpha_1)f\left(\frac{a+b}{2}\right) + (b-\alpha_2)f(b)\right]\right| \\ \leq\ & \left[\frac{1}{8}(b-a)^2 + \left(\alpha_1 - \frac{3a+b}{4}\right)^2 + \left(\alpha_2 - \frac{a+3b}{4}\right)^2\right]\|f'\|_\infty . \end{aligned} \tag{1.67}$$

The proof is obvious by Proposition 1.27 by choosing $x_1 = \frac{a+b}{2}$.

REMARK 1.8. The best estimation we can obtain from (1.67) is that one for which $\alpha_1 = \frac{3a+b}{4}$ and $\alpha_2 = \frac{a+3b}{4}$, obtaining the inequality [**3**]

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{1}{8}(b-a)^2 \|f'\|_\infty . \tag{1.68}$$

The following proposition generalizes the *"three-eights rule"* of Newton-Cotes:

PROPOSITION 1.31. Let f be as above and $a \leq x_1 \leq x_2 \leq b$ and $\alpha_1 \in [a, x_1]$, $\alpha_2 \in [x_1, x_2]$, $\alpha_3 \in [x_2, b]$. Then we have the inequality

$$\begin{aligned} & \left| \int_a^b f(x)\,dx - [(\alpha_1 - a)f(a) + (\alpha_2 - \alpha_1)f(x_1) \right. \\ & \left. + (\alpha_3 - \alpha_2)f(x_2) + (b-\alpha_3)f(b)]\right| \\ \leq\ & [\frac{1}{4}[(x_1-a)^2 + (x_2-x_1)^2 + (b-x_2)^2] \\ & + \left(\alpha_1 - \frac{a+x_1}{2}\right)^2 + \left(\alpha_2 - \frac{x_1+x_2}{2}\right)^2 + \left(\alpha_3 - \frac{x_2+b}{2}\right)^2]\, \|f'\|_\infty . \end{aligned} \tag{1.69}$$

The proof is obvious by Theorem 1.21.

The next corollary contains a generalization of the *"three-eights rule"* of Newton-Cotes in the following way:

Corollary 1.32. Let f be as above and $a \leq \alpha_1 \leq \frac{2a+b}{3} \leq \alpha_2 \leq \frac{2b+a}{3} \leq \alpha_3 \leq b$. Then we have the inequality:

$$\begin{aligned}
&\left| \int_a^b f(x)\,dx - \left[(\alpha_1 - a) f(a) + (\alpha_2 - \alpha_1) f\left(\frac{2a+b}{3}\right) \right.\right. \\
&\left.\left. +(\alpha_3 - \alpha_2) f\left(\frac{a+2b}{3}\right) + (b - \alpha_3) f(b) \right] \right| \\
\leq\; & \left[\frac{(b-a)^2}{12} + \left(\alpha_1 - \frac{5a+b}{6}\right)^2 \right. \\
& \left. + \left(\alpha_2 - \frac{a+b}{2}\right)^2 + \left(\alpha_3 - \frac{a+5b}{6}\right)^2 \right] \|f'\|_\infty .
\end{aligned} \tag{1.70}$$

The proof follows by the above proposition by choosing $x_1 = \frac{2a+b}{3}$ and $x_2 = \frac{a+2b}{3}$.

Remark 1.9. (1)

a) Now, if we choose $\alpha_1 = \frac{b+7a}{8}, \alpha_2 = \frac{a+b}{2}$ and $\alpha_3 = \frac{a+7b}{8}$ in (1.70), then we get the *"three-eights rule"* of Newton-Cotes

$$\begin{aligned}
&\left| \int_a^b f(x)\,dx - \frac{b-a}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right| \\
\leq\; & \frac{25}{288} (b-a)^2 \|f'\|_\infty .
\end{aligned} \tag{1.71}$$

b) The best estimation we can get from (1.70) is that one for which $\alpha_1 = \frac{5a+b}{6}, \alpha_2 = \frac{a+b}{2}$, $\alpha_3 = \frac{a+5b}{6}$ obtaining the inequality

$$\begin{aligned}
&\left| \int_a^b f(x)\,dx - \frac{b-a}{6} \left[f(a) + 2f\left(\frac{2a+b}{3}\right) + 2f\left(\frac{a+2b}{3}\right) + f(b) \right] \right| \\
\leq\; & \frac{1}{12} (b-a)^2 \|f'\|_\infty .
\end{aligned} \tag{1.72}$$

1.3.4. Particular Quadrature Formulae. Let us consider the partitioning of the interval $[a,b]$ given by $\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ and put $h_i := x_{i+1} - x_i (i = 0, ..., n-1)$ and $\nu(h) := \max\{h_i |\ i = 0, ..., n-1\}$.

The following theorem holds [**5**]:

Theorem 1.33. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$ and $k \geq 1$. Then we have the composite quadrature formula*

$$\int_a^b f(x)\,dx = A_k(\Delta_n, f) + R_k(\Delta_n, f), \tag{1.73}$$

where

$$A_k(\Delta_n, f) := \frac{1}{k} \left[T(\Delta_n, f) + \sum_{i=0}^{n-1} \sum_{j=1}^{k-1} f\left[\frac{(k-j)x_i + jx_{i+1}}{k} \right] h_i \right] \tag{1.74}$$

and

$$T(\Delta_n, f) := \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(x_i\right) + f\left(x_{i+1}\right)\right]h_i \tag{1.75}$$

is the trapezoid quadrature formula.

The remainder $R_k(\Delta_n, f)$ *satisfies the estimate*

$$|R_k(\Delta_n, f)| \le \frac{1}{4k}\left\|f'\right\|_\infty \sum_{i=0}^{n-1} h_i^2. \tag{1.76}$$

Proof. Applying Corollary 1.22 on the intervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$, we obtain

$$\begin{aligned} & \left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \left[\frac{1}{k}\cdot\frac{f\left(x_i\right)+f\left(x_{i+1}\right)}{2}h_i + \frac{h_i}{k}\sum_{j=1}^{k-1} f\left[\frac{(k-j)x_i + jx_{i+1}}{k}\right]\right]\right| \\ \le\ & \frac{1}{4k}h_i^2\left\|f'\right\|_\infty. \end{aligned}$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality, we get the desired estimation (1.76). ∎

The following corollary holds:

Corollary 1.34. Let f, Δ_n be as above. Then we have the quadrature formula

$$\int_a^b f(x)\,dx = \frac{1}{2}\left[T(\Delta_n, f) + M(\Delta_n, f)\right] + R_2(\Delta_n, f),$$

where $M(\Delta_n, f)$ is the midpoint rule:

$$M(\Delta_n, f) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)h_i.$$

The remainder satisfies the estimate:

$$|R_2(\Delta_n, f)| \le \frac{1}{8}\left\|f'\right\|_\infty \sum_{i=0}^{n-1} h_i^2. \tag{1.77}$$

The following corollary also holds:

Corollary 1.35. Let f, Δ_n be as above. Then we have the quadrature formula

$$\begin{aligned} \int_a^b f(x)\,dx &= \frac{1}{2}\left[T(\Delta_n, f) + \sum_{i=0}^{n-1} f\left(\frac{2x_i + x_{i+1}}{3}\right)h_i\right. \\ &\quad \left. + \sum_{i=0}^{n-1} f\left(\frac{x_i + 2x_{i+1}}{3}\right)h_i\right] + R_3\left(\Delta_n, f\right), \end{aligned} \tag{1.78}$$

where the remainder $R_3(\Delta_n, f)$ satisfies the bound:

$$|R_2(\Delta_n, f)| \le \frac{1}{12}\left\|f'\right\|_\infty \sum_{i=0}^{n-1} h_i^2. \tag{1.79}$$

The following theorem holds as well [**5**] (see also [**4**]):

THEOREM 1.36. *Let f and Δ_n be as above. Suppose that $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$. Then we have the formula*

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \left[(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})\right] + R(\xi, \Delta_n, f). \tag{1.80}$$

The remainder $R(\xi_n, \Delta_n, f)$ satisfies the estimate:

$$\begin{aligned} |R(\xi, \Delta_n, f)| &\leq \left[\frac{1}{4}\sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right] \|f'\|_\infty \\ &\leq \frac{1}{2}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2. \end{aligned} \tag{1.81}$$

PROOF. Apply Proposition 1.26 on the intervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ to get

$$\begin{aligned} &\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \left[(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})\right]\right| \\ &\leq \left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right] \|f'\|_\infty \leq \frac{1}{2}\|f'\|_\infty h_i^2. \end{aligned}$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality we deduce the desired inequality (1.81). ∎

COROLLARY 1.37. Let f and Δ_n be as above. Then we have

(*i*) The "left rectangle rule":

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(x_i)\,h_i + R_l(\Delta_n, f); \tag{1.82}$$

(*ii*) The "right rectangle rule":

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(x_{i+1})\,h_i + R_r(\Delta_n, f); \tag{1.83}$$

(*iii*) The "trapezoid rule":

$$\int_a^b f(x)\,dx = T(\Delta_n, f) + R_T(\Delta_n, f) \tag{1.84}$$

where

$$|R_l(\Delta_n, f)|, |R_r(\Delta_n, f)| \leq \frac{1}{2}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2$$

and

$$|R_T(\Delta_n, f)| \leq \frac{1}{4}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2.$$

The following theorem also holds [**5**].

THEOREM 1.38. *Let f and Δ_n be as above. If $x_i \leq \alpha_i^{(1)} \leq \xi_i \leq \alpha_i^{(2)} \leq x_{i+1}$ $(i = 0, ..., n-1)$, then we have the formula:*

$$\begin{aligned}
(1.85)\qquad & \int_a^b f(x)\,dx \\
&= \sum_{i=0}^{n-1} \left(\alpha_i^{(1)} - x_i\right) f(x_i) + \sum_{i=0}^{n-1} \left(\alpha_i^{(2)} - \alpha_i^{(1)}\right) f(\xi_i) \\
&\quad + \sum_{i=0}^{n-1} \left(x_{i+1} - \alpha_i^{(2)}\right) f(x_{i+1}) + R\left(\xi, \alpha^{(1)}, \alpha^{(2)}, \Delta_n, f\right),
\end{aligned}$$

where the remainder satisfies the bound

$$\begin{aligned}
(1.86)\qquad & \left|R\left(\xi, \alpha^{(1)}, \alpha^{(2)}, \Delta_n, f\right)\right| \\
&\leq \left[\frac{1}{8}\sum_{i=0}^{n-1} h_i^2 + \frac{1}{2}\sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right. \\
&\quad \left. + \sum_{i=0}^{n-1}\left(\alpha_i^{(1)} - \frac{x_i + \xi_i}{2}\right)^2 + \sum_{i=0}^{n-1}\left(\alpha_i^{(2)} - \frac{\xi_i + x_{i+1}}{2}\right)^2\right] \|f'\|_\infty .
\end{aligned}$$

The proof follows by Proposition 1.27 applied for the intervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ and we omit the details.

The following corollary of the above theorem holds [**20**].

COROLLARY 1.39. Let f, Δ_n be as above and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$. Then we have the formula of Riemann's type:

$$(1.87)\qquad \int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(\xi_i)\,h_i + R_R(\xi, \Delta_n, f)$$

where the remainder, $R_R(\xi, \Delta_n, f)$, satisfies the estimate

$$\begin{aligned}
(1.88)\qquad |R_R(\xi, \Delta_n, f)| &\leq \left[\frac{1}{4}\sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]\|f'\|_\infty \\
&\leq \frac{1}{2}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2.
\end{aligned}$$

REMARK 1.10. If we choose in (1.87), $\xi_i = \frac{x_i + x_{i+1}}{2}$, then we get the midpoint quadrature formula [**20**]

$$\int_a^b f(x)\,dx = M(\Delta_n, f) + R_M(\Delta_n, f),$$

where

$$|R_M(\Delta_n, f)| \leq \frac{1}{4}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2.$$

The following corollary holds as well [**5**] (see also [**3**]).

Corollary 1.40. Let f,Δ_n be as above and $\xi_i \in \left[\frac{5x_i+x_{i+1}}{6},\frac{x_i+5x_{i+1}}{6}\right]$ $(i=0,...,n-1)$. Then we have the formula

$$\int_a^b f(x)\,dx = \frac{1}{6}\sum_{i=0}^{n-1}\left[f(x_i)+f(x_{i+1})\right]h_i + \frac{2}{3}\sum_{i=0}^{n-1} f(\xi_i)\,h_i + R_s(\xi,\Delta_n,f), \tag{1.89}$$

where the remainder, $R_s(\xi,\Delta_n,f)$, satisfies the inequality:

$$|R_s(\xi,\Delta_n,f)| \le \left[\frac{5}{36}\sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^2\right]\|f'\|. \tag{1.90}$$

Remark 1.11. If we choose above $\xi_i = \frac{x_i+x_{i+1}}{2}$ $(i=0,...,n-1)$, then we obtain

$$\int_a^b f(x)\,dx = \frac{1}{6}\sum_{i=0}^{n-1}\left[f(x_i)+f(x_{i+1})\right]h_i + \frac{2}{3}\sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)h_i + R_S(\Delta_n,f), \tag{1.91}$$

where the remainder satisfies the bound [**10**]

$$|R_S(\Delta_n,f)| \le \frac{5}{36}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2. \tag{1.92}$$

The following corollary holds too.

Corollary 1.41. Let f,Δ_n be as above and $x_i \le \alpha_i^{(1)} \le \frac{x_i+x_{i+1}}{2} \le \alpha_i^{(2)} \le x_{i+1}$ $(i=0,...,n-1)$. Then we have the formula

$$\begin{aligned}\int_a^b f(x)\,dx &= \sum_{i=0}^{n-1}\left(\alpha_i^{(1)}-x_i\right)f(x_i) + \sum_{i=0}^{n-1}\left(\alpha_i^{(2)}-\alpha_i^{(1)}\right)f\left(\frac{x_i+x_{i+1}}{2}\right) \\ &\quad + \sum_{i=0}^{n-1}\left(x_i-\alpha_i^{(2)}\right)f(x_{i+1}) + R_B\left(\alpha^{(1)},\alpha^{(2)},\Delta_n,f\right),\end{aligned} \tag{1.93}$$

where the remainder satisfies the estimate:

$$\begin{aligned}\left|R_B\left(\alpha_n^{(1)},\alpha_n^{(2)},\Delta_n,f\right)\right| &\le \left[\frac{1}{8}\sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1}\left(\alpha_i^{(1)} - \frac{3x_i+x_{i+1}}{4}\right)^2\right. \\ &\quad \left. + \sum_{i=0}^{n-1}\left(\alpha_i^{(2)} - \frac{x_i+3x_{i+1}}{4}\right)^2\right]\|f'\|_\infty .\end{aligned}$$

Finally, the following theorem holds [**5**]:

THEOREM 1.42. *Let f, Δ_n be as above and $x_i \le \xi_i^{(1)} \le \xi_i^{(2)} \le x_{i+1}$ and $\alpha_i^{(1)} \in \left[x_i, \xi_i^{(1)}\right], \alpha_i^{(2)} \in \left[\xi_i^{(1)}, \xi_i^{(2)}\right]$ and $\alpha_i^{(3)} \in \left[\xi_i^{(2)}, x_{i+1}\right]$ for $i = 0, ..., n-1$. Then we have formula:*

$$
\begin{aligned}
(1.94)\qquad & \int_a^b f(x)\,dx \\
= {} & \sum_{i=0}^{n-1} \left(\alpha_i^{(1)} - x_i\right) f(x_i) + \sum_{i=0}^{n-1} \left(\alpha_i^{(2)} - \alpha_i^{(1)}\right) f\left(\xi_i^{(1)}\right) \\
& + \sum_{i=0}^{n-1} \left(\alpha_i^{(3)} - \alpha_i^{(2)}\right) f\left(\xi_i^{(2)}\right) + \sum_{i=0}^{n-1} \left(x_{i+1} - \alpha_i^{(3)}\right) f(x_{i+1}) \\
& + R\left(\xi^{(1)}, \xi^{(2)}, \alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \Delta_n, f\right)
\end{aligned}
$$

and the remainder satisfies the estimation

$$
\begin{aligned}
& \left| R\left(\xi^{(1)}, \xi^{(2)}, \alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \Delta_n, f\right) \right| \\
\le & \left[\frac{1}{4} \left[\sum_{i=0}^{n-1} \left(\xi_i^{(1)} - x_i\right)^2 + \sum_{i=0}^{n-1} \left(\xi_i^{(2)} - \xi_i^{(1)}\right)^2 + \sum_{i=0}^{n-1} \left(x_{i+1} - \xi_i^{(1)}\right)^2 \right] \right. \\
& + \sum_{i=0}^{n-1} \left(\alpha_i^{(1)} - \frac{x_{i+1} + \xi_i^{(1)}}{2}\right)^2 + \sum_{i=0}^{n-1} \left(\alpha_i^{(2)} - \frac{\xi_i^{(1)} + \xi_i^{(2)}}{2}\right) \\
& \left. + \sum_{i=0}^{n-1} \left(\alpha_i^{(3)} - \frac{\xi_i^{(2)} + x_{i+1}}{2}\right) \right] \|f'\|_\infty .
\end{aligned}
$$

The proof follows by Proposition 1.31. We omit the details.

REMARK 1.12. We note only that if we choose $\alpha_i^{(1)} = \frac{x_{i+1}+7x_i}{8}$, $\alpha_i^{(2)} = \frac{x_i+x_{i+1}}{2}$, $\alpha_i^{(3)} = \frac{x_i+7x_{i+1}}{8}$, $\xi_i^{(1)} = \frac{2x_i+x_{i+1}}{3}$ and $\xi_i^{(2)} = \frac{x_i+2x_{i+1}}{3}$ $(i = 0, ..., n-2)$ then we get the "*three-eights formula*" of Newton-Cotes:

$$
\begin{aligned}
\int_a^b f(x)\,dx = {} & \frac{1}{8} \sum_{i=0}^{n-1} \left[f(x_i) + 3f\left(\frac{2x_i + x_{i+1}}{3}\right) + 3f\left(\frac{x_i + 2x_{i+1}}{3}\right) + f(x_{i+1}) \right] \\
& + R_{N-C}(\Delta_n, f),
\end{aligned}
$$

where the remainder satisfies the bound

$$
|R_{N-C}(\Delta_n, f)| \le \frac{25}{288} \|f'\|_\infty \sum_{i=0}^{n-1} h_i^2.
$$

1.4. Generalisation for Functions whose Derivatives are in L_p

1.4.1. Some Inequalities.

We start with the following result [**7**],

THEOREM 1.43. *Let $I_k : a = x_0 < x_1 < ... < x_{k-1} < x_k = b$ be a division of the interval $[a, b]$ and α_i $(i = 0, ..., k+1)$ be "$k + 2$" points such that $\alpha_0 = a$,*

$\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, ..., k)$ and $\alpha_{k+1} = b$. If $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$, then we have the inequality:

$$\begin{aligned}
(1.95)\qquad & \left| \int_a^b f(x)\,dx - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \\
\le\ & \frac{1}{(q+1)^{\frac{1}{q}}} \|f'\|_p \left[\sum_{i=0}^{k-1} \left[(\alpha_{i+1} - x_i)^{q+1} + (x_{i+1} - \alpha_{i+1})^{q+1} \right] \right]^{\frac{1}{q}} \\
\le\ & \frac{1}{(q+1)^{\frac{1}{q}}} \|f'\|_p \left[\sum_{i=0}^{k-1} h_i^{q+1} \right]^{\frac{1}{q}} \le \frac{\nu(h)(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|f'\|_p
\end{aligned}$$

where $h_i := x_{i+1} - x_i \, (i = 0, ..., k-1)$, $\nu(h) := \max\{h_i | i = 0, ..., n\}$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\|\cdot\|_p$ is the usual $L_p[a, b]$ $-$norm.

Proof. Consider the kernel $K : [a, b] \to \mathbb{R}$ given by (see also Theorem 1.21):

$$K(t) := \begin{cases} t - \alpha_1, & t \in [a, x_1); \\ t - \alpha_2, & t \in [x_1, x_2); \\ & \vdots \\ t - \alpha_{k-1}, & t \in [x_{k-2}, x_{k-1}); \\ t - \alpha_k, & t \in [x_{k-1}, b]. \end{cases}$$

Integrating by parts, we have the identity:

$$(1.96)\qquad \int_a^b f(t)\,dt = \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) - \int_a^b K(t) f'(t)\,dt.$$

On the other hand, we have

$$\begin{aligned}
(1.97)\qquad \left| \int_a^b K(t) f'(t)\,dt \right| &= \left| \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} K(t) f'(t)\,dt \right| \\
&\le \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} |K(t)| |f'(t)|\,dt \\
&= \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} |t - \alpha_{i+1}| |f'(t)|\,dt.
\end{aligned}$$

Using Hölder's integral inequality for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, we can write

(1.98)

$$\int_{x_i}^{x_{i+1}} |t - \alpha_{i+1}| |f'(t)|\,dt \le \left(\int_{x_i}^{x_{i+1}} |t - \alpha_{i+1}|^q\,dt \right)^{\frac{1}{q}} \left(\int_{x_i}^{x_{i+1}} |f'(t)|^p\,dt \right)^{\frac{1}{q}}.$$

However,

$$
\begin{aligned}
& \int_{x_i}^{x_{i+1}} |t-\alpha_{i+1}|^q \, dt \\
= \; & \int_{x_i}^{\alpha_{i+1}} (\alpha_{i+1}-t)^q \, dt + \int_{\alpha_{i+1}}^{x_{i+1}} (t-\alpha_{i+1})^q \, dt \\
= \; & \frac{1}{(q+1)} \left[(\alpha_{i+1}-x_i)^{q+1} + (x_{i+1}-\alpha_{i+1})^{q+1} \right]
\end{aligned}
$$

and then, by the inequality (1.98), we deduce

$$
\begin{aligned}
& \int_{x_i}^{x_{i+1}} |t-\alpha_{i+1}| \, |f'(t)| \, dt \\
\leq \; & \frac{1}{(q+1)^{\frac{1}{q}}} \left[(\alpha_{i+1}-x_i)^{q+1} + (x_{i+1}-\alpha_{i+1})^{q+1} \right]^{\frac{1}{q}} \left(\int_{x_i}^{x_{i+1}} |f'(t)|^p \, dt \right)^{\frac{1}{q}} .
\end{aligned} \tag{1.99}
$$

By relations (1.97)-(1.99) and by Hölder's discrete inequality, we obtain

$$
\begin{aligned}
& \left| \int_a^b K(t) f'(t) \, dt \right| \\
\leq \; & \frac{1}{(q+1)^{\frac{1}{q}}} \sum_{i=0}^{k-1} \left[(\alpha_{i+1}-x_i)^{q+1} + (x_{i+1}-\alpha_{i+1})^{q+1} \right]^{\frac{1}{q}} \times \left(\int_{x_i}^{x_{i+1}} |f'(t)|^p \, dt \right)^{\frac{1}{p}} \\
\leq \; & \frac{1}{(q+1)^{\frac{1}{q}}} \left[\sum_{i=0}^{k-1} \left(\left[(\alpha_{i+1}-x_i)^{q+1} + (x_{i+1}-\alpha_{i+1})^{q+1} \right]^{\frac{1}{q}} \right)^q \right]^{\frac{1}{q}} \\
& \times \left[\sum_{i=0}^{k-1} \left(\left(\int_{x_i}^{x_{i+1}} |f'(t)|^p \, dt \right)^{\frac{1}{p}} \right)^p \right]^{\frac{1}{p}} \\
= \; & \frac{1}{(q+1)^{\frac{1}{q}}} \left(\sum_{i=0}^{k-1} \left[(\alpha_{i+1}-x_i)^{q+1} + (x_{i+1}-\alpha_{i+1})^{q+1} \right] \right)^{\frac{1}{q}} \|f'\|_p
\end{aligned}
$$

and the first inequality in (1.95) is proved.

Now, consider the function $g : [\alpha, \beta] \to \mathbb{R}$, $g(t) = (t-\alpha)^{q+1} + (\beta - t)^{q+1}$.

Then

$$
g'(t) = (q+1) \left[(t-\alpha)^q - (\beta-t)^q \right],
$$

$g'(t) = 0$ iff $t = \frac{\alpha+\beta}{2}$ and $g'(t) < 0$ if $t \in [\alpha, \frac{\alpha+\beta}{2})$ and $g'(t) > 0$ if $t \in (\frac{\alpha+\beta}{2}, \beta]$, which shows that

$$
\inf_{t \in [\alpha,\beta]} g(t) = g\left(\frac{\alpha+\beta}{2} \right) = \frac{(\beta-\alpha)^{q+1}}{2^q} \tag{1.100}
$$

and

$$
\sup_{t \in [\alpha,\beta]} g(t) = g(\alpha) = g(\beta) = (\beta-\alpha)^{q+1} . \tag{1.101}
$$

Using the above bound (1.101), we may write

$$\sum_{i=0}^{k-1}\left[(\alpha_{i+1}-x_i)^{q+1}+(x_{i+1}-\alpha_{i+1})^{q+1}\right]\leq\sum_{i=0}^{k-1}h_i^{q+1},$$

and the second inequality in (1.95) is obtained.

For the last inequality we only remark that

$$\sum_{i=0}^{k-1}h_i^{q+1}\leq\nu^q(h)\sum_{i=0}^{k-1}h_i=(b-a)\,\nu^q(h).$$

The theorem is completely proved. ∎

Now, if we assume that the points of the division I_k are fixed, then the best inequality we can obtain from Theorem 1.43 is embodied in the following corollary.

Corollary 1.44. Let $I_k : a = x_0 < x_1 < ... < x_{k-1} < x_k = b$ be a division of the interval $[a,b]$. If f is as above, then we have the inequality:

$$\begin{aligned}(1.102)\quad &\left|\int_a^b f(x)\,dx\right.\\ &\left.-\frac{1}{2}\left[(x_1-a)f(a)+\sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})f(x_i)+(b-x_{k-1})f(b)\right]\right|\\ \leq\ &\frac{1}{2(q+1)^{\frac{1}{q}}}\|f'\|_p\left[\sum_{i=0}^{k-1}h_i^{q+1}\right]^{\frac{1}{q}}\leq\frac{\nu(h)(b-a)^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\|f'\|_p.\end{aligned}$$

The case of equidistant partitioning is important in practice.

Corollary 1.45. Let

$$I_k : x_i := a + i\cdot\frac{b-a}{k}\quad(i=0,...,k),$$

be an equidistant partitioning of $[a,b]$. If f is as above, then we have the inequality:

$$\begin{aligned}(1.103)\quad &\left|\int_a^b f(x)\,dx-\left[\frac{1}{k}\cdot\frac{f(a)+f(b)}{2}(b-a)+\frac{(b-a)}{k}\sum_{i=1}^{k-1}f\left[\frac{(k-i)a+ib}{k}\right]\right]\right|\\ \leq\ &\frac{(b-a)^{1+\frac{1}{q}}}{2k(q+1)^{\frac{1}{q}}}\|f'\|_p.\end{aligned}$$

1.4.2. General Quadrature Formulae. Let $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ be a sequence of division of $[a,b]$ and consider the sequence of numerical integration formulae (see Subsection 1.4.1)

$$I_n(f,\Delta_n,w_n):=\sum_{j=0}^{n}w_j^{(n)}f\left(x_j^{(n)}\right),\tag{1.104}$$

where $w_j^{(n)}$ $(j=0,...,n)$ are the quadrature weights and $\sum_{j=0}^{n}w_j^{(n)}=b-a$.

The following theorem contains a sufficient condition for the weights $w_j^{(n)}$ such that $I_n(f,\Delta_n,w_n)$ approximates the integral $\int_a^b f(x)\,dx$ with an error expressed in terms of $\|f'\|_p,\ p\in(1,\infty)$, [**7**].

Theorem 1.46. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping on $[a,b]$. If the quadrature weights $w_j^{(n)}$ satisfy the condition*

$$x_i^{(n)}-a\le\sum_{j=0}^{i}w_j^{(n)}\le x_{i+1}^{(n)}-a \text{ for all } i=0,...,n-1, \tag{1.105}$$

then we have the estimate

$$\begin{aligned}
&\left|I_n(f,\Delta_n,w_n)-\int_a^b f(x)\,dx\right| \qquad (1.106)\\
\le\ &\frac{1}{(q+1)^{\frac{1}{q}}}\|f'\|_p\\
&\times\left[\sum_{i=0}^{n-1}\left[\left(a+\sum_{j=0}^{i}w_j^{(n)}-x_i^{(n)}\right)^{q+1}+\left(x_{i+1}^{(n)}-a-\sum_{j=0}^{i}w_j^{(n)}\right)^{q+1}\right]\right]^{1/q}\\
\le\ &\frac{1}{(q+1)^{\frac{1}{q}}}\|f'\|_p\left[\sum_{i=0}^{n-1}\left[h_i^{(n)}\right]^{q+1}\right]^{\frac{1}{q}}\le\frac{\nu\left(h^{(n)}\right)(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_p,
\end{aligned}$$

where $\nu\left(h^{(n)}\right):=\max\left\{h_i^{(n)}|i=0,...,n-1\right\}$ and $h_i^{(n)}:=x_{i+1}^{(n)}-x_i^{(n)}$. In particular, if $\|f'\|_p<\infty$, then

$$\lim_{\nu\left(h^{(n)}\right)\to 0}I_n(f,\Delta_n,w_n)=\int_a^b f(x)\,dx$$

uniformly by rapport of the weight w_n.

Proof. Similar to the proof of Theorem 1.9 and we omit the details. ∎

The case when the partitioning is equidistant is important in practice. Consider, then, the partitioning

$$E_n: x_i^{(n)}:=a+i\cdot\frac{b-a}{2}\ (i=0,...,n)$$

and define the sequence of numerical quadrature formulae

$$I_n(f,w_n):=\sum_{i=0}^{n}w_i^{(n)}f\left[a+\frac{i}{n}\cdot(b-a)\right]\ \sum_{j=0}^{n}w_j^{(n)}=b-a.$$

The following result holds:

Corollary 1.47. Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. If the quadrature weights $w_j^{(n)}$ satisfy the condition:

$$\frac{i}{n}\le\sum_{j=0}^{i}w_j^{(n)}\le\frac{i+1}{n},\ i=0,...,n-1; \tag{1.107}$$

then we have the estimate:

$$\begin{aligned}
&\left|I_n(f,w_n)-\int_a^b f(x)\,dx\right| \qquad (1.108)\\
\le\ & \frac{1}{(q+1)^{\frac{1}{q}}}\|f'\|_p\\
&\times\left(\sum_{i=0}^{n-1}\left[\left[\sum_{j=0}^{i}w_j^{(n)}-\frac{i}{n}\cdot(b-a)\right]^{q+1}+\left[\frac{i+1}{n}\cdot(b-a)-\sum_{j=0}^{i}w_j^{(n)}\right]^{q+1}\right]\right)^{\frac{1}{q}}\\
\le\ & \frac{(b-a)^{1+\frac{1}{q}}}{n(q+1)^{\frac{1}{q}}}\|f'\|_p.
\end{aligned}$$

In particular, if $\|f'\|_p<\infty$, then

$$\lim_{n\to\infty}I_n(f,w_n)=\int_a^b f(x)\,dx \tag{1.109}$$

uniformly by the influence of the weights w_n.

1.4.3. Particular Inequalities. In this sub-section we point out particular inequalities which generalize some classical results such as: Rectangle Inequality, Trapezoid Inequality, Ostrowski's Inequality, Midpoint Inequality, Simpson's Inequality and others in terms of the $p-$norm of the derivative [**7**].

Proposition 1.48. Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$ and $\alpha\in[a,b]$. Then we have the inequality (see also [**4**]):

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-[(\alpha-a)f(a)+(b-\alpha)f(b)]\right| \qquad (1.110)\\
\le\ & \frac{1}{(q+1)^{\frac{1}{q}}}\|f'\|_p\left[(\alpha-a)^{q+1}+(b-\alpha)^{q+1}\right]^{\frac{1}{q}}\\
\le\ & \frac{(b-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_p.
\end{aligned}$$

Proof. Follows from Theorem 1.43 by choosing $x_0=a,x_1=b,\alpha_0=a,\alpha_1=\alpha\in[a,b]$ and $\alpha_2=b$. ∎

Remark 1.13. (1)

a) If in (1.110) we put $\alpha=b$, then we obtain the "*left rectangle inequality*"

$$\left|\int_a^b f(x)\,dx-(b-a)f(a)\right|\le\frac{(b-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_p. \tag{1.111}$$

b) If $\alpha = a$, then by (1.110) we have the "*right rectangle inequality*"

$$\left| \int_a^b f(x)\, dx - (b-a) f(b) \right| \leq \frac{(b-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|f'\|_p . \tag{1.112}$$

c) It is clear that the best estimate we can have in (1.110) is for $\alpha = \frac{a+b}{2}$ getting the "*trapezoid inequality*" (see also [**4**]):

$$\left| \int_a^b f(x)\, dx - \frac{f(a)+f(b)}{2}(b-a) \right| \leq \frac{1}{2} \frac{(b-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|f'\|_p . \tag{1.113}$$

Another particular integral inequality with many applications is the following one (see also [**3**]):

Proposition 1.49. Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a,b]$ and $a \leq x_1 \leq b$, $a \leq \alpha_1 \leq x_1 \leq \alpha_2 \leq b$. Then we have the inequality:

$$\begin{aligned}
&\left| \int_a^b f(x)\, dx - [(\alpha_1 - a) f(a) + (\alpha_2 - \alpha_1) f(x_1) + (b - \alpha_2) f(b)] \right| \qquad (1.114)\\
\leq\ & \frac{1}{(q+1)^{\frac{1}{q}}} \|f'\|_p \\
& \times \left[(\alpha_1 - a)^{q+1} + (x_1 - \alpha_1)^{q+1} + (\alpha_2 - x_1)^{q+1} + (b - \alpha_2)^{q+1} \right]^{\frac{1}{q}} \\
\leq\ & \frac{1}{(q+1)^{\frac{1}{q}}} \|f'\|_p \left[(x_1 - a)^{q+1} + (b - x_1)^{q+1} \right]^{\frac{1}{q}} \\
\leq\ & \frac{(b-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|f'\|_p .
\end{aligned}$$

Proof. Consider the division $a = x_0 \leq x_1 \leq x_2 = b$ and the numbers $\alpha_0 = a, \alpha_1 \in [a, x_1], \alpha_2 \in [x_1, b], \alpha_3 = b$. Applying Theorem 1.43 for these particular choices, we easily obtain the desired inequalities. We omit the details. ∎

Corollary 1.50. Let f be as above and $x_1 \in [a,b]$. Then we have Ostrowski's inequality (see also [**18**]):

$$\begin{aligned}
&\left| \int_a^b f(x)\, dx - (b-a) f(x_1) \right| \qquad (1.115)\\
\leq\ & \frac{1}{(q+1)^{\frac{1}{q}}} \|f'\|_p \left[(x_1 - a)^{q+1} + (b - x_1)^{q+1} \right]^{\frac{1}{q}} \\
\leq\ & \frac{(b-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|f'\|_p .
\end{aligned}$$

The proof follows by the above theorem choosing $\alpha_1 = a, \alpha_2 = b$.

REMARK 1.14. If we choose $x_1 = \frac{a+b}{2}$ in (1.115), then we get the "*midpoint inequality*" [**18**]

$$\left| \int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)(b-a) \right| \le \frac{(b-a)^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}} \left\| f' \right\|_p . \tag{1.116}$$

The following corollary generalizing Simpson's inequality holds as well:

COROLLARY 1.51. Let f be as above and $x_1 \in \left[\frac{5a+b}{6}, \frac{a+5b}{6}\right]$. Then we have the inequality:

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \frac{b-a}{3} \cdot \left[\frac{f(a)+f(b)}{2} + 2f(x_1) \right] \right| \\ \le\ & \frac{1}{(q+1)^{\frac{1}{q}}} \left\| f' \right\|_p \\ &\times \left[2 \cdot \frac{(b-a)^{q+1}}{6^{q+1}} + \left(x_1 - \frac{5a+b}{6} \right)^{q+1} + \left(\frac{a+5b}{6} - x_1 \right)^{q+1} \right]^{1/q} . \end{aligned} \tag{1.117}$$

The proof follows by Proposition 1.49 by choosing $\alpha_1 = \frac{5a+b}{2}$ and $\alpha_2 = \frac{a+5b}{2}$.

REMARK 1.15. Now, if in (1.117) we choose $x_1 = \frac{a+b}{2}$, then we get "*Simpson's inequality*" [**8**]

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \frac{b-a}{3} \cdot \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\ \le\ & \frac{1}{6(q+1)^{\frac{1}{q}}} \left(\frac{2^{q+1}+1}{3} \right)^{\frac{1}{q}} (b-a)^{1+\frac{1}{q}} \left\| f' \right\|_p . \end{aligned} \tag{1.118}$$

The following corollary also holds [**7**] (see also [**3**]):

COROLLARY 1.52. Let f be as above and $a \le \alpha_1 \le \frac{a+b}{2} \le \alpha_2 \le b$. Then we have the inequality:

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \left[(\alpha_1 - a) f(a) + (\alpha_2 - \alpha_1) f\left(\frac{a+b}{2}\right) + (b-\alpha_2) f(b) \right] \right| \\ \le\ & \frac{1}{(q+1)^{\frac{1}{q}}} \left\| f' \right\|_p \\ &\times \left[(\alpha_1 - a)^{q+1} + \left(\frac{a+b}{2} - \alpha_1 \right)^{q+1} + \left(\alpha_2 - \frac{a+b}{2} \right)^{q+1} + (b-\alpha_2)^{q+1} \right]^{\frac{1}{q}} \\ \le\ & \frac{1}{2(q+1)^{\frac{1}{q}}} \left\| f' \right\|_p (b-a)^{1+\frac{1}{q}} . \end{aligned} \tag{1.119}$$

Finally, we have [**7**] (see also [**3**]):

REMARK 1.16. Now, if we choose in (1.119), $\alpha_1 = \frac{3a+b}{4}$ and $\alpha_2 = \frac{a+3b}{4}$, then we get the inequality

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{1}{4(q+1)^{\frac{1}{q}}}\|f'\|_p (b-a)^{1+\frac{1}{q}}. \tag{1.120}$$

1.4.4. Particular Quadrature Formulae. Let us consider the partitioning of the interval $[a,b]$ given by $\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ and put $h_i := x_{i+1} - x_i$ $(i = 0, ..., n-1)$ and $\nu(h) := \max\{h_i | i = 0, ..., n-1\}$.

The following theorem holds [**7**]:

THEOREM 1.53. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$ and $k \geq 1$. Then we have the composite quadrature formula*

$$\int_a^b f(x)\,dx = A_k(\Delta_n, f) + R_k(\Delta_n, f), \tag{1.121}$$

where

$$A_k(\Delta_n, f) := \frac{1}{k}\left[T(\Delta_n, f) + \sum_{i=0}^{n}\sum_{j=1}^{k-1} f\left[\frac{(k-j)x_i + jx_{i+1}}{k}\right] h_i\right] \tag{1.122}$$

and

$$T(\Delta_n, f) := \frac{1}{2}\sum_{i=0}^{n-1}\left[f(x_i) + f(x_{i+1})\right] h_i \tag{1.123}$$

is the trapezoid quadrature formula.

The remainder $R_k(\Delta_n, f)$ satisfies the estimate

$$|R_k(\Delta_n, f)| \leq \frac{1}{2k(q+1)^{\frac{1}{q}}}\|f'\|_p \left(\sum_{i=0}^{n-1} h_i^{q+1}\right)^{\frac{1}{q}}, p > 1, \frac{1}{p} + \frac{1}{q} = 1. \tag{1.124}$$

PROOF. Applying Corollary 1.45 on the intervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$, we get

$$\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \left[\frac{1}{k}\frac{f(x_i)+f(x_{i+1})}{2}h_i + \frac{h_i}{k}\sum_{j=1}^{k} f\left[\frac{(k-j)x_i + jx_{i+1}}{k}\right]\right]\right| \leq \frac{1}{2k(q+1)^{\frac{1}{q}}} h_i^{1+\frac{1}{q}} \left(\int_{x_i}^{x_{i+1}} |f'(t)\,dt|\right)^{\frac{1}{p}}.$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality, we have:

$$\begin{aligned}
& \left|R_k\left(\Delta_n, f\right)\right| \\
\leq & \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)\, dx\right. \\
& \left.-\left[\frac{1}{k} \cdot \frac{f\left(x_i\right)+f\left(x_{i+1}\right)}{2} h_i+\frac{h_i}{k} \sum_{j=1}^{k-1} f\left[\frac{(k-j) x_i+j x_{i+1}}{k}\right]\right]\right| \\
\leq & \frac{1}{2k(q+1)^{\frac{1}{q}}} \sum_{i=0}^{n-1}\left[h_i^{1+\frac{1}{q}} \times\left(\int_{x_i}^{x_{i+1}}\left|f'(t)^p\right| dt\right)^{\frac{1}{p}}\right].
\end{aligned}$$

By Hölder's discrete inequality, we have

$$\begin{aligned}
& \sum_{i=0}^{n-1}\left[h_i^{1+\frac{1}{q}} \times\left(\int_{x_i}^{x_{i+1}}\left|f'(t)^p\right| dt\right)^{\frac{1}{p}}\right] \\
\leq & \left[\sum_{i=0}^{n-1}\left(h_i^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}} \times\left[\sum_{i=0}^{n-1}\left(\left(\int_{x_i}^{x_{i+1}}\left|f'(t)^p\right| dt\right)^{\frac{1}{p}}\right)^p\right]^{\frac{1}{q}} \\
= & \left(\sum_{i=0}^{n-1} h_i^{q+1}\right)^{\frac{1}{q}}\left\|f'\right\|_p,
\end{aligned}$$

and the theorem is proved. ∎

The following corollary holds:

Corollary 1.54. Let f, Δ_n be as above. Then we have the quadrature formula:

$$\int_a^b f(x)\, dx=\frac{1}{2}\left[T\left(\Delta_n, f\right)+M\left(\Delta_n, f\right)\right]+R_2\left(\Delta_n, f\right), \tag{1.125}$$

where $M\left(\Delta_n, f\right)$ is the midpoint rule, namely,

$$M\left(\Delta_n, f\right):=\sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right) h_i .$$

The remainder $R_2\left(\Delta_n, f\right)$ satisfies the estimate:

$$\left|R_2\left(\Delta_n, f\right)\right| \leq \frac{1}{4(q+1)^{\frac{1}{q}}}\left\|f'\right\|_p\left(\sum_{i=0}^{n-1} h_i^{q+1}\right)^{\frac{1}{q}}. \tag{1.126}$$

The following corollary holds as well.

Corollary 1.55. Let f, Δ_n be as above. Then we have the formula

$$\begin{aligned}
& \int_a^b f(x)\, dx \\
= & \frac{1}{3}\left[T\left(\Delta_n, f\right)+\sum_{i=0}^{n-1} f\left(\frac{2 x_i+x_{i+1}}{3}\right) h_i+\sum_{i=0}^{n-1} f\left(\frac{x_i+2 x_{i+1}}{3}\right) h_i\right] \\
& +R_3\left(\Delta_n, f\right).
\end{aligned} \tag{1.127}$$

The remainder $R_3(\Delta_n, f)$ satisfies the bound:

$$|R_3(\Delta_n, f)| \leq \frac{1}{6(q+1)^{\frac{1}{q}}} \|f'\|_p \left(\sum_{i=0}^{n-1} h_i^{q+1} \right)^{\frac{1}{q}}. \tag{1.128}$$

The following theorem also holds [**7**] (see also [**4**]):

THEOREM 1.56. *Let f and Δ_n be as above. Suppose that $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$. Then we have the formula:*

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \left[(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1}) \right] + R(\xi, \Delta_n, f). \tag{1.129}$$

The remainder $R(\xi, \Delta_n, f)$ satisfies the inequality:

$$\begin{aligned} &|R(\xi, \Delta_n, f)| \\ \leq\ & \frac{1}{(q+1)^{\frac{1}{q}}} \|f'\|_p \left[\sum_{i=0}^{n-1} (\xi_i - x_i)^{q+1} + \sum_{i=0}^{n-1} (x_{i+1} - \xi_i)^{q+1} \right]^{\frac{1}{q}} \\ \leq\ & \frac{1}{(q+1)^{\frac{1}{q}}} \|f'\|_p \left(\sum_{i=0}^{n-1} h_i^{q+1} \right)^{\frac{1}{q}}. \end{aligned} \tag{1.130}$$

PROOF. Apply Proposition 1.48 on the intervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ to get

$$\begin{aligned} &\left| \int_{x_i}^{x_{i+1}} f(x)\,dx - \left[(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1}) \right] \right| \\ \leq\ & \frac{1}{(q+1)^{\frac{1}{q}}} \left[(\xi_i - x_i)^{q+1} + (x_{i+1} - \xi_i)^{q+1} \right]^{\frac{1}{q}} \left(\int_{x_i}^{x_{i+1}} |f'(t)|^p\,dt \right)^{\frac{1}{p}}. \end{aligned}$$

Summing over i from 0 to $n-1$, using the generalized triangle inequality and Hölder's discrete inequality, we may state

$$
\begin{aligned}
& \left|R\left(\xi, \Delta_n, f\right)\right| \\
\leq\ & \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)\, dx-\left[\left(\xi_i-x_i\right) f\left(x_i\right)+\left(x_{i+1}-\xi_i\right) f\left(x_{i+1}\right)\right]\right| \\
\leq\ & \frac{1}{(q+1)^{\frac{1}{q}}} \sum_{i=0}^{n-1}\left\{\left[\left(\xi_i-x_i\right)^{q+1}+\left(x_{i+1}-\xi_i\right)^{q+1}\right]^{\frac{1}{q}}\right. \\
& \left.\times\left(\int_{x_i}^{x_{i+1}}\left|f'(t)\right|^p dt\right)^{\frac{1}{p}}\right\} \\
\leq\ & \frac{1}{(q+1)^{\frac{1}{q}}}\left(\sum_{i=0}^{n-1}\left(\left[\left(\xi_i-x_i\right)^{q+1}+\left(x_{i+1}-\xi_i\right)^{q+1}\right]^{\frac{1}{q}}\right)^q\right)^{\frac{1}{q}} \\
& \times\left[\sum_{i=0}^{n-1}\left(\left(\int_{x_i}^{x_{i+1}}\left|f'(t)\right|^p dt\right)^{\frac{1}{p}}\right)^p\right]^{\frac{1}{p}} \\
=\ & \frac{1}{(q+1)^{\frac{1}{q}}}\left[\sum_{i=0}^{n-1}\left(\xi_i-x_i\right)^{q+1}+\sum_{i=0}^{n-1}\left(x_{i+1}-\xi_i\right)^{q+1}\right]^{\frac{1}{q}}\left\|f'\right\|_p
\end{aligned}
$$

and the first inequality in (1.130) is proved.

The second inequality is obvious by taking into account that

$$\left(\xi_i-x_i\right)^{q+1}+\left(x_{i+1}-\xi_i\right)^{q+1} \leq h_i^{q+1}$$

for all $i=0,\ldots,n-1$. ∎

The following corollary contains some particular well known quadrature formulae:

Corollary 1.57. Let f and Δ_n be as above. Then we have

(1) The "*left rectangle rule*"

$$\int_a^b f(x)\, dx=\sum_{i=0}^{n-1} f\left(x_i\right) h_i+R_l\left(\Delta_n, f\right); \tag{1.131}$$

(2) The "*right rectangle rule*"

$$\int_a^b f(x)\, dx=\sum_{i=0}^{n-1} f\left(x_{i+1}\right) h_i+R_r\left(\Delta_n, f\right); \tag{1.132}$$

(3) The "*trapezoid rule*"

$$\int_a^b f(x)\, dx=\sum_{i=0}^{n-1} \frac{f\left(x_i\right)+f\left(x_{i+1}\right)}{2} h_i+R_T\left(\Delta_n, f\right), \tag{1.133}$$

where

$$\left|R_l\left(\Delta_n, f\right)\right|,\left|R_r\left(\Delta_n, f\right)\right| \leq \frac{1}{(q+1)^{\frac{1}{q}}}\left\|f'\right\|_p\left(\sum_{i=0}^{n-1} h_i^{q+1}\right)^{\frac{1}{q}} \tag{1.134}$$

and

$$\left|R_T\left(\Delta_n, f\right)\right| \leq \frac{1}{2\left(q+1\right)^{\frac{1}{q}}}\left\|f'\right\|_p\left(\sum_{i=0}^{n-1} h_i^{q+1}\right)^{\frac{1}{q}}.$$

The following theorem holds as well [**7**].

Theorem 1.58. *Let f and Δ_n be as above. If $x_i \leq \alpha_i^{(1)} \leq \xi_i \leq \alpha_i^{(2)} \leq x_{i+1}$ $(i=0,...,n-1)$, then we have the formula:*

$$\begin{aligned} &\int_a^b f\left(x\right)dx \\ = &\sum_{i=0}^{n-1}\left(\alpha_i^{(1)}-x_i\right)f\left(x_i\right)+\sum_{i=0}^{n-1}\left(\alpha_i^{(2)}-\alpha_i^{(1)}\right)f\left(\xi_i\right) \\ &+\sum_{i=0}^{n-1}\left(x_{i+1}-\alpha_i^{(2)}\right)f\left(x_{i+1}\right)+R\left(\xi,\alpha^{(1)},\alpha^{(2)},\Delta_n,f\right), \end{aligned} \tag{1.135}$$

where the remainder satisfies the estimate

$$\begin{aligned} &\left|R\left(\xi,\alpha^{(1)},\alpha^{(2)},\Delta_n,f\right)\right| \\ \leq &\frac{1}{\left(q+1\right)^{\frac{1}{q}}}\left\|f'\right\|_p\left[\sum_{i=0}^{n-1}\left(\alpha_i^{(1)}-x_i\right)^{q+1}+\sum_{i=0}^{n-1}\left(\xi_i-\alpha_i^{(1)}\right)^{q+1}\right. \\ &\left.+\sum_{i=0}^{n-1}\left(\alpha_i^{(2)}-\xi_i\right)^{q+1}+\sum_{i=0}^{n-1}\left(x_{i+1}-\alpha_i^{(2)}\right)^{q+1}\right]^{\frac{1}{q}} \\ \leq &\frac{1}{\left(q+1\right)^{\frac{1}{q}}}\left\|f'\right\|_p\left[\sum_{i=0}^{n-1}\left(\xi_i-x_i\right)^{q+1}+\sum_{i=0}^{n-1}\left(x_{i+1}-\xi_i\right)^{q+1}\right]^{\frac{1}{q}} \\ \leq &\frac{1}{\left(q+1\right)^{\frac{1}{q}}}\left\|f'\right\|_p\left(\sum_{i=0}^{n-1} h_i^{q+1}\right)^{\frac{1}{q}}. \end{aligned} \tag{1.136}$$

Proof. The proof follows by Proposition 1.49 applied on the intervals $[x_i, x_{i+1}]$ $(i=0,...,n-1)$. We omit the details. ∎

The following corollary of the above theorem holds (see also [**18**])

Corollary 1.59. Let f, Δ_n be as above and $\xi_i \in [x_i, x_{i+1}]$ $(i=0,...,n-1)$. Then we have the formula of Riemann's type:

$$\int_a^b f\left(x\right)dx = \sum_{i=0}^{n-1} f\left(\xi_i\right)h_i + R_R\left(\xi,\Delta_n,f\right). \tag{1.137}$$

The remainder $R_R(\xi, \Delta_n, f)$ satisfies the estimate

$$
\begin{aligned}
(1.138) \qquad & |R_R(\xi, \Delta_n, f)| \\
&\leq \frac{1}{(q+1)^{\frac{1}{q}}} \|f'\|_p \left[\sum_{i=0}^{n-1} (\xi_i - x_i)^{q+1} + \sum_{i=0}^{n-1} (x_{i+1} - \xi_i)^{q+1} \right]^{\frac{1}{q}} \\
&\leq \frac{1}{(q+1)^{\frac{1}{q}}} \|f'\|_p \left(\sum_{i=0}^{n-1} h_i^{q+1} \right)^{\frac{1}{q}}.
\end{aligned}
$$

Remark 1.17. If we choose in (1.137), $\xi_i = \frac{x_i + x_{i+1}}{2}$, then we get the midpoint quadrature formula

$$
\int_a^b f(x)\, dx = M(\Delta_n, f) + R_M(\Delta_n, f),
$$

where

$$
|R_M(\Delta_n, f)| \leq \frac{1}{2(q+1)^{\frac{1}{q}}} \|f'\|_p \left(\sum_{i=0}^{n-1} h_i^{q+1} \right)^{\frac{1}{q}}.
$$

The following corollary also holds

Corollary 1.60. Let f, Δ_n be as above and $\xi_i \in \left[\frac{5x_i + x_{i+1}}{6}, \frac{x_i + 5x_{i+1}}{6} \right]$ $(i = 0, \dots, n-1)$. Then we have the formula:

$$
(1.139) \qquad \int_a^b f(x)\, dx = \frac{1}{6} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})] h_i + \frac{2}{3} \sum_{i=0}^{n-1} f(\xi_i) h_i + R_S(\xi, \Delta_n, f),
$$

where the remainder, $R_S(\xi, \Delta_n, f)$, satisfies the estimate:

$$
\begin{aligned}
(1.140) \qquad & |R_S(\xi, \Delta_n, f)| \\
&\leq \frac{1}{(q+1)^{\frac{1}{q}}} \|f'\|_p \left[\frac{1}{3 \cdot 6^q} \sum_{i=0}^{n-1} h_i^{q+1} + \sum_{i=0}^{n-1} \left(\xi_i - \frac{5x_i + x_{i+1}}{6} \right)^{q+1} \right. \\
&\quad \left. + \sum_{i=0}^{n-1} \left(\frac{x_i + 5x_{i+1}}{6} - \xi_i \right)^{q+1} \right]^{\frac{1}{q}}.
\end{aligned}
$$

Remark 1.18. Now, if in (1.139) we choose $\xi_i = \frac{x_i + x_{i+1}}{2}$, then we obtain "*Simpson's quadrature formula*"

$$
\begin{aligned}
(1.141) \qquad & \int_a^b f(x)\, dx \\
&= \frac{1}{6} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})] h_i + \frac{2}{3} \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2} \right) h_i + R_S(\Delta_n, f),
\end{aligned}
$$

where the remainder term $R_S(\Delta_n, f)$ satisfies the inequality [**8**]:

$$
(1.142) \qquad |R_S(\Delta_n, f)| \leq \frac{1}{6(q+1)^{\frac{1}{q}}} \left(\frac{2^{q+1} + 1}{3} \right)^{\frac{1}{q}} \|f'\|_p \left(\sum_{i=0}^{n-1} h_i^{q+1} \right)^{\frac{1}{q}}.
$$

The following corollary also holds.

Corollary 1.61. Let f, Δ_n be as above and $x_i \leq \alpha_i^{(1)} \leq \frac{x_i+x_{i+1}}{2} \leq \alpha_i^{(2)} \leq x_{i+1}$ $(i = 0, ..., n-1)$. Then we have the formula

$$\begin{aligned} &\int_a^b f(x)\,dx \\ = \; & \sum_{i=0}^{n-1} \left(\alpha_i^{(1)} - x_i\right) f(x_i) + \sum_{i=0}^{n-1} \left(\alpha_i^{(2)} - \alpha_i^{(1)}\right) f\left(\frac{x_i + x_{i+1}}{2}\right) \\ & + \sum_{i=0}^{n-1} \left(x_{i+1} - \alpha_i^{(2)}\right) f(x_{i+1}) + R_B\left(\alpha^{(1)}, \alpha^{(2)}, \Delta_n, f\right). \end{aligned} \tag{1.143}$$

The remainder satisfies the estimate

$$\begin{aligned} &\left| R_B\left(\alpha^{(1)}, \alpha^{(2)}, \Delta_n, f\right)\right| \\ \leq \; & \frac{1}{(q+1)^{\frac{1}{q}}} \|f'\|_p \left[\sum_{i=0}^{n-1} \left(\alpha_i^{(1)} - x_i\right)^{q+1} + \sum_{i=0}^{n-1} \left(\frac{x_i + x_{i+1}}{2} - \alpha_i^{(1)}\right)^{q+1} \right. \\ & \left. + \sum_{i=0}^{n-1} \left(\alpha_i^{(2)} - \frac{x_i + x_{i+1}}{2}\right)^{q+1} + \sum_{i=0}^{n-1} \left(x_{i+1} - \alpha_i^{(2)}\right)^{q+1}\right]^{\frac{1}{q}} \\ \leq \; & \frac{1}{2(q+1)^{\frac{1}{q}}} \|f'\|_p \left(\sum_{i=0}^{n-1} h_i^{q+1}\right)^{\frac{1}{q}}. \end{aligned} \tag{1.144}$$

Finally, we have

Remark 1.19. If we choose in (1.143), $\alpha_i^{(1)} = \frac{3x_i + x_{i+1}}{4}$ and $\alpha_i^{(2)} = \frac{x_i + 3x_{i+1}}{4}$, then we get the formula:

$$\int_a^b f(x)\,dx = \frac{1}{2}\left[T(\Delta_n, f) + M(\Delta_n, f)\right] + R_B(\Delta_n, f). \tag{1.145}$$

The remainder $R_B(\Delta_n, f)$ satisfies the bound:

$$|R_B(\Delta_n, f)| \leq \frac{1}{4(q+1)^{\frac{1}{q}}} \|f'\|_p \left(\sum_{i=0}^{n-1} h_i^{q+1}\right)^{\frac{1}{q}}. \tag{1.146}$$

1.5. Generalisations in Terms of L_1-norm

1.5.1. Some Inequalities. We start with the following theorem [**6**].

Theorem 1.62. *Let $I_k : a = x_0 < x_1 < ... < x_{k-1} < x_k = b$ be a division of the interval $[a, b]$ and α_i $(i = 0, ..., k+1)$ be "$k+2$" points such that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, ..., k)$ and $\alpha_{k+1} = b$. If $f : [a, b] \to \mathbb{R}$ is absolutely continuous* *on* $[a,b]$*, then we have the inequality:*

$$
\begin{aligned}
(1.147)\qquad & \left|\int_a^b f(x)\,dx-\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\,f(x_i)\right| \\
\leq\ & \left[\frac{1}{2}\nu(h)+\max\left\{\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|,i=0,...,k-1\right\}\right]\|f'\|_1 \\
\leq\ & \nu(h)\|f'\|_1,
\end{aligned}
$$

where $\nu(h):=\max\{h_i|i=0,...,k-1\}$*,* $h_i:=x_{i+1}-x_i\ (i=0,...,k-1)$ *and* $\|f'\|_1:=\int_a^b|f'(t)|\,dt$*, is the usual* $L_1[a,b]-$*norm.*

Proof. Integrating by parts, we have (see also Theorem 1.21):

$$
(1.148)\qquad \int_a^b f(t)\,dt=\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\,f(x_i)-\int_a^b K(t)\,f'(t)\,dt.
$$

On the other hand, we have

$$
\left|\int_a^b K(t)\,f'(t)\,dt\right|\leq\sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}|t-\alpha_{i+1}|\,|f'(t)|\,dt=:T.
$$

However,

$$
\begin{aligned}
\int_{x_i}^{x_{i+1}}|t-\alpha_{i+1}|\,|f'(t)|\,dt &\leq \sup_{t\in[x_i,x_{i+1}]}|t-\alpha_{i+1}|\int_{x_i}^{x_{i+1}}|f'(t)|\,dt \\
&= \max\{\alpha_{i+1}-x_i,x_{i+1}-\alpha_{i+1}\}\int_{x_i}^{x_{i+1}}|f'(t)|\,dt \\
&= \left[\frac{1}{2}(x_{i+1}-x_i)+\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right]\int_{x_i}^{x_{i+1}}|f'(t)|\,dt.
\end{aligned}
$$

Then

$$
\begin{aligned}
T &\leq \sum_{i=0}^{k-1}\left[\frac{1}{2}h_i+\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right]\int_{x_i}^{x_{i+1}}|f'(t)|\,dt \\
&\leq \max_{i=0,\dots,k-1}\left[\frac{1}{2}h_i+\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right]\sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}|f'(t)|\,dt \\
&\leq \left[\frac{1}{2}\nu(h)+\max\left\{\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|,i=0,...,k-1\right\}\right]\|f'\|_1=:V
\end{aligned}
$$

Now, as

$$
\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\leq\frac{1}{2}h_i,
$$

then

$$
\max\left\{\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|,i=0,...,k-1\right\}\leq\frac{1}{2}\nu(h)
$$

and, consequently,

$$
V\leq\nu(h)\|f'\|_1.
$$

The theorem is completely proved. ∎

Now, if we assume that the points of the division I_k are given, then the best inequality we can obtain from Theorem 1.43 is embodied in the following corollary.

COROLLARY 1.63. Let f and I_k be as above. Then we have the inequality:

$$\begin{aligned} &\left| \int_a^b f(x)\,dx \right. \\ &\left. -\frac{1}{2}\left[(x_1 - a) f(a) + \sum_{i=1}^{k-1} (x_{i+1} - x_{i-1}) f(x_i) + (b - x_{k-1}) f(b) \right] \right| \\ \leq\; &\frac{1}{2}\nu(h)\|f'\|_1 \end{aligned} \tag{1.149}$$

PROOF. The proof is obvious by the above theorem and we omit the details. ∎

The following corollary for equidistant partitioning is useful in practice.

COROLLARY 1.64. Let

$$I_k : x_i := a + (b-a)\cdot\frac{i}{k}, \quad (i = 0, ..., k)$$

be an equidistant partitioning of $[a,b]$. If f is as above, then we have the inequality:

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \left[\frac{1}{k}\cdot\frac{f(a)+f(b)}{2}(b-a) + \frac{(b-a)}{k}\sum_{i=1}^{k-1} f\left[\frac{(k-i)a + ib}{k}\right]\right]\right| \\ \leq\; &\frac{1}{2k}(b-a)\|f'\|_1 . \end{aligned} \tag{1.150}$$

1.5.2. A General Quadrature Formula. Let $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ be a sequence of divisions of $[a,b]$ and consider the sequence of numerical integration formulae (see Subsection 1.3.2 and 1.4.2)

$$I_n(f, \Delta_n, w_n) := \sum_{j=0}^{n} w_j^{(n)} f\left(x_j^{(n)}\right),$$

where $w_j^{(n)}$ $(j = 0, ..., n)$ are the quadrature weights and $\sum_{j=0}^{n} w_j^{(n)} = b - a$.

The following theorem provides a sufficient condition for the weights $w_j^{(n)}$ such that $I_n(f, \Delta_n, w_n)$ approximates the integral $\int_a^b f(x)\,dx$ with an error expressed in terms of $\|f'\|_1$, [**6**].

THEOREM 1.65. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a,b]$. If the quadrature weights $w_j^{(n)}$ satisfy the condition*

$$x_i^{(n)} - a \leq \sum_{j=0}^{i} w_j^{(n)} \leq x_{i+1}^{(n)} - a \text{ for all } i = 0, ..., n-1,$$

then we have the estimate

$$
\begin{aligned}
&(1.151)\quad \left| I_n(f,\Delta_n,w_n) - \int_a^b f(x)\,dx \right| \\
&\leq \left[\frac{1}{2}\nu\left(h^{(n)}\right) + \max\left\{ \left| a + \sum_{j=0}^{i} w_j^{(n)} - \frac{x_i^{(n)} + x_{i+1}^{(n)}}{2} \right|, i = 0,...,n-1 \right\} \right] \|f'\|_1 \\
&\leq \nu\left(h^{(n)}\right) \|f'\|_1 ,
\end{aligned}
$$

where $\nu\left(h^{(n)}\right) := \max\left\{ h_i^{(n)} | i = 0,...,n-1 \right\}$ *and* $h_i^{(n)} := x_{i+1}^{(n)} - x_i^{(n)}$.
In particular,

$$
\lim_{\nu\left(h^{(n)}\right)\to 0} I_n(f,\Delta_n,w_n) = \int_a^b f(x)\,dx \tag{1.152}
$$

uniformly by the influence of the w_n.

Proof. The proof is similar to that of Theorem 1.46 and we omit it. ∎

Now, consider the equidistant partitioning of $[a,b]$ given by

$$
E_n : x_i^{(n)} := a + \frac{i}{n}\cdot(b-a) \quad (i = 0,...,n);
$$

and define the sequence of numerical quadrature formulae by

$$
I_n(f,w_n) := \sum_{i=0}^{n} w_i^{(n)} f\left[a + \frac{i}{n}\cdot(b-a) \right].
$$

The following corollary which can be more useful in practice holds:

Corollary 1.66. Let f be as above. If the quadrature weights $w_j^{(n)}$ satisfy the condition:

$$
\frac{i}{n} \leq \frac{1}{b-a}\sum_{j=0}^{i} w_j^{(n)} \leq \frac{i+1}{n},\ i = 0,...,n-1; \tag{1.153}
$$

then we have:

$$
\begin{aligned}
&(1.154)\quad \left| I_n(f,w_n) - \int_a^b f(x)\,dx \right| \\
&\leq \left[\frac{b-a}{2n} + \max\left\{ \left| a + \sum_{j=0}^{i} w_j^{(n)} - \frac{2i+1}{2}\cdot\frac{(b-a)}{n} \right|, i = 0,...,n-1 \right\} \right] \|f'\|_1 \\
&\leq \frac{(b-a)}{n}\|f'\|_1 .
\end{aligned}
$$

In particular, we have the limit

$$
\lim_{n\to\infty} I_n(f,w_n) = \int_a^b f(x)\,dx,
$$

uniformly by the influence of the weights w_n.

1.5.3. Particular Inequalities. The following proposition holds [**6**] (see also [**4**]).

PROPOSITION 1.67. Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a,b]$. Then we have the inequality:

$$\left| \int_a^b f(x)\,dx - [(\alpha - a) f(a) + (b - \alpha) f(b)] \right| \leq \left[\frac{1}{2}(b-a) + \left| \alpha - \frac{a+b}{2} \right| \right] \|f'\|_1 \tag{1.155}$$

for all $\alpha \in [a,b]$.

The proof follows by Theorem 1.43 choosing $x_0 = a, x_1 = b, \alpha_0 = a, \alpha_1 = \alpha \in [a,b]$ and $\alpha_2 = b$.

REMARK 1.20. a) If in (1.155) we put $\alpha = b$, then we get the "*left rectangle inequality*"

$$\left| \int_a^b f(x)\,dx - (b-a) f(a) \right| \leq (b-a) \|f'\|_1 ; \tag{1.156}$$

b) If $\alpha = a$, then, by (1.155), we get the "*right rectangle inequality*"

$$\left| \int_a^b f(x)\,dx - (b-a) f(b) \right| \leq (b-a) \|f'\|_1 ; \tag{1.157}$$

c) It is easy to see that the best inequality we can get from (1.155) is for $\alpha = \frac{a+b}{2}$ obtaining the "*trapezoid inequality*" [**4**]:

$$\left| \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a) \right| \leq \frac{1}{2}(b-a) \|f'\|_1 . \tag{1.158}$$

Another proposition with many interesting particular cases is the following one [**6**] (see also [**3**]):

PROPOSITION 1.68. Let f be as above and $a \leq x_1 \leq b$, $a \leq \alpha_1 \leq x_1 \leq \alpha_2 \leq b$. Then we have

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - [(\alpha_1 - a) f(a) + (\alpha_2 - \alpha_1) f(x_1) + (b - \alpha_2) f(b)] \right| \\ \leq\ & \frac{1}{2} \left[\frac{1}{2}(b-a) + \left| x_1 - \frac{a+b}{2} \right| + \left| \alpha_1 - \frac{a+x_1}{2} \right| \right. \\ & \left. + \left| \alpha_2 - \frac{x_1+b}{2} \right| + \left| \left| \alpha_1 - \frac{a+x_1}{2} \right| - \left| \alpha_2 - \frac{x_1+b}{2} \right| \right| \right] \|f'\|_1 \\ \leq\ & \left[\frac{(b-a)}{2} + \left| x_1 - \frac{a+b}{2} \right| \right] \|f'\|_1 \leq (b-a) \|f'\|_1 . \end{aligned} \tag{1.159}$$

REMARK 1.21. If we choose above $\alpha_1 = a, \alpha_2 = b$, then we get the following Ostrowski's type inequality obtained by Dragomir-Wang in the recent paper [**17**]:

$$\left| \int_a^b f(x)\,dx - (b-a) f(x_1) \right| \leq \left[\frac{1}{2}(b-a) + \left| x_1 - \frac{a+b}{2} \right| \right] \|f'\|_1 \tag{1.160}$$

for all $x_1 \in [a, b]$.

We note that the best inequality we can get in (1.115) is for $x_1 = \frac{a+b}{2}$ obtaining the "*midpoint inequality*" (see also [**2**])

$$\left| \int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)(b-a) \right| \leq \frac{1}{2}(b-a)\|f'\|_1 \tag{1.161}$$

b) If we choose in (1.159) $\alpha_1 = \frac{5a+b}{6}, \alpha_2 = \frac{a+5b}{6}$ and $x_1 \in \left[\frac{5a+b}{6}, \frac{a+5b}{6}\right]$, then we get

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f(x_1)\right] \right| \\ \leq\ & \frac{1}{2}\left[\frac{1}{2}\cdot(b-a) + \left|x_1 - \frac{a+b}{2}\right| \right. \\ & \left. + \max\left\{\left|x_1 - \frac{2a+b}{3}\right|, \left|\frac{a+2b}{3} - x_1\right|\right\}\right]. \end{aligned} \tag{1.162}$$

In particular, if we choose in (1.162), $x_1 = \frac{a+b}{2}$, then we get the following "*Simpson's inequality*" [**9**]

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] \right| \\ \leq\ & \frac{1}{3}(b-a)\|f'\|_1. \end{aligned} \tag{1.163}$$

1.5.4. Particular Quadrature Formulae. Let us consider the partitioning of the interval $[a, b]$ given by $\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ and put $h_i := x_{i+1} - x_i$ $(i = 0, ..., n-1)$ and $\nu(h) := \max\{h_i | i = 0, ..., n-1\}$.

The following theorem holds [**6**]:

THEOREM 1.69. *Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous on $[a, b]$ and $k \geq 1$. Then we have the composite quadrature formula*

$$\int_a^b f(x)\,dx = A_k(\Delta_n, f) + R_k(\Delta_n, f), \tag{1.164}$$

where

$$A_k(\Delta_n, f) := \frac{1}{k}\left[T(\Delta_n, f) + \sum_{i=0}^{n}\sum_{j=1}^{k-1} f\left[\frac{(k-j)x_i + jx_{i+1}}{k}\right] h_i\right] \tag{1.165}$$

and

$$T(\Delta_n, f) := \frac{1}{2}\sum_{i=0}^{n-1}[f(x_i) + f(x_{i+1})] h_i \tag{1.166}$$

is the trapezoid quadrature rule.

The remainder $R_k(\Delta_n, f)$ *satisfies the estimate*

$$|R_k(\Delta_n, f)| \leq \frac{1}{2k}\nu(h)\|f'\|_1. \tag{1.167}$$

Proof. Applying Corollary 1.45 on the intervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$, we obtain

$$\begin{aligned} &\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \left[\frac{1}{k}\cdot\frac{f(x_i)+f(x_{i+1})}{2}h_i + \frac{h_i}{k}\sum_{j=1}^{k} f\left[\frac{(k-j)x_i + jx_{i+1}}{k}\right]\right]\right| \\ \leq\ & \frac{1}{2k}h_i\int_{x_i}^{x_{i+1}} |f'(t)|\,dt. \end{aligned}$$

Now, using the generalized triangle inequality, we get:

$$\begin{aligned} &|R_k(\Delta_n, f)| \\ \leq\ & \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)\,dx \right.\\ & \left. -\left[\frac{1}{k}\cdot\frac{f(x_i)+f(x_{i+1})}{2}h_i + \frac{h_i}{k}\sum_{j=1}^{k-1} f\left[\frac{(k-j)x_i + jx_{i+1}}{k}\right]\right]\right| \\ \leq\ & \frac{1}{2k}\sum_{i=0}^{n-1} h_i\int_{x_i}^{x_{i+1}} |f'(t)|\,dt \leq \frac{\nu(h)}{2k}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} |f'(t)|\,dt = \frac{\nu(h)}{2k}\|f'\|_1, \end{aligned}$$

and the theorem is proved. ∎

The following corollaries hold:

Corollary 1.70. Let f be as above. Then we have the formula:

$$\int_a^b f(x)\,dx = \frac{1}{2}[T_n(\Delta_n, f) + M_n(\Delta_n, f)] + R_2(\Delta_n, f), \tag{1.168}$$

where $M_n(\Delta_n, f)$ is the midpoint quadrature formula,

$$M_n(\Delta_n, f) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)h_i$$

and the remainder $R_2(\Delta_n, f)$ satisfies the inequality:

$$|R_2(\Delta_n, f)| \leq \frac{1}{4}\nu(h)\|f'\|_1. \tag{1.169}$$

Corollary 1.71. Under the above assumptions, we have

$$\begin{aligned} &\int_a^b f(x)\,dx \\ =\ & \frac{1}{3}\left[T_n(\Delta_n, f) + \sum_{i=0}^{n-1} f\left(\frac{2x_i + x_{i+1}}{3}\right)h_i + \sum_{i=0}^{n-1} f\left(\frac{x_i + 2x_{i+1}}{3}\right)h_i\right] \\ & +R_3(\Delta_n, f), \end{aligned} \tag{1.170}$$

where the remainder, $R_3(\Delta_n, f)$, satisfies the bound:

$$|R_3(\Delta_n, f)| \leq \frac{1}{6}\nu(h)\|f'\|_1. \tag{1.171}$$

The following theorem holds [**6**] (see also [**4**]):

THEOREM 1.72. *Let f and Δ_n be as above and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$. Then we have the quadrature formula:*

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1}\left[(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})\right] + R(\xi, \Delta_n, f), \tag{1.172}$$

where the remainder, $R(\xi, \Delta_n, f)$, satisfies the estimation:

$$\begin{aligned} & |R(\xi, \Delta_n, f)| \\ \leq & \left[\frac{1}{2}\nu(h) + \max\left\{\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|,\ i = 0, ..., n-1\right\}\right]\|f'\|_1 \\ \leq & \nu(h)\|f'\|_1. \end{aligned} \tag{1.173}$$

for all ξ_i as above.

PROOF. Follows by Proposition 1.48 and we omit the details. ∎

COROLLARY 1.73. Let f and Δ_n be as above. Then we have

(1) The "*left rectangle rule*"

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(x_i) h_i + R_l(\Delta_n, f); \tag{1.174}$$

(2) The "*right rectangle rule*"

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(x_{i+1}) h_i + R_r(\Delta_n, f); \tag{1.175}$$

(3) The "*trapezoid rule*"

$$\int_a^b f(x)\,dx = T(\Delta_n, f) + R_T(\Delta_n, f), \tag{1.176}$$

where

$$|R_l(\Delta_n, f)|\,|R_r(\Delta_n, f)| \leq \nu(h)\|f'\|_1$$

and

$$|R_T(\Delta_n, f)| \leq \frac{1}{2}\nu(h)\|f'\|_1.$$

The following theorem also holds [**6**].

THEOREM 1.74. *Let f and Δ_n be as above and $\xi_i \in [x_i, x_{i+1}]$, $x_i \le \alpha_i^{(1)} \le \xi_i \le \alpha_i^{(2)} \le x_{i+1}$, then we have the quadrature formula:*

$$\begin{aligned}
&\int_a^b f(x)\,dx \\
&= \sum_{i=0}^{n-1} \left(\alpha_i^{(1)} - x_i\right) f(x_i) + \sum_{i=0}^{n-1} \left(\alpha_i^{(2)} - \alpha_i^{(1)}\right) f(\xi_i) \\
&\quad + \sum_{i=0}^{n-1} \left(x_{i+1} - \alpha_i^{(2)}\right) f(x_{i+1}) + R\left(\xi, \alpha^{(1)}, \alpha^{(2)}, \Delta_n, f\right),
\end{aligned} \tag{1.177}$$

where the remainder, $R\left(\xi, \alpha^{(1)}, \alpha^{(2)}, \Delta_n, f\right)$, satisfies the estimate

$$\begin{aligned}
&\left|R\left(\xi, \alpha^{(1)}, \alpha^{(2)}, \Delta_n, f\right)\right| \\
\le\ & \left\{\frac{1}{2}\left[\frac{1}{2}\nu(h) + \max_{i=0,\dots,n-1}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\right. \\
& \left. + \max\left\{\max_{i=0,\dots,n-1}\left|\alpha_i^{(1)} - \frac{x_i + \xi_i}{2}\right|, \max_{i=0,\dots,n-1}\left|\alpha_i^{(2)} - \frac{\xi_i + x_{i+1}}{2}\right|\right\}\right\} \|f'\|_1 \\
\le\ & \left[\frac{1}{2}\nu(h) + \max_{i=0,\dots n-1}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \|f'\|_1 \le \nu(h)\|f'\|_1 .
\end{aligned} \tag{1.178}$$

PROOF. Follows by Proposition 1.49 and we omit the details. ∎

The following corollary is the result of Dragomir-Wang from the recent paper [**17**]

COROLLARY 1.75. Under the above assumptions, we have the Riemann's quadrature formula:

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(\xi_i)\,h_i + R_R(\xi, \Delta_n, f). \tag{1.179}$$

The remainder $R_R(\xi, \Delta_n, f)$ satisfies the bound

$$\begin{aligned}
&|R_R(\xi, \Delta_n, f)| \\
\le\ & \left[\frac{1}{2}\nu(h) + \max\left\{\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|, i = 0, \dots, n-1\right\}\right] \|f'\|_1 \\
\le\ & \nu(h)\|f'\|_1
\end{aligned} \tag{1.180}$$

for all $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, \dots, n)$.

Finally, the following corollary which generalizes Simpson's quadrature formula holds.

COROLLARY 1.76. Under the above assumptions and if $\xi_i \in \left[\frac{x_{i+1}+5x_i}{6}, \frac{x_i+5x_{i+1}}{6}\right]$ $(i = 0, \dots, n-1)$, then we have the formula:

$$\int_a^b f(x)\,dx = \frac{1}{6}\sum_{i=0}^{n-1}\left[f(x_i) + f(x_{i+1})\right] h_i + \frac{2}{3}\sum_{i=0}^{n-1} f(\xi_i)\,h_i + S(f, \Delta_n, \xi), \tag{1.181}$$

where the remainder, $S(f,\Delta_n,\xi)$, satisfies the estimate:

$$|S(f,\Delta_n,\xi)| \leq \left\{\frac{1}{2}\left[\frac{\nu(h)}{2}+\max_{i=0,\dots,n-1}\left\{\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right\} + \max\left\{\max_{i=0,\dots,n-1}\left|\xi_i-\frac{2x_i+x_{i+1}}{3}\right|, \max_{i=0,\dots,n-1}\left|\frac{x_i+2x_{i+1}}{3}-\xi_i\right|\right\}\right]\right\}\|f'\|_1. \tag{1.182}$$

The proof follows by the inequality (1.162) and we omit the details.

Remark 1.22. Now, if we choose in (1.181), $\xi_i=\frac{x_i+x_{i+1}}{2}$, then we get "*Simpson's quadrature formula*" [**9**]

$$\int_a^b f(x)\,dx = \frac{1}{6}\sum_{i=0}^{n-1}\left[f(x_i)+f(x_{i+1})\right]h_i + \frac{2}{3}\sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)h_i + S(f,\Delta_n), \tag{1.183}$$

where the remainder term $S(f,\Delta_n)$ satisfies the bound:

$$|S(f,\Delta_n)| \leq \frac{1}{3}\nu(h)\|f'\|_1. \tag{1.184}$$

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CHAPTER 2

Integral Inequalities for $n-$Times Differentiable Mappings

by

A. SOFO

Abstract This chapter investigates generalisations of Integral Inequalities for n-times differentiable mappings. With the aid of the modern theory of inequalities and by the use of a general Peano kernel, explicit bounds for interior point rules are obtained.
Integral equalities are obtained which are then used to obtain inequalities for n-times differentiable mappings on the three norms $\|\cdot\|_{\infty}$, $\|\cdot\|_{p}$ and $\|\cdot\|_{1}$. Some particular inequalities are investigated which include explicit bounds for perturbed trapezoid, midpoint, Simpson's, Newton-Cotes and left and right rectangle rules. The inequalities are also applied to various composite quadrature rules and the analysis allows the determination of the partition required that would assure that the accuracy of the result would be within a prescribed error tolerance.

2.1. Introduction

In 1938 Ostrowski [**29**] obtained a bound for the absolute value of the difference of a function to its average over a finite interval. The theorem is as follows.

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on $[a,b]$ and let $|f'(t)|\le M$ for all $t\in(a,b)$, then the following bound is valid*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\,M \tag{2.1}$$

for all $x\in[a,b]$.
The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

Dragomir and Wang [**19, 20, 21, 22**] extended the result (2.1) and applied the extended result to numerical quadrature rules and to the estimation of error bounds for some special means.

Dragomir [**14, 15, 16**] further extended the result (2.1) to incorporate mappings of bounded variation, Lipschitzian mappings and monotonic mappings.

Cerone, Dragomir and Roumeliotis [**6**] as well as Dedić, Matić and Pečarić [**8**] and Pearce, Pečarić, Ujević and Varošanec [**30**] further extended the result (2.1) by considering $n-$times differentiable mappings on an interior point $x \in [a,b]$.

Cerone and Dragomir [**1, 2**] have subsequently given a number of other trapezoidal and midpoint rules based on the Peano kernel approach.

Dragomir [**9, 10, 11**], further refined the inequality (2.1) by considering an interval $[a,b]$ with a multiple number of subdivisions.

In this current work we extend, subsume and generalise some previous results, by considering $n-$times differentiable mappings. We investigate interior point rules by taking into consideration multiple subdivisions of an interval $[a,b]$. Moreover, we obtain explicit bounds through the use of a Peano kernel approach and the modern theory of inequalities. This approach also permits the investigation of quadrature rules that place fewer restrictions on the behaviour of the integrand and thus admits a larger class of functions.

In Section 2.2 we develop a number of integral identities, for $n-$time differentiable mappings, which are of interest in themselves and utilise them, in Section 2.3, to obtain integral inequalities on the Lebesgue spaces, $L_\infty[a,b]$, $L_p[a,b]$ and $L_1[a,b]$.

In Section 2.4 we investigate the convergence of a general quadrature formula that permits the approximation of the integral of a function over a finite interval.

In Section 2.5 we employ the pre-Grüss relationship to obtain more integral inequalities for $n-$times differentiable mappings.

In Section 2.6 we point out a number of particular special cases that incorporate the generalised left and right rectangle inequalities, the perturbed trapezoid and midpoint inequalities, Simpson's inequality, the generalised Newton-Cotes three eighths inequality and a Boole type relationship.

Finally, in Section 2.7, we apply some of the inequalities to numerical quadrature rules.

2.2. Integral Identities

THEOREM 2.2. *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a,b]$ and α_i $(i = 0, \ldots, k+1)$ be '$k+2$' points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, k)$ and $\alpha_{k+1} = b$. If $f : [a,b] \to \mathbb{R}$ is a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$, then for all $x_i \in [a,b]$ we have the identity:*

$$\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \sum_{i=0}^{k-1} \Big\{ (x_{i+1} - \alpha_{i+1})^j f^{(j-1)}(x_{i+1}) - (x_i - \alpha_{i+1})^j f^{(j-1)}(x_i) \Big\} = (-1)^n \int_a^b K_{n,k}(t) f^{(n)}(t)\,dt, \tag{2.2}$$

where the Peano kernel

$$K_{n,k}(t) := \begin{cases} \dfrac{(t-\alpha_1)^n}{n!}, & t \in [a, x_1) \\ \dfrac{(t-\alpha_2)^n}{n!}, & t \in [x_1, x_2) \\ \vdots & \\ \dfrac{(t-\alpha_{k-1})^n}{n!}, & t \in [x_{k-2}, x_{k-1}) \\ \dfrac{(t-\alpha_k)^n}{n!}, & t \in [x_{k-1}, b], \end{cases} \tag{2.3}$$

n and k are natural numbers, $n \geq 1$, $k \geq 1$ and $f^{(0)}(x) = f(x)$.

Proof. The proof is by mathematical induction. For $n = 1$, from (2.2) we have the equality

$$\begin{aligned} \int_a^b f(t)\,dt &= \sum_{i=0}^{k-1} \Big[(x_{i+1} - \alpha_{i+1})^j f(x_{i+1}) - (x_i - \alpha_{i+1})^j f(x_i) \Big] \\ &\quad - \int_a^b K_{1,k}(t) f'(t)\,dt, \end{aligned} \tag{2.4}$$

where

$$K_{1,k}(t) := \begin{cases} (t-\alpha_1), & t \in [a, x_1) \\ (t-\alpha_2), & t \in [x_1, x_2) \\ \vdots & \\ (t-\alpha_{k-1}), & t \in [x_{k-2}, x_{k-1}) \\ (t-\alpha_k), & t \in [x_{k-1}, b]. \end{cases}$$

To prove (2.4), we integrate by parts as follows

$$\begin{aligned} &\int_a^b K_{1,k}(t) f'(t)\,dt = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} (t - \alpha_{i+1}) f'(t)\,dt \\ = &\sum_{i=0}^{k-1} \left[(t - \alpha_{i+1}) f(t) \Big|_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} f(t)\,dt \right] \\ = &\sum_{i=0}^{k-1} \left[(x_{i+1} - \alpha_{i+1}) f(x_{i+1}) - (x_i - \alpha_{i+1}) f(x_i) \right] - \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} f(t)\,dt. \end{aligned}$$

$$\begin{aligned} &\int_a^b f(t)\,dt + \int_a^b K_{1,k}(t) f'(t)\,dt \\ = &\sum_{i=0}^{k-1} \left[(x_{i+1} - \alpha_{i+1}) f(x_{i+1}) - (x_i - \alpha_{i+1}) f(x_i) \right]. \end{aligned}$$

Hence (2.4) is proved.

Assume that (2.2) holds for 'n' and let us prove it for '$n+1$'. We need to prove the equality

$$
\begin{aligned}
(2.5) \qquad & \int_a^b f(t)\,dt + \sum_{i=0}^{k-1}\sum_{j=1}^{n+1} \frac{(-1)^j}{j!}\Big\{ (x_{i+1}-\alpha_{i+1})^j f^{(j-1)}(x_{i+1}) \\
& - (x_i-\alpha_{i+1})^j f^{(j-1)}(x_i) \Big\} \\
= \; & (-1)^{n+1} \int_a^b K_{n+1,k}(t) f^{(n+1)}(t)\,dt,
\end{aligned}
$$

where from (2.3)

$$
K_{n+1,k}(t) := \begin{cases}
\dfrac{(t-\alpha_1)^{n+1}}{(n+1)!}, & t \in [a, x_1) \\[2ex]
\dfrac{(t-\alpha_2)^{n+1}}{(n+1)!}, & t \in [x_1, x_2) \\[2ex]
\qquad \vdots & \\[2ex]
\dfrac{(t-\alpha_{k-1})^{n+1}}{(n+1)!}, & t \in [x_{k-2}, x_{k-1}) \\[2ex]
\dfrac{(t-\alpha_k)^{n+1}}{(n+1)!}, & t \in [x_{k-1}, b].
\end{cases}
$$

Consider

$$
\int_a^b K_{n+1,k}(t) f^{(n+1)}(t)\,dt = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{(t-\alpha_{i+1})^{n+1}}{(n+1)!} f^{(n+1)}(t)\,dt
$$

and upon integrating by parts we have

$$
\begin{aligned}
& \int_a^b K_{n+1,k}(t) f^{(n+1)}(t)\,dt \\
= \; & \sum_{i=0}^{k-1} \left[\frac{(t-\alpha_{i+1})^{n+1}}{(n+1)!} f^{(n)}(t) \Big|_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} \frac{(t-\alpha_{i+1})^{n}}{n!} f^{(n)}(t)\,dt \right] \\
= \; & \sum_{i=0}^{k-1} \left\{ \frac{(x_{i+1}-\alpha_{i+1})^{n+1}}{(n+1)!} f^{(n)}(x_{i+1}) - \frac{(x_i-\alpha_{i+1})^{n+1}}{(n+1)!} f^{(n)}(x_i) \right\} \\
& - \int_a^b K_{n,k}(t) f^{(n)}(t)\,dt.
\end{aligned}
$$

Upon rearrangement we may write

$$
\begin{aligned}
& \int_a^b K_{n,k}(t)\, f^{(n)}(t)\, dt \\
= & \sum_{i=0}^{k-1} \left\{ \frac{(x_{i+1}-\alpha_{i+1})^{n+1}}{(n+1)!} f^{(n)}(x_{i+1}) - \frac{(x_i-\alpha_{i+1})^{n+1}}{(n+1)!} f^{(n)}(x_i) \right\} \\
& - \int_a^b K_{n+1,k}(t)\, f^{(n+1)}(t)\, dt.
\end{aligned}
$$

Now substitute $\int_a^b K_{n,k}(t)\, f^{(n)}(t)\, dt$ from the induction hypothesis (2.2) such that

$$
\begin{aligned}
& (-1)^n \int_a^b f(t)\, dt + (-1)^n \sum_{i=0}^{k-1} \left[\sum_{j=1}^{n} \frac{(-1)^j}{j!} \left\{ (x_{i+1}-\alpha_{i+1})^j \right.\right. \\
& \left.\left. \times f^{(j-1)}(x_{i+1}) - \ (x_i-\alpha_{i+1})^j f^{(j-1)}(x_i) \right\} \right] \\
= & \sum_{i=0}^{k-1} \left\{ \frac{(x_{i+1}-\alpha_{i+1})^{n+1}}{(n+1)!} f^{(n)}(x_{i+1}) - \frac{(x_i-\alpha_{i+1})^{n+1}}{(n+1)!} f^{(n)}(x_i) \right\} \\
& - \int_a^b K_{n+1,k}(t)\, f^{(n+1)}(t)\, dt.
\end{aligned}
$$

Collecting the second and third terms and rearranging, we can state

$$
\begin{aligned}
& \int_a^b f(t)\, dt + \sum_{i=0}^{k-1} \sum_{j=1}^{n+1} \frac{(-1)^j}{j!} \left\{ (x_{i+1}-\alpha_{i+1})^j f^{(j-1)}(x_{i+1}) \right. \\
& \left. - (x_i-\alpha_{i+1})^j f^{(j-1)}(x_i) \right\} \\
= & \ (-1)^{n+1} \int_a^b K_{n+1,k}(t)\, f^{(n+1)}(t)\, dt,
\end{aligned}
$$

which is identical to (2.5), hence Theorem 2.2 is proved. ∎

The following corollary gives a slightly different representation of Theorem 2.2, which will be useful in the following work.

Corollary 2.3. From Theorem 2.2, the equality (2.2) may be represented as

$$
\begin{aligned}
& \int_a^b f(t)\, dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\sum_{i=0}^{k} \left\{ (x_i-\alpha_i)^j - (x_i-\alpha_{i+1})^j \right\} f^{(j-1)}(x_i) \right] \\
= & \ (-1)^n \int_a^b K_{n,k}(t)\, f^{(n)}(t)\, dt.
\end{aligned} \tag{2.6}
$$

PROOF. From (2.2) consider the second term and rewrite it as

$$
\begin{aligned}
S_1+S_2 \quad : \quad &= \sum_{i=0}^{k-1}\left\{-\left(x_{i+1}-\alpha_{i+1}\right) f\left(x_{i+1}\right)+\left(x_i-\alpha_{i+1}\right) f\left(x_i\right)\right\} \\
&+\sum_{i=0}^{k-1}\left[\sum_{j=2}^{n} \frac{(-1)^j}{j!}\left\{\left(x_{i+1}-\alpha_{i+1}\right)^j f^{(j-1)}\left(x_{i+1}\right)\right.\right. \\
&\left.\left.-\left(x_i-\alpha_{i+1}\right)^j f^{(j-1)}\left(x_i\right)\right\}\right].
\end{aligned} \tag{2.7}
$$

Now

$$
\begin{aligned}
S_1 &= (a-\alpha_1) f(a)+\sum_{i=1}^{k-1}\left(x_i-\alpha_{i+1}\right) f\left(x_i\right) \\
&\quad +\sum_{i=0}^{k-2}\left\{-\left(x_{i+1}-\alpha_{i+1}\right) f\left(x_{i+1}\right)\right\}-\left(b-\alpha_k\right) f(b) \\
&= (a-\alpha_1) f(a)+\sum_{i=1}^{k-1}\left(x_i-\alpha_{i+1}\right) f\left(x_i\right) \\
&\quad +\sum_{i=1}^{k-1}\left\{-\left(x_i-\alpha_i\right) f\left(x_i\right)\right\}-\left(b-\alpha_k\right) f(b) \\
&= -\left(\alpha_1-a\right) f(a)-\sum_{i=1}^{k-1}\left(\alpha_{i+1}-\alpha_i\right) f\left(x_i\right)-\left(b-\alpha_k\right) f(b).
\end{aligned}
$$

Also,

$$
\begin{aligned}
S_2 &= \sum_{i=0}^{k-2}\left[\sum_{j=2}^{n} \frac{(-1)^j}{j!}\left\{\left(x_{i+1}-\alpha_{i+1}\right)^j f^{(j-1)}\left(x_{i+1}\right)\right\}\right] \\
&\quad +\sum_{j=2}^{n} \frac{(-1)^j}{j!}\left\{\left(x_k-\alpha_k\right)^j f^{(j-1)}\left(x_k\right)\right\} \\
&\quad -\sum_{j=2}^{n} \frac{(-1)^j}{j!}\left\{\left(x_0-\alpha_1\right)^j f^{(j-1)}\left(x_0\right)\right\} \\
&\quad -\sum_{i=1}^{k-1}\left[\sum_{j=2}^{n} \frac{(-1)^j}{j!}\left\{\left(x_i-\alpha_{i+1}\right)^j f^{(j-1)}\left(x_i\right)\right\}\right] \\
&= \sum_{j=2}^{n} \frac{(-1)^j}{j!}\left(b-\alpha_k\right)^j f^{(j-1)}(b)-\sum_{j=2}^{n} \frac{(-1)^j}{j!}\left(a-\alpha_1\right)^j f^{(j-1)}(a) \\
&\quad +\sum_{i=1}^{k-1}\left[\sum_{j=2}^{n} \frac{(-1)^j}{j!}\left\{\left(x_i-\alpha_i\right)^j-\left(x_i-\alpha_{i+1}\right)^j\right\} f^{(j-1)}\left(x_i\right)\right].
\end{aligned}
$$

From (2.7)

$$\begin{aligned} S_1+S_2 \quad : \quad &= -\left\{ (b-\alpha_k) f(b) + (\alpha_1 - a) f(a) + \sum_{i=1}^{k-1} (\alpha_{i+1}-\alpha_i) f(x_i) \right\} \\ &\quad + \sum_{j=2}^{n} \frac{(-1)^j}{j!} \left\{ (b-\alpha_k)^j f^{(j-1)}(b) - (a-\alpha_1)^j f^{(j-1)}(a) \right\} \\ &\quad + \sum_{i=1}^{k-1} \left[\sum_{j=2}^{n} \frac{(-1)^j}{j!} \left\{ (x_i-\alpha_i)^j - (x_i - \alpha_{i+1})^j \right\} f^{(j-1)}(x_i) \right] \\ &= -\left\{ (\alpha_1 - a) f(a) + \sum_{i=1}^{k-1} (\alpha_{i+1}-\alpha_i) f(x_i) + (b-\alpha_k) f(b) \right\} \end{aligned}$$

$$\begin{aligned} &+ \sum_{j=2}^{n} \frac{(-1)^j}{j!} \Bigg[-(a-\alpha_1)^j f^{(j-1)}(a) \\ &+ \sum_{i=1}^{k-1} \left\{ (x_i-\alpha_i)^j - (x_i-\alpha_{i+1})^j \right\} f^{(j-1)}(x_i) + (b-\alpha_k)^j f^{(j-1)}(b) \Bigg]. \end{aligned}$$

Keeping in mind that $x_0 = a$, $\alpha_0 = 0$, $x_k = b$ and $\alpha_{k+1} = b$ we may write

$$\begin{aligned} S_1+S_2 &= -\sum_{i=0}^{k} (\alpha_{i+1}-\alpha_i) f(x_i) \\ &\quad + \sum_{j=2}^{n} \frac{(-1)^j}{j!} \left[\sum_{i=0}^{k} \left\{ (x_i-\alpha_i)^j - (x_i-\alpha_{i+1})^j \right\} f^{(j-1)}(x_i) \right] \\ &= \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\sum_{i=0}^{k} \left\{ (x_i-\alpha_i)^j - (x_i-\alpha_{i+1})^j \right\} f^{(j-1)}(x_i) \right]. \end{aligned}$$

And substituting S_1+S_2 into the second term of (2.2) we obtain the identity (2.6). ∎

If we now assume that the points of the division I_k are fixed, we obtain the following corollary.

Corollary 2.4. Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a,b]$. If $f : [a,b] \to \mathbb{R}$ is as defined in Theorem 2.2, then we have the equality

$$\begin{aligned} (2.8) \qquad & \int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{1}{2^j j!} \left[\sum_{i=0}^{k} \left\{ -h_i^j + (-1)^j h_{i-1}^j \right\} f^{(j-1)}(x_i) \right] \\ &= (-1)^n \int_a^b K_{n,k}(t) f^{(n)}(t)\,dt, \end{aligned}$$

where $h_i := x_{i+1} - x_i$, $h_{-1} := 0$ and $h_k := 0$.

Proof. Choose

$$\begin{aligned}\alpha_0 &= a, \quad \alpha_1 = \frac{a+x_1}{2}, \quad \alpha_2 = \frac{x_1+x_2}{2}, \ \ldots, \\ \alpha_{k-1} &= \frac{x_{k-2}+x_{k-1}}{2}, \ \alpha_k = \frac{x_{k-1}+x_k}{2} \quad \text{and} \quad \alpha_{k+1} = b.\end{aligned}$$

From Corollary 2.3, the term

$$\begin{aligned}& (b-\alpha_k) f(b) + (\alpha_1 - a) f(a) + \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i) f(x_i) \\ = \ & \frac{1}{2}\left\{ h_0 f(a) + \sum_{i=1}^{k-1} (h_i + h_{i-1}) f(x_i) + h_{k-1} f(b) \right\},\end{aligned}$$

the term

$$\begin{aligned}& \sum_{j=2}^{n} \frac{(-1)^j}{j!}\left\{ (b-\alpha_k)^j f^{(j-1)}(b) - (a-\alpha_1)^j f^{(j-1)}(a) \right\} \\ = \ & \sum_{j=2}^{n} \frac{(-1)^j}{j!2^j}\left\{ h_{k-1}^j f^{(j-1)}(b) - (-1)^j h_0^j f^{(j-1)}(a) \right\}\end{aligned}$$

and the term

$$\begin{aligned}& \sum_{i=1}^{k-1}\left[\sum_{j=2}^{n} \frac{(-1)^j}{j!}\left\{ (x_i-\alpha_i)^j - (x_i - \alpha_{i+1})^j \right\} f^{(j-1)}(x_i) \right] \\ = \ & \sum_{i=1}^{k-1}\left[\sum_{j=2}^{n} \frac{(-1)^j}{j!2^j}\left\{ h_{i-1}^j - (-1)^j h_i^j \right\} f^{(j-1)}(x_i) \right].\end{aligned}$$

Putting the last three terms in (2.6) we obtain

$$\begin{aligned}& \int_a^b f(t)\,dt - \frac{1}{2}\left\{ h_0 f(a) + \sum_{i=1}^{k-1} (h_i + h_{i-1}) f(x_i) + h_{k-1} f(b) \right\} \\ & + \sum_{j=2}^{n} \frac{(-1)^j}{j!2^j}\left\{ h_{k-1}^j f^{(j-1)}(b) - (-1)^j h_0^j f^{(j-1)}(a) \right\} \\ & + \sum_{i=1}^{k-1}\left[\sum_{j=2}^{n} \frac{(-1)^j}{j!2^j}\left\{ h_{i-1}^j - (-1)^j h_i^j \right\} f^{(j-1)}(x_i) \right] \\ = \ & (-1)^n \int_a^b K_{n,k}(t) f^{(n)}(t)\,dt.\end{aligned}$$

Collecting the inner three terms of the last expression, we have

$$\begin{aligned}& \int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!2^j} \sum_{i=0}^{k}\left\{ h_{i-1}^j - (-1)^j h_i^j \right\} f^{(j-1)}(x_i) \\ = \ & (-1)^n \int_a^b K_{n,k}(t) f^{(n)}(t)\,dt,\end{aligned}$$

which is equivalent to the identity (2.8). ∎

The case of equidistant partitioning is important in practice, and with this in mind we obtain the following corollary.

Corollary 2.5. Let

$$I_k : x_i = a + i\left(\frac{b-a}{k}\right), \quad i = 0, \dots, k \tag{2.9}$$

be an equidistant partitioning of $[a,b]$, then we have the equality

$$\begin{aligned}
&\int_a^b f(t)\,dt + \sum_{j=1}^{n}\left(\frac{b-a}{2k}\right)^j \frac{1}{j!}\Bigg[-f^{(j-1)}(a) \\
&+\sum_{i=1}^{k-1}\left\{(-1)^j - 1\right\} f^{(j-1)}(x_i) + (-1)^j f^{(j-1)}(b)\Bigg] \\
=\;& (-1)^n \int_a^b K_{n,k}(t)\, f^{(n)}(t)\,dt.
\end{aligned} \tag{2.10}$$

It is of some interest to note that the second term of (2.10) involves only even derivatives at all interior points x_i, $i = 1, \dots, k-1$.

Proof. Using (2.9) we note that

$$\begin{aligned}
h_0 &= x_1 - x_0 = \frac{b-a}{k}, \quad h_{k-1} = (x_k - x_{k-1}) = \frac{b-a}{k}, \\
h_i &= x_{i+1} - x_i = \frac{b-a}{k} \text{ and } h_{i-1} = x_i - x_{i-1} = \frac{b-a}{k}, \quad (i = 1, \dots, k-1)
\end{aligned}$$

and substituting into (2.8) we have

$$\begin{aligned}
&\int_a^b f(t)\,dt + \sum_{j=1}^{n}\frac{1}{j!2^j}\Bigg[-\left(\frac{b-a}{2k}\right)^j f^{(j-1)}(a) + \sum_{i=0}^{k-1}\Bigg\{-\left(\frac{b-a}{k}\right)^j \\
&+(-1)^j\left(\frac{b-a}{k}\right)^j\Bigg\} f^{(j-1)}(x_i) + (-1)^j\left(\frac{b-a}{k}\right)^j f^{(j-1)}(b)\Bigg] \\
=\;& (-1)^n \int_a^b K_{n,k}(t)\, f^{(n)}(t)\,dt,
\end{aligned}$$

which simplifies to (2.10) after some minor manipulation. ∎

The following Taylor-like formula with integral remainder also holds.

Corollary 2.6. Let $g : [a,y] \to \mathbb{R}$ be a mapping such that $g^{(n)}$ is absolutely continuous on $[a,y]$. Then for all $x_i \in [a,y]$ we have the identity

$$\begin{aligned}
&g(y) \\
=\;& g(a) - \sum_{i=0}^{k-1}\Bigg[\sum_{j=1}^{n}\frac{(-1)^j}{j!}\Big\{(x_{i+1} - \alpha_{i+1})^j g^{(j)}(x_{i+1}) \\
&- (x_i - \alpha_{i+1})^j g^{(j)}(x_i)\Big\}\Bigg] + (-1)^n \int_a^y K_{n,k}(y,t)\, g^{(n+1)}(t)\,dt
\end{aligned} \tag{2.11}$$

or

$$\begin{aligned} g(y) &\\ = \; & g(a)-\sum_{j=1}^{n}\frac{(-1)^{j}}{j!}\left[\sum_{i=0}^{k}\left\{(x_i-\alpha_i)^{j}-(x_i-\alpha_{i+1})^{j}\right\}g^{(j)}(x_i)\right] \\ & +(-1)^{n}\int_a^y K_{n,k}(y,t)\,g^{(n+1)}(t)\,dt. \end{aligned} \tag{2.12}$$

The proof of (2.11) and (2.12) follows directly from (2.2) and (2.6) respectively upon choosing $b=y$ and $f=g'$.

2.3. Integral Inequalities

In this section we utilise the equalities of Section 2.2 and develop inequalities for the representation of the integral of a function with respect to its derivatives at a multiple number of points within some interval. In particular, we develop inequalities which depend on the spaces $L_\infty\in[a,b]$, $L_p\in[a,b]$ and $L_1\in[a,b]$.

Theorem 2.7. *Let $I_k: a=x_0<x_1<\cdots<x_{k-1}<x_k=b$ be a division of the interval $[a,b]$ and α_i $(i=0,\ldots,k+1)$ be '$k+2$' points so that $\alpha_0=a$, $\alpha_i\in[x_{i-1},x_i]$ $(i=1,\ldots,k)$ and $\alpha_{k+1}=b$. If $f:[a,b]\to\mathbb{R}$ is a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$, then for all $x_i\in[a,b]$ we have the inequality:*

$$\left|\int_a^b f(t)\,dt+\sum_{j=1}^{n}\frac{(-1)^{j}}{j!}\left[\sum_{i=0}^{k}\left\{(x_i-\alpha_i)^{j}-(x_i-\alpha_{i+1})^{j}\right\}f^{(j-1)}(x_i)\right]\right| \tag{2.13}$$

$$\leq\begin{cases}\dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\displaystyle\sum_{i=0}^{k-1}\left\{(\alpha_{i+1}-x_i)^{n+1}+(x_{i+1}-\alpha_{i+1})^{n+1}\right\}\\[2ex] \leq\dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\displaystyle\sum_{i=0}^{k-1}h_i^{n+1}\\[2ex] \leq\dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}(b-a)\,\nu^{n}(h)\quad \textit{if}\quad f^{(n)}\in L_\infty[a,b],\end{cases}$$

where

$$\begin{aligned}\left\|f^{(n)}\right\|_\infty &:=\sup_{t\in[a,b]}\left|f^{(n)}(t)\right|<\infty,\\ h_i &:=x_{i+1}-x_i\quad \textit{and}\\ \nu(h) &:=\max\{h_i|i=0,\ldots,k-1\}.\end{aligned}$$

Proof. From Corollary 2.3 we may write

$$\left|\int_a^b f(t)\,dt+\sum_{j=1}^{n}\frac{(-1)^j}{j!}\left[\sum_{i=0}^{k}\left\{(x_i-\alpha_i)^j-(x_i-\alpha_{i+1})^j\right\}f^{(j-1)}(x_i)\right]\right| \tag{2.14}$$
$$=\left|(-1)^n\int_a^b K_{n,k}(t)\,f^{(n)}(t)\,dt\right|,$$

and

$$\left|(-1)^n\int_a^b K_{n,k}(t)\,f^{(n)}(t)\,dt\right|\le\left\|f^{(n)}\right\|_\infty\int_a^b\left|K_{n,k}(t)\right|dt,$$

$$\begin{aligned}\int_a^b\left|K_{n,k}(t)\right|dt &= \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}\frac{\left|t-\alpha_{i+1}\right|^n}{n!}dt\\ &= \sum_{i=0}^{k-1}\left[\int_{x_i}^{\alpha_{i+1}}\frac{(\alpha_{i+1}-t)^n}{n!}dt+\int_{\alpha_{i+1}}^{x_{i+1}}\frac{(t-\alpha_{i+1})^n}{n!}dt\right]\\ &= \frac{1}{(n+1)!}\sum_{i=0}^{k-1}\left\{(\alpha_{i+1}-x_i)^{n+1}+(x_{i+1}-\alpha_{i+1})^{n+1}\right\}\end{aligned}$$

and thus

$$\begin{aligned}&\left|(-1)^n\int_a^b K_{n,k}(t)\,f^{(n)}(t)\,dt\right|\\ \le\ &\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\sum_{i=0}^{k-1}\left\{(\alpha_{i+1}-x_i)^{n+1}+(x_{i+1}-\alpha_{i+1})^{n+1}\right\}.\end{aligned}$$

Hence, from (2.14), the first part of the inequality (2.13) is proved. The second and third lines follow by noting that

$$\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\sum_{i=0}^{k-1}\left\{(\alpha_{i+1}-x_i)^{n+1}+(x_{i+1}-\alpha_{i+1})^{n+1}\right\}\le\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\sum_{i=0}^{k-1}h_i^{n+1},$$

since for $0<B<A<C$ it is well known that

$$(A-B)^{n+1}+(C-A)^{n+1}\le(C-B)^{n+1}. \tag{2.15}$$

Also

$$\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\sum_{i=0}^{k-1}h_i^{n+1}\le\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\nu^n(h)\sum_{i=0}^{k-1}h_i=\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}(b-a)\,\nu^n(h),$$

where $\nu(h):=\max\{h_i|i=0,\ldots,k-1\}$ and therefore the third line of the inequality (2.13) follows, hence Theorem 2.7 is proved. ∎

Theorem 2.8. *Let $I_k:a=x_0<x_1<\cdots<x_{k-1}<x_k=b$ be a division of the interval $[a,b]$ and α_i $(i=0,\ldots,k+1)$ be '$k+2$' points so that $\alpha_0=a$, $\alpha_i\in[x_{i-1},x_i]$ $(i=1,\ldots,k)$ and $\alpha_{k+1}=b$. If $f:[a,b]\to\mathbb{R}$ is a mapping such*

that $f^{(n-1)}$ is absolutely continuous on $[a,b]$, then for all $x_i \in [a,b]$ we have the inequality:

$$\left| \int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\sum_{i=0}^{k} \left\{ (x_i - \alpha_i)^j - (x_i - \alpha_{i+1})^j \right\} f^{(j-1)}(x_i) \right] \right| \tag{2.16}$$

$$\leq \begin{cases} \dfrac{\left\| f^{(n)} \right\|_p}{n!\,(nq+1)^{\frac{1}{q}}} \left(\displaystyle\sum_{i=0}^{k-1} \left\{ (\alpha_{i+1} - x_i)^{nq+1} + (x_{i+1} - \alpha_{i+1})^{nq+1} \right\} \right)^{\frac{1}{q}} \\ \leq \dfrac{\left\| f^{(n)} \right\|_p}{n!\,(nq+1)^{\frac{1}{q}}} \left(\displaystyle\sum_{i=0}^{k-1} h_i^{nq+1} \right)^{\frac{1}{q}} \\ \leq \dfrac{\left\| f^{(n)} \right\|_p}{n!} \nu^n(h) \left(\dfrac{b-a}{nq+1} \right)^{\frac{1}{q}} \end{cases}$$

if $f^{(n)} \in L_p[a,b]$ and $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, where

$$\left\| f^{(n)} \right\|_p := \left(\int_a^b \left| f^{(n)}(t) \right|^p dt \right)^{\frac{1}{p}}.$$

PROOF. From Corollary 2.3 we may use (2.14) and by Hölder's integral inequality write

$$\left| (-1)^n \int_a^b K_{n,k}(t)\, f^{(n)}(t)\,dt \right| \leq \left\| f^{(n)} \right\|_p \left(\int_a^b |K_{n,k}(t)|^q\,dt \right)^{\frac{1}{q}}, \tag{2.17}$$

$$\begin{aligned} \int_a^b |K_{n,k}(t)|^q\,dt &= \sum_{i=0}^{k-1} \left[\int_{x_i}^{\alpha_{i+1}} \frac{(\alpha_{i+1}-t)^{nq}}{(n!)^q} dt + \int_{\alpha_{i+1}}^{x_{i+1}} \frac{(t-\alpha_{i+1})^{nq}}{(n!)^q} dt \right] \\ &= \frac{1}{(n!)^q (nq+1)} \sum_{i=0}^{k-1} \left\{ (\alpha_{i+1} - x_i)^{nq+1} + (x_{i+1} - \alpha_{i+1})^{nq+1} \right\} \end{aligned}$$

From (2.17)

$$\begin{aligned} &\left(\int_a^b |K_{n,k}(t)|^q\,dt \right)^{\frac{1}{q}} \\ &= \frac{1}{n!\,(nq+1)^{\frac{1}{q}}} \left(\sum_{i=0}^{k-1} \left\{ (\alpha_{i+1} - x_i)^{nq+1} + (x_{i+1} - \alpha_{i+1})^{nq+1} \right\} \right)^{\frac{1}{q}} \end{aligned}$$

and utilising (2.14) we obtain the first line of the inequality (2.16).

Using the inequality (2.14) we know that

$$\sum_{i=0}^{k-1} \left\{ (\alpha_{i+1} - x_i)^{nq+1} + (x_{i+1} - \alpha_{i+1})^{nq+1} \right\} \leq \sum_{i=0}^{k-1} h_i^{nq+1}$$

and the second line of the inequality (2.16) follows. From (2.16)

$$\begin{aligned}
\frac{\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}}\left(\sum_{i=0}^{k-1} h_i^{nq+1}\right)^{\frac{1}{q}} &\leq \frac{\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}}\nu^n(h)\left(\sum_{i=0}^{k-1} h_i\right)^{\frac{1}{q}} \\
&= \frac{\left\|f^{(n)}\right\|_p}{n!}\nu^n(h)\left(\frac{b-a}{nq+1}\right)^{\frac{1}{q}}
\end{aligned}$$

and the third line in the inequality (2.16) follows, hence Theorem 2.8 is proved. ∎

THEOREM 2.9. *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a,b]$ and α_i $(i = 0, \ldots, k+1)$ be '$k+2$' points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, k)$ and $\alpha_{k+1} = b$. If $f : [a,b] \to \mathbb{R}$ is a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$, then for all $x_i \in [a,b]$ we have the inequality:*

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt + \sum_{j=1}^{n}\frac{(-1)^j}{j!}\left[\sum_{i=0}^{k}\left\{(x_i-\alpha_i)^j - (x_i-\alpha_{i+1})^j\right\} f^{(j-1)}(x_i)\right]\right| \\
&\leq \frac{\left\|f^{(n)}\right\|_1}{n!}[\nu(h)]^n, \quad \text{if } f^{(n)} \in L_1[a,b],
\end{aligned} \tag{2.18}$$

where

$$\left\|f^{(n)}\right\|_1 = \int_a^b \left|f^{(n)}(t)\right| dt.$$

PROOF. From (2.14)

$$\begin{aligned}
\left|(-1)^n \int_a^b K_{n,k}(t)\, f^{(n)}(t)\,dt\right| &\leq \left\|K_{n,k}(t)\right\|_\infty \left\|f^{(n)}\right\|_1 \\
&= \left\|f^{(n)}\right\|_1 \sup_{t\in[a,b]}\left|K_{n,k}(t)\right|,
\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}
|K_{n,k}(t)| &= \left|\frac{(t-\alpha_{i+1})^n}{n!}\right| \le \sup_{t\in[x_i,x_{i+1}]}\left|\frac{(t-\alpha_{i+1})^n}{n!}\right| \\
&= \frac{1}{n!}\max_{i=0,\dots,k-1}\left\{|(\alpha_{i+1}-x_i)|^n, |(x_{i+1}-\alpha_{i+1})|^n\right\} \\
&\le \frac{1}{n!}\left[\max_{i=0,\dots,k-1}\left\{(\alpha_{i+1}-x_i),(x_{i+1}-\alpha_{i+1})\right\}\right]^n \\
&= \frac{1}{n!}\left[\max_{i=0,\dots,k-1}\left\{\frac{x_{i+1}-x_i}{2}+\left|\alpha_{i+1}-\frac{x_{i+1}+x_i}{2}\right|\right\}\right]^n \\
&= \frac{1}{n!}\left[\max_{i=0,\dots,k-1}\left\{\frac{h_i}{2}\right\}+\max_{i=0,\dots,k-1}|\delta_i|\right]^n \\
&\le \frac{1}{n!}\left[\max_{i=0,\dots,k-1}\left\{\frac{h_i}{2}\right\}+\max_{i=0,\dots,k-1}\left\{\frac{h_i}{2}\right\}\right]^n \\
&\qquad (\text{since } \delta_i := \alpha_{i+1}-\frac{x_{i+1}+x_i}{2} \text{ and therefore } |\delta_i| \le \frac{h_i}{2}) \\
&= \frac{1}{n!}\left[\max_{i=0,\dots,k-1}\{h_i\}\right]^n = \frac{1}{n!}[\nu(h)]^n,
\end{aligned}$$

hence the proof of inequality (2.18) follows. ∎

When the points of the division I_k are fixed, we obtain the following inequality.

Corollary 2.10. Let f and I_k be defined as in Corollary 2.4, then

$$\left|\int_a^b f(t)\,dt+\sum_{j=1}^n\frac{(-1)^j}{2^j j!}\left[\sum_{i=0}^k\left\{-h_i^j+(-1)^j h_{i+1}^j\right\}f^{(j-1)}(x_i)\right]\right| \tag{2.20}$$

$$\le\begin{cases}\dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!2^n}\displaystyle\sum_{i=0}^{k-1}h_i^{n+1} & \text{if } f^{(n)}\in L_\infty[a,b],\\[2ex] \dfrac{\left\|f^{(n)}\right\|_p}{n!(nq+1)^{\frac{1}{q}}2^n}\left(\displaystyle\sum_{i=0}^{k-1}h_i^{nq+1}\right)^{\frac{1}{q}} & \text{if } f^{(n)}\in L_p[a,b],\\ & p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1,\\ \dfrac{\left\|f^{(n)}\right\|_1}{n!}[\nu(h)]^n & \text{if } f^{(n)}\in L_1[a,b].\end{cases}$$

Proof. from Corollary 2.4 we choose

$$\begin{aligned}
\alpha_0 &= a,\ \ \alpha_1=\frac{a+x_1}{2},\ \dots, \\
\alpha_{k-1} &= \frac{x_{k-2}+x_{k-1}}{2},\ \ \alpha_k=\frac{x_{k-1}+x_k}{2}\ \text{ and } \alpha_{k+1}=b.
\end{aligned}$$

Now utilising the first line of the inequality (2.13), we may evaluate

$$\sum_{i=0}^{k-1}\left\{(\alpha_{i+1}-x_i)^{n+1}+(x_{i+1}-\alpha_{i+1})^{n+1}\right\}=\sum_{i=0}^{k-1}2\left(\frac{h_i}{2}\right)^{n+1}$$

and therefore the first part of the inequality (2.20) follows. From (2.16) and (2.18) we obtain the second and third lines of (2.20), hence the corollary is proved. ∎

For the equidistant partitioning case we have the following inequality.

COROLLARY 2.11. Let f be as defined in Theorem 2.2 and let I_k be defined by (2.9). Then

$$\begin{aligned}(2.21)\quad &\left|\int_a^b f(t)\,dt+\sum_{j=1}^{n}\left(\frac{b-a}{2k}\right)^j\frac{1}{j!}\right.\\ &\left.\times\left[-f^{(j-1)}(a)+\sum_{i=1}^{k-1}\left\{(-1)^j-1\right\}f^{(j-1)}(x_i)+(-1)^j f^{(j-1)}(b)\right]\right|\\ &\leq\begin{cases}\dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!\,(2k)^n}(b-a)^{n+1} & \text{if}\quad f^{(n)}\in L_\infty[a,b],\\[2ex] \dfrac{\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}(2k)^n}(b-a)^{n+\frac{1}{q}} & \text{if}\quad f^{(n)}\in L_p[a,b],\\ & \quad p>1\text{ and }\frac{1}{p}+\frac{1}{q}=1,\\ \dfrac{\left\|f^{(n)}\right\|_1}{n!}\left(\dfrac{b-a}{k}\right)^n & \text{if}\quad f^{(n)}\in L_1[a,b].\end{cases}\end{aligned}$$

PROOF. We may utilise (2.10) and from (2.13), (2.16) and (2.18) note that

$$h_0=x_1-x_0=\frac{b-a}{k}\quad\text{and}\quad h_i=x_{i+1}-x_i=\frac{b-a}{k},\ i=1,\ldots,k-1$$

in which case (2.21) follows. ∎

The following inequalities for Taylor-like expansions also hold.

COROLLARY 2.12. Let g be defined as in Corollary 2.6. Then we have the inequality

$$\begin{aligned}(2.22)\quad &\left|g(y)-g(a)+\sum_{j=1}^{n}\frac{(-1)^j}{j!}\left[\sum_{i=0}^{k}\left\{(x_i-\alpha_i)^j-(x_i-\alpha_{i+1})^j\right\}g^{(j)}(x_i)\right]\right|\\ &\leq\begin{cases}\dfrac{\left\|g^{(n+1)}\right\|_\infty}{(n+1)!}\displaystyle\sum_{i=0}^{k-1}\left\{(\alpha_{i+1}-x_i)^{n+1}+(x_{i+1}-\alpha_{i+1})^{n+1}\right\}\text{ if }g^{(n+1)}\in L_\infty[a,b]\\[2ex] \dfrac{\left\|g^{(n+1)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}}\left(\displaystyle\sum_{i=0}^{k-1}\left\{(\alpha_{i+1}-x_i)^{nq+1}+(x_{i+1}-\alpha_{i+1})^{nq+1}\right\}\right)^{\frac{1}{q}}\\ \qquad\qquad\text{if }g^{(n+1)}\in L_p[a,b],\ \ p>1\text{ and }\frac{1}{p}+\frac{1}{q}=1,\\ \dfrac{\left\|g^{(n+1)}\right\|_1}{n!}[\nu(h)]^n\text{ if }g^{(n+1)}\in L_1[a,b]\end{cases}\end{aligned}$$

for all $x_i \in [a, y]$ where

$$\begin{aligned}
\nu(h) &: = \max\{h_i | i = 0, \dots, k-1\}, \\
h_i &: = x_{i+1} - x_i, \\
\left\| g^{(n+1)} \right\|_\infty &: = \sup_{t \in [a,y]} \left| g^{(n+1)}(t) \right| < \infty, \\
\left\| g^{(n+1)} \right\|_p &: = \left(\int_a^y \left| g^{(n+1)}(t) \right|^p dt \right)^{\frac{1}{p}} \quad \text{and} \\
\left\| g^{(n+1)} \right\|_1 &: = \int_a^y \left| g^{(n+1)}(t) \right| dt.
\end{aligned}$$

Proof. Follows directly from (2.12) and using the norms as in (2.13), (2.16) and (2.18). ∎

When the points of the division I_k are fixed we obtain the following.

Corollary 2.13. Let g be defined as in Corollary 2.6 and $I_k : a = x_0 < x_1 < \dots < x_{k-1} < x_k = y$ be a division of the interval $[a, y]$. Then we have the inequality

$$\left| g(y) - g(a) + \sum_{j=1}^{n} \frac{1}{2^j j!} \left[\sum_{i=0}^{k} \left\{ -h_i^j + (-1)^j h_{i+1}^j \right\} g^{(j)}(x_i) \right] \right| \tag{2.23}$$

$$\leq \begin{cases} \dfrac{\left\| g^{(n+1)} \right\|_\infty}{(n+1)! 2^n} \displaystyle\sum_{i=0}^{k-1} h_i^{n+1} & \text{if} \quad g^{(n+1)} \in L_\infty[a, y] \\[2ex] \dfrac{\left\| g^{(n+1)} \right\|_p}{n! (nq+1)^{\frac{1}{q}} 2^n} \left(\displaystyle\sum_{i=0}^{k-1} h_i^{nq+1} \right)^{\frac{1}{q}} & \text{if} \quad g^{(n+1)} \in L_p[a, y], \\ & \quad p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\ \dfrac{\left\| g^{(n+1)} \right\|_1}{n!} [\nu(h)]^n & \text{if} \quad g^{(n+1)} \in L_1[a, y]. \end{cases}$$

Proof. The proof follows directly from using (2.8) with the appropriate norms. ∎

For the equidistant partitioning case we have:

Corollary 2.14. Let g be defined as in Corollary 2.6 and

$$I_k : x_i = a + i \cdot \left(\frac{y-a}{k} \right), \quad i = 0, \dots, k$$

be an equidistant partitioning of $[a,y]$, then we have the inequality:

$$\left| g(y) - g(a) + \sum_{j=1}^{n} \left(\frac{y-a}{2k}\right)^{j} \frac{1}{j!} \times \left[-g^{(j)}(a) + \sum_{i=1}^{k-1} \left\{ (-1)^{j} - 1 \right\} g^{(j)}(x_i) + (-1)^{j} g^{(j)}(y) \right] \right| \tag{2.24}$$
$$\leq \begin{cases} \dfrac{\left\| g^{(n+1)} \right\|_{\infty}}{(n+1)!\,(2k)^{n}} (y-a)^{n+1} & \text{if} \quad g^{(n+1)} \in L_{\infty}[a,y], \\[2ex] \dfrac{\left\| g^{(n+1)} \right\|_{p}}{n!\,(nq+1)^{\frac{1}{q}}\,(2k)^{n}} (y-a)^{n+\frac{1}{q}} & \text{if} \quad g^{(n+1)} \in L_{p}[a,y], \\ & \quad p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \dfrac{\left\| g^{(n+1)} \right\|_{1}}{n!} \left(\dfrac{y-a}{k}\right)^{n} & \text{if} \quad g^{(n+1)} \in L_{1}[a,y]. \end{cases}$$

PROOF. The proof follows directly upon using (2.10) with $f' = g$ and $b = y$. ∎

2.4. The Convergence of a General Quadrature Formula

Let

$$\Delta_m : a = x_0^{(m)} < x_1^{(m)} < \cdots < x_{m-1}^{(m)} < x_m^{(m)} = b$$

be a sequence of division of $[a,b]$ and consider the sequence of real numerical integration formula

$$\begin{aligned} & I_m\left(f, f', \ldots, f^{(n)}, \Delta_m, w_m\right) \\ := & \sum_{j=0}^{m} w_j^{(m)} f\left(x_j^{(m)}\right) - \sum_{r=2}^{n} \frac{(-1)^{r}}{r!} \left[\sum_{j=0}^{m} \left\{ \left(x_j^{(m)} - a - \sum_{s=0}^{j-1} w_s^{(m)} \right)^{r} \right.\right. \\ & \left.\left. - \left(x_j^{(m)} - a - \sum_{s=0}^{j} w_s^{(m)} \right)^{r} \right\} f^{(r-1)}\left(x_j^{(m)}\right) \right], \end{aligned} \tag{2.25}$$

where w_j $(j = 0, \ldots, m)$ are the quadrature weights and assume that $\sum_{j=0}^{m} w_j^{(m)} = b - a$.

The following theorem contains a sufficient condition for the weights $w_j^{(m)}$ so that $I_m\left(f, f', \ldots, f^{(n)}, \Delta_m, w_m\right)$ approximates the integral $\int_a^b f(x)\,dx$ with an error expressed in terms of $\left\| f^{(n)} \right\|_{\infty}$.

THEOREM 2.15. *Let $f : [a,b] \to \mathbb{R}$ be a continuous mapping on $[a,b]$. If the quadrature weights, $w_j^{(m)}$ satisfy the condition*

$$x_i^{(m)} - a \leq \sum_{j=0}^{i} w_j^{(m)} \leq x_{i+1}^{(m)} - a \quad \textit{for all } i = 0, \ldots, m-1 \tag{2.26}$$

then we have the estimation

$$\left| I_m\left(f, f', \dots, f^{(n)}, \Delta_m, w_m\right) - \int_a^b f(t)\,dt \right| \tag{2.27}$$

$$\leq \begin{cases} \dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} \sum\limits_{i=0}^{m-1} \left[\left(a + \sum\limits_{j=0}^{i} w_j^{(m)} - x_i^{(m)} \right)^{n+1} - \left(x_{i+1}^{(m)} - a - \sum\limits_{j=0}^{i} w_j^{(m)} \right)^{n+1} \right] \\ \\ \dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} \sum\limits_{i=0}^{m-1} \left(h_i^{(m)}\right)^{n+1} \\ \\ \dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} \left[\nu\left(h^{(m)}\right)\right](b-a), \quad \text{where} \quad f^{(n)} \in L_\infty[a,b]. \end{cases}$$

Also

$$\begin{aligned} \nu\left(h^{(m)}\right) &:= \max_{i=0,\dots,m-1} \left\{h_i^{(m)}\right\} \\ h_i^{(m)} &:= x_{i+1}^{(m)} - x_i^{(m)}. \end{aligned}$$

In particular, if $\left\|f^{(n)}\right\|_\infty < \infty$, *then*

$$\lim_{\nu\left(h^{(m)}\right)\to 0} I_m\left(f, f', \dots, f^{(n)}, \Delta_m, w_m\right) = \int_a^b f(t)\,dt$$

uniformly by the influence of the weights w_m.

Proof. Define the sequence of real numbers

$$\alpha_{i+1}^{(m)} := a + \sum_{j=0}^{i} w_j^{(m)}, \quad i = 0, \dots, m.$$

Note that

$$\alpha_{i+1}^{(m)} = a + \sum_{j=0}^{m} w_j^{(m)} = a + b - a = b.$$

By the assumption (2.26), we have

$$\alpha_{i+1}^{(m)} \in \left[x_i^{(m)}, x_{i+1}^{(m)}\right] \quad \text{for all} \quad i = 0, \dots, m-1.$$

Define $\alpha_0^{(m)} = a$ and compute

$$\begin{aligned} \alpha_1^{(m)} - \alpha_0^{(m)} &= w_0^{(m)} \\ \alpha_{i+1}^{(m)} - \alpha_i^{(m)} &= a + \sum_{j=0}^{i} w_j^{(m)} - a - \sum_{j=0}^{i-1} w_j^{(m)} = w_i^{(m)} \quad (i = 0, \dots, m-1) \end{aligned}$$

and

$$\alpha_{m+1}^{(m)} - \alpha_m^{(m)} = a + \sum_{j=0}^{m} w_j^{(m)} - a - \sum_{j=0}^{m-1} w_j^{(m)} = w_m^{(m)}.$$

Consequently

$$\sum_{i=0}^{m}\left(\alpha_{i+1}^{(m)}-\alpha_{i}^{(m)}\right)f\left(x_{i}^{(m)}\right)=\sum_{i=0}^{m}w_{i}^{(m)}f\left(x_{i}^{(m)}\right),$$

and let

$$\begin{aligned}&\sum_{j=0}^{m}w_{j}^{(m)}f\left(x_{j}^{(m)}\right)-\sum_{r=2}^{n}\frac{(-1)^{r}}{r!}\left[\sum_{j=0}^{m}\left\{\left(x_{j}^{(m)}-a-\sum_{s=0}^{j-1}w_{s}^{(m)}\right)^{r}\right.\right.\\&\left.\left.-\left(x_{j}^{(m)}-a-\sum_{s=0}^{j}w_{s}^{(m)}\right)^{r}\right\}f^{(r-1)}\left(x_{j}^{(m)}\right)\right]\\:&=I_{m}\left(f,f',\ldots,f^{(n)},\Delta_{m},w_{m}\right).\end{aligned}$$

Applying the inequality (2.13) we obtain the estimate (2.27). ∎

On the L_p norm the following theorem holds.

Theorem 2.16. *Let the conditions of Theorem 2.15 apply. Then we have the estimation*

$$\left|I_{m}\left(f,f',\ldots,f^{(n)},\Delta_{m},w_{m}\right)-\int_{a}^{b}f(t)\,dt\right| \tag{2.28}$$

$$\leq\begin{cases}\dfrac{\left\|f^{(n)}\right\|_{p}}{n!\,(nq+1)^{\frac{1}{q}}}\sum_{i=0}^{m-1}\left[\left(a+\sum_{j=0}^{i}w_{j}^{(m)}-x_{i}^{(m)}\right)^{nq+1}\right.\\ \qquad\left.+\left(x_{i+1}^{(m)}-a-\sum_{j=0}^{i}w_{j}^{(m)}\right)^{nq+1}\right]^{\frac{1}{q}}\\ \dfrac{\left\|f^{(n)}\right\|_{p}}{n!\,(nq+1)^{\frac{1}{q}}}\left(\sum_{i=0}^{m-1}\left(h_{i}^{(m)}\right)^{nq+1}\right)^{\frac{1}{q}}\\ \dfrac{\left\|f^{(n)}\right\|_{p}}{n!}\left(\nu\left(h^{(m)}\right)\right)^{n}\left(\dfrac{b-a}{nq+1}\right)^{\frac{1}{q}},\end{cases}$$

where $f^{(n)}\in L_{p}[a,b]$, $p>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *In particular, if* $\left\|f^{(n)}\right\|_{p}<\infty$, *then*

$$\lim_{\nu\left(h^{(m)}\right)\to 0}I_{m}\left(f,f',\ldots,f^{(n)},\Delta_{m},w_{m}\right)=\int_{a}^{b}f(t)\,dt$$

uniformly by the influence of the weights w_m.

The proof of this theorem follows the same pattern as that of Theorem 2.15 and will not be given here.

Similarly on the L_1 norm the following theorem also holds.

THEOREM 2.17. *Let the conditions of Theorem 2.15 apply. Then we have the estimation*

$$\left|I_m\left(f,f',\dots,f^{(n)},\Delta_m,w_m\right)-\int_a^b f(t)\,dt\right|\le\frac{\left\|f^{(n)}\right\|_1}{n!}\left[\nu\left(h^{(m)}\right)\right]^n, \tag{2.29}$$

where $f^{(n)}\in L_1[a,b]$.
In particular, if $\left\|f^{(n)}\right\|_1<\infty$, *then*

$$\lim_{\nu\left(h^{(m)}\right)\to 0} I_m\left(f,f',\dots,f^{(n)},\Delta_m,w_m\right)=\int_a^b f(t)\,dt$$

uniformly by the influence of the weights w_m.

The proof of Theorem 2.17 follows the same pattern as Theorem 2.15, and will not be given here.

The case when the partitioning is equidistant is important in practice. Consider the equidistant partition

$$E_m := x_i^{(m)} := a+i\frac{b-a}{m},\ \ (i=0,\dots,m)$$

and define the sequence of numerical quadrature formulae

$$\begin{aligned}&I_m\left(f,f',\dots,f^{(n)},\Delta_m,w_m\right)\\ :&=\sum_{j=0}^m w_j^{(m)}f\left(a+\frac{j(b-a)}{n}\right)-\sum_{r=2}^n\frac{(-1)^r}{r!}\left[\sum_{j=0}^m\left\{\left(\frac{j(b-a)}{n}-\sum_{s=0}^{j-1}w_s^{(m)}\right)^r\right.\right.\\ &\left.\left.-\left(\frac{j(b-a)}{n}-\sum_{s=0}^{j}w_s^{(m)}\right)^r\right\}f^{(r-1)}\left(a+\frac{j(b-a)}{n}\right)\right].\end{aligned}$$

The following corollary holds.

COROLLARY 2.18. Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. If the quadrature weights $w_j^{(m)}$ satisfy the condition

$$\frac{i}{m}\le\frac{1}{b-a}\sum_{j=0}^i w_j^{(m)}\le\frac{i+1}{m},\ \ i=0,1,\dots,n-1,$$

then the following bound holds:

$$\left|I_m\left(f,f',\dots,f^{(n)},\Delta_m,w_m\right)-\int_a^b f(t)\,dt\right|$$

$$
\leq \begin{cases}
\dfrac{\left\| f^{(n)} \right\|_{\infty}}{(n+1)!} \displaystyle\sum_{i=0}^{m-1} \left[\left(\sum_{j=0}^{i} w_j^{(m)} - i \left(\frac{b-a}{m} \right) \right)^{n+1} \right. \\
\qquad\qquad \left. - \left((i+1) \left(\dfrac{b-a}{m} \right) - \displaystyle\sum_{j=0}^{i} w_j^{(m)} \right)^{n+1} \right] \\
\leq \dfrac{\left\| f^{(n)} \right\|_{\infty}}{(n+1)!} \left(\dfrac{b-a}{m} \right)^{n+1}, \quad \text{where } f^{(n)} \in L_{\infty} [a,b], \\
\dfrac{\left\| f^{(n)} \right\|_{p}}{n! \, (nq+1)^{\frac{1}{q}}} \displaystyle\sum_{i=0}^{m-1} \left[\left(\sum_{j=0}^{i} w_j^{(m)} - i \left(\frac{b-a}{m} \right) \right)^{nq+1} \right. \\
\qquad\qquad \left. + \left((i+1) \left(\dfrac{b-a}{m} \right) - \displaystyle\sum_{j=0}^{i} w_j^{(m)} \right)^{nq+1} \right]^{\frac{1}{q}} \\
\leq \dfrac{\left\| f^{(n)} \right\|_{p}}{n! \, (nq+1)^{\frac{1}{q}}} \left(\dfrac{b-a}{m\,(nq+1)} \right)^{n+\frac{1}{q}}, \\
\qquad\qquad \text{where } f^{(n)} \in L_p [a,b], \ p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\
\dfrac{\left\| f^{(n)} \right\|_{1}}{n!} \left[\nu \left(h^{(m)} \right) \right]^n \leq \dfrac{\left\| f^{(n)} \right\|_{1}}{n!} \left(\dfrac{b-a}{m} \right)^n, \quad \text{where } f^{(n)} \in L_1 [a,b],
\end{cases}
$$

In particular, if $\left\| f^{(n)} \right\|_{\infty,p,1} < \infty$, then

$$
\lim_{m\to\infty} I_m \left(f, f', \ldots, f^{(n)}, w_m \right) = \int_a^b f(t)\, dt
$$

uniformly by the influence of the weights w_m.

The proof of Corollary 2.18 follows directly from Theorem 2.15.

2.5. Grüss Type Inequalities

The Grüss inequality [**25**], is well known in the literature. It is an integral inequality which establishes a connection between the integral of a product of two functions and the product of the integrals of the two functions.

THEOREM 2.19. *Let $h, g : [a,b] \to \mathbb{R}$ be two integrable functions such that $\phi \leq h(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a,b]$, ϕ, Φ, γ and Γ are constants. Then we have*

$$
|T(h,g)| \leq \frac{1}{4} (\Phi - \phi)(\Gamma - \gamma), \tag{2.30}
$$

where

$$T(h,g) \quad : \quad = \frac{1}{b-a}\int_a^b h(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b h(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx \tag{2.31}$$

and the inequality (2.30) is sharp, in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

For a simple proof of this fact as well as generalisations, discrete variants, extensions and associated material, see [**28**]. The Grüss inequality is also utilised in the papers [**12, 13, 18, 24**] and the references contained therein.

A *premature* Grüss inequality is the following.

Theorem 2.20. *Let f and g be integrable functions defined on $[a,b]$ and let $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a,b]$. Then*

$$|T(h,g)| \leq \frac{\Gamma-\gamma}{2}\left(T(f,f)\right)^{\frac{1}{2}}, \tag{2.32}$$

where $T(f,f)$ is as defined in (2.31).

Theorem 2.20 was proved in 1999 by Matić, Pečarić and Ujević [**27**] and it provides a sharper bound than the Grüss inequality (2.30). The term *premature* is used to highlight the fact that the result (2.32) is obtained by not fully completing the proof of the Grüss inequality. The premature Grüss inequality is completed if one of the functions, f or g, is explicitly known.

We now give the following theorem based on the premature Grüss inequality (2.32).

Theorem 2.21. *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a,b]$, α_i $(i = 0,\ldots,k+1)$ be '$k+1$' points such that $\alpha_0 = a$, $\alpha_i \in [x_{i-1},x_i]$ $(i = 1,\ldots,k)$ and $\alpha_k = b$. If $f : [a,b] \to \mathbb{R}$ is absolutely continuous and n time differentiable on $[a,b]$, then assuming that the n derivative $f^{(n)} : (a,b) \to \mathbb{R}$ satisfies the condition*

$$m \leq f^{(n)} \leq M \quad \text{for all } x \in (a,b),$$

we have the inequality

$$\begin{aligned} &\Bigg|(-1)^n \int_a^b f(t)\,dt + (-1)^n \sum_{j=1}^{n} \frac{(-1)^j}{j!} \times \\ &\left[\sum_{i=0}^{k-1}\left\{\left(\frac{h_i}{2}-\delta_i\right)^j f^{(j-1)}(x_{i+1}) - \left(\frac{h_i}{2}+\delta_i\right)^j f^{(j-1)}(x_i)\right\}\right] \\ &-\left(\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)(n+1)!}\right)\sum_{i=0}^{k-1}\left(\frac{h_i}{2}\right)^{n+1} \\ &\times\left[\sum_{r=0}^{n+1}\binom{n+1}{r}\left(\frac{2\delta_i}{h_i}\right)^r\left\{1+(-1)^r\right\}\right]\Bigg| \end{aligned} \tag{2.33}$$

$$\leq \quad \frac{M-m}{2}\left[\frac{b-a}{(2n+1)(n!)^2}\sum_{i=0}^{k-1}\left(\frac{h_i}{2}\right)^{2n+1}\right.$$

$$\times\left[\sum_{r=0}^{2n+1}\binom{2n+1}{r}\left(\frac{2\delta_i}{h_i}\right)^r\{1+(-1)^r\}\right]$$

$$\left.-\left(\frac{1}{(n+1)!}\sum_{i=0}^{k-1}\left(\frac{h_i}{2}\right)^{n+1}\left[\sum_{r=0}^{n+1}\binom{n+1}{r}\left(\frac{2\delta_i}{h_i}\right)^r\{1+(-1)^r\}\right]\right)^2\right]^{\frac{1}{2}}$$

where

$$\begin{aligned} h_i &:= x_{i+1}-x_i \quad \textit{and} \\ \delta_i &:= \alpha_{i+1}-\frac{x_{i+1}+x_i}{2}, \quad i=0,\ldots,k-1. \end{aligned}$$

Proof. We utilise (2.31) and (2.32), multiply through by $(b-a)$ and choose $h(t) := K_{n,k}(t)$ as defined by (2.3) and $g(t) := f^{(n)}(t)$, $t\in[a,b]$ such that

$$\left|\int_a^b K_{n,k}(t)\,f^{(n)}(t)\,dt-\frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt\cdot\int_a^b K_{n,k}(t)\,dt\right| \tag{2.34}$$

$$\leq \frac{\Gamma-\gamma}{2}\left[(b-a)\int_a^b K_{n,k}^2(t)\,dt-\left(\int_a^b K_{n,k}(t)\,dt\right)^2\right]^{\frac{1}{2}}.$$

Now we may evaluate

$$\int_a^b f^{(n)}(t)\,dt = f^{(n-1)}(b)-f^{(n-1)}(a)$$

and

$$\begin{aligned} G_1 &:= \int_a^b K_{n,k}(t)\,dt=\sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}\frac{1}{n!}(t-\alpha_{i+1})^n\,dt \\ &= \frac{1}{(n+1)!}\sum_{i=0}^{k-1}\left\{(x_{i+1}-\alpha_{i+1})^{n+1}+(\alpha_{i+1}-x_i)^{n+1}\right\}. \end{aligned}$$

Using the definitions of h_i and δ_i we have

$$x_{i+1}-\alpha_{i+1}=\frac{h_i}{2}-\delta_i$$

and

$$\alpha_{i+1}-x_i=\frac{h_i}{2}+\delta_i$$

such that

$$G_1=\frac{1}{(n+1)!}\sum_{i=0}^{k-1}\left\{\left(\frac{h_i}{2}-\delta_i\right)^{n+1}+\left(\frac{h_i}{2}+\delta_i\right)^{n+1}\right\}$$

$$
\begin{aligned}
&= \frac{1}{(n+1)!}\sum_{i=0}^{k-1}\left[\sum_{r=0}^{n+1}\binom{n+1}{r}(-\delta_i)^r\left(\frac{h_i}{2}\right)^{n+1-r}\right.\\
&\quad\left.+\sum_{r=0}^{n+1}\binom{n+1}{r}\delta_i^r\left(\frac{h_i}{2}\right)^{n+1-r}\right]\\
&= \frac{1}{(n+1)!}\sum_{i=0}^{k-1}\left(\frac{h_i}{2}\right)^{n+1}\left[\sum_{r=0}^{n+1}\binom{n+1}{r}\left(\frac{2\delta_i}{h_i}\right)^r\left\{1+(-1)^r\right\}\right].
\end{aligned}
$$

Also,

$$
\begin{aligned}
G_2 \quad : &= \int_a^b K_{n,k}^2(t)\,dt = \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}\frac{(t-\alpha_{i+1})^{2n}}{(n!)^2}dt\\
&= \frac{1}{(2n+1)(n!)^2}\sum_{i=0}^{k-1}\left\{(x_{i+1}-\alpha_{i+1})^{2n+1}+(\alpha_{i+1}-x_i)^{2n+1}\right\}\\
&= \frac{1}{(2n+1)(n!)^2}\sum_{i=0}^{k-1}\left\{\left(\frac{h_i}{2}-\delta_i\right)^{2n+1}+\left(\frac{h_i}{2}+\delta_i\right)^{2n+1}\right\}\\
&= \frac{1}{(2n+1)(n!)^2}\sum_{i=0}^{k-1}\left(\frac{h_i}{2}\right)^{2n+1}\left[\sum_{r=0}^{2n+1}\binom{2n+1}{r}\left(\frac{2\delta_i}{h_i}\right)^r\left\{1+(-1)^r\right\}\right].
\end{aligned}
$$

From identity (2.2), we may write

$$
\begin{aligned}
&\int_a^b K_{n,k}(t)\,f^{(n)}(t)\,dt = (-1)^n\int_a^b f(t)\,dt + (-1)^n\sum_{j=1}^{n}\frac{(-1)^j}{j!}\\
&\times\left[\sum_{i=0}^{k-1}\left\{(x_{i+1}-\alpha_{i+1})^j f^{(j-1)}(x_{i+1}) - (\alpha_{i+1}-x_i)^j f^{(j-1)}(x_i)\right\}\right]
\end{aligned}
$$

and from the left hand side of (2.34) we obtain

$$
\begin{aligned}
G_3 \quad : &= (-1)^n\int_a^b f(t)\,dt + (-1)^n\sum_{j=1}^{n}\frac{(-1)^j}{j!}\\
&\times\left[\sum_{i=0}^{k-1}\left\{\left(\frac{h_i}{2}-\delta_i\right)^j f^{(j-1)}(x_{i+1}) - \left(\frac{h_i}{2}+\delta_i\right)^j f^{(j-1)}(x_i)\right\}\right]\\
&-\left(\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)(n+1)!}\right)\sum_{i=0}^{k-1}\left(\frac{h_i}{2}\right)^{n+1}\\
&\times\left[\sum_{r=0}^{n+1}\binom{n+1}{r}\left(\frac{2\delta_i}{h_i}\right)^r\left\{1+(-1)^r\right\}\right]
\end{aligned}
$$

after substituting for G_1.

From the right hand side of (2.34) we substitute for G_1 and G_2 so that

$$\begin{aligned} G_4 \quad : \quad &= (b-a)\int_a^b K_{n,k}^2(t)\,dt - \left(\int_a^b K_{n,k}(t)\,dt\right)^2 \\ &= \frac{b-a}{(2n+1)(n!)^2}\sum_{i=0}^{k-1}\left(\frac{h_i}{2}\right)^{2n+1}\left[\sum_{r=0}^{2n+1}\binom{2n+1}{r}\left(\frac{2\delta_i}{h_i}\right)^r\{1+(-1)^r\}\right] \\ &\quad -\left(\frac{1}{(n+1)!}\sum_{i=0}^{k-1}\left(\frac{h_i}{2}\right)^{n+1}\left[\sum_{r=0}^{n+1}\binom{n+1}{r}\left(\frac{2\delta_i}{h_i}\right)^r\{1+(-1)^r\}\right]\right)^2. \end{aligned}$$

Hence,

$$|G_3| \le \frac{M-m}{2}(G_4)^{\frac{1}{2}}.$$

and Theorem 2.21 has been proved. ∎

COROLLARY 2.22. Let f, I_k and α_k be defined as in Theorem 2.21 and further define

$$\delta = \alpha_{i+1} - \frac{x_{i+1}+x_i}{2} \tag{2.35}$$

for all $i = 0, \ldots, k-1$ such that

$$|\delta| \le \frac{1}{2}\min\{h_i | i = 1, \ldots, k\}. \tag{2.36}$$

The following inequality applies.

$$\begin{aligned} &\Bigg|(-1)^n\int_a^b f(t)\,dt + (-1)^n\sum_{j=1}^{n}\frac{1}{j!}\times \\ &\left[\sum_{i=0}^{k-1}\left\{(-i)^j\left(\frac{h_i}{2}-\delta_i\right)^j f^{(j-1)}(x_{i+1}) - \left(\frac{h_i}{2}+\delta_i\right)^j f^{(j-1)}(x_i)\right\}\right] \\ &-\left(\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)(n+1)!}\right)\sum_{i=0}^{k-1}\sum_{r=0}^{n+1}\delta^r\left(\frac{h_i}{2}\right)^{n+1-r}\{1+(-1)^r\}\Bigg| \\ \le\ &\frac{M-m}{2}\Bigg[\frac{b-a}{(2n+1)(n!)^2} \\ &\times\sum_{i=0}^{k-1}\sum_{r=0}^{2n+1}\left[\binom{2n+1}{r}\delta^r\left(\frac{h_i}{2}\right)^{2n+1-r}\{1+(-1)^r\}\right] \\ &-\left(\frac{1}{(n+1)!}\sum_{i=0}^{k-1}\sum_{r=0}^{n+1}\binom{n+1}{r}\delta^r\left(\frac{h_i}{2}\right)^{n+1-r}\{1+(-1)^r\}\right)^2\Bigg]^{\frac{1}{2}}. \end{aligned} \tag{2.37}$$

The proof follows directly from (2.33) upon the substitution of (2.35) and some minor simplification.

REMARK 2.1. If for any division $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ of the interval $[a, b]$, we choose $\delta = 0$ in (2.35), we have the inequality

$$
\begin{aligned}
(2.38)\quad & \left| (-1)^n \int_a^b f(t)\,dt + (-1)^n \sum_{j=1}^{n} \frac{1}{j!} \right. \\
& \times \sum_{i=0}^{k-1} \left(\frac{h_i}{2} \right)^j \left\{ (-1)^j f^{(j-1)}(x_{i+1}) - f^{(j-1)}(x_i) \right\} \\
& \left. -2 \left(\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(b-a)(n+1)!} \right) \sum_{i=0}^{k-1} \left(\frac{h_i}{2} \right)^{n+1} \right| \\
\leq\ & \frac{M-m}{2} \left[\frac{2(b-a)}{(2n+1)(n!)^2} \sum_{i=0}^{k-1} \left(\frac{h_i}{2} \right)^{2n+1} - \left(\frac{2}{(n+1)!} \sum_{i=0}^{k-1} \left(\frac{h_i}{2} \right)^{n+1} \right)^2 \right]^{\frac{1}{2}}.
\end{aligned}
$$

The proof follows directly from (2.37).

REMARK 2.2. Let $f^{(n)}$ be defined as in Theorem 2.21 and consider an equidistant partitioning E_k of the interval $[a, b]$, where

$$E_k := x_i = a + i \left(\frac{b-a}{k} \right), \quad i = 0, \ldots, k.$$

The following inequality applies

$$
\begin{aligned}
(2.39)\quad & \left| (-1)^n \int_a^b f(t)\,dt + (-1)^n \sum_{j=1}^{n} \frac{1}{j!} \left(\frac{b-a}{2k} \right)^j \right. \\
& \times \left\{ (-1)^j f^{(j-1)} \left(\frac{a(k-i-1) + b(i-1)}{k} \right) - f^{(j-1)} \left(\frac{a(k-i) + ib}{k} \right) \right\} \\
& \left. -2 \left(\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(b-a)(n+1)!} \right) \left(\frac{b-a}{2k} \right)^{n+1} \right| \\
\leq\ & (M-m) \cdot \frac{nk}{(n+1)!\sqrt{2n+1}} \left(\frac{b-a}{2k} \right)^{n+1}.
\end{aligned}
$$

PROOF. The proof follows upon noting that $h_i = x_{i+1} - x_i = \left(\frac{b-a}{k}\right)$, $i = 0, \ldots, k$. ∎

2.6. Some Particular Integral Inequalities

In this subsection we point out some special cases of the integral inequalities in Section 2.3. In doing so, we shall recover, subsume and extend the results of a number of previous published papers [**3, 4, 5**].

We shall recover the left and right rectangle inequalities, the perturbed trapezoid inequality, the midpoint and Simpson's inequalities and the Newton-Cotes three eighths inequality, and a Boole type inequality.

In the case when $n = 1$, for the kernel $K_{1,k}(t)$ of (2.4), the inequalities (2.13), (2.16) and (2.18) reduce to the results obtained by Dragomir [**9, 10, 11**] for the cases when $f : [a, b] \to \mathbb{R}$ is absolutely continuous and f' belongs, respectively to the $L_\infty[a, b]$, $L_p[a, b]$ and $L_1[a, b]$ spaces.

Similarly, for $n = 1$, Dragomir [**17**] extended Theorem 2.7 for the case when $f[a, b] \to \mathbb{R}$ is a function of bounded variation on $[a, b]$.

For $n = 2$, and the kernel $K_{2,k}(t)$ of (2.4), the following theorem is obtained.

Theorem 2.23. *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a, b]$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, k)$ and $\alpha_{k+1} = b$. If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then we have the inequality*

$$\begin{aligned}(2.40)\qquad &\left| \int_a^b f(t)\,dt - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right.\\ &\left. + \frac{1}{2} \sum_{i=0}^{k} \left\{ (x_{i+1} - \alpha_{i+1})^2 f'(x_{i+1}) - (x_i - \alpha_{i+1})^2 f'(x_i) \right\} \right|\end{aligned}$$

$$\leq \begin{cases} \dfrac{\|f''\|_\infty}{6} \displaystyle\sum_{i=0}^{k-1} \left[(\alpha_{i+1} - x_i)^3 + (x_{i+1} - \alpha_{i+1})^3 \right] & \text{if } f'' \in L_\infty[a, b], \\[2ex] \dfrac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \left(\displaystyle\sum_{i=0}^{k-1} \left[(\alpha_{i+1} - x_i)^{2q+1} + (x_{i+1} - \alpha_{i+1})^{2q+1} \right] \right)^{\frac{1}{q}} & \\ \qquad\qquad \text{if } f'' \in L_p[a, b],\ p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, & \\[2ex] \left(\frac{\nu^2(h)}{16} + \frac{\rho^2(\delta)}{4} + \frac{\nu(h)}{2} \max\limits_{i \in [0, k-1]} |\delta_i| \right) \|f''\|_1 & \\ \qquad \leq \frac{3}{8} \|f''\|_1 \nu^2(h) & \text{if } f'' \in L_1[a, b], \end{cases}$$

where $h_i := x_{i+1} - x_i$, $\nu(h) = \max\{h_i | i = 0, \ldots, k-1\}$,

$$\delta_i = \alpha_{i+1} - \frac{x_{i+1} + x_i}{2}$$

and $\rho(\delta) = \max\{\delta_i | i = 0, \ldots, k-1\}$.

Proof. The first and second part of the inequality (2.40) can be obtained directly from Theorem 2.7 and Theorem 2.8 respectively. We now prove the third line of the inequality (2.40), on the $L_1[a, b]$ space, which is an improvement of the inequality (2.18) for the case $n = 2$.

From (2.14)

$$\begin{aligned}
& \left|\int_a^b K_{2,k}(t) f''(t)\,dt\right| \\
\leq\ & \int_a^b |K_{2,k}(t)|\,|f''(t)|\,dt = \left|\sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}} K_{2,k}(t) f''(t)\,dt\right| \\
\leq\ & \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}} |K_{2,k}(t)|\,|f''(t)|\,dt = \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}} \left|\frac{(t-\alpha_{i+1})^2}{2}\right| |f''(t)|\,dt =: W.
\end{aligned}$$

Now

$$\begin{aligned}
& \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}} \left|\frac{(t-\alpha_{i+1})^2}{2}\right| |f''(t)|\,dt \\
\leq\ & \sup_{t\in[x_i,x_{i+1}]}\left|\frac{(t-\alpha_{i+1})^2}{2}\right| \int_{x_i}^{x_{i+1}} |f''(t)|\,dt \\
=\ & \max_{i\in[0,k-1]}\left\{\frac{(\alpha_{i+1}-x_i)^2}{2}, \frac{(x_{i+1}-\alpha_{i+1})^2}{2}\right\}\int_{x_i}^{x_{i+1}} |f''(t)|\,dt.
\end{aligned}$$

Now, we may observe that

$$\begin{aligned}
& \max_{i\in[0,k-1]}\left\{\frac{(\alpha_{i+1}-x_i)^2}{2}, \frac{(x_{i+1}-\alpha_{i+1})^2}{2}\right\} \\
=\ & \max_{i\in[0,k-1]}\frac{1}{2}\left\{\frac{(x_{i+1}-\alpha_{i+1})^2+(\alpha_{i+1}-x_i)^2}{2} + \left|\frac{(\alpha_{i+1}-x_i)^2-(x_{i+1}-\alpha_{i+1})^2}{2}\right|\right\}.
\end{aligned}$$

Using the identity

$$A^2+B^2 = 2\left(\left(\frac{A+B}{2}\right)^2+\left(\frac{A-B}{2}\right)^2\right)$$

reduces the previous line to

$$\begin{aligned}
& \frac{1}{2}\max_{i\in[0,k-1]}\left\{\frac{(x_{i+1}-x_i)^2}{8}+\frac{1}{2}\left(\alpha_{i+1}-\frac{x_{i+1}+x_i}{2}\right)^2\right. \\
& \left.+(x_{i+1}-x_i)\left|\alpha_{i+1}-\frac{x_{i+1}+x_i}{2}\right|\right\} \\
=\ & \max_{i\in[0,k-1]}\left(\frac{h_i^2}{16}\right)+\max_{i\in[0,k-1]}\left(\frac{\delta_i^2}{4}\right)+\max_{i\in[0,k-1]}\left(\frac{h_i}{2}|\delta_i|\right) \\
=\ & \frac{\nu^2(h)}{16}+\frac{\rho^2(\delta)}{4}+\frac{\nu(h)}{2}\max_{i\in[0,k-1]}|\delta_i|.
\end{aligned}$$

Hence, from

$$W = \left(\frac{\nu^2(h)}{16}+\frac{\rho^2(\delta)}{4}+\frac{\nu(h)}{2}\max_{i\in[0,k-1]}|\delta_i|\right)\|f''\|_1$$

and the third line of the inequality (2.40) is thus proved.

We may also note that

$$\begin{aligned} |\delta_i| &= \left|\alpha_{i+1} - \frac{x_{i+1}+x_i}{2}\right| \leq \frac{h_i}{2}, \\ \delta_i^2 &\leq \left(\frac{h_i}{2}\right)^2, \end{aligned}$$

and hence

$$W \leq \left(\frac{\nu^2(h)}{16} + \frac{\nu^2(h)}{16} + \frac{\nu^2(h)}{4}\right)\|f''\|_1 = \frac{3}{8}\nu^2(h)\|f''\|_1$$

and Theorem 2.23 is completely proved. ∎

For the two branch Peano kernel,

$$K_{n,2}(t) := \begin{cases} \dfrac{1}{n!}(t-a)^n, & t \in [a,x) \\[2ex] \dfrac{1}{n!}(t-b)^n, & t \in (x,b] \end{cases} \tag{2.41}$$

the following results were obtained by Cerone, Dragomir and Roumeliotis [**6**] which follow directly from Theorems 2.7, 2.8 and 2.9.

THEOREM 2.24. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the derivatives $f^{(n-1)}$ $(n \geq 1)$ are absolutely continuous on $[a,b]$ and $x \in [a,b]$. Then*

$$\left|\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\left\{(x-a)^j - (-1)^j(b-x)^j\right\} f^{(j-1)}(x)\right| \tag{2.42}$$

$$\leq \begin{cases} \dfrac{\|f''\|_\infty}{(n+1)!}\left\{(x-a)^{n+1} + (b-x)^{n+1}\right\} & \text{if } f^{(n)} \in L_\infty[a,b], \\[2ex] \dfrac{\|f''\|_p}{n!\,(nq+1)^{\frac{1}{q}}}\left((x-a)^{nq+1} + (b-x)^{nq+1}\right)^{\frac{1}{q}} & \text{if } f^{(n)} \in L_p[a,b], \\ & \quad p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\[1ex] \dfrac{\left\|f^{(n)}\right\|_1}{n!}\left(\dfrac{b-a}{2} + \left|x - \dfrac{a+b}{2}\right|\right)^n & \\[2ex] \qquad \leq \dfrac{\left\|f^{(n)}\right\|_1}{n!}(b-a)^n & \text{if } f^{(n)} \in L_1[a,b], \end{cases}$$

PROOF. The proof follows from Theorems 2.7, 2.8 and 2.9, and using the kernel (2.41) after substituting $\alpha_0 = \alpha_1 = x_0 = a$, $\alpha_2 = \alpha_3 = x_2 = b$ and $x = x_1 \in [a,b]$.

From Theorem 2.9 we note that

$$\begin{aligned}
& \left|\int_a^b K_{n,2}(t) f^{(n)}(t)\,dt\right| \\
\leq\ & \|K_{n,2}(t)\|_\infty \left\|f^{(n)}\right\|_1 = \left\|f^{(n)}\right\|_1 \sup_{t\in[a,b]} |K_{n,2}(t)| \\
=\ & \frac{\left\|f^{(n)}\right\|_1}{n!} (\max\{x-a, b-x\})^n \\
=\ & \frac{\left\|f^{(n)}\right\|_1}{n!} \left(\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right)^n \leq \frac{\left\|f^{(n)}\right\|_1}{n!} (b-a)^n,
\end{aligned}$$

which completes the proof of Theorem 2.24. ∎

Corollary 2.25. If in (2.42) we substitute $x = b$, we obtain the 'perturbed right' rectangle inequality

$$\left|\int_a^b f(t)\,dt + \sum_{j=1}^n \frac{(-1)^j (b-a)^j}{j!} f^{(j-1)}(b)\right| \tag{2.43}$$

$$\leq \begin{cases} \dfrac{\|f''\|_\infty}{(n+1)!}(b-a)^{n+1} & \text{if } f^{(n)} \in L_\infty[a,b], \\[2ex] \dfrac{\|f''\|_p}{n!} \cdot \dfrac{(b-a)^{n+\frac{1}{q}}}{(nq+1)^{\frac{1}{q}}} & \text{if } f^{(n)} \in L_p[a,b], \\ & p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \dfrac{\left\|f^{(n)}\right\|_1}{n!}(b-a)^n & \text{if } f^{(n)} \in L_1[a,b]. \end{cases}$$

If in (2.42) we substitute $x = a$, we obtain the 'perturbed left' rectangle inequality

$$\left|\int_a^b f(t)\,dt - \sum_{j=1}^n \frac{(b-a)^j}{j!} f^{(j-1)}(a)\right| \tag{2.44}$$

$$\leq \begin{cases} \dfrac{\|f''\|_\infty}{(n+1)!}(b-a)^{n+1} & \text{if } f^{(n)} \in L_\infty[a,b], \\[2ex] \dfrac{\|f''\|_p}{n!} \cdot \dfrac{(b-a)^{n+\frac{1}{q}}}{(nq+1)^{\frac{1}{q}}} & \text{if } f^{(n)} \in L_p[a,b], \\ & p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \dfrac{\left\|f^{(n)}\right\|_1}{n!}(b-a)^n & \text{if } f^{(n)} \in L_1[a,b]. \end{cases}$$

If in (2.42) we substitute $x = \frac{a+b}{2}$, we obtain the best estimate, a 'perturbed trapezoid' inequality

$$\left| \int_a^b f(t)\,dt - \sum_{j=1}^{n} \left(\frac{b-a}{2}\right)^j \frac{1}{j!} \left\{(-1)^j - 1\right\} f^{(j-1)}\left(\frac{a+b}{2}\right) \right| \tag{2.45}$$

$$\leq \begin{cases} \dfrac{\|f''\|_\infty}{(n+1)!2^n}(b-a)^{n+1} & \text{if} \quad f^{(n)} \in L_\infty[a,b], \\ \dfrac{\|f''\|_p (b-a)^{n+\frac{1}{q}}}{n!\,(nq+1)^{\frac{1}{q}}\,2^n} & \text{if} \quad f^{(n)} \in L_p[a,b], \\ & \quad p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \dfrac{\|f^{(n)}\|_1 (b-a)^n}{n!2^n} & \text{if} \quad f^{(n)} \in L_1[a,b]. \end{cases}$$

Remark 2.3. It is of interest to note that only the even derivatives occur in the left hand side of (2.45). Taking $n = 1$ in Theorem 2.24 reproduces some of the results obtained by Dragomir and Wang [**19, 20, 22**] and for $n = 2$ we recover the results obtained by Cerone, Dragomir and Roumeliotis [**3, 4, 5**]. Moreover, we may observe that the bounds given in (2.42) for a generalised interior point method obtained from investigating various norms of the kernel (2.41) are the same as the bounds obtained from the generalised trapezoidal type rule resulting from various norms of the Peano kernel given by $\frac{(x-t)^n}{n!}$.

The following corollary may be obtained from (2.43) and (2.44).

Corollary 2.26. Let f be defined as given in Theorem 2.24. Then

$$\left| \int_a^b f(t)\,dt + \frac{1}{2}\sum_{j=1}^{n} \frac{(b-a)^j}{j!} \left\{(-1)^j f^{(j-1)}(b) - f^{(j-1)}(a)\right\} \right| \tag{2.46}$$

$$\leq \begin{cases} \dfrac{\|f^{(n)}\|_\infty}{(n+1)!}(b-a)^{n+1} \begin{cases} 1 & \text{if} \quad n=2r \\ \frac{2^{2r+1}-1}{2^{2r}} & \text{if} \quad n=2r+1 \end{cases} & \text{if} \quad f^{(n)} \in L_\infty[a,b], \\ \dfrac{\|f^{(n)}\|_p}{n!} \cdot \dfrac{(b-a)^{n+\frac{1}{q}}}{(nq+1)^{\frac{1}{q}}} & \text{if} \quad f^{(n)} \in L_p[a,b], \\ & \quad p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \dfrac{\|f^{(n)}\|_1}{n!}(b-a)^n & \text{if} \quad f^{(n)} \in L_1[a,b]. \end{cases}$$

Proof. Using the identity (2.6) with the kernel (2.41) at the left and right corners, we obtain (2.46). Details may also be seen in the paper by Cerone, Dragomir and Roumeliotis [**3**] for the $\|\cdot\|_\infty$ norm. ∎

COROLLARY 2.27. Let f be defined as above. Then

$$\left|\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\left[(b-\alpha)^j f^{(j-1)}(b) - (-1)^j (\alpha-a)^j f^{(j-1)}(a)\right]\right| \tag{2.47}$$

$$\leq \begin{cases} \dfrac{\|f''\|_\infty}{(n+1)!}\left((\alpha-a)^{n+1} + (b-\alpha)^{n+1}\right), \\[2ex] \dfrac{\|f''\|_p}{n!\,(nq+1)^{\frac{1}{q}}}\left((\alpha-a)^{nq+1} + (b-\alpha)^{nq+1}\right)^{\frac{1}{q}}, \\[2ex] \dfrac{\left\|f^{(n)}\right\|_1}{n!}\left(\dfrac{b-a}{2} + \left|\alpha - \dfrac{a+b}{2}\right|\right)^n. \end{cases}$$

PROOF. Follows from Theorem 2.7 with $\alpha_0 = a$, $x_0 = a$, $x_1 = b$, $\alpha_2 = b$ and $\alpha_1 = \alpha \in [a,b]$. ∎

The following inequalities relate to Taylor like expansions.

COROLLARY 2.28. Let g be defined as in Corollary 2.12. Then we have the inequality

$$\left|g(y) - g(a) + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\left\{(x-a)^j - (-1)^j (y-x)^j\right\} g^{(j)}(x)\right| \tag{2.48}$$

$$\leq \begin{cases} \dfrac{\left\|g^{(n+1)}\right\|_\infty}{(n+1)!}\left\{(x-a)^{n+1} + (y-x)^{n+1}\right\} \text{ if } g^{(n+1)} \in L_\infty[a,y] \\[2ex] \dfrac{\left\|g^{(n+1)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}}\left((x-a)^{nq+1} + (y-x)^{nq+1}\right)^{\frac{1}{q}} \\ \qquad\qquad \text{if } g^{(n+1)} \in L_p[a,y],\ \ p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\[2ex] \dfrac{\left\|g^{(n+1)}\right\|_1}{n!}\left[\dfrac{y-a}{2} + \left|x - \dfrac{a+y}{2}\right|\right]^n \text{ if } g^{(n+1)} \in L_1[a,y] \end{cases}$$

for all $a \leq x \leq y$.

PROOF. Follows directly from (2.42) by choosing $b = y$ and $f = g'$.

Perturbed 'right' and 'left' inequalities may also be obtained from (2.43) and (2.44) respectively by putting $x = a$ and $x = y$. ∎

It is also well known that for the classical Taylor expansion around some point a we have the following inequality,

$$\left|g(y) - \sum_{j=0}^{n} \frac{(y-a)^j}{j!} g^{(j)}(a)\right| \leq \frac{(y-a)^{n+1}}{(n+1)!}\left\|g^{(n+1)}\right\|_\infty \tag{2.49}$$

for all $y \geq a$.

We therefore may see that the approximation (2.48) around the arbitrary point $x \in [a, y]$ provides a better approximation for the mapping g at a point y than the classical Taylor expansion (2.49) around a point a. The inequality (2.48) attains its optimum when $x = \frac{a+b}{2}$ in (2.48), so that we have the inequality

$$\left| g(y) - g(a) + \sum_{j=1}^{n} \left(\frac{y-a}{2}\right)^{j} \frac{1}{j!} \left\{(-1)^{j} - 1\right\} g^{(j)}\left(\frac{a+y}{2}\right) \right| \tag{2.50}$$

$$\leq \begin{cases} \dfrac{\left\| g^{(n+1)} \right\|_{\infty}}{(n+1)!} \cdot \frac{(y-a)^{n+1}}{2^{n}} & \text{if } \ g^{(n+1)} \in L_{\infty}[a, y] \\[2ex] \dfrac{\left\| g^{(n+1)} \right\|_{p}}{n!\,(nq+1)^{\frac{1}{q}}} \cdot \dfrac{(y-a)^{n+\frac{1}{q}}}{2^{n}} & \text{if } \ g^{(n+1)} \in L_{p}[a, y], \\ & \quad p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\ \dfrac{\left\| g^{(n+1)} \right\|_{1}}{n!} \left(\dfrac{y-a}{2}\right)^{n} & \text{if } \ g^{(n+1)} \in L_{1}[a, y]. \end{cases}$$

The inequality (2.50) shows that for $g \in C^{\infty}[a, y]$ the series

$$g(a) - \sum_{j=1}^{\infty} \left(\frac{y-a}{2}\right)^{j} \frac{1}{j!} \left\{(-1)^{j} - 1\right\} g^{(j)}\left(\frac{a+y}{2}\right)$$

converges more rapidly to $g(y)$ than the usual one,

$$\sum_{j=0}^{n} \left(\frac{y-a}{2}\right)^{j} g^{(j)}(a)$$

from (2.49).

If we choose $n = 1$ in (2.42) we obtain the Ostrowski inequality

$$\left| \int_{a}^{b} f(t)\,dt - (b-a) f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^{2}}{(b-a)^{2}} \right] (b-a)^{2} \left\| f' \right\|_{\infty}$$

also the midpoint inequality

$$\left| \int_{a}^{b} f(t)\,dt - (b-a) f\left(\frac{a+b}{2}\right) \right| \leq \left(\frac{b-a}{2}\right)^{2} \left\| f' \right\|_{\infty}$$

and the trapezoid inequality

$$\left| \int_{a}^{b} f(t)\,dt - \frac{b-a}{2} \left(f(a) + f(b)\right) \right| \leq \frac{1}{2} (b-a)^{2} \left\| f' \right\|_{\infty}.$$

For $n = 2$ in (2.42) we deduce the following results, also obtained by Cerone, Dragomir and Roumeliotis [**3**].

$$\left| \int_{a}^{b} f(t)\,dt - (b-a) f(x) + (b-a)\left(x - \frac{a+b}{2}\right) f'(x) \right|$$

$$\leq \left[\frac{1}{24} + \frac{1}{2} \frac{\left(x - \frac{a+b}{2}\right)^{2}}{(b-a)^{2}} \right] (b-a)^{3} \left\| f'' \right\|_{\infty}.$$

The classical midpoint inequality,

$$\left|\int_a^b f(t)\,dt-(b-a)\,f\left(\frac{a+b}{2}\right)\right|\le\frac{1}{24}(b-a)^3\|f''\|_\infty,$$

and the perturbed trapezoid inequality

$$\begin{aligned}&\left|\int_a^b f(t)\,dt-\frac{b-a}{2}(f(a)+f(b))-\left(\frac{b-a}{2}\right)^2(f'(b)-f'(a))\right|\\ \le\;&\frac{(b-a)^3}{6}\|f''\|_\infty.\end{aligned}$$

From (2.48) we obtain

$$\left|g(y)-g(a)+(y-a)\,g'(x)+(y-a)\left(x-\frac{a+y}{2}\right)g''(x)\right|$$

$$\le\begin{cases}\|g'''\|_\infty(y-a)^3\left(\dfrac{1}{24}+\dfrac{1}{2}\dfrac{\left(x-\frac{a+y}{2}\right)^2}{(y-a)^2}\right) & \text{if } g'''\in L_\infty[a,y],\\[2ex] \dfrac{\|g'''\|_p}{6(2q+1)^{\frac{1}{q}}}\left((x-a)^{2q+1}+(y-x)^{2q+1}\right)^{\frac{1}{q}} & \text{if } g'''\in L_p[a,y],\\ & p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1,\\ \dfrac{\|g'''\|_1}{2}\left(\dfrac{y-a}{2}+\left|x-\dfrac{a+y}{2}\right|\right)^2 & \text{if } g'''\in L_1[a,y].\end{cases}$$

The following theorem produces another integral inequality with many applications.

Theorem 2.29. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping on $[a,b]$ and $a\le x_1\le b$, $a\le\alpha_1\le x_1\le\alpha_2\le b$. Then we have*

$$\begin{aligned}(2.51)\qquad&\left|\int_a^b f(t)\,dt+\sum_{j=1}^n\frac{(-1)^j}{j!}\Big[-(a-\alpha_1)^j f^{(j-1)}(a)\right.\\ &\left.+\left\{(x_1-\alpha_1)^j-(x_1-\alpha_2)^j\right\}f^{(j-1)}(x_1)+(b-\alpha_2)^j f^{(j-1)}(b)\Big]\right|\end{aligned}$$

$$\leq \begin{cases} \dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left((\alpha_1-a)^{n+1}+(x_1-\alpha_1)^{n+1}+(\alpha_2-x_1)^{n+1}+(b-\alpha_2)^{n+1}\right) \\ \leq \dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left((x_1-a)^{n+1}+(b-x_1)^{n+1}\right) \\ \leq \dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}(b-a)^{n+1}, \qquad f^{(n)}\in L_\infty[a,b], \\ \\ \dfrac{\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}}\left((\alpha_1-a)^{nq+1}+(x_1-\alpha_1)^{nq+1}+(\alpha_2-x_1)^{nq+1}+(b-\alpha_2)^{nq+1}\right)^{\frac{1}{q}}, \\ f^{(n)}\in L_p[a,b],\ \ p>1 \ \textit{and}\ \frac{1}{p}+\frac{1}{q}=1 \\ \\ \dfrac{\left\|f^{(n)}\right\|_1}{n!}(b-a)^n, \qquad f^{(n)}\in L_1[a,b]. \end{cases}$$

Proof. Consider the division $a=x_0\leq x_1\leq x_2=b$ and the numbers $\alpha_0=a$, $\alpha_1\in[a,x_1)$, $\alpha_2\in(x_1,b]$ and $\alpha_3=b$.

From the left hand side of (2.13) we obtain

$$\begin{aligned} & \sum_{j=1}^{n}\frac{(-1)^j}{j!}\sum_{i=0}^{2}\left\{(x_i-\alpha_i)^j-(x_i-\alpha_{i+1})^j\right\}f^{(j-1)}(x_i) \\ = & \sum_{j=1}^{n}\frac{(-1)^j}{j!}\Big[-(a-\alpha_1)^j f^{(j-1)}(a)+\left\{(x_1-\alpha_1)^j-(x_1-\alpha_2)^j\right\}f^{(j-1)}(x_1) \\ & +(b-\alpha_2)^j f^{(j-1)}(b)\Big]. \end{aligned}$$

From the right hand side of (2.13) we obtain

$$\begin{aligned} & \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\sum_{i=0}^{1}\left\{(\alpha_{i+1}-x_i)^{n+1}+(x_{i+1}-\alpha_{i+1})^{n+1}\right\} \\ = & \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left((\alpha_1-a)^{n+1}+(x_1-\alpha_1)^{n+1}+(\alpha_2-x_1)^{n+1}+(b-\alpha_2)^{n+1}\right) \end{aligned}$$

and hence the first line of the inequality (2.51) follows.

The $\|\cdot\|_p$ norm and $\|\cdot\|_1$ norm inequalities follow from (2.16) and (2.18) respectively. Hence Theorem 2.29 is proved.

Notice that if we choose $\alpha_1=a$ and $\alpha_2=b$ in Theorem 2.29 we obtain the inequality (2.42) of Theorem 2.24. ∎

The following proposition embodies a number of results, including the Ostrowski inequality, the midpoint and Simpson's inequalities and the three-eighths Newton-Cotes inequality including its generalisation.

Proposition 2.30. Let f be defined as in Theorem 2.29 and let $a \leq x_1 \leq b$, and $a \leq \frac{(m-1)a+b}{m} \leq x_1 \leq \frac{a+(m-1)b}{m} \leq b$ for m a natural number, $m \geq 2$, then we have the inequality

$$
\begin{aligned}
(2.52)\qquad & |P_{m,n}| \\
:= & \left| \int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\left(\frac{b-a}{m}\right)^j \left\{ f^{(j-1)}(b) - (-1)^j f^{(j-1)}(a) \right\} \right. \right. \\
& \left. \left. + \left\{ \left((x_1-a) - \frac{b-a}{m} \right)^j - \left(\frac{b-a}{m} - (b-x_1) \right)^j \right\} f^{(j-1)}(x_1) \right] \right| \\
\leq & \begin{cases}
\frac{\left\| f^{(n)} \right\|_\infty}{(n+1)!} \left(2\left(\frac{b-a}{m}\right)^{n+1} + \left(x_1 - a - \left(\frac{b-a}{m}\right) \right)^{n+1} \right. \\
\qquad \left. + \left(b - x_1 - \left(\frac{b-a}{m}\right) \right)^{n+1} \right) \\
\text{if} \quad f^{(n)} \in L_\infty[a,b], \\[2ex]
\frac{\left\| f^{(n)} \right\|_p}{n!\,(nq+1)^{\frac{1}{q}}} \left(2\left(\frac{b-a}{m}\right)^{nq+1} + \left(x_1 - a - \left(\frac{b-a}{m}\right) \right)^{nq+1} \right. \\
\qquad \left. + \left(b - x_1 - \left(\frac{b-a}{m}\right) \right)^{nq+1} \right)^{\frac{1}{q}}, \\
\text{if} \quad f^{(n)} \in L_p[a,b], \ \ p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1 \\[2ex]
\frac{\left\| f^{(n)} \right\|_1}{n!} (b-a)^n, \quad \text{if} \quad f^{(n)} \in L_1[a,b].
\end{cases}
\end{aligned}
$$

Proof. From Theorem 2.29 we note that

$$\alpha_1 = \frac{(m-1)a+b}{m} \quad \text{and} \quad \alpha_2 = \frac{a+(m-1)b}{m}$$

so that

$$
\begin{aligned}
a-\alpha_1 &= -\left(\frac{b-a}{m}\right), \ b-\alpha_2 = \frac{b-a}{m}, \\
x_1-\alpha_1 &= x_1 - a - \left(\frac{b-a}{m}\right) \quad \text{and} \quad x_1-\alpha_2 = \left(\frac{b-a}{m}\right) - (b-x_1).
\end{aligned}
$$

From the left hand side of (2.51) we have

$$
\begin{aligned}
& -(a-\alpha_1)^j f^{(j-1)}(a) + \left\{ (x_1-\alpha_1)^j - (x_1-\alpha_2)^j \right\} f^{(j-1)}(x_1) \\
& + (b-\alpha_2)^j f^{(j-1)}(b) \\
= & \left(\frac{b-a}{m}\right)^j \left\{ f^{(j-1)}(b) - f^{(j-1)}(a) \right\} \\
& + \left\{ \left(x_1 - a - \left(\frac{b-a}{m}\right) \right)^j - \left(\left(\frac{b-a}{m}\right) - (b-x_1) \right)^j \right\} f^{(j-1)}(x_1).
\end{aligned}
$$

From the right hand side of (2.51),

$$\begin{aligned}
& (\alpha_1 - a)^{n+1} + (x_1 - \alpha_1)^{n+1} + (\alpha_2 - x_1)^{n+1} + (b - \alpha_2)^{n+1} \\
= \; & 2\left(\frac{b-a}{m}\right)^{n+1} + \left(x_1 - a - \left(\frac{b-a}{m}\right)\right)^{n+1} + \left(b - x_1 - \left(\frac{b-a}{m}\right)\right)^{n+1}
\end{aligned}$$

and the first line of the inequality (2.52) follows. The second and third lines of the inequality (2.52) follow directly from (2.51), hence the proof is complete. ∎

The following corollary points out that the optimum of Proposition 2.30 occurs at $x_1 = \frac{\alpha_1+\alpha_2}{2} = \frac{a+b}{2}$ in which case we have:

Corollary 2.31. Let f be defined as in Proposition 2.30 and let $x_1 = \frac{a+b}{2}$ in which case we have the inequality

$$\left| P_{m,n}\left(\frac{a+b}{2}\right)\right| \tag{2.53}$$

$$\begin{aligned}
: \; = & \left| \int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\left[\left(\frac{b-a}{m}\right)^j \left\{ f^{(j-1)}(b) - (-1)^j f^{(j-1)}(a) \right\} \right.\right. \\
& \left.\left. + \left(\frac{(m-2)(b-a)}{2m}\right)^j \left(1-(-1)^j\right) f^{(j-1)}\left(\frac{a+b}{2}\right)\right]\right| \\
\leq & \begin{cases}
\dfrac{2\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left(\dfrac{b-a}{m}\right)^{n+1}\left(1+\left(\dfrac{m-2}{2}\right)^{n+1}\right) \\
\text{if} \quad f^{(n)} \in L_\infty[a,b], \\[2ex]
\dfrac{\left\|f^{(n)}\right\|_p}{n!}\left[\left(\dfrac{2}{nq+1}\right)^{\frac{1}{q}}\left(\dfrac{b-a}{m}\right)^{n+\frac{1}{q}}\right]\left(1+\left(\dfrac{m-2}{2}\right)^{nq+1}\right)^{\frac{1}{q}} \\
\text{if} \quad f^{(n)} \in L_p[a,b], \;\; p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1 \\[2ex]
\dfrac{\left\|f^{(n)}\right\|_1}{n!}(b-a)^n, \quad \text{if} \quad f^{(n)} \in L_1[a,b].
\end{cases}
\end{aligned}$$

The proof follows directly from (2.52) upon substituting $x_1 = \frac{a+b}{2}$.

A number of other corollaries follow naturally from Proposition 2.30 and Corollary 2.31 and will now be investigated.

COROLLARY 2.32. Let f be defined as in Proposition 2.30. Then

$$\lim_{m\to\infty}|P_{m,n}| \tag{2.54}$$
$$= \left|\int_a^b f(t)\,dt + \sum_{j=1}^{n}\frac{(-1)^j}{j!}\left\{(x_1-a)^j-(x_1-b)^j\right\}f^{(j-1)}(x_1)\right|$$
$$\leq \begin{cases} \dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left((x_1-a)^{n+1}+(b-x_1)^{n+1}\right) \\[2ex] \dfrac{\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}}\left((x_1-a)^{nq+1}+(b-x_1)^{nq+1}\right)^{\frac{1}{q}} \\[2ex] \dfrac{\left\|f^{(n)}\right\|_1}{n!}\left(\dfrac{b-a}{2}+\left|x_1-\dfrac{a+b}{2}\right|\right)^n \end{cases}$$

and this is the result obtained in Theorem 2.24. In this case we see that $\alpha_1 = a$ and $\alpha_2 = b$ and the optimum of (2.53) occurs when $x_1 = \frac{a+b}{2}$ in which case we recover the result (2.45).

The following two corollaries generalise the Simpson inequality and follow directly from (2.52) and (2.53) for $m = 6$.

COROLLARY 2.33. Let the conditions of Corollary 2.31 hold and put $m = 6$. Then we have the inequality

$$|P_{6,n}| \tag{2.55}$$
$$: = \left|\int_a^b f(t)\,dt + \sum_{j=1}^{n}\frac{(-1)^j}{j!}\left[\left(\frac{b-a}{6}\right)^j\left\{f^{(j-1)}(b)-(-1)^j f^{(j-1)}(a)\right\}\right.\right.$$
$$\left.\left.+\left\{\left(x_1-\frac{5a+b}{6}\right)^j-\left(x_1-\frac{a+5b}{6}\right)^j\right\}f^{(j-1)}(x_1)\right]\right|$$
$$\leq \begin{cases} \dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left(2\left(\dfrac{b-a}{6}\right)^{n+1}+\left(x_1-\dfrac{5a+b}{6}\right)^{n+1}\right. \\ \qquad\left.+\left(-x_1+\dfrac{a+5b}{6}\right)^{n+1}\right), \quad f^{(n)}\in L_\infty[a,b], \\[2ex] \dfrac{\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}}\left(2\left(\dfrac{b-a}{6}\right)^{nq+1}+\left(x_1-\dfrac{5a+b}{6}\right)^{nq+1}\right. \\ \qquad\left.+\left(-x_1+\dfrac{a+5b}{6}\right)^{n+1}\right)^{\frac{1}{q}}, \\ \qquad\qquad \text{if } f^{(n)}\in L_p[a,b],\ p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1 \\[2ex] \dfrac{\left\|f^{(n)}\right\|_1}{n!}(b-a)^n, \quad \text{if } f^{(n)}\in L_1[a,b], \end{cases}$$

which is the generalised Simpson inequality.

Corollary 2.34. Let the conditions of Corollary 2.31 hold and put $m = 6$. Then at the midpoint $x_1 = \frac{a+b}{2}$ we have the inequality

$$\left|P_{6,n}\left(\frac{a+b}{2}\right)\right| \tag{2.56}$$

$$: \quad = \left|\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\left[\left(\frac{b-a}{6}\right)^j \left\{f^{(j-1)}(b) - (-1)^j f^{(j-1)}(a)\right\}\right.\right.$$

$$\left.\left. + \left(\frac{b-a}{3}\right)^j \left(1-(-1)^j\right) f^{(j-1)}\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \begin{cases} \dfrac{2\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left(\dfrac{b-a}{6}\right)^{n+1}\left(1+2^{n+1}\right), \quad f^{(n)} \in L_\infty[a,b], \\[2ex] \dfrac{\left\|f^{(n)}\right\|_p}{n!}\left[\left(\dfrac{2}{nq+1}\right)^{\frac{1}{q}}\left(\dfrac{b-a}{6}\right)^{n+\frac{1}{q}}\right]\left(1+2^{nq+1}\right)^{\frac{1}{q}}, \\ \text{if} \quad f^{(n)} \in L_p[a,b], \ \ p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1 \\[2ex] \dfrac{\left\|f^{(n)}\right\|_1}{n!}(b-a)^n, \quad \text{if} \quad f^{(n)} \in L_1[a,b]. \end{cases}$$

Remark 2.4. Choosing $n = 1$ in (2.56) we have

$$\left|P_{6,1}\left(\frac{a+b}{2}\right)\right| := \left|\int_a^b f(t)\,dt - \left(\frac{b-a}{6}\right)\left\{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right\}\right| \tag{2.57}$$

$$\leq \begin{cases} \dfrac{5}{36}\left\|f'\right\|_\infty (b-a)^2, \quad f' \in L_\infty[a,b], \\[2ex] \left\|f'\right\|_p \left(\dfrac{2}{q+1}\right)^{\frac{1}{q}}\left(\dfrac{b-a}{6}\right)^{1+\frac{1}{q}}\left(1+2^{q+1}\right)^{\frac{1}{q}}, \\ \text{if} \quad f' \in L_p[a,b], \ \ p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1 \\[2ex] \left\|f'\right\|_1 (b-a), \quad \text{if} \quad f^{(n)} \in L_1[a,b]. \end{cases} \tag{2.58}$$

Remark 2.5. Choosing $n = 2$ in (2.40) we have a perturbed Simpson type inequality

$$\left|P_{6,2}\left(\frac{a+b}{2}\right)\right| \tag{2.59}$$

$$: \quad = \left|\int_a^b f(t)\,dt - \left(\frac{b-a}{6}\right)\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right.$$

$$\left. + \left(\frac{b-a}{6}\right)^2 \left(\frac{f'(b) - f'(a)}{2}\right)\right|$$

$$\leq \begin{cases} \dfrac{(b-a)^3}{72}\left\|f''\right\|_{\infty}, & f'' \in L_{\infty}[a,b], \\ \left(\dfrac{b-a}{6}\right)^{2+\frac{1}{q}} \left(\dfrac{2}{2q+1}\right)^{\frac{1}{q}} \left(1+2^{2q+1}\right)^{\frac{1}{q}} \dfrac{\left\|f''\right\|_p}{2}, & f'' \in L_p[a,b], \\ & p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1 \\ \dfrac{(b-a)^2}{2}\left\|f''\right\|_1, & f'' \in L_1[a,b]. \end{cases}$$

Corollary 2.35. Let f be defined as in Corollary 2.31 and let $m=4$, then we have the inequality

$$\left|P_{4,n}\left(\frac{a+b}{2}\right)\right| \tag{2.60}$$

$$: = \left|\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\left(\frac{b-a}{4}\right)^j \right.$$
$$\left. \times \left[\left(f^{(j-1)}(b) - (-1)^j f^{(j-1)}(a)\right) + \left(1-(-1)^j\right) f^{(j-1)}\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \begin{cases} \dfrac{\left\|f^{(n)}\right\|_{\infty}}{(n+1)!4^n}(b-a)^{n+1}, & f^{(n)} \in L_{\infty}[a,b], \\ \dfrac{\left\|f^{(n)}\right\|_p}{n!}\left[\left(\dfrac{4}{nq+1}\right)^{\frac{1}{q}}\left(\dfrac{b-a}{4}\right)^{n+\frac{1}{q}}\right], & f^{(n)} \in L_p[a,b], \ p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1 \\ \dfrac{\left\|f^{(n)}\right\|_1}{n!}(b-a)^n, & f^{(n)} \in L_1[a,b]. \end{cases}$$

Remark 2.6. From (2.60) we choose $n=2$ and we have the inequality

$$\left|P_{4,2}\left(\frac{a+b}{2}\right)\right| \tag{2.61}$$

$$: = \left|\int_a^b f(t)\,dt - \left(\frac{b-a}{4}\right)\left(f(b)+f(a)+2f\left(\frac{a+b}{2}\right)\right) \right.$$
$$\left. + \left(\frac{b-a}{4}\right)^2\left(\frac{f'(b)-f'(a)}{2}\right)\right|$$

$$\leq \begin{cases} \dfrac{\left\|f''\right\|_{\infty}}{96}(b-a)^3, & f'' \in L_{\infty}[a,b], \\ \dfrac{\left\|f''\right\|_p}{2}\left(\dfrac{4}{2q+1}\right)^{\frac{1}{q}}\left(\dfrac{b-a}{4}\right)^{2+\frac{1}{q}}, & f'' \in L_p[a,b], \ p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1 \\ \dfrac{(b-a)^2}{2}\left\|f''\right\|_1, & f'' \in L_1[a,b]. \end{cases}$$

Theorem 2.36. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping on $[a,b]$ and let $a<x_1\leq x_2\leq b$ and $\alpha_1\in[a,x_1)$, $\alpha_2\in[x_1,x_2)$ and $\alpha_3\in[x_2,b]$. Then we*

have the inequality

$$\left|\int_a^b f(t)\,dt+\sum_{j=1}^{n}\frac{(-1)^j}{j!}\left[-(a-\alpha_1)^j f^{(j-1)}(a)\right.\right.$$
$$+\left((x_1-\alpha_1)^j-(x_1-\alpha_2)^j\right)f^{(j-1)}(x_1)$$
$$\left.\left.+\left((x_2-\alpha_2)^j-(x_2-\alpha_3)^j\right)f^{(j-1)}(x_2)+(b-\alpha_3)^j f^{(j-1)}(b)\right]\right| \tag{2.62}$$
$$\leq\begin{cases}\dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\Big((\alpha_1-a)^{n+1}+(x_1-\alpha_1)^{n+1}+(\alpha_2-x_1)^{n+1}\\ \qquad+(x_2-\alpha_2)^{n+1}+(\alpha_3-x_2)^{n+1}+(b-\alpha_3)^{n+1}\Big)\\ \quad f^{(n)}\in L_\infty[a,b],\\[2ex] \dfrac{\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}}\Big((\alpha_1-a)^{nq+1}+(x_1-\alpha_1)^{nq+1}+(\alpha_2-x_1)^{nq+1}\\ \qquad+(x_2-\alpha_2)^{nq+1}+(\alpha_3-x_2)^{nq+1}+(b-\alpha_3)^{nq+1}\Big)^{\frac{1}{q}},\\ f^{(n)}\in L_p[a,b],\;\; p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1\\[2ex] \dfrac{\left\|f^{(n)}\right\|_1}{n!}(b-a)^n,\qquad f^{(n)}\in L_1[a,b].\end{cases}$$

Proof. Consider the division $a=x_0<x_1<x_2=b$, $\alpha_1\in[a,x_1)$, $\alpha_2\in[x_1,x_2)$, $\alpha_3\in[x_2,b]$, $\alpha_0=a$, $x_0=a$, $x_3=b$ and put $\alpha_4=b$. From the left hand side of (2.13) we obtain

$$\sum_{j=1}^{n}\frac{(-1)^j}{j!}\sum_{i=0}^{3}\left\{(x_i-\alpha_i)^j-(x_i-\alpha_{i+1})^j\right\}f^{(j-1)}(x_i)$$
$$=\sum_{j=1}^{n}\frac{(-1)^j}{j!}\left[-(a-\alpha_1)^j f^{(j-1)}(a)\right.$$
$$+\left((x_1-\alpha_1)^j-(x_1-\alpha_2)^j\right)f^{(j-1)}(x_1)$$
$$\left.+\left((x_2-\alpha_2)^j-(x_2-\alpha_3)^j\right)f^{(j-1)}(x_2)+(b-\alpha_3)^j f^{(j-1)}(b)\right].$$

From the right hand side of (2.13) we obtain

$$\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\sum_{i=0}^{2}\left\{(\alpha_{i+1}-x_i)^{n+1}+(x_{i+1}-\alpha_{i+1})^{n+1}\right\}$$
$$=\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\Big((\alpha_1-a)^{n+1}+(x_1-\alpha_1)^{n+1}+(\alpha_2-x_1)^{n+1}$$
$$+(x_2-\alpha_2)^{n+1}+(\alpha_3-x_2)^{n+1}+(b-\alpha_3)^{n+1}\Big)$$

and hence the first line of the inequality (2.62) follows. The $\|\cdot\|_p$ norm and $\|\cdot\|_1$ norm inequalities follow from (2.16) and (2.18) respectively, hence Theorem 2.36 is proved. ∎

COROLLARY 2.37. Let f be defined as in Theorem 2.36 and consider the division

$$a \leq \alpha_1 \leq \frac{(m-1)\,a+b}{m} \leq \alpha_2 \leq \frac{a+(m-1)\,b}{m} \leq \alpha_3 \leq b$$

for m a natural number, $m \geq 2$. Then we have the inequality

$$\begin{aligned}(2.63)\qquad |Q_{m,n}| &:= \left|\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\left[-(a-\alpha_1)^j f^{(j-1)}(a)\right.\right.\\
&+ \left\{\left(\frac{(m-1)\,a+b}{m}-\alpha_1\right)^j - \left(\frac{(m-1)\,a+b}{m}-\alpha_2\right)^j\right\} f^{(j-1)}(x_1)\\
&+ \left\{\left(\frac{a+(m-1)\,b}{m}-\alpha_2\right)^j - \left(\frac{a+(m-1)\,b}{m}-\alpha_3\right)^j\right\} f^{(j-1)}(x_2)\\
&+ \left.\left.(b-\alpha_3)^j f^{(j-1)}(b)\right]\right|\end{aligned}$$

$$\leq \begin{cases} \dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} R_{m,n} := \dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left((\alpha_1-a)^{n+1} + \left(\dfrac{(m-1)\,a+b}{m}-\alpha_1\right)^{n+1}\right. \\ \qquad + \left(\alpha_2 - \dfrac{(m-1)\,a+b}{m}\right)^{n+1} + \left(\dfrac{a+(m-1)\,b}{m}-\alpha_2\right)^{n+1} \\ \qquad + \left.\left(\alpha_3 - \dfrac{a+(m-1)\,b}{m}\right)^{n+1} + (b-\alpha_3)^{n+1}\right), \quad f^{(n)} \in L_\infty[a,b], \\ \dfrac{\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}}\left(R_{m,nq}\right)^{\frac{1}{q}}, \quad f^{(n)} \in L_p[a,b],\; p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \dfrac{\left\|f^{(n)}\right\|_1}{n!}(b-a)^n, \quad f^{(n)} \in L_1[a,b]. \end{cases}$$

PROOF. Choose in Theorem 2.36, $x_1 = \frac{(m-1)a+b}{m}$, and $x_2 = \frac{a+(m-1)b}{m}$, hence the theorem is proved. ∎

REMARK 2.7. For particular choices of the parameters m and n, Corollary 2.37 contains a generalisation of the three-eighths rule of Newton and Cotes.

The following corollary is a consequence of Corollary 2.37.

COROLLARY 2.38. Let f be defined as in Theorem 2.36 and choose $\alpha_2 = \frac{a+b}{2} = \frac{x_1+x_2}{2}$, then we have the inequality

$$\begin{aligned}(2.64)\quad \left|\bar{Q}_{m,n}\right| \;:&= \left|\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\left[-(a-\alpha_1)^j f^{(j-1)}(a)\right.\right.\\
&+ \left\{(x_1-\alpha_1)^j - (-1)^j\left(\frac{(m-2)\,(b-a)}{2m}\right)^j\right\} f^{(j-1)}(x_1)\\
&+ \left\{\left(\frac{(m-2)\,(b-a)}{2m}\right)^j - (x_2-\alpha_3)^j\right\} f^{(j-1)}(x_2)\\
&+ \left.\left.(b-\alpha_3)^j f^{(j-1)}(b)\right]\right|\end{aligned}$$

$$\leq \begin{cases} \dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\bar{R}_{m,n} := \dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left((\alpha_1-a)^{n+1}+\left(\dfrac{(m-1)\,a+b}{m}-\alpha_1\right)^{n+1}\right. \\ \qquad +2\left((b-a)\dfrac{(m-2)}{2m}\right)^{n+1}+\left(\alpha_3-\dfrac{a+(m-1)\,b}{m}\right)^{n+1} \\ \qquad \left.+(b-\alpha_3)^{n+1}\right), \qquad f^{(n)}\in L_\infty[a,b], \\ \dfrac{\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}}\left(\bar{R}_{m,nq}\right)^{\frac{1}{q}}, \quad f^{(n)}\in L_p[a,b],\ \ p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \dfrac{\left\|f^{(n)}\right\|_1}{n!}(b-a)^n, \quad f^{(n)}\in L_1[a,b]. \end{cases}$$

Proof. If we put $\alpha_2=\frac{a+b}{2}=\frac{x_1+x_2}{2}$ into (2.63), we obtain the inequality (2.64) and the corollary is proved. ∎

The following corollary contains an optimum estimate for the inequality (2.64).

Corollary 2.39. Let f be defined as in Theorem 2.36 and make the choices

$$\begin{aligned} \alpha_1 &= \left(\frac{3m-4}{2m}\right)a+\left(\frac{4-m}{2m}\right)b \quad \text{and} \\ \alpha_3 &= \left(\frac{4-m}{2m}\right)a+\left(\frac{3m-4}{2m}\right)b \end{aligned}$$

then we have the best estimate

$$\begin{aligned} &\left|\hat{Q}_{m,n}\right| \\ &: = \left|\int_a^b f(t)\,dt+\sum_{j=1}^{n}\frac{(-1)^j}{j!}\right. \\ &\times\left[\left(\frac{(b-a)(4-m)}{2m}\right)^j\left\{f^{(j-1)}(b)-(-1)^j f^{(j-1)}(a)\right\}\right. \\ &+\left.\left.\left(\frac{(b-a)(m-2)}{2m}\right)^j\left\{1-(-1)^j\right\}\left(f^{(j-1)}(x_1)+f^{(j-1)}(x_2)\right)\right]\right| \end{aligned} \tag{2.65}$$

$$\leq \begin{cases} \dfrac{2\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left(\dfrac{b-a}{2m}\right)^{n+1}\left((4-m)^{n+1}+2\,(m-2)^{n+1}\right) \\ \qquad f^{(n)}\in L_\infty[a,b], \\ \dfrac{2\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}}\left(\dfrac{b-a}{2m}\right)^{n+\frac{1}{q}}\left((4-m)^{nq+1}+2\,(m-2)^{nq+1}\right)^{\frac{1}{q}}, \\ \qquad f^{(n)}\in L_p[a,b],\ \ p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \dfrac{\left\|f^{(n)}\right\|_1}{n!}(b-a)^n, \quad f^{(n)}\in L_1[a,b]. \end{cases}$$

PROOF. Using the choice $\alpha_2 = \frac{a+b}{2} = \frac{x_1+x_2}{2}$, $x_1 = \frac{(m-1)a+b}{m}$ and $x_2 = \frac{a+(m-1)b}{m}$ we may calculate

$$(\alpha_1 - a) = \frac{(4-m)(b-a)}{2m} = (b - \alpha_3)$$

and

$$\begin{aligned}(x_1 - \alpha_1) &= (\alpha_2 - x_1) = (x_2 - \alpha_2) = (\alpha_3 - x_2) \\ &= \frac{(m-2)(b-a)}{2m}.\end{aligned}$$

Substituting in the inequality (2.64) we obtain the proof of (2.65). ∎

REMARK 2.8. For $m = 3$, we have the best estimation of (2.65) such that

$$\begin{aligned}(2.66)\quad \left|\hat{Q}_{3,n}\right| \quad : \quad &= \Bigg| \int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \\ &\times \left[\left(\frac{b-a}{6}\right)^j \left\{ f^{(j-1)}(b) - (-1)^j f^{(j-1)}(a) \right\} + \left(\frac{b-a}{6}\right)^j \right. \\ &\left. \times \left\{1 - (-1)^j\right\} \left(f^{(j-1)}\left(\frac{2a+b}{2}\right) + f^{(j-1)}\left(\frac{a+2b}{2}\right)\right)\right] \Bigg| \\ &\leq \begin{cases} \dfrac{\left\|f^{(n)}\right\|_\infty}{6^n (n+1)!} (b-a)^{n+1}, & f^{(n)} \in L_\infty[a,b], \\[2ex] \dfrac{2\left\|f^{(n)}\right\|_p}{n!} \left(\frac{3}{nq+1}\right)^{\frac{1}{q}} \left(\dfrac{b-a}{6}\right)^{n+\frac{1}{q}}, & \\ \qquad f^{(n)} \in L_p[a,b], \; p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, & \\[2ex] \dfrac{\left\|f^{(n)}\right\|_1}{n!} (b-a)^n, \quad f^{(n)} \in L_1[a,b]. & \end{cases}\end{aligned}$$

PROOF. From the right hand side of (2.65), consider the mapping

$$M_{m,n} := \left(\frac{4-m}{2m}\right)^{n+1} + 2\left(\frac{m-2}{2m}\right)^{n+1}$$

then

$$M'_{m,n} = \frac{2(n+1)}{m^2}\left(-\left(\frac{2}{m} - \frac{1}{2}\right)^n + \left(\frac{1}{2} - \frac{1}{m}\right)^n\right)$$

and $M_{m,n}$ attains its optimum when

$$\frac{2}{m} - \frac{1}{2} = \frac{1}{2} - \frac{1}{m},$$

in which case $m = 3$. Substituting $m = 3$ into (2.65), we obtain (2.66) and the corollary is proved. ∎

When $n=2$, then from (2.66) we have

$$\begin{aligned}\left|\hat{Q}_{3,2}\right| &:= \left|\int_a^b f(t)\,dt - \left[\left(\frac{b-a}{6}\right)(f(b)+f(a))\right.\right. \\ &+\frac{1}{2}\left(\frac{b-a}{6}\right)^2 (f'(b)-f'(a)) \\ &\left.\left.+\left(\frac{b-a}{3}\right)\left(f\left(\frac{2a+b}{2}\right)+f\left(\frac{a+2b}{2}\right)\right)\right]\right|\end{aligned}$$

$$\leq \begin{cases} \dfrac{\|f''\|_\infty}{216}(b-a)^3, & f''\in L_\infty[a,b], \\ \|f''\|_p\left(\dfrac{3}{2q+1}\right)^{\frac{1}{q}}\left(\dfrac{b-a}{6}\right)^{2+\frac{1}{q}}, & \\ \qquad f''\in L_p[a,b],\ \ p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, & \\ \dfrac{\|f''\|_1}{2}(b-a)^2, \quad f''\in L_1[a,b]. & \end{cases}$$

The next corollary encapsulates the generalised Newton-Cotes inequality.

Corollary 2.40. Let f be defined as in Theorem 2.36 and choose

$$\begin{aligned} x_1 &= \frac{2a+b}{3},\ x_2=\frac{a+2b}{3},\ \alpha_2=\frac{a+b}{2}, \\ \alpha_1 &= \frac{(r-1)a+b}{r} \text{ and } \alpha_3=\frac{a+(r-1)b}{r}.\end{aligned}$$

Then for r a natural number, $r\geq 3$, we have the inequality

$$\begin{aligned} &|T_{r,n}| \\ &:= \left|\int_a^b f(t)\,dt+\sum_{j=1}^{n}\frac{(-1)^j}{j!}\left[\left(\frac{b-a}{r}\right)^j\left\{f^{(j-1)}(b)-(-1)^j f^{(j-1)}(a)\right\}\right.\right. \\ &+\left\{\left(\frac{(b-a)(r-3)}{3r}\right)^j-(-1)^j\left(\frac{b-a}{6}\right)^j\right\}f^{(j-1)}(x_1) \\ &\left.\left.+\left\{\left(\frac{b-a}{6}\right)^j-(-1)^j\left(\frac{(b-a)(r-3)}{3r}\right)^j\right\}f^{(j-1)}(x_2)\right]\right| \end{aligned}\tag{2.67}$$

$$\leq \begin{cases} \dfrac{2\left\|f^{(n)}\right\|_{\infty}}{(n+1)!}(b-a)^{n+1}\left(\dfrac{1}{r^{n+1}}+\left(\dfrac{r-3}{3r}\right)^{n+1}+\dfrac{1}{6^{n+1}}\right), \\ \quad f^{(n)} \in L_{\infty}[a,b], \\ \\ \dfrac{2\left\|f^{(n)}\right\|_{p}}{n!\,(nq+1)^{\frac{1}{q}}}\left(\dfrac{b-a}{6r}\right)^{n+\frac{1}{q}}\left(6^{nq+1}+(2r(r-3))^{nq+1}+r^{nq+1}\right)^{\frac{1}{q}}, \\ f^{(n)} \in L_p[a,b], \quad p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \dfrac{\left\|f^{(n)}\right\|_{1}}{n!}(b-a)^{n}, \qquad f^{(n)} \in L_1[a,b]. \end{cases}$$

PROOF. From Theorem 2.36, we put $x_1 = \frac{2a+b}{3}$, $x_2 = \frac{a+2b}{3}$, $\alpha_2 = \frac{a+b}{2}$, $\alpha_1 = \frac{(r-1)a+b}{r}$ and $\alpha_3 = \frac{a+(r-1)b}{r}$. Then (2.67) follows. ∎

REMARK 2.9. The optimum estimate of the inequality (2.67) occurs when $r=6$. from (2.67) consider the mapping

$$M_{r,n} := \frac{1}{r^{n+1}}+\left(\frac{r-3}{3r}\right)^{n+1}+\frac{1}{6^{n+1}}$$

the $M'_{r,n} = -(n+1)\,r^{-n-2}+\left(\frac{n+1}{r^2}\right)\left(\frac{1}{3}-\frac{1}{r}\right)^{n}$ and $M_{r,n}$ attains its optimum when $\frac{1}{r}=\frac{1}{3}-\frac{1}{r}$, in which case $r=6$. In this case, we obtain the inequality (2.66) and specifically for $n=1$, we obtain a Simpson type inequality

$$\begin{aligned} (2.68) \qquad & \left|\hat{Q}_{3,1}\right| \\ : \quad &= \left|\int_a^b f(t)\,dt-\left(\frac{b-a}{6}\right)(f(a)+f(b))\right. \\ & \quad \left. -\left(\frac{b-a}{3}\right)\left(f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right)\right| \\ & \leq \begin{cases} \dfrac{\|f'\|_{\infty}}{12}(b-a)^2 & f' \in L_{\infty}[a,b], \\ \\ 2\|f'\|_p\left(\dfrac{3}{q+1}\right)^{\frac{1}{q}}\left(\dfrac{b-a}{6}\right)^{2+\frac{1}{q}}, & f' \in L_p[a,b], \\ & p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1 \\ \|f'\|_1 (b-a)^2, & f' \in L_1[a,b]. \end{cases} \end{aligned}$$

COROLLARY 2.41. Let f be defined as in Theorem 2.36 and choose $m=8$ such that $\alpha_1 = \frac{7a+b}{8}$, $\alpha_2 = \frac{a+b}{8}$ and $\alpha_3 = \frac{a+7b}{8}$ with $x_1 = \frac{2a+b}{3}$ and $x_2 = \frac{a+2b}{3}$. Then we have the inequality

$$
\begin{aligned}
(2.69) \quad & |T_{8,n}| \\
: \quad & = \left| \int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\left(\frac{b-a}{8}\right)^j \left(f^{(j-1)}(b) - (-1)^j f^{(j-1)}(a)\right) \right.\right. \\
& \left.\left. + \left(\frac{b-a}{6}\right)^j \left(\left(\frac{5}{4}\right)^j - (-1)^j\right) \left(f^{(j-1)}(x_1) - (-1)^j f^{(j-1)}(x_2)\right) \right] \right|
\end{aligned}
$$

$$
\leq \begin{cases}
\dfrac{2\left\|f^{(n)}\right\|_\infty}{(n+1)!} S_n := \dfrac{2\left\|f^{(n)}\right\|_\infty}{(n+1)!} \left(\dfrac{b-a}{24}\right)^{n+1} \left(3^{n+1} + 4^{n+1} + 5^{n+1}\right), \\
f^{(n)} \in L_\infty[a,b], \\
\\
\dfrac{2\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}} (S_{nq})^{\frac{1}{q}}, \\
f^{(n)} \in L_p[a,b], \quad p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1 \\
\\
\dfrac{\left\|f^{(n)}\right\|_1}{n!} (b-a)^n, \qquad f^{(n)} \in L_1[a,b].
\end{cases}
$$

where

$$
S_n := \left(\frac{b-a}{24}\right)^{n+1} \left(3^{n+1} + 4^{n+1} + 5^{n+1}\right)
$$

and

$$
S_{nq} := \left(\frac{b-a}{24}\right)^{nq+1} \left(3^{nq+1} + 4^{nq+1} + 5^{nq+1}\right).
$$

Proof. From Theorem 2.36 we put

$$
x_1 = \frac{2a+b}{3}, \quad x_2 = \frac{a+2b}{3}, \quad \alpha_1 = \frac{7a+b}{8}, \quad \alpha_3 = \frac{a+7b}{8}
$$

and $\alpha_2 = \frac{a+b}{2}$ and the inequality (2.69) is obtained. ∎

When $n = 1$ we obtain from (2.69) the ‘three-eighths rule’ of Newton-Cotes.

REMARK 2.10. From (2.67) with $r=3$ we have

$$
\begin{aligned}
|T_{3,n}| \quad : \quad & = \left| \int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\left(\frac{b-a}{3} \right)^j \left(f^{(j-1)}(b) - (-1)^j f^{(j-1)}(a) \right) \right.\right. \\
& \left.\left. + \left(\frac{b-a}{6} \right)^j \left\{ f^{(j-1)}(x_2) - f^{(j-1)}(x_1) \right\} \right] \right| \\
& \leq \begin{cases}
\dfrac{2\left\| f^{(n)} \right\|_\infty}{(n+1)!} (b-a)^{n+1} \left(\dfrac{1}{3^{n+1}} + \dfrac{1}{6^{n+1}} \right), \\
f^{(n)} \in L_\infty [a,b], \\[2ex]
\dfrac{2\left\| f^{(n)} \right\|_p}{n!\,(nq+1)^{\frac{1}{q}}} \left(\dfrac{b-a}{6} \right)^{n+\frac{1}{q}} \left(1 + 2^{nq+1} \right)^{\frac{1}{q}}, \\
f^{(n)} \in L_p [a,b], \quad p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1 \\[2ex]
\dfrac{\left\| f^{(n)} \right\|_1}{n!} (b-a)^n, \qquad f^{(n)} \in L_1 [a,b].
\end{cases}
\end{aligned}
$$

In particular, for $n=2$, we have the inequality

$$
\begin{aligned}
|T_{3,2}| \quad : \quad & = \left| \int_a^b f(t)\,dt - \left(\frac{b-a}{3} \right) (f(b)+f(a)) + \left(\frac{b-a}{3} \right)^2 \left(\frac{f'(b)-f'(a)}{2} \right) \right. \\
& - \left(\frac{b-a}{6} \right) \left(f\left(\frac{a+2b}{3} \right) + f\left(\frac{2a+b}{3} \right) \right) \\
& \left. + \left(\frac{b-a}{6} \right)^2 \left(\frac{f'\left(\frac{a+2b}{3}\right) - f'\left(\frac{2a+b}{3}\right)}{2} \right) \right| \\
& \leq \begin{cases}
\dfrac{\|f''\|_\infty}{72} (b-a)^3, & f'' \in L_\infty [a,b], \\[2ex]
\|f''\|_p \left(\dfrac{b-a}{6} \right)^{2+\frac{1}{q}} \left(\dfrac{1+2^{2q+1}}{2q+1} \right)^{\frac{1}{q}}, & f'' \in L_p [a,b], \\
 & p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1 \\
\dfrac{\|f''\|_1}{2} (b-a)^2, & f'' \in L_1 [a,b].
\end{cases}
\end{aligned}
$$

The following theorem encapsulates Boole's rule.

THEOREM 2.42. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping on $[a,b]$ and let $a<x_1<x_2<x_3<b$ and $\alpha_1\in[a,x_1)$, $\alpha_2\in[x_1,x_2)$, $\alpha_3\in[x_2,x_3)$ and*

$\alpha_4 \in [x_3, b]$. *Then we have the inequality*

$$\begin{aligned}(2.70)\qquad &\left|\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\left[-(a-\alpha_1)^j f^{(j-1)}(a)\right.\right.\\ &+\left((x_1-\alpha_1)^j-(x_1-\alpha_2)^j\right) f^{(j-1)}(x_1)\\ &+\left((x_2-\alpha_2)^j-(x_2-\alpha_3)^j\right) f^{(j-1)}(x_2)\\ &\left.\left.+\left((x_3-\alpha_3)^j-(x_3-\alpha_4)^j\right) f^{(j-1)}(x_3)+(b-\alpha_4)^j f^{(j-1)}(b)\right]\right|\end{aligned}$$

$$\leq \begin{cases} \dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\Big((\alpha_1-a)^{n+1}+(x_1-\alpha_1)^{n+1}+(\alpha_2-x_1)^{n+1} \\ \qquad +(x_2-\alpha_2)^{n+1}+(\alpha_3-x_2)^{n+1}+(x_3-\alpha_3)^{n+1} \\ \qquad +(\alpha_4-x_3)^{n+1}+(b-\alpha_4)^{n+1}\Big) \quad \text{if} \quad f^{(n)} \in L_\infty[a,b], \\ \\ \dfrac{\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}}\Big[(\alpha_1-a)^{nq+1}+(x_1-\alpha_1)^{nq+1}+(\alpha_2-x_1)^{nq+1} \\ \quad +(x_2-\alpha_2)^{nq+1}+(\alpha_3-x_2)^{nq+1}+(x_3-\alpha_3)^{nq+1}+(\alpha_4-x_3)^{nq+1} \\ \quad +(b-\alpha_4)^{nq+1}\Big]^{\frac{1}{q}}, \quad \text{if} \quad f^{(n)} \in L_p[a,b], \quad p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \dfrac{\left\|f^{(n)}\right\|_1}{n!}(b-a)^n, \qquad\qquad f^{(n)} \in L_1[a,b]. \end{cases}$$

Proof. Follows directly from (2.13) with the points $\alpha_0 = x_0 = a$, $x_4 = \alpha_5 = b$ and the division $a = x_0 < x_1 < x_2 < x_3 = b$, $\alpha_1 \in [a, x_1)$, $\alpha_2 \in [x_1, x_2)$, $\alpha_3 \in [x_2, x_3)$ and $\alpha_4 \in [x_3, b]$. ∎

The following inequality arises from Theorem 2.42.

Corollary 2.43. Let f be defined as in Theorem 2.36 and choose $\alpha_1 = \frac{11a+b}{12}$, $\alpha_2 = \frac{11a+7b}{18}$, $\alpha_3 = \frac{7a+11b}{18}$, $\alpha_4 = \frac{a+11b}{12}$, $x_1 = \frac{7a+2b}{9}$, $x_3 = \frac{2a+7b}{9}$ and $x_2 = \frac{x_1+x_3}{2} = \frac{a+b}{2}$, then we can state:

$$\begin{aligned}&\left|\int_a^b f(t)\,dt + \sum_{j=1}^{n}\frac{(-1)^j}{j!}\left[\left(\frac{b-a}{12}\right)^j\left\{f^{(j-1)}(b)-(-1)^j f^{(j-1)}(a)\right\}\right.\right.\\ &+\left(\frac{b-a}{6}\right)^j\left\{\left(\frac{5}{6}\right)^j-(-1)^j\right\}\left\{f^{(j-1)}\left(\frac{7a+2b}{9}\right)-(-1)^j f^{(j-1)}\left(\frac{2a+7b}{9}\right)\right\}\\ &\left.\left.+\left(\frac{b-a}{9}\right)^j\left\{1-(-1)^j\right\} f^{(j-1)}\left(\frac{a+b}{2}\right)\right]\right|\end{aligned}$$

$$\leq \begin{cases} \dfrac{2\left\|f^{(n)}\right\|_{\infty}}{(n+1)!}\left(\dfrac{b-a}{36}\right)^{n+1}\left(3^{n+1}+4^{n+1}+5^{n+1}+6^{n+1}\right), \\ \text{if} \quad f^{(n)} \in L_{\infty}[a,b], \\ \\ \dfrac{2\left\|f^{(n)}\right\|_{p}}{n!\,(nq+1)^{\frac{1}{q}}}\left(\dfrac{b-a}{36}\right)^{n+\frac{1}{q}}\left(3^{nq+1}+4^{nq+1}+5^{nq+1}+6^{nq+1}\right)^{\frac{1}{q}}, \\ \text{if} \quad f^{(n)} \in L_{p}[a,b], \; p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \dfrac{\left\|f^{(n)}\right\|_{1}}{n!}(b-a)^{n}, \qquad \text{if} \quad f^{(n)} \in L_{1}[a,b]. \end{cases}$$

2.7. Applications for Numerical Integration

In this section we utilise the particular inequalities of the previous sections and apply them to numerical integration.

Consider the partitioning of the interval $[a,b]$ given by $\Delta_m : a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b$, put $h_i := x_{i+1} - x_i$ $(i = 0, \ldots, m-1)$ and put $\nu(h) := \max(h_i | i = 0, \ldots, m-1)$. The following theorem holds.

Theorem 2.44. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$, $k \geq 1$ and $m \geq 1$. Then we have the composite quadrature formula*

$$\int_a^b f(t)\,dt = A_k(\Delta_m, f) + R_k(\Delta_m, f) \tag{2.71}$$

where

$$A_k(\Delta_m, f) := -T_k(\Delta_m, f) - U_k(\Delta_m, f), \tag{2.72}$$

$$T_k(\Delta_m, f) := \sum_{i=0}^{m-1}\sum_{j=1}^{n}\left(\frac{h_i}{2k}\right)^j \frac{1}{j!}\left[-f^{(j-1)}(x_i) + (-1)^j f^{(j-1)}(x_{i+1})\right] \tag{2.73}$$

and

$$\begin{aligned} U_k(\Delta_m, f) \quad : \quad &= \sum_{i=0}^{m-1}\sum_{j=1}^{n}\left(\frac{h_i}{2k}\right)^j \frac{1}{j!} \\ &\times \left[\sum_{r=1}^{k-1}\left\{(-1)^j - 1\right\} f^{(j-1)}\left(\frac{(k-r)x_i + r x_{i+1}}{k}\right)\right] \end{aligned} \tag{2.74}$$

is a perturbed quadrature formula. The remainder $R_k(\Delta_m, f)$ satisfies the estimation

$$|R_k(\Delta_m, f)| \tag{2.75}$$

$$\leq \begin{cases} \dfrac{\left\|f^{(n)}\right\|_{\infty}}{2^n (n+1)!k^{n+1}} \displaystyle\sum_{i=0}^{m-1} h_i^{n+1}, & \textit{if} \quad f^{(n)} \in L_{\infty}[a,b], \\ \dfrac{\left\|f^{(n)}\right\|_{p}}{n! \left(k(nq+1)\right)^{\frac{1}{q}} (2k)^n} \left(\displaystyle\sum_{i=0}^{m-1} h_i^{nq+1}\right)^{\frac{1}{q}}, & \textit{if} \quad f^{(n)} \in L_p[a,b], \\ & p>1 \ \textit{and} \ \frac{1}{p}+\frac{1}{q}=1, \\ \dfrac{\left\|f^{(n)}\right\|_{1}}{n!k^n} \displaystyle\sum_{i=0}^{m-1} h_i^{n} \leq \dfrac{\left\|f^{(n)}\right\|_{1}}{n!k^n} \nu^n(h), & \textit{if} \quad f^{(n)} \in L_1[a,b]. \end{cases}$$

where $\nu(h) := \max(h_i | i = 0, \ldots, m-1)$.

Proof. We shall apply Corollary 2.11 on the interval $[x_i, x_{i+1}]$, $(i = 0, \ldots, m-1)$. Thus we obtain

$$\begin{aligned} &\left| \int_{x_i}^{x_{i+1}} f(t)\,dt + \sum_{j=1}^{n} \left(\frac{h_i}{2k}\right)^j \frac{1}{j!} \left[-f^{(j-1)}(x_i) + (-1)^j f^{(j-1)}(x_{i+1}) \right.\right. \\ &\left.\left. + \sum_{r=1}^{k-1} \left\{(-1)^j - 1\right\} f^{(j-1)}\left(\frac{(k-r)x_i + r x_{i+1}}{k}\right)\right]\right| \\ &\leq \begin{cases} \dfrac{1}{(n+1)!2^n} \displaystyle\sup_{t \in [x_i, x_{i+1}]} \left|f^{(n)}(t)\right| \left(\dfrac{x_{i+1}-x_i}{k}\right)^{n+1}, \\ \dfrac{1}{n!(2k)^n \left(k(nq+1)\right)^{\frac{1}{q}}} (x_{i+1}-x_i)^{n+\frac{1}{q}} \left(\displaystyle\int_{x_i}^{x_{i+1}} \left|f^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}}, \\ \dfrac{1}{n!k^n} (x_{i+1}-x_i)^n \displaystyle\int_{x_i}^{x_{i+1}} \left|f^{(n)}(t)\right| dt. \end{cases} \end{aligned}$$

Summing over i from 0 to $m-1$ and using the generalised triangle inequality, we have

$$\sum_{i=0}^{m-1}\left|\int_{x_i}^{x_{i+1}} f(t)\,dt+\sum_{j=1}^{n}\left(\frac{h_i}{2k}\right)^j\frac{1}{j!}\left[-f^{(j-1)}(x_i)+(-1)^j f^{(j-1)}(x_{i+1})\right.\right.$$
$$\left.\left.+\sum_{r=1}^{k-1}\left\{(-1)^j-1\right\}f^{(j-1)}\left(\frac{(k-r)x_i+rx_{i+1}}{k}\right)\right]\right|$$
$$\leq\begin{cases}\dfrac{1}{(n+1)!2^n}\displaystyle\sum_{i=0}^{m-1}\frac{h_i^{n+1}}{k^{n+1}}\sup_{t\in[x_i,x_{i+1}]}\left|f^{(n)}(t)\right|,\\[2ex] \dfrac{1}{n!\,(2k)^n\,(k\,(nq+1))^{\frac{1}{q}}}\displaystyle\sum_{i=0}^{m-1}h_i^{n+\frac{1}{q}}\left(\int_{x_i}^{x_{i+1}}\left|f^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}},\\[2ex] \dfrac{1}{n!k^n}\displaystyle\sum_{i=0}^{m-1}h_i^n\int_{x_i}^{x_{i+1}}\left|f^{(n)}(t)\right|dt.\end{cases}$$

Now,

$$\left|\int_a^b f(t)\,dt+\sum_{i=0}^{m-1}\sum_{j=1}^{n}\left(\frac{h_i}{2k}\right)^j\frac{1}{j!}\left[-f^{(j-1)}(x_i)+(-1)^j f^{(j-1)}(x_{i+1})\right]\right.\tag{2.76}$$
$$\left.+\sum_{i=0}^{m-1}\sum_{j=1}^{n}\left(\frac{h_i}{2k}\right)^j\frac{1}{j!}\left[\sum_{r=1}^{k-1}\left\{(-1)^j-1\right\}f^{(j-1)}\left(\frac{(k-r)x_i+rx_{i+1}}{k}\right)\right]\right|$$
$$\leq\quad R_k(\Delta_m,f).$$

As $\sup_{t\in[x_i,x_{i+1}]}\left|f^{(n)}(t)\right|\leq\left\|f^{(n)}\right\|_\infty$, the first inequality in (2.75) follows.

Using the discrete Hölder inequality, we have, from

$$\frac{1}{n!\,(2k)^n\,(k\,(nq+1))^{\frac{1}{q}}}\left(\sum_{i=0}^{m-1}h_i^{n+\frac{1}{q}}\left(\int_{x_i}^{x_{i+1}}\left|f^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}}\right)$$
$$\leq\quad\frac{1}{n!\,(2k)^n\,(k\,(nq+1))^{\frac{1}{q}}}\left(\sum_{i=0}^{m-1}\left(h_i^{n+\frac{1}{q}}\right)^q\right)^{\frac{1}{q}}$$
$$\times\left[\sum_{i=0}^{m-1}\left(\left(\int_{x_i}^{x_{i+1}}\left|f^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}}\right)^p\right]^{\frac{1}{p}}$$
$$=\quad\frac{1}{n!\,(k\,(nq+1))^{\frac{1}{q}}\,(2k)^n}\left(\sum_{i=0}^{m-1}h_i^{nq+1}\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p$$

and the second line of (2.75) follows.

Now, we may observe that

$$\frac{1}{n!k^n}\sum_{i=0}^{m-1} h_i^n \int\limits_{x_i}^{x_{i+1}} \left|f^{(n)}(t)\right| dt \leq \frac{1}{n!k^n} \max_{(i=0,\ldots,m-1)} (h_i)^n \left(\sum_{i=0}^{m-1} \int\limits_{x_i}^{x_{i+1}} \left|f^{(n)}(t)\right| dt\right)$$
$$= \frac{1}{n!k^n} (\nu(h))^n \left\|f^{(n)}\right\|_1$$

Hence, from (2.76) we see that

$$\left|\int_a^b f(t)\,dt + T_k(\Delta_m, f) + U_k(\Delta_m, f)\right| \leq |R_k(\Delta_m, f)|$$

and the theorem is proved. ∎

The following corollary holds.

COROLLARY 2.45. Let f be defined as above, then we have the equality

$$\int_a^b f(t)\,dt = -T_2(\Delta_m, f) - U_2(\Delta_m, f) + R_2(\Delta_m, f) \tag{2.77}$$

where

$$T_2(\Delta_m, f) := \sum_{i=0}^{m-1}\sum_{j=1}^{n} \left(\frac{h_i}{4}\right)^j \frac{1}{j!}\left[-f^{(j-1)}(x_i) + (-1)^j f^{(j-1)}(x_{i+1})\right]$$

$U_2(\Delta_m, f)$ is the perturbed midpoint quadrature rule, containing only even derivatives

$$U_2(\Delta_m, f) := \sum_{i=0}^{m-1}\sum_{j=1}^{n} \left(\frac{h_i}{4}\right)^j \frac{1}{j!}\left\{(-1)^j - 1\right\} f^{(j-1)}\left(\frac{x_i + x_{i+1}}{2}\right)$$

and the remainder, $R_2(\Delta_m, f)$ satisfies the estimation

$$|R_2(\Delta_m, f)| \leq \begin{cases} \dfrac{\left\|f^{(n)}\right\|_\infty}{2^{2n+1}(n+1)!}\displaystyle\sum_{i=0}^{m-1} h_i^{n+1}, & \text{if } f^{(n)} \in L_\infty[a,b] \\[2ex] \dfrac{\left\|f^{(n)}\right\|_p}{n!\,(2(nq+1))^{\frac{1}{q}}\,2^{2n}}\left(\displaystyle\sum_{i=0}^{m-1} h_i^{nq+1}\right)^{\frac{1}{q}}, & \text{if } f^{(n)} \in L_p[a,b], \\ & p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \dfrac{\left\|f^{(n)}\right\|_1}{n!2^n}\nu^n(h), & \text{if } f^{(n)} \in L_1[a,b]. \end{cases}$$

COROLLARY 2.46. Let f and Δ_m be defined as above. Then we have the equality

$$\int_a^b f(t)\,dt = -T_3(\Delta_m, f) - U_3(\Delta_m, f) + R_3(\Delta_m, f), \tag{2.78}$$

where

$$T_3(\Delta_m, f) := \sum_{i=0}^{m-1} \sum_{j=1}^{n} \left(\frac{h_i}{6}\right)^j \frac{1}{j!} \left[-f^{(j-1)}(x_i) + (-1)^j f^{(j-1)}(x_{i+1})\right]$$

and

$$\begin{aligned} U_3(\Delta_m, f) \quad : \quad &= \sum_{i=0}^{m-1} \sum_{j=1}^{n} \frac{\left((-1)^j - 1\right)}{j!} \left(\frac{h_i}{6}\right)^j \\ &\times \left[f^{(j-1)}\left(\frac{2x_i + x_{i+1}}{3}\right) + f^{(j-1)}\left(\frac{x_i + 2x_{i+1}}{3}\right)\right] \end{aligned}$$

and the remainder satisfies the bound

$$\begin{aligned} &|R_3(\Delta_m, f)| \\ \leq &\begin{cases} \dfrac{\left\|f^{(n)}\right\|_\infty}{3 \cdot 6^n (n+1)!} \displaystyle\sum_{i=0}^{m-1} h_i^{n+1}, & \text{if} \quad f^{(n)} \in L_\infty[a,b] \\ \dfrac{\left\|f^{(n)}\right\|_p}{n! (3(nq+1))^{\frac{1}{q}} 6^n} \left(\displaystyle\sum_{i=0}^{m-1} h_i^{nq+1}\right)^{\frac{1}{q}}, & \text{if} \quad f^{(n)} \in L_p[a,b], \\ & p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\ \dfrac{\left\|f^{(n)}\right\|_1}{n! 3^n} \displaystyle\sum_{i=0}^{m-1} h_i^n, & \text{if} \quad f^{(n)} \in L_1[a,b]. \end{cases} \end{aligned}$$

THEOREM 2.47. *Let f and Δ_m be defined as above and suppose that $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, \ldots, m-1)$. Then we have the quadrature formula:*

$$\begin{aligned} (2.79) \quad \int_a^b f(t)\, dt \quad = \quad &\sum_{i=0}^{m-1} \sum_{j=1}^{n} \frac{(-1)^j}{j!} \Big\{(\xi_i - x_i)^j f^{(j-1)}(x_i) \\ &- (-1)^j (x_{i+1} - \xi_i)^j f^{(j-1)}(x_{i+1})\Big\} + R(\xi, \Delta_m, f) \end{aligned}$$

and the remainder, $R(\xi, \Delta_m, f)$ satisfies the inequality

$$\begin{aligned} &|R(\xi, \Delta_m, f)| \\ \leq &\begin{cases} \dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} \displaystyle\sum_{i=0}^{m-1} \left((\xi_i - x_i)^{n+1} + (x_{i+1} - \xi_i)^{n+1}\right), \\ \qquad\qquad\qquad\qquad \textit{if } f^{(n)} \in L_\infty[a,b] \\ \dfrac{\left\|f^{(n)}\right\|_p}{n! (nq+1)^{\frac{1}{q}}} \left(\displaystyle\sum_{i=0}^{m-1} \left\{(\xi_i - x_i)^{nq+1} + (x_{i+1} - \xi_i)^{nq+1}\right\}\right)^{\frac{1}{q}}, \\ \qquad\qquad \textit{if } f^{(n)} \in L_p[a,b], \ p > 1 \textit{ and } \frac{1}{p} + \frac{1}{q} = 1, \\ \dfrac{\left\|f^{(n)}\right\|_1}{n!} \displaystyle\sum_{i=0}^{m-1} h_i^n, \qquad\qquad \textit{if } f^{(n)} \in L_1[a,b]. \end{cases} \end{aligned}$$

Proof. From Theorems 2.7, 2.8 and 2.9 we put $\alpha_0 = a$, $x_0 = a$, $x_1 = b$, $\alpha_2 = b$ and $\alpha_1 = \alpha \in [a, b]$ such that

$$\begin{aligned}
&\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\left[-(a-\alpha)^j f^{(j-1)}(a) + (b-\alpha)^j f^{(j-1)}(b)\right] \\
= \;& R(\xi, \Delta_m, f).
\end{aligned}$$

Over the interval $[x_i, x_{i+1}]$ $(i = 0, \ldots, m-1)$, we have

$$\begin{aligned}
&\int_{x_i}^{x_{i+1}} f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\Big[(x_{i+1}-\xi_i)^j f^{(j-1)}(x_{i+1}) \\
&-(-1)^j (\xi_i - x_i)^j f^{(j-1)}(x_i)\Big] \\
= \;& R(\xi, \Delta_m, f)
\end{aligned}$$

and therefore, using the generalised triangle inequality

$$\begin{aligned}
&|R(\xi, \Delta_m, f)| \\
\leq\;& \sum_{i=0}^{m-1}\left|\int_{x_i}^{x_{i+1}} f(t)\,dt + \sum_{i=0}^{m-1}\sum_{j=1}^{n}\frac{(-1)^j}{j!}\Big[(x_{i+1}-\xi_i)^j f^{(j-1)}(x_{i+1})\right. \\
&\left.-(-1)^j(\xi_i - x_i)^j f^{(j-1)}(x_i)\Big]\right| \\
\leq\;& \begin{cases}
\dfrac{1}{(n+1)!}\displaystyle\sum_{i=0}^{m-1}\sup_{t\in[x_i,x_{i+1}]}\left|f^{(n)}(t)\right|\left((\xi_i - x_i)^{n+1} + (x_{i+1}-\xi_i)^{n+1}\right), \\[2ex]
\dfrac{1}{n!\,(nq+1)^{\frac{1}{q}}}\displaystyle\sum_{i=0}^{m-1}\left(\int_{x_i}^{x_{i+1}}\left|f^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}} \\
\qquad\qquad\times\left\{(\xi_i - x_i)^{nq+1} + (x_{i+1}-\xi_i)^{nq+1}\right\}^{\frac{1}{q}}, \\[2ex]
\dfrac{1}{n!}(x_{i+1}-x_i)^n \displaystyle\int_{x_i}^{x_{i+1}}\left|f^{(n)}(t)\right| dt.
\end{cases}
\end{aligned}$$

The first part of the inequality (2.80) follows, since we have

$$\sup_{t\in[x_i,x_{i+1}]}\left|f^{(n)}(t)\right| \leq \left\|f^{(n)}\right\|_\infty .$$

In the case that $f^{(n)} \in L_p[a,b]$ we utilise the discrete Hölder inequality and for the second part of (2.80) we have

$$\begin{aligned}
& \frac{1}{n!\,(nq+1)^{\frac{1}{q}}} \sum_{i=0}^{m-1} \left\{ (\xi_i - x_i)^{nq+1} + (x_{i+1} - \xi_i)^{nq+1} \right\}^{\frac{1}{q}} \left(\int_{x_i}^{x_{i+1}} \left| f^{(n)}(t) \right|^p dt \right)^{\frac{1}{p}} \\
\leq\ & \frac{1}{n!\,(nq+1)^{\frac{1}{q}}} \left(\sum_{i=0}^{m-1} \left\{ (\xi_i - x_i)^{nq+1} + (x_{i+1} - \xi_i)^{nq+1} \right\} \right)^{\frac{1}{q}} \\
& \times \left[\sum_{i=0}^{m-1} \left(\left(\int_{x_i}^{x_{i+1}} \left| f^{(n)}(t) \right|^p dt \right)^{\frac{1}{p}} \right)^p \right]^{\frac{1}{p}} \\
=\ & \frac{1}{n!\,(nq+1)^{\frac{1}{q}}} \left\| f^{(n)} \right\|_p \left(\sum_{i=0}^{m-1} \left\{ (\xi_i - x_i)^{nq+1} + (x_{i+1} - \xi_i)^{nq+1} \right\} \right)^{\frac{1}{q}}.
\end{aligned}$$

Finally, observe that

$$\frac{\left\| f^{(n)} \right\|_1}{n!} \sum_{i=0}^{m-1} h_i^n \leq \frac{1}{n!} \sum_{i=0}^{m-1} h_i^n \left(\int_{x_i}^{x_{i+1}} \left| f^{(n)}(t) \right| dt \right)$$

and Theorem 2.47 is proved. ∎

The following corollary is a consequence of Theorem 2.47.

Corollary 2.48. Let f and Δ_m be as defined above. The following estimates apply.

(*i*) The n^{th} order left rectangle rule

$$\int_a^b f(t)\,dt = \sum_{i=0}^{m-1} \sum_{j=1}^{n} \frac{(-h_i)^j}{j!} f^{(j-1)}(x_i) + R_l(\Delta_m, f).$$

(*ii*) The n^{th} order right rectangle rule

$$\int_a^b f(t)\,dt = -\sum_{i=0}^{m-1} \sum_{j=1}^{n} \frac{(h_i)^j}{j!} f^{(j-1)}(x_{i+1}) + R_r(\Delta_m, f).$$

(*iii*) The n^{th} order trapezoidal rule

$$\begin{aligned}
\int_a^b f(t)\,dt \ =\ & \sum_{i=0}^{m-1} \sum_{j=1}^{n} \left(-\frac{h_i}{2} \right)^j \frac{1}{j!} \left\{ f^{(j-1)}(x_i) - (-1)^j f^{(j-1)}(x_{i+1}) \right\} \\
& + R_T(\Delta_m, f),
\end{aligned}$$

where

$$
\begin{aligned}
|R_l(\Delta_m, f)| &= |R_r(\Delta_m, f)| \\
&\leq \begin{cases}
\dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} \displaystyle\sum_{i=0}^{m-1} h_i^{n+1}, & \text{if} \quad f^{(n)} \in L_\infty[a,b] \\
\dfrac{\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}} \left(\displaystyle\sum_{i=0}^{m-1} h_i^{nq+1}\right)^{\frac{1}{q}}, & \text{if} \quad f^{(n)} \in L_p[a,b], \\
& \quad p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\
\dfrac{\left\|f^{(n)}\right\|_1}{n!} \displaystyle\sum_{i=0}^{m-1} h_i^{n}, & \text{if} \quad f^{(n)} \in L_1[a,b].
\end{cases}
\end{aligned}
$$

and

$$
|R_T(\Delta_m, f)| \leq \begin{cases}
\dfrac{\left\|f^{(n)}\right\|_\infty}{2^n\,(n+1)!} \displaystyle\sum_{i=0}^{m-1} h_i^{n+1}, & \text{if} \quad f^{(n)} \in L_\infty[a,b], \\
\dfrac{\left\|f^{(n)}\right\|_p}{2^n\,(nq+1)^{\frac{1}{q}}\,n!} \left(\displaystyle\sum_{i=0}^{m-1} h_i^{nq+1}\right)^{\frac{1}{q}}, & \text{if} \quad f^{(n)} \in L_p[a,b], \\
& \quad p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\
\dfrac{\left\|f^{(n)}\right\|_1}{n!} \displaystyle\sum_{i=0}^{m-1} h_i^{n}, & \text{if} \quad f^{(n)} \in L_1[a,b].
\end{cases}
$$

Theorem 2.49. *Consider the interval $x_i \leq \alpha_i^{(1)} \leq \xi_i \leq \alpha_i^{(2)} \leq x_{i+1}$, $i = 0, \ldots, m-1$, and let f and Δ_m be defined as above. Then we have the equality*

$$
\begin{aligned}
&\int_a^b f(t)\,dt \\
&= \sum_{i=0}^{m-1} \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left\{ \left(x_i - \alpha_i^{(i)}\right)^j f^{(j-1)}(x_i) \right. \\
&\quad - \left\{ \left(\xi_i - \alpha_i^{(1)}\right)^j - \left(\xi_i - \alpha_i^{(2)}\right)^j \right\} f^{(j-1)}(\xi_i) \\
&\quad \left. - \left(x_{i+1} - \alpha_i^{(2)}\right)^j f^{(j-1)}(x_{i+1}) \right\} + R\left(\xi, \alpha_i^{(1)}, \alpha_i^{(2)}, \Delta_m, f\right)
\end{aligned} \tag{2.80}
$$

and the remainder satisfies the estimation

$$
\begin{aligned}
&\left|R\left(\xi,\alpha_i^{(1)},\alpha_i^{(2)},\Delta_m,f\right)\right| \\
&\leq \begin{cases}
\dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\displaystyle\sum_{i=0}^{m-1}\left\{\left(\alpha_i^{(1)}-x_i\right)^{n+1}+\left(\xi_i-\alpha_i^{(1)}\right)^{n+1}\right. \\
\qquad\qquad \left.+\left(\alpha_i^{(2)}-\xi_i\right)^{n+1}+\left(x_{i+1}-\alpha_i^{(2)}\right)^{n+1}\right\}, \ \text{if } f^{(n)}\in L_\infty[a,b], \\
\dfrac{\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}}\left(\displaystyle\sum_{i=0}^{m-1}\left\{\left(\alpha_i^{(1)}-x_i\right)^{nq+1}+\left(\xi_i-\alpha_i^{(1)}\right)^{nq+1}\right.\right. \\
\qquad\qquad \left.\left.+\left(\alpha_i^{(2)}-\xi_i\right)^{nq+1}+\left(x_{i+1}-\alpha_i^{(2)}\right)^{nq+1}\right\}\right)^{\frac{1}{q}}, \\
\text{if } f^{(n)}\in L_p[a,b],\ p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\
\dfrac{\left\|f^{(n)}\right\|_1}{n!}\displaystyle\sum_{i=0}^{m-1}(x_{i+1}-x_i)^n, \quad \text{if } f^{(n)}\in L_1[a,b].
\end{cases}
\end{aligned}
$$

The proof follows directly from Theorem 2.29 on the intervals $[x_i, x_{i+1}]$, $(i = 0, \ldots, m-1)$.

The following Riemann type formula also holds.

COROLLARY 2.50. Let f and Δ_m be defined as above and choose $\xi_i \in [x_i, x_{i+1}]$, $(i = 0, \ldots, m-1)$. Then we have the equality

$$
\begin{aligned}
\int_a^b f(t)\,dt &= \sum_{i=0}^{m-1}\sum_{j=1}^{n}\frac{(-1)^j}{j!}\left\{(\xi_i-x_i)^{n+1}-(\xi_i-x_{i+1})^{n+1}\right\} \\
&\times f^{(j-1)}(\xi_i)+R_R(\xi,\Delta_m,f)
\end{aligned}
\tag{2.81}
$$

and the remainder satisfies the estimation

$$
\begin{aligned}
&|R_R(\xi,\Delta_m,f)| \\
&\leq \begin{cases}
\dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\displaystyle\sum_{i=0}^{m-1}\left((\xi_i-x_i)^{n+1}+(x_{i+1}-\xi_i)^{n+1}\right), \ \text{ if } f^{(n)}\in L_\infty[a,b], \\
\dfrac{\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}}\left(\displaystyle\sum_{i=0}^{m-1}\left\{(\xi_i-x_i)^{nq+1}+(x_{i+1}-\xi_i)^{nq+1}\right\}\right)^{\frac{1}{q}}, \\
\qquad\qquad \text{if } f^{(n)}\in L_p[a,b],\ p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\
\dfrac{\left\|f^{(n)}\right\|_1}{n!}\displaystyle\sum_{i=0}^{m-1}h_i^n, \qquad \text{if } f^{(n)}\in L_1[a,b].
\end{cases}
\end{aligned}
$$

The proof follows from (2.80) where $\alpha_i^{(1)} = x_i$ and $\alpha_i^{(2)} = x_{i+1}$.

Remark 2.11. If in (2.81) we choose the midpoint $2\xi_i = x_{i+1} + x_i$ we obtain the generalised midpoint quadrature formula

$$\int_a^b f(t)\,dt = \sum_{i=0}^{m-1}\sum_{j=1}^{n} \frac{(-1)^{j+1}}{j!}\left(\frac{h_i}{2}\right)^j \times \left\{1-(-1)^j\right\} f^{(j-1)}\left(\frac{x_i+x_{i+1}}{2}\right) + R_M(\Delta_m, f) \tag{2.82}$$

and $R_M(\Delta_m, f)$ is bounded by

$$|R_M(\Delta_m, f)| \le \begin{cases} \dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!2^n}\displaystyle\sum_{i=0}^{m-1} h_i^{n+1}, & \text{if } f^{(n)} \in L_\infty[a,b], \\ \dfrac{\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}\cdot 2^n}\left(\displaystyle\sum_{i=0}^{m-1} h_i^{nq+1}\right)^{\frac{1}{q}}, & \\ \qquad \text{if } f^{(n)} \in L_p[a,b],\ p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, & \\ \dfrac{\left\|f^{(n)}\right\|_1}{n!}\displaystyle\sum_{i=0}^{m-1} h_i^{n}, & \text{if } f^{(n)} \in L_1[a,b]. \end{cases}$$

Corollary 2.51. Consider a set of points

$$\xi_i \in \left[\frac{5x_i+x_{i+1}}{6}, \frac{x_i+5x_{i+1}}{6}\right] \quad (i=0,\dots,m-1)$$

and let f and Δ_m be defined as above. Then we have the equality

$$\int_a^b f(t)\,dt = \sum_{i=0}^{m-1}\sum_{j=1}^{n}\frac{(-1)^j}{j!}\left[\left(\frac{h_i}{6}\right)^j\left\{(-1)^j f^{(j-1)}(x_i) - f^{(j-1)}(x_{i+1})\right\} - \left\{\left(\xi_i - \frac{5x_i+x_{i+1}}{6}\right)^j - \left(\xi_i - \frac{x_i+5x_{i+1}}{6}\right)^j\right\} f^{(j-1)}(\xi_i)\right] + R_s(\Delta_m, f) \tag{2.83}$$

and the remainder, $R_s(\Delta_m, f)$ satisfies the bound

$$|R_s(\Delta_m, f)|$$

$$\leq \begin{cases} \dfrac{\left\|f^{(n)}\right\|_{\infty}}{(n+1)!}\displaystyle\sum_{i=0}^{m-1}\left\{2\left(\frac{h_i}{6}\right)^{n+1}+\left(\xi_i-\frac{5x_i+x_{i+1}}{6}\right)^{n+1}\right. \\ \qquad\qquad \left.+\left(\dfrac{x_i+5x_{i+1}}{6}-\xi_i\right)^{n+1}\right\}, \quad \text{if } f^{(n)}\in L_{\infty}\left[a,b\right], \\ \\ \dfrac{\left\|f^{(n)}\right\|_{p}}{n!\left(nq+1\right)^{\frac{1}{q}}}\left(\displaystyle\sum_{i=0}^{m-1}\left\{2\left(\frac{h_i}{6}\right)^{nq+1}+\left(\xi_i-\frac{5x_i+x_{i+1}}{6}\right)^{nq+1}\right.\right. \\ \qquad\qquad \left.\left.+\left(\dfrac{x_i+5x_{i+1}}{6}-\xi_i\right)^{nq+1}\right\}\right)^{\frac{1}{q}}, \\ \qquad\qquad \text{if } f^{(n)}\in L_{p}\left[a,b\right],\ p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \dfrac{\left\|f^{(n)}\right\|_{1}}{n!}\displaystyle\sum_{i=0}^{m-1}h_i^n, \quad \text{if } f^{(n)}\in L_{1}\left[a,b\right]. \end{cases}$$

REMARK 2.12. If in (2.83) we choose the midpoint $\xi_i=\frac{x_{i+1}+x_i}{2}$ we obtain a generalised Simpson formula:

$$\begin{aligned} \int_a^b f(t)\,dt &= \sum_{i=0}^{m-1}\sum_{j=1}^{n}\frac{(-1)^j}{j!}\left[\left(\frac{h_i}{6}\right)^j\left\{(-1)^j f^{(j-1)}(x_i)-f^{(j-1)}(x_{i+1})\right\}\right. \\ &\quad \left.-\left(\frac{h_i}{3}\right)^j\left\{1-(-1)^j\right\}f^{(j-1)}\left(\frac{x_{i+1}+x_i}{2}\right)\right]+R_s\left(\Delta_m,f\right) \end{aligned}$$

and $R_s\left(\Delta_m,f\right)$ is bounded by

$$\left|R_s\left(\Delta_m,f\right)\right|\leq\begin{cases} \dfrac{\left\|f^{(n)}\right\|_{\infty}}{(n+1)!}2\left(1+2^{n+1}\right)\displaystyle\sum_{i=0}^{m-1}\left(\frac{h_i}{6}\right)^{n+1}, \quad \text{if } f^{(n)}\in L_{\infty}\left[a,b\right], \\ \\ \dfrac{\left\|f^{(n)}\right\|_{p}}{n!}\left(\dfrac{2\left(1+2^{n+1}\right)}{nq+1}\right)^{\frac{1}{q}}\left(\displaystyle\sum_{i=0}^{m-1}\left(\frac{h_i}{6}\right)^{nq+1}\right)^{\frac{1}{q}}, \\ \qquad\qquad \text{if } f^{(n)}\in L_{p}\left[a,b\right],\ p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \dfrac{\left\|f^{(n)}\right\|_{1}}{n!}\displaystyle\sum_{i=0}^{m-1}h_i^n, \quad \text{if } f^{(n)}\in L_{1}\left[a,b\right]. \end{cases}$$

The following is a consequence of Theorem 2.49.

COROLLARY 2.52. Consider the interval

$$x_i\leq\alpha_i^{(1)}\leq\frac{x_{i+1}+x_i}{2}\leq\alpha_i^{(2)}\leq x_{i+1}\quad\left(i=0,\ldots,m-1\right),$$

and let f and Δ_m be defined as above.
The following equality is obtained:-

$$\begin{aligned}(2.84)\quad \int_a^b f(t)\,dt &= \sum_{i=0}^{m-1}\sum_{j=1}^{n}\frac{(-1)^j}{j!}\left\{\left(x_i-\alpha_i^{(i)}\right)^j f^{(j-1)}(x_i)\right.\\ &-\left\{\left(\frac{x_{i+1}+x_i}{2}-\alpha_i^{(1)}\right)^j-\left(\frac{x_{i+1}+x_i}{2}-\alpha_i^{(2)}\right)^j\right\}f^{(j-1)}\left(\frac{x_{i+1}+x_i}{2}\right)\\ &-\left.\left(x_{i+1}-\alpha_i^{(2)}\right)^j f^{(j-1)}(x_{i+1})\right\}+R_B\left(\alpha_i^{(1)},\alpha_i^{(2)},\Delta_m,f\right),\end{aligned}$$

where the remainder satisfies the bound

$$\left|R_B\left(\alpha_i^{(1)},\alpha_i^{(2)},\Delta_m,f\right)\right|$$

$$\leq\begin{cases}\dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\displaystyle\sum_{i=0}^{m-1}\left\{\left(\alpha_i^{(1)}-x_i\right)^{n+1}+\left(\frac{x_{i+1}+x_i}{2}-\alpha_i^{(1)}\right)^{n+1}\right.\\ \qquad\left.+\left(\alpha_i^{(2)}-\frac{x_{i+1}+x_i}{2}\right)^{n+1}+\left(x_{i+1}-\alpha_i^{(2)}\right)^{n+1}\right\},\\[2ex] \dfrac{\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}}\left(\displaystyle\sum_{i=0}^{m-1}\left\{\left(\alpha_i^{(1)}-x_i\right)^{nq+1}+\left(\frac{x_{i+1}+x_i}{2}-\alpha_i^{(1)}\right)^{nq+1}\right.\right.\\ \qquad\left.\left.+\left(\alpha_i^{(2)}-\frac{x_{i+1}+x_i}{2}\right)^{nq+1}+\left(x_{i+1}-\alpha_i^{(2)}\right)^{nq+1}\right\}\right)^{\frac{1}{q}},\\[2ex] \dfrac{\left\|f^{(n)}\right\|_1}{n!}\displaystyle\sum_{i=0}^{m-1}h_i^n.\end{cases}$$

The following remark applies to Corollary 2.52.

Remark 2.13. If in (2.84) we choose

$$\alpha_i^{(1)}=\frac{3x_i+x_{i+1}}{4}\quad\text{and}\quad\alpha_i^{(2)}=\frac{x_i+3x_{i+1}}{4},$$

we have the formula:

$$\begin{aligned}(2.85)\qquad &\int_a^b f(t)\,dt\\ &=\sum_{i=0}^{m-1}\sum_{j=1}^{n}\frac{(-1)^j}{j!}\left(\frac{h_i}{4}\right)^j\left[(-1)^j f^{(j-1)}(x_i)-f^{(j-1)}(x_{i+1})\right.\\ &\qquad\left.-\left\{1-(-1)^j\right\}f^{(j-1)}\left(\frac{x_{i+1}+x_i}{2}\right)\right]+R_B(\Delta_m,f).\end{aligned}$$

The remainder, $R_B(\Delta_m, f)$ satisfies the bound

$$|R_B(\Delta_m, f)| \le \begin{cases} \dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} \times 4 \displaystyle\sum_{i=0}^{m-1} \left(\frac{h_i}{4}\right)^{n+1}, & \text{if } \ f^{(n)} \in L_\infty[a,b] \\ \dfrac{\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}} \left(4 \displaystyle\sum_{i=0}^{m-1} \left(\frac{h_i}{4}\right)^{nq+1}\right)^{\frac{1}{q}}, & \text{if } \ f^{(n)} \in L_p[a,b], \\ & \quad p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \dfrac{\left\|f^{(n)}\right\|_1}{n!} \displaystyle\sum_{i=0}^{m-1} h_i^n, & \text{if } \ f^{(n)} \in L_1[a,b]. \end{cases}$$

The following theorem incorporates the Newton-Cotes formula.

THEOREM 2.53. *Consider the interval*

$$x_i \le \alpha_i^{(1)} \le \xi_i^{(1)} \le \alpha_i^{(2)} \le \xi_i^{(2)} \le \alpha_i^{(3)} \le x_{i+1} \quad (i = 0, \ldots, m-1),$$

and let Δ_m *and* f *be defined as above. This consideration gives us the equality*

$$\begin{aligned} \int_a^b f(t)\,dt &= \sum_{i=0}^{m-1} \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\left(x_i - \alpha_i^{(1)}\right)^j f^{(j-1)}(x_i) - \left(x_{i+1} - \alpha_i^{(3)}\right)^j \right. \\ &\times f^{(j-1)}(x_{i+1}) - \left\{ \left(\xi_i^{(1)} - \alpha_i^{(1)}\right)^j - \left(\xi_i^{(1)} - \alpha_i^{(2)}\right)^j \right\} f^{(j-1)}\left(\xi_i^{(1)}\right) \\ &\left. - \left\{ \left(\xi_i^{(2)} - \alpha_i^{(2)}\right)^j - \left(\xi_i^{(2)} - \alpha_i^{(3)}\right)^j \right\} f^{(j-1)}\left(\xi_i^{(2)}\right) \right] \\ &+ R\left(\alpha_i^{(1)}, \alpha_i^{(2)}, \alpha_i^{(3)}, \xi_i^{(1)}, \xi_i^{(2)}, \Delta_m, f\right). \end{aligned} \tag{2.86}$$

The remainder satisfies the bound

$$\left| R\left(\alpha_i^{(1)}, \alpha_i^{(2)}, \alpha_i^{(3)}, \xi_i^{(1)}, \xi_i^{(2)}, \Delta_m, f\right) \right|$$

$$\le \begin{cases} \dfrac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} \displaystyle\sum_{i=0}^{m-1} \left\{ \left(\alpha_i^{(1)} - x_i\right)^{n+1} + \left(\xi_i^{(1)} - \alpha_i^{(1)}\right)^{n+1} + \left(\alpha_i^{(2)} - \xi_i^{(1)}\right)^{n+1} \right. \\ \qquad \left. + \left(\xi_i^{(2)} - \alpha_i^{(2)}\right)^{n+1} + \left(\alpha_i^{(3)} - \xi_i^{(2)}\right)^{n+1} + \left(x_{i+1} - \alpha_i^{(3)}\right)^{n+1} \right\}, \\ \qquad\qquad \textit{if } f^{(n)} \in L_\infty[a,b], \\ \dfrac{\left\|f^{(n)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}} \left(\displaystyle\sum_{i=0}^{m-1} \left\{ \left(\alpha_i^{(1)} - x_i\right)^{nq+1} + \left(\xi_i^{(1)} - \alpha_i^{(1)}\right)^{nq+1} + \left(\alpha_i^{(2)} - \xi_i^{(1)}\right)^{nq+1} \right. \right. \\ \qquad \left. \left. + \left(\xi_i^{(2)} - \alpha_i^{(2)}\right)^{nq+1} + \left(\alpha_i^{(3)} - \xi_i^{(2)}\right)^{nq+1} + \left(x_{i+1} - \alpha_i^{(3)}\right)^{nq+1} \right\} \right)^{\frac{1}{q}}, \\ \qquad\qquad \textit{if } f^{(n)} \in L_p[a,b],\ p>1 \textit{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \dfrac{\left\|f^{(n)}\right\|_1}{n!} \displaystyle\sum_{i=0}^{m-1} h_i^n, \qquad\qquad \textit{if } f^{(n)} \in L_1[a,b]. \end{cases}$$

The following is a consequence of Theorem 2.53.

COROLLARY 2.54. Let f and Δ_m be defined as above and make the choices $\alpha_i^{(1)} = \frac{7x_i+x_{i+1}}{8}$, $\alpha_i^{(2)} = \frac{x_i+x_{i+1}}{2}$, $\alpha_i^{(3)} = \frac{x_i+7x_{i+1}}{8}$, $\xi_i^{(1)} = \frac{2x_i+x_{i+1}}{3}$ and $\xi_i^{(2)} = \frac{x_i+2x_{i+1}}{3}$, then we have the equality:

$$
\begin{aligned}
&\int_a^b f(t)\,dt \\
&= \sum_{i=0}^{m-1} \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\left(\frac{h_i}{8}\right)^j \left\{ (-1)^j f^{(j-1)}(x_i) - f^{(j-1)}(x_{i+1}) \right\} \right. \\
&\quad - \left(\frac{h_i}{24}\right)^j \left\{ 5^j - (-4)^j \right\} f^{(j-1)}\left(\frac{2x_i+x_{i+1}}{3}\right) \\
&\quad \left. - \left(\frac{h_i}{24}\right)^j \left\{ 4^j - (-5)^j \right\} f^{(j-1)}\left(\frac{x_i+2x_{i+1}}{3}\right) \right] \\
&\quad + R_N(\Delta_m, f),
\end{aligned} \tag{2.87}
$$

where the remainder satisfies the bound

$$
|R_N(\Delta_m, f)| \leq \begin{cases}
\dfrac{2\left\|f^{(n)}\right\|_\infty}{(n+1)!} \left(3^{n+1}+4^{n+1}+5^{n+1}\right) \displaystyle\sum_{i=0}^{m-1} \left(\frac{h_i}{24}\right)^{n+1}, & \text{if } f^{(n)} \in L_\infty[a,b], \\[2ex]
\dfrac{2\left\|f^{(n)}\right\|_p}{n!} \left(\dfrac{3^{nq+1}+4^{nq+1}+5^{nq+1}}{nq+1}\right)^{\frac{1}{q}} \left(\displaystyle\sum_{i=0}^{m-1} \left(\frac{h_i}{24}\right)^{nq+1}\right)^{\frac{1}{q}}, & \\
\qquad \text{if } f^{(n)} \in L_p[a,b],\ p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, & \\[2ex]
\dfrac{\left\|f^{(n)}\right\|_1}{n!} \displaystyle\sum_{i=0}^{m-1} h_i^n, & \text{if } f^{(n)} \in L_1[a,b].
\end{cases}
$$

When $n = 1$, we obtain from (2.87) the three-eighths rule of Newton-Cotes.

REMARK 2.14. For $n = 2$ from (2.87), we obtained a perturbed three-eighths Newton-Cotes formula:

$$
\begin{aligned}
\int_a^b f(t)\,dt &= \sum_{i=0}^{m-1} \left(\left(\frac{h_i}{8}\right) \left(f(x_i)+f(x_{i+1})\right) + \left(\frac{h_i}{8}\right)^2 \left(\frac{f'(x_i)-f'(x_{i+1})}{2}\right) \right. \\
&\quad + \left(\frac{3h_i}{8}\right) \left(f\left(\frac{2x_i+x_{i+1}}{3}\right) + f\left(\frac{x_i+2x_{i+1}}{3}\right) \right) \\
&\quad \left. - \left(\frac{3h_i}{8}\right)^2 \left\{ \frac{f'\left(\frac{2x_i+x_{i+1}}{3}\right) - f'\left(\frac{x_i+2x_{i+1}}{3}\right)}{2} \right\} \right] \\
&\quad + R_N(\Delta_m, f),
\end{aligned}
$$

where the remainder satisfies the bound

$$
\begin{aligned}
&\left|R_N\left(\Delta_m, f\right)\right| \\
\leq &\left\{\begin{array}{l}
\dfrac{\left\|f''\right\|_{\infty}}{192} \displaystyle\sum_{i=0}^{m-1} h_i^3, \qquad \text{if } \ f'' \in L_{\infty}[a, b] \\[3ex]
\left\|f''\right\|_p\left(\dfrac{3^{2q+1}+4^{2q+1}+5^{2q+1}}{2q+1}\right)^{\frac{1}{q}}\left(\displaystyle\sum_{i=0}^{m-1}\left(\dfrac{h_i}{24}\right)^{2q+1}\right)^{\frac{1}{q}}, \\
\qquad\qquad \text{if } \ f'' \in L_p[a, b], \ \ p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\
\dfrac{\left\|f''\right\|_1}{n!} \displaystyle\sum_{i=0}^{m-1} h_i^2, \qquad \text{if } \ f'' \in L_1[a, b].
\end{array}\right.
\end{aligned}
$$

2.8. Concluding Remarks

This work has subsumed, extended and generalised many previous Ostrowski type results. Integral inequalities for $n-$times differentiable mappings have been obtained by the use of a generalised Peano kernel. Some particular integral inequalities, including the trapezoid, midpoint, Simpson and Newton-Cotes rules have been obtained and further developed into composite quadrature rules.

Moreover we have brought together interior point rules giving explicit error bounds, using Peano type kernels and results from the modern theory of inequalities. Work on obtaining bounds through the use of Peano kernels has also been, briefly treated, in the classical review books on numerical integration, by Stroud [**31**], Engels [**23**] and Davis and Rabinowitz [**7**]. However, in some other recent works, see for example Krommer and Ueberhuber [**26**], a constructive approach is taken, via Taylor or interpolating polynomials, to obtain quadrature results. This approach gives the order of approximation to the quadrature rule rather than readily providing explicit error bounds.

Further research in this area will be undertaken by considering the Chebychev and Lupaş inequalities. Similarly, the following alternate Grüss type results may be used to examine all the interior point rules of this chapter.

Let $\sigma\left(h\left(x\right)\right)=h\left(x\right)-M\left(g\right)$ where

$$M\left(h\right)=\frac{1}{b-a}\int_a^b h\left(t\right)dt.$$

Then from (2.31)

$$T\left(h,g\right)=M\left(hg\right)-M\left(h\right)M\left(g\right).$$

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CHAPTER 3

Three Point Quadrature Rules

by

P. CERONE and S.S. DRAGOMIR

ABSTRACT A unified treatment of three point quadrature rules is presented in which the classical rules of mid-point, trapezoidal and Simpson type are recaptured as particular cases. Riemann integrals are approximated for the derivative of the integrand belonging to a variety of norms. The Grüss inequality and a number of variants are also presented which provide a variety of inequalities that are suitatable for numerical implementation. Mappings that are of bounded total variation, Lipschitzian and monotonic are also investigated with relation to Riemann-Stieltjes integrals. Explicit *a priori* bounds are provided allowing the determination of the partition required to achieve a prescribed error tolerance.
It is demonstrated that with the above classes of functions, the average of a mid-point and trapezoidal type rule produces the best bounds.

3.1. Introduction

Three point quadrature rules of Newton-Cotes type have been examined extensively in the literature. In particular, the mid-point, trapezoidal and Simpson rules have been investigated more recently [**33**]-[**20**] with the view of obtaining bounds on the quadrature rule in terms of a variety of norms involving, at most, the first derivative. The bounds that have been obtained more recently also depend on the Peano kernel used in obtaining the quadrature rule. The general approach used in the past involves the assumption of bounded derivatives of degree higher than one. The partitioning is halved until the desired accuracy is obtained (see for example Atkinson [**1**]). The work in papers [**33**]-[**20**] aims at obtaining *a priori* estimates of the partition required in order to attain a particular bound on the error.

The current work employs the modern theory of inequalities to obtain bounds for three-point quadrature rules consisting of an interior point and boundary points. The mid-point, trapezoidal and Simpson rules are recaptured as particular instances of the current development. Riemann integrals are approximated for the derivative of the integrand belonging to a variety of norms. An inequality due to Grüss together with a number of extensions and variants is used to obtain perturbed three-point rules which produce tight bounds suitable for numerical quadrature. The approximation of Riemann-Stieltjes integrals is also investigated for which the mappings belong to a variety of classes including: total bounded variation, Lipschitzian and monotonic.

The chapter is divided in two sections. The first contains estimates of the error in terms of at most the first derivative while the second deals with the remainder when the function is n - differentiable.

The first section is arranged in the following manner.

In Subsection 3.2.1, an identity is derived that involves a three-point rule whose bound may be obtained in terms of the first derivative, $f' \in L_\infty [a, b]$. Application of the results in numerical integration are presented in Subsection 3.2.2. An Ostrowski- Grüss inequality is developed in Subsection 3.2.3, as is a *premature* Grüss which produces perturbed three-point rules. A further Ostrowski-Grüss inequality is developed in Subsection 3.2.4 which produces bounds that are even sharper than those obtained from the *premature* Grüss results.

Results and numerical implementation of inequalities in which the first derivative $f' \in L_1 [a, b]$ are developed in Subsection 3.2.5, while perturbed three-point rules are obtained in Subsection 3.2.6 through the analysis of some new Grüss-type results.

Three-point Lobatto rules are obtained in Subsection 3.2.7 when $f' \in L_p [a, b]$, while perturbed rules through the development of Grüss-type rules are investigated in Subsection 3.2.8.

Subsection 3.2.9 is reserved for functions that are not necessarily differentiable and so inequalities involving Riemann-Stieltjes integrals that are suitable for numerical implementation are investigated in which the functions are assumed to be either of total bounded variation, Lipschitzian or monotonic.

The work repeatedly demonstrates that a Newton-Cotes rule that is equivalent to the average of a mid-point and trapezoidal rule consistently gives tighter bounds than a Simpson-type rule. Some concluding remarks to the section and discussion are given in Subsection 3.2.10.

The second section is structured as follows.

A variety of identities are obtained in Subsection 3.3.2 for $f^{(n-1)}$ absolutely continuous for a generalisation of the kernel (3.198). Specific forms are highlighted and a generalised Taylor-like expansion is obtained. Inequalities are developed in Subsection 3.3.3 and perturbed results through Grüss inequalities and premature variants are discussed in Subsection 3.3.4. Subsection 3.3.5 demonstrates the applicability of the inequalities to numerical integration. Concluding remarks to the section are given in Subsection 3.3.7.

3.2. Bounds Involving at most a First Derivative

3.2.1. Inequalities Involving the First Derivative. We start with the following results in terms of sup-norms.

THEOREM 3.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative is bounded on (a,b) and denote $\|f'\|_\infty=\sup_{t\in(a,b)}|f'(t)|<\infty$. Further, let $\alpha:[a,b]\to\mathbb{R}$ and $\beta:(a,b]\to[a,b]$, $\alpha(x)\le x$, $\beta(x)\ge x$. Then, for all $x\in[a,b]$ we have the inequality*

$$\left|\int_a^b f(t)\,dt-[(\beta(x)-\alpha(x))f(x)+(b-\beta(x))f(b)+(\alpha(x)-a)f(a)]\right| \tag{3.1}$$
$$\le \|f'\|_\infty\left\{\frac{1}{2}\left[\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2\right]\right.$$
$$\left.+\left(\alpha(x)-\frac{a+x}{2}\right)^2+\left(\beta(x)-\frac{b+x}{2}\right)^2\right\}.$$

PROOF. Let

$$K(x,t)=\begin{cases} t-\alpha(x), & t\in[a,x] \\ t-\beta(x), & t\in(x,b]\end{cases}, \tag{3.2}$$

and consider

$$\int_a^b K(x,t)f'(t)\,dt.$$

Now, from (3.2),

$$\int_a^b K(x,t)f'(t)\,dt=\int_a^x (t-\alpha(x))f'(t)\,dt+\int_x^b (t-\beta(x))f'(t)\,dt,$$

and integrating by parts produces the identity

$$\int_a^b K(x,t)f'(t)\,dt \tag{3.3}$$
$$=(\beta(x)-\alpha(x))f(x)+(b-\beta(x))f(b)+(\alpha(x)-a)f(a)-\int_a^b f(t)\,dt.$$

Thus,

$$\left|\int_a^b f(t)\,dt-[(\beta(x)-\alpha(x))f(x)+(b-\beta(x))f(b)+(\alpha(x)-a)f(a)]\right| \tag{3.4}$$
$$\le \|f'\|_\infty\int_a^b |K(x,t)|\,dt.$$

Let $Q(x) = \int_a^b |K(x,t)|\,dt$ and so

$$\begin{aligned} Q(x) &= -\int_a^{\alpha(x)} (t-\alpha(x))\,dt + \int_{\alpha(x)}^{x} (t-\alpha(x))\,dt \\ &\quad -\int_x^{\beta(x)} (t-\beta(x))\,dt + \int_{\beta(x)}^{b} (t-\beta(x))\,dt \\ &= \frac{1}{2}\left\{(a-\alpha(x))^2 + (x-\alpha(x))^2 + (x-\beta(x))^2 + (b-\beta(x))^2\right\}. \end{aligned}$$

If we use the identity

$$\frac{X^2+Y^2}{2} = \left(\frac{X+Y}{2}\right)^2 + \left(\frac{X-Y}{2}\right)^2 \tag{3.5}$$

we may write $Q(x)$ as

$$Q(x) = \left(\frac{x-a}{2}\right)^2 + \left(\alpha(x) - \frac{a+x}{2}\right)^2 + \left(\frac{b-x}{2}\right)^2 + \left(\beta(x) - \frac{b+x}{2}\right)^2. \tag{3.6}$$

The reutilizing of identity (3.5) on $\frac{1}{2}\left[(x-a)^2 + (b-x)^2\right]$ in (3.6) and substitution into (3.4) will produce the result (3.1) and thus the theorem is proved. ∎

Corollary 3.2. Let f satisfy the conditions of Theorem 3.1. Then $\alpha(x) = \frac{a+x}{2}$ and $\beta(x) = \frac{b+x}{2}$ give the best bound for any $x \in [a,b]$ and so

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \frac{b-a}{2}\left[f(x) + \left(\frac{x-a}{b-a}\right)f(a) + \left(\frac{b-x}{b-a}\right)f(b)\right]\right| \\ \leq\ & \frac{\|f'\|_\infty}{2}\left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right]. \end{aligned} \tag{3.7}$$

Proof. The proof is trivial since (3.1) is a sum of squares and the minimum occurs when each of the terms are zero. ∎

Remark 3.1. Result (3.7) is similar to that obtained by Milanović and Pečarić [**45**, p. 470], although their bound relies on the second derivative being bounded. This is not always possible so that the weaker assumption of the first derivative being bounded as in (3.4) may prove to be useful.

Remark 3.2. An even more accurate quadrature formula is obtained when $x = \frac{a+b}{2}$, giving from (3.7) :

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\right| \\ \leq\ & \frac{\|f'\|_\infty}{2}\left(\frac{b-a}{2}\right)^2. \end{aligned} \tag{3.8}$$

This is equivalent to approximating an integral as the average of a mid-point and trapezoidal rule. The bound in (3.7) however, only requires the first derivative of the function f to be bounded.

Motivated by the results of Theorem 3.1 and Corollary 3.2, we consider a kernel of the form (3.2) where $\alpha(x)$ and $\beta(x)$ are convex combinations of the end points.

THEOREM 3.3. *Let f satisfy the conditions as stated in Theorem 3.1. Then the following inequality holds for any $\gamma \in [0,1]$ and $x \in [a,b]$:*

$$\begin{aligned} & \left| \int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f(x) \right.\right. \\ & \left.\left. + \gamma \left[\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b) \right] \right\} \right| \\ \leq\ & 2\|f'\|_\infty \left[\frac{1}{4} + \left(\gamma - \frac{1}{2}\right)^2 \right] \left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2 \right] \qquad (3.9) \\ \leq\ & \frac{(b-a)^2}{2} \|f'\|_\infty . \qquad (3.10) \end{aligned}$$

PROOF. Let

$$\alpha(x) = \gamma x + (1-\gamma) a \text{ and } \beta(x) = \gamma x + (1-\gamma) b. \qquad (3.11)$$

Thus, from Theorem 3.1 and its proof utilizing (3.3) where

$$Q(x) = \int_a^b |K(x,t)|\,dt \qquad (3.12)$$

we have from (3.5), on substituting for $\alpha(x)$ and $\beta(x)$ from (3.11), that

$$Q(x) = \left(\frac{x-a}{2}\right)^2 + \left[\left(\gamma - \frac{1}{2}\right)(x-a)\right]^2 + \left(\frac{b-x}{2}\right)^2 + \left[\left(\gamma - \frac{1}{2}\right)(b-x)\right]^2 .$$

Thus,

$$\begin{aligned} Q(x) &= \left[\frac{1}{4} + \left(\gamma - \frac{1}{2}\right)^2 \right] \left[(x-a)^2 + (b-x)^2 \right] \qquad (3.13) \\ &= 2\left[\frac{1}{4} + \left(\gamma - \frac{1}{2}\right)^2 \right] \left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2 \right], \end{aligned}$$

upon using the identity (3.5). Now, utilizing (3.12) and (3.13) in (3.4) will give the first part of the theorem, namely, equation (3.8). Inequality (3.10) can easily be ascertained since (3.9) attains its maximum at $\gamma = 0$ or 1, and at $x = a$ or b. ∎

REMARK 3.3. Corollary 3.2 may be recovered of γ is set at its optimal value of $\frac{1}{2}$ in Theorem 3.3.

REMARK 3.4. $\gamma = 0$ in (3.9) reproduces Ostrowski's inequality [**45**, p. 468] whose bound is sharpest when $x = \frac{a+b}{2}$, giving the midpoint rule.

REMARK 3.5. $\gamma = 1$ produces the generalized trapezoidal rule for which again the best bound occurs when $x = \frac{a+b}{2}$ giving the standard trapezoidal type rule.

Corollary 3.4. Let the conditions on f be as in Theorem 3.1. Then, the following inequality holds

$$
\begin{aligned}
(3.14)\qquad & \left|\int_a^b f(t)\,dt-(b-a)\left\{(1-\gamma)f\left(\frac{a+b}{2}\right)+\gamma\left[\frac{f(a)+f(b)}{2}\right]\right\}\right| \\
\leq\; & \frac{\|f'\|_\infty}{2}(b-a)^2\left[\frac{1}{4}+\left(\gamma-\frac{1}{2}\right)^2\right].
\end{aligned}
$$

Proof. Placing the optimal value of $x=\frac{a+b}{2}$ in (3.8) produces the result (3.14). ∎

Remark 3.6. Result (3.14) gives a linear combination between a mid-point and a trapezoidal rule. The optimal result is obtained by taking $\gamma=\frac{1}{2}$ in (3.14), giving the optimal bound when only the assumption of a bounded first derivative is used. This gives the result from (3.14)

$$
\begin{aligned}
(3.15)\qquad & \left|\int_a^b f(t)\,dt-\frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right| \\
\leq\; & \frac{(b-a)^2}{8}\|f'\|_\infty,
\end{aligned}
$$

which is equivalent to (3.8).
It should be noted that taking $\gamma=\frac{1}{3}$ in (3.14) gives a Simpson-type rule that is worse than (3.15), remembering that here we are only using the assumption of a bounded first derivative rather than the more restrictive (though more accurate) result of a bounded fourth derivative.

3.2.2. Application in Numerical Integration.

The following quadrature result holds.

Theorem 3.5. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $\|f'\|_\infty=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then for any partition $I_n: a=x_0<x_1<...<x_{n-1}<x_n=b$ of $[a,b]$ and any intermediate point vector $\xi=(\xi_0,\xi_1,...,\xi_{n-1})$ such that $\xi_i\in[x_i,x_{i+1}]$ for $i=0,1,...,n-1$, we have:*

$$(3.16)\qquad \int_a^b f(x)\,dx=A_c(f,I_n,\xi)+R_c(f,I_n,\xi),$$

where

$$
\begin{aligned}
& A_c(f,I_n,\xi) \\
=\;& (1-\gamma)\sum_{i=0}^{n-1}h_i f(\xi_i)+\gamma\left[\sum_{i=0}^{n-1}(\xi_i-x_i)f(x_i)+\sum_{i=0}^{n-1}(x_{i+1}-\xi_i)f(x_{i+1})\right] \\
=\;& (1-\gamma)\sum_{i=0}^{n-1}h_i f(\xi_i)+\gamma\left[\sum_{i=0}^{n-1}\xi_i\left(f(x_{i+1})-f(x_i)\right)+bf(b)-af(a)\right]
\end{aligned}
$$

and the remainder

$$\begin{aligned}|R_c(f,I_n,\xi)| &\leq 2\|f'\|_\infty\left[\frac{1}{4}+\left(\gamma-\frac{1}{2}\right)^2\right]\sum_{i=0}^{n-1}\left[\left(\frac{h_i}{2}\right)^2+\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right] \\ &\leq \frac{\|f'\|_\infty}{2}\sum_{i=0}^{n-1}h_i^2=\frac{\|f'\|_\infty}{2}n\nu^2(h),\end{aligned}$$

where $\nu(h)=\max_{i=0,\ldots,n-1}h_i$.

Proof. Applying inequality (3.8) on the interval $[x_i,x_{i+1}]$ for $i=0,1,2,\ldots,n-1$ we have

$$\begin{aligned}&\left|\int_{x_i}^{x_{i+1}}f(x)\,dx\right.\\ &\left.-\{(1-\gamma)f(\xi_i)h_i+\gamma[(\xi_i-x_i)f(x_i)+(x_{i+1}-\xi_i)f(x_{i+1})]\}\right| \\ \leq\ & 2\|f'\|_\infty\left[\frac{1}{4}+\left(\gamma-\frac{1}{2}\right)^2\right]\left[\left(\frac{h_i}{2}\right)^2+\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right] \\ \leq\ & \frac{\|f'\|_\infty}{2}h_i^2,\end{aligned}$$

since the coarsest bound is obtained at $\gamma=0$ or 1 and $\xi_i\in[x_{li},x_{i+1}]$.
Summing over i for $i=0$ to $n-1$ we may deduce (3.16) and its subsequent elucidation. ∎

Corollary 3.6. Let the assumptions of Theorem 3.5 hold. Then we have

$$\int_a^b f(x)\,dx=A_c(f,I_n)+R_c(f,I_n)$$

where

$$A_c(f,I_n)=(1-\gamma)\sum_{i=0}^{n-1}h_if\left(\frac{x_i+x_{i+1}}{2}\right)+\frac{\gamma}{2}\sum_{i=0}^{n-1}h_i\left(f(x_i)+f(x_{i+1})\right),$$

which is a linear combination of the mid-point and trapezoidal rules and the remainder $R(f,I_n)$ satisfies the relation

$$\begin{aligned}|R_c(f,I_n)| &\leq \frac{\|f'\|_\infty}{2}\left[\frac{1}{4}+\left(\gamma-\frac{1}{2}\right)^2\right]\sum_{i=0}^{n-1}h_i^2 \\ &= \frac{\|f'\|_\infty}{2}\left[\frac{1}{4}+\left(\gamma-\frac{1}{2}\right)^2\right]n\nu^2(h),\end{aligned}$$

where $\nu(h)=\max_{i=0,\ldots,n-1}h_i$.

Proof. Similar to Theorem 3.5 with $\xi_i=\frac{x_i+x_{i+1}}{2}$. ∎

3.2.3. A Generalized Ostrowski-Grüss Inequality Using Cauchy-Schwartz. The following result is known as the Grüss inequality which was proved by Grüss in 1935 (see for example [**44**, p. 296]).

THEOREM 3.7. *Let f, g be two integrable functions defined on $[a, b]$, satisfying the conditions*

$$c \le f(t) \le C \text{ and } d \le g(t) \le D$$

for all $t \in [a, b]$. Then

$$|\mathfrak{T}(f, g)| \le \frac{1}{4}(C - c)(D - d), \tag{3.17}$$

where

$$\mathfrak{T}(f, g) = \frac{1}{b-a}\int_a^b f(t)\, g(t)\, dt - \frac{1}{b-a}\int_a^b f(t)\, dt \cdot \frac{1}{b-a}\int_a^b g(t)\, dt, \tag{3.18}$$

and the constant $\frac{1}{4}$ is the best possible.

The proof of this theorem is an extension of that for Theorem 3.14 and discussion will be delayed until then. See also Remark 3.14.

THEOREM 3.8. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping in $\mathring{I}$ and let $a, b \in \mathring{I}$ with $a < b$. Further, let $f' \in L_1[a, b]$ and $d \le f'(x) \le D$, $\forall x \in [a, b]$. We have, then, the following inequality*

$$\begin{aligned} &\left| \int_a^b f(t)\, dt - (b-a)\left\{(1-\gamma) f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right) f(a) \right.\right.\right. \\ &\left.\left.\left. + \left(\frac{b-x}{b-a}\right) f(b)\right]\right\} + (b-a)(1-2\gamma)\left(x - \frac{a+b}{2}\right) S \right| \\ \le\ & \frac{(D-d)}{4}(b-a)\left\{\frac{b-a}{2} + \frac{1}{2}\left[\left|x - \frac{a+b}{2} + \left(\gamma - \frac{1}{2}\right)(b-a)\right| \right.\right. \\ &\left.\left. + \left|x - \frac{a+b}{2} - \left(\gamma - \frac{1}{2}\right)(b-a)\right|\right]\right\} \end{aligned} \tag{3.19}$$

where $S = \frac{f(b) - f(a)}{b-a}$ and $\gamma \in [0, 1]$.

PROOF. From the identity (3.3) with $\alpha(x)$ and $\beta(x)$ as defined in (3.11) we have

$$\begin{aligned} &\int_a^b f(t)\, dt - (b-a)\left\{(1-\gamma) f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b)\right]\right\} \\ =\ & -\int_a^b K(x, t)\, f'(t)\, dt \end{aligned} \tag{3.20}$$

where

$$K(x, t) = \begin{cases} t - [\gamma x + (1-\gamma) a], & t \in [a, x] \\ t - [\gamma x + (1-\gamma) b], & t \in (x, b] \end{cases}. \tag{3.21}$$

Now it is clear that for all $x \in [a, b]$ and $t \in [a, b]$ we have that

$$\phi(x) \le K(x, t) \le \Phi(x)$$

where

$$\phi(x) = -\max\{\gamma(x-a), (1-\gamma)(b-x)\}$$

and

$$\Phi(x) = \max\{(1-\gamma)(x-a), \gamma(b-x)\}.$$

Using the result that $\max\{X, Y\} = \frac{X+Y}{2} + \frac{1}{2}|Y - X|$ we have that

$$\Phi(x) = \frac{1}{2}[\gamma b - (1-\gamma)a + (1-2\gamma)x] + \frac{1}{2}|\gamma b + (1-\gamma)a - x|$$

and

$$-\phi(x) = \frac{1}{2}[(2\gamma-1)x + (1-\gamma)b - \gamma a] + \frac{1}{2}|\gamma a + (1-\gamma)b - x|.$$

Thus,

$$\begin{aligned} \Phi(x) - \phi(x) &= \frac{b-a}{2} + \frac{1}{2}\left\{\left|x - \frac{a+b}{2} + \left(\gamma - \frac{1}{2}\right)(b-a)\right| \right. \\ &\quad \left. + \left|x - \frac{a+b}{2} - \left(\gamma - \frac{1}{2}\right)(b-a)\right|\right\}. \end{aligned} \tag{3.22}$$

Now,

$$\begin{aligned} &\int_a^b K(x,t)\,dt \\ &= \int_a^x [t - [\gamma x + (1-\gamma)a]]\,dt + \int_x^b [t - [\gamma x + (1-\gamma)b]]\,dt \\ &= \int_{-\gamma(x-a)}^{(1-\gamma)(x-a)} u du + \int_{-(1-\gamma)(b-x)}^{\gamma(b-x)} v dv \\ &= \frac{1}{2}\left[(1-\gamma)^2 - \gamma^2\right]\left[(x-a)^2 - (b-x)^2\right] \\ &= (b-a)(1-2\gamma)\left(x - \frac{a+b}{2}\right). \end{aligned} \tag{3.23}$$

Applying Theorem 3.7 of Grüss to the mappings $K(x,\cdot)$ and $f'(t)$, and using $S = \frac{1}{b-a}\int_a^b f'(t)\,dt$, we obtain from (3.22) and (3.23),

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b K(x,t) f'(t)\,dt - (1-2\gamma)\left(x - \frac{a+b}{2}\right)S\right| \\ &\le \frac{(D-d)}{4}\left\{\frac{b-a}{2} + \frac{1}{2}\left[\left|x - \frac{a+b}{2} + \left(\gamma - \frac{1}{2}\right)(b-a)\right| \right.\right. \\ &\quad \left.\left. + \left|x - \frac{a+b}{2} - \left(\gamma - \frac{1}{2}\right)(b-a)\right|\right]\right\}. \end{aligned}$$

Then, using the identity (3.20) we obtain the result (3.19) as stated in the theorem. Hence, the theorem is completely proved. ∎

Remark 3.7. It should be noted that the shifted quadrature rule that is obtained through the Grüss inequality still involves function evaluations at the end points and an interior point x. Thus, a simple grouping of terms would produce the left hand side of (3.19) in an alternate form

$$\int_a^b f(t)\,dt - \left\{(1-\gamma)(b-a) f(x) + \left[\gamma(b-x) + \left(x - \frac{a+b}{2}\right)\right] f(a) + \left[\gamma(x-a) - \left(x - \frac{a+b}{2}\right)\right] f(b)\right\}.$$

Therefore, it is argued, the above quadrature rule is no more difficult to implement than the rule as given in Theorem 3.3 for example.

Corollary 3.9. Let the conditions be as in Theorem 3.8. Then the following inequality holds for any $x \in [a,b]$,

$$\left|\int_a^b f(t)\,dt - \frac{1}{2}\left[(b-a) f(x) + (x-a) f(a) + (b-x) f(b)\right]\right| \leq \frac{(D-d)(b-a)}{4}\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]. \tag{3.24}$$

Proof. Setting $\gamma = \frac{1}{2}$ in (3.19) readily produces the result (3.24). ∎

Corollary 3.10. Let the conditions be as in Theorem 3.8. Then the following inequality holds for any $\gamma \in [0,1]$,

$$\left|\int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f\left(\frac{a+b}{2}\right) + \frac{\gamma}{2}\left[f(a) + f(b)\right]\right\}\right| \leq \frac{(D-d)(b-a)^2}{4}\left[\frac{1}{2} + \left(\gamma - \frac{1}{2}\right)^2\right]. \tag{3.25}$$

Proof. Choosing x to be at the mid-point of $[a,b]$ in (3.19) gives the result (3.25). ∎

Remark 3.8. Placing $\gamma = 0$ in (3.19) produces an adjusted Ostrowski type rule, namely:

$$\left|\int_a^b f(t)\,dt - \left[f(x) - \left(x - \frac{a+b}{2}\right) S\right]\right| \leq \frac{(D-d)}{8} \cdot (b-a) + \left[|x-b| + |x-a|\right].$$

This bound is sharpest at $x = \frac{a+b}{2}$, thus producing the mid-point type rule

$$\left|\int_a^b f(t)\,dt - (b-a) f\left(\frac{a+b}{2}\right)\right| \leq \frac{(D-d)(b-a)^2}{4}. \tag{3.26}$$

Remark 3.9. Placing $\gamma = 1$ in (3.19) gives an adjusted generalized trapezoidal rule, namely:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t)\,dt - \left[\left(\frac{x-a}{b-a} \right) f(a) + \left(\frac{b-x}{b-a} \right) f(b) \right] - \left(x - \frac{a+b}{2} \right) S \right| \\ \leq \quad & \frac{(D-d)}{8} \left[b - a + \left[|x-b| + |x-a| \right] \right]. \end{aligned}$$

This bound is sharpest at $x = \frac{a+b}{2}$ giving the trapezoidal type rule

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{2} \left[f(a) + f(b) \right] \right| \leq \frac{(D-d)(b-a)^2}{4}. \tag{3.27}$$

Remark 3.10. The sharpest bound on (3.24) and (3.25) are at $x = \frac{a+b}{2}$ and $\gamma = \frac{1}{2}$ respectively. The same result can be obtained directly from (3.19), giving as the best quadrature rule of this type

$$\begin{aligned} & \left| \int_a^b f(t)\,dt - \frac{(b-a)}{2} \left[f\left(\frac{a+b}{2} \right) + \frac{f(a) + f(b)}{2} \right] \right| \\ \leq \quad & \frac{(b-a)^2}{8} (D-d) \end{aligned} \tag{3.28}$$

which is an averaged mid-point and trapezoidal rule.

Again, as noted in Subsection 3.2.1 when it was assumed that $f' \in L_\infty(a,b)$, $\gamma = \frac{1}{3}$ (in (3.25)) produces a Simpson type rule which is worse than the optimal rule given by (3.28). Here, we are only assuming that $d < f'(x) < D$ rather than the more restrictive, though more accurate, assumptions in the development of a traditional Simpson's rule of a bounded fourth derivative.

The following two results by Ostrowski will be needed for the proof of the theorem that follows. An improvement by Lupas is also presented. These will be presented as theorems which are generalizations of the Grüss inequality. The notation of Pečarić, Proschan and Tong [**47**] will be used.

Theorem 3.11. *Let f be a bounded measurable function on $I = (a,b)$ such that $c_1 \leq f(t) \leq c_2$ for $t \in I$ and assume $g'(t)$ exists and is bounded on I. Then,*

$$|\mathfrak{T}(f,g)| \leq \frac{b-a}{8} (c_2 - c_1) \sup_{t \in I} |g'(t)|$$

and $\frac{1}{8}$ is the best constant possible.

Theorem 3.12. *Let g be locally absolutely continuous on I with $g' \in L_2(I)$, and let f be bounded and measurable on $I = (a,b)$ such that $c_1 \leq f(t) \leq c_2$ for $t \in I$. Then*

$$|\mathfrak{T}(f,g)| \leq \frac{b-a}{4\sqrt{2}} (c_2 - c_1) \|g'\|_2, \tag{3.29}$$

and, the improved result,

$$|\mathfrak{T}(f,g)| \leq \frac{b-a}{2\pi} (c_2 - c_1) \|g'\|_2, \tag{3.30}$$

where

$$\|g'\|_2 = \left(\frac{1}{b-a} \int_a^b |g'(t)|^2 \, dt \right)^{\frac{1}{2}}.$$

THEOREM 3.13. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$ and let $a, b \in \mathring{I}$ with $a < b$. Further, let $f' \in L_1[a,b]$ and $d \leq f'(x) \leq D$, $\forall x \in [a,b]$. Then, the following inequality holds.*

$$\begin{aligned} (3.31) \quad & \left| \int_a^b f(t) \, dt - (b-a) \left\{ (1-\gamma) f(x) + \gamma \left[\left(\frac{x-a}{b-a} \right) f(a) + \left(\frac{b-x}{b-a} \right) f(b) \right] \right\} \right. \\ & \left. + (b-a)(1-2\gamma) \left(x - \frac{a+b}{2} \right) S \right| \\ \leq \; & \frac{D-d}{8} (b-a) \end{aligned}$$

where $S = \frac{f(b)-f(a)}{b-a}$ and $\gamma \in [0,1]$.

PROOF. Let $K(x,t)$ be as given by (3.21) and consider the interval $[a,x]$. Let $d_1 \leq f'(t) \leq D_1$ for $t \in [a,x]$. Then, from Theorem 3.7,

$$\left| \mathfrak{T}_{[a,x]}(f', K) \right| \leq \frac{x-a}{8} (D_1 - d_1),$$

since

$$\sup_{t \in [a,x]} |K'(x,t)| = 1, \text{ as } K' \equiv 1.$$

Let $d_2 \leq f'(t) \leq D_2$ for $t \in (x,b]$. Then, in a similar fashion

$$\left| \mathfrak{T}_{(x,b]}(f', K) \right| \leq \frac{b-x}{8} (D_2 - d_2).$$

Now, using the triangle inequality readily produces

$$\begin{aligned} \left| \mathfrak{T}_{[a,b]}(f', K) \right| & \leq \frac{(x-a)}{8} (D_1 - d_1) + \frac{(b-x)}{8} (D_2 - d_2) \\ & \leq \frac{b-a}{8} (D - d). \end{aligned}$$

Thus, from (3.23) and (3.21),

$$\begin{aligned} & \left| \mathfrak{T}_{[a,b]}(f', K) \right| \\ = & \left| \frac{1}{b-a} \int_a^b K(x,t) f'(t) \, dt - (1-2\gamma) \left(x - \frac{a+b}{2} \right) S \right| \\ \leq & \frac{b-a}{8} (D-d). \end{aligned}$$

Using identity (3.20) readily produces (3.31), and the theorem is proved. ∎

REMARK 3.11. On each of the intervals $[a,x]$ and $(x,b]$

$$\sup_{t \in I} |k'(t)| = 1 = \|k'\|_2,$$

where $k(t) \equiv K(x,t)$. Thus, using Theorem 3.11 is superior to using either of the two results of Theorem 3.12.

Remark 3.12. The bound obtained by (3.31) is uniform. The bound given by (3.19) attains its sharpest bound when $x = \frac{a+b}{2}$ and $\gamma = \frac{1}{2}$, producing $\frac{(D-d)}{8}(b-a)^2$. Thus, the current bound is better for $b-a>1$ and for all x.

Remark 3.13. If Theorem 3.11 is used and $T(K, f')$ is considered, then a result similar to (3.19) would be obtained with $\frac{D-d}{4}$ being replaced by $\frac{\|f''\|_\infty}{8}$. This will not be investigated further since the second derivative is involved, thus placing it outside the scope of the work of this section.

The following result shall be termed as a **premature** *Grüss inequality* in that the proof of the Grüss inequality is not taken to its final conclusion but is stopped prematurely.

Theorem 3.14. *Let f, g be integrable functions defined on $[a,b]$, and let $d \le g(t) \le D$. Then*

$$|\mathfrak{T}(f,g)| \le \frac{D-d}{2}\left[\mathfrak{T}(f,f)\right]^{\frac{1}{2}}, \tag{3.32}$$

where

$$\mathfrak{T}(f,f) = \mathfrak{M}\left[f^2\right] - \left[\mathfrak{M}(f)\right]^2 \tag{3.33}$$

with

$$\mathfrak{M}(f) = \frac{1}{b-a}\int_a^b f(t)\,dt \tag{3.34}$$

and $\frac{1}{2}$ is the best possible constant.

Proof. The proof follows that of the Grüss inequality as given in [**44**, p. 296]. The identity

$$\mathfrak{T}(f,g) = \frac{1}{2(b-a)^2}\int_a^b\int_a^b (f(t)-f(\tau))(g(t)-g(\tau))\,dt d\tau \tag{3.35}$$

may easily be shown to be valid.
Now, applying the Cauchy-Schwartz-Buniakowsky integral inequality for double integrals, we have, on denoting the right hand side of (3.35) by $\mathfrak{T}_2(f,g)$,

$$\mathfrak{T}_2^2(f,g) \le \mathfrak{T}_2(f,f)\cdot\mathfrak{T}_2(g,g).$$

Therefore, from (3.35)

$$\mathfrak{T}^2(f,g) \le \mathfrak{T}(f,f)\cdot\mathfrak{T}(g,g). \tag{3.36}$$

Now,

$$\begin{aligned} &\mathfrak{T}(g,g) \\ =\;& (D-\mathfrak{M}(g))(\mathfrak{M}(g)-d) - \frac{1}{b-a}\int_a^b (D-g(t))(g(t)-d)\,dt \\ \le\;& (D-\mathfrak{M}(g))(\mathfrak{M}(g)-d) \end{aligned} \tag{3.37}$$

since $d \le g(t) \le D$.
In addition, using the elementary inequality for any real numbers p and q,

$$pq \le \left(\frac{p+q}{2}\right)^2,$$

we have, from (3.37),

$$\mathfrak{T}(g,g)\leq\left(\frac{D-d}{2}\right)^{2}. \tag{3.38}$$

Combining (3.38) and (3.36), the results $(3.32-3.34)$ are obtained and the theorem is proved. To prove the sharpness of (3.32) simply take $f(t)=g(t)=sgn\left(t-\frac{a+b}{2}\right)$. ∎

Remark 3.14. To prove (3.17), the bound $\left(\frac{C-c}{2}\right)^{2}$ for $\mathfrak{T}(f,f)$ would be obtained in a similar fashion to that of $\mathfrak{T}(g,g)$, and hence the term **premature**.

Theorem 3.15. *Let the conditions be as in Theorem 3.8. The following sharper inequality holds:*

$$\begin{aligned}&\left|\int_{a}^{b}f(t)\,dt-(b-a)\left\{(1-\gamma)f(x)+\gamma\left[\left(\frac{x-a}{b-a}\right)f(a)\right.\right.\right.\\&\left.\left.\left.+\left(\frac{b-x}{b-a}\right)f(b)\right]\right\}-(b-a)(1-2\gamma)\left(x-\frac{a+b}{2}\right)S\right|\\&\leq\frac{(D-d)}{2\sqrt{3}}(b-a)\left\{\left(\frac{b-a}{2}\right)^{2}\left[\frac{1}{4}+3\left(\gamma-\frac{1}{2}\right)^{2}\right]\right.\\&\left.+3\left(x-\frac{a+b}{2}\right)^{2}\left[\frac{1}{4}-\left(\gamma-\frac{1}{2}\right)^{2}\right]\right\}^{\frac{1}{2}}\end{aligned} \tag{3.39}$$

where $S=\frac{f(b)-f(a)}{b-a}$, *the secant slope.*

Proof. The proof of the current theorem follows along similar lines to that of Theorem 3.8 with the exception that a *premature* Grüss theorem (Theorem 3.11) is used.
From the identity (3.20) the function $K(x,t)$ is known and it is as given by (3.21). Thus, applying the *premature* Grüss theorem (Theorem 3.11) to the mappings $K(x,\cdot)$ and $f'(\cdot)$ we obtain

$$\begin{aligned}&\left|\frac{1}{b-a}\int_{a}^{b}K(x,t)f'(t)\,dt-\frac{1}{b-a}\int_{a}^{b}K(x,t)\,dt\cdot\frac{1}{b-a}\int_{a}^{b}f'(t)\,dt\right|\\&\leq\frac{D-d}{2}\cdot\left[\frac{1}{b-a}\int_{a}^{b}K^{2}(x,t)\,dt-\left(\frac{1}{b-a}\int_{a}^{b}K(x,t)\,dt\right)^{2}\right]^{\frac{1}{2}}.\end{aligned} \tag{3.40}$$

Now, from (3.21),

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b K^2(x,t)\,dt \\
=\ &\frac{1}{b-a}\left\{\int_a^x [t-(\gamma x+(1-\gamma)a)]^2\,dt+\int_x^b [t-(\gamma x+(1-\gamma)b)]^2\,dt\right\} \\
=\ &\frac{1}{b-a}\left\{\int_{-\gamma(x-a)}^{(1-\gamma)(x-a)} u^2du+\int_{-(1-\gamma)(b-x)}^{\gamma(b-x)} v^2dv\right\} \\
=\ &\frac{1}{3(b-a)}\left[\gamma^3+(1-\gamma)^3\right]\left[(x-a)^3+(b-x)^3\right].
\end{aligned}$$

The well-known identity

$$X^3+Y^3=(X+Y)\left[\left(\frac{X+Y}{2}\right)^2+3\left(\frac{X-Y}{2}\right)^2\right] \tag{3.41}$$

may be utilized to give

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b K^2(x,t)\,dt \\
=\ &\frac{1}{3}\left[\frac{1}{4}+3\left(\gamma-\frac{1}{2}\right)^2\right]\left[\left(\frac{b-a}{2}\right)^2+3\left(x-\frac{a+b}{2}\right)^2\right].
\end{aligned} \tag{3.42}$$

Thus, using the fact that

$$\frac{1}{b-a}\int_a^b f'(t)\,dt=\frac{f(b)-f(a)}{b-a}=S,$$

the secant slope together with (3.23) and (3.42) gives, from (3.40):

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b K(x,t)\,f'(t)\,dt-(1-2\gamma)\left(x-\frac{a+b}{2}\right)S\right| \\
\le\ &\frac{(D-d)}{2}\left\{\frac{1}{3}\left[\frac{1}{4}+3\left(\gamma-\frac{1}{2}\right)^2\right]\left[\left(\frac{b-a}{2}\right)^2+3\left(x-\frac{a+b}{2}\right)^2\right]\right. \\
&\left.-4\left(\gamma-\frac{1}{2}\right)^2\left(x-\frac{a+b}{2}\right)^2\right\}^{\frac{1}{2}} \\
=\ &\frac{(D-d)}{2\sqrt{3}}\left\{\left(\frac{b-a}{2}\right)^2\left[\frac{1}{4}+3\left(\gamma-\frac{1}{2}\right)^2\right]\right. \\
&\left.+3\left(x-\frac{a+b}{2}\right)^2\left[\frac{1}{4}-\left(\gamma-\frac{1}{2}\right)^2\right]\right\}^{\frac{1}{2}}.
\end{aligned} \tag{3.43}$$

The term in the braces is, of course, positive since $b>a,\gamma\in[0,1]\,,x\in[a,b]$ and

$$\frac{1}{4}-\left(\gamma-\frac{1}{2}\right)^2=\gamma(1-\gamma).$$

Utilizing the identity (3.20) in (3.43) produces the result (3.39). Thus, the theorem is proved. ∎

Corollary 3.16. Let the conditions be as in Theorem 3.8. Then the following inequality holds for all $x \in [a,b]$,

$$\left| \int_a^b f(t)\,dt - \frac{1}{2}\left[(b-a) f(x) + (x-a) f(a) + (b-x) f(b)\right] \right| \tag{3.44}$$
$$\leq \frac{(D-d)}{4\sqrt{3}}(b-a)\left[\left(\frac{b-a}{2}\right)^2 + 3\left(x - \frac{a+b}{2}\right)^2\right]^{\frac{1}{2}}.$$

Proof. The result (3.44) is readily obtained from (3.39) by substituting $\gamma = \frac{1}{2}$. ∎

Corollary 3.17. Let the conditions be as in Theorem 3.8. Then the following inequality holds for all $\gamma \in [0,1]$

$$\left| \int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f\left(\frac{a+b}{2}\right) + \frac{\gamma}{2}\left[f(a) + f(b)\right]\right\} \right| \tag{3.45}$$
$$\leq \frac{(D-d)}{4\sqrt{3}}(b-a)^2\left[\frac{1}{4} + 3\left(\gamma - \frac{1}{2}\right)^2\right]^{\frac{1}{2}}.$$

Proof. Taking $x = \frac{a+b}{2}$ in (3.39) together with a minor rearrangement gives (3.45). ∎

Remark 3.15. Result (3.39) is sharper than result (3.19) since the premature Grüss theorem is sharper than the Grüss theorem utilized to obtain (3.19).

Remark 3.16. Substituting $\gamma = 0$ into (3.39) gives an adjusted Ostrowski type rule, namely

$$\left| \int_a^b f(t)\,dt - (b-a)\left[f(x) - \left(x - \frac{a+b}{2}\right)S\right] \right| \leq \frac{(D-d)}{4\sqrt{3}}(b-a)^2.$$

This is a uniform bound which does not depend on the value of x. Thus, a mid-point rule would have the same bound as evaluating the function at any $x \in [a,b]$ together with an adjustment factor. Evaluation of the above result at $x = a$ or $x = b$ produces the standard trapezoidal rule.

Remark 3.17. Taking $\gamma = 1$ in (3.39) gives

$$\left| \int_a^b f(t)\,dt - (b-a)\left[\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b) + \left(x - \frac{a+b}{2}\right)S\right] \right|$$
$$\leq \frac{(D-d)}{4\sqrt{3}}(b-a)^2.$$

That is, using the fact that $S = \frac{f(b)-f(a)}{b-a}$, the trapezoidal rule,

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{2}\left[f(a) + f(b)\right] \right| \leq \frac{(D-d)}{4\sqrt{3}}(b-a)^2,$$

is recovered.

Remark 3.18. The sharpest bounds for (3.44) and (3.45) are at $x = \frac{a+b}{2}$ and $\gamma = \frac{1}{2}$ respectively. This result can be obtained directly from (3.39) by taking x and γ at the mid-point, giving the best quadrature rule of this type as

$$\left| \int_a^b f(t)\,dt - \frac{(b-a)}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\right| \leq \frac{(D-d)}{8\sqrt{3}}(b-a)^2. \tag{3.46}$$

If $\gamma = \frac{1}{3}$ is taken in (3.45), then a Simpson type rule is obtained, giving

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{3}\left[2f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\right| \leq \frac{(D-d)}{8\sqrt{3}}(b-a)^2 \cdot \frac{2}{\sqrt{3}}.$$

This bound is worse than the optimal rule (3.46) by a relative amount of $\left(\frac{2}{\sqrt{3}} - 1\right)$ which is approximately 15.5%. Computationally, the quadrature rule (3.46) is just as easy to apply as Simpson's rule since the only difference is the weights.

Remark 3.19. The bound in (3.46) is $\frac{1}{\sqrt{3}}$ times better than that in (3.28). That is, the bound in (3.28) is worse than that in (3.46) by a relative amount of $\left(1 - \frac{1}{\sqrt{3}}\right)$.

The *optimal* quadrature rule of this subsection will now be applied from (3.46) and it will be denoted by A_o.

Theorem 3.18. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping in $\mathring{I}$ (the interior of I) and let $a, b \in \mathring{I}$ with $b > a$. Let $f' \in L_1[a,b]$ and $d \leq f'(x) \leq D, \forall x \in [a,b]$. Further, let I_n be any partition of $[a,b]$ such that $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$. Then we have*

$$\int_a^b f(x)\,dx = A_o(f, I_n) + R_o(f, I_n)$$

where

$$A_o(f, I_n) = \frac{1}{2}\sum_{i=0}^{n-1} h_i f\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{1}{4}\sum_{i=0}^{n-1} h_i\left[f(x_i) + f(x_{i+1})\right],$$

and

$$\begin{aligned} |R_o(f, I_n)| &\leq \frac{(D-d)}{8\sqrt{3}}\sum_{i=0}^{n-1} h_i^2 \\ &\leq \frac{(D-d)}{8\sqrt{3}} n\nu^2(h) \end{aligned}$$

with $\nu(h) = \max_{i=0,\dots,n-1} h_i$.

Proof. Applying inequality (3.46) on the interval $[x_i, x_{i+1}]$ for $i = 0, 1, ..., n-1$ we have

$$\left| \int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{h_i}{4}\left[2f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_i) + f(x_{i+1})\right]\right| \leq \frac{(D-d)}{8\sqrt{3}} h_i^2$$

where $h_i = x_{i+1} - x_i$.

Summing over i for $i = 0$ to $n-1$ gives $A_o(f, I_n)$ and $|R_o(f, I_n)|$. ∎

Corollary 3.19. Let the conditions of Theorem 3.18 hold. In addition, let I_n be the equidistant partition of $[a,b]$, $I_n : x_i = a + \left(\frac{b-a}{n}\right) i, i = 0, 1, ..., n$ then

$$\left| \int_a^b f(x)\,dx - A_o(f, I_n) \right| \leq \frac{(D-d)}{8\sqrt{3}} \frac{(b-a)^2}{n}.$$

Proof. From Theorem 3.18 with $h_i = \frac{b-a}{n}$ for all i such that

$$|R_o(f, I_n)| \leq \frac{(D-d)}{8\sqrt{3}} \sum_{i=0}^{n-1} \left(\frac{b-a}{n}\right)^2 = \frac{(D-d)}{8\sqrt{3}} \cdot \frac{(b-a)^2}{n}$$

and hence the result is proved. ∎

Remark 3.20. If we wish to qpproximate the integral $\int_a^b f(x)\,dx$ using the quadrature rule $A_o(f, I_n)$ of Corollary 3.19 with an accuracy of $\varepsilon > 0$, then we need $n_\varepsilon \in \mathbb{N}$ points for the equispaced partition I_n where

$$n_\varepsilon \geq \left[\frac{(D-d)(b-a)^2}{8\sqrt{3}\,\varepsilon} \right] + 1$$

where $[x]$ denotes the integer part of x.

It should further be noted that the application of Corollary 3.19, in practice, is costly as it stands. The following corollary is more appropriate as it is more efficient.

Corollary 3.20. Let the conditions of Theorem 3.18 hold and let I_{2m} be the equidistant partition of $[a,b]$, $I_{2m} : x_i = a + ih,\ i = 0, 1, ..., 2m$ with $h = \frac{b-a}{2m}$. Then

$$\left| \int_a^b f(x)\,dx - \frac{h}{4}\left[f(x_0) + f(x_{2m})\right] - \frac{h}{2} \sum_{i=1}^{2m-1} f(x_i) \right| \leq \frac{D-d}{16\sqrt{3}} \frac{(b-a)^2}{m}.$$

Proof. From Theorem 3.18

$$A_o(f, I_{2m}) = \frac{h}{2} \sum_{i=0}^{m-1} f(x_{2i+1}) + \frac{h}{4} \sum_{i=0}^{m-1} \left[f(x_{2i}) + f\left(x_{2(i+1)}\right)\right]$$

since

$$\frac{x_{2(i+1)} + x_{2i}}{2} = a + h(2i+1) = x_{2i+1}.$$

Now,

$$\begin{aligned} \sum_{i=0}^{m-1} \left[f(x_{2i}) + f\left(x_{2(i+1)}\right)\right] &= f(x_0) + f(x_{2m}) + \sum_{i=1}^{m-1} f(x_{2i}) + \sum_{i=0}^{m-2} f\left(x_{2(i+1)}\right) \\ &= f(x_0) + f(x_{2m}) + 2 \sum_{i=1}^{m-1} f(x_{2i}). \end{aligned}$$

Thus

$$A_o(f, I_{2m}) = \frac{h}{4}\left[f(x_0) + f(x_{2m})\right] + \frac{h}{2} \sum_{i=1}^{2m-1} f(x_i),$$

where $h = \frac{b-a}{2m}$.
Further, from Theorem 3.18 with $h_i = \frac{b-a}{2m}$ for $i = 0, 1, ..., 2m-1$,

$$|R_o(f, I_{2m})| \le \frac{(D-d)}{8\sqrt{3}} \sum_{i=0}^{2m-1} \left(\frac{b-a}{2m}\right)^2 = \frac{D-d}{16\sqrt{3}} \frac{(b-a)^2}{m}.$$

The corollary is thus proved. ∎

3.2.4. A Generalized Ostrowski-Grüss Inequality Via a New Identity.

Traditionally, the Grüss inequality was effectively obtained by seeking a bound on $\mathfrak{T}^2(f,g)$ via a double integral identity and the Cauchy-Schwartz-Buniakowsky integral inequality to reduce the problem down to obtaining bounds for $\mathfrak{T}(f,f)$.

Recently, Dragomir and McAndrew [**30**] have obtained bounds on $\mathfrak{T}(f,g)$ as defined in (3.18), where f and g are integrable, by using the identity

$$\mathfrak{T}(f,g) = \frac{1}{b-a}\int_a^b [f(t) - \mathfrak{M}(f)][g(t) - \mathfrak{M}(g)]\,dt. \tag{3.47}$$

Hence

$$|\mathfrak{T}(f,g)| \le \frac{1}{b-a}\int_a^b |(f(t) - \mathfrak{M}(f))(g(t) - \mathfrak{M}(g))|\,dt. \tag{3.48}$$

In particular, they apply the inequality when one of the functions is known and so effectively (although not explicitly stated) use

$$|\mathfrak{T}(f,g)| \le \sup_{t\in[a,b]} |g(t) - \mathfrak{M}(g)| \cdot \frac{1}{b-a}\int_a^b |f(t) - \mathfrak{M}(f)|\,dt, \tag{3.49}$$

where f is known.

THEOREM 3.21. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping in $\mathring{I}$ and let $a, b \in \mathring{I}$ with $a < b$. Furthermore, let $f' \in L_1[a,b]$ and $d \le f'(x) \le D$, $\forall x \in [a,b]$. Then the following inequality holds:*

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma)f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right)f(a) + \left(\frac{b-x}{b-a}\right)f(b)\right]\right\}\right. \\ &\quad \left. + (b-a)(1-2\gamma)\left(x - \frac{a+b}{2}\right)S\right| \\ &\le \left[\frac{D-d}{2} + \left|S - \frac{d+D}{2}\right|\right] I(\gamma,x) \\ &\le (D-d)\, I(\gamma,x), \end{aligned} \tag{3.50}$$

where

$$I(\gamma,x) = \int_a^b \left|K(x,t) - (1-2\gamma)\left(x - \frac{a+b}{2}\right)\right| dt, \tag{3.51}$$

$$K(x,t) = \begin{cases} t - (\gamma x + (1-\gamma)a), & t \le x \\ t - (\gamma x + (1-\gamma)b), & t > x \end{cases}, \tag{3.52}$$

$$\gamma \in [0,1],$$

and

$$S=\frac{f(b)-f(a)}{b-a}.$$

Proof. Applying (3.49) on the mappings $K(x,\cdot)$ and $f'(\cdot)$ gives
(3.53)

$$(b-a)\left|\mathfrak{T}(K,f')\right|\leq\sup_{t\in[a,b]}\left|f'(t)-S\right|\cdot\int_a^b\left|K(x,t)-\frac{1}{b-a}\int_a^bK(x,u)\,du\right|dt.$$

Now,

$$\max\{D-S,S-d\}=\frac{D-d}{2}+\left|S-\frac{d+D}{2}\right|$$

and, from (3.23),

$$\frac{1}{b-a}\int_a^bK(x,u)\,du=(1-2\gamma)\left(x-\frac{a+b}{2}\right),$$

and so

$$\int_a^b\left|K(x,t)-\frac{1}{b-a}\int_a^bK(x,u)\,du\right|dt=I(\gamma,x).$$

Hence,

$$\left|\mathfrak{T}(K,f')\right|\leq\left[\frac{D-d}{2}+\left|S-\frac{d+D}{2}\right|\right]I(\gamma,x). \tag{3.54}$$

Furthermore, using identity (3.20), (3.54) and the fact that $S=\frac{1}{b-a}\int_a^bf'(t)\,dt$, (3.50) results and the first part of the theorem is proved. Taking $S=d$ or D provides the upper bound given by the second inequality. ∎

We now wish to determine a closed form expression for $I(\gamma,x)$ as given by (3.51) where $K(x,t)$ is from (3.52).

Now, $I(\gamma,x)$ may be written as

$$I(\gamma,x)=\int_a^x|t-\phi(x)|\,dt+\int_x^b|t-\psi(x)|\,dt \tag{3.55}$$

where

$$\phi(x)=(1-\gamma)x+\gamma b-\frac{b-a}{2},\;\;\psi(x)=(1-\gamma)x+\gamma a+\frac{b-a}{2}. \tag{3.56}$$

In (3.55), let $(b-a)u=t-\phi(x)$ and $(b-a)v=t-\psi(x)$, such that

$$I(\gamma,x)=(b-a)^2\left\{\int_{\frac{a-\phi(x)}{b-a}}^{\frac{x-\phi(x)}{b-a}}|u|\,du+\int_{\frac{x-\psi(x)}{b-a}}^{\frac{b-\psi(x)}{b-a}}|v|\,dv\right\}. \tag{3.57}$$

To simplify the problem it is worthwhile to parameterize the partitioning of the interval $[a,b]$. To that end let

$$x=\delta b+(1-\delta)a,\;\;\delta\in[0,1]$$

so that

$$\delta=\frac{x-a}{b-a},\;1-\delta=\frac{b-x}{b-a}\text{ and }x-\frac{a+b}{2}=(b-a)\left(\delta-\frac{1}{2}\right). \tag{3.58}$$

Now, from (3.56)

$$\begin{aligned}
\frac{a-\phi(x)}{b-a} &= \left(\frac{\frac{a+b}{2}-x}{b-a}\right)-\gamma\left(\frac{b-x}{b-a}\right) \\
&= \frac{1}{2}-\delta-\gamma(1-\delta) \\
&= (1-\gamma)\left[1-\frac{1}{2(1-\gamma)}-\delta\right],
\end{aligned} \tag{3.59}$$

$$\begin{aligned}
\frac{x-\phi(x)}{b-a} &= \frac{1}{2}-\left(\frac{b-x}{b-a}\right)\gamma \\
&= \frac{1}{2}-(1-\delta)\gamma \\
&= \gamma\left[\delta-\left(1-\frac{1}{2\gamma}\right)\right],
\end{aligned} \tag{3.60}$$

$$\begin{aligned}
\frac{x-\psi(x)}{b-a} &= \gamma\left(\frac{x-a}{b-a}\right)-\frac{1}{2} \\
&= \gamma\delta-\frac{1}{2} \\
&= \gamma\left[\delta-\frac{1}{2\gamma}\right]
\end{aligned} \tag{3.61}$$

and

$$\begin{aligned}
\frac{b-\psi(x)}{b-a} &= \frac{\left(\frac{a+b}{2}-x\right)}{b-a}+\gamma\left(\frac{x-a}{b-a}\right) \\
&= \frac{1}{2}-\delta+\gamma\delta \\
&= (1-\gamma)\left[\frac{1}{2(1-\gamma)}-\delta\right].
\end{aligned} \tag{3.62}$$

Thus, (3.57) becomes, for $x=\delta b+(1-\delta)a$, on using $(3.59)-(3.62)$,

$$I(\gamma,x)=J(\gamma,\delta)=(b-a)^2\left[J_1(\gamma,\delta)+J_2(\gamma,\delta)\right], \tag{3.63}$$

where

$$J_1(\gamma,\delta)=\int_{(1-\gamma)\left[1-\frac{1}{2(1-\gamma)}-\delta\right]}^{\gamma\left[\delta-\left(1-\frac{1}{2\gamma}\right)\right]}|u|\,du \tag{3.64}$$

and

$$J_2(\gamma,\delta)=\int_{\gamma\left[\delta-\frac{1}{2\gamma}\right]}^{(1-\gamma)\left[\frac{1}{2(1-\gamma)}-\delta\right]}|v|\,dv. \tag{3.65}$$

It should be noted that

$$J_2(\gamma,\delta)=J_1(1-\gamma,1-\delta)$$

and

$$J_1(\gamma,\delta)=J_2(1-\gamma,1-\delta), \tag{3.66}$$

so that only one of (3.64) or (3.65) need be evaluated explicitly and the other may be obtained in terms of it.

We shall consider $J_2(\gamma,\delta)$ in some detail. There are three possibilities to investigate. The limits in (3.65) are either both negative, one negative and one positive, or are both positive. We note that the top limit is always greater than the bottom since $\frac{1}{2}-(1-\gamma)\delta > \gamma\delta-\frac{1}{2}$.

Thus, over the three different regions we have:

$$J_2(\gamma,\delta) = \begin{cases} \int_{\gamma\left[\delta-\frac{1}{2\gamma}\right]}^{(1-\gamma)\left[\frac{1}{2(1-\gamma)}-\delta\right]} -v\,dv, & \frac{1}{2(1-\gamma)}<\delta<\frac{1}{2\gamma} \\ \int_{\gamma\left[\delta-\frac{1}{2\gamma}\right]}^{0} -v\,dv + \int_0^{(1-\gamma)\left[\frac{1}{2(1-\gamma)}-\delta\right]} v\,dv, & \delta<\frac{1}{2\gamma},\ \delta<\frac{1}{2(1-\gamma)} \\ \int_{\gamma\left[\delta-\frac{1}{2\gamma}\right]}^{(1-\gamma)\left[\frac{1}{2(1-\gamma)}-\delta\right]} v\,dv, & \frac{1}{2\gamma}<\delta<\frac{1}{2(1-\gamma)} \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left[\left(\gamma\delta-\frac{1}{2}\right)^2-\left(\frac{1}{2}-(1-\gamma)\delta\right)^2\right], & \frac{1}{2(1-\gamma)}<\delta<\frac{1}{2\gamma} \\ \frac{1}{2}\left[\left(\gamma\delta-\frac{1}{2}\right)^2+\left(\frac{1}{2}-(1-\gamma)\delta\right)^2\right], & \delta<\frac{1}{2\gamma},\ \delta<\frac{1}{2(1-\gamma)} \\ \frac{1}{2}\left[\left(\frac{1}{2}-(1-\gamma)\delta\right)^2-\left(\gamma\delta-\frac{1}{2}\right)^2\right], & \frac{1}{2\gamma}<\delta<\frac{1}{2(1-\gamma)}. \end{cases}$$

Now, using the result $\frac{X^2-Y^2}{2}=\frac{1}{2}(X-Y)(X+Y)$ and (3.5), the above expressions may be simplified to give

$$J_2(\gamma,\delta) = \begin{cases} -(1-\delta)\left(\gamma-\frac{1}{2}\right)\delta, & \frac{1}{2(1-\gamma)}<\delta<\frac{1}{2\gamma} \\ \left(\frac{1-\delta}{2}\right)^2+\left(\gamma-\frac{1}{2}\right)^2\delta^2, & \delta<\frac{1}{2\gamma},\ \delta<\frac{1}{2(1-\gamma)} \\ (1-\delta)\left(\gamma-\frac{1}{2}\right)\delta, & \frac{1}{2\gamma}<\delta<\frac{1}{2(1-\gamma)}. \end{cases} \tag{3.67}$$

Further, using (3.66), an expression for $J_1(\gamma,\delta)$ may be readily obtained from (3.67) to give

$$J_1(\gamma,\delta) = \begin{cases} -\delta\left(\frac{1}{2}-\gamma\right)(1-\delta), & 1-\frac{1}{2(1-\gamma)}<\delta<1-\frac{1}{2\gamma} \\ \left(\frac{\delta}{2}\right)^2+\left(\frac{1}{2}-\gamma\right)^2(1-\delta)^2, & \delta>1-\frac{1}{2\gamma},\ \delta>1-\frac{1}{2(1-\gamma)} \\ \delta\left(\frac{1}{2}-\gamma\right)(1-\delta), & 1-\frac{1}{2\gamma}<\delta<1-\frac{1}{2(1-\gamma)}. \end{cases} \tag{3.68}$$

For an explicit evaluation of $I(\gamma,x)$, (3.63) needs to be determined. This involves the addition of $J_1(\gamma,\delta)$ and $J_2(\gamma,\delta)$. This may best be accomplished by reference to a diagram. Figure 3.1 shows five regions on the $\gamma\delta-$plane defined by the curves $\delta=\frac{1}{2(1-\gamma)},\ \delta=\frac{1}{2\gamma},\ \delta=1-\frac{1}{2\gamma},\ \delta=1-\frac{1}{2(1-\gamma)}$, where $\gamma=0,\ \gamma=1,\ \delta=0,\ \delta=1$

define the outside boundary. The regions are defined as follows

$$
(3.69)\qquad \begin{cases}
A: & \delta > \frac{1}{2(1-\gamma)}, & \delta < \frac{1}{2\gamma}, & \delta > 1-\frac{1}{2\gamma}, & \delta > 1-\frac{1}{2(1-\gamma)}; \\
B: & \delta > \frac{1}{2\gamma}, & \delta < \frac{1}{2(1-\gamma)}, & \delta > 1-\frac{1}{2\gamma}, & \delta > 1-\frac{1}{2(1-\gamma)}; \\
C: & \delta < \frac{1}{2(1-\gamma)}, & \delta < \frac{1}{2\gamma}, & \delta < 1-\frac{1}{2\gamma}, & \delta > 1-\frac{1}{2(1-\gamma)}; \\
D: & \delta < \frac{1}{2\gamma}, & \delta < \frac{1}{2(1-\gamma)}, & \delta < 1-\frac{1}{2(1-\gamma)}, & \delta > 1-\frac{1}{2\gamma}; \\
\text{and} & & & & \\
E: & \delta < \frac{1}{2(1-\gamma)}, & \delta < \frac{1}{2\gamma}, & \delta < 1-\frac{1}{2\gamma}, & \delta > 1-\frac{1}{2(1-\gamma)}.
\end{cases} .
$$

It is important to note that the first two inequalities in each of the regions define the contributions from $J_2(\gamma,\delta)$ and the second two, that of $J_1(\gamma,\delta)$. Thus using $(3.67)-(3.69)$, (3.63) is given by

$$
(3.70)\qquad \frac{J(\gamma,\delta)}{(b-a)^2} = \begin{cases}
\left(\gamma-\frac{1}{2}\right)(1-\delta)\left[\left(\gamma-\frac{1}{2}\right)(1-\delta)-\delta\right]+\left(\frac{\delta}{2}\right)^2 & \text{on} \quad A \\
\left(\gamma-\frac{1}{2}\right)(1-\delta)\left[\left(\gamma-\frac{1}{2}\right)(1-\delta)+\delta\right]+\left(\frac{\delta}{2}\right)^2 & \text{on} \quad B \\
\left(\gamma-\frac{1}{2}\right)\delta\left[\left(\gamma-\frac{1}{2}\right)\delta+(1-\delta)\right]+\left(\frac{1-\delta}{2}\right)^2 & \text{on} \quad C \\
\left(\gamma-\frac{1}{2}\right)\delta\left[\left(\gamma-\frac{1}{2}\right)\delta-(1-\delta)\right]+\left(\frac{1-\delta}{2}\right)^2 & \text{on} \quad D \\
\left[\frac{1}{4}+\left(\gamma-\frac{1}{2}\right)^2\right]\left[\frac{1}{4}+\left(\delta-\frac{1}{2}\right)^2\right] & \text{on} \quad E
\end{cases} .
$$

Remark 3.21. We may now proceed in one of two ways. One approach is to transform to an expression involving x, thus giving $I(\gamma,x)$, and so (3.50) may be used. The second approach is to work in terms of δ so that Theorem 3.21 would be converted to an expression involving δ. We will take a modification of the first approach. Once a particular value γ is determined which dictates the type of rule δ is transformed in terms of x using the relation $J(\gamma,\delta)=I(\gamma,x)$, where $\delta=\frac{x-a}{b-a}$.

Remark 3.22. Taking different values of γ will produce bounds for various inequalities.

For $\gamma=0$, then from Figure 3.1 it may be seen that we are on the left boundary of region A and D and obtain, from (3.70), a uniform bound independent of δ,

$$
J(0,\delta)=\frac{(b-a)^2}{4}.
$$

Thus, from (3.50), a perturbed Ostrowski inequality is obtained on noting from (3.63) that $I(\gamma,x)=J(\gamma,\delta)$,

$$
(3.71)\qquad \left|\int_a^b f(t)\,dt-(b-a)\left[f(x)-\left(x-\frac{a+b}{2}\right)S\right]\right| \\
\leq \frac{(b-a)^2}{4}\left[\frac{D-d}{2}+\left|S-\frac{d+D}{2}\right|\right].
$$

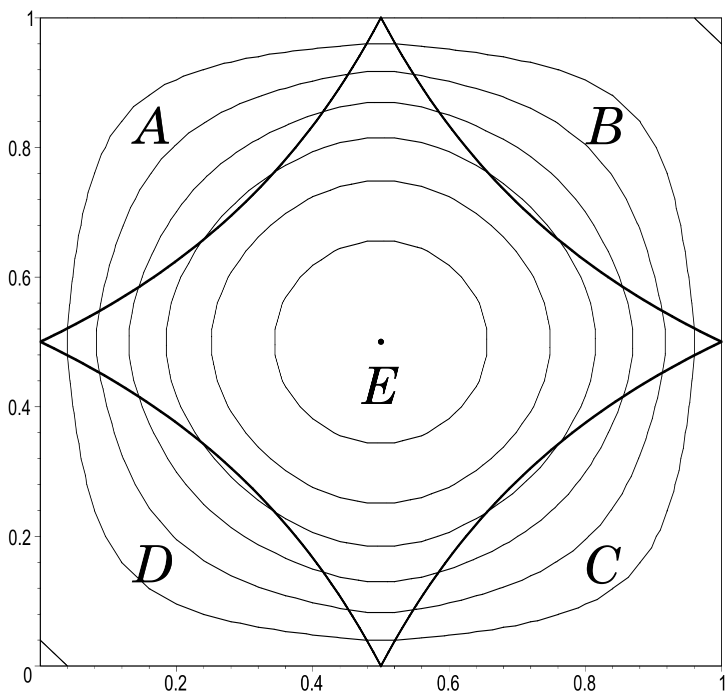

FIGURE 3.1. Diagram showing regions of validity for $\frac{J(\gamma,\delta)}{(b-a)^2}$ as given by (3.69) and (3.70), as well as its contours.

For $\gamma = 1$, it may be noticed from Figure 3.1 that we are now on the right boundary of B and D so that from (3.70), a uniform bound independent of δ is obtained viz.,

$$J(1,\delta) = \frac{(b-a)^2}{4}.$$

Thus, from (3.50), a perturbed generalized trapezoidal inequality is obtained, namely

$$\text{(3.72)} \quad \left| \int_a^b f(t)\,dt - (b-a)\left[\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b) - \left(x - \frac{a+b}{2}\right) S \right] \right|$$
$$\leq \frac{(b-a)^2}{4} \left[\frac{D-d}{2} + \left| S - \frac{d+D}{2} \right| \right].$$

Taking $x = \frac{a+b}{2}$ reproduces the result of Dragomir and McAndrew [**30**]. Again, it may be noticed that the above result is a uniform bound for any $x \in [a,b]$.

Corollary 3.22. Let the conditions of f be as in Theorem3.21. Then the following inequality holds for any $x \in [a,b]$:

$$\begin{aligned} & \left| \int_a^b f(t)\,dt - \frac{1}{2}\left[(b-a) f(x) + (x-a) f(a) + (b-x) f(b) \right] \right| \\ & \leq \frac{1}{4}\left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2 \right] \left[\frac{D-d}{2} + \left| S - \frac{d+D}{2} \right| \right], \end{aligned} \tag{3.73}$$

where $S = \frac{f(b)-f(a)}{b-a}$.

Proof. Letting $\gamma = \frac{1}{2}$ in (3.50) readily produces the result (3.73) from (3.70), on noting that $I\left(\frac{1}{2}, x\right) = J\left(\frac{1}{2}, \delta\right) = \frac{b-a}{4}\left[\frac{1}{4} + \left(\delta - \frac{1}{2}\right)^2\right]$ where $(b-a)\left(\delta - \frac{1}{2}\right) = x - \frac{a+b}{2}$. ∎

Corollary 3.23. Let the conditions of f be as in Theorem 3.21. Then the following inequality holds for any $\gamma \in [0,1]$:

$$\begin{aligned} & \left| \int_a^b f(t)\,dt - (b-a)\left\{ (1-\gamma) f\left(\frac{a+b}{2}\right) + \frac{\gamma}{2}\left[f(a) + f(b) \right] \right\} \right| \\ & \leq \frac{(b-a)^2}{4}\left[\frac{1}{4} + \left(\gamma - \frac{1}{2}\right)^2 \right] \left[\frac{D-d}{2} + \left| S - \frac{d+D}{2} \right| \right]. \end{aligned} \tag{3.74}$$

Proof. Letting $x = \frac{a+b}{2}$ in (3.50) produces the result (3.74) from (3.70) on noting $I\left(\gamma, \frac{a+b}{2}\right) = J\left(\gamma, \frac{1}{2}\right) = \frac{(b-a)^2}{4}\left[\frac{1}{4} + \left(\gamma - \frac{1}{2}\right)^2\right]$ in region E. ∎

Remark 3.23. Taking $x = \frac{a+b}{2}$ in (3.73) or $\gamma = \frac{1}{2}$ in (3.74) is equivalent to taking both these values in (3.50). This produces the sharpest bound in this class, giving

$$\begin{aligned} & \left| \int_a^b f(t)\,dt - \frac{b-a}{2}\left\{ f\left(\frac{a+b}{2}\right) + \frac{1}{2}\left[f(a) + f(b) \right] \right\} \right| \\ & \leq \frac{(b-a)^2}{16}\left[\frac{D-d}{2} + \left| S - \frac{d+D}{2} \right| \right]. \end{aligned} \tag{3.75}$$

For the bound, it is equivalent to taking the point $\left(\frac{1}{2}, \frac{1}{2}\right)$ in region E from (3.70) and Figure 3.1, thus giving $J\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{16}$. For a Simpson type rule, taking the point $\left(\frac{1}{3}, \frac{1}{2}\right)$ in region E from (3.70), and Figure 3.1 gives $J\left(\frac{1}{3}, \frac{1}{2}\right) = \frac{1}{16} + \frac{1}{144}$ which is a coarser bound than $J\left(\frac{1}{2}, \frac{1}{2}\right)$ at which the minimum occurs (the centre point in Figure 3.1).

Remark 3.24. It should be noted that the best bound possible with the **premature** Grüss is given by (3.46). This may be compared with the current bound (3.75). Now, (3.75) is computationally more expensive, but even the **worst** bound, $\frac{(b-a)^2}{16}(D-d)$ in (3.75) is **better** than that of (3.46).

Remark 3.25. A generalized Simpson type rule may be obtained by taking $\gamma = \frac{1}{3}$ for unprescribed x. Thus, from (3.70),

$$\frac{J\left(\frac{1}{3},\delta\right)}{(b-a)^2} = \begin{cases} \frac{1-\delta}{6}\cdot\frac{1+5\delta}{6} + \left(\frac{\delta}{2}\right)^2 & \frac{3}{4} < \delta < 1 \\ \frac{5}{18}\left[\frac{1}{4} + \left(\delta - \frac{1}{2}\right)^2\right] & \frac{1}{4} < \delta < \frac{3}{4} \\ \frac{\delta}{6}\cdot\frac{6-5\delta}{6} + \left(\frac{1-\delta}{2}\right)^2 & 0 < \delta < \frac{1}{4} \end{cases}, \tag{3.76}$$

and so from (3.50) :

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \frac{(b-a)}{3}\left[2f(x) + \left(\frac{x-a}{b-a}\right)f(a) + \left(\frac{b-x}{b-a}\right)f(b)\right]\right. \\ &\left. + \frac{b-a}{3}\left(x - \frac{a+b}{2}\right)S\right| \\ \leq\ &\left[\frac{D-d}{2} + \left|S - \frac{d+D}{2}\right|\right] I\left(\frac{1}{3},x\right), \end{aligned} \tag{3.77}$$

where

$$I\left(\frac{1}{3},x\right) = (b-a)^2 J\left(\frac{1}{3},\delta\right)$$

with

$$\delta = \frac{x-a}{b-a}.$$

That is, from (3.76),

$$I\left(\frac{1}{3},x\right) = \begin{cases} \frac{(b-x)}{6}\frac{[(b-a)+5(x-a)]}{6} + \left(\frac{x-a}{2}\right)^2, & \frac{3}{4} < \frac{x-a}{b-a} \\ \frac{5}{18}\left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right], & \frac{1}{4} < \frac{x-a}{b-a} < \frac{3}{4} \\ \frac{x-a}{6}\cdot\frac{6(b-a)-5(x-a)}{6} + \left(\frac{b-x}{2}\right)^2, & 0 < \frac{x-a}{b-a} < \frac{1}{4} \end{cases}.$$

Remark 3.26. It may have been noticed from Figure 3.1 or, for that matter, directly from (3.70). Replacing $1-\delta$ by δ in A and B would give the regions D and C respectively. Also, replacing $1-\gamma$ by γ in B and C would give the regions A and D respectively. Thus, it would have been possible to investigate the region $\frac{1}{2} \leq \gamma \leq 1$ and $\frac{1}{2} \leq \delta \leq 1$ since we may readily transform any point (γ', δ') in the $\gamma\delta-$plane to one in this region. Thus, only the regions B and E^* would need to be analyzed where for $\frac{1}{2} \leq \gamma, \delta \leq 1$,

$$B : \delta > \frac{1}{2\gamma},$$

and

$$E^* : \delta < \frac{1}{2\gamma}.$$

This approach was not followed since we were more interested in evaluation along lines perpendicular to the axes.

Remark 3.27. For practical implementation of the above in numerical integration it would be expensive to calculate the bounds as given. However, instead of $\frac{D-d}{2} + \left|S - \frac{d+D}{2}\right|$ being used, the coarser bound of $D-d$ may be more suitable.

The optimal quadrature rule of this subsection will now be applied from (3.75) and it will be denoted by A_0.

Theorem 3.24. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$ (the interior of I) and let $a, b \in \mathring{I}$ with $b > a$. Let $f' \in L_1[a,b]$ and $d \leq f'(x) \leq D$, $\forall x \in [a,b]$. In addition, let I_n be a partition of $[a,b]$ such that $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$. Then we have*

$$\int_a^b f(x)\,dx = A_0(f, I_n) + R_0(f, I_n),$$

where

$$A_0(f, I_n) = \frac{1}{2}\sum_{i=0}^{n-1} h_i f\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{1}{4}\sum_{i=0}^{n-1} h_i\left[f(x_i) + f(x_{i+1})\right]$$

and

$$\begin{aligned}|R_0(f, I_n)| &\leq \frac{D-d}{32}\sum_{i=0}^{n-1} h_i^2 + \frac{1}{16}\sum_{i=0}^{n-1} h_i\sigma_i \\ &\leq \frac{D-d}{16}\sum_{i=0}^{n-1} h_i^2 \leq \left(\frac{D-d}{16}\right) n\nu^2(h),\end{aligned}$$

where

$$\sigma_i = \left|f(x_{i+1}) - f(x_i) - h_i(d+D)\right|$$

and

$$\nu(h) = \max_{i=0,\ldots,n-1} h_i.$$

Proof. Applying inequality (3.75) on the interval $[x_i, x_{i+1}]$ for $i = 0, ..., n-1$ we have

$$\begin{aligned}&\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{h_i}{4}\left\{2f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_i) + f(x_{i+1})\right\}\right| \\ \leq\ & \frac{h_i^2}{16}\left[\frac{D-d}{2} + \left|S_i - \frac{d+D}{2}\right|\right],\end{aligned}$$

where

$$S_i = \frac{f(x_{i+1}) - f(x_i)}{h_i},\ h_i = x_{i+1} - x_i.$$

Summing over i for $i = 0, 1, ..., n-1$ gives $A_0(f, I_n)$ and the first bound for $|R_0(f, I_n)|$.

Now, consider the right hand side of the inequality above. Then

$$\frac{h_i^2}{16}\left[\frac{D-d}{2} + \left|S_i - \frac{d+D}{2}\right|\right] \leq \frac{h_i^2}{16}(D-d),$$

since

$$\left|S_i - \frac{d+D}{2}\right| \leq \frac{D-d}{2}.$$

Summing over i produces the last two upper bounds for the error. ∎

COROLLARY 3.25. Let the conditions of Theorem 3.24 hold. Also, let I_{2m} be the equidistant partition of $[a,b]$, $I_{2m} : x_i = a + ih,\ i = 0, 1, ..., 2m$ with $h = \frac{b-a}{2m}$. Then

$$\left| \int_a^b f(x)\,dx - \frac{h}{4}\left[f(x_0) + f(x_{2m})\right] - \frac{h}{2}\sum_{i=1}^{2m-1} f(x_i) \right| \leq \frac{D-d}{32} \cdot \frac{(b-a)^2}{m}.$$

PROOF. From Theorem 3.24 with $h_i = \frac{b-a}{2m}$ for all i and using the expression for $A_0(f, I_{2m})$ as given in Corollary 3.20 produces the desired result. ∎

REMARK 3.28. If we wish to approximate the integral $\int_a^b f(t)\,dt$ using the above quadrature rule in Corollary 3.25, with an accuracy of $\varepsilon > 0$, then we need $2m_\varepsilon \in \mathbb{N}$ points for the equispaced partition I_{2m},

$$m_\varepsilon \geq \left[\frac{D-d}{32} \frac{(b-a)^2}{\varepsilon} \right] + 1,$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$.

3.2.5. Inequalities for which the First Derivative Belongs to $L_1[a,b]$. In this subsection we discuss the situation in which $f' \in L_1[a,b]$ which is a linear space of all absolutely integrable functions on $[a,b]$. We use the usual norm notation $\|\cdot\|_1$, where, we recall, $\|g\|_1 := \int_a^b |g(s)|\,ds,\ g \in L_1[a,b]$.

THEOREM 3.26. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$ (the interior of I) and $a, b \in \mathring{I}$ are such that $b > a$. If $f' \in L_1[a,b]$, then the following inequality holds for all $x \in [a,b]$, $\alpha(x) \in [a,x]$ and $\beta(x) \in [x,b]$,*

$$\begin{aligned} (3.78)\quad & \left| \int_a^b f(t)\,dt - \left[(\beta(x) - \alpha(x)) f(x) + (\alpha(x) - a) f(a) + (b - \beta(x)) f(b)\right] \right| \\ \leq\ & \frac{\|f'\|_1}{2} \left\{ \frac{b-a}{2} + \left|\alpha(x) - \frac{a+x}{2}\right| + \left|\beta(x) - \frac{x+b}{2}\right| \right. \\ & \left. + \left| x - \frac{a+b}{2} + \left|\alpha(x) - \frac{a+x}{2}\right| - \left|\beta(x) - \frac{b+x}{2}\right| \right| \right\}. \end{aligned}$$

PROOF. Let $K(x,t)$ be as defined in (3.2). An integration by parts produces the identity as given by (3.3). Thus, from (3.3),

$$\begin{aligned} (3.79)\quad & \left| \int_a^b f(t)\,dt - \left[(\beta(x) - \alpha(x)) f(x) + (\alpha(x) - a) f(a) + (b - \beta(x)) f(b)\right] \right| \\ =\ & \left| \int_a^b K(x,t) f'(t)\,dt \right|. \end{aligned}$$

Now, using (3.2),

$$\begin{aligned}
(3.80)\qquad & \left|\int_a^b K(x,t) f'(t)\,dt\right| \\
&\leq \int_a^x |t-\alpha(x)|\,|f'(t)|\,dt + \int_x^b |t-\beta(x)|\,|f'(t)|\,dt \\
&= \int_a^{\alpha(x)} (\alpha(x)-t)\,|f'(t)|\,dt + \int_{\alpha(x)}^x (t-\alpha(x))\,|f'(t)|\,dt \\
&\quad + \int_x^{\beta(x)} (\beta(x)-t)\,|f'(t)|\,dt + \int_{\beta(x)}^b (t-\beta(x))\,|f'(t)|\,dt \\
&\leq (\alpha(x)-a)\int_a^{\alpha(x)} |f'(t)|\,dt + (x-\alpha(x))\int_{\alpha(x)}^x |f'(t)|\,dt \\
&\quad + (\beta(x)-x)\int_x^{\beta(x)} |f'(t)|\,dt + (b-\beta(x))\int_{\beta(x)}^b |f'(t)|\,dt \\
&\leq M(x)\,\|f'\|_1
\end{aligned}$$

where

$$M(x) = \max\{M_1(x), M_2(x)\}$$

with

$$M_1(x) = \max\{\alpha(x)-a, x-\alpha(x)\}$$

and

$$M_2(x) = \max\{\beta(x)-x, b-\beta(x)\}.$$

The well-known identity

$$\max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{X-Y}{2}\right|$$

may be used to give

$$M_1(x) = \frac{x-a}{2} + \left|\alpha(x) - \frac{a+x}{2}\right|$$

and

$$M_2(x) = \frac{b-x}{2} + \left|\beta(x) - \frac{x+b}{2}\right|.$$

Thus, using the identity again gives

$$\begin{aligned}
(3.81)\qquad M(x) &= \frac{M_1(x)+M_2(x)}{2} + \left|\frac{M_1(x)-M_2(x)}{2}\right| \\
&= \frac{1}{2}\left\{\frac{b-a}{2} + \left|\alpha(x) - \frac{a+x}{2}\right| + \left|\beta(x) - \frac{x+b}{2}\right|\right. \\
&\quad \left. + \left|x - \frac{a+b}{2} + \left|\alpha(x) - \frac{a+x}{2}\right| - \left|\beta(x) - \frac{b+x}{2}\right|\right|\right\}.
\end{aligned}$$

On substituting (3.81) into (3.80) and using (3.79), result (3.78) is produced and thus the theorem is proved. ∎

Corollary 3.27. Let f satisfy the conditions of Theorem 3.26. Then $\alpha(x) = \frac{a+x}{2}$ and $\beta(x) = \frac{b+x}{2}$ give the best bound for any $x \in [a,b]$ and so

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{2}\left[f(x) + \left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b)\right] \right| \leq \frac{\|f'\|_1}{2}\left[\frac{b-a}{2} + \left| x - \frac{a+b}{2}\right|\right]. \tag{3.82}$$

Proof. From (3.78) the minimal value each of the moduli can take is zero. Hence the result. ∎

Remark 3.29. An even tighter bound may be obtained from (3.117) if x is taken to be at the mid-point giving

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] \right| \leq \frac{b-a}{4}\|f'\|_1 . \tag{3.83}$$

This result corresponds to the average of a mid-point and trapezoidal quadrature rule for which $f' \in L_1[a,b]$.

Theorem 3.28. *Let f satisfy the conditions as stated in Theorem 3.36. Then the following inequality holds for any $\gamma \in [0,1]$ and $x \in [a,b]$:*

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f(x) \right.\right. \\ &\left.\left. + \gamma\left[\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b)\right]\right\}\right| \\ \leq\ & \|f'\|_1 \left[\frac{1}{2} + \left|\gamma - \frac{1}{2}\right|\right]\left[\frac{b-a}{2} + \left| x - \frac{a+b}{2}\right|\right]. \end{aligned} \tag{3.84}$$

Proof. Let $\alpha(x)$ and $\beta(x)$ be as in (3.11). Then,

$$\beta(x) - \alpha(x) = (1-\gamma)(b-a),$$

$$\alpha(x) - a = \gamma(x-a),$$

$$b - \beta(x) = \gamma(b-x),$$

$$\alpha(x) - \frac{a+x}{2} = \left(\gamma - \frac{1}{2}\right)(x-a)$$

and

$$\beta(x) - \frac{x+b}{2} = -\left(\gamma - \frac{1}{2}\right)(b-x).$$

Now,

$$\begin{aligned} & \frac{b-a}{2} + \left|\alpha(x) - \frac{a+x}{2}\right| + \left|\beta(x) - \frac{x+b}{2}\right| \\ =\ & (b-a)\left[\frac{1}{2} + \left|\gamma - \frac{1}{2}\right|\right] \end{aligned}$$

and

$$\begin{aligned}
& \left| x-\frac{a+b}{2}+\left| \alpha\left(x\right)-\frac{a+x}{2}\right| -\left| \beta\left(x\right)-\frac{x+b}{2}\right| \right| \\
= & \left| x-\frac{a+b}{2}+\left| \gamma-\frac{1}{2}\right| \left(x-a\right)-\left| \gamma-\frac{1}{2}\right| \left(b-x\right)\right| \\
= & \left| x-\frac{a+b}{2}+2\left| \gamma-\frac{1}{2}\right| \left(x-\frac{a+b}{2}\right)\right| \\
= & 2\left| x-\frac{a+b}{2}\right| \left[\frac{1}{2}+\left| \gamma-\frac{1}{2}\right| \right].
\end{aligned}$$

Substitution of the above results into (3.78) gives (3.84), thus proving the theorem. ∎

Remark 3.30. If $\gamma=\frac{1}{2}$ in (3.84) then $\alpha\left(x\right)=\frac{a+x}{2}$ and $\beta\left(x\right)=\frac{x+b}{2}$ and so result (3.82) is rightly recovered. The best quadrature rule of this type is given by (3.83) which is obtained by taking the optimal γ and x values at their respective mid-points of $\frac{1}{2}$ and $\frac{a+b}{2}$ in (3.84).

Remark 3.31. Taking $\gamma=0$ in (3.84) gives Ostrowski's inequality for $f'\in L_1\left[a,b\right]$ as obtained by Dragomir and Wang [**33**]. If $\gamma=1$ in (3.84), then a generalized trapezoidal rule is obtained for which the best bound occurs when $x=\frac{a+b}{2}$ giving the classical trapezoidal type rule for functions $f'\in L_1\left[a,b\right]$.

Corollary 3.29. Let f satisfy the conditions as stated in Theorem 3.26. Then the following inequality holds for $\gamma\in\left[0,1\right]$:

$$\begin{aligned}
(3.85)\qquad & \left| \int_a^b f\left(t\right)dt-\left(b-a\right)\left\{ \left(1-\gamma\right)f\left(\frac{a+b}{2}\right)+\frac{\gamma}{2}\left[f\left(a\right)+f\left(b\right)\right]\right\} \right| \\
\leq & \left\| f'\right\|_1\frac{b-a}{2}\left[\frac{1}{2}+\left| \gamma-\frac{1}{2}\right| \right].
\end{aligned}$$

Proof. Simply evaluating (3.119) at $x=\frac{a+b}{2}$ gives (3.85). ∎

Remark 3.32. Taking $\gamma=0$ and 1 into (3.120) gives the mid-point and trapezoidal type rules respectively.

Remark 3.33. Taking $\gamma=\frac{1}{2}$ in (3.85) gives the optimal quadrature rule shown in (3.83). Placing $\gamma=\frac{1}{3}$ gives a Simpson type rule with an error bound of $\left\| f'\right\|_1\cdot\frac{b-a}{3}$. Thus, a Simpson type rule is relatively worse (by $\frac{1}{3}$) when compared with the optimal rule (3.83). In addition, the optimal rule is just as easy to implement as the Simpson rule. All that is different are the weights.

The following results investigate the implementation of the above inequalities to numerical integration.

Theorem 3.30. *For any $a,b\in\mathbb{R}$ with $a<b$ let $f:\left(a,b\right)\rightarrow\mathbb{R}$ be a differentiable mapping. Let $f'\in L_1\left[a,b\right]$, then, for any partition $I_n:a=x_0<x_1<...<x_{n-1}<x_n=b$ of $\left[a,b\right]$ and any intermediate point vector $\xi=\left(\xi_0,\xi_1,...,\xi_{n-1}\right)$ such that*

$\xi_i \in [x_i, x_{i+1}]$ *for* $i = 0, 1, ..., n-1,$ *we have, for* $\gamma \in [0,1]$,

$$\begin{aligned}
(3.86) \qquad & \left| \int_a^b f(x)\,dx - A_c(f, I_n, \xi) \right| \\
& \leq \|f'\|_1 \left[\frac{1}{2} + \left| \gamma - \frac{1}{2} \right| \right] \max_{0 \leq i \leq n-1} \left\{ \frac{h_i}{2} + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right\} \\
& \leq \|f'\|_1 \left[\frac{1}{2} + \left| \gamma - \frac{1}{2} \right| \right] \nu(h)
\end{aligned}$$

where $h_i = x_{i+1} - x_i, \nu(h) = \max_{0 \leq i \leq n-1} h_i$ *and* $A_c(f, I_n, \xi)$ *is given by*

$$\begin{aligned}
A_c(f, I_n, \xi) &= (1-\gamma) \sum_{i=0}^{n-1} h_i f(\xi_i) \\
&\quad + \gamma \left[\sum_{i=0}^{n-1} (\xi_i - x_i) f(x_i) + \sum_{i=0}^{n-1} (x_{i+1} - \xi_i) f(x_{i+1}) \right].
\end{aligned}$$

Proof. Applying inequality (3.84) on the interval $[x_i, x_{i+1}]$ for $i = 0, 1, ..., n-1$ we have:

$$\begin{aligned}
& \left| \int_{x_i}^{x_{i+1}} f(x)\,dx \right. \\
& \left. - \{(1-\gamma) f(\xi_i) h_i + \gamma [(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})]\} \right| \\
\leq & \left[\frac{1}{2} + \left| \gamma - \frac{1}{2} \right| \right] \max_{0 \leq i \leq n-1} \left[\frac{h_i}{2} + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \int_{x_i}^{x_{i+1}} |f'(x)|\,dx.
\end{aligned}$$

Summing the above inequality, we have (3.86). Furthermore, observe that

$$\left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \leq \frac{h_i}{2}$$

for $i = 0, 1, ..., n-1$. Therefore,

$$\max_{0 \leq i \leq n-1} \left[\frac{h_i}{2} + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \leq \max_{0 \leq i \leq n-1} h_i = \nu(h)$$

and hence the theorem is proved. ∎

Remark 3.34. The coefficient of the γ term in $A_c(f, I_n, \xi)$ may be simplified to give

$$\begin{aligned}
& \sum_{i=0}^{n-1} (\xi_i - x_i) f(x_i) + \sum_{i=0}^{n-1} (x_{i+1} - \xi_i) f(x_{i+1}) \\
= & \sum_{i=0}^{n-1} \xi_i [f(x_i) - f(x_{i+1})] + \sum_{i=0}^{n-1} [x_{i+1} f(x_{i+1}) - x_i f(x_i)] \\
= & \sum_{i=1}^{n-1} \left(\xi_i - \xi_{i-1}\right) f(x_i) + \xi_0 f(a) - \xi_{n-1} f(b) + b f(b) - a f(a) \\
= & \sum_{i=1}^{n-1} \left(\xi_i - \xi_{i-1}\right) f(x_i) + (\xi_0 - a) f(a) + \left(b - \xi_{n-1}\right) f(b).
\end{aligned}$$

This version has the advantage in that the number of function evaluations is minimized. Thus,

$$A_c(f, I_n, \xi) = (1-\gamma)\sum_{i=0}^{n-1} h_i f(\xi_i) + \gamma\left\{\sum_{i=1}^{n-1}\left(\xi_i - \xi_{i-1}\right) f(x_i) + (\xi_0 - a) f(a) + \left(b - \xi_{n-1}\right) f(b)\right\} \tag{3.87}$$

Corollary 3.31. Let $a, b \in \mathbb{R}$ with $a < b$ and the mapping $f : (a, b) \to \mathbb{R}$ be differentiable. Further, let $f' \in L_1[a, b]$. Then, for any partition $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ of $[a, b]$ we have for any $0 \le \gamma \le 1$,

$$\int_a^b f(x)\, dx = A_1(f, I_n) + R_1(f, I_n) \tag{3.88}$$

where

$$A_1(f, I_n) = (1-\gamma)\sum_{i=0}^{n-1} h_i f\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{\gamma}{2}\sum_{i=0}^{n-1} h_i\left[f(x_i) + f(x_{i+1})\right],$$

and

$$|R_1(f, I_n)| \le \frac{\|f'\|_1}{2}\left[\frac{1}{2} + \left|\gamma - \frac{1}{2}\right|\right]\nu(h).$$

Proof. The proof is straightforward. We either start with Corollary 3.29 and follow the procedure of Theorem 3.30, or we can take the easier option of placing $\xi_i = \frac{x_i + x_{i+1}}{2}$ in Theorem 3.30 to immediately produce the result. ∎

Remark 3.35. The quadrature rule given by (3.88) is a composite mid-point and trapezoidal rule with γ determining the relative weighting of the two. The optimal rule is obtained when the composition is a straightforward average which is obtained by taking $\gamma = \frac{1}{2}$.

Corollary 3.32. Let the conditions of Corollary 3.31 hold, taking in particular $\gamma = \frac{1}{2}$ and the partition to be equidistant so that $I_{2m} : x_i = a + ih, i = 0, 1, ..., 2m$ with $h = \frac{b-a}{2m}$. Then

$$\int_a^b f(x)\, dx = A_o(f, I_{2m}) + R_o(f, I_{2m}) \tag{3.89}$$

where

$$A_o(f, I_{2m}) = \frac{h}{4}\left[f(a) + f(b) + 2\sum_{i=1}^{2m-1} f(x_i)\right]$$

and

$$|R_o(f, I_{2m})| \le \frac{\|f'\|_1}{8}\left(\frac{b-a}{m}\right).$$

Proof. From Corollary 3.31, let a subscript of o signify the optimal quadrature rule obtained when $\gamma = \frac{1}{2}$ and so

$$A_o(f, I_{2m}) = \frac{h}{2}\sum_{i=0}^{m-1} f\left(\frac{x_{2i} + x_{2(i+1)}}{2}\right) + \frac{h}{4}\sum_{i=0}^{m-1}\left[f(x_{2i}) + f\left(x_{2(i+1)}\right)\right],$$

where

$$\frac{x_{2i}+x_{2(i+1)}}{2}=a+h\left(2i+1\right)=x_{2i+1}.$$

Now

$$\begin{aligned}\sum_{i=0}^{m-1}\left[f\left(x_{2i}\right)+f\left(x_{2(i+1)}\right)\right] &= f\left(x_{0}\right)+f\left(x_{2m}\right)+\sum_{i=1}^{m-1}f\left(x_{2i}\right)+\sum_{i=0}^{m-2}f\left(x_{2(i+1)}\right)\\ &= f\left(x_{0}\right)+f\left(x_{2m}\right)+2\sum_{i=1}^{m-1}f\left(x_{2i}\right).\end{aligned}$$

Thus,

$$\begin{aligned}A_{o}\left(f,I_{2m}\right) &= \frac{h}{2}\left[\sum_{i=0}^{m-1}f\left(x_{2i+1}\right)+\sum_{i=1}^{m-1}f\left(x_{2i}\right)\right]+\frac{h}{4}\left[f\left(x_{0}\right)+f\left(x_{2m}\right)\right]\\ &= \frac{h}{4}\left[f\left(x_{0}\right)+f\left(x_{2m}\right)+2\sum_{i=1}^{2m-1}f\left(x_{i}\right)\right].\end{aligned}$$

Further, from Corollary 3.31 with $h_i=\frac{b-a}{2m}$ for $i=0,1,...,2m$ and $\gamma=\frac{1}{2}$ we obtain

$$\left|R_{o}\left(f,I_{2m}\right)\right|\leq\frac{\left\|f'\right\|_{1}}{8}\left(\frac{b-a}{m}\right),$$

and hence the corollary is proved. ∎

Remark 3.36. If we wish to approximate the integral $\int_a^b f\left(t\right)dt$ using the quadrature rule $A_o\left(f,I_{2m}\right)$ and (3.89) with an accuracy of $\varepsilon>0$, then we need $2m_\varepsilon\in\mathbb{N}$ points for the equispaced partition I_{2m} where

$$m_{\varepsilon}\geq\left[\frac{\left\|f'\right\|_{1}}{8}\frac{\left(b-a\right)}{\varepsilon}\right]+1,$$

where $[x]$ denotes the integer part of $x\in\mathbb{R}$.

3.2.6. Grüss-type Inequalities for Functions whose First Derivative Belongs to $L_1[a,b]$. The identity of Dragomir and McAndrew [**30**] as given by (3.47) will now be utilized to obtain further inequalities. Define an operator σ such that

$$\sigma\left(f\right)=f-\mathfrak{M}\left(f\right),\tag{3.90}$$

where $\mathfrak{M}\left(f\right)=\frac{1}{b-a}\int_a^b f\left(u\right)du$.

Then, (3.47) may be written as

$$\mathfrak{T}\left(f,g\right)=\mathfrak{T}\left(\sigma\left(f\right),\sigma\left(g\right)\right),\tag{3.91}$$

where

$$\mathfrak{T}\left(f,g\right)=\mathfrak{M}\left(fg\right)-\mathfrak{M}\left(f\right)\mathfrak{M}\left(g\right).$$

It may be noticed from (3.90) and (3.91) that $\mathfrak{M}\left(\sigma\left(f\right)\right)=\mathfrak{M}\left(\sigma\left(g\right)\right)=0$, so that (3.91) may be written in the alternative form:

$$\mathfrak{M}\left(fg\right)-\mathfrak{M}\left(f\right)\mathfrak{M}\left(g\right)=\mathfrak{M}\left(\sigma\left(f\right)\sigma\left(g\right)\right).\tag{3.92}$$

THEOREM 3.33. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$ (the interior of I) and $a, b \in \mathring{I}$ are such that $b > a$. If $f' \in L_1[a,b]$, then the following inequality holds for all $x \in [a,b]$ and $\gamma \in [0,1]$:*

$$\begin{aligned} (3.93)\quad & \left| \int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b)\right]\right\} \right. \\ & \left. + (b-a)(1-2\gamma)\left(x - \frac{a+b}{2}\right) S \right| \\ \leq\ & \|\sigma(f')\|_1 \,\theta(\gamma, x), \end{aligned}$$

where

$$\theta(\gamma, x) = \sup_{t\in[a,b]} \left| K(x,t) - (1-2\gamma)\left(x - \frac{a+b}{2}\right)\right|, \tag{3.94}$$

$K(x,t)$ is as given by (3.52) and $S = \mathfrak{M}(f')$ with $\mathfrak{M}(\cdot)$ and $\sigma(\cdot)$ as given by (3.90).

PROOF. Applying (3.91) or (3.92) on the mappings $K(x,\cdot)$ and $f'(\cdot)$ gives

$$\begin{aligned} \mathfrak{T}(K, f') &= \mathfrak{T}(\sigma(K), \sigma(f')) \\ &= \mathfrak{M}(\sigma(K)\,\sigma(f')). \end{aligned} \tag{3.95}$$

Thus,

$$(b-a)\left|\mathfrak{T}(K, f')\right| \leq \|\sigma(f')\|_1 \sup_{t\in[a,b]} |\sigma(K)|. \tag{3.96}$$

Now,

$$\theta(\gamma, x) = \sup_{t\in[a,b]} |\sigma(K)| = \sup_{t\in[a,b]} |K(x,t) - \mathfrak{M}(K)|, \tag{3.97}$$

where, from (3.23),

$$\mathfrak{M}(K) = \frac{1}{b-a}\int_a^b K(x,u)\,du = (1-2\gamma)\left(x - \frac{a+b}{2}\right).$$

Further, using identity (3.20), (3.96) and (3.97), the inequality (3.93) is derived and the theorem is hence proved. ∎

We now wish to obtain an explicit expression for $\theta(\gamma, x)$ as given by (3.94). Using (3.52) in (3.94) gives

$$\theta(\gamma, x) = \sup_{t\in[a,b]} |k(x,t)|, \tag{3.98}$$

where

$$k(x,t) = \begin{cases} t - \phi(x), & t\,[a,x] \\ t - \psi(x), & t \in (x,b] \end{cases} \tag{3.99}$$

and $\phi(x)$, $\psi(x)$ are as given by (3.56).

Therefore,

$$\theta(\gamma, x) = \max\{|a - \phi(x)|, |x - \phi(x)|, |x - \psi(x)|, |b - \psi(x)|\} \tag{3.100}$$

since the extremum points from (3.98) and (3.99) are obtained at the ends of the intervals as $k(x,t)$ is piecewise linear.

The representation (3.100) may be explicit enough, but it is possible to proceed further, as in Subsection 3.2.4, by making the transformation

$$x = \delta b + (1-\delta)\,a,\ \ \delta \in [0,1]\,. \tag{3.101}$$

Using (3.58) – (3.62) gives

$$\begin{aligned} \theta(\gamma,x) &= \Theta(\gamma,\delta) \\ &= (b-a)\max\left\{\left|\frac{1}{2}-\delta-\gamma(1-\delta)\right|,\left|\frac{1}{2}-\gamma(1-\delta)\right|,\right. \\ &\left.\left|\gamma\delta-\frac{1}{2}\right|,\left|\frac{1}{2}-\delta+\gamma(\delta)\right|\right\}. \end{aligned} \tag{3.102}$$

Now, the expressions in (3.102) can be either positive or negative depending on the region A, $B, ..., E$ as defined by (3.69) and depicted in Figure 3.1. The well known result

$$\max\{X,Y\} = \frac{X+Y}{2}+\frac{1}{2}|X-Y|$$

may be applied twice to give

$$\begin{aligned} &\max\{X,Y,Z,W\} \\ =\ &\frac{1}{2}\left[\frac{X+Y+Z+W}{2}+\left|\frac{X-Y}{2}\right|+\left|\frac{Z-W}{2}\right|\right] \\ &+\frac{1}{2}\left|\frac{(X+Y)-(Z+W)}{2}+\left|\frac{X-Y}{2}\right|-\left|\frac{Z-W}{2}\right|\right|. \end{aligned} \tag{3.103}$$

Taking heed of Remark 3.26 then, since we are now dealing with the maximum in (3.102), that is, a point, then it is possible to investigate the regions B and E_B for $\frac{1}{2} \leq \gamma,\ \delta \leq 1$ where $B : \delta > \frac{1}{2\gamma}$ and $E_B : \delta < \frac{1}{2\gamma}$.

In region B, from (3.102),

(3.104) $\Theta_B(\gamma,\delta)$

$$= (b-a)\max\left\{(1-\gamma)\,\delta+\left(\gamma-\frac{1}{2}\right),\gamma\delta-\left(\gamma-\frac{1}{2}\right),\gamma\delta-\frac{1}{2},\frac{1}{2}-(1-\gamma)\,\delta\right\}$$

and associating these elements in order with those of (3.103) gives

$$\begin{aligned} X+Y &= \delta,\ X-Y=(2\gamma-1)(1-\delta) \\ Z+W &= (2\gamma-1)\,\delta,\ Z-W=-(1-\delta)\,. \end{aligned}$$

Thus, after some simplification,

$$\frac{\Theta_B}{b-a}(\gamma,\delta) = \frac{\gamma}{2}+(1-\gamma)\left(\delta-\frac{1}{2}\right). \tag{3.105}$$

Similarly, in region E_B

$$\begin{aligned} &\Theta_{E_B}(\gamma,\delta) \\ =\ &(b-a)\max\left\{(1-\gamma)\,\delta+\gamma-\frac{1}{2},\gamma\delta-\left(\gamma-\frac{1}{2}\right),\frac{1}{2}-\gamma\delta,\frac{1}{2}-(1-\gamma)\,\delta\right\} \end{aligned}$$

and again associating these elements in order with those of (3.103) gives

$$\begin{aligned} X+Y &= \delta,\ X-Y=(2\gamma-1)(1-\delta) \\ Z+W &= 1-\delta,\ Z-W=(1-2\gamma)\,\delta. \end{aligned}$$

Therefore, after some simplification, we obtain

$$\frac{\Theta_{E_B}(\gamma,\delta)}{b-a}=\frac{\gamma-\delta}{2}+(1-\gamma)\left|\delta-\frac{1}{2}\right|. \tag{3.106}$$

Now

$$\Theta_A(\gamma,\delta)=\Theta_B(1-\gamma,\delta),\ \Theta_C(\gamma,\delta)=\Theta_B(\gamma,1-\delta) \tag{3.107}$$

and

$$\Theta_D(\gamma,\delta)=\Theta_B(1-\gamma,1-\delta).$$

Let $E=E_1\cup E_2$ where E_1 represents the region of E for which $\gamma\le\frac{1}{2}$ and E_2 represents the region of E for which $\gamma>\frac{1}{2}$. That is, $E_1=E_A\cup E_D$ and $E_2=E_B\cup E_C$ where E_k is the remainder of the square region containing region $k=A,B,C,D$.

Hence, using (3.105) – (3.107) in (3.102) gives

$$\frac{\Theta(\gamma,\delta)}{b-a}=\begin{cases}\frac{1-\gamma}{2}+\gamma\left(\delta-\frac{1}{2}\right) & \text{on} \quad A,\\ \frac{\gamma}{2}+(1-\gamma)\left(\delta-\frac{1}{2}\right) & \text{on} \quad B,\\ \frac{\gamma}{2}+(1-\gamma)\left(\frac{1}{2}-\delta\right) & \text{on} \quad C,\\ \frac{1-\gamma}{2}+\gamma\left(\frac{1}{2}-\delta\right) & \text{on} \quad D,\\ \frac{1-\gamma}{2}+\gamma\left|\delta-\frac{1}{2}\right| & \text{on} \quad E_1,\\ \frac{\gamma}{2}+(1-\gamma)\left|\delta-\frac{1}{2}\right| & \text{on} \quad E_2.\end{cases} \tag{3.108}$$

Remark 3.37. It may be noticed that (3.108) may be simplified to give

$$\frac{\Theta(\gamma,\delta)}{b-a}=\begin{cases}\frac{1-\gamma}{2}+\gamma\left|\delta-\frac{1}{2}\right|, & \gamma\le\frac{1}{2}\\ \frac{\gamma}{2}+(1-\gamma)\left|\delta-\frac{1}{2}\right|, & \gamma\ge\frac{1}{2}\end{cases} \tag{3.109}$$

and so, using the fact that $(b-a)\left(\gamma-\frac{1}{2}\right)=x-\frac{a+b}{2}$ in (3.109) gives,

$$\theta(\gamma,x)=\begin{cases}\frac{b-a}{2}\cdot(1-\gamma)+\gamma\left|x-\frac{a+b}{2}\right|, & \gamma\le\frac{1}{2}\\ \frac{b-a}{2}\cdot\gamma+(1-\gamma)\left|x-\frac{a+b}{2}\right|, & \gamma\ge\frac{1}{2}.\end{cases} \tag{3.110}$$

Thus the bound in Theorem 3.33, namely (3.94), is explicitly given by (3.110).

Remark 3.38. Taking different values if γ will produce bounds for various inequalities.

For $\gamma=0$ in (3.93) and (3.110), a perturbed Ostrowski type inequality is obtained with a uniform bound. Namely,

$$\left|\int_a^b f(t)\,dt-(b-a)\left[f(x)-\left(x-\frac{a+b}{2}\right)S\right]\right|\le\frac{b-a}{2}\left\|\sigma(f')\right\|_1.$$

For $\gamma = 1$ in (3.93) and (3.110) a generalized perturbed trapezoidal rule is obtained with the same uniform bound viz.

$$\left|\int_a^b f(t)\,dt - (b-a)\left[\left(\frac{x-a}{b-a}\right)f(a) + \left(\frac{b-x}{b-a}\right)f(b) - \left(x - \frac{a+b}{2}\right)S\right]\right|$$
$$\leq \frac{b-a}{2}\left\|\sigma(f')\right\|_1.$$

Corollary 3.34. Let the conditions on f be as in Theorem 3.33. Then the following inequality holds for any $x \in [a,b]$

$$\left|\int_a^b f(t)\,dt - \frac{1}{2}\left[(b-a)f(x) + (x-a)f(a) + (b-x)f(b)\right]\right| \tag{3.111}$$
$$\leq \frac{1}{2}\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]\left\|\sigma(f')\right\|_1.$$

Proof. Letting $\gamma = \frac{1}{2}$ in (3.93) and (3.110) readily produces the result. ∎

Corollary 3.35. Let the conditions on f be as in Theorem 3.33. Then the following inequality holds for any $\gamma \in [0,1]$

$$\left|\int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma)f\left(\frac{a+b}{2}\right) + \frac{\gamma}{2}\left[f(a)+f(b)\right]\right\}\right| \tag{3.112}$$
$$\leq \left\|\sigma(f')\right\|_1 \begin{cases} \frac{b-a}{2}(1-\gamma), & \gamma \leq \frac{1}{2} \\ \frac{b-a}{2}\gamma, & \gamma \geq \frac{1}{2}. \end{cases}.$$

Proof. Taking $x = \frac{a+b}{2}$ in (3.93) and (3.110) gives the result as stated. ∎

Remark 3.39. Taking $x = \frac{a+b}{2}$ in (3.111) or $\gamma = \frac{1}{2}$ in (3.112) is equivalent to taking both these in (3.93) and (3.110). This produces the sharpest bound in this case, giving

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}\left\{f\left(\frac{a+b}{2}\right) + \frac{1}{2}\left[f(a)+f(b)\right]\right\}\right|$$
$$\leq \frac{b-a}{4}\left\|\sigma(f')\right\|_1.$$

A Simpson type rule is obtained from (3.112) if $\gamma = \frac{1}{3}$ is taken, giving a bound consisting of $\frac{b-a}{3}$ rather than the $\frac{b-a}{4}$ obtained above.
A perturbed generalized Simpson type rule may be demonstrated directly from (3.93) and its bound from (3.110) by taking $\gamma = \frac{1}{3}$ to give

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{3}\left\{2f(a) + \left(\frac{x-a}{b-a}\right)f(a)\right.\right.$$
$$\left.\left. + \left(\frac{b-x}{b-a}\right)f(b) - \left(x - \frac{a+b}{2}\right)S\right\}\right|$$
$$\leq \frac{1}{3}\left[b-a + \left|x - \frac{a+b}{2}\right|\right]\left\|\sigma(f')\right\|_1.$$

Remark 3.40. The numerical implementation of the inequalities obtained in the current subsection will not be followed up since they follow those of Subsection 3.2.5. It may be noticed that Corollaries 3.34 and 3.35 are similar to Corollaries 3.27 and 3.29 with $\|\sigma(f')\|_1$ replacing $\|f'\|_1$. In a similar fashion, the implementation of the average of the mid-point and trapezoidal rules as developed in Corollary 3.32 may similarly be developed here with $\|\sigma(f')\|_1$ replacing $\|f'\|_1$. Each of these norms may be better for differing functions f.

3.2.7. Inequalities for which the First Derivative Belongs to $L_p[a,b]$.

Theorem 3.36. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) and $f'\in L_p(a,b)$ where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds for all $x\in[a,b]$, $\alpha(x)\in[a,x]$ and $\beta(x)\in[x,b]$,*

$$\begin{aligned}(3.113)\quad &\left|\int_a^b f(t)\,dt-[(\beta(x)-\alpha(x))f(x)+(\alpha(x)-a)f(a)+(b-\beta(x))f(b)]\right|\\ \leq\;& \Big[(\alpha(x)-a)^{q+1}+(x-\alpha(x))^{q+1}\\ &+(\beta(x)-x)^{q+1}+(b-\beta(x))^{q+1}\Big]^{\frac{1}{q}}(q+1)^{-\frac{1}{q}}\|f'\|_p\\ \leq\;& \left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}\|f'\|_p\\ \leq\;& (b-a)\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\|f'\|_p\end{aligned}$$

where $\|f'\|_p:=\left(\int_a^b|f'(t)|^p\,dt\right)^{\frac{1}{p}}$.

Proof. Let $K(x,t)$ be as defined by (3.2). Then an integration by parts of $\int_a^b K(x,t)f'(t)\,dt$ produces the identity as given by (3.3). Thus, from (3.3) :

$$\begin{aligned}(3.114)\quad &\left|\int_a^b f(t)\,dt-[(\beta(x)-\alpha(x))f(x)+(\alpha(x)-a)f(a)+(b-\beta(x))f(b)]\right|\\ =\;&\left|\int_a^b K(x,t)f'(t)\,dt\right|.\end{aligned}$$

Now, by Hölder's inequality we have:

$$(3.115)\qquad \left|\int_a^b K(x,t)f'(t)\,dt\right|\leq\left(\int_a^b|K(x,t)|^q\,dt\right)^{\frac{1}{q}}\|f'\|_p.$$

Now, from (3.2),

$$
\begin{aligned}
& \int_a^b |K(x,t)|^q \, dt \\
= & \int_a^{\alpha(x)} |t-\alpha(x)|^q \, dt + \int_{\alpha(x)}^x |t-\alpha(x)|^q \, dt \\
& + \int_x^{\beta(x)} |t-\beta(x)|^q \, dt + \int_{\beta(x)}^b |t-\beta(x)|^q \, dt \\
= & \int_a^{\alpha(x)} (\alpha(x)-t)^q \, dt + \int_{\alpha(x)}^x (t-\alpha(x))^q \, dt \\
& + \int_x^{\beta(x)} (\beta(x)-t)^q \, dt + \int_{\beta(x)}^b (t-\beta(x))^q \, dt.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& (q+1) \int_a^b |K(x,t)|^q \, dt \\
= & (\alpha(x)-a)^{q+1} + (x-\alpha(x))^{q+1} + (\beta(x)-x)^{q+1} + (b-\beta(x))^{q+1}. \qquad (3.116)
\end{aligned}
$$

Thus, using (3.114), (3.115) and (3.116) gives the first inequality in (3.113).
Now, using the inequality

$$(z-x)^n + (y-z)^n \leq (y-x)^n \qquad (3.117)$$

with $z \in [x,y]$ and $n > 1$, in (3.116) twice, taking $z = \alpha(x)$ and then $z = \beta(x)$, we have

$$
\begin{aligned}
(q+1) \int_a^b |K(x,t)|^q \, dt & \leq (x-a)^{q+1} + (b-x)^{q+1} \qquad (3.118) \\
& \leq (b-a)^{q+1} \qquad (3.119)
\end{aligned}
$$

upon using (3.117) once more.
Hence, by utilizing (3.114), (3.115) with (3.118) and (3.119) we obtain the second bound and the third inequality in (3.113). ∎

Corollary 3.37. Let the conditions on f of Theorem 3.36 hold. Then $\alpha(x) = \frac{a+x}{2}$ and $\beta(x) = \frac{b+x}{2}$ give the best bound for any $x \in [a,b]$ and so

$$
\begin{aligned}
& \left| \int_a^b f(t) \, dt - \frac{b-a}{2} \left[f(x) + \left(\frac{x-a}{b-a} \right) f(a) + \left(\frac{b-x}{b-a} \right) f(b) \right] \right| \qquad (3.120) \\
\leq & \frac{1}{2} \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \|f'\|_p.
\end{aligned}
$$

Proof. The inequality (3.117) produces an upper bound obtained with $z = x$ or y. For $z \in [x,y]$ and $n > 1$

$$(z-x)^n + (y-z)^n \geq 2 \left(\frac{y-x}{2} \right)^n \qquad (3.121)$$

where the lower bound is realized when $z = \frac{x+y}{2}$. Thus a tighter bound than the first inequality in (3.113) is obtained when, from (3.116) and using (3.121), $\alpha(x) = \frac{a+x}{2}$ and $\beta(x) = \frac{x+b}{2}$. Hence, (3.120) is obtained and the corollary is proved. ∎

Remark 3.41. The best inequality we may obtain from (3.120) results from utilizing (3.121) again, giving, with $x = \frac{a+b}{2}$,

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{2}\left\{ f\left(\frac{a+b}{2}\right) + \frac{1}{2}\left[f(a)+f(b)\right]\right\}\right| \leq \frac{(b-a)}{4}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \|f'\|_p . \tag{3.122}$$

Motivated by Theorem 3.36 and Corollary 3.37 we now take $\alpha(x)$ and $\beta(x)$ to be convex combinations of the end points so that they are as defined in (3.11). The following theorem then holds.

Theorem 3.38. *Let f satisfy the conditions as stated in Theorem 3.36. Then the following inequality holds for any $\gamma \in [0,1]$ and $x \in [a,b]$:*

$$\left| \int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b)\right]\right\}\right| \leq \left[\gamma^{q+1} + (1-\gamma)^{q+1}\right]^{\frac{1}{q}} \left[(x-a)^{q+1} + (b-x)^{q+1}\right]^{\frac{1}{q}} (q+1)^{-\frac{1}{q}} \|f'\|_p . \tag{3.123}$$

Proof. Using $\alpha(x)$ and $\beta(x)$ as defined in (3.11), then

$$\beta(x) - \alpha(x) = (1-\gamma)(b-a),$$
$$\alpha(x) - a = \gamma(x-a),$$
$$b - \beta(x) = \gamma(b-x),$$
$$x - \alpha(x) = (1-\gamma)(x-a)$$

and

$$\beta(x) - x = (1-\gamma)(b-x).$$

Substituting these results into the first inequality of Theorem 3.36 gives the stated result. ∎

Remark 3.42. Taking $\gamma = 0$ or 1 in (3.123) produces the coarser upper bound as obtained in the second inequality of Theorem 3.36. In addition, taking $x = a$ or b in (3.123) gives the even coarser bound as given by the third inequality of Theorem 3.36. Here we are utilizing (3.117) where the upper bound is attained at $z = x$ or y, the end points.

Corollary 3.39. Let f satisfy the conditions of Theorems 3.36 and 3.38. Then, the following inequality holds for any $\gamma \in [0,1]$:

$$\left| \int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f\left(\frac{a+b}{2}\right) + \frac{\gamma}{2}\left[f(a)+f(b)\right]\right\}\right| \leq \left[\gamma^{q+1} + (1-\gamma)^{q+1}\right]^{\frac{1}{q}} \left(\frac{b-a}{2}\right)\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \|f'\|_p . \tag{3.124}$$

PROOF. From identity (3.121) the minimum is obtained at the mid-point. Therefore, from (3.123),

$$\inf_{x\in[a,b]}\left[(x-a)^{q+1}+(b-x)^{q+1}\right]=2\left(\frac{b-a}{2}\right)^{q+1},$$

when $x=\frac{a+b}{2}$. Hence the result (3.124). ∎

REMARK 3.43. Corollary 3.37 is recaptured if (3.123) is evaluated at $\gamma=\frac{1}{2}$, the mid-point.

REMARK 3.44. Taking $\gamma=0$ in (3.123) produces an Ostrowski type inequality for which $f'\in L_p[a,b]$ as obtained by Dragomir and Wang [**34**]. Furthermore, taking $x=\frac{a+b}{2}$ gives a mid-point rule.

REMARK 3.45. Taking $\gamma=1$ in (3.123) produces a generalized trapezoidal rule for which the best bound occurs when $x=\frac{a+b}{2}$, giving the standard trapezoidal rule with a bound of $\frac{b-a}{2}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}$. This bound is twice as sharp as that obtained by Dragomir and Wang [**34**] since they used an Ostrowski type rule and obtained results at $x=a,x=b$ and utilized the triangle inequality.

REMARK 3.46. Taking $\gamma=\frac{1}{2}$ and $x=\frac{a+b}{2}$ in (3.123) gives the best inequality as given by (3.122). Taking $\gamma=\frac{1}{3}$ in (3.124) produces a Simpson type rule with a bound on the error of

$$\left(\frac{1}{3}\right)^{1+\frac{1}{q}}\left[1+2^{q+1}\right]^{\frac{1}{q}}\left(\frac{b-a}{2}\right)\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\|f'\|_p.$$

Taking $\gamma=\frac{1}{2}$ in (3.124) gives the optimal rule with a bound on the error of

$$\frac{1}{2}\left(\frac{b-a}{2}\right)\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\|f'\|_p.$$

Thus, there is a relative difference of

$$\left|\frac{2}{3}\left(\frac{1+2^{q+1}}{3}\right)^{\frac{1}{q}}-1\right|$$

between a Simpson type rule and the optimal. When $q=2$ for example, the relative difference is $\frac{2}{\sqrt{3}}-1\approx0.1547$. The greatest the relative difference can be is $\frac{1}{3}$.

The following particular instance for Euclidean norms is of interest.

COROLLARY 3.40. Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) and $f'\in L_2(a,b)$. Then the following inequality holds for all $x\in[a,b]$ and $\gamma\in[0,1]$:

$$\begin{aligned}(3.125)\quad&\left|\int_a^b f(t)\,dt-(b-a)\left\{(1-\gamma)f(x)+\gamma\left[\left(\frac{x-a}{b-a}\right)f(a)+\left(\frac{b-x}{b-a}\right)f(b)\right]\right\}\right|\\ \le\;&\left(\frac{b-a}{3}\right)^{\frac{1}{2}}\left[\frac{1}{4}+3\left(\gamma-\frac{1}{2}\right)^2\right]^{\frac{1}{2}}\\ &\times\left[\left(\frac{b-a}{2}\right)^2+3\left(x-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}}\|f'\|_2.\end{aligned}$$

Proof. Applying Theorem 3.38 for $p = q = 2$ immediately gives the left hand side of (3.125) with a bound of

$$\left[\gamma^3 + (1-\gamma)^3\right]^{\frac{1}{2}} \left[(x-a)^3 + (b-x)^3\right]^{\frac{1}{2}} \frac{\|f'\|_2}{\sqrt{3}}. \tag{3.126}$$

Now, using identity (3.41) we get:

$$\left[\gamma^3 + (1-\gamma)^3\right]^{\frac{1}{2}} = \left[\frac{1}{4} + 3\left(\gamma - \frac{1}{2}\right)^2\right]^{\frac{1}{2}}$$

and

$$\left[(x-a)^3 + (b-x)^3\right]^{\frac{1}{2}} = \sqrt{b-a}\left[\left(\frac{b-a}{2}\right)^2 + 3\left(x - \frac{a+b}{2}\right)^2\right]^{\frac{1}{2}},$$

which, upon substitution into (3.126) gives (3.125) . ∎

Remark 3.47. The numerical implementation of the inequalities in this subsection follows along similar lines as treated previously. The only difference is in the approximation of the bound and knowledge of $\|f'\|_p$, which need to be determined **a priori** in order that the coarseness of the partition may be calculated, given a particular error tolerance.

3.2.8. Grüss-type Inequalities for Functions whose First Derivative Belongs to $L_p[a,b]$.

From (3.91) and (3.92) we have

$$\mathfrak{T}(f,g) = \mathfrak{M}(\sigma(f)\,\sigma(g)), \tag{3.127}$$

where $\sigma(f)$ represents a shift of the function by its mean, $\mathfrak{M}$ as given in (3.90) .

Thus, using Hölder's inequality from (3.127) gives

$$(b-a)\,|\mathfrak{T}(f,g)| \le \|\sigma(f)\|_q \|\sigma(g)\|_p, \tag{3.128}$$

where

$$\|h\|_p := \left(\int_a^b |h(t)|^p\, dt\right)^{\frac{1}{p}}$$

and we say $h \in L_p[a,b]$.

Theorem 3.41. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) and $f' \in L_p(a,b)$ where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. The following inequality then, holds for all $x \in [a,b]$,*

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma)f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right)f(a) + \left(\frac{b-x}{b-a}\right)f(b)\right]\right\}\right. \\ &\quad \left.(b-a)(1-2\gamma)\left(x - \frac{a+b}{2}\right)S\right| \\ &\le \|\sigma(K(x,\cdot))\|_q \|\sigma(f')\|_p, \end{aligned} \tag{3.129}$$

where $\sigma(\cdot)$ is as given in (3.90) and $K(x,\cdot)$, S are as in (3.52) .

PROOF. Identifying $K(x,\cdot)$ with $f(\cdot)$ and $f'(\cdot)$ with $g(\cdot)$ in (3.128) gives

$$(b-a)\left|\mathfrak{T}\left(K(x,\cdot),f'\right)\right| \leq \left\|\sigma\left(K(x,\cdot)\right)\right\|_q \left\|\sigma\left(f'\right)\right\|_p . \tag{3.130}$$

Further, using identities (3.21), (3.24) and the fact that $S = \frac{1}{b-a}\int_a^b f'(t)\,dt$ in (3.130) readily produces (3.129) and hence the theorem is proved. ∎

We now wish to obtain a closed form expression for $\left\|\sigma\left(K(x,\cdot)\right)\right\|_q$.

Notice that

$$\sigma\left(K(x,t)\right) = K(x,t) - \frac{1}{b-a}\int_a^b K(x,u)\,du,$$

where $K(x,t)$ is as given by (3.52) and so using (3.24)

$$K_s(x,t) = \sigma\left(K(x,t)\right) = \begin{cases} t-\phi(x), & t\in[a,x] \\ t-\psi(x), & t\in(x,b] \end{cases}, \tag{3.131}$$

where ϕ and ψ are as presented in (3.56).

Thus,

$$\int_a^b \left|K_s(x,t)\right|^q dt = \int_a^x \left|t-\phi(x)\right|^q dt + \int_x^b \left|t-\psi(x)\right|^q dt. \tag{3.132}$$

Using (3.131) in (3.132) gives

$$\left\|\sigma\left(K(x,\cdot)\right)\right\|_q = \left\|K_s(x,\cdot)\right\|_q = \left(\int_a^b \left|K_s(x,t)\right|^q dt\right)^{\frac{1}{q}},$$

and upon making the respective substitutions $(b-a)u = t-\phi(x)$ and $(b-a)v = t-\psi(x)$ for the integrals on the right hand side,

$$\int_a^b \left|K_s(x,t)\right|^q dt = (b-a)^{q+1}\left\{\int_{\frac{a-\phi(x)}{b-a}}^{\frac{x-\phi(x)}{b-a}} |u|^q\,du + \int_{\frac{x-\phi(x)}{b-a}}^{\frac{b-\phi(x)}{b-a}} |v|^q\,dv\right\}. \tag{3.133}$$

Following the procedure of Subsection 3.2.4, we may make the substitution

$$x = \delta b + (1-\delta)a, \tag{3.134}$$

to give, from (3.133), and using (3.58) – (3.62),

$$I^{(q)}(\gamma,x) = (b-a)^{q+1} J^{(q)}(\gamma,\delta) = (b-a)^{q+1}\left[J_1^{(q)}(\gamma,\delta) + J_2^{(q)}(\gamma,\delta)\right], \tag{3.135}$$

where

$$I^{(q)}(\gamma,x) = \int_a^b \left|K_s(x,t)\right|^q dt, \tag{3.136}$$

$$\begin{aligned} J_2^{(q)}(\gamma,\delta) &= \int_w^{w+1-\delta} |v|^q\,dv, \\ w &= \gamma\delta - \frac{1}{2} \end{aligned} \tag{3.137}$$

and

$$J_1^{(q)}(\gamma,\delta) = J_2^{(q)}(1-\gamma,1-\delta). \tag{3.138}$$

Now, from (3.136), the limits may be both negative, one negative and one positive, or both positive. Therefore, with $w = \gamma\delta - \frac{1}{2}$,

$$(3.139)\qquad (q+1)\, J_2^{(q)}(\gamma,\delta) = \begin{cases} (-w)^{q+1} - (\delta - 1 - w)^{q+1}, & \frac{1}{2(1-\gamma)} < \delta < \frac{1}{2\gamma}; \\ (-w)^{q+1} + (w - 1 - \delta)^{q+1}, & \delta < \frac{1}{2\gamma},\ \delta < \frac{1}{2(1-\gamma)}; \\ (w + 1 - \delta)^{q+1} - w^{q+1}, & \frac{1}{2\gamma} < \delta < \frac{1}{2(1-\gamma)}. \end{cases}$$

Further, from (3.136) and (3.137)

$$(3.140)\qquad (q+1)\, J_1^{(q)}(\gamma,\delta) = \begin{cases} (-\tilde{w})^{q+1} - (-\delta - \tilde{w})^{q+1}, & 1 - \frac{1}{2(1-\gamma)} < \delta < 1 - \frac{1}{2\gamma}; \\ (-\tilde{w})^{q+1} + (\tilde{w} + \delta)^{q+1}, & \delta > 1 - \frac{1}{2\gamma},\ \delta > 1 - \frac{1}{2(1-\gamma)}; \\ (\tilde{w} + \delta)^{q+1} - \tilde{w}^{q+1}, & 1 - \frac{1}{2\gamma} < \delta < \frac{1}{2(1-\gamma)}, \end{cases}$$

where $\tilde{w} = (1-\gamma)(1-\delta) - \frac{1}{2}$.

We are now in a position to combine (3.139) and (3.140) by using the result (3.135) on each of the regions $A, ..., E$ as given by (3.69) and depicted in Figure 3.1. Thus,

$$(3.141)\qquad (q+1)\, J^{(q)}(\gamma,\delta) = \begin{cases} (-w)^{q+1} - (\delta - 1 - w)^{q+1} + (-\tilde{w})^{q+1} + (\tilde{w} + \delta)^{q+1} & \text{on} \quad A; \\ (w - 1 - \delta)^{q+1} - w^{q+1} + (-\tilde{w})^{q+1} + (\tilde{w} + \delta)^{q+1} & \text{on} \quad B; \\ (-w)^{q+1} + (w + 1 - \delta)^{q+1} + (-\tilde{w})^{q+1} - (-\delta - \tilde{w})^{q+1} & \text{on} \quad C; \\ (-w)^{q+1} + (w - 1 - \delta)^{q+1} + (\tilde{w} + \delta)^{q+1} - (-\tilde{w})^{q+1} & \text{on} \quad D; \\ (-w)^{q+1} + (w + 1 - \delta)^{q+1} + (-\tilde{w})^{q+1} + (\tilde{w} + \delta)^{q+1} & \text{on} \quad E. \end{cases}$$

Hence $\|\sigma(K(x,\cdot))\|_q$ is explicitly determined from (3.132), (3.135) and (3.140) on using (3.134) and the fact that $w = \gamma\delta - \frac{1}{2}$ and $\tilde{w} = (1-\gamma)(1-\delta) - \frac{1}{2}$.

Remark 3.48. It is instructive to take different values of γ to obtain various inequalities that lead to a variety of quadrature rules.
For $\gamma = 0$ then, from Figure 3.1 it may be seen that we are on the left boundary of regions A and D so that, from (3.141), a uniform bound independent of δ is obtained to give

$$(q+1)\, J^{(q)}(0,\delta) = \left(\frac{1}{2}\right)^q.$$

Using (3.135), (3.132) and (3.129) produces a perturbed Ostrowski inequality

$$\left|\int_a^b f(t)\,dt-(b-a)\left[f(x)-\left(x-\frac{a+b}{2}\right)S\right]\right| \leq \frac{b-a}{2}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\|\sigma(f')\|_p . \tag{3.142}$$

Evaluation at $x=\frac{a+b}{2}$ gives the mid-point rule.
In a similar fashion, for $\gamma=1$ the right hand boundary of B and C results to produce a perturbed generalized trapezoidal inequality

$$\left|\int_a^b f(t)\,dt-(b-a)\left[\left(\frac{x-a}{b-a}\right)f(a)+\left(\frac{b-x}{b-a}\right)f(b)-\left(x-\frac{a+b}{2}\right)S\right]\right| \leq \frac{b-a}{2}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\|\sigma(f')\|_p . \tag{3.143}$$

Taking $x=\frac{a+b}{2}$ produces the trapezoidal rule for which $\sigma(f')\in L_p[a,b]$.

Corollary 3.42. Let the conditions on f be as in Theorem 3.41. Then the following inequality holds for any $x\in[a,b]$.

$$\begin{aligned}\left|\int_a^b f(t)\,dt-\frac{1}{2}\left[(b-a)f(x)+(x-a)f(a)+(b-x)f(b)\right]\right| &\leq \frac{1}{2}\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}\|\sigma(f')\|_p \\ &\leq \frac{b-a}{2}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\|\sigma(f')\|_p .\end{aligned} \tag{3.144}$$

Proof. Placing $\gamma=\frac{1}{2}$ in (3.141) gives, after some simplification,

$$(q+1)\,J^{(q)}\left(\frac{1}{2},\delta\right)=\left(\frac{1}{2}\right)^q\left[\delta^{q+1}+(1-\delta)^{q+1}\right].$$

Hence, from (3.134) and (3.135),

$$I^{(q)}\left(\frac{1}{2},x\right)=\left(\frac{1}{2}\right)^q\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right].$$

From (3.132) and (3.136), taking the q^{th} root of the above expression gives (3.144) from (3.129) on taking $\gamma=\frac{1}{2}$.
The second inequality is obtained on using (3.117). ∎

Corollary 3.43. Let the conditions on f be as in Theorem 3.41. Then the following inequality holds for any $\gamma \in [0,1]$.

$$(3.145)\qquad \left| \int_a^b f(t)\,dt - (b-a)\left\{ (1-\gamma) f\left(\frac{a+b}{2}\right) + \frac{\gamma}{2}[f(a)+f(b)] \right\} \right|$$
$$\le \frac{b-a}{2}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \left[\gamma^{q+1} + (1-\gamma)^{q+1}\right]^{\frac{1}{q}} \|\sigma(f')\|_p$$
$$\le \frac{b-a}{2}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \|\sigma(f')\|_p .$$

Proof. Taking $\gamma = \frac{1}{2}$ in (3.141) places us in the region E and so

$$(q+1)\,J^{(q)}\left(\gamma, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^q \left[\gamma^{q+1} + (1-\gamma)^{q+1}\right]^{\frac{1}{q}}$$

and, from (3.134) and (3.135),

$$I^{(q)}\left(\gamma, \frac{a+b}{2}\right) = \frac{(b-a)^{q+1}}{q+1}\left(\frac{1}{2}\right)^q \left[\gamma^{q+1} + (1-\gamma)^{q+1}\right].$$

From (3.132) and (3.136), taking the q^{th} root of the above expression produces (3.145) from (3.129) on taking $x = \frac{a+b}{2}$. The second inequality is easily obtained on using (3.117). ∎

Remark 3.49. Taking $x = \frac{a+b}{2}$ in (3.144) or $\gamma = \frac{1}{2}$ in (3.145) is equivalent to taking both of these in (3.129) and using (3.132), (3.134) – (3.136) and

$$(q+1)\,J^{(q)}\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{4}\right)^q$$

from (3.141). This produces the sharpest inequality (see (3.131)) in the class. Namely,

$$(3.146)\qquad \left| \int_a^b f(t)\,dt - \frac{b-a}{2}\left\{ f\left(\frac{a+b}{2}\right) + \frac{1}{2}[f(a)+f(b)] \right\} \right|$$
$$\le \frac{b-a}{4}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \|\sigma(f')\|_p .$$

Result (3.146) may be compared with (3.122) and it may be seen that either one may be better, depending on the behaviour of f.
A Simpson type rule is obtained from (3.145) if $\gamma = \frac{1}{3}$, giving a bound consisting of $\frac{2}{3}\left[\frac{1+2^{q+1}}{3}\right]^{\frac{1}{q}}$ times the above bound for the average of a midpoint and trapezoidal rule. For the Euclidean norm, $q = 2$ and so Simpson's rule has a bound of $\frac{2}{\sqrt{3}}$ times that of the average of the midpoint and trapezoidal rule. A generalized Simpson type rule may be obtained by taking $\gamma = \frac{1}{3}$ in (3.129) and using (3.132), (3.134) – (3.136) and (3.141) in much the same way as Remark 3.39.

Remark 3.50. Corollaries 3.42 and 3.43 may be implemented in a straight forward fashion as carried out in earlier subsections. The bounds involve determining $\|\sigma(f')\|$ in advance to decide on the refinement of the grid that is required in order to achieve a particular accuracy.

3.2.9. Three Point Inequalities for Mappings of Bounded Variation, Lipschitzian or Monotonic. The following result involving a Riemann-Stieltjes integral is well known. It will be proved here for completeness.

LEMMA 3.44. Let $g,\, v : [a,b] \to \mathbb{R}$ be such that g is continuous on $[a,b]$ and v is of bounded variation on $[a,b]$. Then $\int_a^b g(t)\,dv(t)$ exists and is such that

$$\left| \int_a^b g(t)\,dv(t) \right| \leq \sup_{t\in[a,b]} |g(t)| \bigvee_a^b (v), \tag{3.147}$$

where $\bigvee_a^b (v)$ is the total variation of v on $[a,b]$.

PROOF. We only prove the inequality (3.147). Let $\Delta_n : a < x_0^{(n)} < x_1^{(n)} < \ldots < x_{n-1}^{(n)} < x_n^{(n)} = b$ be a sequence of partitions of $[a,b]$ such that $\nu(\Delta_n) \to 0$ as $n \to \infty$ where $\nu(\Delta_n) := \max_{i\in\{0,1,\ldots,n-1\}} h_i^{(n)}$ with $h_i^{(n)} = x_{i+1}^{(n)} - x_i^{(n)}$. Let $\xi_i^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)}\right]$ for $i - 0, 1, \ldots, n-1$ then

$$\begin{aligned} \left| \int_a^b g(t)\,dv(t) \right| &= \left| \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} g\left(\xi_i^{(n)}\right) \left[v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right) \right] \right| \\ &\leq \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left| g\left(\xi_i^{(n)}\right) \right| \left| v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right) \right| \\ &\leq \sup_{t\in[a,b]} |g(t)| \cdot \bigvee_a^b (v), \end{aligned}$$

where

$$\bigvee_a^b (v) = \sup_{\Delta_n} \sum_{i=0}^{n-1} \left| v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right) \right|, \tag{3.148}$$

and Δ_n is any partition of $[a,b]$. ∎

THEOREM 3.45. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then the following inequality holds*

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - \left[(\beta(x) - \alpha(x)) f(x) + (\alpha(x) - a) f(a) + (b - \beta(x)) f(b) \right] \right| \\ &\leq \frac{\bigvee_a^b (f)}{2} \left\{ \frac{b-a}{2} + \left| \alpha(x) - \frac{a+x}{2} \right| + \left| \beta(x) - \frac{x+b}{2} \right| \right. \\ &\quad \left. + \left| \frac{b+a}{2} - x + \left| \beta(x) - \frac{x+b}{2} \right| - \left| \alpha(x) - \frac{a+x}{2} \right| \right| \right\}, \end{aligned} \tag{3.149}$$

where $\alpha(x) \in [a,x]$ and $\beta(x) \in [x,b]$.

Proof. Let the Peano kernel be as defined in (3.2) , then consider the Riemann-Stieltjes integral $\int_a^b K(x,t)\,df(t)$ giving

$$\begin{aligned}\int_a^b K(x,t)\,df(t) &= \int_a^x (t-\alpha(x))\,df(t) + \int_x^b (t-\beta(x))\,df(t)\\ &= (t-\alpha(x))\,f(t)]_{t=a}^{x} - \int_a^x f(t)\,dt\\ &\quad + (t-\beta(x))\,f(t)]_{t=x}^{b} - \int_x^b f(t)\,dt.\end{aligned}$$

Simplifying and grouping some of the terms together produces the identity

$$\begin{aligned}&\int_a^b K(x,t)\,df(t)\\ &= [\beta(x)-\alpha(x)]\,f(x) + [\alpha(x)-a]\,f(a) + [b-\beta(x)]\,f(b) - \int_a^b f(t)\,dt.\end{aligned} \tag{3.150}$$

Now, to obtain the bounds from our identity (3.150) ,

$$\begin{aligned}\left|\int_a^b K(x,t)\,df(t)\right| &= \left|\int_a^x (t-\alpha(x))\,df(t) + \int_x^b (t-\beta(x))\,df(t)\right|\\ &\le \left|\int_a^x (t-\alpha(x))\,df(t)\right| + \left|\int_x^b (t-\beta(x))\,df(t)\right|.\end{aligned}$$

Further, using the result of Lemma 3.44, namely (3.147) on each of the intervals $[a,x]$ and $[x,b]$ by associating $g(t)$ with $t-\alpha(x)$ and $t-\beta(x)$ respectively gives, on taking $dv(t) \equiv df(t)$,

$$\begin{aligned}&\left|\int_a^b K(x,t)\,df(t)\right|\\ &\le \sup_{t\in[a,x]} |t-\alpha(x)| \bigvee_a^x (f) + \sup_{t\in[x,b]} |t-\beta(x)| \bigvee_x^b (f)\\ &\le m(x) \bigvee_a^b (f).\end{aligned} \tag{3.151}$$

Let

$$m_1(x) = \sup_{t\in[a,x]} |t-\alpha(x)| = \max\{\alpha(x)-a, x-\alpha(x)\}$$

and so

$$m_1(x) = \frac{x-a}{2} + \left|\alpha(x) - \frac{a+x}{2}\right|. \tag{3.152}$$

Similarly,

$$m_2(x) = \sup_{t\in[x,b]} |t-\beta(x)| = \max\{\beta(x)-b, b-\beta(x)\}$$

and so

$$m_2(x) = \frac{b-x}{2} + \left|\beta(x) - \frac{x+b}{2}\right|. \tag{3.153}$$

Thus, from (3.151)

$$\left|\int_a^b K(x,t)\,df(t)\right| \le m_1(x)\bigvee_a^x(f) + m_2(x)\bigvee_x^b(f) \le m(x)\bigvee_a^b(f), \tag{3.154}$$

where

$$m(x) = \max\{m_1(x), m_2(x)\}.$$

Therefore,

$$m(x) = \frac{m_1(x)+m_2(x)}{2} + \left|\frac{m_1(x)-m_2(x)}{2}\right|. \tag{3.155}$$

Substitution of $m_1(x)$ and $m_2(x)$ from (3.152) and (3.153) into (3.154) and using (3.150) gives inequality (3.149), and the theorem is proved. ∎

It should be noted that it is now possible to take various $\alpha(x)$ and $\beta(x)$ to obtain the previous results. For example, taking $\alpha(x) = \beta(x) = x$ produces the results of Dragomir, Cerone and Pearce [**25**] involving the generalized trapezoidal rule. Further, evaluation at $x = \frac{a+b}{2}$ of this result gives the classical trapezoidal type rule as obtained by Dragomir [**20**]. Taking $\alpha(x) = a$ and $\beta(x) = b$ reproduces the Ostrowski rule for functions of bounded variation [**19**]. In particular, we shall take $\alpha(x)$ and $\beta(x)$ to be convex combinations of the end points to obtain the following theorem.

THEOREM 3.46. *Let f satisfy the conditions of Theorem 3.45. Then the following inequality holds for any $\gamma \in [0,1]$ and $x \in [a,b]$:*

$$\begin{aligned} &\left|\int_a^b f(t)\,dt \right.\\ &\left. -(b-a)\left\{(1-\gamma)f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right)f(a) + \left(\frac{b-x}{b-a}\right)f(b)\right]\right\}\right| \\ \le\ & \left[\frac{1}{2} + \left|\gamma - \frac{1}{2}\right|\right]\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]\bigvee_a^b(f). \end{aligned} \tag{3.156}$$

PROOF. Let $\alpha(x)$, $\beta(x)$ be as in (3.11). Then, from Theorem 3.45

$$\beta(x) - \alpha(x) = (1-\gamma)(b-a),$$

$$\alpha(x) - a = \gamma(x-a),$$

and

$$b - \beta(x) = \gamma(b-x).$$

Further, from (3.152), (3.153), and (3.155),

$$m_1(x) = \left(\frac{1}{2} + \left|\gamma - \frac{1}{2}\right|\right)(x-a)$$

$$m_2(x) = \left(\frac{1}{2} + \left|\gamma - \frac{1}{2}\right|\right)(b-x)$$

and

$$m(x)=\left[\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|\right]\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right].$$

Substitution of $m(x)$ into (3.154), and using the identity (3.150) gives the result (3.156) and the theorem is thus proved. ∎

Remark 3.51. We note that coarser uniform bounds may be obtained on using the fact that

$$\max_{X\in[A,B]}\left|X-\frac{A+B}{2}\right|=\frac{B-A}{2}.$$

Remark 3.52. A tighter bound is obtained when

$$\min_{X\in[A,B]}\left|X-\frac{A+B}{2}\right|.$$

The minimum of 0 is attained when $X=\frac{A+B}{2}$.

Corollary 3.47. Let f satisfy the conditions of Theorems 3.45 and 3.46. Then the following inequality holds for any $\gamma\in[0,1]$:

$$\left|\int_a^b f(t)\,dt-(b-a)\left\{(1-\gamma)f\left(\frac{a+b}{2}\right)+\frac{\gamma}{2}\left[f(a)+f(b)\right]\right\}\right| \leq \frac{b-a}{2}\left[\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|\right]\bigvee_a^b(f). \tag{3.157}$$

Proof. From (3.154) and Remark 3.52,

$$\min_{x\in[a,b]}\left|x-\frac{a+b}{2}\right|=0$$

when $x=\frac{a+b}{2}$. Hence the result (3.155) is obtained. ∎

Corollary 3.48. Let f satisfy the conditions of Theorems 3.45 and 3.46. Then the following inequality holds for all $x\in[a,b]$,

$$\left|\int_a^b f(t)\,dt-\frac{1}{2}\left\{(b-a)f(x)+(x-a)f(a)+(b-x)f(b)\right\}\right| \leq \frac{1}{2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f). \tag{3.158}$$

Proof. From (3.156) and Remark 3.52,

$$\min_{\gamma\in[0,1]}\left|\gamma-\frac{1}{2}\right|=0,$$

when $\gamma=\frac{1}{2}$. Thus, placing $\gamma=\frac{1}{2}$ in (3.156) gives the result (3.158). ∎

Remark 3.53. The sharpest bounds on (3.157) and (3.158) occur when $\gamma=\frac{1}{2}$ and $x=\frac{a+b}{2}$ as may be concluded from the result of Remark 3.52. The same result can be obtained directly from (3.156), giving the quadrature rule with the sharpest bound as

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\right| \le \frac{b-a}{4}\bigvee_a^b (f). \tag{3.159}$$

It should be noted, as previously on similar occasions, that taking $\gamma = \frac{1}{3}$ in (3.157) produces a Simpson-type rule as obtained by Dragomir [**18**] which is worse than the optimal 3 point Lobatto rule as given by (3.159).
Namely,

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{3}\left[2f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\right| \le \frac{b-a}{3}\bigvee_a^b (f), \tag{3.160}$$

which is worse than (3.159) by an absolute amount of $\frac{1}{12}$.
Computationally speaking, the Simpson type rule (3.160) is just as efficient and easy to apply as the optimal rule (3.159) which is the average of a trapezoidal and mid- point rule.

Remark 3.54. Taking various values of $\gamma \in [0,1]$ and/or $x \in [a,b]$ will reproduce earlier results.
Taking $\gamma = 0$ in (3.156) will reproduce the results of Dragomir [**19**], giving an Ostrowski integral inequality for mappings of bounded variation. In addition, taking $x = \frac{a+b}{2}$ would give a mid-point rule.
If $\gamma = 1$ is substituted into (3.157), then the results of Dragomir, Cerone and Pearce [**25**] are recovered, giving a generalized trapezoidal inequality for any $x \in [a,b]$. Furthermore, fixing x at its optimal value of $\frac{a+b}{2}$ would give the results of Dragomir [**19**].
Putting $\gamma = \frac{1}{3}$ in (3.156), we obtain

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \frac{1}{3}\left[2(b-a)f(x) + (x-a)f(a) + (b-x)f(b)\right]\right| \\ \le\ & \frac{2}{3}\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]\bigvee_a^b (f), \end{aligned}$$

which is a generalized Simpson type rule. Also, taking $x = \frac{a+b}{2}$ gives the result (3.160), which was also produced in Dragomir [**18**].

Remark 3.55. If f is absolutely continuous on $[a,b]$ and $f' \in L_1[a,b]$, then f is of bounded variation. By applying the theorems of this subsection, the theorems of Subsection 3.2.5 are hence recovered. Thus, replacing $\bigvee_a^b (f)$ by $\|f'\|_1$ in this subsection reproduces the results of Subsection 3.2.5 and vice versa, provided that the conditions on f are satisfied.
Further, the perturbed three point quadrature rules obtained in Subsection 3.2.6 through Grüss-type inequalities may be obtained here, where, instead of $\|\sigma(f')\|_1$ in identity (3.93), we would have $\bigvee_a^b (\sigma(f)) \equiv \bigvee_a^b (f)$. Thus the following theorem would result.

Theorem 3.49. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then the following inequality holds*

$$\begin{aligned} (3.161) \qquad & \left| \int_a^b f(t)\,dt \right. \\ & - (b-a)\left\{(1-\gamma) f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b)\right]\right\} \\ & \left. + (b-a)(1-2\gamma)\left(x - \frac{a+b}{2}\right) S \right| \\ \leq \; & \theta(\gamma, x) \bigvee_a^b (f), \end{aligned}$$

where $\theta(\gamma, x)$ is as given by (3.110) *and S is the secant slope.*

Proof. Identifying $\sigma(K(x,\cdot))$ with $g(\cdot)$ and $\sigma(f(\cdot))$ with $v(\cdot)$ in (3.147) gives, upon noting that $\bigvee_a^b (\sigma(f)) \equiv \bigvee_a^b (f)$, and $d\sigma(f) = df$ since the σ operator merely shifts a function by its mean,

$$(3.162) \qquad \left| \int_a^b \sigma(K(x,t))\,df(t) \right| \leq \sup_{t\in[a,b]} |\sigma(K(x,t))| \bigvee_a^b (f),$$

where

$$\sigma(K(x,t)) = \begin{cases} t - \phi(x), & t \in [a,x], \\ t - \psi(x), & t \in (x,b] \end{cases}$$

and $\phi(x)$, $\psi(x)$ are as given in (3.56).

The Riemann-Stieltjes integral may be integrated by parts to produce an identity similar to (3.150) with $\alpha(x)$ and $\beta(x)$ replaced by $\phi(x)$ and $\psi(x)$ respectively, since we are now considering $\sigma(K(x,\cdot))$ rather than $K(x,\cdot)$. In other words,

$$\begin{aligned} (3.163) \qquad & \int_a^b \sigma(K(x,t))\,df(t) \\ = \; & [\psi(x) - \phi(x)] f(x) + [\phi(x) - a] f(a) + [b - \psi(x)] f(b) - \int_a^b f(t)\,dt, \end{aligned}$$

which becomes, on using (3.56),

$$\begin{aligned} (3.164) \qquad & \int_a^b \sigma(K(x,t))\,df(t) \\ = \; & (b-a)(1-\gamma) f(x) + \left[\gamma(b-x) + \left(x - \frac{a+b}{2}\right)\right] f(a) \\ & + \left[\gamma(x-a) - \left(x - \frac{a+b}{2}\right)\right] f(b) - \int_a^b f(t)\,dt. \end{aligned}$$

A straightforward reorganization of (3.163), on noting that $S = \frac{f(b)-f(a)}{b-a}$ and using (3.162) readily produces (3.161) where $\theta(\gamma, x) = \sup_{t\in[a,b]} |\sigma(K(x,t))|$, and hence the theorem is proved. ∎

REMARK 3.56. Identity (3.163) (or indeed (3.150)) demonstrates that a three point quadrature rule may be obtained for arbitrary functions $\phi(\cdot)$ and $\psi(\cdot)$ (or $\alpha(\cdot)$ and $\beta(\cdot)$).

DEFINITION 1. The mapping $u : [a,b] \to \mathbb{R}$ is said to be $L-$Lipschitzian on $[a,b]$ if

$$|u(x) - u(y)| \leq L|x-y| \text{ for all } x, y \in [a,b]. \tag{3.165}$$

The following lemma holds.

LEMMA 3.50. Let $g, v : [a,b] \to \mathbb{R}$ be such that g is Riemann integrable on $[a,b]$ and v is L-Lipschitzian on $[a,b]$. Then

$$\left| \int_a^b g(t)\, dv(t) \right| \leq L \int_a^b |g(t)|\, dt. \tag{3.166}$$

PROOF. Let $\Delta_n : a < x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ be a sequence of partitions of $[a,b]$ such that $\nu(\Delta_n) \to 0$ as $n \to \infty$, where $\nu(\Delta_n) := \max_{i \in \{0,1,\dots,n-1\}} h_i^{(n)}$ with $h_i^{(n)} = x_{i+1}^{(n)} - x_i^{(n)}$. Further, let $\xi_i^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)}\right]$ such that

$$\begin{aligned}
\left| \int_a^b g(t)\, dv(t) \right| &= \left| \lim_{\nu(\Delta_n) \to 0} \sum_{i=0}^{n-1} g\left(\xi_i^{(n)}\right) \left[v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right) \right] \right| \\
&\leq \lim_{\nu(\Delta_n) \to 0} \sum_{i=0}^{n-1} \left| g\left(\xi_i^{(n)}\right) \right| \left| \frac{v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)}{x_{i+1}^{(n)} - x_i^{(n)}} \right| \left(x_{i+1}^{(n)} - x_i^{(n)} \right) \\
&\leq L \lim_{\nu(\Delta_n) \to 0} \sum_{i=0}^{n-1} \left| g\left(\xi_i^{(n)}\right) \right| \left(x_{i+1}^{(n)} - x_i^{(n)} \right) \\
&= L \int_a^b |g(t)|\, dt.
\end{aligned}$$

Hence the lemma is proved. ∎

THEOREM 3.51. *Let $f : [a,b] \to \mathbb{R}$ be L-Lipschitzian on $[a,b]$. Then the following inequality holds*

$$\begin{aligned}
&\left| \int_a^b f(t)\, dt - \left[(\beta(x) - \alpha(x)) f(x) + (\alpha(x) - a) f(a) + (b - \beta(x)) f(b) \right] \right| \qquad (3.167) \\
&\leq L \left\{ \frac{1}{2} \left[\left(\frac{b-a}{2} \right)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \right. \\
&\quad \left. + \left(\alpha(x) - \frac{a+x}{2} \right)^2 + \left(\beta(x) - \frac{x+b}{2} \right)^2 \right\},
\end{aligned}$$

where $\alpha(x)$, $\beta(x)$ are as given by (3.11).

Proof. The proof is straightforward from identity (3.148), giving, after taking the absolute value

$$\left|\int_a^b K(x,t)\,df(t)\right| \le L\int_a^b |K(x,t)|\,dt,$$

since f is L-Lipschitzian, and thus Lemma 3.50 may be used. Now, $K(x,t)$ is as given by (3.2) and $\int_a^b |K(x,t)|\,dt = Q(x)$ given in (3.6). Using identity (3.5) simplifies the expression for $Q(x)$ in (3.6) to give result (3.167). Hence the theorem is proved. ∎

Remark 3.57. If f is L-Lipschitzian on $[a,b]$, then the bound on the Riemann-Stieltjes integral

$$\left|\int_a^b K(x,t)\,df(t)\right| \le L\int_a^b |K(x,t)|\,dt. \tag{3.168}$$

On the other hand, if f is differentiable on $[a,b]$ and $f' \in L_\infty[a,b]$, then the Riemann integral

$$\left|\int_a^b K(x,t)\,f'(t)\,dt\right| \le \|f'\|_\infty \int_a^b |K(x,t)|\,dt.$$

Thus, all the theorems and bounds obtained in Subsection 3.2.1 are applicable here if f is L-Lipschitzian. The $\|f'\|_\infty$ norm is simply replaced by L.

Theorem 3.52. *Let $f:[a,b]\to\mathbb{R}$ be $L-$Lipschitzian on $[a,b]$. Then*

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - [(\psi(x)-\phi(x))f(x)+(\phi(x)-a)f(a)+(b-\psi(x))f(b)]\right| \\ &\le L\|\sigma(K(x,\cdot))\|_1, \end{aligned} \tag{3.169}$$

where

$$\sigma(K(x,t)) = \begin{cases} t-\phi(x), & t\in[a,x], \\ t-\psi(x), & t\in(x,b]. \end{cases}$$

Proof. Consider

$$\int_a^b \sigma(K(x,t))\,df(t),$$

giving identity (3.163) and using (3.166) readily produces result (3.169). Thus, the theorem is proved. ∎

Remark 3.58. If $\phi(x)$ and $\psi(x)$ are taken as in (3.56), then

$$\begin{aligned} &\left|\int_a^b f(t)\,dt-(b-a)\left\{(1-\gamma)f(x)+\gamma\left[\left(\frac{x-a}{b-a}\right)f(a)\right.\right.\right. \\ &\left.\left.\left.+\left(\frac{b-x}{b-a}\right)f(b)\right]\right\}+(b-a)(1-2\gamma)\left(x-\frac{a+b}{2}\right)S\right| \\ &\le L\cdot I(\gamma,x), \end{aligned} \tag{3.170}$$

where

$$I(\gamma,x) = \|\sigma(K(x,\cdot))\|_1 = J\left(\gamma,\frac{x-a}{b-a}\right)$$

which is given in (3.70) and $S=\frac{f(b)-f(a)}{b-a}$.

LEMMA 3.53. Let $g,v\in[a,b]\to\mathbb{R}$ be such that g is Riemann integrable on $[a,b]$ and v is monotonic nondecreasing on $[a,b]$. Then

$$\left|\int_a^b g(t)\,dv(t)\right|\le\int_a^b|g(t)|\,dv(t)\,. \tag{3.171}$$

PROOF. Let $\Delta_n : a<x_0^{(n)}<x_1^{(n)}<...<x_{n-1}^{(n)}<x_n^{(n)}=b$ be a sequence of partitions of $[a,b]$ such that $\nu(\Delta_n)\to 0$ as $n\to\infty$ where $\nu(\Delta_n):=\max_{i\in\{0,1,\ldots,n-1\}}h_i^{(n)}$ with $h_i^{(n)}=x_{i+1}^{(n)}-x_i^{(n)}$. Now, let $\xi_i^{(n)}\in\left[x_i^{(n)},x_{i+1}^{(n)}\right]$ so that

$$\begin{aligned}\left|\int_a^b g(t)\,dv(t)\right| &= \left|\lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1}g\left(\xi_i^{(n)}\right)\left[v\left(x_{i+1}^{(n)}\right)-v\left(x_i^{(n)}\right)\right]\right|\\ &\le \lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1}\left|g\left(\xi_i^{(n)}\right)\right|\left|v\left(x_{i+1}^{(n)}\right)-v\left(x_i^{(n)}\right)\right|.\end{aligned}$$

Now, using the fact that v is monotonic nondecreasing, then

$$\left|\int_a^b g(t)\,dv(t)\right|\le\lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1}\left|g\left(\xi_i^{(n)}\right)\right|\left(v\left(x_{i+1}^{(n)}\right)-v\left(x_i^{(n)}\right)\right).$$

Making use of the definition of the integral, the lemma is proved. ∎

THEOREM 3.54. *Let $f:[a,b]\to\mathbb{R}$ be a monotonic nondecreasing mapping on $[a,b]$. Then the following inequality holds:*

$$\begin{aligned}&\left|\int_a^b f(t)\,dt-[(\beta(x)-\alpha(x))f(x)+(\alpha(x)-a)f(a)+(b-\beta(x))f(b)]\right| \quad (3.172)\\ &\le [2x-(\alpha(x)+\beta(x))]f(x)+(b-\beta(x))f(b)-(\alpha(x)-a)f(a)\\ &\quad -\int_a^b sgn(K(x,t))f(t)\,dt\end{aligned}$$

$$\begin{aligned}&\le [2x-(\alpha(x)+\beta(x))]f(x)+(b-\beta(x))f(b)-(\alpha(x)-a)f(a) \quad (3.173)\\ &\quad +[2\alpha(x)-(a+x)]f(\alpha(x))+[2\beta(x)-(x+b)]f(\beta(x)),\end{aligned}$$

where $K(x,t)$ is as given by (3.2) *and $\alpha(x)\in[a,x]$, $\beta(x)\in[x,b]$.*

PROOF. Let the Peano kernel $K(x,t)$ be as given by (3.2). Then the identity (3.150) is obtained upon integration by parts of the Riemann-Stieltjes integral $\int_a^b K(x,t)\,df(t)$. Now, since f is monotonic nondecreasing, then, using Lemma 3.53 and identifying $f(\cdot)$ with $v(\cdot)$ and $K(x,\cdot)$ with $g(\cdot)$ in (3.171) gives:

$$\left|\int_a^b K(x,t)\,df(t)\right|\le\int_a^b|K(x,t)|\,df(t)\,. \tag{3.174}$$

Now, from (3.2),

$$\begin{aligned}\left|\int_a^b K(x,t)\,df(t)\right| &\leq \int_a^x |t-\alpha(x)|\,df(t)+\int_x^b |t-\beta(x)|\,df(t)\\ &= \int_a^{\alpha(x)} (\alpha(x)-t)\,df(t)+\int_{\alpha(x)}^x (t-\alpha(x))\,df(t)\\ &\quad+\int_x^{\beta(x)} (\beta(x)-t)\,df(t)+\int_{\beta(x)}^b (t-\beta(x))\,df(t).\end{aligned}$$

Integration by parts and some grouping of terms gives

$$\begin{aligned}&\left|\int_a^b K(x,t)\,df(t)\right| \qquad (3.175)\\ &\leq [2x-(\alpha(x)+\beta(x))]\,f(x)+(b-\beta(x))\,f(b)-(\alpha(x)-a)\,f(a)\\ &\quad+\left\{\int_a^{\alpha(x)} f(t)\,dt-\int_{\alpha(x)}^x f(t)\,dt+\int_x^{\beta(x)} f(t)\,dt-\int_{\beta(x)}^b f(t)\,dt\right\}.\end{aligned}$$

Using the fact that the terms in the braces are equal to $-\int_a^b sgn\,(K(x,t))\,f(t)\,dt$, where

$$sgn_{x\in[a,b]}u(x)=\begin{cases}1 \text{ if } & u>0\\ -1 \text{ if } & u<0\end{cases},$$

then, from identity (3.150) and equation (3.174) we obtain (3.172) and thus, the first part of the theorem is proved.

Now for the second part. Since $f(\cdot)$ is monotonic nondecreasing,

$$\int_a^{\alpha(x)} f(t)\,dt \leq (\alpha(x)-a)\,f(\alpha(x)),$$

$$\int_{\alpha(x)}^x f(t)\,dt \geq (x-\alpha(x))\,f(\alpha(x)),$$

$$\int_x^{\beta(x)} f(t)\,dt \leq (\beta(x)-b)\,f(\beta(x)),$$

and

$$\int_{\beta(x)}^b f(t)\,dt \geq (b-\beta(x))\,f(\beta(x)).$$

Thus, from (3.175)

$$\begin{aligned}&\int_a^b |K(x,t)|\,df(t) \qquad (3.176)\\ &\leq [2x-(\alpha(x)+\beta(x))]\,f(x)+(b-\beta(x))\,f(b)-(\alpha(x)-a)\,f(a)\\ &\quad+[2\alpha(x)-(a+x)]\,f(\alpha(x))+[2\beta(x)-(x+b)]\,f(\beta(x)).\end{aligned}$$

Substituting (3.176) into (3.174) and utilizing (3.150), we obtain (3.173). Thus, the second part of the theorem is proved. ∎

REMARK 3.59. It is now possible to recapture previous results for monotonic nondecreasing mappings. If $\alpha(x) = \beta(x) = x$, then the result obtained by Dragomir, Cerone and Pearce [**25**] for the generalized trapezoidal rule is recovered. Moreover, taking $x = \frac{a+b}{2}$ gives the trapezoidal-type rule. Taking $\alpha(x) = a$ and $\beta(x) = b$ reproduces an Ostrowski inequality for monotonic nondecreasing mappings which was developed by Dragomir [**21**]. Taking $\alpha(x) = \frac{a+x}{2}$ and $\beta(x) = \frac{x+b}{2}$ gives from (3.173) :

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - \frac{1}{2}\left[(b-a) f(x) + (x-a) f(a) + (b-x) f(b)\right] \right| \\ \leq\; & \left(x - \frac{a+b}{2} \right) f(x) + (b-x) f(b) - (x-a) f(a), \end{aligned}$$

which is the Lobatto type rule obtained by Milovanović and Pečarić [**45**, p. 470]. However, here it is for monotonic functions.

As discussed earlier, it is much more enlightening to take $\alpha(x)$ and $\beta(x)$ to be a linear combination of the end points, and so the following theorem can be shown to hold.

THEOREM 3.55. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing mapping on $[a,b]$. Then the following inequality exists*

$$\begin{aligned} & \Bigg| \int_a^b f(t)\,dt \\ & \quad - (b-a) \left\{ (1-\gamma) f(x) + \gamma \left[\left(\frac{x-a}{b-a} \right) f(a) + \left(\frac{b-x}{b-a} \right) f(b) \right] \right\} \Bigg| \\ \leq\; & 2(1-\gamma) \left(x - \frac{a+b}{2} \right) f(x) + \gamma \left[(b-x) f(b) - (x-a) f(a) \right] \\ & - \int_a^b sgn\,(K(x,t))\, f(t)\,dt \end{aligned} \tag{3.177}$$

$$\begin{aligned} \leq\; & (x-a) \{ (1-\gamma) [f(x) - f(\alpha(x))] + \gamma [f(\alpha(x)) - f(a)] \} \\ & + (b-x) \{ (1-\gamma) [f(\beta(x)) - f(x)] + \gamma [f(b) - f(\beta(x))] \} \end{aligned} \tag{3.178}$$

$$\leq \left[\frac{1}{2} + \left| \gamma - \frac{1}{2} \right| \right] \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] (f(b) - f(a)), \tag{3.179}$$

where $K(x,t)$ is as given by (3.2) *and $\alpha(x), \beta(x)$ by* (3.11).

PROOF. Let $\alpha(x), \beta(x)$ be as in (3.11). Then, from Theorem 3.54,

$$\begin{aligned} \beta(x) - \alpha(x) &= (1-\gamma)(b-a), \\ \alpha(x) - a &= \gamma(x-a) \text{ and} \\ b - \beta(x) &= \gamma(b-x). \end{aligned}$$

In addition,

$$2x - (\alpha(x) + \beta(x)) = 2(1-\gamma)\left(x - \frac{a+b}{2} \right),$$

and so using these results in (3.172), (3.177) is obtained and the first part is proved. Now for the second part. Note that

$$2\alpha(x)-(a+x)=(2\gamma-1)(x-a)$$

and

$$2\beta(x)-(x+b)=(1-2\gamma)(b-x).$$

Substituting the above expressions into the right hand side of (3.173) gives

$$\begin{aligned}&2(1-\gamma)\left(x-\frac{a+b}{2}\right)f(x)+\gamma\left[(b-x)f(b)-(x-a)f(a)\right]\\&+(2\gamma-1)(x-a)f(\alpha(x))+(1-2\gamma)(b-x)f(\beta(x)),\end{aligned}$$

which, upon rearrangement, produces (3.178).
Now, to prove (3.179), the well known result for the maximum may be used, namely

$$\max\{X,Y\}=\frac{X+Y}{2}+\left|\frac{X-Y}{2}\right|.$$

Thus, from the right hand side of (3.178), using

$$\max\{\gamma,(1-\gamma)\}=\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|,$$

gives

$$\left[\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|\right]\{(x-a)(f(x)-f(a))+(b-x)(f(b)-f(x))\}.$$

Furthermore, using

$$\max\{x-a,b-x\}=\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|$$

readily produces (3.179) where $f(b)>f(a)$, since f is monotonic nondecreasing. Hence the theorem is completely proved. ∎

Remark 3.60. Taking $\alpha(x)$ and $\beta(x)$ to be a convex combination of the endpoints produces, it is argued, a more elegant bound (3.179) than would otherwise be the case. The right hand side of (3.176) may easily be shown to equal

$$\begin{aligned}&(\alpha(x)-a)[f(\alpha(x))-f(a)]+(x-\alpha(x))[f(x)-f(\alpha(x))]\\&+(\beta(x)-x)[f(\beta(x))-f(x)]+(b-\beta(x))[f(b)-f(\beta(x))]\\\leq\;&M(x)[f(b)-f(a)]\end{aligned}$$

where

$$M(x)=\max\{\alpha(x)-a,x-\alpha(x),\beta(x)-x,b-\beta(x)\}$$

which is as given in (3.81).

It is further argued that the product form bound in $(3.177)-(3.179)$ when a convex combination of the end points for $\alpha(x)$ and $\beta(x)$ is taken, it is much more enlightening than the bound given by (3.172) and (3.173).

Corollary 3.56. Let f satisfy the conditions as stated in Theorem 3.55. Then the following inequalities hold for any $x \in [a,b]$:

$$\left|\int_a^b f(t)\,dt - \frac{1}{2}\left[(b-a) f(x) + (x-a) f(a) + (b-x) f(b)\right]\right|$$

$$\leq \frac{(x-a)}{2}\left[f(x) - f(a)\right] + \frac{(b-x)}{2}\left[f(b) - f(x)\right] \tag{3.180}$$

$$\leq \frac{1}{2}\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]\left(f(b) - f(a)\right). \tag{3.181}$$

Proof. Placing $\gamma = \frac{1}{2}$ in (3.178) and (3.179) readily produces (3.180) and (3.181) . ∎

Corollary 3.57. Let f satisfy the conditions of Theorem 3.55. Then the following inequalities hold for all $\gamma \in [0,1]$.

$$\left|\int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f\left(\frac{a+b}{2}\right) + \frac{\gamma}{2}\left[f(a) + f(b)\right]\right\}\right|$$

$$\leq \frac{b-a}{2}\left\{\gamma\left[f(b) - f(a)\right]\right. \tag{3.182}$$

$$\left. + (1-2\gamma)\left[f\left(\beta\left(\frac{a+b}{2}\right)\right) - f\left(\alpha\left(\frac{a+b}{2}\right)\right)\right]\right\}$$

$$\leq \frac{(b-a)}{2}\left[\frac{1}{2} + \left|\gamma - \frac{1}{2}\right|\right]\left(f(b) - f(a)\right). \tag{3.183}$$

Proof. Taking $x = \frac{a+b}{2}$ into (3.178) readily produces (3.182) after some minor simplification. Placing $x = \frac{a+b}{2}$ into (3.179) gives (3.183) . ∎

Remark 3.61. The monotonicity properties of $f(\cdot)$ may be used to obtain bounds from (3.178) (or indeed, (3.180) and (3.182)).
Now, since f is monotonic nondecreasing, then

$$f(a) \leq f(\alpha(x)) \leq f(x) \leq f(\beta(x)) \leq f(b),$$

for any $x \in [a,b]$ and $\alpha(x) \in [a,x]$, $\beta(x) \in [x,b]$. Hence, the right hand side of (3.178) is bounded by

$$(x-a)\left[f(x) - f(a)\right] + (b-x)\left[f(b) - f(x)\right],$$

for any $\gamma \in [0,1]$ (that is a uniform bound). This is further bounded by

$$\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]\left(f(b) - f(a)\right),$$

upon using the maximum identity, which is the coarsest bound possible from (3.179), and is obtained by only controlling the γ parameter.

Remark 3.62. Although (3.177) , (3.178) and its particularizations (3.180) and (3.182) are of academic interest, their practical applicability in numerical quadrature is computationally restrictive. Hence, the bound (3.179) and its specializations (3.181) and (3.183) are emphasized.

REMARK 3.63. In the foregoing work we have assumed that f is monotonic nondecreasing. If f is assumed to be simply monotonic, then the modulus sign is required for function differences. Thus, for example, in (3.179), $|f(b) - f(a)|$ would be required rather than simply $f(b) - f(a)$. Alternatively, if $f(\cdot)$ were monotonically nondecreasing, then $-f(\cdot)$ would be monotonically nonincreasing.

REMARK 3.64. Taking various values of $\gamma \in [0, 1]$ and/or $x \in [a, b]$ will produce some specific special cases.
Placing $\gamma = 0$ in (3.179) gives a generalized trapezoidal rule for monotonic mappings and the results of Dragomir, Cerone and Pearce [**25**] are recovered. If $x = \frac{a+b}{2}$, then the trapezoidal rule results.

REMARK 3.65. The optimum result from inequality (3.179) is obtained when both γ and x are taken to be at the midpoints of their respective intervals. Thus, the best quadrature rule is:

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - \frac{1}{2}\left[(b-a) f\left(\frac{a+b}{2}\right) + \frac{1}{2}[f(a) + f(b)]\right] \right| \\ &\leq \frac{b-a}{4}(f(b) - f(a)). \end{aligned} \tag{3.184}$$

This result could equivalently be obtained by taking $\gamma = \frac{1}{2}$ in (3.183) or $x = \frac{a+b}{2}$ in (3.181).
Taking $\gamma = \frac{1}{3}$ in (3.179) gives a generalized Simpson-type rule in which the interior point is unspecified. Namely,

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - \frac{1}{3}[2(b-a) f(x) + (x-a) f(a) + (b-x) f(b)] \right| \\ &\leq \frac{2}{3}\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right](f(b) - f(a)). \end{aligned}$$

If x is taken at the midpoint, then the Simpson-type rule is obtained viz.,

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - \frac{(b-a)}{3}\left[2f\left(\frac{a+b}{2}\right) + \frac{1}{2}(f(a) + f(b))\right] \right| \\ &\leq \frac{b-a}{3}(f(b) - f(a)), \end{aligned} \tag{3.185}$$

which is a worse bound than (3.184). Computationally, there is no difference in implementing (3.184) or (3.185), and yet (3.185) is worse by an absolute amount of $\frac{1}{12}$.

THEOREM 3.58. *Let $f:[a,b]\to\mathbb{R}$ be a monotonic non-decreasing mapping on $[a,b]$. Then the following inequality holds.*

$$
\begin{aligned}
&\left|\int_a^b f(t)\,dt-(b-a)\left\{(1-\gamma)f(x)+\gamma\left[\left(\frac{x-a}{b-a}\right)f(a)\right.\right.\right.\\
&\left.\left.\left.+\left(\frac{b-x}{b-a}\right)f(b)\right]\right\}+(b-a)(1-2\gamma)\left(x-\frac{a+b}{2}\right)S\right|\\
\leq\;& 2\gamma\left(x-\frac{a+b}{2}\right)f(x)+\left[\gamma(x-a)-\left(x-\frac{a+b}{2}\right)\right]f(b)\\
&-\left[\gamma(b-x)+\left(x-\frac{a+b}{2}\right)\right]f(a)\\
&-\int_a^b sgn(\sigma(K(x,t)))f(t)\,dt \qquad (3.186)\\
\leq\;& 2\gamma\left(x-\frac{a+b}{2}\right)f(x)+\left[\gamma(x-a)-\left(x-\frac{a+b}{2}\right)\right]f(b)\\
&-\left[\gamma(b-x)+\left(x-\frac{a+b}{2}\right)\right]f(a)+(2\gamma-1)(b-x)f(\phi(x))\\
&+(1-2\gamma)(x-a)f(\psi(x)) \qquad (3.187)\\
\leq\;& \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\{\gamma(f(b)-f(a))+(1-2\gamma)(f(\psi)-f(\phi))\}. \qquad (3.188)
\end{aligned}
$$

PROOF. From (3.154), and identifying $f(\cdot)$ with $v(\cdot)$ and $\sigma(K(x,\cdot))$ with $g(\cdot)$ gives

$$\left|\int_a^b \sigma(K(x,t))\,df(t)\right|\leq\int_a^b|\sigma(K(x,t))|\,df(t), \tag{3.189}$$

which is equivalent to (3.174) with $\sigma(K(x,t))$ replacing $K(x,t)$. The results of Theorem 3.54 are obtained, namely, by (3.158) and (3.173) with $\phi(\cdot)$, $\psi(\cdot)$ and $\sigma(K(x,\cdot))$ replacing $\alpha(\cdot)$, $\beta(\cdot)$ and $K(x,\cdot)$ respectively. Taking $\phi(x)$ and $\psi(x)$ as given by (3.56) then gives

$$
\begin{aligned}
\psi(x)-\phi(x) &= (1-\gamma)(b-a),\\
\phi(x)-a &= \gamma(b-x)+\left(x-\frac{a+b}{2}\right),\\
b-\psi(x) &= \gamma(x-a)-\left(x-\frac{a+b}{2}\right),\\
2x-(\phi(x)+\psi(x)) &= 2\gamma\left(x-\frac{a+b}{2}\right).
\end{aligned}
$$

Thus, from (3.172), with the appropriate changes to ϕ, ψ and $\sigma(K)$, we have

$$\begin{aligned}
(3.190) \qquad & (1-\gamma)(b-a)f(x)+\left[\gamma(b-x)+\left(x-\frac{a+b}{2}\right)\right]f(a) \\
& +\left[\gamma(x-a)-\left(x-\frac{a+b}{2}\right)\right]f(b) \\
= \ & (1-\gamma)(b-a)f(x)+\gamma\left[(x-a)f(a)+(b-x)f(b)\right] \\
& -(b-a)(1-2\gamma)\left(x-\frac{a+b}{2}\right)S
\end{aligned}$$

and

$$\begin{aligned}
(3.191) \qquad & [2x-(\phi(x)+\psi(x))]f(x)+(b-\psi(x))f(b)-(\phi(x)-a)f(a) \\
= \ & 2\gamma\left(x-\frac{a+b}{2}\right)f(x)+\left[\gamma(x-a)-\left(x-\frac{a+b}{2}\right)\right]f(b) \\
& -\left[\gamma(b-x)+\left(x-\frac{a+b}{2}\right)\right]f(a).
\end{aligned}$$

Hence, combining (3.190) and (3.191) readily gives the first inequality.

Now, for the second inequality, we have from (3.56),

$$\begin{aligned}
2\phi(x)-(a+x) &= (2\gamma-1)(b-x) \\
\text{and } 2\psi(x)-(x+b) &= (1-2\gamma)(x-a).
\end{aligned}$$

Thus, from (3.173) and identifying α with ϕ and β with ψ readily gives the second inequality of the theorem.

The third inequality (3.188) is obtained by grouping the terms in (3.187) as the coefficients of $x-a$ and $b-x$, and then using the fact that

$$\max\{x-a,b-x\}=\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|.$$

Thus, the theorem is now completely proved. ∎

Remark 3.66. From (3.189), we have that

$$\begin{aligned}
\left|\int_a^b \sigma(K(x,t))\,df(t)\right| &\le \int_a^b |\sigma(K(x,t))|\,df(t) \\
&\le \sup_{t\in[a,b]}|\sigma(K(x,t))|\int_a^b df(t) \\
&= \theta(\gamma,x)(f(b)-f(a)),
\end{aligned}$$

where $\theta(\gamma,x)$ is as given by (3.110). Thus, result (3.161) is obtained because, for monotonic nondecreasing functions, $\bigvee_a^b(f)=f(b)-f(a)$.

Remark 3.67. Applications in probability theory are worthy of a mention.
For X a random variable taking on values in the finite interval $[a,b]$, the cumulative distribution function $F(x)$ is defined by $F(x)=\Pr(X\le x)=\int_a^x f(t)\,dt$. Thus, from (3.156), (3.169) with (3.11), we obtain rules for evaluating the cumulative

distribution function in terms of function evaluation of the density. Namely, for any $y \in [a, x]$

$$\begin{aligned} & \left| F(x) - \left\{ (1-\gamma)(x-a) f(y) + \gamma \left[(y-a) f(a) + (x-y) f(x) \right] \right\} \right| \\ \leq & \left\{ \begin{array}{c} \left[\frac{1}{2} + \left| \gamma - \frac{1}{2} \right| \right] \left[\frac{x-a}{2} + \left| y - \frac{a+x}{2} \right| \right] \bigvee_a^x (f) \\ \\ 2L \left[\frac{1}{4} + \left(\gamma - \frac{1}{2} \right)^2 \right] \left[\left(\frac{x-a}{2} \right)^2 + \left(y - \frac{a+x}{2} \right)^2 \right]. \end{array} \right. \end{aligned}$$

The above results could be used to approximate $\Pr(c \leq X \leq d)$ where $[c, d] \subseteq [a, b]$.

If $\gamma = 0$, then the results of Barnett and Dragomir [**2**] are recaptured.

If $f(t) \equiv F(t)$, then $\int_a^b F(t)\, dt = b - E[X]$ and so $F(t)$ is monotonic non-decreasing. In addition, $F(a) = 0,\ F(b) = 1$ give, from Theorem 3.58,

$$\begin{aligned} & |\beta(x) - E[X] - (1-\gamma)(b-a) F(x)| \\ \leq\ & 2(1-\gamma)\left(x - \frac{a+b}{2} \right) F(x) + \gamma (b-x) - \int_a^b sgn(K(x,t)) f(t)\, dt \\ \leq\ & (x-a)\left[(1-\gamma) F(x) + (2\gamma - 1) F(\alpha(x)) \right] \\ & + (b-x)\left[(\gamma - 1) F(x) + (1-2\gamma) F(\beta(x)) + \gamma \right] \\ \leq\ & \left[\frac{1}{2} - \left| \gamma - \frac{1}{2} \right| \right] \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right], \end{aligned}$$

where $\alpha(x)$ and $\beta(x)$ are as given in (3.11) and $K(x,t)$ by (3.2). Noting that $R(x) = \Pr\{X \geq x\} = 1 - F(x)$, then bounds for

$$|\alpha(x) - E[X] + (1-\gamma)(b-a) R(x)|$$

could be obtained from the above development. This is more suitable for work in reliability where $f(x)$ is a failure density. Taking various values of $\gamma \in [0,1]$ and $x \in [a,b]$ gives a variety of results, some of which $(\gamma = 0)$ have been obtained in Barnett and Dragomir [**2**].

3.2.10. Conclusion and Discussion. The work of this section has investigated three-point quadrature rules in which, at most, the first derivative is involved. The major thrust of the work aims at providing **a priori** error bounds so that a suitable partition may be determined that will provide an approximation which is within a particular specified tolerance. The work contains, as special cases, both open and closed Newton-Cotes formulae such as the mid-point, trapezoidal and Simpson rules. The results mainly involve Ostrowski-type rules which contain an arbitrary point $x \in [a,b]$. These rules may be utilised when data is only known at discrete points, which may be non- uniform, without first interpolating.

The approach taken has been through the use of appropriate Peano kernels, resulting in an identity. The identity is then exploited through the Theory of Inequalities to obtain bounds on the error, subject to a variety of norms. The results developed in the current work provide both Riemann and Riemann-Stieltjes quadrature rules.

Grüss-type results are obtained, giving perturbed quadrature rules. A **premature** Grüss approach has produced rules that have tighter bounds. A new identity introduced recently by Dragomir and McAndrew [**30**] is exploited in Subsections 3.2.4, 3.2.6, 3.2.8 and within Subsection 3.2.9 to produce Ostrowski-Grüss type results that seem like a perturbation of the original three-point quadrature rule. A simple reorganisation of the rule to incorporate the perturbation produces a different three- point rule. In effect, the following identity holds,

$$\mathfrak{T}(f,g)=\mathfrak{T}(\sigma(f),g)=\mathfrak{T}(f,\sigma(g))=\mathfrak{T}(\sigma(f),\sigma(g)), \tag{3.192}$$

where

$$\begin{aligned}\mathfrak{T}(f,g) &= \mathfrak{M}(fg)-\mathfrak{M}(f)\,\mathfrak{M}(g),\\ \sigma(f) &= f-\mathfrak{M}(f)\end{aligned}$$

and

$$\mathfrak{M}(f)=\frac{1}{b-a}\int_a^b f(u)\,du.$$

That is,

$$\mathfrak{M}(fg)-\mathfrak{M}(f)\,\mathfrak{M}(g)=\mathfrak{M}(\sigma(f)\,g)=\mathfrak{M}(f\sigma(g))=\mathfrak{M}(\sigma(f)\,\sigma(g)), \tag{3.193}$$

since $\mathfrak{M}(\sigma(f))=\mathfrak{M}(\sigma(g))=0$.

Relation (3.192) (or indeed (3.193)) does not seem to have been **fully** realised in the literature. Dragomir and McAndrew [**30**] used only the equality of the outside two terms in (3.192) (or equivalently (3.193)) to obtain results for a trapezoidal rule. As a matter of fact, there are only two rules that have been used in the current article in which $f(\cdot)\equiv K(x,\cdot)$ and $g(\cdot)\equiv f'(\cdot)$. For $K(x,t)$ as given by (3.2), identity (3.3) is obtained. A similar identity to (3.3) would be obtained if $\sigma(K(x,\cdot))$ were to be considered with $\alpha(x)$ and $\beta(x)$ being replaced by $\phi(x)$ and $\psi(x)$ respectively, where

$$\sigma(K(x,t))=\begin{cases} t-\phi(x), & t\in[a,x]\\ t-\psi(x), & t\in(x,b]. \end{cases} \tag{3.194}$$

Hence, from (3.2),

$$\sigma(K(x,t))=K(x,t)-\mathfrak{M}(K(x,\cdot)),$$

where

$$\begin{aligned}
&\mathfrak{M}(K(x,\cdot))\\
&= \frac{1}{b-a}\int_a^b K(x,u)\,du\\
&= \frac{1}{b-a}\left[\int_a^x (u-\alpha(x))\,du+\frac{1}{b-a}\int_x^b (u-\beta(x))\,du\right]\\
&= \left(\frac{a+x}{2}-\alpha(x)\right)\left(\frac{x-a}{b-a}\right)+\left(\frac{x+b}{2}-\beta(x)\right)\left(\frac{b-x}{b-a}\right)\\
&= \frac{a+b}{2}-\alpha(x)\left(\frac{x-a}{b-a}\right)-\beta(x)\left(\frac{b-x}{b-a}\right).
\end{aligned}$$

Therefore,

$$\phi(x)=\alpha(x)+\mathfrak{M}(K(x,\cdot))=\frac{a+b}{2}+(\alpha(x)-\beta(x))\left(\frac{b-x}{b-a}\right) \tag{3.195}$$

with

$$\psi(x) = \beta(x) + \mathfrak{M}(K(x,\cdot)) = \frac{a+b}{2} + (\beta(x) - \alpha(x))\left(\frac{x-a}{b-a}\right). \tag{3.196}$$

Thus, from (3.3) and (3.194) we have, on using (3.195) and (3.196) and identifying $\phi(\cdot)$ with $\alpha(\cdot)$ and $\psi(\cdot)$ with $\beta(\cdot)$,

$$\begin{aligned}
&\int_a^b \sigma(K(x,t)) f'(t)\, dt \\
&= (\psi(x) - \phi(x)) f(x) + (\phi(x) - a) f(a) + (b - \psi(x)) f(b) - \int_a^b f(t)\, dt \\
&= (\beta(x) - \alpha(x)) f(x) + (\alpha(x) - a) f(a) + (b - \beta(x)) f(b) \\
&\quad + \mathfrak{M}(K(x,\cdot))[f(b) - f(a)] - \int_a^b f(t)\, dt.
\end{aligned}$$

Therefore, using $\mathfrak{T}(K(x,\cdot), f')$, giving rise to what seems to be a perturbed quadrature rule, is equivalent to considering a Peano kernel shifted by its mean. Namely,

$$\mathfrak{T}(\sigma(K(x,\cdot)), f') = \mathfrak{M}(\sigma(K(x,\cdot)), f').$$

Bounds on the above quadrature rule may be obtained using a variety of norms as shown in the section. Using the above identity, bounds involving the first derivative result. If in (3.192) or (3.193), $\sigma(f')$ is used rather than f', then bounds involving the norms of $\sigma(f')$ would result. Either choice may be superior depending on the particular function $f(\cdot)$. For the Riemann-Stieltjes integral $df \equiv d\sigma(f)$ and so the two cases are equivalent.

The work of this section has provided a means for estimating the partition required in order to be guaranteed a certain accuracy for Newton- Cotes quadrature rules. The efficiency, mainly in terms of the number of function evaluations to achieve a particular accuracy, is a very important practical consideration.

3.3. Bounds for $n-$Time Differentiable Functions

3.3.1. Introduction.

Recently, Cerone and Dragomir [**5**] (see also Section 3.1, (3.150)) obtained the following three point identity for $f : [a,b] \to [a,b]$ and $\alpha : [a,b] \to [a,b]$, $\beta : [a,b] \to [a,b]$, $\alpha(x) \le x$, $\beta(x) \ge x$, then

$$\begin{aligned}
&\int_a^b f(t)\, dt - [(\beta(x) - \alpha(x)) f(x) + (\alpha(x) - a) f(a) + (b - \beta(x)) f(b)] \\
&= -\int_a^b K(x,t)\, df(t),
\end{aligned} \tag{3.197}$$

where

$$K(x,t) = \begin{cases} t - \alpha(x), & t \in [a,x] \\ \\ t - \beta(x), & t \in (x,b]. \end{cases} \tag{3.198}$$

They obtained a variety of inequalities for f satisfying different conditions such as bounded variation, Lipschitzian or monotonic. For f absolutely continuous then the above Riemann-Stieltjes integral would be equivalent to a Riemann integral and again a variety of bounds were obtained for $f \in L_p[a,b]$, $p \geq 1$.

Inequalities of Grüss type and a number of premature variants were examined fully in Section 3.2 covering the situation in which f exhibits at most a first derivative. Applications to numerical quadrature were investigated covering rules of Newton-Cotes type containing the evaluation of the function at three possible points: the interior and extremities. The development included the midpoint, trapezoidal and Simpson type rules. However, unlike the classical rules, the results were not as restrictive in that the bounds are derived in terms of the behaviour of at most the first derivative and the Peano kernel (3.198).

It is the aim of the current section to obtain inequalities for $f^{(n)} \in L_p[a,b]$, $p \geq 1$ where $f^{(n)}$ are again evaluated at most at an interior point x and the end points. Results that involve the evaluation only at an interior point are termed Ostrowski type and those that involve only the boundary points will be referred to as trapezoidal type. In the numerical analysis literature these are also termed as Open and Closed Newton-Cotes rules (Atkinson [**1**]) respectively.

In 1938, Ostrowski (see for example [**44**, p. 468]) proved the following integral inequality:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$ ($\mathring{I}$ is the interior of I), and let $a, b \in \mathring{I}$ with $a < b$. If $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have the inequality:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty \tag{3.199}$$

for all $x \in [a,b]$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

For applications of Ostrowski's inequality to some special means and some numerical quadrature rules, we refer the reader to the recent paper [**36**] by S.S. Dragomir and S. Wang who used integration by parts from $\int_a^b p(x,t)\,f'(t)\,dt$ to prove Ostrowski's inequality (3.199) where $p(x,t)$ is a peano kernel given by

$$p(x,t) = \begin{cases} t-a, & t \in [a,x] \\ \\ t-b, & t \in (x,b]. \end{cases} \tag{3.200}$$

Fink [**38**] used the integral remainder from a Taylor series expansion to show that for $f^{(n-1)}$ absolutely continuous on $[a,b]$, then the identity

$$\int_a^b f(t)\,dt - \frac{1}{n}\left((b-a)\,f(x) + \sum_{k=1}^{n-1} F_k(x) \right) = \int_a^b K_F(x,t)\,f^{(n)}(t)\,dt \tag{3.201}$$

is shown to hold where

$$K_F(x,t) = \frac{(x-t)^{n-1}}{(n-1)!}\cdot\frac{p(x,t)}{n} \quad \text{with } \ p(x,t) \ \text{ being given by (3.200)} \tag{3.202}$$

and

$$F_k(x) = \frac{n-k}{k!}\left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b)\right].$$

Fink then proceeds to obtain a variety of bounds from (3.201), (3.202) for $f^{(n)} \in L_p[a,b]$. Milovanović and Pečarić [**43**] earlier obtained a result for $f^{(n)} \in L_\infty[a,b]$ although they did not use the integral form of the remainder. It may be noticed that (3.201) is again an identity that involves function evaluations at three points to approximate the integral from the resulting inequalities. See Mitrinović, Pečarić and Fink [**6**, Chapter XV] for further related results and papers [**28**], [**31**] and [**32**].

A number of other authors have obtained results in the literature that may be recaptured under the general formulation of the current work. These will be highlighted throughout the section.

The section is structured as follows.

A variety of identities are obtained in Subsection 3.3.2 for $f^{(n-1)}$ absolutely continuous for a generalisation of the kernel (3.198). Specific forms are highlighted and a generalised Taylor-like expansion is obtained. Inequalities are developed in Subsection 3.3.3 and perturbed results through Grüss inequalities and premature variants are discussed in Subsection 3.3.4. Subsection 3.3.5 demonstrates the applicability of the inequalities to numerical integration. Concluding remarks for the section are given in Subsection 3.3.7.

3.3.2. Some Integral Identities. In this subsection, identities are obtained involving n-time differentiable functions with evaluation at an interior point and at the end points.

THEOREM 3.59. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Further, let $\alpha:[a,b]\to[a,b]$ and $\beta:[a,b]\to[a,b]$, $\alpha(x)\le x$, $\beta(x)\ge x$. Then, for all $x\in[a,b]$ the following identity holds,*

$$(3.203)\qquad (-1)^n \int_a^b K_n(x,t) f^{(n)}(t)\,dt = \int_a^b f(t)\,dt - \sum_{k=1}^{n}\frac{1}{k!}\left[R_k(x) f^{(k-1)}(x) + S_k(x)\right],$$

where the kernel $K_n:[a,b]^2\to\mathbb{R}$ is given by

$$(3.204)\qquad K_n(x,t) := \begin{cases} \frac{(t-\alpha(x))^n}{n!}, & t\in[a,x] \\ \frac{(t-\beta(x))^n}{n!}, & t\in(x,b], \end{cases}$$

$$(3.205)\qquad \begin{cases} R_k(x) = (\beta(x)-x)^k + (-1)^{k-1}(x-\alpha(x))^k \\ \text{and}\ \ S_k(x) = (\alpha(x)-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-\beta(x))^k f^{(k-1)}(b) \end{cases}.$$

Proof. Let

$$I_n(x) = (-1)^n \int_a^b K_n(x,t) f^{(n)}(t)\,dt = (-1)^n J_n(a,x,b) \tag{3.206}$$

then from (3.205)

$$J_n(a,x,x) = \int_a^x \frac{(t-\alpha(x))^n}{n!} f^{(n)}(t)\,dt$$

giving, upon using integration by parts

$$\begin{aligned} J_n(a,x,x) &= \frac{(x-\alpha(x))^n}{n!} f^{(n-1)}(x) \\ &\quad + (-1)^{n-1} \frac{(\alpha(x)-a)^n f^{(n-1)}(a)}{n!} - J_{n-1}(a,x,x). \end{aligned} \tag{3.207}$$

Similarly,

$$\begin{aligned} J_n(x,x,b) &= (-1)^{n-1} \frac{(\beta(x)-x)^n}{n!} f^{(n-1)}(x) \\ &\quad + \frac{(b-\beta(x))^n}{n!} f^{(n-1)}(b) - J_{n-1}(x,x,b) \end{aligned}$$

and so upon adding to (3.207) gives from (3.206) the recurrence relation

$$I_n(x) - I_{n-1}(x) = -\omega_n(x), \tag{3.208}$$

where

$$n!\,\omega_n(x) = \left[R_n(x) f^{(n-1)}(x) + S_n(x)\right] \tag{3.209}$$

with $R_n(x)$ and $S_n(x)$ being given by (3.205).

It may easily be shown that

$$I_n(x) = -\sum_{k=1}^{n} \omega_k(x) + I_0(x) \tag{3.210}$$

is a solution of (3.208) and so the theorem is proven since (3.210) is equivalent to (3.203) and $I_n(x)$ is as given by (3.206). ∎

Remark 3.68. If we take $n = 1$ then an identity obtained by Cerone and Dragomir [**5**] results. In the same paper Riemann-Stieltjes integrals were also considered.

Remark 3.69. If $\alpha(x) = a$ and $\beta(x) = b$ then $S_k(x) \equiv 0$ and the Ostrowski type results for n-time differentiable functions of Cerone et al. [**11**] are recaptured. Merkle [**42**] also obtains Ostrowski type results. For $\alpha(x) = \beta(x) = x$ then $R(x) \equiv 0$ and the generalized trapezoidal type rules for n-time differentiable functions of Cerone et al. [**12**] are obtained. Qi [**48**] used a Taylor series whose remainder was not expressed in integral form so that only the supremum norm was possible. If the integral form of the remainder were used, then similar to Fink [**38**], the other $L_p(a,b)$ norms for $p \geq 1$ would be possible. However, this will not be pursued further here. For $\alpha(x)$ and $\beta(x)$ at their respective midpoints, then the identity

$$(3.211)\qquad (-1)^n \int_a^b K_n(x,t) f^{(n)}(t)\,dt$$
$$= \int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{2^{-k}}{k!} \left\{ \left[(b-x)^k + (-1)^{k-1}(x-a)^k \right] f^{(k-1)}(x) \right.$$
$$\left. + \left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b) \right] \right\}$$

results, where

$$(3.212)\qquad K_n(x,t) = \begin{cases} \frac{\left(t-\frac{a+x}{2}\right)^n}{n!}, & t \in [a,x] \\ \frac{\left(t-\frac{x+b}{2}\right)^n}{n!}, & t \in (x,b]. \end{cases}$$

As demonstrated in the above remarks, different choices of $\alpha(x)$ and $\beta(x)$ give a variety of identities. The following corollary allows for $\alpha(x)$ and $\beta(x)$ to be in the same relative position within their respective intervals.

Corollary 3.60. Let f satisfy the conditions as stated in Theorem 3.59. Then the following identity holds for any $\gamma \in [0,1]$ and $x \in [a,b]$. Namely,

$$(3.213)\qquad (-1)^n \int_a^b C_n(x,t) f^{(n)}(t)\,dt$$
$$= \int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{1}{k!} \left\{ (1-\gamma)^k \left[(b-x)^k + (-1)^{k-1}(x-a)^k \right] f^{(k-1)}(x) \right.$$
$$\left. + \gamma^k \left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b) \right] \right\},$$

where

$$(3.214)\qquad C_n(x,t) = \begin{cases} \frac{[t-(\gamma x+(1-\gamma)a)]^n}{n!}, & t \in [a,x] \\ \frac{[t-(\gamma x+(1-\gamma)b)]^n}{n!}, & t \in (x,b]. \end{cases}$$

Proof. Let

$$(3.215)\qquad \alpha(x) = \gamma x + (1-\gamma)a \quad \text{and} \quad \beta(x) = \gamma x + (1-\gamma)b,$$

then

$$(3.216)\qquad \begin{cases} x-\alpha(x) = (1-\gamma)(x-a), & \alpha(x)-a = \gamma(x-a) \\ \text{and}\ \ \beta(x)-x = (1-\gamma)(b-x), & b-\beta(x) = \gamma(b-x) \end{cases}$$

so that from (3.205)

$$R_k(x) = (1-\gamma)^k \left[(b-x)^k + (-1)^{k-1}(x-a)^k \right]$$

and

$$S_k(x) = \gamma^k \left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b) \right].$$

In addition, $C_n(x,t)$ is the same as $K_n(x,t)$ in (3.204) with $\alpha(x)$ and $\beta(x)$ as given by (3.215) and hence the corollary is proven. ∎

The following Taylor-like formula with integral remainder also holds.

Corollary 3.61. Let $g : [a, y] \to \mathbb{R}$ be a mapping such that $g^{(n)}$ is absolutely continuous on $[a, y]$. Then for all $x \in [a, y]$ we have the identity

$$\begin{aligned} (3.217) \qquad & g(y) \\ = \quad & g(a) + \sum_{k=1}^{n} \frac{1}{k!} \left\{ \left[(\beta(x) - x)^k + (-1)^{k-1} (x - \alpha(x))^k \right] g^{(k)}(x) \right. \\ & \left. + \left[(\alpha(x) - a)^k g^{(k)}(a) + (-1)^{k-1} (y - \beta(x))^k g^{(k)}(y) \right] \right\} \\ & + (-1)^n \int_a^y \tau_n(x, t) g^{(n+1)}(t)\, dt \end{aligned}$$

where

$$(3.218) \qquad \tau_n(x, t) = \begin{cases} \frac{(t - \alpha(x))^n}{n!}, & t \in [a, x] \\ \\ \frac{(t - \beta(x))^n}{n!}, & t \in (x, y]. \end{cases}$$

Proof. The proof is straight forward from Theorem 3.59 on taking $f \equiv g'$ and $b = y$ so that $\beta(x) \in (x, y]$ and $\tau_n(x, t) \equiv K_n(x, t)$ for $t \in [a, y]$. ∎

Remark 3.70. If $\alpha(x) = \beta(x) = x$ then we recapture the results of Cerone et al. [**12**], a trapezoidal type series expansion. That is, an expansion involving the end points. For $\alpha(x) = a$, $\beta(x) = b$ then a Taylor-like expansion of Cerone et al. [**11**] is reproduced as are the results of Merkle [**42**].

3.3.3. Integral Inequalities. In this subsection we develop some inequalities from using the identities obtained in Subsection 3.3.2.

Theorem 3.62. *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$ and, let $\alpha : [a, b] \to [a, b]$ and $\beta : [a, b] \to [a, b]$, $\alpha(x) \le x$, $\beta(x) \ge x$. Then the following inequalities hold for all $x \in [a, b]$*

$$\begin{aligned} (3.219) \quad |P_n(x)| \quad : \quad & = \left| \int_a^b f(t)\, dt - \sum_{k=1}^{n} \frac{1}{k!} \left[R_k(x) f^{(k-1)}(x) + S_k(x) \right] \right| \\ & \le \begin{cases} \frac{\|f^{(n)}\|_\infty}{n!} Q_n(1, x) & \text{if } f^{(n)} \in L_\infty[a, b], \\ \\ \frac{\|f^{(n)}\|_p}{n!} [Q_n(q, x)]^{\frac{1}{q}} & \text{if } f^{(n)} \in L_p[a, b] \\ & \text{with } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{\|f^{(n)}\|_1}{n!} M^n(x), & \text{if } f^{(n)} \in L_1[a, b], \end{cases} \end{aligned}$$

where

$$\begin{aligned} (3.220) \qquad Q_n(q, x) \quad = \quad & \frac{1}{nq + 1} \left[(\alpha(x) - a)^{nq+1} + (x - \alpha(x))^{nq+1} \right. \\ & \left. + (\beta(x) - x)^{nq+1} + (b - \beta(x))^{nq+1} \right], \end{aligned}$$

$$\begin{aligned}(3.221)\quad M(x) &= \frac{1}{2}\left\{\frac{b-a}{2}+\left|\alpha(x)-\frac{a+x}{2}\right|+\left|\beta(x)-\frac{x+b}{2}\right|\right.\\ &\quad\left.+\left|x-\frac{a+b}{2}+\left|\alpha(x)-\frac{a+x}{2}\right|+\left|\beta(x)-\frac{x+b}{2}\right|\right|\right\}\\ &= \max\{\alpha(x)-a, x-\alpha(x), x-\beta(x), b-\beta(x)\},\end{aligned}$$

$R_k(x)$, $S_k(x)$ are given by (3.205), and

$$\left\|f^{(n)}\right\|_\infty := ess\sup_{t\in[a,b]}\left|f^{(n)}(t)\right|<\infty \quad and \quad \left\|f^{(n)}\right\|_p := \left(\int_a^b \left|f^{(n)}(t)\right|^p\right)^{\frac{1}{p}},\ p\geq 1.$$

Proof. Taking the modulus of (3.203) then

$$(3.222)\qquad |P_n(x)| = |I_n(x)|,$$

where $P_n(x)$ is as defined by the left hand side of (3.219) and

$$(3.223)\qquad |I_n(x)| = \left|\int_a^b K_n(x,t)\,f^{(n)}(t)\,dt\right|,$$

with $K_n(x,t)$ given by (3.204).

Now, observe that

$$\begin{aligned}(3.224)\qquad |I_n(x)| &\leq \left\|f^{(n)}\right\|_\infty \|K_n(x,\cdot)\|_1\\ &= \left\|f^{(n)}\right\|_\infty \int_a^b |K_n(x,t)|\,dt,\end{aligned}$$

where, from (3.204),

$$\begin{aligned}(3.225)\int_a^b |K_n(x,t)|\,dt &= \frac{1}{n!}\left\{\int_a^{\alpha(x)}|t-\alpha(x)|^n\,dt+\int_{\alpha(x)}^x|t-\alpha(x)|^n\,dt\right.\\ &\quad\left.+\int_x^{\beta(x)}|t-\beta(x)|^n\,dt+\int_{\beta(x)}^b|t-\beta(x)|^n\,dt\right\}\\ &= \frac{1}{(n+1)!}\left[(\alpha(x)-a)^{n+1}+(x-\alpha(x))^{n+1}\right.\\ &\quad\left.+(\beta(x)-x)^{n+1}+(b-\beta(x))^{n+1}\right].\end{aligned}$$

Thus, on combining (3.222), (3.224) and (3.225), the first inequality in (3.219) is obtained.

Further, using Hölder's integral inequality we have the result

$$\begin{aligned}(3.226)\quad |I_n(x)| &\leq \left\|f^{(n)}\right\|_p \|K_n(x,\cdot)\|_q \quad\text{where } \frac{1}{p}+\frac{1}{q}=1, \text{ with } p>1\\ &= \left\|f^{(n)}\right\|_p \left(\int_a^b |K_n(x,t)|^q\,dt\right)^{\frac{1}{q}}.\end{aligned}$$

Now,

$$\begin{aligned}(3.227)\quad \int_a^b |K_n(x,t)|^q\,dt &= \frac{1}{n!}\left\{\int_a^{\alpha(x)} |t-\alpha(x)|^{nq}\,dt + \int_{\alpha(x)}^x |t-\alpha(x)|^{nq}\,dt\right.\\ &\quad \left.+\int_x^{\beta(x)} |t-\beta(x)|^{nq}\,dt + \int_{\beta(x)}^b |t-\beta(x)|^{nq}\,dt\right\}\\ &= \frac{1}{n!}Q_n(q,x),\end{aligned}$$

where $Q_n(q,x)$ is as given by (3.220).

Combing (3.227) with (3.226) gives the second inequality in (3.219).

Finally, let us observe that from (3.222)

$$\begin{aligned}(3.228)\quad & |I_n(x)|\\ \le\ & \|K_n(x,\cdot)\|_\infty \left\|f^{(n)}\right\|_1 = \left\|f^{(n)}\right\|_1 \sup_{t\in[a,b]} |K_n(x,t)|\\ =\ & \frac{\left\|f^{(n)}\right\|_1}{n!}\max\left\{|a-\alpha(x)|^n, |x-\alpha(x)|^n, |b-\beta(x)|^n, |x-\beta(x)|^n\right\}\\ =\ & \frac{\left\|f^{(n)}\right\|_1}{n!}M^n(x),\end{aligned}$$

where

$$(3.229)\qquad M(x) = \max\{M_1(x), M_2(x)\}$$

with

$$M_1(x) = \max\{\alpha(x)-a, x-\alpha(x)\}$$

and

$$M_2(x) = \max\{\beta(x)-x, b-\beta(x)\}.$$

The well known identity

$$(3.230)\qquad \max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{X-Y}{2}\right|$$

may be used to give

$$(3.231)\qquad \begin{aligned} M_1(x) &= \tfrac{x-a}{2} + \left|\alpha(x) - \tfrac{a+x}{2}\right|,\\ \text{and}\quad M_2(x) &= \tfrac{b-x}{2} + \left|\beta(x) - \tfrac{x+b}{2}\right|.\end{aligned}$$

Using the identity (3.230) again gives, from (3.229),

$$M(x) = \frac{M_1(x)+M_2(x)}{2} + \left|\frac{M_1(x)-M_2(x)}{2}\right|$$

which on substituting (3.231) gives (3.221) and so from (3.228) and (3.222) readily results in the third inequality in (3.219) and the theorem is completely proved. ∎

Remark 3.71. Various choices of $\alpha(\cdot)$ and $\beta(\cdot)$ allow us to reproduce many of the earlier inequalities involving function and derivative evaluations at an interior point and/or boundary points. For other related results see Chapter XV of [**44**].

If $\alpha(x) = a$ and $\beta(x) = b$ then $S_k(x) \equiv 0$ and Ostrowski type results for n-time differentiable functions of Cerone et al. [**11**] are reproduced (see also [**49**]). Further, taking $n = 1$ recaptures the results of Dragomir and Wang [**35**]-[**34**] and $n = 2$ gives the results of Cerone, Dragomir and Roumeliotis [**8**]-[**10**]. The $n = 2$ case is of importance since with $x = \frac{a+b}{2}$ the classic midpoint rule is obtained. However, here the bound is obtained for $f'' \in L_p[a,b]$ for $p \geq 1$ rather than the traditional $f'' \in L_\infty[a,b]$, see for example [**24**] and [**26**].

If $\alpha(x) = \beta(x) = x$ then $R(x) \equiv 0$ and inequalities are obtained for a generalised trapezoidal type rule in which functions are assumed to be n-time differentiable, recapturing the results in Cerone et al. [**12**]. Taking $n = 2$ the classic trapezoidal rule in which the bound involves the behaviour of the second derivative is recaptured as presented in Dragomir et al. [**27**].

Taking $\alpha(\cdot)$ and $\beta(\cdot)$ to be other than at the extremities results in three point inequalities for $n-$time differentiable functions. Cerone and Dragomir [**5**] presented results for functions that at most admit a first derivative.

Remark 3.72. It should be noted that the bounds in (3.219) may themselves be bounded since $\alpha(\cdot)$, $\beta(\cdot)$ and x have not been explicitly specified.

To demonstrate, consider the mappings, for $t \in [A,B]$,

$$\begin{cases} h_1(t) = (t-A)^\theta + (B-t)^\theta, \ \theta > 1 \\ \text{and } h_2(t) = \frac{B-A}{2} + \left|t - \frac{A+B}{2}\right|. \end{cases} \tag{3.232}$$

Now, both these functions attain their maximum values at the ends of the interval and their minimums at the midpoints. That is, they are symmetric and convex. Thus,

$$\begin{cases} \sup_{t\in[A,B]} h_1(t) = h_1(A) = h_1(B) = (B-A)^\theta, \\ \sup_{t\in[A,B]} h_2(t) = h_2(A) = h_2(B) = B-A, \\ \inf_{t\in[A,B]} h_1(t) = h_1\left(\frac{A+B}{2}\right) = 2\left(\frac{B-A}{2}\right)^\theta, \\ \text{and } \inf_{t\in[A,B]} h_2(t) = h_2\left(\frac{A+B}{2}\right) = \frac{B-A}{2}. \end{cases} \tag{3.233}$$

Using (3.232) and (3.233) then from (3.220) and (3.221), on taking $\alpha(\cdot)$ and $\beta(\cdot)$ at either of their extremities gives

$$Q_n(q,x) \leq Q_n^U(q,x) = \frac{1}{nq+1}\left[(x-a)^{nq+1} + (b-x)^{nq+1}\right] \leq \frac{(b-a)^{nq+1}}{nq+1}$$

and

$$M(x) \leq M^U(x) = \frac{1}{2}\left[b-a+|x-a|\right] \leq b-a,$$

where the coarsest bounds are obtained from taking x at its extremities.

The following corollary holds.

COROLLARY 3.63. Let the conditions of Theorem 3.62 hold. Then the following result is valid for any $x \in [a, b]$. Namely,

$$\begin{aligned}
(3.234)\quad & \left| \int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{2^{-k}}{k!} \left\{ \left[(b-x)^k + (-1)^{k-1} (x-a)^k \right] f^{(k-1)}(x) \right.\right. \\
& \left.\left. + \left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1} (b-x)^k f^{(k-1)}(b) \right] \right\} \right| \\
& \leq \begin{cases}
\frac{\|f^{(n)}\|_\infty}{n!} 2^{-n} \left[(x-a)^{n+1} + (b-x)^{n+1} \right], & f^{(n)} \in L_\infty [a,b], \\
\frac{\|f^{(n)}\|_p}{n!} \frac{2^{-n}}{(nq+1)^{\frac{1}{q}}} \left[(x-a)^{nq+1} + (b-x)^{nq+1} \right]^{\frac{1}{q}} & f^{(n)} \in L_p [a,b] \\
& \text{with } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\
\frac{\|f^{(n)}\|_1}{n!} 2^{-n} \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^n, & f^{(n)} \in L_1 [a,b].
\end{cases}
\end{aligned}$$

PROOF. Taking $\alpha(\cdot)$ and $\beta(\cdot)$ at their respective midpoints, namely $\alpha(x) = \frac{a+x}{2}$ and $\beta(x) = \frac{x+b}{2}$ in (3.219)-(3.221) and using (3.205) readily gives (3.234) ∎

REMARK 3.73. Corollary 3.63 could have equivalently been proven using (3.211) and (3.212) following essentially the same proof of Theorem 3.62 from using identity (3.211). The more general setting however, allows greater flexibility and, it is argued, is no more difficult to prove.

COROLLARY 3.64. Let the conditions on f of Theorem 3.62 hold. Then the following result for any $x \in [a, b]$, is valid. Namely, for any $\gamma \in [0, 1]$

$$(3.235)\qquad \left| \int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{(-1)^k}{k!} \left[(1-\gamma)^k r_k(x) f^{(k-1)}(x) + \gamma^k s_k(x) \right] \right|$$

$$\leq \begin{cases}
\frac{\|f^{(n)}\|_\infty}{(n+1)!} H_1(\gamma) G_1(x), & f^{(n)} \in L_\infty [a,b], \\
\frac{\|f^{(n)}\|_p}{n!(nq+1)^{\frac{1}{q}}} H_q^{\frac{1}{q}}(\gamma) G_q^{\frac{1}{q}}(x), & f^{(n)} \in L_p [a,b] \\
& p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\
\frac{\|f^{(n)}\|_1}{n!} \nu^n(x), & f^{(n)} \in L_1 [a,b],
\end{cases}$$

where

$$(3.236)\qquad \begin{cases}
H_q(\gamma) = \gamma^{nq+1} + (1-\gamma)^{nq+1}, \\
G_q(x) = (x-a)^{nq+1} + (b-x)^{nq+1}, \\
\nu(x) = \left[\frac{1}{2} + \left| \gamma - \frac{1}{2} \right| \right] \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right], \\
r_k(x) = (b-x)^k + (-1)^{k-1} (x-a)^k, \\
\text{and } s_k(x) = (x-a)^k f^{(k-1)}(a) + (-1)^{k-1} (b-x)^k f^{(k-1)}(b).
\end{cases}$$

PROOF. Take $\alpha(\cdot)$ and $\beta(\cdot)$ to be a convex combination of their respective boundary points as given by (3.215) then from (3.219)-(3.221) and using (3.216) and (3.205) readily produces the stated result. We omit any further details. ∎

REMARK 3.74. It is instructive to note that the relative location of $\alpha(\cdot)$ and $\beta(\cdot)$ is the same in Corollary 3.64 and is determined through the parameter γ as defined in (3.215). Theorem 3.62 is much more general. From (3.215) it may be seen that $\alpha(x) = \beta(x) = x$ is equivalent to $\gamma = 1$, giving trapezoidal type rules while $\alpha(x) = a$, $\beta(x) = b$ corresponds to $\gamma = 0$ which produces interior point rules. Taking $\gamma = 0$ and $\gamma = 1$ reproduces the results of Cerone et al. [**11**] and [**12**] respectively.

Taking $\gamma = \frac{1}{2}$ in (3.235) produces the optimal rule while keeping x general and thus reproducing the result of Corollary 3.63. Following the discussion in Remark 3.72 and as may be ascertained from (3.236) the optimal rules, in the sense of providing the tightest bounds, are obtained by taking γ and x at their respective midpoints.

The following two corollaries may thus be stated.

COROLLARY 3.65. Let the conditions on f of Theorem 3.62 be valid. Then for any $\gamma \in [0,1]$, the following inequalities hold

$$\left|\int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{(-1)^k}{k!}\left[(1-\gamma)^k r_k\left(\frac{a+b}{2}\right) f^{(k-1)}\left(\frac{a+b}{2}\right) + \gamma^k s_k\left(\frac{a+b}{2}\right)\right]\right| \tag{3.237}$$

$$\leq \begin{cases} \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} H_1(\gamma)\, G_1\left(\frac{a+b}{2}\right), & f^{(n)} \in L_\infty[a,b], \\ \frac{\left\|f^{(n)}\right\|_p}{n!(nq+1)^{\frac{1}{q}}} H_q^{\frac{1}{q}}(\gamma)\, G_q^{\frac{1}{q}}\left(\frac{a+b}{2}\right), & f^{(n)} \in L_p[a,b] \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{\left\|f^{(n)}\right\|_1}{n!} \nu^n\left(\frac{a+b}{2}\right), & f^{(n)} \in L_1[a,b], \end{cases} \tag{3.238}$$

where

$$\begin{cases} H_q(\gamma) \text{ is as given by (3.236)}, \\ G_q\left(\frac{a+b}{2}\right) = 2\left(\frac{b-a}{2}\right)^{nq+1}, \\ \nu\left(\frac{a+b}{2}\right) = \left(\frac{b-a}{2}\right)\left(\frac{1}{2} + \left|\gamma - \frac{1}{2}\right|\right), \\ r_k\left(\frac{a+b}{2}\right) = \left(\frac{b-a}{2}\right)^k\left[1 + (-1)^{k-1}\right] \\ \text{and } s_k\left(\frac{a+b}{2}\right) = \left(\frac{b-a}{2}\right)^k\left[f^{(k-1)}(a) + (-1)^{k-1} f^{(k-1)}(b)\right]. \end{cases} \tag{3.239}$$

PROOF. The proof is trivial. Taking $x = \frac{a+b}{2}$ in (3.235)-(3.236) readily produces the result. ∎

REMARK 3.75. It is of interest to note from (3.239) that

$$r_k\left(\frac{a+b}{2}\right)=\begin{cases}2\left(\frac{b-a}{2}\right)^k, & k \text{ odd}\\ 0, & k \text{ even}\end{cases} \tag{3.240}$$

so that only the evaluation of even order derivatives are involved in (3.237). Further, for $f^{(k-1)}(a)=f^{(k-1)}(b)$ then

$$s_k\left(\frac{a+b}{2}\right)=f^{(k-1)}(a)\,r_k\left(\frac{a+b}{2}\right)=f^{(k-1)}(b)\,r_k\left(\frac{a+b}{2}\right) \tag{3.241}$$

so that only evaluation of even order derivatives at the end points are present.

COROLLARY 3.66. Let the conditions on f of Theorem 3.62 hold. Then the following inequalities are valid

$$\left|\int_a^b f(t)\,dt-\sum_{k=1}^{n}\frac{(-1)^k}{k!}2^{-k}\left[r_k\left(\frac{a+b}{2}\right)f^{(k-1)}\left(\frac{a+b}{2}\right)+s_k\left(\frac{a+b}{2}\right)\right]\right| \tag{3.242}$$
$$\leq\begin{cases}\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\cdot 2^{-(n-1)}\left(\frac{b-a}{2}\right)^{n+1}, & f^{(n)}\in L_\infty[a,b],\\ \frac{\left\|f^{(n)}\right\|_p}{n!(nq+1)^{\frac{1}{q}}}\cdot 2^{-n}\left(\frac{b-a}{nq+1}\right)^{\frac{1}{q}}\left(\frac{b-a}{2}\right)^n, & f^{(n)}\in L_p[a,b]\\ & \text{with } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \frac{\left\|f^{(n)}\right\|_1}{n!}\left(\frac{b-a}{4}\right)^n, & f^{(n)}\in L_1[a,b],\end{cases}$$

where $r_k\left(\frac{a+b}{2}\right)$ and $s_k\left(\frac{a+b}{2}\right)$ are as given by (3.239).

PROOF. Taking $\gamma=\frac{1}{2}$ in Corollary 3.65 will produce inequalities with the tightest bounds as given in (3.242). Alternatively, taking $\gamma=\frac{1}{2}$ and $x=\frac{a+b}{2}$ in Corollary 3.64 will produce the results (3.242). ∎

The results (3.240) and (3.241) together with the discussion in Remark 3.75 are also valid for Corollary 3.66.

The following are Taylor-like inequalities which are of interest (see [**14**] and [**16**] for related results).

Corollary 3.67. Let $g:[a,y]\to\mathbb{R}$ be a mapping such that $g^{(n)}$ is absolutely continuous on $[a,y]$. Then for all $x\in[a,y]$

$$
\begin{aligned}
(3.243)\quad & \left| g(y)-g(a)-\sum_{k=1}^{n}\frac{1}{k!}\left\{ \left[(\beta(x)-x)^{k}+(-1)^{k-1}(x-\alpha(x))^{k}\right] g^{(k)}(x)\right.\right. \\
& \left.\left. +\left[(\alpha(x)-a)^{k}g^{(k)}(a)+(-1)^{k-1}(y-\beta(x))^{k}g^{(k)}(y)\right] \right\} \right| \\
& \leq \begin{cases}
\frac{\left\Vert g^{(n+1)}\right\Vert_{\infty}}{n!}\tilde{Q}_{n}(1,x), & g^{(n+1)}\in L_{\infty}[a,b], \\[2mm]
\frac{\left\Vert g^{(n+1)}\right\Vert_{p}}{n!}\left[\tilde{Q}_{n}(q,x)\right]^{\frac{1}{q}}, & g^{(n+1)}\in L_{p}[a,b] \\
& \text{with } \ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\
\frac{\left\Vert g^{(n+1)}\right\Vert_{1}}{n!}\tilde{M}^{n}(x), & g^{(n+1)}\in L_{1}[a,b],
\end{cases}
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{Q}_{n}(q,x) &= \frac{1}{nq+1}\left[(\alpha(x)-a)^{nq+1}+(x-\alpha(x))^{nq+1}\right. \\
&\quad \left. +(\beta(x)-a)^{nq+1}+(y-\beta(x))^{nq+1}\right], \\
\tilde{M}(x) &= \frac{1}{2}\left\{ \frac{y-a}{2}+\left| \alpha(x)-\frac{a+x}{2}\right| +\left| \beta(x)-\frac{x+y}{2}\right| \right. \\
&\quad \left. +\left| x-\frac{a+y}{2}+\left| \alpha(x)-\frac{a+x}{2}\right| +\left| \beta(x)-\frac{x+y}{2}\right| \right| \right\}.
\end{aligned}
$$

Proof. The proof follows from Theorem 3.62 on taking $f(\cdot)\equiv g'(\cdot)$ and $b=y$ so that $\beta(x)\in(x,y]$. Alternatively, starting from (3.217) and (3.218) and, following the proof of Theorem 3.62 with b replaced by y and $f(\cdot)$ replaced by $g'(\cdot)$ readily produces the results shown and the corollary is thus proven. ∎

Remark 3.76. Similar corollaries to 3.63, 3.64, 3.74 and 3.75 could be determined from the Taylor-like inequalities given in Corollary 3.66. This would simply be done by taking specific forms of $\alpha(\cdot)$, $\beta(\cdot)$ or values of x as appropriate.

Remark 3.77. If in particular we take $\alpha(x)=a$ and $\beta(x)=y$ in (3.243) then for any $x\in[a,y]$

$$
\begin{aligned}
(3.244)\quad & \left| g(y)-g(a)-\sum_{k=1}^{n}\frac{1}{k!}\left[(y-x)^{k}+(-1)^{k-1}(x-a)^{k}\right] g^{(k)}(x)\right| \\
& \leq e_{n}(x,y)
\end{aligned}
$$

(3.245)

$$:= \begin{cases} \frac{\left\|g^{(n+1)}\right\|_{\infty}}{(n+1)!}\left[(x-a)^{n+1}+(y-x)^{n+1}\right], & g^{(n+1)} \in L_{\infty}[a, y], \\ \frac{\left\|g^{(n+1)}\right\|_{p}}{n!(nq+1)^{\frac{1}{q}}}\left[(x-a)^{nq+1}+(y-x)^{nq+1}\right]^{\frac{1}{q}}, & g^{(n+1)} \in L_{p}[a, y] \\ & \text{with } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{\left\|g^{(n+1)}\right\|_{1}}{n!}\left[\frac{y-a}{2}+\left|x-\frac{a+y}{2}\right|\right]^{n}, & g^{(n+1)} \in L_{1}[a, y]. \end{cases}$$

Merkle [**42**] effectively obtains the first bound in (3.244).

It is well known (see for example, Dragomir [**16**]) that the classical Taylor expansion around a point satisfies the inequality

$$\left|g(y)-\sum_{k=1}^{n} \frac{(y-a)^{k}}{k!} g^{(k)}(a)\right| \leq \left|\frac{1}{n!} \int_{a}^{y}(y-u)^{n} g^{(n+1)}(u)\, du\right|:=E_{n}(y), \tag{3.246}$$

where

$$E_{n}(y) \leq \begin{cases} \frac{(y-a)^{n+1}}{(n+1)!}\left\|g^{(n+1)}\right\|_{\infty}, & g^{(n+1)} \in L_{\infty}[a, y], \\ \frac{(y-a)^{n+\frac{1}{q}}}{n!(nq+1)^{\frac{1}{q}}}\left\|g^{(n+1)}\right\|_{p}, & g^{(n+1)} \in L_{p}[a, y] \\ & \text{with } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{(y-a)^{n}}{n!}\left\|g^{(n+1)}\right\|_{1}, & g^{(n+1)} \in L_{1}[a, y], \end{cases} \tag{3.247}$$

for $y \geq a$ and $y \in I \subset \mathbb{R}$.

Now, it may readily be noticed that if $x=a$ in (3.244), then the classical result as given by (3.246) is regained. As discussed in Remark 3.72 the bounds are convex so that a coarse bound is obtained at the end points and the best at the midpoint. Thus, taking $x=\frac{a+y}{2}$ gives

$$\begin{aligned} &\left|g(y)-g(a)-\sum_{k=1}^{n} \frac{\left[1+(-1)^{k-1}\right]}{k!} 2^{-k}(y-a)^{k} g^{(k)}\left(\frac{a+y}{2}\right)\right| \qquad (3.248)\\ &\leq e_{n}\left(\frac{a+y}{2}, y\right) \\ &= \begin{cases} \frac{\left\|g^{(n+1)}\right\|_{\infty}}{(n+1)!} 2^{-n}(y-a)^{n+1}, & g^{(n+1)} \in L_{\infty}[a, y], \\ \frac{\left\|g^{(n+1)}\right\|_{p}}{n!(nq+1)^{\frac{1}{q}}} 2^{-n}(y-a)^{n+\frac{1}{q}}, & g^{(n+1)} \in L_{p}[a, y] \\ & \text{with } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{\left\|g^{(n+1)}\right\|_{1}}{n!} 2^{-n}(y-a)^{n}, & g^{(n+1)} \in L_{1}[a, y]. \end{cases} \end{aligned}$$

The above inequalities (3.248) show that for $g \in C^{\infty}[a,b]$ the series

$$g(a) + \sum_{k=1}^{\infty} \frac{\left[1 + (-1)^{k-1}\right]}{k! 2^k} (y-a)^k g^{(k)}\left(\frac{a+y}{2}\right)$$

converges more rapidly to $g(y)$ than the usual one

$$\sum_{k=0}^{\infty} \frac{(y-a)^k}{k!} g^{(k)}(a),$$

which comes from Taylor's expansion (3.246). It should further be noted that (3.247) only involves the odd derivatives of $g(\cdot)$ evaluated at the midpoint of the interval under consideration.

REMARK 3.78. If $\alpha(x) = \beta(x) = x$ in (3.243), then for any $x \in [a,y]$

$$\left| g(y) - g(a) - \sum_{k=1}^{n} \frac{1}{k!} \left[(x-a)^k g^{(k)}(a) + (-1)^{k-1} (y-x)^k g^{(k)}(y) \right] \right| \leq e_n(x,y), \tag{3.249}$$

where $e_n(x,y)$ is as defined by (3.244). See Cerone et al. [**12**] for related results.

3.3.4. Perturbed Rules Through Grüss Type Inequalities.

In 1935, G. Grüss (see for example [**44**]), proved the following integral inequality which gives an approximation for the integral of a product in terms of the product of integrals.

THEOREM 3.68. *Let $h, g : [a,b] \to \mathbb{R}$ be two integrable mappings so that $\phi \leq h(x) \leq \Phi(x)$ and $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a,b]$, where $\phi, \Phi, \gamma, \Gamma$ are real numbers. Then we have*

$$|T(h,g)| \leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma), \tag{3.250}$$

where

$$T(h,g) = \frac{1}{b-a}\int_a^b h(x) g(x)\,dx - \frac{1}{b-a}\int_a^b h(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx \tag{3.251}$$

and the inequality is sharp, in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

For a simple proof of this fact as well as for extensions, generalisations, discrete variants and other associated material, see [**44**], and the papers [**7**], [**13**], [**15**], [**22**] and [**35**] where further references are given.

A *premature* Grüss inequality is embodied in the following theorem which was proved in the paper [**41**]. It provides a sharper bound than the above Grüss inequality. The term *premature* is used to denote the fact that the result is obtained from not completing the proof of the Grüss inequality if one of the functions is known explicitly. See also [**5**] for further details.

THEOREM 3.69. *Let h, g be integrable functions defined on $[a,b]$ and let $d \leq g(t) \leq D$. Then*

$$|T(h,g)| \leq \frac{D-d}{2}\left[T(h,h)\right]^{\frac{1}{2}}, \tag{3.252}$$

where $T(h,g)$ *is as defined in (3.251).*

The above Theorem 3.69 will now be used to provide a perturbed generalised three point rule.

3.3.5. Perturbed Rules From Premature Inequalities. We start with the following result.

THEOREM 3.70. *Let* $f:[a,b]\to\mathbb{R}$ *be such that the derivative* $f^{(n-1)}$, $n\geq 1$ *is absolutely continuous on* $[a,b]$. *Assume that there exist constants* $\gamma,\Gamma\in\mathbb{R}$ *such that* $\gamma\leq f^{(n)}(t)\leq\Gamma$ *a.e. on* $[a,b]$. *Then the following inequality holds*

$$
\begin{aligned}
(3.253)\quad |\rho_n(x)| \quad : &= \left|\int_a^b f(t)\,dt-\sum_{k=1}^{n}\frac{1}{k!}\left[R_k(x)f^{(k-1)}(x)+S_k(x)\right]\right.\\
&\qquad\left.-(-1)^n\frac{\theta_n(x)}{n+1}\cdot\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right|\\
&\leq \frac{\Gamma-\gamma}{2}\cdot\frac{1}{n!}I(x,n)\\
&\leq \frac{\Gamma-\gamma}{\sqrt{2}}\cdot\frac{n}{(n+1)!}\cdot\frac{(b-a)^{n+1}}{\sqrt{2n+1}},
\end{aligned}
$$

where

$$
\begin{aligned}
(3.254)\qquad I(x,n) &= \frac{1}{(n+1)\sqrt{2n+1}}\left\{n^2(b-a)\hat{Q}_n(2,x)\right.\\
&\qquad\left.+(2n+1)\sum_{\substack{i=1\\ j>i}}^{4}z_iz_j\left[z_i^n-(-z_j)^n\right]^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
Z &= \{\alpha(x)-a,x-\alpha(x),\beta(x)-x,b-\beta(x)\},\ z_i\in Z,\ i=1,...,4,\\
\hat{Q}_n(\cdot,x) &= (2n+1)Q_n(\cdot,x)\ \text{ with } Q_n(\cdot,x)\ \text{ being as defined in (3.220)},\\
\theta_n(x) &= (-1)^n z_1^{n+1}+z_2^{n+1}+(-1)^n z_3^{n+1}+z_4^{n+1},
\end{aligned}
$$

and $R_k(x)$, $S_k(x)$ *are as given by (3.205).*

PROOF. Applying the premature Grüss result (3.252) by associating $f^{(n)}(t)$ with $g(t)$ and $h(t)$ with $K_n(x,t)$, from (3.204) gives

$$
\begin{aligned}
(3.255)\qquad &\left|(-1)^n\int_a^b K_n(x,t)f^{(n)}(t)\,dt\right.\\
&\qquad\left.-\left((-1)^n\int_a^b K_n(x,t)\,dt\right)\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right|\\
&\leq (b-a)\frac{\Gamma-\gamma}{2}\left[T(K_n,K_n)\right]^{\frac{1}{2}},
\end{aligned}
$$

where from (3.251)

$$
T(K_n,K_n)=\frac{1}{b-a}\int_a^b K_n^2(x,t)\,dt-\left(\frac{1}{b-a}\int_a^b K_n(x,t)\,dt\right)^2.
$$

Now, from (3.204),

$$\begin{aligned}(3.256)\qquad & \frac{1}{b-a}\int_a^b K_n(x,t)\,dt \\ &= \frac{1}{b-a}\left[\int_a^x \frac{(t-\alpha(x))^n}{n!}dt + \int_x^b \frac{(t-\beta(x))^n}{n!}dt\right] \\ &= \frac{1}{(b-a)(n+1)!}\Big[(x-\alpha(x))^{n+1} + (-1)^n(\alpha(x)-a)^{n+1} \\ &\quad (b-\beta(x))^{n+1} + (-1)^n(\beta(x)-x)^{n+1}\Big] \\ &:= \frac{1}{(b-a)(n+1)!}\theta_n(x)\end{aligned}$$

and

$$\begin{aligned}(3.257)\qquad & \frac{1}{b-a}\int_a^b K_n^2(x,t)\,dt \\ &= \frac{1}{(b-a)(n!)^2}\left[\int_a^x (t-\alpha(x))^{2n}\,dt + \int_x^b (t-\beta(x))^{2n}\,dt\right] \\ &= \frac{1}{(b-a)(n!)^2(2n+1)}\Big[(x-\alpha(x))^{2n+1} + (\alpha(x)-a)^{2n+1} \\ &\quad (b-\beta(x))^{2n+1} + (\beta(x)-x)^{2n+1}\Big] \\ &= \frac{1}{(b-a)(n!)^2(2n+1)}\hat{Q}_n(2,x)\end{aligned}$$

on using (3.220).

Hence, substitution of (3.256) and (3.257) into (3.255) gives

$$\begin{aligned}(3.258)\qquad & \left|\int_a^b K_n(x,t)\,f^{(n)}(t)\,dt - (-1)^n\frac{\theta_n(x)}{(n+1)!}\cdot\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right| \\ &\leq \frac{\Gamma-\gamma}{2}\cdot\frac{1}{n!}J(x,n),\end{aligned}$$

where

$$\begin{aligned}(3.259)\qquad & (2n+1)(n+1)^2 J^2(x,n) \\ &= (n+1)^2(b-a)\hat{Q}_n(2,x) - (2n+1)\theta_n^2(x).\end{aligned}$$

Now, let

$$(3.260)\qquad A=\alpha(x)-a,\ X=x-\alpha(x),\ Y=\beta(x)-x \text{ and } B=b-\beta(x),$$

then (3.256) and (3.257) imply that

$$\hat{Q}_n(2,x) = A^{2n+1}+X^{2n+1}+Y^{2n+1}+B^{2n+1}$$

and

$$\theta_n(x) = (-1)^n A^{n+1} + X^{n+1} + (-1)^n Y^{n+1} + B^{n+1}.$$

Hence, from (3.259) and using the fact that $b-a=A+X+Y+B$,

$$
\begin{aligned}
(3.261)\qquad & (n+1)^2(b-a)\hat{Q}_n(2,x)-(2n+1)\theta_n^2(x)\\
&= n^2\hat{Q}_n(2,x)+(2n+1)\left[(A+X+Y+B)Q_n(2,x)-\theta_n^2(x)\right]\\
&= n^2\hat{Q}_n(2,x)+(2n+1)\sum_{\substack{i=1\\ j>i}}^{4} z_i z_j\left[z_i^n-(-z_j)^n\right]^2
\end{aligned}
$$

after some straight forward algebra, where $Z=\{A,X,Y,B\}$, $z_i\in Z$, $i=1,...,4$.

Substitution of (3.261) into (3.259) gives $I(x,n)=\frac{J(x,n)}{(n+1)\sqrt{2n+1}}$ as presented by (3.254). Utilising identity (3.203) in (3.255) gives (3.253) and the first part of the theorem is proved.

The upper bound is obtained by taking $\alpha(\cdot)$, $\beta(\cdot)$, x at their end points since $I(x,n)$ is convex and symmetric. The second term for $I(x,n)$ is then zero and $\hat{Q}_n(2,x)<2(b-a)^{2n+1}$ and hence after some simplification, the theorem is completely proven. ∎

Corollary 3.71. Let the conditions of Theorem 3.68 hold. Then the following result is valid,

$$
\begin{aligned}
(3.262)\quad & \left|\int_a^b f(t)\,dt-\sum_{k=1}^{n}\frac{2^{-k}}{k!}\left[r_k(x)f^{(k-1)}(x)+s_k(x)\right]\right.\\
&\left.-2^{-n}\left[1+(-1)^n\right](A^n+B^n)\cdot\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right|\\
\leq\ & \frac{\Gamma-\gamma}{2}\cdot\frac{2^{-2(n+1)}}{n!}\left\{\left[4n^2+\left(1+(-1)^{n-1}\right)(2n+1)\right]\left[A^{2(n+1)}+B^{2(n+1)}\right]\right.\\
&\left.+\left[4n^2+2(2n+1)\right]AB\left(A^{2n}+B^{2n}\right)+4(2n+1)(-1)^{n-1}(A-B)^{n+1}\right\},
\end{aligned}
$$

where $r_m(x)$ and $s_m(x)$ are as given by (3.236) and $A=x-a$, $B=b-x$.

Proof. Let $\alpha(x)=\frac{a+x}{2}$ and $\beta(x)=\frac{x+b}{2}$ in (3.253), readily giving the left hand side of (3.262). Now, for the right hand side. Taking $A=x-a$, $B=b-x$, we have

$$\hat{Q}_n(2,x)=2^{-2n}\left[A^{2n+1}+B^{2n+1}\right]$$

and

$$
\begin{aligned}
\sum_{\substack{i=1\\ j>i}}^{4} z_i z_j\left[z_i^n-(-z_j)^n\right]^2 &= 2^{-2(n+1)}\left[A^{2(n+1)}+B^{2(n+1)}\right]\left(1+(-1)^{n-1}\right)\\
&\quad +2AB\left(A^n+(-1)^{n-1}B^n\right)^2
\end{aligned}
$$

so that from (3.254) and using the fact that $b-a=A+B$,

$$
\begin{aligned}
(3.263)\quad & (n+1)\sqrt{2n+1}I(x,n) \\
= & \ 2^{-2(n+1)}\Big\{4n^2(A+B)\left[A^{2n+1}+B^{2n+1}\right] \\
& +(2n+1)\Big[\left(A^{2(n+1)}+B^{2(n+1)}\right)\left(1+(-1)^{n-1}\right) \\
& +2AB\left(A^n+(-1)^{n-1}B^n\right)^2\Big]\Big\} \\
= & \ 2^{-2(n+1)}\left\{\left[4n^2+\left(1+(-1)^{n-1}\right)(2n+1)\right]\left[A^{2(n+1)}+B^{2(n+1)}\right]\right. \\
& \left.+\left[4n^2+2(2n+1)\right]AB\left(A^{2n}+B^{2n}\right)+4(2n+1)(-1)^{n-1}(AB)^{n+1}\right\}.
\end{aligned}
$$

A simple substitution in (3.253) of (3.263) completes the proof. ∎

Corollary 3.72. Let the conditions of Theorem 3.68 and Corollary 3.71 hold. Then the following inequality results,

$$
\begin{aligned}
(3.264)\quad & \left|\int_a^b f(t)\,dt-\sum_{k=1}^{n}\frac{2^{-k}}{k!}\left[r_k\left(\frac{a+b}{2}\right)f^{(k-1)}\left(\frac{a+b}{2}\right)+s_k\left(\frac{a+b}{2}\right)\right]\right. \\
& \left.-2\cdot 4^{-n}\left(1+(-1)^n\right)\left[f^{(n-1)}(b)-f^{(n-1)}(a)\right]\right| \\
\leq & \ \frac{\Gamma-\gamma}{n!}\left(\frac{b-a}{4}\right)^{2(n+1)}\left[8n^2+3(2n+1)\left(1+(-1)^{n-1}\right)\right],
\end{aligned}
$$

where $r_m\left(\frac{a+b}{2}\right)$ and $s_m\left(\frac{a+b}{2}\right)$ are as given in (3.239).

Proof. The proof follows directly from (3.262) with $x=\frac{a+b}{2}$ so that $A=B=\frac{b-a}{2}$, giving for the braces on the right hand side

$$
2\left(\frac{b-a}{4}\right)^{2(n+1)}\left[8n^2+3(2n+1)\left(1+(-1)^{n-1}\right)\right].
$$

Some straight forward simplification produces the result (3.264). ∎

Remark 3.79. It may be noticed (See also Remark 3.75) that only even order degrees are involved, in (3.264), at the midpoint while this is only the case at the endpoints if the further restriction $f^{(k-1)}(a)=f^{(k-1)}(b)$ is imposed. Further, if n is odd, then there is no perturbation arising from the Grüss type result (3.264).

Theorem 3.73. *Let the conditions of Theorem 3.68 be satisfied. Further, suppose that $f^{(n)}$ is absolutely continuous and is such that*

$$
\left\|f^{(n+1)}\right\|_\infty := ess\sup_{t\in[a,b]}\left|f^{(n+1)}(t)\right|<\infty.
$$

Then

$$
(3.265)\qquad |\rho_n(x)|\leq\frac{b-a}{\sqrt{12}}\left\|f^{(n+1)}\right\|_\infty\cdot\frac{1}{n!}I(x,n),
$$

where $\rho_n(x)$ is the perturbed interior point rule given by the left hand side of (3.253) and $I(x,n)$ is as given by (3.254).

Proof. Let $h, g : [a,b] \to \mathbb{R}$ be absolutely continuous and h', g' be bounded. Then Chebychev's inequality holds (see [**47**, p. 207])

$$|T(h,g)| \le \frac{(b-a)^2}{\sqrt{12}} \sup_{t\in[a,b]} |h'(t)| \cdot \sup_{t\in[a,b]} |g'(t)| .$$

Matić, Pečarić and Ujević [**41**] using a *premature* Grüss type argument proved that

$$|T(h,g)| \le \frac{(b-a)}{\sqrt{12}} \sup_{t\in[a,b]} |g'(t)| \sqrt{T(h,h)}. \tag{3.266}$$

Thus, associating $f^{(n)}(\cdot)$ with $g(\cdot)$ and $K(x,\cdot)$, from (3.204), with $h(\cdot)$ in (3.266) produces (3.265) where $I(x,n)$ is as given by (3.254). ∎

Theorem 3.74. *Let the conditions of Theorem 3.68 be satisfied. Further, suppose that $f^{(n)}$ is locally absolutely continuous on (a,b) and let $f^{(n+1)} \in L_2(a,b)$. Then*

$$|\rho_n(x)| \le \frac{b-a}{\pi} \left\| f^{(n+1)} \right\|_2 \cdot \frac{1}{n!} I(x,n) , \tag{3.267}$$

where $\rho_n(x)$ is the perturbed generalised interior point rule given by the left hand side of (3.253) and $I(x,n)$ is as given in (3.254).

Proof. The following result was obtained by Lupaş (see [**47**, p. 210]). For $h, g : (a,b) \to \mathbb{R}$ locally absolutely continuous on (a,b) and $h', g' \in L_2(a,b)$, then

$$|T(h,g)| \le \frac{(b-a)^2}{\pi^2} \|h'\|_2 \|g'\|_2 ,$$

where

$$\|k\|_2 := \left(\frac{1}{b-a} \int_a^b |k(t)|^2 \right)^{\frac{1}{2}} \quad \text{for} \quad k \in L_2(a,b) .$$

Matić, Pečarić and Ujević [**41**] further show that

$$|T(h,g)| \le \frac{b-a}{\pi} \|g'\|_2 \sqrt{T(h,h)}. \tag{3.268}$$

Now, associating $f^{(n)}(\cdot)$ with $g(\cdot)$ and $K(x,\cdot)$, from (3.204) with $h(\cdot)$ in (3.268) gives (3.267) where $I(x,n)$ is as found in (3.254). ∎

Remark 3.80. Results (3.265) and (3.267) are not readily comparable to that obtained in Theorem 3.68 since the bound now involves the behaviour of $f^{(n+1)}(\cdot)$ rather than $f^{(n)}(\cdot)$.

Remark 3.81. Premature results presented in this subsection may also be obtained, producing bounds for generalized Taylor-like series expansion by taking $f \equiv g'$ and $b = y$. See also Matić et al. [**41**] for related results.

3.3.6. Applications in Numerical Integration. Any of the inequalities in Subsections 3.3.3 and 3.3.4 may be utilised for numerical implementation. Here we illustrate the procedure by giving details for the implementation of Corollary 3.64.

Consider the partition $I_m : a = x_0 < x_1 < ... < x_{m-1} < x_m = b$ of the interval $[a,b]$ and let the intermediate points $\boldsymbol{\xi} = (\xi_0, ..., \xi_{m-1})$ where $\xi_j \in [x_j, x_{j+1}]$ for $j = 0, 1, ..., m-1$. Define the formula for $\gamma \in [0,1]$,

$$\begin{aligned} \mathcal{A}_{m,n}(f, I_m, \boldsymbol{\xi}) &= \sum_{j=0}^{m-1} \sum_{k=1}^{n} \frac{(-1)^k}{k!} \Big\{ (1-\gamma)^k r_k(\xi_j) f^{(k-1)}(\xi_j) \\ &\quad + \gamma^k \left[A_j^k f^{(k-1)}(x_j) + (-1)^{k-1} B_j^k f^{(k-1)}(x_{j+1}) \right] \Big\}, \end{aligned} \tag{3.269}$$

where

$$\begin{cases} r_k(\xi_j) = B_j^k + (-1)^{k-1} A_j^k \\ A_j = \xi_j - x_j, \quad B_j = x_{j+1} - \xi_j, \\ \text{and} \quad h_j = A_j + B_j = x_{j+1} - x_j \quad \text{for} \quad j = 0, 1, ..., m-1. \end{cases} \tag{3.270}$$

The following theorem holds involving (3.269).

Theorem 3.75. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and I_m be a partition of $[a,b]$ as described above. Then the following quadrature rule holds. Namely,*

$$\int_a^b f(x)\,dx = \mathcal{A}_{m,n}(f, I_m, \boldsymbol{\xi}) + \mathcal{R}_{m,n}(f, I_m, \boldsymbol{\xi}), \tag{3.271}$$

where $\mathcal{A}_{m,n}$ is as defined by (3.269)-(3.270) and the remainder $\mathcal{R}_{m,n}(f, I_m, \boldsymbol{\xi})$ satisfies the estimation

$$\begin{aligned} &|\mathcal{R}_{m,n}(f, I_m, \boldsymbol{\xi})| \\ &\le \begin{cases} \frac{\|f^{(n)}\|_\infty}{(n+1)!} H_1(\gamma) \sum_{j=0}^{m-1} \left(A_j^{n+1} + B_j^{n+1}\right), & \textit{for} \quad f^{(n)} \in L_\infty[a,b], \\ \frac{\|f^{(n)}\|_p}{n!} \frac{H_q(\gamma)}{(nq+1)^{\frac{1}{q}}} \left[\sum_{j=0}^{m-1} \left(A_j^{nq+1} + B_j^{nq+1} \right) \right]^{\frac{1}{q}}, & \textit{for} \quad f^{(n)} \in L_p[a,b], \\ & \qquad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{\|f^{(n)}\|_1}{n!} \left(\frac{1}{2} + \left|\gamma - \frac{1}{2}\right| \right)^n \times & \\ \quad \left[\frac{\nu(h)}{2} + \max_{j=0,...,m-1} \left| \xi_j - \frac{x_j + x_{j+1}}{2} \right| \right]^n, & \textit{for} \quad f^{(n)} \in L_1[a,b], \end{cases} \end{aligned} \tag{3.272}$$

where $H_q(\gamma)$ is given by (3.236), $\nu(h) = \max\{h_j | j = 0, ..., m-1\}$, and the rest of the terms are as given in (3.270).

Proof. Apply Corollary 3.64 on the interval $[x_j, x_{j+1}]$ to give

$$\left|\int_{x_j}^{x_{j+1}} f(t)\,dt-\sum_{k=1}^{n}\frac{(-1)^k}{k!}\left\{(1-\gamma)^k r_k(\xi_j) f^{(k-1)}(\xi_j)\right.\right.$$
$$\left.\left.+\gamma^k\left[A_j^k f^{(k-1)}(x_j)+(-1)^{k-1}B_j^k f^{(k-1)}(x_{j+1})\right]\right\}\right| \tag{3.273}$$
$$\leq \begin{cases} \frac{H_1(\gamma)}{(n+1)!}\sup\limits_{t\in[x_j,x_{j+1}]}\left|f^{(n)}(t)\right|\left(A_j^{n+1}+B_j^{n+1}\right), \\ \frac{H_q(\gamma)}{n!}\left[\int_{x_j}^{x_{j+1}}\left|f^{(n)}(u)\right|^p du\right]^{\frac{1}{p}}\left(\frac{A_j^{nq+1}+B_j^{nq+1}}{nq+1}\right)^{\frac{1}{q}}, \\ \frac{\left(\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|\right)^n}{n!}\left[\int_{x_j}^{x_{j+1}}\left|f^{(n)}(u)\right|du\right]\left(\frac{h_j}{2}+\left|\xi_j-\frac{x_j+x_{j+1}}{2}\right|\right)^n, \end{cases}$$

where the parameters are as defined in (3.270) and $H_q(\gamma)$ is as given in (3.236). Summing over j from 0 to $m-1$ and using the generalised triangle inequality gives

$$\left|\mathcal{R}_{m,n}(f,I_m,\boldsymbol{\xi})\right| \tag{3.274}$$
$$\leq \begin{cases} \frac{H_1(\gamma)}{(n+1)!}\sum\limits_{j=0}^{m-1}\sup\limits_{t\in[x_j,x_{j+1}]}\left|f^{(n)}(t)\right|\left(A_j^{n+1}+B_j^{n+1}\right), \\ \frac{H_q(\gamma)}{n!}\sum\limits_{j=0}^{m-1}\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}(u)\right|^p du\right)^{\frac{1}{p}}\left(\frac{A_j^{nq+1}+B_j^{nq+1}}{nq+1}\right)^{\frac{1}{q}}, \\ \frac{\left(\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|\right)^n}{n!}\sum\limits_{j=0}^{m-1}\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}(u)\right|du\right)\left(\frac{h_j}{2}+\left|\xi_j-\frac{x_j+x_{j+1}}{2}\right|\right)^n. \end{cases}$$

Now, since $\sup\limits_{t\in[x_j,x_{j+1}]}\left|f^{(n)}(t)\right| \leq \left\|f^{(n)}\right\|_\infty$, the first inequality in (3.272) readily follows.

Further, using the discrete Hölder inequality, we have

$$\sum_{j=0}^{m-1}\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}(u)\right|^p du\right)^{\frac{1}{p}}\left(\frac{A_j^{nq+1}+B_j^{nq+1}}{nq+1}\right)^{\frac{1}{q}}$$
$$\leq \left(\frac{1}{nq+1}\right)^{\frac{1}{q}}\left[\sum_{j=0}^{m-1}\left[\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}(u)\right|^p du\right)^{\frac{1}{p}}\right]^p\right]^{\frac{1}{p}}$$
$$\times\left[\sum_{j=0}^{m-1}\left[\left(A_j^{nq+1}+B_j^{nq+1}\right)^{\frac{1}{q}}\right]^q\right]^{\frac{1}{q}}$$
$$= \frac{\left\|f^{(n)}\right\|_p}{(nq+1)^{\frac{1}{q}}}\left[\sum_{j=0}^{m-1}\left(A_j^{nq+1}+B_j^{nq+1}\right)\right]^{\frac{1}{q}}$$

and thus the second inequality in (3.272) is proven.

Finally, let us observe from (3.273) that

$$\begin{aligned}
&\sum_{j=0}^{m-1}\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}(u)\right|du\right)\left(\frac{h_j}{2}+\left|\xi_j-\frac{x_j+x_{j+1}}{2}\right|\right)^n \\
\leq\;& \max_{j=0,\ldots,m-1}\left(\frac{h_j}{2}+\left|\xi_j-\frac{x_j+x_{j+1}}{2}\right|\right)^n\sum_{j=0}^{m-1}\int_{x_j}^{x_{j+1}}\left|f^{(n)}(u)\right|du \\
\leq\;& \left(\frac{\nu(h)}{2}+\max_{j=0,\ldots,m-1}\left|\xi_j-\frac{x_j+x_{j+1}}{2}\right|\right)^n\left\|f^{(n)}\right\|_1.
\end{aligned}$$

Hence, the theorem is completely proved. ∎

Remark 3.82. Following the discussion in Remark 3.72, coarser upper bounds to those in (3.272) are obtained by taking ξ_j at either extremity of its interval, giving

$$\begin{aligned}
&\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}H_1(\gamma)\sum_{j=0}^{m-1}h_j^{n+1},\quad \frac{\left\|f^{(n)}\right\|_p}{n!}\frac{H_q(\gamma)}{(nq+1)^{\frac{1}{q}}}\left(\sum_{j=0}^{m-1}h_j^{nq+1}\right)^{\frac{1}{q}}, \\
&\frac{\left\|f^{(n)}\right\|_1}{n!}\nu^n(h)
\end{aligned}$$

for $f^{(n)}$ belonging to the obvious $L_p[a,b]$, $1\leq p\leq\infty$. These are uniform bounds relative to the intermediate points $\boldsymbol{\xi}$.

Corollary 3.76. Let the conditions of Theorem 3.75 hold. Then we have

$$\int_a^b f(x)\,dx=\mathcal{A}_{m,n}(f,I_m)+\mathcal{R}_{m,n}(f,I_m),$$

where

$$\begin{aligned}
\mathcal{A}_{m,n}(f,I_m) \;=\;& \sum_{j=0}^{m-1}\sum_{k=1}^{n}\frac{(-1)^k}{k!}\Big\{(1-\gamma)^k r_k(\delta_j)\,f^{(k-1)}(\delta_j) \\
&+\gamma^k h_j^k\left[f^{(k-1)}(x_j)+(-1)^{k-1}f^{(k-1)}(x_{j+1})\right]\Big\},
\end{aligned}$$

with

$$\delta_j=\frac{x_j+x_{j+1}}{2}\quad\text{and}\quad r_k(\delta_j)=\frac{h_j^k}{2}\left(1+(-1)^{k-1}\right)$$

and the remainder $\mathcal{R}_{m,n}(f,I_m)$ satisfies the inequality

$$
\begin{aligned}
&|\mathcal{R}_{m,n}(f,I_m)| \\
&\leq \begin{cases}
\frac{\|f^{(n)}\|_\infty}{(n+1)!} 2\sum_{j=0}^{m-1}\left(\frac{h_j}{2}\right)^k, & \text{for } f^{(n)} \in L_\infty[a,b], \\[2ex]
\frac{\|f^{(n)}\|_p}{n!}\frac{H_q(\gamma)}{(nq+1)^{\frac{1}{q}}}\left[\sum_{j=0}^{m-1} 2\left(\frac{h_j}{2}\right)^{nq+1}\right]^{\frac{1}{q}}, & \text{for } f^{(n)} \in L_p[a,b] \\
 & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\
\frac{\|f^{(n)}\|_1}{n!}\left(\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|\right)^n\left(\frac{\nu(h)}{2}\right)^n, & \text{for } f^{(n)} \in L_1[a,b].
\end{cases}
\end{aligned}
$$

Proof. The proof is trivial from Theorem 3.75. Taking $\xi_j = \frac{x_j+x_{j+1}}{2}$ gives $A_j = B_j = \frac{h_j}{2}$ and the results stated follow. ∎

3.3.7. Concluding Remarks. Taking $\gamma = 0$ in Corollary 3.60 gives a generalised Ostrowski type identity which has bounds given by Corollary 3.64 with $\gamma = 0$, reproducing the results of Cerone and Dragomir [**3**]. This gives a coarse upper bound as discussed in Remark 3.72 since the bound is convex and symmetric in both γ and x. Let the identity be denoted by $M_n(x)$ which is produced from taking a Peano kernel of

$$
k_M(x,t) = \begin{cases} \frac{(t-a)^n}{n!}, & t \in [a,x] \\[1ex] \frac{(t-b)^n}{n!}, & t \in (x,b]. \end{cases} \tag{3.275}
$$

Further, taking $\gamma = 1$ in Corollary 3.60 gives a generalised Trapezoidal type identity with bounds given by Corollary 3.64 with $\gamma = 1$ reproducing the results of Cerone and Dragomir [**6**]. This choice of γ again gives the coarsest bound as discussed in Remark 3.72. Let the resulting identity be denoted by $T_n(x)$, which results from taking a Peano kernel of

$$
k_T(x,t) = \frac{(x-t)^n}{n!}. \tag{3.276}
$$

It was shown in Cerone and Dragomir [**3**] that $\|k_M(x,t)\|_q = \|k_T(x,t)\|_q$. Let

$$
I_L(x) = \lambda M_n(x) + (1-\lambda) T_n(x)
$$

which is obtained from the kernel

$$
k(x,t) = \lambda k_M(x,t) + (1-\lambda) k_T(x,t)
$$

where $k_M(x,t)$ and $k_T(x,t)$ are given by (3.275) and (3.276) respectively. The best one can do for $q > 1$, $q \neq 2$ with such a kernel when determining bounds is to use the triangle inequality and so

$$
\begin{aligned}
\|k(x,t)\|_q &\leq \lambda\|k_M(x,t)\|_q + (1-\lambda)\|k_T(x,t)\|_q \\
&= \|k_M(x,t)\|_q = \|k_T(x,t)\|_q.
\end{aligned} \tag{3.277}
$$

The results thus obtained would be given by, for $f^{(n)} \in L_p[a,b]$, $p \geq 1$,

$$\left| \int_a^b f(t)\,dt - \lambda \sum_{k=1}^{n} (-1)^k r_k(x) f^{(k-1)}(x) - (1-\lambda) \sum_{k=1}^{n} (-1)^k s_k(x) \right|$$

$$\leq \begin{cases} \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} G_1(x), & \text{for } f^{(n)} \in L_\infty[a,b], \\ \\ \frac{\left\|f^{(n)}\right\|_p}{n!(nq+1)^{\frac{1}{q}}} G_q(x), & \text{for } f^{(n)} \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{\left\|f^{(n)}\right\|_1}{n!} \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right], & \text{for } f^{(n)} \in L_1[a,b], \end{cases}$$

where $r_k(x)$, $s_k(x)$, $G_q(x)$ are given by (3.236) and it should be noted that the bound is independent of λ. This result would be no more difficult to implement than (3.235) in Corollary 3.64, with the best bounds resulting from $\gamma = \frac{1}{2}$. For $q = 1, 2$ or infinity, $\|k(x,t)\|_q$ may be evaluated explicitly without using the triangle inequality at which stage comparison with the results of Corollary 3.64 would be less conclusive. This will not be discussed further here.

In the application of the current work to quadrature, if we wished to approximate the integral $\int_a^b f(x)\,dx$ using a rule $Q(f, I_m)$ with bound $E(m)$, where I_m is a uniform partition for example, with an accuracy of $\varepsilon > 0$, then we require $m_\varepsilon \in \mathbb{N}$ where

$$m_\varepsilon \geq \left[E^{-1}(\varepsilon)\right] + 1,$$

with $[w]$ denoting the integer part of $w \in \mathbb{R}$.

The approach thus described enables the user to predetermine the partition required to *assure* the result to be within a certain error tolerance. This approach is somewhat different from that commonly used of systematic mesh refinement followed by a comparison of successive approximations which forms the basis of a stopping rule. See [**1**], [**37**] and [**39**] for a comprehensive treatment of traditional methods.

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CHAPTER 4

Product Branches of Peano Kernels and Numerical Integration

by

P. CERONE

ABSTRACT Product branches of Peano kernels are used to obtain results suitable for numerical integration. In particular, identities and inequalities are obtained involving evaluations at an interior and at the end points. It is shown how previous work and rules in numerical integration are recaptured as particular instances of the current development. Explicit *a priori* bounds are provided allowing the determination of the partition required for achieving a prescribed error tolerance. In the main, Ostrowski-Grüss type inequalities are used to obtain bounds on the rules in terms of a variety of norms.

4.1. Introduction

Recently, Cerone and Dragomir [**7**] obtained the following identity involving $n-$time differentiable functions with evaluation at an interior point and at the end points.

For $f:[a,b]\to\mathbb{R}$ a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ with $\alpha:[a,b]\to\mathbb{R}$, $\beta:[a,b]\to\mathbb{R}$, $\alpha(x)\le x$ and $\beta(x)\ge x$, then for all $x\in[a,b]$ the following identity holds

$$(-1)^n\int_a^b \tilde{K}_n(x,t)\,f^{(n)}(t)\,dt = \int_a^b f(t)\,dt-\sum_{k=1}^{n}\frac{1}{k!}\left[R_k(x)\,f^{(k-1)}(x)+S_k(x)\right], \tag{4.1}$$

where the kernel $\tilde{K}_n:[a,b]^2\to\mathbb{R}$ is given by

$$\tilde{K}_n(x,t):=\begin{cases}\frac{(t-\alpha(x))^n}{n!}, & t\in[a,x]\\ \frac{(t-\beta(x))^n}{n!}, & t\in(x,b],\end{cases} \tag{4.2}$$

$$(4.3)\quad \begin{cases} & R_k(x) = (\beta(x) - x)^k + (-1)^{k-1}(x - \alpha(x))^k \\ \text{and} & \\ & S_k(x) = (\alpha(x) - a)^k f^{(k-1)}(a) + (-1)^{k-1}(b - \beta(x))^k f^{(k-1)}(b) \end{cases}.$$

They obtained inequalities for $f^{(n)} \in L_p[a,b]$, $p \geq 1$. In an earlier paper [**6**] the same authors treated the case $n = 1$ but also examined the results eminating from the Riemann-Stieltjes integral $\int_a^b K_1(x,t)\,df(t)$ and obtained bounds for f being of bounded variation, Lipschitzian or monotonic. Applications to numerical quadrature were investigated covering rules of Newton-Cotes type containing the evaluation of the function at three possible points: the interior and extremities. The development included the midpoint, trapezoidal and Simpson type rules. However, unlike the classical rules, the results were not as restrictive in that the bounds were derived in terms of the behaviour of at most the first derivative and the Peano kernel $K_1(x,t)$. Perturbed rules were also obtained using Grüss type inequalities.

In 1938, Ostrowski (see for example [**16**, p. 468]) proved the following integral inequality:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$ ($\mathring{I}$ is the interior of I), and let $a, b \in \mathring{I}$ with $a < b$. If $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have the inequality:

$$(4.4)\qquad \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty$$

for all $x \in [a,b]$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

For applications of Ostrowski's inequality and its companions to some special means and some numerical quadrature rules, we refer the reader to the recent paper [**13**] by S.S. Dragomir and S. Wang.

Fink [**14**] used the integral remainder from a Taylor series expansion to show that for $f^{(n-1)}$ absolutely continuous on $[a,b]$, then the identity

$$(4.5)\qquad \int_a^b f(t)\,dt = \frac{1}{n}\left((b-a) f(x) + \sum_{k=1}^{n-1} F_k(x) \right) + \int_a^b K_F(x,t) f^{(n)}(t)\,dt$$

is shown to hold where

$$(4.6)\qquad K_F(x,t) = \frac{(x-t)^{n-1}}{(n-1)!} \cdot \frac{p(x,t)}{n}$$

with $p(x,t)$ being given by

$$p(x,t) = \begin{cases} t - a, & t \in [a,x] \\ t - b, & t \in (x,b]. \end{cases}$$

and

$$F_k(x) = \frac{n-k}{k!}\left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b)\right].$$

Fink then proceeds to obtain a variety of bounds from (4.5), (4.6) for $f^{(n)} \in L_p[a,b]$. It may be noticed that (4.5) is again an identity that involves function evaluations at three points to approximate the integral from the resulting inequalities. See Mitrinović, Pečarić and Fink [**16**, Chapter XV] for further related results.

In the current article a split Peano kernel is utilised in which the branches are products of two Appell-like polynomials, unlike Pearce et al. [**17**] who assumed that each of the branches themselves were Appell-like polynomials. We shall focus, in this chapter, on the Peano kernel with only two branches giving rise to rules involving evaluation at three points. Thus, the branches of the Peano kernel are of product form. The general result then allows for great flexibility in deriving integration rules which involve evaluation at an interior point and two end points. Bounds are obtained in terms of the $L_p[a,b]$ norms and thus allowing, in advance, for the determination of a partition required to achieve a given error tolerance. Simpson type formulae are obtained in Section 4.3 and perturbed rules using Grüss type inequalities involving the Chebychev functional are investigated in Section 4.4.

More perturbed results are obtained in Section 4.5 where the idea of bounds in terms of $\Delta-$seminorms is utilised. These can in turn be bounded in terms of Lebesgue norms by assuming stronger conditions on the function and/or the Peano kernel. It should be noted that Pearce et al. [**17**] call the Appell-like polynomials by the term harmonic polynomials.

4.2. Fundamental Results

The following lemma is paramount to the development of the subsequent work (see also Cerone [**3**]).

Lemma 4.1. Let r_k, s_k, u_k, $v_k \in \mathcal{H}$ for $k \in \mathbb{N}$ be sequences of polynomials which are such that $w_k \in \mathcal{H}$ if

$$w_k'(t) = w_{k-1}(t), \quad w_0(t) = 1, \quad t \in \mathbb{R}. \tag{4.7}$$

Further, define $K_n(x,t)$, $p_n(t)$ and $q_n(t)$ by

$$K_n(x,t) = \begin{cases} p_n(t) = r_{n-m}(t)\, s_m(t), & t \in [a,x] \\ \\ q_n(t) = u_{n-m}(t)\, v_m(t), & t \in (x,b]. \end{cases} \tag{4.8}$$

Then for $f : [a,b] \to \mathbb{R}$ and $f^{(n-1)}$ absolutely continuous on $[a,b]$, the identity

$$\begin{aligned}(4.9)\qquad & (-1)^n \int_a^b K_n(x,t)\, f^{(n)}(t)\, dt \\ = \; & \binom{n}{m}\int_a^b f(t)\, dt + \sum_{k=0}^{n-1}(-1)^k \left\{\left[p_n^{(k)}(x) - q_n^{(k)}(x)\right] f^{(n-1-k)}(x)\right. \\ & \left. + q_n^{(k)}(b)\, f^{(n-1-k)}(b) - p_n^{(k)}(a)\, f^{(n-1-k)}(a)\right\}\end{aligned}$$

holds, where

$$\left\{\begin{array}{l} p_n^{(k)}(\cdot) = \sum\limits_{j=L}^{U} \binom{k}{j} r_{n-m-j}(\cdot)\, s_{m-k+j}(\cdot), \\ \\ q_n^{(k)}(\cdot) = \sum\limits_{j=L}^{U} \binom{k}{j} u_{n-m-j}(\cdot)\, v_{m-k+j}(\cdot) \end{array}\right. \tag{4.10}$$

with

$$U = \min\{k, n-m\}, \ L = \max\{0, k-m\}. \tag{4.11}$$

Proof. Let

$$J_n(a,x,b) = \int_a^b K_n(x,t) f^{(n)}(t)\, dt. \tag{4.12}$$

Then integration by parts from utilising (4.8) gives

$$J_n(a,x,x) = \sum_{k=0}^{n-1} (-1)^{n+k} p_n^{(k)}(t) f^{(n-1-k)}(t)\Big|_a^x + (-1)^n \int_a^x p_n^{(n)}(t) f(t)\, dt. \tag{4.13}$$

Now, using the Leibniz rule for differentiation of a product gives from (4.8)

$$p_n^{(k)}(t) = \sum_{j=0}^{k} \binom{k}{j} \frac{d^j r_{n-m}(t)}{dt^j} \frac{d^{k-j} s_m(t)}{dt^{k-j}}.$$

It should be noted from (4.7) that

$$w_k^{(j)}(t) = \begin{cases} w_{k-j}(t), & j \le k \\ \\ 0, & j > k \end{cases} \tag{4.14}$$

and so from (4.13), since $r_{n-m}(t)$ and $s_m(t)$ are polynomials satisfying (4.7) and using (4.14) gives

$$p_n^{(k)}(t) = \sum_{j=\max\{0,k-m\}}^{\min\{k,n-m\}} \binom{k}{j} r_{n-m-j}(t)\, s_{m-k+j}(t). \tag{4.15}$$

Similarly, from (4.12)

$$\begin{aligned} J_n(x,x,b) \quad &= \quad \sum_{k=0}^{n-1} (-1)^{n+k} q_n^{(k)}(t) f^{(n-1-k)}(t)\Big|_x^b \\ &\quad + (-1)^n \int_x^b q_n^{(n)}(t) f(t)\, dt, \end{aligned} \tag{4.16}$$

where

$$q_n^{(k)}(t) = \sum_{j=\max\{0,k-m\}}^{\min\{k,n-m\}} \binom{k}{j} u_{n-m-j}(t)\, v_{m-k+j}(t). \tag{4.17}$$

Further, from (4.15) and (4.17) we deduce that $j = n-m$ since the subscripts of u and v are nonnegative and thus

$$p_n^{(n)}(t) = q_n^{(n)}(t) = \binom{n}{n-m} = \binom{n}{m}. \tag{4.18}$$

Combining (4.12), (4.13) and (4.16) on using (4.18) readily produces the identity (4.9) where (4.10) is obtained from (4.15) and (4.17). ∎

Remark 4.1. If we allow $w_k \in \mathcal{P}$ provided

$$w_k'(t) = \xi_k w_{k-1}(t), \ w_0(t) = 1, \ t \in \mathbb{R}, \tag{4.19}$$

then an appropriately modified Lemma 4.1 would still hold.
When $\xi_k = k$, then such functions satisfying (4.19) were defined by Appell in 1880, [**1**] and are now known as Appell polynomials (see, for example, [**10**] for an extensive treatment of related results). For $\xi_k \equiv 1$, that is satisfying (4.7), Pearce et al. [**17**] call them harmonic polynomials. Polynomials satisfying (4.19) will be termed Appell-like polynomials.

Theorem 4.2. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. The following inequalities hold for $f^{(n)} \in L_p[a,b]$, $p \geq 1$ and for all $x \in [a,b]$,*

$$\begin{aligned} |\tau_n(x)| &:= \Bigg| \binom{n}{m} \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} (-1)^{k+1} \Big\{ \Big[p_n^{(k)}(x) \\ &\quad - q_n^{(k)}(x) \Big] f^{(n-1-k)}(x) + q_n^{(k)}(b) f^{(n-1-k)}(b) - p_n^{(k)}(a) f^{(n-1-k)}(a) \Big\} \Bigg| \\ &\leq \begin{cases} \left\| f^{(n)} \right\|_\infty B_n(1,x) & \text{for} \quad f^{(n)} \in L_\infty[a,b]; \\ \left\| f^{(n)} \right\|_p [B_n(q,x)]^{\frac{1}{q}} & \text{for} \quad f^{(n)} \in L_p[a,b], \\ & \text{with} \quad p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \left\| f^{(n)} \right\|_1 \theta_n(x) & \text{for} \quad f^{(n)} \in L_1[a,b], \end{cases} \end{aligned} \tag{4.20}$$

where

$$B_n(q,x) = \int_a^x |p_n(t)|^q\,dt + \int_x^b |q_n(t)|^q\,dt \tag{4.21}$$

$$\theta_n(x) = \frac{M_n(a,x) + M_n(x,b)}{2} + \frac{|M_n(x,b) - M_n(a,x)|}{2}, \tag{4.22}$$

with $M_n(a,x) = \sup_{t \in [a,x]} |p_n(t)|, \ M_n(x,b) = \sup_{t \in (x,b]} |q_n(t)|,$

$p_n(t), \ q_n(t)$ *are defined by (4.8),*

$p_n^{(k)}(t), \ q_n^{(k)}(t)$ *are as given by (4.10),*
and the Lebesgue norms are defined by

$$\begin{aligned} \left\| f^{(n)} \right\|_\infty &:= ess \sup_{t \in [a,b]} \left| f^{(n)}(t) \right| < \infty, \\ \left\| f^{(n)} \right\|_p &:= \left(\int_a^b \left| f^{(n)}(t) \right|^p dt \right)^{\frac{1}{p}}, \ p \geq 1. \end{aligned}$$

Proof. Taking the modulus of (4.9) gives

$$|\tau_n(x)| = |J_n(a,x,b)|, \tag{4.23}$$

where $|\tau_n(x)|$ is as defined by the left hand side of (4.20) and $J_n(a,x,b)$ by (4.12).

Now, observe that, from (4.12),

$$|J_n(a,x,b)| \le \left\| f^{(n)} \right\|_\infty \int_a^b |K_n(x,t)|\, dt, \tag{4.24}$$

where from (4.8)

$$\int_a^b |K_n(x,t)|\, dt = \int_a^x |p_n(t)|\, dt + \int_x^b |q_n(t)|\, dt. \tag{4.25}$$

Thus, combining (4.23), (4.24) and (4.25) gives the first inequality in (4.20).

Further, using Hölder's integral inequality, we have the result, from (4.12),

$$|J_n(a,x,b)| \le \left\| f^{(n)} \right\|_p \left(\int_a^b |K_n(x,t)|^q\, dt \right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p > 1,$$

which, upon using (4.8) produces the second inequality in (4.20) from (4.23).

Finally, observe that

$$|J_n(a,x,b)| \le \left\| f^{(n)} \right\|_1 \sup_{t\in[a,b]} |K_n(x,t)|,$$

where, from (4.12)

$$\sup_{t\in[a,b]} |K_n(x,t)| = \max \left\{ \sup_{t\in[a,x]} |p_n(t)|, \sup_{t\in(x,b]} |q_n(t)| \right\}$$

and so using the well known identity

$$\max\{X,Y\} = \frac{X+Y}{2} + \frac{|Y-X|}{2} \tag{4.26}$$

produces, from (4.23), the third identity and the theorem is completely proved. ∎

Remark 4.2. Lemma 4.1 recaptures an identity obtained by Pearce et al. [**18**] if both $p_n(t)$ and $q_n(t)$ satisfy (4.7). However, here $p_n(t)$ and $q_n(t)$ do not in themselves satisfy (4.7) but are the product of two polynomials that do.

Although in some sense the following result is a specialisation of Theorem 4.2, it is deemed to be of such importance, recapturing a number of past works as particular cases, that it is itself referred to as a Theorem in its own right.

Theorem 4.3. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. The following inequalities hold for $f^{(n)} \in L_p[a,b]$, $p \ge 1$ and* *for* $\alpha, x, \beta \in [a,b]$ *with* $a \leq \alpha,\ x,\ \beta \leq b$,

$$(4.27)\quad \left|\tau_n^C(x)\right| := \left|\int_a^b f(t)\,dt - \frac{1}{\binom{n}{m}} \sum_{k=0}^{n-1} (-1)^{k+1} \left\{ \left[p_n^{(k)}(x) - q_n^{(k)}(x) \right] f^{(n-1-k)}(x) + q_n^{(k)}(b) f^{(n-1-k)}(b) - p_n^{(k)}(a) f^{(n-1-k)}(a) \right\} \right|$$

$$\leq \begin{cases} \frac{\|f^{(n)}\|_\infty}{n!} \tilde{B}_n^C(1,x) & \text{for} \quad f^{(n)} \in L_\infty[a,b]; \\ \frac{\|f^{(n)}\|_p}{n!} \left[\tilde{B}_n^C(q,x)\right]^{\frac{1}{q}} & \text{for} \quad f^{(n)} \in L_p[a,b], \\ & \text{with} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f^{(n)}\|_1}{n!} \tilde{\theta}_n^C(x) & \text{for} \quad f^{(n)} \in L_1[a,b], \end{cases}$$

where

$$(4.28)\quad \tilde{B}_n^C(q,x) = \begin{cases} (\alpha-a)^{nq+1} B((n-m)q+1, mq+1) \\ +(x-\alpha)^{nq+1} \Psi\left(mq, (n-m)q; \frac{\alpha-a}{x-\alpha}\right) \\ +(\beta-x)^{nq+1} \Psi\left(mq, (n-m)q; \frac{b-\beta}{\beta-x}\right) \\ +(b-\beta)^{nq+1} B(mq+1, (n-m)q+1), \quad a \leq \alpha < x < \beta \leq b \\ \\ (x-a)^{nq+1} \chi\left((n-m)q, mq, \frac{\alpha-a}{x-a}\right) \\ +(b-x)^{nq+1} \chi\left((n-m)q, mq, \frac{b-\beta}{\beta-x}\right), \qquad a \leq \beta < x < \alpha \leq b \end{cases}$$

$$(4.29)\quad \tilde{\theta}_n^C(x) = \begin{cases} \max\left\{ A^{n-m}\left[\frac{A}{2} + \left|C - \frac{A}{2}\right|\right]^m, B^{n-m}\left[\frac{B}{2} + \left|D - \frac{B}{2}\right|\right]^m \right\}, \\ a \leq \alpha < x < \beta \leq b \\ \\ \max\left\{A^{n-m}(A-C)^m, B^{n-m}(B-D)^m\right\}, \quad a \leq \beta \leq x \leq \alpha \leq b \end{cases}$$

$$(4.30)\quad \begin{cases} p_n^{(k)}(t) = \sum_{j=L}^{U} \binom{k}{j} \frac{(t-a)^{n-m-j}}{(n-m-j)!} \cdot \frac{(t-\alpha)^{m-k+j}}{(m-k+j)!} \\ \text{and} \quad q_n^{(k)}(t) = \sum_{j=L}^{U} \binom{k}{j} \frac{(t-b)^{n-m-j}}{(n-m-j)!} \cdot \frac{(t-\beta)^{m-k+j}}{(m-k+j)!} \end{cases}$$

with U *and* L *as given by (4.11),*

$$(4.31)\quad \begin{cases} \Psi(j,k;X) = \int_0^1 u^j (X+u)^k\,du, \\ \\ \chi(j,k;X) = \int_0^1 u^j (X-u)^k\,du, \\ \\ B(j,k) = \chi(j-1,k-1;1) \ \text{is the Euler beta function,} \end{cases}$$

and

$$(4.32)\quad A = x-a,\ \ C = x-\alpha,\ \ B = b-x,\ \ D = \beta - x.$$

Proof. Let, for $n, m \in \mathbb{N}$ and $m \leq n$

$$\begin{array}{ll} & r_{n-m}^{C}(t)=\frac{(t-a)^{n-m}}{(n-m)!}, \quad s_{m}^{C}(t)=\frac{(t-\alpha)^{m}}{m!} \\ \text{and} & u_{n-m}^{C}(t)=\frac{(t-b)^{n-m}}{(n-m)!}, \quad v_{m}^{C}(t)=\frac{(t-\beta)^{m}}{m!}, \end{array} \tag{4.33}$$

where $a \leq \alpha,\ x,\ \beta \leq b$ and then, from (4.8), and using the superscript C to identify the above polynomials, which satisfy (4.7), giving

$$p_{n}^{C}(t)=\frac{(t-a)^{n-m}}{(n-m)!} \cdot \frac{(t-\alpha)^{m}}{m!} \text{ and } q_{n}^{C}(t)=\frac{(t-b)^{n-m}}{(n-m)!} \cdot \frac{(t-\beta)^{m}}{m!}. \tag{4.34}$$

Now, from (4.21) and (4.34)

$$\begin{aligned} \frac{B_{n}^{C}(q, x)}{\binom{n}{m}} &= \frac{1}{(n!)^{q}}\left\{\int_{a}^{x}(t-a)^{(n-m) q}|t-\alpha|^{m q} d t\right. \\ & \quad \left.+(b-t)^{(n-m) q}|t-\beta|^{m q} d t\right\} \\ : &= \frac{\tilde{B}_{n}^{C}(q, x)}{(n!)^{q}}. \end{aligned} \tag{4.35}$$

Now, for $\alpha \in [a, x)$ and $\beta \in (x, b]$, then from (4.35)

$$\begin{aligned} \tilde{B}_{n}^{C}(q, x) &= \int_{a}^{\alpha}(t-a)^{(n-m) q}(\alpha-t)^{m q} d t+\int_{\alpha}^{x}(t-a)^{(n-m) q}(t-\alpha)^{m q} d t \\ & \quad +\int_{x}^{\beta}(b-t)^{(n-m) q}(\beta-t)^{m q} d t+\int_{\beta}^{b}(b-t)^{(n-m) q}(t-\beta)^{m q} d t \end{aligned}$$

which upon substitution $u=\frac{t-a}{\alpha-a}, \frac{t-a}{x-\alpha}, \frac{b-t}{\beta-x}$ and $\frac{t-\beta}{b-\beta}$ respectively, produces the first part of (4.28) in terms of (4.31).

Similarly, for $\alpha \in [x, b]$ and $\beta \in [a, x]$ then

$$\tilde{B}_{n}^{C}(q, x)=\int_{a}^{x}(t-a)^{(n-m) q}(\alpha-t)^{m q} d t+\int_{x}^{b}(b-t)^{(n-m) q}(t-\beta)^{m q} d t$$

which on substituting $u=\frac{t-a}{x-a}$ and $\frac{b-t}{b-x}$ respectively gives the second result in (4.28).

Further, from (4.22) and (4.31)

$$\begin{aligned} & \frac{\theta_{n}^{C}(x)}{\binom{n}{m}} \\ & = \frac{1}{n!} \max \left\{\sup_{t \in[a, x]}(t-a)^{n-m}|t-\alpha|^{m}, \sup_{t \in[x, b]}(b-t)^{n-m}|t-\beta|^{m}\right\} \\ & := \frac{\tilde{\theta}_{n}^{C}(x)}{n!}:=\frac{1}{n!} \max \left\{\tilde{M}_{n}^{C}(a, x), \tilde{M}_{n}^{C}(x, b)\right\}. \end{aligned} \tag{4.36}$$

Now, from (4.36)

$$\begin{aligned}\tilde{M}_n^C(a,x) &= (x-a)^{n-m}\sup_{t\in[a,x]}|t-\alpha|^m\\ &= (x-a)^{n-m}\begin{cases}[\max\{\alpha-a,x-\alpha\}]^m, & \alpha\in[a,x],\\ (\alpha-a)^m, & \alpha\in[x,b]\end{cases}\end{aligned}$$

and

$$\begin{aligned}\tilde{M}_n^C(x,b) &= (b-x)^{n-m}\sup_{t\in[x,b]}|t-\beta|^m\\ &= (b-x)^{n-m}\begin{cases}[\max\{\beta-x,b-\beta\}]^m, & \beta\in[x,b],\\ (b-\beta)^m, & \beta\in[a,x]\end{cases}\end{aligned}$$

which upon using (4.26) produces (4.29), where A,B,C,D are as defined in (4.32) and the theorem is proved. ∎

Remark 4.3. The results of Theorem 4.3 are quite general, giving previously reported results as special cases. It is perhaps best to examine the Peano kernel (4.8) where $p_n(t)$ and $q_n(t)$ are as given by (4.34), where the superscript C is used to denote that the kernel involves coupled kernels resulting in function evaluation of at most three points. For $m=0$ the results of Cerone et al. [**8**] involving $n-$time differentiable Ostrowski results are obtained. Taking $m=n$ reproduces the results of Cerone and Dragomir [**7**] involving three point results for $n-$time differentiable functions consisting of evaluations at the end points and an interior point. If $m=n$ and $\alpha=\beta=x$, then the generalised trapezoidal results for the $n-$time differentiable results of Cerone et al. [**9**] are recovered. If we take $m=n-1$ and $\alpha=\beta=x$, then the results of Fink [**14**] are obtained within this quite general framework.

The following corollary gives a more elegant representation of the results through a parametrisation of the intervals under consideration through a representation of α and β as a convex combination of the end points.

Corollary 4.4. Let the conditions of Theorem 4.3 hold. Then

$$\left|\tau_n^{C^*}(x)\right| := \left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}(-1)^{k+1}d_n^{(k)}(x)f^{(n-1-k)}(x) - \sum_{k=n-m}^{n-1}(-1)^{k+1}\left[Q_n^{(k)}(b)f^{(n-1-k)}(b)-P_n^{(k)}(a)f^{(n-1-k)}(a)\right]\right| \tag{4.37}$$

$$\le\begin{cases}\frac{B_n^{C^*}(1,x)}{n!}\left\|f^{(n)}\right\|_\infty, & \text{for } f^{(n)}\in L_\infty[a,b];\\ \frac{\left[B_n^{C^*}(q,x)\right]^{\frac{1}{q}}}{n!}\left\|f^{(n)}\right\|_p, & \text{for } f^{(n)}\in L_p[a,b],\\ & \text{with } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{\theta_n^{C^*}(x)}{n!}\left\|f^{(n)}\right\|_1, & \text{for } f^{(n)}\in L_1[a,b],\end{cases}$$

where

$$B_n^{C^*}(q,x) \tag{4.38}$$
$$= \left[A^{nq+1} + B^{nq+1}\right] \begin{cases} \lambda^{nq+1} B\left((n-m)q+1, mq+1\right) + \\ (1-\lambda)^{nq+1} \Psi\left(mq, (n-m)q, \frac{\lambda}{1-\lambda}\right), & 0 \le \lambda < 1; \\ \\ \chi\left((n-m)q, mq, \lambda\right), & \lambda \ge 1, \end{cases}$$

$$\theta_n^{C^*}(x) \tag{4.39}$$
$$= \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^n \begin{cases} \left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right]^m, & 0 \le \lambda < 1; \\ \\ \lambda^m, & \lambda \ge 1, \end{cases}$$

with $\Psi(j,k;X)$, $\chi(j,k;X)$ and $B(j,k)$ as given by (4.31) and $A = x-a$, $B = b-x$. Further,

$$d_n^{(k)}(x) = \left[A^{n-k} - (-B)^{n-k}\right] \frac{1}{\binom{n}{m}} \sum_{j=L}^{U} \frac{\binom{k}{j}(1-\lambda)^{m-k+j}}{(n-m-j)!(m-k+j)!},$$

$$P_n^{(k)}(a) = A^{n-k}(-\lambda)^{n-k} \frac{\binom{k}{n-m}}{\binom{n}{m}}, \quad k \ge n-m \tag{4.40}$$

$$\text{and} \quad Q_n^{(k)}(b) = B^{n-k}\lambda^{n-k} \frac{\binom{k}{n-m}}{\binom{n}{m}}, \quad k \ge n-m$$

with, from (4.12), $U = \min\{k, n-m\}$ and $L = \max\{0, k-m\}$.

Proof. Taking $\alpha = \lambda x + (1-\lambda)a$ and $\beta = \lambda x + (1-\lambda)b$ in Theorem 4.3 gives the above results after some simplification. ∎

Remark 4.4. Taking $\lambda = 0$ in Corollary 4.4 implies that $\alpha = a$ and $\beta = b$, producing an Ostrowski type result similar to those obtained in Cerone et al. [**8**]. Taking $\lambda = 1$ gives generalized trapezoidal results for $n-$time differentiable functions of Cerone et al. [**9**].

4.3. Simpson Type Formulae

The inequality

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \le \frac{(b-a)^5}{2880}\left\|f^{(4)}\right\|_\infty \tag{4.41}$$

is well known as the Simpson inequality where the mapping $f : [a,b] \to \mathbb{R}$ is assumed to have $f^{(4)}$ bounded on (a,b). A review article by Dragomir et al. [**12**] presents bounds in terms of at most a first derivative while Pečarić and Varošanec [**18**] obtain bounds in terms of $f^{(k)} \in L_p[a,b]$, $p \ge 1$ and $k \le 4$. The current work looks at recapturing the above results as special cases of the development presented in the previous section.

Let $m = 1$ in (4.34), let α and β be shown to depend on x and subscripted to denote their dependence on n. Further, a superscript of S is used to depict the relationship with Simpson's rule. Then

$$p_n^S(t) = \frac{(t-a)^{n-1}}{(n-1)!}(t-\alpha_n(x)), \quad q_n^S(t) = \frac{(t-b)^{n-1}}{(n-1)!}(t-\beta_n(x)), \tag{4.42}$$

with

$$\alpha_n(x) = \lambda_n x + (1-\lambda_n)a, \quad \beta_n(x) = \lambda_n x + (1-\lambda_n)b, \quad \lambda_n = \frac{n}{3}. \tag{4.43}$$

Corollary 4.5. Let the conditions of Corollary 4.4 hold, then

$$\begin{aligned} \left|\tau_n^S(x)\right| \quad : \quad &= \Bigg| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1}(-1)^{k+1} d_n^{(k)}(x) f^{(n-1-k)}(x) - \\ &\quad \frac{(-1)^n \lambda_n}{n}\left[B\cdot f(b) + A\cdot f(a)\right]\Bigg| \\ &\le \begin{cases} \frac{B_n^S(1,x)}{n!}\left\|f^{(n)}\right\|_\infty, & \text{for} \quad f^{(n)} \in L_\infty[a,b]; \\ \frac{\left[B_n^S(q,x)\right]^{\frac{1}{q}}}{n!}\left\|f^{(n)}\right\|_p, & \text{for} \quad f^{(n)} \in L_p[a,b], \\ & \text{with} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\theta_n^S(x)}{n!}\left\|f^{(n)}\right\|_1, & \text{for} \quad f^{(n)} \in L_1[a,b], \end{cases} \end{aligned} \tag{4.44}$$

where

$$\begin{aligned} & B_n^S(q,x) \\ &= \left[A^{nq+1}+B^{nq+1}\right]\begin{cases} \lambda_n^{nq+1} B((n-1)q+1,q+1) + \\ (1-\lambda_n)^{nq+1}\,\Psi\left(q,(n-1)q,\frac{\lambda_n}{1-\lambda_n}\right), & 0 \le \lambda_n < 1; \\ \chi((n-1)q,q,\lambda_n), & \lambda_n \ge 1, \end{cases} \end{aligned} \tag{4.45}$$

$$\theta_n^S(x) = \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^n \begin{cases} \frac{1}{2} + \left|\lambda_n - \frac{1}{2}\right|, & 0 \le \lambda_n < 1; \\ \lambda_n, & \lambda_n \ge 1, \end{cases} \tag{4.46}$$

$$d_n^{(k)}(x) = \left[A^{n-k} - (-B)^{n-k}\right]\frac{1}{n}\sum_{j=\max\{0,k-1\}}^{\min\{k,n-1\}} \frac{\binom{k}{j}(1-\lambda_n)}{(n-1-j)!\,(1-k+j)!}, \tag{4.47}$$

with $\Psi(j,k;X)$, $\chi(j,k;X)$ and $B(j,k)$ as given by (4.31) and $A = x-a$, $B = b-x$, $\lambda_n = \frac{x}{3}$.

Proof. Taking $p_n^S(t)$, $q_n^S(t)$, $\alpha_n(x)$, $\beta_n(x)$ and $\lambda_n = \frac{n}{3}$ produces the above results after some simplification. ∎

If $n = 1, 2, 3$ and 4, then we obtain the Simpson type results of Dragomir et al. [**12**] and Pečarić and Varošanec [**18**], provided that x is taken at the midpoint, that is, at $x = \frac{a+b}{2}$. It may be seen that $d_n^{(k)}(x) = 0$ from (4.40) and (4.47) when $n - k$ is even so that only even derivatives are present when $x = \frac{a+b}{2}$. Further, from (4.47), $d_4^{(1)} = 0$ and so there is no $f''(x)$ term in the quadrature rule. Taking, for example, $n = 4$, $q = 1$ and $\lambda_4 = \frac{4}{3}$ in (4.42) – (4.47) reproduces (4.41).

4.4. Perturbed Results

Perturbed versions of the results of the previous sections may be obtained by using Grüss type results involving the Chebychev functional

$$T(f,g) = \mathfrak{M}(fg) - \mathfrak{M}(f)\,\mathfrak{M}(g) \tag{4.48}$$

with

$$\mathfrak{M}(f) = \frac{1}{b-a}\int_a^b f(t)\,dt. \tag{4.49}$$

For $f, g : [a,b] \to \mathbb{R}$ and integrable on $[a,b]$, as is their product, then

$$\begin{aligned} |T(f,g)| &\le [T(f,f)]^{\frac{1}{2}}[T(g,g)]^{\frac{1}{2}}, && \text{Dragomir [11]} \\ & && \text{for } f,g \in L_2[a,b]; \\ &= \tfrac{\Gamma-\gamma}{2}[T(f,f)]^{\frac{1}{2}}, && \text{Matić et al. [15]} \\ & && \text{for } \gamma \le g(t) \le \Gamma,\ t \in [a,b]; \\ &= \tfrac{(\Gamma-\gamma)(\Phi-\phi)}{4} && \text{Grüss (see [16, pp. 295-310])} \\ & && \phi \le f \le \Phi,\ t \in [a,b]. \end{aligned} \tag{4.50}$$

Dragomir [**11**] obtains numerous results if either f or g or both are known, although the first inequality in (4.50) has a long history (see for example [**16**], pp. 295-310). The inequalities in (4.50) when proceeding from top to bottom are in the order of increasing coarseness. See also Cerone [**2**] for the perturbed results and their implementation as composite rules.

Theorem 4.6. *Let the mapping $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous. Then the following inequality holds. Namely,*

$$\begin{aligned} &|\tau_n(x) - (-1)^n U_n(x) S_{n-1}(f;a,b)| \\ &\le (b-a)\,\kappa_n(x)\left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2 - S_{n-1}^2(f;a,b)\right]^{\frac{1}{2}}, \quad f^{(n)} \in L_2[a,b], \\ &\le (b-a)\,\kappa_n(x)\left(\frac{\Gamma_n-\gamma_n}{2}\right), \quad \gamma_n \le f^{(n)}(t) \le \Gamma_n,\ t \in [a,b], \\ &\le (b-a)\,\frac{(\Phi_n(x)-\phi_n(x))}{4}(\Gamma_n-\gamma_n), \\ &\qquad \phi_n(x) \le K_n(x,t) \le \Phi_n(x),\ t \in [a,b], \end{aligned} \tag{4.51}$$

where $\tau_n(x)$ is as defined in (4.20),

$$U_n(x) = \sum_{k=0}^{m}(-1)^{m-k}\left[P_{n,m,k}(x) - P_{n,m,k}(a) + Q_{n,m,k}(b) - Q_{n,m,k}(x)\right], \tag{4.52}$$

$$P_{n,m,k}(t) = r_{n-(m-k)}(t)\, s_k(t)\,, \quad Q_{n,m,k}(t) = u_{n-(m-k)}(t)\, v_k(t) \tag{4.53}$$

with $r_{\cdot}(\cdot)$, $s_{\cdot}(\cdot)$, $u_{\cdot}(\cdot)$, $v_{\cdot}(\cdot)$ *satisfying (4.7),*

$$S_n(f;a,b) = \frac{f^{(n)}(b) - f^{(n)}(a)}{b-a}, \tag{4.54}$$

and

$$\kappa_n(x) = \left[\frac{1}{b-a}\int_a^b K_n^2(x,t)\,dt - \left(\frac{U_n(x)}{b-a}\right)^2\right]^{\frac{1}{2}}, \tag{4.55}$$

with $K_n(x,t)$ *being as defined by (4.8).*

Proof. Associating $f(t)$ with $(-1)^n K_n(x,t)$ and $g(t)$ with $f^{(n)}(t)$, then from (4.9), (4.48) and (4.49), we obtain

$$\begin{aligned} & T\left((-1)^n K_n(x,\cdot)\,, f^{(n)}(\cdot)\right) \\ = \; & \mathfrak{M}\left((-1)^n K_n(x,\cdot)\,, f^{(n)}(\cdot)\right) - \mathfrak{M}\left((-1)^n K_n(x,\cdot)\right)\mathfrak{M}\left(f^{(n)}(\cdot)\right) \end{aligned}$$

and thus

$$\begin{aligned} & (b-a)\, T\left((-1)^n K_n(x,\cdot)\,, f^{(n)}(\cdot)\right) \\ = \; & \tau_n(x) - (-1)^n U_n(x)\left(\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\right), \end{aligned} \tag{4.56}$$

where $\tau_n(x)$ is the left hand side of (4.20) and

$$U_n(x) = \int_a^x p_n(t)\,dt + \int_x^b q_n(t)\,dt. \tag{4.57}$$

Now, using (4.8)

$$\int_a^x p_n(t)\,dt = \int_a^x r_{n-m}(t)\, s_m(t)\,dt$$

from which repeated integration by parts and using the fact that $r_{\cdot}(\cdot)$ and $s_{\cdot}(\cdot)$ satisfy (4.7) gives on using (4.53)

$$\int_a^x p_n(t)\,dt = \sum_{k=0}^{m} (-1)^{m-k}\left[P_{n,m,k}(t)\right]_a^x. \tag{4.58}$$

A similar argument gives on using (4.56)

$$\int_a^b q_n(t)\,dt = \sum_{k=0}^{m} (-1)^{m-k}\left[Q_{n,m,k}(t)\right]_x^b \tag{4.59}$$

and so combining (4.58) and (4.59) in (4.56) gives (4.52).

Now for the bounds in (4.51).

From (4.48) we have, for $K_n(x,t)$ as defined by (4.8),

$$\begin{aligned} (4.60) \qquad & T\left((-1)^n K_n(x,\cdot),(-1)^n K_n(x,\cdot)\right) \\ = & \left[\mathfrak{M}\left(K_n^2(x,\cdot)\right)-\mathfrak{M}^2\left(K_n(x,\cdot)\right)\right]^{\frac{1}{2}} \\ = & \left[\frac{1}{b-a}\int_a^b K_n^2(x,t)\,dt-\left(\frac{1}{b-a}\int_a^b K_n(x,t)\,dt\right)^2\right]^{\frac{1}{2}} \\ = & \kappa_n(x), \end{aligned}$$

as defined in (4.55) since $U_n(x)=\int_a^b K_n(x,t)\,dt$.

Further, using (4.48), (4.51) and (4.54) gives

$$\begin{aligned} T\left(f^{(n)}(t),f^{(n)}(t)\right) &= \left[\mathfrak{M}\left(\left[f^{(n)}(t)\right]^2\right)-\mathfrak{M}^2\left(f^{(n)}(t)\right)\right]^{\frac{1}{2}} \\ &= \left\{\frac{1}{b-a}\int_a^b\left[f^{(n)}(t)\right]^2dt-\left[\frac{\int_a^b f^{(n)}(t)\,dt}{b-a}\right]^2\right\}^{\frac{1}{2}} \\ &= \left\{\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2-S_{n-1}^2(f;a,b)\right\}^{\frac{1}{2}} \end{aligned}$$

and so combining the above result with (4.60) produces the first inequality.

For $f^{(n)}\in L_\infty[a,b]$ ($\subset L_2[a,b]$ with strict inclusion) then

$$\begin{aligned} (4.61) \qquad 0 &\le T\left(f^{(n)}(t),f^{(n)}(t)\right) \\ &= \frac{1}{b-a}\int_a^b\left|f^{(n)}(t)\right|^2dt-\left[\frac{\int_a^b f^{(n)}(t)\,dt}{b-a}\right]^2 \\ &\le \left(\frac{\Gamma_n-\gamma_n}{2}\right)^2, \quad \text{where } \gamma_n\le f^{(n)}(t)\le\Gamma_n,\ t\in[a,b], \end{aligned}$$

and the third inequality in (4.52), is due to Grüss.

Hence the second inequality in (4.51) is obtained which is coarser than the first.

Further, from (4.60)

$$0\le\kappa_n^2\le\left(\frac{\Phi_n(x)-\phi_n(x)}{2}\right)^2, \quad \text{where } \phi_n(x)\le K_n(x,t)\le\Phi_n(x),\ t\in[a,b],$$

which, when combined with (4.61), produces the third bound in (4.51) which is the coarsest bound of all. The proof of the theorem is thus complete. ∎

Remark 4.5. The second result in (4.51) is a generalisation of a result by Pearce et al. [**17**] in which each of the branches of the Peano kernel (4.8) satisfy (4.7). That is,

$$(4.62) \qquad K_n^*(x,t)=\begin{cases} P_n(t), & t\in[a,x]; \\ Q_n(t), & t\in(x,b], \end{cases}$$

where $P_n(t)$ and $Q_n(t)$ satisfy (4.8). With the Peano kernel (4.62), they obtained the result

THEOREM 4.7. *Assume that $f:[a,b]\to\mathbb{R}$ is such that $f^{(n)}$ is integrable and $\gamma_n\le f^{(n)}\le\Gamma_n$ for all $t\in[a,b]$. Put*

$$U_n(x):=\frac{1}{b-a}\left[Q_{n+1}(b)-Q_{n+1}(x)+P_{n+1}(x)-P_{n+1}(a)\right]. \tag{4.63}$$

Then for all $x\in[a,b]$, we have the inequality

$$\begin{aligned}\Bigg|\int_a^b f(t)\,dt-\sum_{k=1}^{n}(-1)^{k+1}\Big[Q_k(b)\,f^{(k-1)}(b)+\left(P_k(x)-Q_k(x)\right)f^{(k-1)}(x)\\ -P_k(a)\,f^{(k-1)}(a)\Big]-(-1)^n U_n(x)\left[f^{(n-1)}(b)-f^{(n-1)}(a)\right]\Bigg|\\ \le\frac{1}{2}K(x)\left(\Gamma_n-\gamma_n\right)(b-a),\end{aligned} \tag{4.64}$$

where

$$K(x):=\left\{\frac{1}{b-a}\int_a^x P_n^2(t)\,dt+\int_x^b Q_n^2(t)\,dt-\left[U_n(x)\right]^2\right\}^{\frac{1}{2}}. \tag{4.65}$$

In the recent paper [**11**], Dragomir proved the following refinement of (4.64).

THEOREM 4.8. *Assume that the mapping $f:[a,b]\to\mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)}\in L_2[a,b]$ $(n\ge 1)$. If we denote*

$$\left[f^{(n-1)};a,b\right]:=\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a},$$

then we have the inequality

$$\begin{aligned}\Bigg|\int_a^b f(t)\,dt-\sum_{k=1}^{n}(-1)^{k+1}\Big[Q_k(b)\,f^{(k-1)}(b)+\left(P_k(x)-Q_k(x)\right)f^{(k-1)}(x)\\ -P_k(a)\,f^{(k-1)}(a)\Big]-(-1)^n\left[Q_{n+1}(b)-Q_{n+1}(x)\right.\\ \left.+P_{n+1}(x)-P_{n+1}(a)\right]\left[f^{(n-1)};a,b\right]\Bigg|\\ \le K(x)(b-a)\left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2-\left(\left[f^{(n)};a,b\right]\right)^2\right]^{\frac{1}{2}}\\ \left(\le\frac{1}{2}K(x)(b-a)\left(\Gamma_n-\gamma_n\right)\quad\text{if}\quad f^{(n)}\in L_\infty(a,b)\right),\end{aligned} \tag{4.66}$$

for all $x\in[a,b]$ and $K(x)$ as is given in (4.65).

The results of Theorem 4.6 are generalisations of these which allow each of the branches themselves to be made up of products of functions satisfying (4.7).

Corollary 4.9. Let the conditions on f of Theorem 4.6 hold and let $\alpha,\, x,\, \beta \in [a,b]$ with $a \le \alpha,\, x < \beta \le b$ then,

$$\left|\binom{n}{m}\tau_n^C(x) - (-1)^n U_n^C(x)\, S_{n-1}(f;a,b)\right| \tag{4.67}$$
$$\le (b-a)\,\kappa_n^C(x)\left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2 - S_{n-1}^2(f;a,b)\right]^{\frac12},\quad f^{(n)} \in L_2[a,b],$$
$$\le (b-a)\,\kappa_n^C(x)\left(\frac{\Gamma_n-\gamma_n}{2}\right),\quad \gamma_n \le f^{(n)}(t) \le \Gamma_n,\ t\in[a,b],$$
$$\le (b-a)\,\frac{\left(\Phi_n^C(x)-\phi_n^C(x)\right)(\Gamma_n-\gamma_n)}{4},$$
$$\phi_n^C(x) \le \kappa_n^C(x,t) \le \Phi_n^C(x),\ t\in[a,b],$$

where $\tau_n^C(x)$ and $S_n(f;a,b)$ are as defined by (4.27) and (4.54) respectively,

$$K_n^C(x,t) = \begin{cases} p_n^C(t) = \dfrac{(t-a)^{n-m}}{(n-m)!}\cdot\dfrac{(t-\alpha)^m}{m!}, & t\in[a,x] \\[2ex] q_n^C(t) = \dfrac{(t-b)^{n-m}}{(n-m)!}\cdot\dfrac{(t-\beta)^m}{m!}, & t\in(x,b], \end{cases} \tag{4.68}$$

$$U_n^C(x) = \frac{1}{(n-m)!m!}\left[\tilde{B}_a(n,m) + \tilde{B}_b(n,m)\right], \tag{4.69}$$

with

$$\tilde{B}_a(n,m) = \begin{cases} \dfrac{(x-a)^{n+1}}{n+1}, & \alpha = a \\[2ex] (-1)^m(\alpha-a)^{n+1} B\left(n-m+1, m+1, \dfrac{x-a}{\alpha-a}\right), & \alpha \ne a \end{cases}$$

$$\tilde{B}_b(n,m) = \begin{cases} \dfrac{(-1)^n(b-x)^{n+1}}{n+1}, & \beta = b \\[2ex] (-1)^{n-m}(b-\beta)^{n+1} B\left(n-m+1, m+1, \frac{b-x}{b-\beta}\right), & \beta \ne b \end{cases}$$

and

$$B(j,k,X) = \int_0^X u^{j-1}(1-u)^{k-1}\,du,\ \text{ the incomplete beta function,}$$

$$\kappa_n^C(x) \tag{4.70}$$
$$= \left\{\frac{1}{(n-m)!m!(b-a)}\left[\tilde{B}_a(2n,2m)+\tilde{B}_b(2n,2m)\right] - \left(\frac{U_n^C(x)}{b-a}\right)^2\right\}^{\frac12}$$

and

$$\phi_n^C(x) = \min\left\{\phi_n^a(x), \phi_n^b(x)\right\},\quad \Phi_n^C(x) = \max\left\{\Phi_n^a(x), \Phi_n^b(x)\right\}, \tag{4.71}$$

with

$$\begin{aligned}\phi_n^a(x) &= \inf_{t\in[a,x]} p_n^C(t), \ \phi_n^b(x) = \inf_{t\in(x,b]} q_n^C(t), \\ \Phi_n^a(x) &= \sup_{t\in[a,x]} p_n^C(t), \ \Phi_n^b(x) = \sup_{t\in(x,b]} q_n^C(t).\end{aligned}$$

PROOF. Let $r_\cdot^C(\cdot)$, $s_\cdot^C(\cdot)$, $u_\cdot^C(\cdot)$ and $v_\cdot^C(\cdot)$ be as defined in (4.33) giving from (4.34) and (4.8) $K_n^C(x,t)$ as shown in (4.68).

Now,

$$U_n^C(x) = \int_a^b K_n^C(x,t)\,dt = \int_a^x p_n^C(t)\,dt + \int_x^b q_n^C(t)\,dt \tag{4.72}$$

and so using (4.68) consider

$$\begin{aligned}(n-m)!m!\int_a^x p_n^C(t)\,dt &= \int_a^x (t-a)^{n-m}(t-\alpha)^m\,dt \\ &= \frac{(x-a)^{n+1}}{n+1}, \ \alpha = a.\end{aligned} \tag{4.73}$$

If $\alpha \neq a$, let $(\alpha-a)u = t-a$, then

$$\int_a^x (t-a)^{n-m}(t-\alpha)^m\,dt = (-1)^n(\alpha-a)^{n+1}\int_0^{\frac{x-a}{\alpha-a}} u^{n-m}(1-u)^m\,du.$$

Similarly,

$$\begin{aligned}(n-m)!m!\int_x^b q_n^C(t)\,dt &= \int_x^b (t-b)^{n-m}(t-\beta)^m\,dt \\ &= \frac{(-1)^n(b-x)^{n+1}}{n+1}, \ \beta = b.\end{aligned} \tag{4.74}$$

If $\beta \neq b$, let $(b-\beta)v = b-t$ to give

$$\int_x^b (t-b)^{n-m}(t-\beta)^m\,dt = (-1)^{n-m}(b-\beta)^{n+1}\int_0^{\frac{b-x}{b-\beta}} v^{n-m}(1-v)^m\,dv.$$

Substitution of the above results into (4.72) gives (4.69).

Further, from (4.68),

$$\begin{aligned}\int_a^b \left[K_n^C(x,t)\right]^2 dt &= \frac{1}{(n-m)!m!}\left\{\int_a^x (t-a)^{2(n-m)}(t-\alpha)^{2m}\,dt\right. \\ &\quad \left. + \int_x^b (t-b)^{2(n-m)}(t-\beta)^{2m}\,dt\right\},\end{aligned}$$

which, on following a similar procedure to the above gives

$$\int_a^b \left[K_n^C(x,t)\right]^2 dt = \frac{1}{(n-m)!m!}\left[\tilde{B}_a(2n,2m) + \tilde{B}_b(2n,2m)\right]. \tag{4.75}$$

Hence, since

$$\kappa_n^C(x) = \left[\frac{1}{b-a}\int_a^b \left[K_n^C(x,t)\right]^2 dt - \left(\frac{U_n^C(x)}{b-a}\right)^2\right]^{\frac{1}{2}}, \tag{4.76}$$

then from (4.75) and (4.69) we obtain (4.70).

For the first inequality we observe that, from (4.68),

$$\Phi_n^C(x)=\sup_{t\in[a,b]}K_n^C(x,t)=\max\left\{\sup_{t\in[a,x]}p_n^C(t),\ \sup_{t\in(x,b]}q_n^C(t)\right\}$$

and

$$\phi_n^C(x)=\inf_{t\in[a,b]}K_n^C(x,t)=\min\left\{\inf_{t\in[a,x]}p_n^C(t),\ \inf_{t\in(x,b]}q_n^C(t)\right\}.$$

The corollary is thus completely proven where the inequalities are in increasing coarseness as discussed in Theorem 4.6. ∎

Remark 4.6. If $\alpha=a$ and $\beta=b$, then a perturbed generalised Ostrowski type result is obtained. If $\alpha=\beta=x$, perturbed trapezoidal type results are obtained for n-time differentiable functions.

Corollary 4.10. Let the conditions of Theorem 4.6 hold and let $\alpha,\ x,\ \beta\in[a,b]$ with $a\leq\alpha,\ x,\ \beta\leq b$, then

$$\begin{aligned}(4.77)\quad&\left|\binom{n}{m}\tau_n^{C^*}(x)-(-1)^nU_n^{C^*}(x)S_{n-1}(f;a,b)\right|\\&\leq(b-a)\kappa_n^{C^*}(x)\left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2-S_{n-1}^2(f;a,b)\right]^{\frac{1}{2}},\quad f^{(n)}\in L_2[a,b],\\&\leq(b-a)\kappa_n^{C^*}(x)\left(\frac{\Gamma_n-\gamma_n}{2}\right),\quad\gamma_n\leq f^{(n)}(t)\leq\Gamma_n,\ t\in[a,b],\\&\leq(b-a)\frac{\left(\Phi_n^{C^*}(x)-\phi_n^{C^*}(x)\right)(\Gamma_n-\gamma_n)}{4},\\&\qquad\phi_n^{C^*}(x)\leq K_n^{C^*}(x,t)\leq\Phi_n^{C^*}(x),\ t\in[a,b],\end{aligned}$$

where $\tau_n^{C^*}(x)$ and $S_n(f;a,b)$ are as defined by (4.37) and (4.54) respectively, (4.78)

$$K_n^{C^*}(x,t)=\begin{cases}p_n^{C^*}(t)=\dfrac{(t-a)^{n-m}}{(n-m)!}\cdot\dfrac{(t-(x\lambda+(1-\lambda)a))^m}{m!}, & t\in[a,x]\\[2ex] q_n^{C^*}(t)=\dfrac{(t-b)^{n-m}}{(n-m)!}\cdot\dfrac{(t-(x\lambda+(1-\lambda)b))^m}{m!}, & t\in(x,b],\end{cases}$$

$$(4.79)\qquad U_n^{C^*}(x)=\frac{\tilde{B}^*(n,m)}{(n-m)!m!}\left[(x-a)^{n+1}+(-1)^n(b-x)^{n+1}\right],$$

with

$$\tilde{B}^*(n,m)=\begin{cases}\dfrac{1}{n+1}, & \lambda=0,\\[2ex](-1)^m\lambda^{n+1}B\left(n-m+1,m+1,\frac{1}{\lambda}\right), & \lambda\neq0,\end{cases}$$

and

$$B(j,k,X)=\int_0^X u^{j-1}(1-u)^{k-1}\,du,\quad\text{the incomplete beta function,}$$

$$\kappa_n^{C^*}(x) = \left\{ \frac{\lambda^{2n+1}\tilde{B}^*(n,m)}{(n-m)!m!(b-a)} \left[(x-a)^{2n+1} + (b-x)^{2n+1} \right] - \left(\frac{U_n^{C^*}(x)}{b-a} \right)^2 \right\}^{\frac{1}{2}} \tag{4.80}$$

and

$$\phi_n^{C^*}(x) = \min\left\{ \inf_{t\in[a,x]} p_n^{C^*}(t), \inf_{t\in(x,b]} q_n^{C^*}(t) \right\},$$

$$\Phi_n^{C^*}(x) = \max\left\{ \sup_{t\in[a,x]} p_n^{C^*}(t), \sup_{t\in(x,b]} q_n^{C^*}(t) \right\}.$$

Proof. Taking $\alpha = \lambda x + (1-\lambda)a$ and $\beta = \lambda x + (1-\lambda)b$ in Corollary 4.9 produces the above results after some simplification. ∎

Remark 4.7. Taking $\lambda = 0$ in Corollary 4.10 implying that $\alpha = a$ and $\beta = b$ produces a perturbed Ostrowski type results. If $\lambda = 1$, then perturbed trapezoidal type results are obtained for $n-$time differentiable functions.

Corollary 4.11. Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then, for α, x, $\beta \in [a,b]$ with $a \le \alpha$, x, $\beta \le b$, the perturbed Simpson rule

$$\begin{aligned} &\left| \tau_n^S(x) - \frac{(-1)^n}{n} U_n^S(x) S_{n-1}(f;a,b) \right| \\ &\le \frac{(b-a)}{n}\kappa_n^S(x)\left[\frac{1}{b-a}\left\| f^{(n)} \right\|_2^2 - S_{n-1}^2(f;a,b) \right]^{\frac{1}{2}}, \quad f^{(n)} \in L_2[a,b], \\ &\le \frac{(b-a)}{n}\kappa_n^S(x)\left(\frac{\Gamma_n - \gamma_n}{2} \right), \quad \gamma_n \le f^{(n)}(t) \le \Gamma_n,\ t\in[a,b], \\ &\le \frac{(b-a)}{n} \cdot \frac{\left(\Phi_n^S(x) - \phi_n^S(x) \right)(\Gamma_n - \gamma_n)}{4}, \\ &\qquad \phi_n^S(x) \le K_n^S(x,t) \le \Phi_n^S(x),\ t\in[a,b], \end{aligned} \tag{4.81}$$

where $\tau_n^S(x)$ and $S_n(f;a,b)$ are as defined by (4.45) and (4.54) respectively,

$$K_n^S(x,t) = \begin{cases} p_n^S(t) = \dfrac{(t-a)^{n-1}}{(n-1)!}(t-\alpha_n(x)), \\[2ex] q_n^S(t) = \dfrac{(t-b)^{n-1}}{(n-1)!}(t-\beta_n(x)), \end{cases} \tag{4.82}$$

with

$$\alpha_n(x) = \lambda_n x + (1-\lambda_n)a, \quad \beta_n(x) = \lambda_n x + (1-\lambda_n)b, \quad \lambda_n = \frac{n}{3}, \tag{4.83}$$

$$U_n^S(x) = \frac{n(1-\lambda_n) - \lambda_n}{(n+1)!}\left[(x-a)^{n+1} + (-1)^n(b-x)^{n+1} \right], \tag{4.84}$$

(4.85)

$$\kappa_n^S(x) = \left\{ \frac{\lambda^{2n+1} B\left(2n-1, 3, \frac{1}{\lambda}\right)}{(n-1)!\,(b-a)} \left[(x-a)^{2n+1} + (b-x)^{2n+1}\right] - \left(\frac{U_n^S(x)}{b-a}\right)^2 \right\}^{\frac{1}{2}}$$

and

$$\begin{aligned} (4.86) \qquad \phi_n^S(x) &= \frac{1}{(n-1)!} \min\left\{ \inf_{U\in[0,x-a]} X(U), \inf_{V\in(0,b-x]} Y(V) \right\}, \\ \Phi_n^S(x) &= \frac{1}{(n-1)!} \min\left\{ \sup_{U\in[0,x-a]} X(U), \sup_{V\in(0,b-x]} Y(V) \right\}, \end{aligned}$$

with

$$X(U) = U^{n-1}\left(U - \lambda_n (x-a)\right), \; Y(V) = (-1)^n V^{n-1}\left(V - \lambda_n (b-x)\right).$$

Proof. Taking $m = 1$ in Corollary 4.10 and (4.69) and (4.83) in place of (4.78) gives

$$(4.87) \qquad U_n^S(x) = \int_a^x p_n^S(t)\,dt + \int_x^b q_n^S(t)\,dt.$$

Taking $m = 1$ and $\lambda \equiv \lambda_n^S$ in (4.79) gives $U_n^S(x)$. Alternatively, direct calculation gives,

$$\begin{aligned} \int_a^x p_n^S(t)\,dt &= \int_a^x \frac{(t-a)^{n-1}}{(n-1)!}\left(t - \alpha_n(x)\right)dt \\ &= \frac{(t-a)^n}{(n+1)!}\left[(n+1)\left(t-\alpha_n(x)\right) - (t-a)\right]\Big]_{t=a}^{x} \\ &= \frac{(x-a)^n}{(n+1)!}\left[(n+1)\left(x-\alpha_n(x)\right) - (x-a)\right] \\ &= \frac{(x-a)^n}{(n+1)!}\left[(n+1)\left(1-\lambda_n\right) - 1\right] \\ &= \frac{(x-a)^n}{(n+1)!}\left[n\left(1-\lambda_n\right) - \lambda_n\right]. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_x^b q_n^S(t)\,dt &= \int_x^b \frac{(t-b)^{n-1}}{(n-1)!}\left(t - \beta_n(x)\right)dt \\ &= (-1)^n \frac{(b-x)^n}{(n+1)!}\left[(n)\left(1-\lambda_n\right) - \lambda_n\right]. \end{aligned}$$

Combining the above results into (4.37) produces (4.84).

Now, to determine $\kappa_n^S(x)$ from (4.76) we have, using (4.82)

$$\frac{1}{b-a}\int_a^b \left[K_n^S(x,t)\right]^2 dt = \frac{1}{b-a}\left[\int_a^x \left(p_n^S(t)\right)^2 dt + \int_x^b \left(q_n^S(t)\right)^2 dt\right].$$

For direct calculation we require integration by parts twice. Alternatively, utilising (4.70) with $m = 1$ and $\alpha_n(x)$, $\beta_n(x)$ being as given by (4.83) produces the stated result (4.85).

Now,

$$\Phi_n^S(x) = \max\left\{\sup_{t\in[a,x]} p_n^S(t), \sup_{t\in(x,b]} q_n^S(t)\right\},$$

and on using (4.83),

$$\begin{aligned}\sup_{t\in[a,x]} p_n^S(t) &= \frac{1}{(n-1)!}\sup_{U\in[0,x-a]} U^{n-1}(U-\lambda_n(x-a)),\\ \sup_{t\in(x,b]} q_n^S(t) &= \frac{1}{(n-1)!}\sup_{V\in[0,b-x)} (-1)^n V^{n-1}(V-\lambda_n(b-x)).\end{aligned}$$

In a similar fashion

$$\phi_n^S(x) = \min\left\{\inf_{t\in[a,x]} p_n^S(t), \inf_{t\in(x,b]} q_n^S(t)\right\},$$

and using the above expressions with sup replaced by inf gives the results as stated in (4.86).

The proof is now complete. ∎

REMARK 4.8. The perturbed results obtained above through the use of the Chebychev functional (4.48) and the resulting bounds given by (4.50) may be advantageous when compared to the first bounds in (4.20), (4.27), (4.36) and (4.44). For functions $g, h : [a,b] \to \mathbb{R}$ and $\gamma \le g(t) \le \Gamma$, then $\frac{\Gamma-\gamma}{2} \le \|g\|_\infty$. It is however, difficult to compare $\|h\|_1 \|g\|_\infty$ obtained for the unperturbed results of previous sections, with the perturbed bounds of the form

$$(b-a)\|h\|_\infty \sigma^{\frac{1}{2}}(g) < (b-a)\|h\|_\infty \frac{\Gamma-\gamma}{2} < (b-a)\frac{(\Phi-\phi)(\Gamma-\gamma)}{2},$$

where

$$\begin{aligned}\sigma(g) &= \frac{1}{b-a}\|g\|_2^2 - S^2(g;a,b),\ S(g;a,b) = \frac{g(a)-g(b)}{b-a}\\ \text{and } \phi &\le h(t) \le \Phi.\end{aligned}$$

This is so since a comparison between $\|h\|_1$ and $\|h\|_\infty$ cannot readily be made.

4.5. More Perturbed Results Using $\Delta-$Seminorms

In a recent article Cerone and Dragomir [**5**] obtained the following results of Grüss type for the Chebychev functional $T(f,g)$. They utilised the notion of a $\Delta-$seminorm introduced by Cerone and Dragomir [**4**] where

$$\left\{\begin{array}{ll}\|f\|_p^\Delta := \left(\int_a^b\int_a^b |f(s)-f(t)|^p\, dsdt\right)^{\frac{1}{p}}, & \text{for } f \in L_p[a,b],\ p\in[1,\infty),\\ \text{and} & \\ \|f\|_\infty^\Delta := ess \sup_{(s,t)\in[a,b]^2} |f(s)-f(t)|, & \text{for } f\in L_\infty[a,b].\end{array}\right. \tag{4.88}$$

If we consider $f_\Delta : [a,b]^2 \to \mathbb{R}$, where $f_\Delta(s,t) = f(s) - f(t)$, then

$$\|f\|_p^\Delta \equiv \|f_\Delta\|_p, \quad p \in [1,\infty] \tag{4.89}$$

with $\|\cdot\|_p$ being the usual Lebesgue $p-$norms on $[a,b]^2$.

Using the properties of the Lebesque $p-$norms, we may deduce the following semi-norm properties for $\|\cdot\|_p^\Delta$:

(i) $\|f\|_p^\Delta \geq 0$ for $f \in L_p[a,b]$ and $\|f\|_p^\Delta = 0$ implies that $f = c$ (c is a constant) a.e. in $[a,b]$;
(ii) $\|f+g\|_p^\Delta \leq \|f\|_p^\Delta + \|g\|_p^\Delta$ if $f,g \in L_p[a,b]$;
(iii) $\|\lambda f\|_p^\Delta = |\lambda| \|f\|_p^\Delta$.

We note that if $p = 2$, then,

$$\begin{aligned}
\|f\|_2^\Delta &= \left(\int_a^b \int_a^b (f(t) - f(s))^2 \, dt ds \right)^{\frac{1}{2}} \\
&= \sqrt{2} \left[(b-a) \|f\|_2^2 - \left(\int_a^b f(t) \, dt \right)^2 \right]^{\frac{1}{2}}.
\end{aligned}$$

The following theorem giving bounds for the Chebychev functional in terms of $\Delta-$seminorms (4.88) holds (see also Cerone and Dragomir [**5**]).

THEOREM 4.12. *Let $f,g : [a,b] \to \mathbb{R}$ be measurable on $[a,b]$. Then the inequality*

$$|T(f,g)| \leq \frac{1}{2(b-a)^2} \|f\|_p^\Delta \ \|g\|_q^\Delta \tag{4.90}$$

holds provided the integrals exist, where $T(f,g)$ is the Chebychev functional given by (4.48) – (4.49), $p = 1$, $q = \infty$ or $q = 1$, $p = \infty$ or $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\|f\|_\cdot^\Delta$ is defined by (4.88).

PROOF. Using Korkine's identity, we have

$$T(f,g) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f(x) - f(y))(g(x) - g(y)) \, dx dy,$$

where $T(f,g)$ is the Chebychev functional defined by (4.48).

Now, if $f \in L_\infty[a,b]$, then

$$\begin{aligned}
|T(f,g)| &\leq \frac{1}{2(b-a)^2} \int_a^b \int_a^b |f(x) - f(y)| \, |g(x) - g(y)| \, dx dy \\
&\leq \frac{1}{2(b-a)^2} ess \sup_{(x,y) \in [a,b]^2} (f(x) - f(y)) \int_a^b \int_a^b |g(x) - g(y)| \, dx dy \\
&= \frac{1}{2(b-a)^2} \|f\|_\infty^\Delta \ \|g\|_1^\Delta,
\end{aligned}$$

and the inequality is proved for $p = \infty$, $q = 1$.

A similar argument applies for $p=1$, $q=\infty$.

If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then applying Holder's integral inequality for double integrals, we deduce that

$$\begin{aligned}
|T(f,g)| &\leq \frac{1}{2(b-a)^2}\int_a^b\int_a^b |f(x)-f(y)|\,|g(x)-g(y)|\,dxdy \\
&\leq \frac{1}{2(b-a)^2}\left(\int_a^b\int_a^b |f(x)-f(y)|^p\,dxdy\right)^{\frac{1}{p}}\left(\int_a^b\int_a^b |g(x)-g(y)|^q\,dxdy\right)^{\frac{1}{q}} \\
&\leq \frac{1}{2(b-a)^2}\|f\|_p^{\Delta}\;\|g\|_q^{\Delta}
\end{aligned}$$

and the theorem is proved. ∎

Using the fact that if $f:[a,b]\to\mathbb{R}$ is absolutely continuous then

$$f(s)-f(t)=\int_t^s f'(u)\,du,$$

the following result was obtained by Cerone and Dragomir [**4**] and the proof is presented here for the sake of completeness.

THEOREM 4.13. *For $f:[a,b]\to\mathbb{R}$ absolutely continuous on $[a,b]$ the following inequalities hold.*

(i) *If $p\in[1,\infty)$, then*

$$\|f\|_p^{\Delta}\leq\begin{cases}\dfrac{2^{\frac{1}{p}}(b-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}}\|f'\|_\infty, & f'\in L_\infty[a,b],\\[2ex] \dfrac{(2\delta^2)^{\frac{1}{p}}(b-a)^{\frac{1}{\delta}+\frac{2}{p}}}{[(p+\delta)(p+2\delta)]^{\frac{1}{p}}}\|f'\|_\gamma, & f'\in L_\gamma[a,b],\\ & \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1\\ (b-a)^{\frac{2}{p}}\|f'\|_1, & \end{cases}\tag{4.91}$$

(ii)

$$\|f\|_\infty^{\Delta}\leq\begin{cases}(b-a)\|f'\|_\infty, & f'\in L_\infty[a,b];\\[1ex] (b-a)^{\frac{1}{\delta}}\|f'\|_\gamma, & f'\in L_\gamma[a,b],\\ & \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1;\\ \|f'\|_1. & \end{cases}\tag{4.92}$$

PROOF. As $f:[a,b]\to\mathbb{R}$ is absolutely continuous, then $f(s)-f(t)=\int_t^s f'(u)\,du$ for all $s,t\in[a,b]$, and then

$$(4.93)\quad |f(s)-f(t)| = \left|\int_t^s f'(u)\,du\right| \le \begin{cases} |s-t|\,\|f'\|_\infty & \text{if } f'\in L_\infty[a,b]; \\ |s-t|^{\frac{1}{\delta}}\,\|f'\|_\gamma & \text{if } f'\in L_\gamma[a,b],\ \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \|f'\|_1 & \text{if } f'\in L_1[a,b] \end{cases}$$

and so for $p\in[1,\infty)$, we may write

$$|f(s)-f(t)|^p \le \begin{cases} |s-t|^p\,\|f'\|_\infty^p & \text{if } f'\in L_\infty[a,b]; \\ |s-t|^{\frac{p}{\delta}}\,\|f'\|_\gamma^p & \text{if } f'\in L_\gamma[a,b],\ \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \|f'\|_1^p & \text{if } f'\in L_1[a,b], \end{cases}$$

and then from (4.89)

$$(4.94)\qquad \|f\|_p^\Delta \le \begin{cases} \|f'\|_\infty\left(\int_a^b\int_a^b |s-t|^p\,dsdt\right)^{\frac{1}{p}} & \text{if } f'\in L_\infty[a,b]; \\ \|f'\|_\gamma\left(\int_a^b\int_a^b |s-t|^{\frac{p}{\delta}}\,dsdt\right)^{\frac{1}{p}} & \text{if } f'\in L_\alpha[a,b], \\ & \quad \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \|f'\|_1\left(\int_a^b\int_a^b dsdt\right)^{\frac{1}{p}} & \text{if } f'\in L_1[a,b]. \end{cases}$$

Further, since

$$\begin{aligned} \left(\int_a^b\int_a^b |s-t|^p\,dsdt\right)^{\frac{1}{p}} &= \left[\int_a^b\left(\int_a^s (s-t)^p\,dt+\int_s^b (t-s)^p\,dt\right)ds\right]^{\frac{1}{p}} \\ &= \left(\int_a^b\left[\frac{(s-a)^{p+1}+(b-s)^{p+1}}{p+1}\right]ds\right)^{\frac{1}{p}} \\ &= \frac{2^{\frac{1}{p}}(b-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}}, \end{aligned}$$

giving

$$\left(\int_a^b\int_a^b |s-t|^{\frac{p}{\beta}}\,dsdt\right)^{\frac{1}{p}} = \frac{\left(2\delta^2\right)^{\frac{1}{p}}(b-a)^{\frac{1}{\delta}+\frac{2}{p}}}{[(p+\delta)(p+2\delta)]^{\frac{1}{p}}},$$

and

$$\left(\int_a^b\int_a^b dsdt\right)^{\frac{1}{p}} = (b-a)^{\frac{2}{p}},$$

we obtain, from (4.94), the stated result (4.91).

Using (4.93) we have (for $p=\infty$) that

$$\|f\|_{\infty}^{\Delta} \leq \begin{cases} \|f'\|_{\infty}\, ess \sup\limits_{(s,t)\in[a,b]^2} |s-t| \\ \\ \|f'\|_{\gamma}\, ess \sup\limits_{(s,t)\in[a,b]} |s-t|^{\frac{1}{\delta}} \\ \\ \|f'\|_1 \end{cases} = \begin{cases} (b-a)\,\|f'\|_{\infty} \\ \\ (b-a)^{\frac{1}{\delta}}\,\|f'\|_{\gamma} \\ \\ \|f'\|_1 \end{cases}$$

and the inequality (4.92) is also proved. ∎

Remark 4.9. If $p=q=2$ is taken in Theorem 4.12 then the first result in (4.50) is obtained.

Theorem 4.14. *Let the mapping $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous, then the following inequality holds, provided the integrals exist. Namely,*

$$\begin{aligned} &\left|\tau_n(x)-(-1)^n U_n(x)\, S_{n-1}(f;a,b)\right| \\ \leq\ & \frac{1}{2(b-a)}\,\|K_n(x,\cdot)\|_q^{\Delta}\left\|f^{(n)}(\cdot)\right\|_p^{\Delta},\ \ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \end{aligned} \tag{4.95}$$

where $K_n(x,t)$ is a Peano kernel defined by (4.8), $U_n(x)=\int_a^b K_n(x,t)\,dt$ and $S_{n-1}(f;a,b)$ is as defined in (4.54).

Proof. Following the proof of Theorem 4.6 and associating $f(t)$ with $(-1)^n K_n(x,t)$ and $g(t)$ with $f^{(n)}(t)$, then from (4.9), (4.48), (4.49) we obtain the result (4.54) giving $(b-a)\,T\left((-1)^n K_n(x,\cdot), f^{(n)}(\cdot)\right)$, the left hand side of (4.95). Now for the bound we have from (4.90) the inequality as given. ∎

Bounds may be obtained that are more easily calculated than those in (4.95) by placing stronger conditions on $K_n(x,\cdot)$ and/or $f^{(n)}(\cdot)$. The following corollary assumes that $f^{(n)}$ is absolutely continuous.

Corollary 4.15. Let the mapping $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous, then

$$\left|\tau_n(x)-(-1)^n U_n(x)\, S_{n-1}(f;a,b)\right| \tag{4.96}$$

$$
\leq \begin{cases}
\frac{1}{2}\left\|K_n(x,\cdot)\right\|_1^{\Delta}\left\|f^{(n+1)}\right\|_{\infty}, & f^{(n+1)} \in L_{\infty}[a,b]; \\
\frac{1}{2}(b-a)^{\frac{1}{\delta}-1}\left\|K_n(x,\cdot)\right\|_1^{\Delta}\left\|f^{(n+1)}\right\|_{\gamma}, & f^{(n+1)} \in L_{\gamma}[a,b],\ \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\
\frac{1}{2(b-a)}\left\|K_n(x,\cdot)\right\|_1^{\Delta}\left\|f^{(n+1)}\right\|_1; & \\
\frac{2^{\frac{1}{p}-1}(b-a)^{\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}}\left\|K_n(x,\cdot)\right\|_q^{\Delta}\left\|f^{(n+1)}\right\|_{\infty}, & f^{(n+1)} \in L_{\infty}[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\frac{2^{\frac{1}{p}-1}\left(\delta^2\right)^{\frac{1}{p}}(b-a)^{\frac{1}{\delta}+\frac{2}{p}-1}}{[(p+\delta)(p+2\delta)]^{\frac{1}{p}}}\left\|K_n(x,\cdot)\right\|_q^{\Delta}\left\|f^{(n+1)}\right\|_{\gamma}, & \\
\quad f^{(n+1)} \in L_{\gamma}[a,b],\ \ \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1, & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\frac{1}{2}(b-a)^{\frac{2}{p}-1}\left\|K_n(x,\cdot)\right\|_q^{\Delta}\left\|f^{(n+1)}\right\|_1, & f^{(n+1)} \in L_1[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\
\frac{(b-a)^2}{6}\left\|K_n(x,\cdot)\right\|_{\infty}^{\Delta}\left\|f^{(n+1)}\right\|_{\infty}, & f^{(n+1)} \in L_{\infty}[a,b]; \\
\frac{\delta^2(b-a)^{\frac{1}{\delta}+1}}{(\delta+1)(2\delta+1)}\left\|K_n(x,\cdot)\right\|_{\infty}^{\Delta}\left\|f^{(n+1)}\right\|_{\gamma}, & f^{(n+1)} \in L_{\gamma}[a,b],\ \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\
\frac{1}{2}(b-a)\left\|K_n(x,\cdot)\right\|_{\infty}^{\Delta}\left\|f^{(n+1)}\right\|_1. &
\end{cases}
$$

Proof. Using result (4.95) of Theorem 4.14, we have that $f^{(n)}(\cdot)$ is absolutely continuous from the assumptions of the current theorem and so the conditions of Theorem 4.13 hold for $f^{(n)}(\cdot)$. ∎

Remark 4.10. If we choose $K_n(x,t)$ from (4.8) such that $K_n(x,t)$ is absolutely continuous, then the results of Theorem 4.13 would hold and $\left\|K_n(x,\cdot)\right\|^{\Delta}$ could be determined in terms of $\frac{\partial K_n(x,t)}{\partial t} \in L_{\cdot}[a,b]$, where the differentiation is taken over each subinterval. This would result in bounds from (4.96) involving 27 branches. Thus, with the assumption of absolute continuity of $K_n(x,\cdot)$ then, using Theorem 4.13 we would have from (4.91) and (4.92), where it is understood that $K_n'(x,\cdot)$ represents differentiation over each subinterval $[a,x]$ and $(x,b]$,

(i) for $q \in [1,\infty)$,

$$
\left\|K_n(x,\cdot)\right\|_q^{\Delta} \leq \begin{cases}
2^{\frac{1}{q}}(b-a)^{1+\frac{2}{q}}\left\|K_n'(x,\cdot)\right\|_{\infty}, & K_n'(x,\cdot) \in L_{\infty}[a,b]; \\
\left(2\eta^2\right)^{\frac{1}{p}}(b-a)^{\frac{1}{\eta}+\frac{2}{q}}\left\|K_n'(x,\cdot)\right\|_{\nu}, & K_n'(x,\cdot) \in L_{\nu}[a,b], \\
 & \nu>1,\ \frac{1}{\nu}+\frac{1}{\eta}=1; \\
(b-a)^{\frac{2}{q}}\left\|K_n'(x,\cdot)\right\|_1, &
\end{cases}
$$

and

(ii)

$$\|K_n(x,\cdot)\|_\infty^\Delta \leq \begin{cases} (b-a)\left\|K_n'(x,\cdot)\right\|_\infty, & K_n'(x,\cdot) \in L_\infty[a,b]; \\ \\ (b-a)^{\frac{1}{\eta}}\left\|K_n'(x,\cdot)\right\|_\nu, & K_n'(x,\cdot) \in L_\nu[a,b], \\ & \nu > 1,\ \frac{1}{\nu}+\frac{1}{\eta}=1; \\ \left\|K_n'(x,\cdot)\right\|_1. \end{cases}$$

Remark 4.11. For $K_n(x,t)$ to be absolutely continuous, it is sufficient to have that, from (4.8), $|p_n(x)| = |q_n(x)|$.
As an example, consider the Simpson type kernel as given by (4.82), then using (4.83) gives

$$p_n^S(x) = \frac{(x-a)^n}{(n-1)!}(1-\lambda_n) \quad \text{and} \quad q_n^S(x) = (-1)^n \frac{(b-x)^n}{(n-1)!}(1-\lambda_n).$$

Thus, the above condition is satisfied for $x = \frac{a+b}{2}$.
Now,

$$\frac{\partial K_n^S}{\partial t}(x,t) = \begin{cases} \dfrac{(t-a)^{n-2}}{(n-1)!}\left[n\left(t-\alpha_n(x)\right)+\lambda_n(x-a)\right], & t \in [a,x] \\ \\ \dfrac{(t-b)^{n-2}}{(n-1)!}\left[n\left(t-\beta_n(x)\right)+\lambda_n(b-x)\right], & t \in (x,b] \end{cases}$$

and so

$$\frac{\partial K_n^S}{\partial t}\left(\frac{a+b}{2},t\right) = \begin{cases} \dfrac{(t-a)^{n-2}}{(n-1)!}\left[n(t-a)-(n-1)\lambda_n\left(\dfrac{b-a}{2}\right)\right], & t \in \left[a,\dfrac{a+b}{2}\right] \\ \\ \dfrac{(t-b)^{n-2}}{(n-1)!}\left[n(t-b)-(n-1)\lambda_n\left(\dfrac{b-a}{2}\right)\right], & t \in \left(\dfrac{a+b}{2},b\right]. \end{cases}$$

from which $\left\|\dfrac{\partial K_n^S}{\partial t}\left(\dfrac{a+b}{2},t\right)\right\|_\nu$, $\nu \in [1,\infty]$ may be obtained explicitly. We omit the details.

4.6. Concluding Remarks

Quadrature rules eminating from Peano kernels involving branches of products of Appell-like polynomials satisfying (4.7) have been investigated in this chapter. The rules involve function evaluations at the boundary points and an interior point. Explicit *a priori* bounds are obtained in terms of a variety of norms so as to allow the possibility of determining the partition required to achieve a prescribed error tolerance.

Perturbed rules are examined and bounds are obtained through detailed study of the Chebychev functional. The work sets the foundation for further investigation of this new class of quadrature rules which includes a great deal of work involving three point rules as special cases.

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CHAPTER 5

Ostrowski Type Inequalities for Multiple Integrals

by

N.S. BARNETT, P. CERONE and S.S. DRAGOMIR

5.1. Introduction

In 1938, A. Ostrowski proved the following integral inequality [**9**, p. 468]

Theorem 5.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)}{(b-a)^2} \right](b-a)\|f'\|_\infty \tag{5.1}$$

for all $x\in[a,b]$.
The constant $\frac{1}{4}$ is the best possible.

For some generalizations of this result see [**9**, p. 468-484] by Mitrinović, Pečarić and Fink. Recent results on Ostrowski's inequality may be found online at: `http://rgmia.vu.edu.au/IneqNumAnaly`

In 1975, G.N. Milovanović generalized Theorem 5.1 to the case where f is a function of several variables.

Following [**8**], let $D=\{(x_1,\dots,x_m)\,|a_i<x_i<b_i \quad (i=1,\dots,m)\}$ and let $\bar{D}$ be the closure of D.

We now propose and prove the following generalisation of Theorem 5.1.

THEOREM 5.2. *Let $f:\mathbb{R}^m\to\mathbb{R}$ be a differentiable function defined on $\bar{D}$ and let $\left|\frac{\partial f}{\partial x_1}\right|\le M_i$ $(M_i>0;\ i=1,\ldots,m)$ in D. Then, for every $X=(x_1,\ldots,x_m)\in\bar{D}$,*

$$\left|f(X)-\frac{1}{\prod_{i=1}^{m}(b_i-a_i)}\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}f(y_1,\ldots,y_m)\,dy_1\cdots dy_m\right| \tag{5.2}$$
$$\le\sum_{i=1}^{m}\left[\frac{1}{4}+\frac{\left(x_i-\frac{a_i+b_i}{2}\right)^2}{(b_i-a_i)^2}\right](b_i-a_i)\,M_i.$$

PROOF. Let $X=(x_1,\ldots,x_m)$ and $Y=(y_1,\ldots,y_m)$ $\big(X\in\bar{D},Y\in D\big)$. According to Taylor's formula, we have

$$f(X)-f(Y)=\sum_{i=1}^{m}\frac{\partial f(C)}{\partial x_i}(x_i-y_i), \tag{5.3}$$

where $C=(y_1+\theta(x_1-y_1),\ldots,y_m+\theta(x_m-y_m))$ $(0<\theta<1)$.

Integrating (5.3), we obtain

$$f(X)\operatorname{mes}D-\int_D\cdots\int f(Y)\,dY=\sum_{i=1}^{m}\int_D\cdots\int\frac{\partial f(C)}{\partial x_i}(x_i-y_i)\,dY, \tag{5.4}$$

where $dY=dy_1\ldots dy_m$ and mes $D=\prod_{i=1}^{m}(b_i-a_i)$.

From (5.4), it follows that,

$$\begin{aligned}\left|f(X)\operatorname{mes}D-\int_D\cdots\int f(Y)\,dY\right| &\le \left|\sum_{i=1}^{m}\int_D\cdots\int\frac{\partial f(C)}{\partial x_i}(x_i-y_i)\,dY\right|\\ &\le \sum_{i=1}^{m}\int_D\cdots\int\left|\frac{\partial f(C)}{\partial x_i}\right|\cdot|x_i-y_i|\,dY,\end{aligned}$$

and

$$\left|f(X)\operatorname{mes}D-\int_D\cdots\int f(Y)\,dY\right|\le\sum_{i=1}^{m}M_i\int_D\cdots\int|x_i-y_i|\,dY, \tag{5.5}$$

respectively, owing to the assumption $\left|\frac{\partial f}{\partial x_i}\right|\le M_i$ $(M_i>0;\ i=1,\ldots,m)$.

Since

$$\int_{a_i}^{b_i}|x_i-y_i|\,dy_i=\frac{1}{4}(b_i-a_i)^2+\left(x_i-\frac{a_i+b_i}{2}\right)^2,$$

we have

$$\begin{aligned}\int_D\cdots\int|x_i-y_i|\,dy_i &= \frac{\operatorname{mes}D}{b_i-a_i}\int_{a_i}^{b_i}|x_i-y_i|\,dy_i\\ &= (\operatorname{mes}D)(b_i-a_i)\left[\frac{1}{4}+\frac{\left(x_i-\frac{a_i+b_i}{2}\right)^2}{(b_i-a_i)^2}\right].\end{aligned}$$

Since mes $D > 0$, inequality (5.5) becomes

$$\begin{aligned} & \left| f\left(x_1,\dots,x_m\right) - \frac{1}{\prod\limits_{i=1}^{m}\left(b_i-a_i\right)} \int_D \cdots \int f\left(Y\right) dY \right| \\ \leq \quad & \sum_{i=1}^{m} \left[\frac{1}{4} + \frac{\left(x_i - \frac{a_i+b_i}{2}\right)^2}{\left(b_i-a_i\right)^2} \right] \left(b_i-a_i\right) M_i, \end{aligned}$$

and the result is proved. ∎

Theorem 5.2 can be generalised as follows (see [**8**]).

THEOREM 5.3. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a differentiable function defined on $\bar{D}$ and $\left|\frac{\partial f}{\partial x_i}\right| \leq M_i$ $(M_i > 0;\ i = 1,\dots,m)$ in D. Furthermore, let function $X \longmapsto p\left(X\right)$ be defined, integrable and $p\left(X\right) > 0$ for every $X \in \bar{D}$. Then, for every $X \in \bar{D}$,*

$$\left| f\left(X\right) - \frac{\int_D \cdots \int p\left(Y\right) f\left(Y\right) dY}{\int_D \cdots \int p\left(Y\right) dY} \right| \leq \frac{\sum\limits_{i=1}^{m} M_i \int_D \cdots \int p\left(Y\right) \left|x_i - y_i\right| dY}{\int_D \cdots \int p\left(Y\right) dY}.$$

This theorem can be proved similarly to Theorem 5.2.

For Theorem 5.4, we use the following notation:

$$\begin{gathered} m, n_i \in \mathbb{N}\ \left(i = 1,\dots,m\right); \\ 0 = a_{i0} < a_{i1} < \cdots < a_{in_i} = 1\ \left(i = 1,\dots,m\right); \\ a_{ik_i-1} \leq x_{ik_i} \leq a_{ik_i},\ \ \lambda_{ik_i} = a_{ik_i} - a_{ik_{i-1}}\ \ \left(k_i = 1,\dots,n_i;\ i = 1,\dots,m\right) \\ k = \left(k_1,\dots,k_m\right),\ \ X = \left(x_1,\dots,x_m\right),\ \ X_k = \left(x_{1k_1},\dots,x_{mk_m}\right); \\ D = \left\{X | 0 < x_i < 1;\ i = 1,\dots,m\right\}; \\ D\left(k\right) = \left\{X_k | a_{ik_{i-1}} < x_{ik_i} < a_{ik_i}\ \left(k_i = 1,\dots,n_i;\ i = 1,\dots,m\right)\right\}; \\ dX = dx_1 \cdots dx_m; \\ E\left(f;k\right) = f\left(X_k\right) - \frac{1}{\prod\limits_{i=1}^{m} \lambda_{ik_i}} \int_{D(k)} \cdots \int f\left(X\right) dX. \end{gathered}$$

THEOREM 5.4. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a differentiable function defined on $\bar{D}$ and $\left|\frac{\partial f}{\partial x_1}\right| \leq M_i$ $(M_i > 0;\ i = 1,\dots,m)$ in D.*
Then

$$\begin{aligned} & \left| \int_D \cdots \int f\left(X\right) dX - \sum_{k_1=1}^{n_i} \cdots \sum_{k_m=1}^{n_m} \lambda_{1k_1} \cdots \lambda_{mk_m} f\left(X_k\right) \right| \\ \leq \quad & \frac{1}{2} \sum_{i=1}^{m} M_i \left(\sum_{k_1=1}^{n_i} H\left(x_{ik_i};k_i\right) \right), \end{aligned} \tag{5.6}$$

where

$$H\left(t;k_i\right) = \left(t - a_{ik_{i-1}}\right)^2 + \left(a_{ik_i} - t\right)^2.$$

PROOF. According to Theorem 5.2, we have

$$|E(f;k)| \leq \frac{1}{2}\sum_{i=1}^{m}\frac{M_i}{\lambda_{ik_i}}H(x_{ik_i};k_i). \tag{5.7}$$

Since $\bigcup_{k_1=1}^{n_1}\cdots\bigcup_{k_m=1}^{n_m}\bar{D}(k)=\bar{D}$, we have

$$\begin{aligned}
&\left|\int_D\cdots\int f(X)\,dX-\sum_{k_1=1}^{n_i}\cdots\sum_{k_m=1}^{n_m}\lambda_{1k_1}\cdots\lambda_{mk_m}f(X_k)\right| \\
= &\left|\sum_{k_1=1}^{n_i}\cdots\sum_{k_m=1}^{n_m}\lambda_{1k_1}\cdots\lambda_{mk_m}E(f;k)\right| \leq \sum_{k_1=1}^{n_i}\cdots\sum_{k_m=1}^{n_m}\lambda_{1k_1}\cdots\lambda_{mk_m}|E(f;k)|.
\end{aligned}$$

Using (5.7), the last inequality becomes

$$\begin{aligned}
&\left|\int_D\cdots\int f(X)\,dX-\sum_{k_1=1}^{n_i}\cdots\sum_{k_m=1}^{n_m}\lambda_{1k_1}\cdots\lambda_{mk_m}f(X_k)\right| \\
\leq\ &\frac{1}{2}\sum_{k_1=1}^{n_i}\cdots\sum_{k_m=1}^{n_m}\lambda_{1k_1}\cdots\lambda_{mk_m}\left(\sum_{i=1}^{m}\frac{M_i}{\lambda_{ik_i}}H(x_{ik_i};k_i)\right) \\
=\ &\frac{1}{2}\sum_{i=1}^{m}M_i\left(\sum_{k_1=1}^{n_i}H(x_{ik_i};k_i)\right).
\end{aligned}$$

The proof is thus completed. ∎

COROLLARY 5.5. If $x_{ik_i}=a_{ik_i}$ or $x_{ik_i}=a_{ik_i-1}$, from (5.4) it follows that

$$\left|\int_D\cdots\int f(X)\,dX-\sum_{k_1=1}^{n_i}\cdots\sum_{k_m=1}^{n_m}\lambda_{1k_1}\cdots\lambda_{mk_m}f(X_k)\right| \leq \frac{1}{2}\sum_{i=1}^{m}M_i\left(\sum_{k_1=1}^{n_i}\lambda_{ik_i}^2\right).$$

Furthermore, if $\lambda_{ik_i}=\frac{1}{n_i}$ holds, then

$$\left|\int_D\cdots\int f(X)\,dX-\frac{1}{n_1\cdots n_m}\sum_{k_1=1}^{n_i}\cdots\sum_{k_m=1}^{n_m}f(X_k)\right| \leq \frac{1}{2}\sum_{i=1}^{m}\frac{M_i}{n_i}.$$

COROLLARY 5.6. If $x_{ik_i}=\frac{1}{2}(a_{ik_i-1}+a_{ik_i})$, holds

$$\left|\int_D\cdots\int f(X)\,dX-\sum_{k_1=1}^{n_i}\cdots\sum_{k_m=1}^{n_m}\lambda_{1k_1}\cdots\lambda_{mk_m}f(X_k)\right| \leq \frac{1}{4}\sum_{i=1}^{m}M_i\left(\sum_{k_1=1}^{n_i}\lambda_{ik_i}^2\right).$$

Furthermore, if $\lambda_{ik_i}=\frac{1}{n_i}$, we have

$$\left|\int_D\cdots\int f(X)\,dX-\frac{1}{n_1\cdots n_m}\sum_{k_1=1}^{n_i}\cdots\sum_{k_m=1}^{n_m}f(X_k)\right| \leq \frac{1}{4}\sum_{i=1}^{m}\frac{M_i}{n_i}.$$

The following result can be found in [**8**].

THEOREM 5.7. *Let $f:\mathbb{R}^m\to\mathbb{R}$ be a differentiable function defined on*

$$D=\{(x_1,...,x_m)\,|a_i\leq x_i\leq b_i\,(i=1,...,m)\}$$

and let $\left|\frac{\partial f}{\partial x_i}\right| \leq M_i \ \ (M_i > 0, i = 1, ..., m)$ *in* D. *Furthermore, let function* $x \longmapsto p(x)$ *be integrable and* $p(x) > 0$ *for every* $x \in D$. *Then for every* $x \in D$, *we have the inequality:*

$$\left| f(x) - \frac{\int_D p(y) f(y) \, dy}{\int_D p(y) \, dy} \right| \leq \frac{\sum_{i=1}^{m} M_i \int_D p(y) |x_i - y_i| \, dy}{\int_D p(y) \, dy}. \tag{5.8}$$

5.2. An Ostrowski Type Inequality for Double Integrals

5.2.1. Some Inequalities in Terms of $\|\cdot\|_\infty$ −Norm. The following inequality of Ostrowski's type for mappings of two variables holds [**1**] (see also [**2**]):

THEOREM 5.8. *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be so that* $f(\cdot,\cdot)$ *is continuous on* $[a,b] \times [c,d]$. *If* $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$ *exists on* $[a,b] \times [c,d]$ *and is in* $L_\infty([a,b] \times [c,d])$, *i.e.,*

$$\left\| f''_{s,t} \right\|_\infty := \underset{(x,y)\in(a,b)\times(c,d)}{ess\sup} \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| < \infty$$

then we have the inequality:

$$\begin{aligned} &\left| \int_a^b \int_c^d f(s,t) \, dtds - \left[(b-a) \int_c^d f(x,t) \, dt + (d-c) \int_a^b f(s,y) \, ds \right.\right. \\ &\left.\left. - (d-c)(b-a) f(x,y) \right] \right| \\ \leq \ & \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2} \right)^2 \right] \left\| f''_{s,t} \right\|_\infty \end{aligned} \tag{5.9}$$

for all $(x,y) \in [a,b] \times [c,d]$.

PROOF. We have the equality:

$$\begin{aligned} & \int_a^x \int_c^y (s-a)(t-c) f''_{s,t}(s,t) \, dtds \\ = \ & \int_a^x (s-a) [f'_s(s,y)(y-c) - \int_c^y f'_s(s,t) \, dt] ds \\ = \ & (y-c) \int_a^x (s-a) f'_s(s,y) \, ds - \int_c^y \left(\int_a^x (s-a) f'_s(s,t) \, ds \right) dt \\ = \ & (y-c) \left[(x-a) f(x,y) - \int_a^x f(s,y) \, ds \right] \\ & - \int_c^y \left[(x-a) f(x,t) - \int_a^x f(s,t) \, ds \right] dt \\ = \ & (y-c)(x-a) f(x,y) - (y-c) \int_a^x f(s,y) \, ds \\ & - (x-a) \int_c^y f(x,t) \, dt + \int_a^x \int_c^y f(s,t) \, dtds. \end{aligned} \tag{5.10}$$

By similar computations we have,

$$
\begin{aligned}
(5.11)\qquad & \int_a^x \int_y^d (s-a)(t-d) f''_{s,t}(s,t)\,dtds \\
&= \int_a^x (s-a)\left[(d-y) f'_s(s,y) - \int_y^d f'_s(s,t)\,dt\right] ds \\
&= (d-y)\int_a^x (s-a) f'_s(s,y)\,ds - \int_y^d \left(\int_a^x (s-a) f'_s(s,t)\,ds\right) dt
\end{aligned}
$$

$$
\begin{aligned}
&= (d-y)\left[(x-a) f(x,y) - \int_a^x f(s,y)\,ds\right] \\
&\quad - \int_y^d \left[(x-a) f(x,t) - \int_a^x f(s,t)\,ds\right] dt \\
&= (x-a)(d-y) f(x,y) - (d-y)\int_a^x f(s,y)\,ds \\
&\quad - (x-a)\int_y^d f(x,t)\,dt + \int_a^x \int_c^y f(s,t)\,dtds.
\end{aligned}
$$

Now,

$$
\begin{aligned}
(5.12)\qquad & \int_x^b \int_y^d (s-b)(t-d) f''_{s,t}(s,t)\,dtds \\
&= \int_x^b (s-b)\left[(d-y) f'_s(s,y) - \int_y^d f'_s(s,t)\,dt\right] ds \\
&= (d-y)\int_x^b (s-b) f'_s(s,y)\,ds - \int_y^d \left(\int_x^b (s-b) f'_s(s,t)\,ds\right) dt \\
&= (d-y)\left[(b-x) f(x,y) - \int_x^b f(s,y)\,ds\right] \\
&\quad - \int_y^d \left[(b-x) f(x,t) - \int_x^b f(s,t)\,ds\right] dt \\
&= (d-y)(b-x) f(x,y) - (d-y)\int_x^b f(s,y)\,ds \\
&\quad - (b-x)\int_y^d f(x,t)\,dt + \int_x^b \int_y^d f(s,t)\,dtds
\end{aligned}
$$

and finally

$$
\begin{aligned}
(5.13)\qquad & \int_x^b \int_c^y (s-b)(t-c) f''_{s,t}(s,t)\, dtds \\
&= \int_x^b (s-b)\left[(y-c) f'_s(s,y) - \int_c^y f'_s(s,t)\, dt\right] ds \\
&= (y-c)\int_x^b (s-b) f'_s(s,y)\, ds - \int_c^y \left(\int_x^b (s-b) f'_s(s,t)\, ds\right) dt \\
&= (y-c)\left[(b-x) f(x,y) - \int_x^b f(s,y)\, ds\right] \\
&\quad - \int_c^y \left[(b-x) f(x,t) - \int_x^b f(s,t)\, ds\right] dt \\
&= (y-c)(b-x) f(x,y) - (y-c)\int_x^b f(s,y)\, ds \\
&\quad - (b-x)\int_c^y f(x,t)\, dt + \int_x^b \int_c^y f(s,t)\, dtds.
\end{aligned}
$$

If we add the equalities (5.10) – (5.13) we get in the right membership:

$$
\begin{aligned}
& [(y-c)(x-a) + (x-a)(d-y) + (d-y)(b-x) + (y-c)(b-x)] f(x,y) \\
& - (d-c)\int_a^x f(s,y)\, ds - (d-c)\int_x^b f(s,y)\, ds - (b-a)\int_c^y f(x,t)\, dt \\
& - (b-a)\int_y^d f(x,t)\, dt + \int_a^x \int_c^y f(s,t)\, dtds + \int_a^x \int_y^d f(s,t)\, dtds \\
& + \int_x^b \int_y^d f(s,t)\, dtds + \int_x^b \int_c^y f(s,t)\, dtds \\
= {} & (d-c)(b-a) f(x,y) - (d-c)\int_a^b f(s,y)\, ds \\
& - (b-a)\int_c^b f(x,t)\, dt + \int_a^b \int_c^d f(s,t)\, dtds.
\end{aligned}
$$

Define the kernels: $p : [a,b]^2 \to \mathbb{R}$, $q : [c,d]^2 \to \mathbb{R}$ given by:

$$
p(x,s) := \begin{cases} s-a & \text{if } s \in [a,x] \\ s-b & \text{if } s \in (x,b] \end{cases}
$$

and

$$
q(y,t) := \begin{cases} t-c & \text{if } t \in [c,y] \\ t-d & \text{if } t \in (y,d] \end{cases}.
$$

Now, using these, we deduce that the left hand side of this sum can be represented as :

$$
\int_a^b \int_c^d p(x,s)\, q(y,t)\, f''_{s,t}(s,t)\, dtds.
$$

Consequently, we get the identity

$$
\begin{aligned}
(5.14) \qquad & \int_a^b \int_c^d p(x,s)\, q(y,t)\, f''_{s,t}(s,t)\, dt ds \\
= \; & (d-c)(b-a) f(x,y) - (d-c)\int_a^b f(x,y)\, ds \\
& - (b-a)\int_c^d f(x,t)\, dt + \int_a^b \int_c^d f(s,t)\, dt ds
\end{aligned}
$$

for all $(x,y) \in [a,b] \times [c,d]$.

Using the identity (5.14) we obtain,

$$
\begin{aligned}
(5.15) \qquad & \left| \int_a^b \int_c^d f(s,t)\, dt ds - \left[(b-a)\int_c^d f(x,t)\, dt \right.\right. \\
& \left.\left. + (d-c)\int_a^b f(x,y)\, ds - (d-c)(b-a) f(x,y) \right] \right| \\
\leq \; & \int_a^b \int_c^d |p(x,s)|\, |q(y,t)| \left| f''_{s,t}(s,t) \right| dt ds \\
\leq \; & \left\| f''_{s,t} \right\|_\infty \int_a^b \int_c^d |p(x,s)|\, |q(y,t)|\, dt ds.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\int_a^b |p(x,s)|\, ds &= \int_a^x (s-a)\, ds + \int_x^b (b-s)\, ds \\
&= \frac{(x-a)^2 + (b-x)^2}{2} = \frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2
\end{aligned}
$$

and, similarly,

$$
\int_c^d |q(y,t)|\, dt = \frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2} \right)^2.
$$

Finally, using (5.15) , we have the desired inequality (5.9) . ∎

Corollary 5.9. Under the above assumptions, we have the inequality:

$$
\begin{aligned}
(5.16) \qquad & \left| \int_a^b \int_c^d f(s,t)\, dt ds - \left[(b-a)\int_c^d f\left(\frac{a+b}{2}, t \right) dt \right.\right. \\
& \left.\left. + (d-c)\int_a^b f\left(s, \frac{c+d}{2} \right) ds - (d-c)(b-a) f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right] \right| \\
\leq \; & \frac{1}{16}(b-a)^2 (d-c)^2 \left\| f''_{s,t} \right\|_\infty .
\end{aligned}
$$

Remark 5.1. The constants $\frac{1}{4}$ from the first and the second bracket are optimal in the sense that not both of them can be less than $\frac{1}{4}$.

Indeed, if we had assumed that there exists $c_1, c_2 \in \left(0, \frac{1}{4}\right)$ so that

$$\begin{aligned} (5.17) \quad & \left| \int_a^b \int_c^d f(s,t)\, dtds - \left[(b-a) \int_c^d f(x,t)\, dt + (d-c) \int_a^b f(s,y)\, ds \right.\right. \\ & \left.\left. - (d-c)(b-a) f(x,y) \right] \right| \\ \leq \; & \left[c_1 (b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \left[c_2 (d-c)^2 + \left(y - \frac{c+d}{2} \right)^2 \right] \left\| f''_{s,t} \right\|_\infty \end{aligned}$$

for all f as in Theorem 5.8 and $(x,y) \in [a,b] \times [c,d]$, then we would have had, for $f(s,t) = st$ and $x = a, y = c$,

$$\begin{aligned} \int_a^b \int_c^d f(s,t)\, dtds &= \frac{\left(b^2 - a^2\right)\left(d^2 - c^2\right)}{4}, \\ \int_c^d f(x,t)\, dt &= a \cdot \frac{d^2 - c^2}{2}, \int_a^b f(s,y)\, ds = c \cdot \frac{b^2 - a^2}{2}, \\ \text{and} \quad \left\| f''_{s,t} \right\|_\infty &= 1. \end{aligned}$$

By (5.17), and

$$\begin{aligned} & \left| \frac{\left(b^2 - a^2\right)\left(d^2 - c^2\right)}{4} - (b-a)\, a \cdot \frac{d^2 - c^2}{2} - (d-c)\, c \cdot \frac{b^2 - a^2}{2} + (d-c)(b-a)\, ac \right| \\ \leq \; & (b-a)^2 \left(c_1 + \frac{1}{4} \right) (d-c)^2 \left(c_2 + \frac{1}{4} \right) \end{aligned}$$

giving

$$\frac{(b-a)^2 (d-c)^2}{4} \leq (b-a)^2 (d-c)^2 \left(c_1 + \frac{1}{4} \right) \left(c_2 + \frac{1}{4} \right)$$

i.e.

$$(5.18) \qquad \frac{1}{4} \leq \left(c_1 + \frac{1}{4} \right) \left(c_2 + \frac{1}{4} \right).$$

As we have assumed that $c_1, c_2 \in \left(0, \frac{1}{4}\right)$, we get

$$c_1 + \frac{1}{4} < \frac{1}{2}, c_2 + \frac{1}{4} < \frac{1}{2}$$

and then $\left(c_1 + \frac{1}{4}\right)\left(c_2 + \frac{1}{4}\right) < \frac{1}{4}$ which contradicts the inequality (5.18), and establishes the remark 5.1.

Remark 5.2. If we assume that $f(s,t) = h(s)\, h(t)$, $h : [a,b] \to \mathbb{R}$ and suppose that $\|h'\|_\infty < \infty$, then from (5.9) we get (for $x = y$)

$$\begin{aligned} & \left| \int_a^b h(s)\, ds \int_a^b h(s)\, ds - h(x)(b-a) \int_a^b h(s)\, ds \right. \\ & \left. - h(x)(b-a) \int_a^b h(s)\, ds + (b-a)^2 h^2(x) \right| \\ \leq \; & \left[\frac{1}{4} (b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right]^2 \|h\|_\infty^2 \end{aligned}$$

i.e.

$$\left[\int_a^b h(s)\,ds - h(x)(b-a)\right]^2 \le \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)\right]^2 \|h\|_\infty^2$$

which is clearly equivalent to Ostrowski's inequality.

Consequently (5.9) can be also regarded as a generalization, for double integrals, of the classical result due to Ostrowski.

5.2.2. Applications for Cubature Formulae. Let us now consider the arbitrary division $I_n : a = x_1 < x_1 < \dots < x_{n-1} < x_n = b$ and $J_m : c = y_0 < y_1 < \dots < y_{m-1} < y_m = b$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, \dots, n-1)$, $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, \dots, m-1)$ be intermediate points. Consider the sum

$$\begin{aligned} C(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta}) &:= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} f(\xi_i, t)\,dt \\ &+ \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f(s, \eta_j)\,ds - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i l_j f(\xi_i, \eta_j) \end{aligned} \tag{5.19}$$

for which we assume that the involved integrals can more easily be computed than the original double integral

$$D := \int_a^b \int_c^d f(s,t)\,dsdt,$$

and

$$h_i := x_{i+1} - x_i\,(i = 0, \dots, n-1), \qquad l_j := y_{j+1} - y_j \quad (j = 0, \dots, m-1).$$

With this assumption, we can state the following cubature formula:

THEOREM 5.10. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be as in Theorem 5.8 and $I_n, J_m, \boldsymbol{\xi}$ and $\boldsymbol{\eta}$ be as above. Then we have the cubature formula:*

$$\int_a^b \int_c^d f(s,t)\,dtds = C(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta}) + R(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta}) \tag{5.20}$$

where the remainder term $R(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta})$ satisfies the estimation:

$$\begin{aligned} &|R(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta})| \\ \le\ & \left\|f''_{s,t}\right\|_\infty \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]\left[\frac{1}{4}l_j^2 + \left(\eta_j - \frac{y_j + y_{j+1}}{2}\right)^2\right] \\ \le\ & \frac{1}{4}\left\|f''_{s,t}\right\|_\infty \sum_{i=0}^{n-1} h_i^2 \sum_{j=0}^{m-1} l_j^2. \end{aligned} \tag{5.21}$$

Proof. Apply Theorem 5.8 on the interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ $(i = 0, ..., n-1; j = 0, ..., m-1)$ to get:

$$\begin{aligned}
&\left| \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} f(s,t)\, dt ds - \left[h_i \int_{y_j}^{y_{j+1}} f(\xi_i, t)\, dt \right.\right. \\
&\left.\left. + \; l_j \int_{x_i}^{x_{i+1}} f(s, \eta_j)\, ds - h_i l_j f(\xi_i, \eta_j) \right] \right| \\
\leq \;& \left[\frac{1}{4} h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \left[\frac{1}{4} l_j^2 + \left(\eta_j - \frac{y_j + y_{j+1}}{2} \right)^2 \right] \left\| f''_{s,t} \right\|_\infty
\end{aligned}$$

for all $i = 0, ..., n-1; \quad j = 0, ..., m-1.$

Summing over i from 0 to $n-1$ and over j from 0 to $m-1$ and using the generalized triangle inequality we deduce the first inequality in (5.21).

For the second part we observe that

$$\left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \leq \frac{1}{2} h_i \quad \text{and} \quad \left| \eta_j - \frac{y_j + y_{j+1}}{2} \right| \leq \frac{1}{2} l_j$$

for all i, j as above. ∎

Remark 5.3. As

$$\sum_{i=0}^{n-1} h_i^2 \leq \nu(h) \sum_{i=0}^{n-1} h_i = (b-a)\, \nu(h)$$

and

$$\sum_{j=0}^{m-1} l_j^2 \leq \mu(l) \sum_{j=0}^{m-1} l_j = (d-c)\, \mu(l)$$

where

$$\nu(h) = \max\{h_i : i = 0, ..., n-1\}$$

and

$$\mu(l) = \max\{l_j : j = 0, ..., m-1\},$$

the right membership of (5.21) can be bounded by

$$\frac{1}{4} \left\| f''_{s,t} \right\|_\infty (b-a)(d-c)\, \nu(h)\, \mu(l),$$

which is of order precision 2.

Define the sum,

$$\begin{aligned}
&C_M(f, I_n, J_m) \\
: \; &= \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} f\left(\frac{x_i + x_{i+1}}{2}, t \right) dt + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f\left(s, \frac{y_j + y_{j+1}}{2} \right) ds \\
&- \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i l_j f\left(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2} \right),
\end{aligned}$$

then we have the best cubature formula possible from (5.20).

Corollary 5.11. Under the above assumptions we have

$$\int_a^b \int_c^d f(s,t)\,dtds = C_M(f, I_n, J_m) + R(f, I_n, J_m) \tag{5.22}$$

when the remainder $R(f, I_n, J_m)$ satisfies the estimation:

$$|R(f, I_n, J_m)| \le \frac{\left\| f''_{s,t} \right\|_\infty}{16} \sum_{i=0}^{n-1} h_i^2 \sum_{j=0}^{m-1} l_j^2.$$

5.2.3. Some Inequalities in Terms of $\|\cdot\|_p$ −Norm. The following inequality of Ostrowski's type for mappings of two variables holds [**3**]:

Theorem 5.12. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a continuous mapping on $[a,b] \times [c,d]$, $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$ exists on $[a,b] \times [c,d]$ and is in $L_p([a,b] \times [c,d])$, i.e.,*

$$\left\| f''_{s,t} \right\|_p := \left(\int_a^b \int_c^d \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right|^p dxdy \right)^{\frac{1}{p}} < \infty, \qquad p > 1$$

then we have the inequality:

$$\begin{aligned} &\left| \int_a^b \int_c^d f(s,t)\,dtds - \left[(b-a) \int_c^d f(x,t)\,dt + (d-c) \int_a^b f(s,y)\,ds \right. \right. \\ &\left. \left. - (d-c)(b-a) f(x,y) \right] \right| \\ \le\ & \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \left[\frac{(y-c)^{q+1} + (d-y)^{q+1}}{q+1} \right]^{\frac{1}{q}} \left\| f''_{s,t} \right\|_p \end{aligned} \tag{5.23}$$

for all $(x,y) \in [a,b] \times [c,d]$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. If we consider the kernels: $p : [a,b]^2 \to \mathbb{R}$, $q : [c,d]^2 \to \mathbb{R}$ given by:

$$p(x,s) := \begin{cases} s-a & \text{if } s \in [a,x] \\ \\ s-b & \text{if } s \in (x,b] \end{cases}$$

and

$$q(y,t) := \begin{cases} t-c & \text{if } t \in [c,y] \\ \\ t-d & \text{if } s \in (y,d] \end{cases}$$

then we have the identity, (see also (5.14))

$$\begin{aligned} & \int_a^b \int_c^d p(x,s)\, q(y,t)\, f''_{s,t}(s,t)\,dsdt \\ =\ & (d-c)(b-a) f(x,y) - (d-c) \int_a^b f(s,y)\,ds \\ & - (b-a) \int_c^d f(x,t)\,dt + \int_a^b \int_c^d f(s,t)\,dsdt \end{aligned} \tag{5.24}$$

for all $(x,y) \in [a,b] \times [c,d]$.

From this we can obtain the following inequality:-

$$\begin{aligned}
&\left|\int_a^b \int_c^d f(s,t)\,dsdt - [(b-a)\int_c^d f(x,t)\,dt + (d-c)\int_a^b f(s,y)\,ds \right.\\
&\left. - (d-c)(b-a) f(x,y)]\right| \\
\leq \; & \int_a^b \int_c^d |p(x,s)\,q(y,t)| \left|f''_{s,t}(s,t)\right| dsdt.
\end{aligned}$$

Using Hölder's integral inequality for double integrals, we further have,

$$\begin{aligned}
&\int_a^b \int_c^d |p(x,s)\,q(y,t)| \left|f''_{s,t}(s,t)\right| dtds \\
\leq \; & \left(\int_a^b \int_c^d |p(x,s)\,q(y,t)|^q\, dtds\right)^{\frac{1}{q}} \left(\int_a^b \int_c^d \left|f''_{s,t}(s,t)\right|^p dtds\right)^{\frac{1}{p}} \\
= \; & \left(\int_a^b |p(x,s)|^q\, ds\right)^{\frac{1}{q}} \left(\int_c^d |q(y,t)|^q\, dt\right)^{\frac{1}{q}} \left\|f''_{s,t}\right\|_p \\
= \; & \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}} \left[\frac{(y-c)^{q+1} + (d-y)^{q+1}}{q+1}\right]^{\frac{1}{q}} \left\|f''_{s,t}\right\|_p
\end{aligned}$$

and the theorem is proved. ∎

Corollary 5.13. Under the above assumptions, we have the inequality:

$$\begin{aligned}
&\left|\int_a^b \int_c^d f(s,t)\,dtds - \left[(b-a)\int_c^d f\left(\frac{a+b}{2},t\right) dt \right.\right.\\
&\left.\left. + (d-c)\int_a^b f\left(s,\frac{c+d}{2}\right) ds - (d-c)(b-a) f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\right| \\
\leq \; & \frac{(b-a)^{1+\frac{1}{q}}(d-c)^{1+\frac{1}{q}}}{4(q+1)^{\frac{2}{q}}} \left\|f''_{s,t}\right\|_p .
\end{aligned} \tag{5.25}$$

Remark 5.4. Consider the mapping $g : [\alpha,\beta] \to \mathbb{R}, g(t) = (t-\alpha)^m + (\beta-t)^m$, $(m \geq 1)$. Taking into account the properties

$$\inf_{t\in[\alpha,\beta]} g(t) = g\left(\frac{\alpha+\beta}{2}\right) = \frac{(\beta-\alpha)^m}{2^{m-1}}$$

and

$$\sup_{t\in[\alpha,\beta]} g(t) = g(\alpha) = g(\beta) = (\beta-\alpha)^m,$$

(5.25) is seen to be the best inequality that can be obtained from (5.23).

REMARK 5.5. Now, if we assume that $f(s,t) = h(s)h(t)$, $h:[a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and suppose that $\|h'\|_p < \infty$, then from (5.23) we get (for $x = y$)

$$\begin{aligned} & \left| \int_a^b h(s)\,ds \int_a^b h(s)\,ds - h(x)(b-a)\int_a^b h(s)\,ds \right. \\ & \left. -h(x)(b-a)\int_a^b h(s)\,ds + (b-a)^2 h^2(x) \right| \\ \leq \; & \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{2}{q}} \|h'\|_p^2, \end{aligned}$$

i.e.,

$$\left[\int_a^b h(s)\,ds - h(x)(b-a) \right]^2 \leq \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{2}{q}} \|h'\|_p^2$$

which is clearly equivalent to Ostrowski's inequality for $p-$norms obtained in [**6**]

5.2.4. Applications For Cubature Formulae. Let us consider the arbitrary division $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ and $J_m : c = y_0 < y_1 < ... < y_{m-1} < y_m = b$ and $\xi_i \in [x_i, x_{i+1}]$ $\quad (i = 0, ..., n-1)$, $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, ..., m-1)$ be intermediate points. Consider the sum

$$\begin{aligned} C(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta}) \quad : \quad & = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} f(\xi_i, t)\,dt \\ & + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f(s, \eta_j)\,ds - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i l_j f(\xi_i, \eta_j) \end{aligned}$$

for which we assume that the involved integrals can more easily be computed than the original double integral

$$D := \int_a^b \int_c^d f(s,t)\,dsdt,$$

and

$$h_i := x_{i+1} - x_i \,(i = 0, ..., n-1), \quad l_j := y_{j+1} - y_j \,(j = 0, ..., m-1).$$

We can state the following cubature formula:

THEOREM 5.14. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be as in Theorem 5.23 and $I_n, J_m, \boldsymbol{\xi}$ and $\boldsymbol{\eta}$ be as above. Then we have,*

$$\int_a^b \int_c^d f(s,t)\,dtds = C(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta}) + R(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta}),$$

where the remainder term $R(f,I_n,J_m,\boldsymbol{\xi},\boldsymbol{\eta})$ satisfies the inequality (estimation),

$$
\begin{aligned}
(5.26)\qquad & |R(f,I_n,J_m,\boldsymbol{\xi},\boldsymbol{\eta})| \\
\leq\ & \left\|f''_{s,t}\right\|_p \left[\sum_{i=0}^{n-1}\left(\frac{(x_{i+1}-\xi_i)^{q+1}+(\xi_i-x_i)^{q+1}}{q+1}\right)\right]^{\frac{1}{q}} \\
& \times\left[\sum_{j=0}^{m-1}\left(\frac{\left(y_{j+1}-\eta_j\right)^{q+1}+\left(\eta_j-y_j\right)^{q+1}}{q+1}\right)\right]^{\frac{1}{q}} \\
\leq\ & \frac{\left\|f''_{s,t}\right\|_p}{(q+1)^{\frac{2}{q}}}\sum_{i=0}^{n-1}h_i^{1+\frac{1}{q}}\sum_{j=0}^{m-1}l_j^{1+\frac{1}{q}}.
\end{aligned}
$$

for all $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ as above.

Proof. Apply Theorem 5.12 to the interval $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$ $(i=0,...,n-1;j=0,...,m-1)$ to get:

$$
\begin{aligned}
& \Bigg|\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f(s,t)\,dtds \\
& -\left[h_i\int_{y_j}^{y_{j+1}} f(\xi_i,t)\,dt + l_j\int_{x_i}^{x_{i+1}} f\left(s,\eta_j\right)ds - h_i l_j f\left(\xi_i,\eta_j\right)\right]\Bigg| \\
\leq\ & \left[\left(\frac{(x_{i+1}-\xi_i)^{q+1}+(\xi_i-x_i)^{q+1}}{q+1}\right)\left(\frac{\left(y_{j+1}-\eta_j\right)^{q+1}+\left(\eta_j-y_j\right)^{q+1}}{q+1}\right)\right]^{\frac{1}{q}} \\
& \times\left(\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} |\, f(s,t)\,|^p\,dtds\right)^{\frac{1}{p}},
\end{aligned}
$$

for all $i=0,...,n-1;j=0,...,m-1$.

Summing over i from 0 to $n-1$ and over j from 0 to $m-1$ and using the generalized triangle inequality and Hölder's discrete inequality for double sums, we deduce

$$\begin{aligned}
&\left|R\left(f,I_n,J_m,\boldsymbol{\xi},\boldsymbol{\eta}\right)\right| \\
\leq\ & \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[\left(\frac{\left(x_{i+1}-\xi_i\right)^{q+1}+\left(\xi_i-x_i\right)^{q+1}}{q+1}\right)\right. \\
&\left.\times\left(\frac{\left(y_{j+1}-\eta_j\right)^{q+1}+\left(\eta_j-y_j\right)^{q+1}}{q+1}\right)\right]^{\frac{1}{q}}\times\left(\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\left|f\left(s,t\right)\right|^p dtds\right)^{\frac{1}{p}} \\
\leq\ & \left[\sum_{i=0}^{n-1}\left(\frac{\left(x_{i+1}-\xi_i\right)^{q+1}+\left(\xi_i-x_i\right)^{q+1}}{q+1}\right)\right]^{\frac{1}{q}} \\
&\times\left[\sum_{j=0}^{m-1}\left(\frac{\left(y_{j+1}-\eta_j\right)^{q+1}+\left(\eta_j-y_j\right)^{q+1}}{q+1}\right)\right]^{\frac{1}{q}} \\
&\times\left[\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\left|f\left(s,t\right)\right|^p dtds\right]^{\frac{1}{p}} \\
=\ & \left[\sum_{i=0}^{n-1}\left(\frac{\left(x_{i+1}-\xi_i\right)^{q+1}+\left(\xi_i-x_i\right)^{q+1}}{q+1}\right)\right. \\
&\left.\sum_{j=0}^{m-1}\left(\frac{\left(y_{j+1}-\eta_j\right)^{q+1}+\left(\eta_j-y_j\right)^{q+1}}{q+1}\right)\right]^{\frac{1}{q}}\times\left\|f''_{s,t}\right\|_p.
\end{aligned}$$

To prove the second part, we observe that

$$\left(x_{i+1}-\xi_i\right)^{q+1}+\left(\xi_i-x_i\right)^{q+1}\leq\left(x_{i+1}-x_i\right)^{q+1}$$

and

$$\left(y_{j+1}-\eta_j\right)^{q+1}+\left(\eta_j-y_j\right)^{q+1}\leq\left(y_{j+1}-y_j\right)^{q+1}$$

for all i,j as above and the intermediate points ξ_i and η_j.

We omit the details. ∎

REMARK 5.6. As

$$\sum_{i=0}^{n-1}h_i^{1+\frac{1}{q}}\leq\left[\nu\left(h\right)\right]^{\frac{1}{q}}\sum_{i=0}^{n-1}h_i=\left(b-a\right)\left[\nu\left(h\right)\right]^{\frac{1}{q}}$$

and

$$\sum_{j=0}^{m-1}l_j^{1+\frac{1}{q}}\leq\left[\mu\left(l\right)\right]^{\frac{1}{q}}\sum_{j=0}^{m-1}l_j=\left(d-c\right)\left[\mu\left(l\right)\right]^{\frac{1}{q}}$$

where

$$\nu\left(h\right)=\max\left\{h_i:i=0,...,n-1\right\},$$

and

$$\mu\left(l\right)=\max\left\{l_j:j=0,...,m-1\right\},$$

the right hand side of (5.26) can be bounded by

$$\frac{1}{(q+1)^{\frac{2}{q}}}\left\|f''_{s,t}\right\|_p (b-a)(d-c)\left[\nu(h)\mu(l)\right]^{\frac{1}{q}}.$$

Defining the sum,

$$\begin{aligned} C_M(f,I_n,J_m) \quad : \quad &= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} f\left(\frac{x_i+x_{i+1}}{2},t\right)dt \\ &+ \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f\left(s,\frac{y_j+y_{j+1}}{2}\right)ds \\ &- \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i l_j f\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right), \end{aligned}$$

we have the best cubature formula possible from Theorem 5.14.

Corollary 5.15. Under the above assumptions we have,

$$\int_a^b\int_c^d f(s,t)\,dsdt = C_M(f,I_n,J_m) + R(f,I_n,J_m),$$

where the remainder $R(f,I_n,J_m)$ satisfies the inequality (estimation),

$$|R(f,I_n,J_m)| \le \frac{1}{4(q+1)^{\frac{2}{q}}}\left\|f''_{s,t}\right\|_p \sum_{i=0}^{n-1} h_i^{1+\frac{1}{q}} \sum_{j=0}^{m-1} l_j^{1+\frac{1}{q}}.$$

5.2.5. Some Inequalities in Terms of $\|\cdot\|_1$ −Norm. The following result of Ostrowski's type also holds.

Theorem 5.16. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a continuous mapping on $[a,b]\times[c,d]$, $f''_{x,y}=\frac{\partial^2 f}{\partial x\partial y}$ exists on $[a,b]\times[c,d]$ and is in $L_1([a,b]\times[c,d])$, i.e.,*

$$\left\|f''_{s,t}\right\|_1 := \int_a^b\int_c^d \left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right| dxdy < \infty,$$

then we have the inequality,

$$\begin{aligned} &\left|\int_a^b\int_c^d f(s,t)\,dtds - \left[(b-a)\int_c^d f(x,t)\,dt\right.\right. \\ &\left.\left. + (d-c)\int_a^b f(s,y)\,ds - (d-c)(b-a)f(x,y)\right]\right| \\ \le\ &\left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\left[\frac{1}{2}+\frac{\left|y-\frac{c+d}{2}\right|}{d-c}\right](b-a)(d-c)\left\|f''_{s,t}\right\|_1 \end{aligned} \tag{5.27}$$

for all $(x,y)\in[a,b]\times[c,d]$.

Proof. As in the proof of Theorem 5.12, we use the inequality

$$\begin{aligned}
&\left|\int_a^b\int_c^d f(s,t)\,dtds-\left[(b-a)\int_c^d f(x,t)\,dt\right.\right. \\
&\left.\left.+(d-c)\int_a^b f(s,y)\,ds-(d-c)(b-a)f(x,y)\right]\right| \\
\leq\ & \int_a^b\int_c^d |p(x,s)\,q(y,t)|\left|f''_{s,t}(s,t)\right|dtds. \qquad (5.28)
\end{aligned}$$

However, it is easy to see that

$$\begin{aligned}
&\int_a^b\int_c^d |p(x,s)|\,|q(y,t)|\left|f''_{s,t}(s,t)\right|dtds \\
=\ & \sup_{s\in[a,b]}|p(x,s)|\cdot\sup_{t\in[c,d]}|q(y,t)|\left\|f''_{s,t}\right\|_1 \\
=\ & \max\{x-a,b-x\}\cdot\max\{d-y,y-c\}\left\|f''_{s,t}\right\|_1 \\
=\ & \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]\left\|f''_{s,t}\right\|_1,
\end{aligned}$$

which, via (5.28), proves the desired inequality (5.27). ∎

The best inequality we can get from Theorem 5.16 is embodied in the following corollary.

Corollary 5.17. With the above assumptions, we have,

$$\begin{aligned}
&\left|\int_a^b\int_c^d f(s,t)\,dtds-\left[(b-a)\int_c^d f\left(\frac{a+b}{2},t\right)dt\right.\right. \\
&\left.\left.+(d-c)\int_a^b f\left(s,\frac{c+d}{2}\right)ds-(d-c)(b-a)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\right| \\
\leq\ & \frac{1}{4}(b-a)(d-c)\left\|f''_{s,t}\right\|_1. \qquad (5.29)
\end{aligned}$$

Remark 5.7. If we assume that $f(s,t)=h(s)\,h(t)$, $h:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and suppose that $h'\in L_1[a,b]$, then from (5.27) we get (for $x=y$)

$$\begin{aligned}
&\left|\int_a^b h(s)\,ds\int_a^b h(s)\,ds-h(x)(b-a)\int_a^b h(s)\,ds\right. \\
&\left.-h(x)(b-a)\int_a^b h(s)\,ds+(b-a)^2h^2(x)\right| \\
\leq\ & \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^2\|h'\|_1^2
\end{aligned}$$

i.e.,

$$\left[\int_a^b h(s)\,ds-h(x)(b-a)\right]^2\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^2\|h'\|_1^2,$$

which is clearly equivalent to Ostrowski's inequality for 1−norms obtained in [**7**].

Applications for cubature formulae can be provided in a similar fashion as above, but we omit the detials.

5.3. Other Ostrowski Type Inequalities

5.3.1. Some Identities. The following theorem holds [**5**].

THEOREM 5.18. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be such that the partial derivatives $\frac{\partial f(t,s)}{\partial t}$, $\frac{\partial f(t,s)}{\partial s}$, $\frac{\partial^2 f(t,s)}{\partial t\partial s}$ exist and are continuous on $[a,b]\times[c,d]$. Then for all $(x,y)\in[a,b]\times[c,d]$, we have the representation*

$$\begin{aligned} f(x,y) &= \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt \\ &+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p(x,t)\frac{\partial f(t,s)}{\partial t}dsdt \\ &+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d q(y,s)\frac{\partial f(t,s)}{\partial s}dsdt \\ &+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p(x,t)\,q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt, \end{aligned} \tag{5.30}$$

where $p:[a,b]^2\to\mathbb{R}$, $q:[c,d]^2\to\mathbb{R}$ and are given by

$$p(x,t):=\begin{cases} t-a & \text{if} \quad t\in[a,x] \\ t-b & \text{if} \quad t\in(x,b] \end{cases}, \tag{5.31}$$

and

$$q(y,s):=\begin{cases} s-c & \text{if} \quad s\in[c,y] \\ s-d & \text{if} \quad s\in(y,d] \end{cases}. \tag{5.32}$$

PROOF. We use the following identity, which can be easily proved using integration by parts

$$g(u)=\frac{1}{\beta-\alpha}\int_\alpha^\beta g(z)\,dz+\frac{1}{\beta-\alpha}\int_\alpha^\beta k(u,z)\,g'(z)\,dz, \tag{5.33}$$

where $k:[\alpha,\beta]^2\to\mathbb{R}$ is given by

$$k(u,z):=\begin{cases} z-\alpha & \text{if} \quad z\in[\alpha,u] \\ z-\beta & \text{if} \quad z\in(u,\beta] \end{cases}$$

and g is absolutely continuous on $[\alpha,\beta]$.

Indeed, we have

$$\int_\alpha^u(z-\alpha)\,g'(z)\,dz=(u-\alpha)\,g(u)-\int_\alpha^u g(z)\,dz$$

and

$$\int_u^\beta(z-\beta)\,g'(z)\,dz=(\beta-u)\,g(u)-\int_u^\beta g(z)\,dz$$

which produces, by summation, the desired identity (5.33).

We can write the identity (5.33) for the partial map $f(\cdot, y)$, $y \in [c,d]$ to obtain

$$f(x,y) = \frac{1}{b-a}\int_a^b f(t,y)\,dt + \frac{1}{b-a}\int_a^b p(x,t)\frac{\partial f(t,y)}{\partial t}dt \tag{5.34}$$

for all $(x,y) \in [a,b] \times [c,d]$.

Similarly,

$$f(t,y) = \frac{1}{d-c}\int_c^d f(t,s)\,ds + \frac{1}{d-c}\int_c^d q(y,s)\frac{\partial f(t,s)}{\partial s}ds, \tag{5.35}$$

for all $(t,y) \in [a,b] \times [c,d]$.

The same formula (5.33) applied for the partial derivative $\frac{\partial f(\cdot,y)}{\partial t}$ gives,

$$\frac{\partial f(t,y)}{\partial t} = \frac{1}{d-c}\int_c^d \frac{\partial f(t,s)}{\partial t}ds + \frac{1}{d-c}\int_c^d q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}ds \tag{5.36}$$

for all $(t,y) \in [a,b] \times [c,d]$.

Substituting (5.35) and (5.36) in (5.34), and using Fubini's theorem, we have

$$\begin{aligned} f(x,y) &= \frac{1}{b-a}\int_a^b \left[\frac{1}{d-c}\int_c^d f(t,s)\,ds + \frac{1}{d-c}\int_c^d q(y,s)\frac{\partial f(t,s)}{\partial s}ds\right]dt \\ &\quad + \frac{1}{b-a}\int_a^b p(x,t)\left[\frac{1}{d-c}\int_c^d \frac{\partial f(t,s)}{\partial t}ds\right. \\ &\quad \left. + \frac{1}{d-c}\int_c^d q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}ds\right]dt \\ &= \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d f(t,s)\,dsdt + \int_a^b\int_c^d q(y,s)\frac{\partial f(t,s)}{\partial s}dsdt\right. \\ &\quad \left. + \int_a^b\int_c^d p(x,t)\frac{\partial f(t,s)}{\partial t}dsdt + \int_a^b\int_c^d p(x,t)\,q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right], \end{aligned}$$

and the identity (5.30) is established. ∎

A particular case which is of interest is embodied in the following corollary.

Corollary 5.19. Let f be as in Theorem 5.18, then we have the identity

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ &= \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d f(t,s)\,dsdt + \int_a^b\int_c^d p_0(t)\frac{\partial f(t,s)}{\partial t}dsdt\right. \\ &\quad \left. + \int_a^b\int_c^d q_0(s)\frac{\partial f(t,s)}{\partial s}dsdt + \int_a^b\int_c^d p_0(t)\,q_0(s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right], \end{aligned} \tag{5.37}$$

where $p_0 : [a,b] \to \mathbb{R},\ q_0 : [c,d] \to \mathbb{R}$ are given by

$$p_0(t) := \begin{cases} t-a & \text{if} \quad t \in \left[a, \frac{a+b}{2}\right] \\ t-b & \text{if} \quad t \in \left(\frac{a+b}{2}, b\right] \end{cases},$$

and

$$q_0(s) := \begin{cases} s-c & \text{if} \quad s \in \left[c, \frac{c+d}{2}\right] \\ s-d & \text{if} \quad s \in \left(\frac{c+d}{2}, d\right] \end{cases}.$$

The following corollary, which provides a trapezoid type identity, is also of interest.

Corollary 5.20. Let f be as in Theorem 5.18. Then we have the identity

$$\begin{aligned} &\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\ &= \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d f(t,s)\,dsdt + \int_a^b\int_c^d \left(t-\frac{a+b}{2}\right)\frac{\partial f(t,s)}{\partial t}dsdt\right. \\ &\quad + \int_a^b\int_c^d \left(s-\frac{c+d}{2}\right)\frac{\partial f(t,s)}{\partial s}dsdt \\ &\quad \left. + \int_a^b\int_c^d \left(t-\frac{a+b}{2}\right)\left(s-\frac{c+d}{2}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right]. \end{aligned} \tag{5.38}$$

Proof. Letting $(x,y) = (a,c),\ (a,d),\ (b,c)$ and (b,d) in (5.30), we obtain successively,

$$\begin{aligned} f(a,c) &= \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d f(t,s)\,dsdt + \int_a^b\int_c^d (t-b)\frac{\partial f(t,s)}{\partial t}dsdt\right. \\ &\quad \left. + \int_a^b\int_c^d (s-d)\frac{\partial f(t,s)}{\partial s}dsdt + \int_a^b\int_c^d (t-b)(s-d)\frac{\partial^2 f(t,s)}{\partial s\partial t}dsdt\right], \end{aligned}$$

$$\begin{aligned} f(a,d) &= \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d f(t,s)\,dsdt + \int_a^b\int_c^d (t-b)\frac{\partial f(t,s)}{\partial t}dsdt\right. \\ &\quad \left. + \int_a^b\int_c^d (s-c)\frac{\partial f(t,s)}{\partial s}dsdt + \int_a^b\int_c^d (t-b)(s-c)\frac{\partial^2 f(t,s)}{\partial s\partial t}dsdt\right], \end{aligned}$$

$$\begin{aligned} f(b,c) &= \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d f(t,s)\,dsdt + \int_a^b\int_c^d (t-a)\frac{\partial f(t,s)}{\partial t}dsdt\right. \\ &\quad \left. + \int_a^b\int_c^d (s-d)\frac{\partial f(t,s)}{\partial s}dsdt + \int_a^b\int_c^d (t-a)(s-d)\frac{\partial^2 f(t,s)}{\partial s\partial t}dsdt\right], \end{aligned}$$

and

$$\begin{aligned} f(b,d) &= \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d f(t,s)\,dsdt + \int_a^b\int_c^d (t-a)\frac{\partial f(t,s)}{\partial t}dsdt\right. \\ &\quad \left. + \int_a^b\int_c^d (s-c)\frac{\partial f(t,s)}{\partial s}dsdt + \int_a^b\int_c^d (t-a)(s-c)\frac{\partial^2 f(t,s)}{\partial s\partial t}dsdt\right]. \end{aligned}$$

After summing over the above equalities, dividing by 4 and some simple computation, we arrive at the desired identity (5.38). ∎

5.3.2. Some Bounds. We can state the following inequality of the Ostrowski type which holds for mappings of two independent variables [**5**].

THEOREM 5.21. *Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be a mapping as in Theorem 5.18. Then we have the inequality:*

$$\begin{aligned} &\left|f(x,y) - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right| \\ &\leq M_1(x) + M_2(y) + M_3(x,y), \end{aligned} \tag{5.39}$$

where

$$M_1(x) = \begin{cases} \dfrac{\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right]}{b-a}\left\|\dfrac{\partial f}{\partial t}\right\|_\infty, & \text{if } \quad \dfrac{\partial f(t,s)}{\partial t} \in L_\infty([a,b]\times[c,d]); \\ \dfrac{\left[\frac{(b-x)^{q_1+1}+(x-a)^{q_1+1}}{q_1+1}\right]^{\frac{1}{q_1}}}{(b-a)\left[(d-c)\right]^{\frac{1}{p_1}}}\left\|\dfrac{\partial f}{\partial t}\right\|_{p_1}, & \text{if } \quad \dfrac{\partial f(t,s)}{\partial t} \in L_{p_1}([a,b]\times[c,d]), \\ & \quad p_1 > 1,\ \frac{1}{p_1} + \frac{1}{q_1} = 1; \\ \dfrac{\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]}{(b-a)(d-c)}\left\|\dfrac{\partial f}{\partial t}\right\|_1, & \text{if } \quad \dfrac{\partial f(t,s)}{\partial t} \in L_1([a,b]\times[c,d]). \end{cases}$$

$$M_2(y) = \begin{cases} \dfrac{\left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2}\right)^2\right]}{d-c}\left\|\dfrac{\partial f}{\partial s}\right\|_\infty, & \text{if } \quad \dfrac{\partial f(t,s)}{\partial s} \in L_\infty([a,b]\times[c,d]); \\ \dfrac{\left[\frac{(d-y)^{q_2+1}+(y-c)^{q_2+1}}{q_2+1}\right]^{\frac{1}{q_2}}}{\left[(b-a)\right]^{\frac{1}{p_2}}(d-c)}\left\|\dfrac{\partial f}{\partial s}\right\|_{p_2}, & \text{if } \quad \dfrac{\partial f(t,s)}{\partial s} \in L_{p_2}([a,b]\times[c,d]); \\ & \quad p_2 > 1,\ \frac{1}{p_2} + \frac{1}{q_2} = 1; \\ \dfrac{\left[\frac{1}{2}(d-c) + \left|y - \frac{c+d}{2}\right|\right]}{(b-a)(d-c)}\left\|\dfrac{\partial f}{\partial s}\right\|_1, & \text{if } \quad \dfrac{\partial f(t,s)}{\partial s} \in L_1([a,b]\times[c,d]); \end{cases}$$

and

$$M_3(x,y) = \begin{cases} \dfrac{\left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2 + \left(y-\frac{c+d}{2}\right)^2\right]}{(b-a)(d-c)} \left\|\dfrac{\partial^2 f}{\partial t \partial s}\right\|_\infty, \\ \text{if} \quad \dfrac{\partial^2 f(t,s)}{\partial s \partial t} \in L_\infty([a,b]\times[c,d]); \\ \dfrac{\left[\frac{(b-x)^{q_3+1}+(x-a)^{q_3+1}}{q_3+1}\right]^{\frac{1}{q_3}}\left[\frac{(d-y)^{q_3+1}+(y-c)^{q_3+1}}{q_3+1}\right]^{\frac{1}{q_3}}}{(b-a)(d-c)} \left\|\dfrac{\partial^2 f}{\partial t \partial s}\right\|_{p_3}, \\ \text{if} \quad \dfrac{\partial^2 f(t,s)}{\partial s \partial t} \in L_{p_3}([a,b]\times[c,d]),\ p_3>1,\ \frac{1}{p_3}+\frac{1}{q_3}=1; \\ \dfrac{\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]}{(b-a)(d-c)} \left\|\dfrac{\partial^2 f}{\partial t \partial s}\right\|_1, \\ \text{if} \quad \dfrac{\partial^2 f(t,s)}{\partial s \partial t} \in L_1([a,b]\times[c,d]); \end{cases}$$

for all $(x,y) \in [a,b]\times[c,d]$, *where* $\|\cdot\|_p$ $(1 \le p \le \infty)$ *are the usual* $p-$*norms on* $[a,b]\times[c,d]$.

Proof. Using the identity (5.30), we can state that

$$\left|f(x,y) - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right| \tag{5.40}$$
$$\le \left|\frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d p(x,t)\frac{\partial f(t,s)}{\partial t}dsdt\right.\right.$$
$$\left.\left.+\int_a^b\int_c^d q(y,s)\frac{\partial f(t,s)}{\partial s}dsdt + \int_a^b\int_c^d p(x,t)\,q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right]\right|$$
$$\le \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d |p(x,t)|\left|\frac{\partial f(t,s)}{\partial t}\right|dsdt\right.$$
$$\left.+\int_a^b\int_c^d |q(y,s)|\left|\frac{\partial f(t,s)}{\partial s}\right|dsdt + \int_a^b\int_c^d |p(x,t)|\,|q(y,s)|\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|dsdt\right].$$

We have,

$$\int_a^b\int_c^d |p(x,t)|\left|\frac{\partial f(t,s)}{\partial t}\right|dsdt \tag{5.41}$$
$$\le \begin{cases} \left\|\frac{\partial f}{\partial t}\right\|_\infty \int_a^b\int_c^d |p(x,t)|\,dsdt, & \text{if} \quad \frac{\partial f(t,s)}{\partial t} \in L_\infty([a,b]\times[c,d]); \\ \left\|\frac{\partial f}{\partial t}\right\|_{p_1}\left(\int_a^b\int_c^d |p(x,t)|^{q_1}dsdt\right)^{\frac{1}{q_1}}, & \text{if} \quad \frac{\partial f(t,s)}{\partial t} \in L_{p_1}([a,b]\times[c,d]), \\ & p_1>1,\ \frac{1}{p_1}+\frac{1}{q_1}=1; \\ \left\|\frac{\partial f}{\partial t}\right\|_1 \sup_{t\in[a,b]}|p(x,t)|, & \text{if} \quad \frac{\partial f(t,s)}{\partial t} \in L_1([a,b]\times[c,d]). \end{cases}$$

and as

$$\begin{aligned}
&\int_a^b \int_c^d |p(x,t)|\, dsdt \\
=\;& \int_c^d \left(\int_a^b |p(x,t)|\, dt \right) ds = (d-c) \left[\int_a^x |p(x,t)|\, dt + \int_x^b |p(x,t)|\, dt \right] \\
=\;& (d-c) \left[\int_a^x (t-a)\, dt + \int_x^b (b-t)\, dt \right] \\
=\;& (d-c) \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right],
\end{aligned}$$

$$\begin{aligned}
\left[\int_a^b \int_c^d |p(x,t)|^{q_1}\, dsdt \right]^{\frac{1}{q_1}} &= \left[\int_c^d \left(\int_a^b |p(x,t)|^{q_1}\, dt \right) ds \right]^{\frac{1}{q_1}} \\
&= (d-c)^{\frac{1}{q_1}} \left[\int_a^x |p(x,t)|^{q_1}\, dt + \int_x^b |p(x,t)|^{q_1}\, dt \right]^{\frac{1}{q_1}} \\
&= (d-c)^{\frac{1}{q_1}} \left[\int_a^x (t-a)^{q_1}\, dt + \int_x^b (b-t)^{q_1}\, dt \right]^{\frac{1}{q_1}} \\
&= (d-c)^{\frac{1}{q_1}} \left[\frac{(b-x)^{q_1+1} + (x-a)^{q_1+1}}{q_1+1} \right]^{\frac{1}{q_1}}.
\end{aligned}$$

$$\sup_{t\in[a,b]} |p(x,t)| = \max\{x-a, b-x\} = \frac{b-a}{2} + \left| x - \frac{a+b}{2} \right|,$$

and then, by (5.41), we obtain

$$\int_a^b \int_c^d |p(x,t)| \left| \frac{\partial f(t,s)}{\partial t} \right| dsdt \tag{5.42}$$

(5.43)

$$\leq \begin{cases} (d-c)\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \left\| \frac{\partial f}{\partial t} \right\|_\infty, & \text{if } \frac{\partial f(t,s)}{\partial t} \in L_\infty([a,b]\times[c,d]); \\ (d-c)^{\frac{1}{q_1}} \left[\frac{(b-x)^{q_1+1}+(x-a)^{q_1+1}}{q_1+1}\right]^{\frac{1}{q_1}} \left\| \frac{\partial f}{\partial t} \right\|_{p_1}, & \text{if } \frac{\partial f(t,s)}{\partial t} \in L_{p_1}([a,b]\times[c,d]), \\ & p_1 > 1,\ \frac{1}{p_1} + \frac{1}{q_1} = 1; \\ \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \left\| \frac{\partial f}{\partial t} \right\|_1. & \text{if } \frac{\partial f(t,s)}{\partial t} \in L_1([a,b]\times[c,d]). \end{cases}$$

In a similar fashion, we can state,

$$\int_a^b \int_c^d |q(y,s)| \left| \frac{\partial f(t,s)}{\partial s} \right| dsdt \tag{5.44}$$

$$(5.45)\quad \leq \begin{cases} (b-a)\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]\left\|\frac{\partial f}{\partial s}\right\|_\infty, & \text{if } \frac{\partial f(t,s)}{\partial s}\in L_\infty([a,b]\times[c,d])\,; \\ (b-a)^{\frac{1}{q_2}}\left[\frac{(d-y)^{q_2+1}+(y-c)^{q_2+1}}{q_2+1}\right]^{\frac{1}{q_2}}\left\|\frac{\partial f}{\partial s}\right\|_{p_2}, & \text{if } \frac{\partial f(t,s)}{\partial s}\in L_{p_2}([a,b]\times[c,d])\,, \\ & p_2>1,\ \frac{1}{p_2}+\frac{1}{q_2}=1; \\ \left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]\left\|\frac{\partial f}{\partial s}\right\|_1. & \text{if } \frac{\partial f(t,s)}{\partial s}\in L_1([a,b]\times[c,d])\,. \end{cases}$$

In addition, we have

$$\int_a^b\int_c^d |p(x,t)|\,|q(y,s)|\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right| dsdt \tag{5.46}$$

$$\leq \begin{cases} \left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty \int_a^b |p(x,t)|\,dt\int_c^d |q(y,s)|\,ds, \\ \text{if } \frac{\partial^2 f(t,s)}{\partial s\partial t}\in L_\infty([a,b]\times[c,d])\,; \\ \left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_{p_3}\left(\int_a^b |p(x,t)|^{q_3}\,dt\right)^{\frac{1}{q_3}}\left(\int_c^d |q(y,s)|^{q_3}\,ds\right)^{\frac{1}{q_3}}, \\ \text{if } \frac{\partial^2 f(t,s)}{\partial s\partial t}\in L_{p_3}([a,b]\times[c,d])\,,\ p_3>1,\ \frac{1}{p_3}+\frac{1}{q_3}=1; \\ \left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_1 \sup_{t\in[a,b]}|p(x,t)|\sup_{s\in[c,d]}|q(y,s)| \\ \text{if } \frac{\partial^2 f(t,s)}{\partial s\partial t}\in L_1([a,b]\times[c,d])\,. \end{cases} \tag{5.47}$$

$$= \begin{cases} \left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty, \\ \text{if } \frac{\partial^2 f(t,s)}{\partial s\partial t}\in L_\infty([a,b]\times[c,d])\,; \\ \left[\frac{(b-x)^{q_3+1}+(x-a)^{q_3+1}}{q_3+1}\right]^{\frac{1}{q_3}}\left[\frac{(d-y)^{q_3+1}+(y-c)^{q_3+1}}{q_3+1}\right]^{\frac{1}{q_3}}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{p_3}, \\ \text{if } \frac{\partial^2 f(t,s)}{\partial s\partial t}\in L_{p_3}([a,b]\times[c,d])\,;\ p_3>1,\ \frac{1}{p_3}+\frac{1}{q_3}=1; \\ \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_1, \\ \text{if } \frac{\partial^2 f(t,s)}{\partial s\partial t}\in L_1([a,b]\times[c,d])\,. \end{cases}$$

and the theorem is proved. ∎

The following corollary holds by taking $x=\frac{a+b}{2}$, $y=\frac{c+d}{2}$.

COROLLARY 5.22. With the assumptions in Theorem 5.18, we have the inequality

$$
\begin{aligned}
(5.48)\qquad & \left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right| \\
\leq\ & \tilde{M}_1 + \tilde{M}_2 + \tilde{M}_3,
\end{aligned}
$$

where

$$
\tilde{M}_1 := \begin{cases}
\frac{1}{4}(b-a)\left\|\frac{\partial f}{\partial t}\right\|_\infty, & \text{if } \frac{\partial f}{\partial t}\in L_\infty([a,b]\times[c,d]) \\
\frac{1}{2}\left[\frac{(b-a)^{\frac{1}{q_1}}}{(q_1+1)^{\frac{1}{q_1}}(d-c)^{\frac{1}{p_1}}}\right]\left\|\frac{\partial f}{\partial t}\right\|_{p_1}, & \text{if } \frac{\partial f}{\partial t}\in L_{p_1}([a,b]\times[c,d]) \\
& p_1>1,\ \frac{1}{p_1}+\frac{1}{q_1}=1; \\
\frac{1}{2(d-c)}\left\|\frac{\partial f}{\partial t}\right\|_1, & \text{if } \frac{\partial f}{\partial t}\in L_1([a,b]\times[c,d])
\end{cases}
$$

$$
\tilde{M}_2 := \begin{cases}
\frac{1}{4}(d-c)\left\|\frac{\partial f}{\partial s}\right\|_\infty, & \text{if } \frac{\partial f}{\partial t}\in L_\infty([a,b]\times[c,d]) \\
\frac{1}{2}\left[\frac{(d-c)^{\frac{1}{q_2}}}{(q_2+1)^{\frac{1}{q_2}}(b-a)^{\frac{1}{p_2}}}\right]\left\|\frac{\partial f}{\partial s}\right\|_{p_2} & \text{if } \frac{\partial f}{\partial t}\in L_{p_2}([a,b]\times[c,d]) \\
& p_2>1,\ \frac{1}{p_2}+\frac{1}{q_2}=1; \\
\frac{1}{2(b-a)}\left\|\frac{\partial f}{\partial s}\right\|_1, & \text{if } \frac{\partial f}{\partial t}\in L_1([a,b]\times[c,d])
\end{cases}
$$

and

$$
\tilde{M}_3 := \begin{cases}
\frac{1}{16}(b-a)(d-c)\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty, & \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_\infty([a,b]\times[c,d]); \\
\frac{1}{4}\cdot\frac{(b-a)^{\frac{1}{q_3}}(d-c)^{\frac{1}{q_3}}}{(q_3+1)^{\frac{2}{q_3}}}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{p_3}, & \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_{p_3}([a,b]\times[c,d]); \\
& p_3>1,\ \frac{1}{p_3}+\frac{1}{q_3}=1, \\
\frac{1}{4}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_1, & \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_1([a,b]\times[c,d]).
\end{cases}
$$

Using the inequality (5.38) in Corollary 5.20 and a similar argument to the one used in Theorem 5.21, we have the following trapezoid type inequality:

COROLLARY 5.23. With the assumption in Theorem 5.18, we have the inequality

$$
\begin{aligned}
(5.49)\qquad & \left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right| \\
\leq\ & \tilde{M}_1+\tilde{M}_2+\tilde{M}_3,
\end{aligned}
$$

where $\tilde{M}_i\ \ (i=1,2,3)$ are as given above.

5.3.3. Applications for Cubature Formulae. Consider the arbitrary divisions $I_n := a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ and $J_m : c = y_0 < y_1 < ... < y_{m-1} < y_m = d$, where $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$, $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, ..., m-1)$ are intermediate points. Consider further the Riemann sum:

$$R(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i l_j f(\xi_i, \eta_j), \tag{5.50}$$

where $h_i := x_{i+1} - x_i$, $l_j := y_{j+1} - y_j$, $i = 0, ..., n-1$, $j = 0, ..., m-1$.

Using Theorem 5.21, we can state twenty-seven different inequalities bounding the quantity

$$\left| f(x, y) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s) \, ds dt \right|, \quad (x, y) \in [a, b] \times [c, d]. \tag{5.51}$$

Consider one, namely, when all the partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial s}$, $\frac{\partial^2 f}{\partial t \partial s}$ are bounded. We then have:-,

$$\begin{aligned} & \left| f(x, y) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s) \, ds dt \right| \\ \leq \; & \frac{1}{b-a} \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \left\| \frac{\partial f}{\partial t} \right\|_\infty \\ & + \frac{1}{d-c} \left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2} \right)^2 \right] \left\| \frac{\partial f}{\partial s} \right\|_\infty \\ & + \frac{1}{(b-a)(d-c)} \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \\ & \times \left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2} \right)^2 \right] \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \end{aligned} \tag{5.52}$$

for all $[x, y] \in [a, b] \times [c, d]$.

Using this inequality, we can state the following theorem [**5**].

THEOREM 5.24. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be as in Theorem 5.18, then we have*

$$\int_a^b \int_c^d f(t, s) \, ds dt = R(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta}) + W(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta}), \tag{5.53}$$

where $R(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta})$ *is the Riemann sum defined by* (5.50) *and the remainder, through the approximation* $W(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta})$, *satisfies,*

$$
\begin{aligned}
(5.54)\quad & |W(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta})| \\
\leq\ & (d-c)\left\|\frac{\partial f}{\partial t}\right\|_\infty \sum_{i=0}^{n-1}\left[\frac{1}{4}h_i^2+\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right] \\
& +(b-a)\left\|\frac{\partial f}{\partial s}\right\|_\infty \sum_{j=0}^{m-1}\left[\frac{1}{4}l_j^2+\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)^2\right] \\
& +\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty \sum_{i=0}^{n-1}\left[\frac{1}{4}h_i^2+\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right] \\
& \times\sum_{j=0}^{m-1}\left[\frac{1}{4}l_j^2+\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)^2\right] \\
\leq\ & \frac{1}{2}(d-c)\left\|\frac{\partial f}{\partial t}\right\|_\infty \sum_{i=0}^{n-1}h_i^2+\frac{1}{2}(b-a)\left\|\frac{\partial f}{\partial s}\right\|_\infty \sum_{j=0}^{m-1}l_j^2 \\
& +\frac{1}{4}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty \sum_{i=0}^{n-1}h_i^2\sum_{j=0}^{m-1}l_j^2 \\
\leq\ & \frac{1}{2}(d-c)(b-a)\left[\nu(h)\left\|\frac{\partial f}{\partial t}\right\|_\infty+\nu(l)\left\|\frac{\partial f}{\partial s}\right\|_\infty+\frac{1}{2}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty \nu(h)\nu(l)\right]
\end{aligned}
$$

for all ξ, η *intermediate points, where* $\nu(h) := \max\{h_i, i=0,...,n-1\}$ *and* $\nu(l) := \max\{l_j, j=0,...,m-1\}$.

Proof. Apply (5.52) in the intervals $[x_i, x_{i+1}]\times[y_j, y_{j+1}]$ to obtain

$$
\begin{aligned}
& \left|\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f(t,s)\,ds dt-h_i l_j f(\xi_i,\eta_j)\right| \\
\leq\ & \left[\frac{1}{4}h_i^2+\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right] l_j\left\|\frac{\partial f}{\partial t}\right\|_\infty \\
& +\left[\frac{1}{4}l_j^2+\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)^2\right] h_i\left\|\frac{\partial f}{\partial s}\right\|_\infty \\
& +\left[\frac{1}{4}h_i^2+\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right]\left[\frac{1}{4}l_j^2+\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)^2\right]\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty
\end{aligned}
$$

for all $i=0,...,n-1,\ j=0,...,m-1$.

Summing over i from 0 to $n-1$ and over j from 0 to $m-1$, we get the desired estimation (5.54). ∎

Consider the mid-point formula:

$$
(5.55)\qquad M(f, I_n, J_m) := \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i\, l_j\, f\left(\frac{x_i+x_{i+1}}{2}, \frac{y_j+y_{j+1}}{2}\right).
$$

The following corollary contains the best quadrature formula we can obtain from (5.54).

Corollary 5.25. Let f be as in Theorem 5.18, then we have:

$$\int_a^b \int_c^d f(t,s)\,dsdt = M(f, I_n, J_m) + L(f, I_n, J_m), \tag{5.56}$$

where $M(f, I_n, J_m)$ is the midpoint formula given by (5.55) and the remainder $L(f, I_n, J_m)$ satisfies the estimate

$$\begin{aligned}
&|L(f, I_n, J_m)| \\
\leq\;& \frac{1}{4}(d-c)\left\|\frac{\partial f}{\partial t}\right\|_\infty \sum_{i=0}^{n-1} h_i^2 + \frac{1}{4}(b-a)\left\|\frac{\partial f}{\partial s}\right\|_\infty \sum_{j=0}^{m-1} l_j^2 \\
&+\frac{1}{16}\left\|\frac{\partial^2 f}{\partial t \partial s}\right\|_\infty \sum_{i=0}^{n-1} h_i^2 \sum_{j=0}^{m-1} l_j^2 \\
:\;&= \mathcal{M}_1(f, I_n, J_m) \\
\leq\;& \frac{1}{4}(d-c)(b-a)\left[\nu(h)\left\|\frac{\partial f}{\partial t}\right\|_\infty + \nu(l)\left\|\frac{\partial f}{\partial s}\right\|_\infty + \frac{1}{4}\nu(h)\nu(l)\left\|\frac{\partial^2 f}{\partial t \partial s}\right\|_\infty\right] \\
:\;&= \mathcal{M}_2(f, I_n, J_m).
\end{aligned} \tag{5.57}$$

We can also consider the trapezoid formula

$$\begin{aligned}
&T(f, I_n, J_m) \\
:\;&= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i\, l_j \cdot \frac{f(x_i, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})}{4}.
\end{aligned} \tag{5.58}$$

Using Corollary 5.23 and a similar argument to the one used in the proof of Theorem 5.24, we can state the following,

Corollary 5.26. Let f be as in Theorem 5.18, then we have

$$\int_a^b \int_c^d f(t,s)\,dsdt = T(f, I_n, J_m) + P(f, I_n, J_m), \tag{5.59}$$

where $T(f, I_n, J_m)$ is the trapezoid formula obtained by (5.58) and the remainder $P(f, I_n, J_m)$ satisfies the estimate

$$|P(f, I_n, J_m)| \leq \mathcal{M}_1(f, I_n, J_m) \leq \mathcal{M}_2(f, I_n, J_m), \tag{5.60}$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are as defined above.

5.4. Ostrowski's Inequality for Hölder Type Functions

5.4.1. The Unweighted Case. We start with the following Ostrowski type inequality for mappings of the $r-$Hölder type (see [**4**]).

THEOREM 5.27. *Assume that the mapping $f : [a_1, b_1] \times ... \times [a_n, b_n] \to \mathbb{R}$ satisfies the following $r-$Hölder type condition:*

$$\text{(H)} \qquad |f(\bar{\mathbf{x}}) - f(\bar{\mathbf{y}})| \le \sum_{i=1}^{n} L_i |x_i - y_i|^{r_i} \, (L_i \ge 0, i = 1, ..., n)$$

for all $\bar{\mathbf{x}} = (x_1, ..., x_n)$, $\bar{\mathbf{y}} = (y_1, ..., y_n) \in [\bar{\mathbf{a}}, \bar{\mathbf{b}}] := [a_1, b_1] \times ... \times [a_n, b_n]$, where $r_i \in (0, 1]$, $i = 1, ..., n$. We have then the Ostrowski type inequality:

$$(5.61) \qquad \begin{aligned} & \left| f(\bar{\mathbf{x}}) - \frac{1}{\prod_{i=1}^{n} (b_i - a_i)} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} f(\bar{\mathbf{t}}) \, d\bar{\mathbf{t}} \right| \\ \le \; & \sum_{i=1}^{n} \frac{L_i}{r_i + 1} \left[\left(\frac{x_i - a_i}{b_i - a_i} \right)^{r_i + 1} + \left(\frac{b_i - x_i}{b_i - a_i} \right)^{r_i + 1} \right] (b_i - a_i)^{r_i} \\ \le \; & \sum_{i=1}^{n} \frac{L_i (b_i - a_i)^{r_i}}{r_i + 1}, \end{aligned}$$

for all $\bar{\mathbf{x}} \in [\bar{\mathbf{a}}, \bar{\mathbf{b}}]$, where $\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} f(\bar{\mathbf{t}}) \, d\bar{\mathbf{t}} = \int_a^b ... \int_a^b f(t_1, ..., t_n) \, dt_n ... dt_1$.

PROOF. By (H) we have

$$|f(\bar{\mathbf{x}}) - f(\bar{\mathbf{t}})| \le \sum_{i=1}^{n} L_i |x_i - t_i|^{r_i}$$

for all $\bar{\mathbf{x}}, \bar{\mathbf{t}} \in [\bar{\mathbf{a}}, \bar{\mathbf{b}}]$.

Integrating over $\bar{\mathbf{t}}$ on $[\bar{\mathbf{a}}, \bar{\mathbf{b}}]$ and using the modulus properties we get

$$(5.62) \quad \begin{aligned} \left| f(\bar{\mathbf{x}}) \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} d\bar{\mathbf{t}} - \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} f(\bar{\mathbf{t}}) \, d\bar{\mathbf{t}} \right| & \le \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |f(\bar{\mathbf{x}}) - f(\bar{\mathbf{t}})| \, d\bar{\mathbf{t}} \\ & \le \sum_{i=1}^{n} L_i \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} |x_i - t_i|^{r_i} \, dt_n ... dt_1. \end{aligned}$$

As

$$\int_{a_1}^{b_1} ... \int_{a_n}^{b_n} dt_n ... dt_1 = \prod_{i=1}^{n} (b_i - a_i)$$

and

$$\begin{aligned}
& \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} |x_i - t_i|^{r_i} \, dt_n \dots dt_1 \\
= & \prod_{\substack{j=1 \\ j \neq i}}^{n} (b_j - a_j) \int_{a_i}^{b_i} |x_i - t_i|^{r_i} \, dt_i \\
= & \prod_{\substack{j=1 \\ j \neq i}}^{n} (b_j - a_j) \left[\frac{(b_i - x_i)^{r_i+1} + (x_i - a_i)^{r_i+1}}{r_i + 1} \right] \\
= & \prod_{j=1}^{n} (b_j - a_j) \frac{1}{r_i + 1} \left[\left(\frac{b_i - x_i}{b_i - a_i} \right)^{r_i+1} + \left(\frac{x_i - a_i}{b_i - a_i} \right)^{r_i+1} \right] (b_i - a_i)^{r_i},
\end{aligned}$$

then, dividing (5.62) by $\prod_{j=1}^{n} (b_j - a_j)$ we get the first part of (5.61).

Using the elementary inequality

$$(y - \alpha)^{p+1} + (\beta - y)^{p+1} \leq (\beta - \alpha)^{p+1}$$

for all $\alpha \leq y \leq \beta$ and $p > 0$, we obtain,

$$\left(\frac{b_i - x_i}{b_i - a_i} \right)^{r_i+1} + \left(\frac{x_i - a_i}{b_i - a_i} \right)^{r_i+1} \leq 1, i = 1, \dots, n$$

and the last part of (5.61) is also proved. ∎

Some particular cases are interesting.

Corollary 5.28. Under the above assumptions, we have the mid-point inequality:

$$\left| f\left(\frac{a_1 + b_1}{2}, \dots, \frac{a_n + b_n}{2} \right) - \frac{1}{\prod_{i=1}^{n} (b_i - a_i)} \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} f(t_1, \dots, t_n) \, dt_n \dots dt_1 \right| \leq \sum_{i=1}^{n} \frac{L_i (b_i - a_i)^{r_i}}{2^{r_i} (r_i + 1)}, \tag{5.63}$$

which is the best inequality we can get from (5.61).

Proof. Note that the mapping $h_p : [\alpha, \beta] \to \mathbb{R}$, $h_p(y) = (y - \alpha)^{p+1} + (\beta - y)^{p+1}$ $(p > 0)$ has its infimum at $y_0 = \frac{\alpha+\beta}{2}$ and

$$\inf_{y \in [\alpha, \beta]} h_p(y) = \frac{(\beta - \alpha)^{p+1}}{2^p}.$$

Consequently, the best inequality we can get from (5.61) is the one for which $x_i = \frac{a_i + b_i}{2}$ giving the desired inequality (5.63). ∎

The following trapezoid type inequaltiy also holds

COROLLARY 5.29. Under the above assumptions, we have:

$$\left|\frac{f(a_1,...,a_n)+f(b_1,...,b_n)}{2}-\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n}f(t_1,...,t_n)\,dt_n...dt_1\right| \le \sum_{i=1}^{n}\frac{L_i(b_i-a_i)^{r_i}}{r_i+1}. \tag{5.64}$$

PROOF. Put in (5.61) $\bar{\mathbf{x}}=\bar{\mathbf{a}}$ and then $\bar{\mathbf{x}}=\bar{\mathbf{b}}$, add the obtained inequalities and use the triangle inequality to get (5.64). ∎

An important particular case is one for which the mapping f is Lipschitzian, i.e.,

$$|f(x_1,...,x_n)-f(y_1,...,y_n)| \le \sum_{i=1}^{n}L_i|x_i-y_i| \tag{5.65}$$

for all $\bar{\mathbf{x}},\bar{\mathbf{y}}\in\left[\bar{\mathbf{a}},\bar{\mathbf{b}}\right]$.

Thus, we have following corollary.

COROLLARY 5.30. Let f be a Lipschitzian mapping with the constants L_i. Then we have

$$\left|f(\bar{\mathbf{x}})-\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}f(\bar{\mathbf{t}})\,d\bar{\mathbf{t}}\right| \le \sum_{i=1}^{n}L_i\left[\frac{1}{4}+\left(\frac{x_i-\frac{a_i+b_i}{2}}{b_i-a_i}\right)^2\right](b_i-a_i) \tag{5.66}$$

for all $\bar{\mathbf{x}}\in\left[\bar{\mathbf{a}},\bar{\mathbf{b}}\right]$.

The constant $\frac{1}{4}$, in all the brackets, is the best possible.

PROOF. Choose $r_i=1\ \ (i=1,...,n)$ in (5.61) to get

$$\left|f(x_1,...,x_n)-\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n}f(t_1,...,t_n)\,dt_n...dt_1\right| \le \frac{1}{2}\sum_{i=1}^{n}L_i\left[\left(\frac{x_i-a_i}{b_i-a_i}\right)^2+\left(\frac{b_i-x_i}{b_i-a_i}\right)^2\right](b_i-a_i).$$

A simple computation shows that

$$\frac{1}{2}\left[\left(\frac{x_i-a_i}{b_i-a_i}\right)^2+\left(\frac{b_i-x_i}{b_i-a_i}\right)^2\right]=\frac{1}{4}+\left(\frac{x_i-\frac{a_i+b_i}{2}}{b_i-a_i}\right)^2, i=1,...,n$$

giving the desired inequality (5.66).

To prove the sharpness of the constants $\frac{1}{4}$, assume that the inequality (5.66) holds for some positive constants $c_i > 0$, i.e.,

(5.67)
$$\left| f(\bar{\mathbf{x}}) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} f(\bar{\mathbf{t}})\, d\bar{\mathbf{t}} \right| \le \sum_{i=1}^{n} L_i \left[c_i + \left(\frac{x_i - \frac{a_i+b_i}{2}}{b_i - a_i} \right)^2 \right] (b_i - a_i),$$

for all $\bar{\mathbf{x}} \in [\bar{\mathbf{a}}, \bar{\mathbf{b}}]$.

Choose $f(x_1, ..., x_n) = x_i \ (i = 1, ..., n)$. Then, by (5.67), we get

$$\left| x_i - \frac{a_i + b_i}{2} \right| \le \left[c_i + \frac{\left(x_i - \frac{a_i+b_i}{2}\right)^2}{(b_i - a_i)^2} \right] (b_i - a_i)$$

for all $x_i \in [a_i, b_i]$. Put $x_i = a_i$, to get

$$\frac{b_i - a_i}{2} \le \left(c_i + \frac{1}{4} \right) (b_i - a_i)$$

from which we deduce $c_i \ge \frac{1}{4}$, and the sharpness of $\frac{1}{4}$ is proved. ∎

Corollary 5.31. If f is as in Corollary 5.30, then we get

a) the mid-point formula

(5.68)
$$\left| f\left(\frac{a_1 + b_1}{2}, ..., \frac{a_n + b_n}{2} \right) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} f(t_1, ..., t_n)\, dt_n ... dt_1 \right| \le \frac{1}{4} \sum_{i=1}^{n} L_i (b_i - a_i),$$

b) the trapezoid formula

$$\left| \frac{f(a_1, ..., a_n) + f(b_1, ..., b_n)}{2} - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} f(t_1, ..., t_n)\, dt_n ... dt_1 \right| \le \frac{1}{2} \sum_{i=1}^{n} L_i (b_i - a_i).$$

Remark 5.8. In practical applications, we assume that the mapping $f : [\bar{\mathbf{a}}, \bar{\mathbf{b}}] \to \mathbb{R}$ has the partial derivatives $\frac{\partial f}{\partial x_i}$ and $\frac{\partial f}{\partial x_i}$ bounded on $[\bar{\mathbf{a}}, \bar{\mathbf{b}}]$, i.e.

$$\left\| \frac{\partial f}{\partial x_i} \right\|_\infty := \sup_{\bar{\mathbf{x}} \in (\bar{\mathbf{a}}, \bar{\mathbf{b}})} \left| \frac{\partial f(x_1, ..., x_n)}{\partial x_i} \right| < \infty.$$

With this assumption, we have the Ostrowski type inequality (see Introduction)

$$\left| f(\bar{\mathbf{x}}) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} f(\bar{\mathbf{t}})\, d\bar{\mathbf{t}} \right| \leq \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{\infty} \left[\frac{1}{4} + \left(\frac{x_i - \frac{a_i+b_i}{2}}{b_i - a_i} \right)^2 \right] (b_i - a_i).$$

The constants, $\frac{1}{4}$, are sharp.

5.4.2. The Weighted Case. The following generalization of Theorem 5.27 holds (see [**4**]).

THEOREM 5.32. *Let $f, w : [\bar{\mathbf{a}}, \bar{\mathbf{b}}] \to \mathbb{R}$ be such that f is of the $r-$Hölder type with the constants L_i and $r_i \in (0,1]\,(i = 1, ..., n)$ and where w is integrable on $[\bar{\mathbf{a}}, \bar{\mathbf{b}}]$, nonnegative on this interval and*

$$\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{x}})\, d\bar{\mathbf{x}} := \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} w(x_1, ..., x_n)\, dx_n ... dx_1 > 0.$$

We then have the inequality,

$$\left| f(\bar{\mathbf{x}}) - \frac{1}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}}) f(\bar{\mathbf{y}})\, d\bar{\mathbf{y}} \right| \leq \sum_{i=1}^{n} L_i \frac{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{r_i} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}}. \tag{5.69}$$

for all $\bar{\mathbf{x}} \in [\bar{\mathbf{a}}, \bar{\mathbf{b}}]$.

PROOF. The proof is similar to that of Theorem 5.27.

As f is of the $r-$Hölder type with the constants L_i and $r_i\,(i = 1, ..., n)$, we can write

$$|f(\bar{\mathbf{x}}) - f(\bar{\mathbf{y}})| \leq \sum_{i=1}^{n} L_i |x_i - y_i|^{r_i}$$

for all $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in [\bar{\mathbf{a}}, \bar{\mathbf{b}}]$.

Multiplying by $w(\bar{\mathbf{y}}) \geq 0$ and integrating over $\bar{\mathbf{y}}$ on $[\bar{\mathbf{a}}, \bar{\mathbf{b}}]$, we get

$$\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |f(\bar{\mathbf{x}}) - f(\bar{\mathbf{y}})|\, w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}} \leq \sum_{i=1}^{n} L_i \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{r_i} w(y_i, ..., y_n)\, dy_n ... dy_1. \tag{5.70}$$

On the other hand, we have

$$\begin{aligned} &\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |f(\bar{\mathbf{x}}) - f(\bar{\mathbf{y}})|\, w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}} \\ \geq\; &\left| \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} (f(\bar{\mathbf{x}}) - f(\bar{\mathbf{y}}))\, w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}} \right| = \left| f(\bar{\mathbf{x}}) \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}} - \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} f(\bar{\mathbf{y}})\, w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}} \right|. \end{aligned} \tag{5.71}$$

Combining (5.70) with (5.71) and dividing by $\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}} > 0$ we get the desired inequality (5.69). ∎

REMARK 5.9. If we assume that the mapping f is Lipschitzian with constants L_i then we get,

$$\left| f(\bar{\mathbf{x}}) - \frac{1}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}}) f(\bar{\mathbf{y}})\, d\bar{\mathbf{y}} \right| \le \sum_{i=1}^{n} L_i \frac{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{r_i} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}},$$

which generalizes Milovanović result from 1975 (see Introduction).

The following corollaries hold.

COROLLARY 5.33. With the assumptions from Theorem 5.32 we have:

$$\begin{aligned} &\left| f(\bar{\mathbf{x}}) - \frac{1}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}}) f(\bar{\mathbf{y}})\, d\bar{\mathbf{y}} \right| \\ \le\ & \sum_{i=1}^{n} L_i \left[\frac{b_i - a_i}{2} + \left| x_i - \frac{a_i + b_i}{2} \right| \right] \end{aligned} \tag{5.72}$$

for all $\bar{\mathbf{x}} \in [\bar{\mathbf{a}}, \bar{\mathbf{b}}]$.

PROOF. We have

$$\begin{aligned} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{r_i} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}} &\le \sup_{\bar{y} \in [\bar{a}, \bar{b}]} |x_i - y_i|^{r_i} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}} \\ &= \max\{|x_i - a_i|, |x_i - b_i|\} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}} \\ &= \max\{x_i - a_i, b_i - x_i\} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}} \\ &= \left[\frac{b_i - a_i}{2} + \left| x_i - \frac{a_i + b_i}{2} \right| \right] \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}. \end{aligned}$$

Then,

$$\sum_{i=1}^{n} L_i \frac{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{r_i} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}} \le \sum_{i=1}^{n} L_i \left[\frac{b_i - a_i}{2} + \left| x_i - \frac{a_i + b_i}{2} \right| \right]$$

and by (5.69), we get the desired estimation (5.72). ∎

Another type of inequality (estimation) is as follows.

COROLLARY 5.34. With the assumptions from Theorem 5.32, we have

$$\begin{aligned} &\left| f(\bar{x}) - \frac{1}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}}) f(\bar{\mathbf{y}}) d\bar{\mathbf{y}} \right| \\ \le\ & \frac{1}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}} \cdot \sup_{\bar{\mathbf{y}} \in [\bar{\mathbf{a}}, \bar{\mathbf{b}}]} w(\bar{\mathbf{y}}) \prod_{j=1}^{n} (b_j - a_j) \\ & \times \sum_{i=1}^{n} L_i \left[\frac{(b_i - x_i)^{r_i+1} + (x_i - a_i)^{r_i+1}}{(r_i + 1)(b_i - a_i)} \right] \end{aligned} \tag{5.73}$$

for all $\bar{\mathbf{x}} \in [\bar{\mathbf{a}}, \bar{\mathbf{b}}]$.

Proof. We observe that

$$\begin{aligned}
\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{r_i} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}} &\le \sup_{\bar{\mathbf{y}} \in [\bar{\mathbf{a}}, \bar{\mathbf{b}}]} w(\bar{\mathbf{y}}) \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{r_i}\, d\bar{\mathbf{y}} \\
&= \prod_{\substack{j=1 \\ j \ne i}}^{n} (b_j - a_j) \int_{a_i}^{b_i} |x_i - y_i|^{r_i}\, dy_i \\
&= \prod_{\substack{j=1 \\ j \ne i}}^{n} (b_j - a_j) \left[\frac{(b_i - x_i)^{r_i+1} + (x_i - a_i)^{r_i+1}}{r_i + 1} \right] \\
&= \prod_{j=1}^{n} (b_j - a_j) \left[\frac{(b_i - x_i)^{r_i+1} + (x_i - a_i)^{r_i+1}}{(r_i + 1)(b_i - a_i)} \right].
\end{aligned}$$

Now, using (5.69) we get (5.73).

Finally, by Hölder's integral inequality we can also state the following corollary:

Corollary 5.35. With the assumptions from Theorem 5.32, we have

$$\left| f(\bar{\mathbf{x}}) - \frac{1}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}}) f(\bar{\mathbf{y}}) d\bar{\mathbf{y}} \right| \tag{5.74}$$

$$\le \frac{\left(\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} [w(\bar{\mathbf{y}})]^q d\bar{\mathbf{y}} \right)^{1/q}}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}} \prod_{j=1}^{n} (b_j - a_j)^{\frac{1}{p}} \sum_{i=1}^{n} L_i \left[\frac{(b_i - x_i)^{pr_i+1} + (x_i - a_i)^{pr_i+1}}{(pr_i + 1)(b_i - a_i)} \right]^{\frac{1}{p}}$$

for all $\bar{\mathbf{x}} \in [\bar{\mathbf{a}}, \bar{\mathbf{b}}]$.

Proof. Using Hölder's integral inequality for multiple integrals, we get

$$\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{r_i} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}} \le \left(\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} [w(\bar{\mathbf{y}})]^q d\bar{\mathbf{y}} \right)^{1/q} \left(\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{pr_i}\, d\bar{\mathbf{y}} \right)^{\frac{1}{p}}. \tag{5.75}$$

However,

$$\begin{aligned}
\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{pr_i}\, d\bar{\mathbf{y}} &= \prod_{\substack{j=1 \\ j \ne i}}^{n} (b_j - a_j) \int_{a_i}^{b_i} |x_i - y_i|^{r_i}\, dy_i \\
&= \prod_{j=1}^{n} (b_j - a_j) \left[\frac{(b_i - x_i)^{pr_i+1} + (x_i - a_i)^{pr_i+1}}{(pr_i + 1)(b_i - a_i)} \right]
\end{aligned}$$

and then, by (5.75), we get

$$\begin{aligned}
&\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{pr_i}\, d\bar{\mathbf{y}} \\
&\le \prod_{j=1}^{n} (b_j - a_j)^{\frac{1}{p}} \left(\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} [w(\bar{\mathbf{y}})]^q d\bar{\mathbf{y}} \right)^{\frac{1}{q}} \left[\frac{(b_i - x_i)^{pr_i+1} + (x_i - a_i)^{pr_i+1}}{(pr_i + 1)(b_i - a_i)} \right]^{\frac{1}{p}}.
\end{aligned}$$

Using (5.69), we deduce the desired estimation(5.74). ∎

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CHAPTER 6

Some Results for Double Integrals Based on an Ostrowski Type Inequality

by

G. HANNA

ABSTRACT An Ostrowski type inequality in two dimensions for double integrals on a rectangle region is developed. The resulting integral inequalities are evaluated for the class of functions with bounded first derivative. They are employed to approximate the double integral by one dimensional integrals and function evaluations using different types of norms. If the one-dimensional integrals are not known, they themselves can be approximated by using a suitable rule, to produce a cubature rule consisting only of sampling points.
In addition, some generalisations of an Ostrowski type inequality in two dimensions for n - time differentiable mappings are given. The result is an integral inequality with bounded n - time derivatives. This is employed to approximate double integrals using one dimensional integrals and function evaluations at the boundary and interior points.

6.1. Introduction

In this chapter three point cubature rules for two-dimensional rectangular regions are developed. An *a priori* error bound is obtained for functions whose first partial derivatives exist and are bounded. The term "three point" is used to draw an analogy with Newton-Cotes type rules where sampling occurs at the boundary and interior points. The rule presented here approximates a two-dimensional integral via application of function evaluations and one-dimensional integrals at the boundary and interior points. And a parameterization, similar to that of [**5**], is employed to distinguish rule type. If the one-dimensional integrals are not known, they themselves can be approximated to produce a cubature rule consisting only of sampling points. An additional three point rule, as in [**5**], may be subsequently used, or indeed any other desired quadrature rule. (For example, the optimal rules in[**7**], [**10**]). As a result the error bound will be larger.
The method presented here is based on Ostrowski's integral inequality, and as such is amenable to the production of error bounds for a variety of norms. In addition

smoother and product integrands may also be considered as has been done for one-dimensional integrals, see for example [**5, 6, 9**].

6.2. The One Dimensional Ostrowski Inequality

The classical Ostrowski integral inequality in one dimension stipulates a bound between a function evaluated at an interior point x and the average of the function of over an interval. The Ostrowski inequality has been extended and generalized in many ways -usually by placing higher demands on the mapping f (smoothness, monotonicity, etc..). Here we focus on two such extensions. In [**5**], where Cerone and Dragomir presented a three point inequality and showed that the tightest bound is an average of the mid-point and trapezoidal rules. In the paper [**4**], Barnett and Dragomir developed a two dimensional version of the Ostrowski inequality. The current work combines the above two results and develops a two dimensional three point integral inequality for functions with bounded first derivatives. An application in the numerical integration of a two-dimensional integral is investigated. Also, some generalisations of an Ostrowski type inequality in two dimensions for $n-$time differentiable mappings are given. The result is an integral inequality with bounded $n-$time derivatives. This is employed to approximate double integrals using one dimensional integrals and function evaluations at the boundary and interior points. The Chapter is arranged in the following manner.

In Section 6.3, an inequality for double integrals is obtained in terms of first derivatives where $\frac{\partial 2}{\partial t_1 \partial t_2}(f_{t_1}, f_{t_2}) \in \; L_\infty[a_1, b_1] \times [a_2, b_2]$. Some Numerical results are computed in Section 6.4. An application for the cubature formula is illustrated in Section 6.5.

In Section 6.6, an inequality is developed for mappings whose first derivatives $\frac{\partial 2}{\partial t_1 \partial t_2}(f_{t_1}, f_{t_2}) \in \; L_p[a_1, b_1] \times [a_2, b_2]$.

An application is demonstrated through numerical results in Section 6.7. Section6.8 is reserved for results involving mappings whose first derivatives belong to $\|.\|_1$-norm.

In Section 6.9 some general identities in two dimention for n-time differentiable mapping are given. These are then applied to produce some generalizations of Ostrowski's type inequalities using different types of norms in Section 6.10. These results are employed to produce applications to numerical integration as in Section 6.11.

6.3. Mapping Whose First Derivatives Belong to $L_\infty(a, b)$

THEOREM 6.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable mapping on $[a_1, b_1] \times [a_2, b_2]$ and let $f''_{t_1,t_2} = \frac{\partial^2 f}{\partial t_1 \partial t_2}$ be bounded on $(a_1, b_1) \times (a_2, b_2)$. That is,*

$$\left\| f''_{t_1,t_2} \right\|_\infty := \sup_{(x_1,x_2) \in (a_1,b_1) \times (a_2,b_2)} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right| < \infty.$$

Furthermore, let $x_i \in (a_i, b_i)$ and introduce the parameterization α_i, β_i defined by

$$\alpha_i = (1 - \gamma_i)\, a_i + \gamma_i x_i, \quad \beta_i = (1 - \gamma_i)\, b_i + \gamma_i x_i, \tag{6.1}$$

where $\gamma_i \in [0,1]$, for $i = 1,2$. Then the following inequality holds

$$
\begin{aligned}
(6.2)\quad & |G(x_1,t_1,x_2,t_2)| \\
&\le \frac{\left\|f''_{t_1,t_2}\right\|_\infty}{4}\left(1+(2\gamma_1-1)^2\right)\left[\left(\frac{b_1-a_1}{2}\right)^2+\left(x_1-\frac{a_1+b_1}{2}\right)^2\right] \\
&\qquad \times\left(1+(2\gamma_2-1)^2\right)\left[\left(\frac{b_2-a_2}{2}\right)^2+\left(x_2-\frac{a_2+b_2}{2}\right)^2\right],
\end{aligned}
$$

given that

$$
\begin{aligned}
(6.3)\quad G(x_1,t_1,x_2,t_2) &= \sum_{k=1}^{3}\sum_{j=1}^{3} C_{k1}C_{j2}f_{jk} - \sum_{j=1}^{3}\left(C_{j1}I_{j2}+C_{j2}I_{j1}\right) \\
&\quad + \int_{a_2}^{b_2}\int_{a_1}^{b_1} f\left(t_1,t_2\right)\, dt_1 dt_2
\end{aligned}
$$

$$
(6.4)\qquad (f_{jk}) = \begin{pmatrix} f\left(a_1,a_2\right) & f\left(x_1,a_2\right) & f\left(b_1,a_2\right) \\ f\left(a_1,x_2\right) & f\left(x_1,x_2\right) & f\left(b_1,x_2\right) \\ f\left(a_1,b_2\right) & f\left(x_1,b_2\right) & f\left(b_1,b_2\right) \end{pmatrix},
$$

$$
(6.5)\qquad (C_{jk}) = \begin{pmatrix} \gamma_1(x_1-a_1) & \gamma_2(x_2-a_2) \\ (1-\gamma_1)\left(b_1-a_1\right) & (1-\gamma_2)\left(b_2-a_2\right) \\ \gamma_1\left(b_1-x_1\right) & \gamma_2(b_2-a_2) \end{pmatrix},
$$

$$
(6.6)\qquad (I_{jk}) = \begin{pmatrix} \int_{a_1}^{b_1} f(t_1,a_2)\,dt_1 & \int_{a_2}^{b_2} f(a_1,t_2)\,dt_2 \\ \int_{a_1}^{b_1} f(t_1,x_2)\,dt_1 & \int_{a_1}^{b_1} f(x_1,t_2)\,dt_2 \\ \int_{a_1}^{b_1} f(t_1,b_2)\,dt_1 & \int_{a_1}^{b_1} f(b_1,t_2)\,dt_2 \end{pmatrix}.
$$

Proof. Define the kernel

$$
(6.7)\qquad p\left(x,t\right) = \begin{cases} t-\alpha, & t\in\left[a,x\right], \\ t-\beta, & t\in\left(x,b\right], \end{cases}
$$

where, as above, $\alpha = \left(1-\gamma\right)a + \gamma x$, and $\beta = \left(1-\gamma\right)b + \gamma x$. Using (6.7) and integrating by parts we obtain, after some simplification, the identity

$$
\begin{aligned}
(6.8)\quad & \int_a^b p\left(x,t\right)F'\left(t\right)dt \\
&= \left(1-\gamma\right)\left(b-a\right)F\left(x\right) + \gamma\left[\left(x-a\right)F\left(a\right)+\left(b-x\right)F\left(b\right)\right] - \int_a^b F\left(t\right)dt.
\end{aligned}
$$

A two dimensional identity can be developed via repeated application of (6.8). To this end, we define the mapping

$$
(6.9)\qquad p_i\left(x_i,t_i\right) = \begin{cases} t_i-\alpha_i, & a_i\le t_i\le x_i, \\ t_i-\beta_i, & x_i<t_i\le b_i, \end{cases} \qquad \text{for } i=1,2.
$$

Substituting p_1 for p and $f(t_1,\cdot)$ for $F(t)$ into (6.8) gives

$$\int_{a_1}^{b_1} p_1(x_1,t_1)\frac{\partial f}{\partial t_1}dt_1 = (1-\gamma_1)(b_1-a_1)f(x_1,t_2) + \gamma_1(x_1-a_1)f(a_1,t_2)+\gamma_1(b_1-x_1)f(b_1,t_2)-\int_{a_1}^{b_1} f(t_1,t_2)\,dt_1. \tag{6.10}$$

Employing (6.8) again with p_2 as the kernel, $F(t_2)=\int_{a_1}^{b_1} p_1(x_1,t_1)\frac{\partial f}{\partial t_1}dt_1$ as the integrand and expanding with (6.10) produces,

$$\int_{a_2}^{b_2} p_2(x_2,t_2)F\prime(t_2)\,dt_2 = \int_{a_2}^{b_2}\int_{a_1}^{b_1} p_2(x_2,t_2)\,p_1(x_1,t_1)\frac{\partial^2 f}{\partial t_1\partial t_2}dt_1dt_2$$

$$\begin{aligned}
&= (1-\gamma_2)(b_2-a_2)F(x_2)+\gamma_2\Big[(b_2-x_2)F_2(b_2)+(x_2-a_2)F_2(a_2)\Big]-\int_{a_2}^{b_2}F_2(t_2)\,dt_2\\
&= (1-\gamma_1)(1-\gamma_2)(b_1-a_1)(b_2-a_2)f(x_1,x_2)+\gamma_1(1-\gamma_2)(b_2-a_2)(b_1-x_1)f(b_1,x_2)\\
&+\gamma_1(1-\gamma_2)(b_2-a_2)(x_1-a_1)f(a_1,x_2)+\gamma_2(1-\gamma_1)(b_1-a_1)(b_2-x_2)f(x_1,b_2)\\
&+\gamma_1\gamma_2(b_2-x_2)(b_1-x_1)f(b_1,b_2)+\gamma_1\gamma_2(b_2-x_2)(x_1-a_1)f(a_1,b_2)\\
&+\gamma_2(1-\gamma_1)(x_2-a_2)(b_1-a_1)f(x_1,a_2)\\
&+\gamma_1\gamma_2(x_2-a_2)(b_1-x_1)f(b_1,a_2)+\gamma_1\gamma_2(x_2-a_2)(x_1-a_1)f(a_1,a_2)\\
&-(1-\gamma_2)(b_2-a_2)\int_{a_1}^{b_1}f(t_1,x_2)\,dt_1-\ \gamma_2(b_2-x_2)\int_{a_1}^{b_1}f(t_1,b_2)\,dt_1\\
&-\gamma_2(x_2-a_2)\int_{a_1}^{b_1}f(t_1,a_2)\,dt_1-(1-\gamma_1)(b_1-a_1)\int_{a_2}^{b_2}f(x_1,t_2)\,dt_2\\
&-\gamma_1(b_1-x_1)\int_{a_2}^{b_2}f(b_1,t_2)\,dt_2-\gamma_1(x_1-a_1)\int_{a_2}^{b_2}f(a_1,t_2)\,dt_2+\int_{a_2}^{b_2}\int_{a_1}^{b_1}f(t_1,t_2)\,dt_1dt_2.\\
&= \sum_{k=1}^{3}\sum_{j=1}^{3}C_{k1}C_{j2}f_{jk}-\sum_{j=1}^{3}(C_{j1}I_{j2}+C_{j2}I_{j1})+\int_{a_2}^{b_2}\int_{a_1}^{b_1}f(t_1,t_2)\,dt_1dt_2.
\end{aligned}$$

and this produces that

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1} p_2(x_2,t_2)\,p_1(x_1,t_1)\frac{\partial^2 f}{\partial t_1\partial t_2}dt_1dt_2 = G(x_1,t_1,x_2,t_2). \tag{6.11}$$

Assuming that both first partial derivatives of f are bounded, we can simply write down the inequality

$$\left|\int_{a_2}^{b_2}\int_{a_1}^{b_1} p_2(x_2,t_2)\,p_1(x_1,t_1)\frac{\partial^2 f}{\partial t_1\partial t_2}dt_1dt_2\right| \le \left\|f''_{t_1,t_2}\right\|_\infty\left(\int_{a_2}^{b_2}|p_2(x_2,t_2)|\,dt_2\right)\left(\int_{a_1}^{b_1}|p_1(x_1,t_1)|\ dt_1\right). \tag{6.12}$$

Now, consider

$$\begin{aligned} G_1(x_1) &= \int_{a_1}^{b_1} |p_1(x_1,t_1)|\,dt_1 \\ &= -\int_{a_1}^{\alpha_1}(t_1-\alpha_1)\,dt_1 + \int_{\alpha_1}^{x_1}(t_1-\alpha_1)\,dt_1 - \int_{x_1}^{\beta_1}(t_1-\beta_1)\,dt_1 + \int_{\beta_1}^{b_1}(t_1-\beta_1)\,dt_1 \\ &= \frac{1}{2}\left[(\alpha_1-a_1)^2+(x_1-\alpha_1)^2+(\beta_1-x_1)^2+(b_1-\beta_1)^2\right] \end{aligned}$$

$$(6.13)\qquad = \frac{1}{2}\left[1+(2\gamma_1-1)^2\right]\left[\left(\frac{b_1-a_1}{2}\right)^2+\left(x_1-\frac{a_1+b_1}{2}\right)^2\right].$$

Similarly, with $G_2(x_2)=\int_{a_2}^{b_2}|p_2(x_2,t_2)|\,dt_2$, we have

$$(6.14)\qquad G_2(x) = \frac{1}{2}\left[1+(2\gamma_2-1)^2\right]\left[\left(\frac{b_2-a_2}{2}\right)^2+\left(x_2-\frac{a_2+b_2}{2}\right)^2\right].$$

Using (6.4) , (6.5) and (6.6) and substituting (6.11), (6.13) and (6.14) into (6.12) will produce the result (6.2) and thus the theorem is proved. ∎

The following result gives an Ostrowski type inequality for double integrals. It involves double and single integrals together with a function evaluation at an interior point.

Corollary 6.2. With the conditions as in Theorem 6.1, then

$$(6.15)\qquad \begin{aligned} &\Bigg|(b_1-a_1)(b_2-a_2)f(x_1,x_2)-(b_2-a_2)\int_{a_1}^{b_1}f(t_1,x_2)\,dt_1 \\ &\qquad -(b_1-a_1)\int_{a_2}^{b_2}f(x_1,t_2)\,dt_2+\int_{a_2}^{b_2}\int_{a_1}^{b_1}f(t_1,t_2)\,dt_1dt_2\Bigg| \\ &\le \left\|f''_{t_1,t_2}\right\|_\infty\left[\left(\frac{b_1-a_1}{2}\right)^2+\left(x_1-\frac{a_1+b_1}{2}\right)^2\right]\left[\left(\frac{b_2-a_2}{2}\right)^2+\left(x_2-\frac{a_2+b_2}{2}\right)^2\right]. \end{aligned}$$

Proof. Place $\gamma_1=\gamma_2=0$ into equation (6.2). ∎

Thus, the earlier results of [**4**] and [**8**, p. 468] are reproduced as a special case of Theorem 6.1. We note that unlike [**4**], the proof for Theorem 6.1 can be easily extended to more than two dimensions.

Different values of the parameters γ_1, γ_2, x_1 and x_2 give rise to Newton-Cotes type inequalities for functions with bounded derivatives. For example $\gamma_1=\gamma_2=0$, $x_1=\frac{a_1+b_1}{2}$ and $x_2=\frac{a_2+b_2}{2}$ produces the two dimensional mid-point inequality; $\gamma_1=\gamma_2=1$ a two dimensional trapezoid-like inequality and $\gamma_1=\gamma_2=\frac{1}{3}$ a two dimensional Simpson's like inequality.

¿From Theorem 3.1 it is a simple matter to show that the tightest bound is obtained when $\gamma_1 = \gamma_2 = \frac{1}{2}$ and x_1 and x_2 are at their mid-points. That is for the average of the mid-point and trapezoid inequalities.

REMARK 6.1. Let $f(t_1, t_2) = g(t_1)\, g(t_2)$ where $g : [a, b] \to \mathbb{R}$. If g is differentiable and satisfies the condition that $\|g'\|_\infty < \infty$, then, for $x_1 = x_2 = x$ and $\gamma_1 = \gamma_2 = \gamma$, we obtain a result from Theorem 6.1 which may be factored to recover the three point rule developed in [**5**], namely

$$\left|\int_a^b g(t)\,dt - \gamma\big((x-a)g(a) + (b-x)g(b)\big) - (1-\gamma)(b-a)g(x)\right| \leq \frac{\|g'\|_\infty}{2}\left(1 + (2\gamma-1)^2\right)\left(\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right). \tag{6.16}$$

In general, cubature formulae are written only in terms of function evaluations, but Theorem 6.1 approximates a double integral in terms of single integrals and function evaluations. Therefore we write down the following corollary which eliminates the one dimensional integrals by approximating them using the 3-point rule in equation (6.16). The resulting inequality has a coarser bound than equation (6.2).

COROLLARY 6.3. Let f be given as in Theorem 6.1. Then

$$\begin{aligned}
&\left|\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(t_1, t_2)\,dt_1 dt_2 - \sum_{k=1}^{3}\sum_{j=1}^{3} C_{k1} C_{j2} f_{jk}\right| \\
&\leq \frac{\left\|f''_{t_1,t_2}\right\|_\infty}{4}\left(1 + (2\gamma_1 - 1)^2\right)\left(1 + (2\gamma_2 - 1)^2\right) \\
&\times\left[\left(\frac{b_1 - a_1}{2}\right)^2 + \left(x_1 - \frac{a_1 + b_1}{2}\right)^2\right]\left[\left(\frac{b_2 - a_2}{2}\right)^2 + \left(x_2 - \frac{a_2 + b_2}{2}\right)^2\right] \\
&+ \frac{1}{2}\left(1 + (2\gamma_1 - 1)^2\right)\left[\left(\frac{b_1 - a_1}{2}\right)^2 + \left(x_1 - \frac{a_1 + b_1}{2}\right)^2\right] \\
&\times\left\{\gamma_2 (x_2 - a_2)\left\|f'_{t_1,a_2}\right\|_\infty + (1-\gamma_2)(b_2 - a_2)\left\|f'_{t_1,x_2}\right\|_\infty + \gamma_2 (b_2 - x_2)\left\|f'_{t_1,b_2}\right\|_\infty\right\} \\
&+ \frac{1}{2}\left(1 + (2\gamma_2 - 1)^2\right)\left[\left(\frac{b_2 - a_2}{2}\right)^2 + \left(x_2 - \frac{a_2 + b_2}{2}\right)^2\right] \\
&\times\left\{\gamma_1 (x_1 - a_1)\left\|f'_{a_1,t_2}\right\|_\infty + (1-\gamma_1)(b_1 - a_1)\left\|f'_{x_1,t_2}\right\|_\infty + \gamma_1 (b_1 - x_1)\left\|f'_{b_1,t_2}\right\|_\infty\right\}
\end{aligned} \tag{6.17}$$

PROOF. Approximating each single integral in (6.2) by (6.16) and applying the triangle inequality produces the desired result. ∎

Remark 6.2. If $\gamma_1 = \gamma_2 = 0$ and $x_i = \frac{a_i+b_i}{2}$, then (6.17) becomes

$$(6.18)\quad \left| \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(t_1,t_2)\,dt_1 dt_2 - (b_1-a_1)(b_2-a_2) f\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right) \right|$$
$$\le \frac{\left\| f''_{t_1,t_2} \right\|_\infty}{16} (b_1-a_1)^2 (b_2-a_2)^2 + \frac{\left\| f'_{t_1,\frac{a_2+b_2}{2}} \right\|_\infty}{4} (b_2-a_2)(b_1-a_1)^2$$
$$+ \frac{\left\| f'_{\frac{a_1+b_1}{2},t_2} \right\|_\infty}{4} (b_1-a_1)(b_2-a_2)^2 .$$

6.4. Numerical Results

In this section the inequalities developed in Section 6.3 are used to approximate the double integral

$$(6.19)\qquad \int_0^1 \int_0^1 1 - e^{-xy} dxdy = 0.203400400702947.$$

This integrand was chosen because integrating once in each direction is trivial. Namely, $\int_0^1 1 - e^{-xy}\,dx = \frac{y+e^{-y}-1}{y}$ and $\int_0^1 1 - e^{-xy}\,dy = \frac{x+e^{-x}-1}{x}$, but the double integral is not. In Table 6.1, results are shown for the approximation to (6.19) using the rule and bound of (6.2). The numerical error is much smaller than the theoretical one and is smallest when Simpson's rule is applied ($\gamma_1 = \gamma_2 = \frac{1}{3}$). The optimal theoretical bound is attained when $\gamma_1 = \gamma_2 = \frac{1}{2}$. It should be noted that $\gamma_1 = \gamma_2 = 0$ approximates (6.19) with the "mid-point" rule and employs one function evaluation (at the midpoint of the region) and two one-dimensional integrals (along the bi-sectors). The "trapezoidal" rule uses four sample points (the boundary corners) and four one-dimensional integrals (along the boundary). All other values, that is $\gamma_1, \gamma_2 \in (0,1)$, produces a rule that is a linear combination of the above and results in the use of nine sample points and six one-dimensional integrals.

γ_1	γ_2	Numerical Error	Theoretical Error
0	0	1.5(-3)	6.3(-2)
$\frac{1}{3}$	$\frac{1}{3}$	5.4(-7)	1.9(-2)
0.5	0.5	4.3(-4)	1.6(-2)
1	1	6.5(-3)	6.3(-2)

Table 6.1. The numerical and theoretical errors in computing (6.19) using (6.2) with $x_1 = x_2 = 0.5$ and various values of γ_1, γ_2.

To approximate (6.19) only in terms of function evaluations we use equation (6.17). The results are presented in Table 6.2. Since (6.17) is an approximation of (6.2), the results are qualitatively similar and quantitatively less accurate than those in Table 6.1. Simpson's rule ($\gamma_1 = \gamma_2 = \frac{1}{3}$, nine sample points) is more accurate than the midpoint rule ($\gamma_1 = \gamma_2 = 0$, one sample point) which in turn is more accurate than the trapezoidal rule ($\gamma_1 = \gamma_2 = 1$, four sample points). We note that the

γ_1	γ_2	Numerical Error	Theoretical Error
0	0	1.8(-2)	4.6(-1)
$\frac{1}{3}$	$\frac{1}{3}$	9.3(-4)	2.2(-1)
0.5	0.5	1.0(-2)	1.9(-1)
1	1	4.5(-2)	3.8(-1)

TABLE 6.2. The numerical and theoretical errors in computing (6.19) using (6.17) with $x_1 = x_2 = 0.5$ and various values of γ_1, γ_2.

theoretical errors are symmetric about $\gamma_1 = \gamma_2 = \frac{1}{2}$ in Table 6.1, but this is not the case in Table 6.2; these properties are easy to see by inspection of (6.2) and (6.17) respectively.

6.5. Application For Cubature Formulae

To illustrate the use of a cubature formula, we form a composite rule from the inequality (6.15). Let us consider the arbitrary division:

$$I_n : a_1 = \xi_0 < \xi_1 < ... < \xi_n = b_1$$

on the interval $[a_1, b_1]$ with $x_i \in \left[\xi_i, \xi_{i+1}\right]$ for $i = 0, 1, ..., n-1$ and $J_m : a_2 = \tau_0 < \tau_1 < ... < \tau_m = b_2$ on the interval $[a_2, b_2]$ with $y_j \in [\tau_j, \tau_{j+1}]$ for $j = 0, 1, ..., m-1$.

Consider the sum

$$\begin{aligned} A\left(f, I_n, J_m, x, y\right) := {} & \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i v_j f\left(x_i, y_j\right) - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i \int_{\tau_j}^{\tau_{j+1}} f\left(x_i, t_2\right) dt_2 \\ & - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} v_j \int_{\xi_i}^{\xi_{i+1}} f\left(t_1, y_j\right) dt_1, \end{aligned} \tag{6.20}$$

where $h_i = \xi_{i+1} - \xi_i$ $(i = 0, 1, ..., n-1)$ and $v_j = \tau_{j+1} - \tau_j$ $(j = 0, 1, ..., m-1)$ and $\gamma_1 = \gamma_2 = 0$.

Under the above assumptions the following theorem holds.

THEOREM 6.4. *Let $f : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ be as in Theorem 6.1 and I_n, J_m, x, y be as above. Then we have the cubature formula*

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1} f\left(t_1, t_2\right) dt_1 dt_2 = A\left(f, I_n, J_m, x, y\right) + R\left(f, I_n, J_m, x, y\right), \tag{6.21}$$

where the remainder term $R\left(f, I_n, J_m, x, y\right)$ satisfies the inequality

$$\begin{aligned} & \left|R\left(f, I_n, J_m, x, y\right)\right| \\ & \leq \left\|f''_{t_1,t_2}\right\|_\infty \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \left[\frac{1}{4}h_i^2 + \left(x_i - \frac{\xi_i + \xi_{i+1}}{2}\right)^2\right]\left[\frac{1}{4}v_j^2 + \left(y_j - \frac{\tau_j + \tau_{j+1}}{2}\right)^2\right]. \end{aligned} \tag{6.22}$$

Proof. Apply Theorem 6.1 on the interval $[\xi_i, \xi_{i+1}] \times [\tau_j, \tau_{j+1}]$, $(i = 0, 1, ..., n-1)$, $(j = 0, 1, ..., m-1)$ to get

$$\begin{aligned} & \Bigg| \overbrace{(\xi_{i+1} - \xi_i)}^{h_i} \overbrace{(\tau_{j+1} - \tau_j)}^{v_j} f(x_i, y_j) - v_j \int_{\xi_i}^{\xi_{i+1}} f(t_1, y_j)\, dt_1 \\ & -h_i \int_{\tau_j}^{\tau_{j+1}} f(x_i, t_2)\, dt_2 + \int_{\xi_i}^{\xi_{i+1}} \int_{\tau_j}^{\tau_{j+1}} f(t_1, t_2)\, dt_1 dt_2 \Bigg| \\ \leq \quad & \left\| f'_{t_1, t_2} \right\|_\infty \left[\frac{1}{4} h_i^2 + \left(x_i - \frac{\xi_i + \xi_{i+1}}{2} \right)^2 \right] \left[\frac{1}{4} v_j^2 + \left(y_j - \frac{\tau_j + \tau_{j+1}}{2} \right)^2 \right] \end{aligned}$$

for all $(i = 0, 1, ..., n-1)$, $(j = 0, 1, ..., m-1)$.
Now, summing over i from 0 to $n-1$ and over j from 0 to $m-1$, and using the generalized triangle inequality, we deduce (6.22). ∎

Corollary 6.5. We know that $\left| x_i - \frac{\xi_i + \xi_{i+1}}{2} \right| \leq \frac{1}{2} h_i$ and $\left| y_j - \frac{\tau_j + \tau_{j+1}}{2} \right| \leq \frac{1}{2} v_j$. Applying these to (6.22), we find that

$$\begin{aligned} |R(f, I_n, J_m, x, y)| & \leq \left\| f'_{t_1, t_2} \right\|_\infty \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[\frac{1}{4} h_i^2 + \frac{1}{4} h_i^2 \right] \left[\frac{1}{4} v_j^2 + \frac{1}{4} v_j^2 \right] \\ & \leq \frac{\left\| f'_{t_1, t_2} \right\|_\infty}{4} \sum_{i=0}^{n-1} h_i^2 \sum_{j=0}^{m-1} v_j^2. \end{aligned}$$

Corollary 6.6. Now, consider the case where x_i and y_i are the midpoints. At the midpoint we have

$$\int_{a_2}^{b_2} \int_{a_1}^{b_1} f(t_1, t_2)\, dt_1 dt_2 = A(f, I_n, J_m) + R(f, I_n, J_m), \tag{6.23}$$

where the remainder term $R(f, I_n, J_m)$ satisfies

$$|R(f, I_n, J_m)| \leq \frac{\left\| f''_{t_1, t_2} \right\|_\infty}{16} \sum_{i=0}^{n-1} h_i^2 \sum_{j=0}^{m-1} v_j^2.$$

Corollary 6.7. Let the conditions of Theorem 6.4 hold. In addition, let I_n be the equidistant partition of $[a_1, b_1]$, $I_n : x_i = a_1 + \left(\frac{b_1 - a_1}{n} \right) i$, $i = 0, 1, ..., n-1$, and J_m be the equidistant partition of $[a_2, b_2]$, $J_m : y_j = a_2 + \left(\frac{b_2 - a_2}{m} \right) j$, $j = 0, 1, ..., m-1$, then

$$\left| \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(t_1, t_2) dt_1 dt_2 - A(f, I_n, J_m) \right| \leq \frac{\left\| f''_{t_1,\, t_2} \right\|_\infty (b_1 - a_1)^2 (b_2 - a_2)^2}{16nm} \tag{6.24}$$

PROOF. ¿From Theorem 6.4 with $h_i = \frac{b_1 - a_1}{n}$ for all i so that

$$\begin{aligned} |R(f, I_n, J_m)| &\le \frac{\left\| f\,''_{t_1,\,t_2} \right\|_\infty}{16} \sum_{i=0}^{n-1} \left(\frac{b_1 - a_1}{n} \right)^2 \sum_{j=0}^{m-1} \left(\frac{b_2 - a_2}{m} \right)^2 \\ &= \frac{\left\| f''_{t_1,\,t_2} \right\|_\infty (b_1 - a_1)^2 (b_2 - a_2)^2}{16nm} \end{aligned}$$

and hence the result is proved. ∎

REMARK 6.3. If we were to use (6.20) to approximate the integral $\int_{a_2}^{b_2} \int_{a_1}^{b_1} f(t_1, t_2)\, dt_1 dt_2$ with a uniform grid and sampling at each mid-point, then the remainder R is bounded by

$$|R(f, I_n, J_m, \mathbf{x}, \mathbf{y})| \le \frac{\left\| f''_{t_1,\,t_2} \right\|_\infty (b_1 - a_1)^2 (b_2 - a_2)^2}{16nm} \tag{6.25}$$

n	m	Numerical Error	Error ratio	Theoretical Error
1	1	1.5(-3)	...	6.3(-2)
2	2	1.0(-4)	14.51	1.6(-2)
4	4	6.7(-6)	15.61	3.9(-3)
8	8	4.2(-7)	15.90	1.0(-3)
16	16	2.6(-8)	15.98	2.0(-4)
32	32	1.6(-9)	15.99	6.1(-5)
64	64	1.0 (-10)	16.00	1.5(-5)
128	128	6.6 (-12)	16.00	3.8(-6)

TABLE 6.3. The numerical and theoretical errors in evaluating (6.19) using the cubature rule in (6.20) for various values of n, m. Sampling occurs at the mid-point of each region.

Table 6.3 shows the numerical and theoretical errors in applying the mid-point cubature rule (6.20) to evaluate the double integral (6.19) for an increasing number of intervals. The numerical error ratio suggests that this composite rule has convergence

$$|R| \sim O\left(\frac{1}{n^2 m^2} \right).$$

This contrasts with (6.24) which predicts a convergence rate of

$$|R| \le \frac{1}{16nm}.$$

It should be noted that the development of the bounds in Section 6.3 assumes that the integrand is once differentiable. This condition admits a wider class of functions than the usual bounds for Newton-Cotes rules, but the error estimate will be more conservative if its applied, as it is here, to an integrand that is infinitely smooth. In addition, theoretical optimality occurs at $\gamma_1 = \gamma_2 = \frac{1}{2}$, while numerically this value seems to be $\gamma_1 = \gamma_2 = \frac{1}{3}$. Because of the behaviour of the integrand, Simpson's rule which is optimal (in the Newton-Cotes sense) for the class of fourth differentiable mappings, will be superior. The methods of Section 6.3 can be applied to smoother [**6**] as well as weighted mappings . Work is continuing in this direction.

6.6. Mapping Whose First Derivatives Belong to $L_p(a,b)$.

For this section we will refer to [**1**] where S. S. Dragomir and S. Wang produced some applications of Ostrowski's inequality to some special means and numerical quadrature rules.

In [**2**], the same author considered another inequality of Ostrowski type for $\|\cdot\|_p$ −norms as follows:

THEOREM 6.8. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$ and $a, b \in \mathring{I}$ with $a < b$. If $f' \in L_p(a,b)$ $\left(p > 1,\ \frac{1}{p} + \frac{1}{q} = 1\right)$, then we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{b-a}\left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1}\right]\|f\|_p \tag{6.26}$$

for all $x \in [a,b]$ where

$$\|f\|_p := \left(\frac{1}{b-a}\int_a^b |f'(t)|^p\,dt\right)^{\frac{1}{p}}$$

is the $L_p(a,b)$ −norm.

In this section we point out an inequality of Ostrowski type for double integrals for the first differentiable mapping in terms of the $\|\cdot\|_p$ −norm.

THEOREM 6.9. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable mapping on $[a_1,b_1] \times [a_2,b_2]$ and let $f''_{t_1,t_2} = \frac{\partial^2 f}{\partial t_1 \partial t_2}$ be bounded on $(a_1,b_1) \times (a_2,b_2)$, that is,*

$$\left\|f''_{t_1,t_2}\right\|_p := \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left|\frac{\partial^2 f}{\partial t_1 \partial t_2}\right|^p dt_1 dt_2\right)^{\frac{1}{p}},$$

and with the same conditions as in Theorem (6.1). We can obtain the following inequality

$$\begin{aligned} &|G(x_1,t_1,x_2,t_2)| \\ \le\ & \frac{\left\|f''_{t_1,t_2}\right\|_p}{(q+1)^{\frac{2}{q}}}\left[\gamma_1^{q+1} + (1-\gamma_1)^{q+1}\right]^{\frac{1}{q}}\left[(x_1-a_1)^{q+1} + (b_1-x_1)^{q+1}\right]^{\frac{1}{q}} \\ & \times\left[\gamma_2^{q+1} + (1-\gamma_2)^{q+1}\right]^{\frac{1}{q}}\left[(x_2-a_2)^{q+1} + (b_2-x_2)^{q+1}\right]^{\frac{1}{q}}, \end{aligned} \tag{6.27}$$

PROOF. As in Theorem (6.1). and by applying Hölder's inequality for double integrals, that is,

$$
\begin{aligned}
(6.28)\quad & \left|\int_{a_2}^{b_2}\int_{a_1}^{b_1} p_2(x_2,t_2)\,p_1(x_1,t_1)\,\frac{\partial^2 f}{\partial t_1\partial t_2}dt_1dt_2\right| \\
\leq & \left(\int_{a_2}^{b_2}\int_{a_1}^{b_1}\left|p_2(x_2,t_2)\,p_1(x_1,t_1)\right|^q dt_1dt_2\right)^{\frac{1}{q}}\left(\int_{a_2}^{b_2}\int_{a_1}^{b_1}\left|\frac{\partial^2 f}{\partial t_1\partial t_2}\right|^p dt_1dt_2\right)^{\frac{1}{p}} \\
= & \left(\int_{a_1}^{b_1}\left|p_1(x_1,t_1)\right|^q dt_1\right)^{\frac{1}{q}}\left(\int_{a_2}^{b_2}\left|p_2(x_2,t_2)\right|^q dt_2\right)^{\frac{1}{q}}\left\|f''_{t_1,t_2}\right\|_p.
\end{aligned}
$$

Consider

$$
\begin{aligned}
G_1(x_1) &= \left(\int_{a_1}^{b_1}\left|p_1(x_1,t_1)\right|^q dt_1\right)^{\frac{1}{q}} \\
&= \left[\left(\int_{a_1}^{\alpha_1}(\alpha_1-t_1)^q dt_1\right)+\left(\int_{\alpha_1}^{x_1}(t_1-\alpha_1)^q dt_1\right)\right. \\
&\quad \left.+\left(\int_{x_1}^{\beta_1}(\beta_1-t_1)^q dt_1\right)+\left(\int_{\beta_1}^{b_1}(t_1-\beta_1)^q dt_1\right)\right]^{\frac{1}{q}}
\end{aligned}
$$

$$
=\frac{1}{2}\left[\frac{(\alpha_1-a_1)^{q+1}+(x_1-\alpha_1)^{q+1}+(\beta_1-x_1)^{q+1}+(b_1-\beta_1)^{q+1}}{q+1}\right]^{\frac{1}{q}}
$$

and we get on using (6.1)

$$
G_1(x_1)=\left[\frac{\left[\gamma_1^{q+1}+(1-\gamma_1)^{q+1}\right]\left[(x_1-a_1)^{q+1}+(b_1-x_1)^{q+1}\right]}{q+1}\right]^{\frac{1}{q}}.
$$

Similarly,

$$
G_2(x_2)=\left[\frac{\left[\gamma_2^{q+1}+(1-\gamma_2)^{q+1}\right]\left[(x_2-a_2)^{q+1}+(b_2-x_2)^{q+1}\right]}{q+1}\right]^{\frac{1}{q}}.
$$

Then substituting into (6.6) will produce the result (6.27) and thus the theorem is proved. ∎

COROLLARY 6.10. With the conditions as in Theorem 6.1, then

$$
\begin{aligned}
(6.29)\quad |G(x_1,t_1,x_2,t_2)| \leq & \frac{\left\|f''_{t_1,t_2}\right\|_p}{4(q+1)^{\frac{2}{q}}}\left[\gamma_1^{q+1}+(1-\gamma_1)^{q+1}\right]^{\frac{1}{q}}\left[(b_1-a_1)\right]^{\frac{q+1}{q}} \\
& \times\left[\gamma_2^{q+1}+(1-\gamma_2)^{q+1}\right]^{\frac{1}{q}}\left[(b_2-a_2)\right]^{\frac{q+1}{q}}
\end{aligned}
$$

PROOF. Place $x_i=\frac{a_i+b_i}{2}$ in equation (6.27) ∎

REMARK 6.4. If $p=q=2$, then (6.29) becomes

$$
\begin{aligned}
(6.30)\qquad |G(x_1,t_1,x_2,t_2)| \leq\; & \frac{\left\|f''_{t_1,t_2}\right\|_2}{12}\left[\gamma_1^3+(1-\gamma_1)^3\right]^{\frac{1}{2}}[b_1-a_1]^{\frac{3}{2}} \\
& \times\left[\gamma_2^3+(1-\gamma_2)^3\right]^{\frac{1}{q}}[b_2-a_2]^{\frac{3}{2}}.
\end{aligned}
$$

REMARK 6.5. If $\gamma_1=\gamma_2=0$, then (6.30) becomes

$$
\begin{aligned}
(6.31)\;& \left|(b_1-a_1)(b_2-a_2) f\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right)-(b_2-a_2)\int_{a_1}^{b_1} f\left(t_1,\frac{a_2+b_2}{2}\right)dt_1\right. \\
& \left.-(b_1-a_1)\int_{a_2}^{b_2} f\left(\frac{a_1+b_1}{2},t_2\right)dt_2-\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(t_1,t_2)\,dt_1dt_2\right| \\
\leq\;& \frac{\left\|f''_{t_1,t_2}\right\|_2}{12}\left[(b_1-a_1)(b_2-a_2)\right]^{\frac{3}{2}}.
\end{aligned}
$$

REMARK 6.6. If $\gamma_1=\gamma_2=1$,(6.30) becomes

$$
\begin{aligned}
(6.32)\qquad & \left|\frac{(b_1-a_1)(b_2-a_2)}{4}\left[f(b_1,b_2)+f(a_1,b_2)+f(b_1,a_2)+f(a_1,a_2)\right]\right. \\
& -\frac{1}{2}\left[(b_2-a_2)\int_{a_1}^{b_1} f(t_1,b_2)\,dt_1+(b_2-a_2)\int_{a_1}^{b_1} f(t_1,a_2)\,dt_1\right. \\
& \left.+(b_1-a_1)\int_{a_2}^{b_2} f(b_1,t_2)\,dt_2+(b_1-a_1)\int_{a_2}^{b_2} f(a_1,t_2)\,dt_2\right] \\
& \left.+\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(t_1,t_2)\,dt_1dt_2\right| \leq \frac{\left\|f''_{t_1,t_2}\right\|_2}{12}\left[(b_1-a_1)(b_2-a_2)\right]^{\frac{3}{2}}.
\end{aligned}
$$

REMARK 6.7. If $\gamma_1=\gamma_2=\frac{1}{2}$, then (6.30) becomes

$$
\begin{aligned}
(6.33)\qquad & \left|\frac{(b_1-a_1)(b_2-a_2)}{4} f\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right)\right. \\
& +\frac{(b_1-a_1)(b_2-a_2)}{8}\left[f\left(b_1,\frac{a_2+b_2}{2}\right)+f\left(a_1,\frac{a_2+b_2}{2}\right)+f\left(\frac{a_1+b_1}{2},a_2\right)\right. \\
& \left.+f\left(\frac{a_1+b_1}{2},a_2\right)\right]+\frac{(b_1-a_1)(b_2-a_2)}{16} \\
& \times\left[f(b_1,b_2)+f(a_1,b_2)+f(b_1,a_2)+f(a_1,a_2)\right] \\
& -\frac{(b_1-a_1)}{4}\left[2\int_{a_2}^{b_2} f\left(\frac{a_1+b_1}{2},t_2\right)dt_2+\int_{a_2}^{b_2} f(b_1,t_2)\,dt_2+\int_{a_2}^{b_2} f(a_1,t_2)\,dt_2\right] \\
& -\frac{(b_2-a_2)}{4}\left[2\int_{a_1}^{b_1} f\left(t_1,\frac{a_2+b_2}{2}\right)dt_1+\int_{a_1}^{b_1} f(t_1,b_2)\,dt_1\right. \\
& \left.\left.+\int_{a_1}^{b_1} f(t_1,a_2)\,dt_1\right]+\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(t_1,t_2)\,dt_1dt_2\right| \\
\leq\;& \frac{\left\|f''_{t_1,t_2}\right\|_2}{48}\left[(b_1-a_1)(b_2-a_2)\right]^{\frac{3}{2}}.
\end{aligned}
$$

REMARK 6.8. Let $f(t_1,t_2) = g(t_1) g(t_2)$ where $g : [a,b] \to \mathbb{R}$. If g is continuous and satisfies the condition that

$$\|g'(t)\|_p = \left(\frac{1}{b-a} \int_a^b |g'(t)|^p \, dt \right)^{\frac{1}{p}},$$

then, for $x_1 = x_2 = x$, we get

$$\begin{aligned} &\Bigg| (b-a)^2 g(x) g(x) - g(x)(b-a) \int_a^b g(t)\, dt \\ &-g(x)(b-a) \int_a^b g(t)\, dt + \int_a^b \int_a^b g(t) g(t)\, dt\, dt \Bigg| \\ \leq\ & \left(\frac{\|g'\|_p}{(q+1)^{\frac{1}{q}}} \right)^2 \left[(x-a)^{q+1} + (b-x)^{q+1} \right]^{\frac{2}{q}}. \end{aligned}$$

Therefore,

$$\left| \int_a^b g(t)\, dt - (b-a) g(x) \right|^2 \leq \left(\frac{\|g'\|_p}{(q+1)^{\frac{1}{q}}} \right)^2 \left(\left[(x-a)^{q+1} + (b-x)^{q+1} \right]^{\frac{1}{q}} \right)^2.$$

This gives

$$\left| \int_a^b g(t)\, dt - (b-a) g(x) \right| \leq \|g'\|_p \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}}$$

which is Ostrowski type inequality for the $\|\cdot\|_p$ −norm.
Thus, (6.27) is a generalization for two dimensional integrals of the Ostrowski type for $\|\cdot\|_p$ −norms. As S.S. Dragomir and S. Wang proved in their paper [**2**].

COROLLARY 6.11. With the conditions as in Theorem 6.1, then

$$|G(x_1,t_1,x_2,t_2)| \leq \frac{\left\| f''_{t_1,t_2} \right\|_p}{(q+1)^{\frac{2}{q}}} \left[(b_1-a_1)(b_2-a_2) \right]^{1+\frac{1}{q}}, \tag{6.34}$$

where $G(x_1,t_1,x_2,t_2)$ is as given in (6.11).

PROOF. $(x_i-a_i)^{q+1} + (b_i-x_i)^{q+1} \leq (b_i-a_i)^{q+1}$ and $\gamma_i^{q+1} + (1-\gamma_i)^{q+1} \leq 1$. ∎

6.7. Application For Cubature Formulae

Let us consider the arbitrary division:

$$I_n : a_1 = \xi_0 < \xi_1 < ... < \xi_n = b_1$$

on the interval $[a_1,b_1]$ with $x_i \in [\xi_i, \xi_{i+1}]$ for $i = 0,1,...,n-1$ and $J_m : a_2 = \tau_0 < \tau_1 < ... < \tau_m = b_2$ on the interval $[a_2,b_2]$ with $y_j \in [\tau_j, \tau_{j+1}]$ for $j = 0,1,...,m-1$.

Consider the sum

$$
\begin{aligned}
(6.35)\quad A(f,I_n,J_m,x,y) &:= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i v_j f(x_i,y_j) - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i \int_{\tau_j}^{\tau_{j+1}} f(x_i,t_2)\,dt_2 \\
&\quad - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} v_j \int_{\xi_i}^{\xi_{i+1}} f(t_1,y_j)\,dt_1,
\end{aligned}
$$

where $h_i = \xi_{i+1}-\xi_i$ $(i=0,1,...,n-1)$ and $v_j = \tau_{j+1}-\tau_j$ $(j=0,1,...,m-1)$.

Under the above assumptions the following theorem holds.

THEOREM 6.12. *Let $f:[a_1,b_1]\times[a_2,b_2]\to\mathbb{R}$ be as in Theorem 6.4 and $I_n,\ J_m,\ x,y$ be as above. Then we have the cubature formula*

$$
(6.36)\qquad \int_{a_2}^{b_2}\int_{a_1}^{b_1} f(t_1,t_2)\,dt_1dt_2 = A(f,I_n,J_m,x,y)+R(f,I_n,J_m,x,y),
$$

where the remainder term $R(f,I_n,J_m,x,y)$ satisfies the inequality

$$
(6.37)\quad |R(f,I_n,J_m,x,y)| \le \frac{\left\|f''_{t_1,t_2}\right\|_p}{(q+1)^{\frac{2}{q}}}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(\left[(x_i-\xi_i)^{q+1} + \left(\xi_{i+1}-x_i\right)^{q+1}\right]^{\frac{1}{q}}\right.
$$

$$
(6.38)\qquad \left.\times\left[(y_i-\zeta_i)^{q+1}+\left(\zeta_{i+1}-y_i\right)^{q+1}\right]^{\frac{1}{q}}\right) \le \frac{\left\|f''_{t_1,t_2}\right\|_p}{(q+1)^{\frac{2}{q}}}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(h_i\ v_j)^{1+\frac{1}{q}}.
$$

PROOF. Apply inequality 6.27 on the interval $\left[\xi_i,\xi_{i+1}\right]\times\left[\zeta_j,\zeta_{j+1}\right]$, $(i=0,1,...,n-1)$, $(j=0,1,...,m-1)$ to get

$$
\begin{aligned}
&\left|\left(\xi_{i+1}-\xi_i\right)\left(\zeta_{j+1}-\zeta_j\right)f(x_i,y_j) - v_j\int_{\xi_i}^{\xi_{i+1}} f(t_1,y_j)\,dt_1\right. \\
&\left. -h_i\int_{\zeta_j}^{\zeta_{j+1}} f(x_i,t_2)\,dt_2 + \int_{\xi_i}^{\xi_{i+1}}\int_{\zeta_j}^{\zeta_{j+1}} f(t_1,t_2)\,dt_1dt_2\right| \\
\le\ & \frac{1}{(q+1)^{\frac{2}{q}}}\left[(x_i-\xi_i)^{q+1}+\left(\xi_{i+1}-x_i\right)^{q+1}\right]^{\frac{1}{q}}\left[(y_i-\zeta_i)^{q+1}+\left(\zeta_{i+1}-y_i\right)^{q+1}\right]^{\frac{1}{q}} \\
&\times\left(\int_{\zeta_j}^{\zeta_{j+1}}\int_{\xi_i}^{\xi_{i+1}}\left|\frac{\partial^2 f}{\partial t_1\partial t_2}\right|^p dt_1dt_2\right)^{\frac{1}{p}}
\end{aligned}
$$

for all $(i=0,1,...,n-1)$, $(j=0,1,...,m-1)$.
Summing over i from 0 to $n-1$ and over j from 0 to $m-1$, and using the generalized

triangle inequality and Hölder's directed inequality, we obtain

$$
\begin{aligned}
& \left|R\left(f, I_n, J_m, x, y\right)\right| \\
\leq\ & \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left| \frac{1}{(q+1)^{\frac{2}{q}}} \left[\left[\left(x_i-\xi_i\right)^{q+1}+\left(\xi_{i+1}-x_i\right)^{q+1} \right]^{\frac{1}{q}} \right.\right. \\
& \left.\left. \times \left[\left(y_i-\zeta_i\right)^{q+1}+\left(\zeta_{i+1}-y_i\right)^{q+1} \right]^{\frac{1}{q}} \right] \times \left(\int_{\zeta_j}^{\zeta_{j+1}} \int_{\xi_i}^{\xi_{i+1}} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right|^p dt_1 dt_2 \right)^{\frac{1}{p}} \right| \\
\leq\ & \frac{1}{(q+1)^{\frac{2}{q}}} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[\left(\left[\left(x_i-\xi_i\right)^{q+1}+\left(\xi_{i+1}-x_i\right)^{q+1} \right]^{\frac{1}{q}} \right.\right. \\
& \left.\left. \times \left[\left(y_i-\zeta_i\right)^{q+1}+\left(\zeta_{i+1}-y_i\right)^{q+1} \right]^{\frac{1}{q}} \right) \times \left(\int_{\zeta_j}^{\zeta_{j+1}} \int_{\xi_i}^{\xi_{i+1}} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right|^p dt_1 dt_2 \right)^{\frac{1}{p}} \right] \\
\leq\ & \frac{1}{(q+1)^{\frac{2}{q}}} \left(\sum_{i=0}^{n-1} \left(\left[\left(x_i-\xi_i\right)^{q+1}+\left(\xi_{i+1}-x_i\right)^{q+1} \right]^{\frac{1}{q}} \right)^q \right)^{\frac{1}{q}} \\
& \times \left(\sum_{j=0}^{m-1} \left(\left[\left(y_i-\zeta_i\right)^{q+1}+\left(\zeta_{i+1}-y_i\right)^{q+1} \right]^{\frac{1}{q}} \right)^q \right)^{\frac{1}{q}} \\
& \times \left(\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left(\left(\int_{\zeta_j}^{\zeta_{j+1}} \int_{\xi_i}^{\xi_{i+1}} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right|^p dt_1 dt_2 \right)^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
=\ & \frac{1}{(q+1)^{\frac{2}{q}}} \left[\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left(\left[\left(x_i-\xi_i\right)^{q+1}+\left(\xi_{i+1}-x_i\right)^{q+1} \right]^{\frac{1}{q}} \right.\right. \\
& \left.\left. \times \left[\left(y_i-\zeta_i\right)^{q+1}+\left(\zeta_{i+1}-y_i\right)^{q+1} \right]^{\frac{1}{q}} \right) \right] \left\| f''_{t_1,t_2} \right\|_p ,
\end{aligned}
$$

and the first inequality in (6.37) is proved.
The second part follows directly from the fact that

$$\left(x_i-\xi_i\right)^{q+1}+\left(\xi_{i+1}-x_i\right)^{q+1} \leq h_i^{q+1} \quad \text{and} \quad \left(y_i-\zeta_i\right)^{q+1}+\left(\zeta_{i+1}-y_i\right)^{q+1} \leq v_j^{q+1}.$$

∎

6.8. Mappings Whose First Derivatives Belong to $L_1(a,b)$.

In this section an inequality of Ostrowski type for two dimensional integrals for functions whose first derivatives belong to L_1 can be produced as shown in the following theorem,

THEOREM 6.13. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable mapping on $[a_1, b_1] \times [a_2, b_2]$ and let $f''_{t_1,t_2} = \frac{\partial^2 f}{\partial t_1 \partial t_2}$ be bounded on $(a_1, b_1) \times (a_2, b_2)$, that is,*

$$\left\| f''_{t_1,t_2} \right\|_1 := \int_{a_1}^{b_1} \int_{a_2}^{b_2} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right| dt_1 dt_2 < \infty \ ,$$

and with the same conditions as in Theorem (6.1). We can obtain the following inequality

$$|G(x_1, t_1, x_2, t_2)| \le \left\| f''_{t_1,t_2} \right\|_1 \prod_{i=1}^{2} M_i, \tag{6.39}$$

where

$$M_i = \frac{(b_i - a_i)}{4} \Big[1 + |2\gamma_i - 1| \Big] + 2 \Big| (x_i - \frac{a_i + b_i}{2})(1 + |2\gamma_i - 1|) \Big|$$

and $G(x_1, t_1, x_2, t_2)$ is as given in (6.11).

PROOF. The proof follows that of Theorem (6.1). we have,

$$\begin{aligned}
&\left| \int_{a_2}^{b_2} \int_{a_1}^{b_1} p_2(x_2, t_2)\, p_1(x_1, t_1)\, \frac{\partial^2 f}{\partial t_1 \partial t_2} dt_1 dt_2 \right| \qquad (6.40)\\
\le\ & \left(\int_{a_2}^{b_2} \int_{a_1}^{b_1} |p_2(x_2, t_2)\, p_1(x_1, t_1)|\, dt_1 dt_2 \right) \left(\int_{a_2}^{b_2} \int_{a_1}^{b_1} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right| dt_1 dt_2 \right) \\
=\ & \sup_{(t_1,t_2) \in [a_1,b_1] \mathrm{X} [a_2,b_2]} \Big| p_2(x_2, t_2) p_1(x_1, t_1) \Big| \left\| f''_{t_1,t_2} \right\|_1 \\
=\ & \sup_{t_2 \in [a_2,b_2]} \Big| p_2(x_2, t_2) \Big| \sup_{t_1 \in [a_1,b_1]} \Big| p_1(x_1, t_1) \Big|
\end{aligned}$$

Now, consider

$$\begin{aligned}
\mathcal{G}_1(x_1) &= \sup_{t_1 \in [a_1,b_1]} \Big| p_1(x_1, t_1) \Big| \\
&= max\{\alpha_1 - a_1, x_1 - \alpha_1, \beta_1 - x_1, b_1 - \beta_1\} \qquad (6.41)
\end{aligned}$$

Let

$$\begin{aligned}
\mathfrak{M}_1(x_1) &= max\{\alpha_1 - a_1, x_1 - \alpha_1\} = \frac{x_1 - a_1}{2} + \Big| \alpha_1 - \frac{a_1 + x_1}{2} \Big| \\
&= \frac{x_1 - a_1}{2} + \Big[1 + |2\gamma_1 - 1| \Big]
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{M}_2(x_1) &= max\{\beta_1 - x_1, b_1 - \beta_1\} = \frac{b_1 - x_1}{2} + \Big| \beta_1 - \frac{b_1 + x_1}{2} \Big| \\
&= \frac{b_1 - x_1}{2} + \Big[1 + |2\gamma_1 - 1| \Big]
\end{aligned}$$

then

$$\begin{aligned}\mathcal{G}_1(x_1) &= max\{\mathfrak{M}_1, \mathfrak{M}_2\}\\ &= \frac{b_1-a_1}{4}\left[1+|2\gamma_1-1|\right]+2\left|(x_1-\frac{a_1+b_1}{2}()1+|2\gamma_1-1|)\right|\end{aligned}$$

and similarly

$$\mathcal{G}_2(x_2) = \frac{b_2-a_2}{4}\left[1+|2\gamma_2-1|\right]+2\left|(x_2-\frac{a_2+b_2}{2})(1+|2\gamma_2-1|)\right|.$$

Substituting into (6.40) will produce the result in (6.39) and thus the proof completed. ∎

Corollary 6.14. With the conditions as in Theorem 6.35, then

$$|G(x_1,t_1,x_2,t_2)| \leq \frac{\left\|f''_{t_1,t_2}\right\|_1}{16}\prod_{i=1}^{2}(b_i-a_i)\left[1+|2\gamma_i-1|\right] \tag{6.42}$$

Proof. Put $x_i = \frac{a_i+b_i}{2}$ in equation (6.39). ∎

Remark 6.9. If $\gamma_1=\gamma_2=0$, then (6.42) becomes

$$\begin{aligned}&\left|(b_1-a_1)(b_2-a_2)f\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right)-(b_2-a_2)\int_{a_1}^{b_1}f\left(t_1,\frac{a_2+b_2}{2}\right)dt_1\right.\\ &\left.\quad-(b_1-a_1)\int_{a_2}^{b_2}f\left(\frac{a_1+b_1}{2},t_2\right)dt_2-\int_{a_2}^{b_2}\int_{a_1}^{b_1}f(t_1,t_2)\,dt_1dt_2\right|\\ &\leq \frac{\left\|f''_{t_1,t_2}\right\|_1}{4}\left[(b_1-a_1)(b_2-a_2)\right].\end{aligned} \tag{6.43}$$

Remark 6.10. If $\gamma_1=\gamma_2=1$, then (6.42) becomes

$$\begin{aligned}&\left|\frac{(b_1-a_1)(b_2-a_2)}{4}\left[f(b_1,b_2)+f(a_1,b_2)+f(b_1,a_2)+f(a_1,a_2)\right]\right.\\ &-\frac{1}{2}\left[(b_2-a_2)\int_{a_1}^{b_1}f(t_1,b_2)\,dt_1+(b_2-a_2)\int_{a_1}^{b_1}f(t_1,a_2)\,dt_1\right.\\ &\left.+(b_1-a_1)\int_{a_2}^{b_2}f(b_1,t_2)\,dt_2+(b_1-a_1)\int_{a_2}^{b_2}f(a_1,t_2)\,dt_2\right]\\ &\left.+\int_{a_2}^{b_2}\int_{a_1}^{b_1}f(t_1,t_2)\,dt_1dt_2\right|\\ &\leq \frac{\left\|f''_{t_1,t_2}\right\|_1}{4}\left[(b_1-a_1)(b_2-a_2)\right].\end{aligned} \tag{6.44}$$

Remark 6.11. If $\gamma_1=\gamma_2=\frac{1}{2}$, then (6.42) becomes

$$\left|\frac{(b_1-a_1)(b_2-a_2)}{4}f\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right)+\frac{(b_1-a_1)(b_2-a_2)}{8}\right. \tag{6.45}$$

$$\times\left[f\left(b_1,\frac{a_2+b_2}{2}\right)+f\left(a_1,\frac{a_2+b_2}{2}\right)+f\left(\frac{a_1+b_1}{2},a_2\right)+f\left(\frac{a_1+b_1}{2},b_2\right)\right]$$
$$+\frac{(b_1-a_1)(b_2-a_2)}{16}\left[f(b_1,b_2)+f(a_1,b_2)+f(b_1,a_2)+f(a_1,a_2)\right]$$
$$-\frac{(b_1-a_1)}{4}\left[2\int_{a_2}^{b_2}f\left(\frac{a_1+b_1}{2},t_2\right)dt_2\right.$$
$$\left.+\int_{a_2}^{b_2}f(b_1,t_2)\,dt_2+\int_{a_2}^{b_2}f(a_1,t_2)\,dt_2\right]-\frac{(b_2-a_2)}{4}\left[2\int_{a_1}^{b_1}f\left(t_1,\frac{a_2+b_2}{2}\right)dt_1\right.$$
$$\left.\left.+\int_{a_1}^{b_1}f(t_1,b_2)\,dt_1+\int_{a_1}^{b_1}f(t_1,a_2)\,dt_1\right]+\int_{a_2}^{b_2}\int_{a_1}^{b_1}f(t_1,t_2)\,dt_1dt_2\right|$$
$$\leq\quad\frac{\left\|f''_{t_1,t_2}\right\|_1}{16}\left[(b_1-a_1)(b_2-a_2)\right].$$

REMARK 6.12. Let $f(t_1,t_2)=g(t_1)\,g(t_2)$ where $g:[a,b]\to\mathbb{R}$. If g is continuous and satisfies the condition that

$$\|g'(t)\|_1=\left(\frac{1}{b-a}\int_a^b|g'(t)|\,dt\right),$$

then, for $x_1=x_2=x$, and $\gamma=1$ we get

$$\left|(b-a)^2g(x)\,g(x)-g(x)(b-a)\int_a^bg(t)\,dt\right.$$
$$\left.-g(x)(b-a)\int_a^bg(t)\,dt+\int_a^b\int_a^bg(t)\,g(t)\,dt\;dt\right|$$
$$\leq(\|g'\|_1)^2\left[\frac{(b-a)}{2}+4\left|x-\frac{a+b}{2}\right|\right]^2$$

Therefore,

$$\left|\int_a^bg(t)\,dt-(b-a)\,g(x)\right|^2\leq(\|g'\|_1)^2\left[\frac{(b-a)}{2}+4\left|x-\frac{a+b}{2}\right|\right]^2$$

This gives

$$\left|\int_a^bg(t)\,dt-(b-a)\,g(x)\right|\leq\|g'\|_1\left[\frac{(b-a)}{2}+4\left|x-\frac{a+b}{2}\right|\right]$$

which is Ostrowski type inequality for the $\|\cdot\|_1$ −norm.
Thus, (6.39) is a generalization for two dimensional integrals of the Ostrowski type inequality for $\|\cdot\|_1$ −norms.

6.9. Integral Identities

In [**11**], P. Cerone, S.S. Dragomir and J. Roumeliotis proved the following Ostrowski type inequality for $n-$time differentiable mappings.

THEOREM 6.15. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)} \in L_\infty [a,b]$. Then for all $x \in [a,b]$, we have the inequality:*

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right| \le \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left[(x-a)^{n+1} + (b-x)^{n+1}\right] \le \frac{\left\|f^{(n)}\right\|_\infty (b-a)^{n+1}}{(n+1)!}, \tag{6.46}$$

where $\left\|f^{(n)}\right\|_\infty := \sup_{t\in[a,b]} \left|f^{(n)}(t)\right| < \infty$.

For other similar results for $n-$time differentiable mappings, see the paper [**17**] by Fink and [**18**] by Anastassiou.

In [**13**] and [**14**] the authors proved some inequalities of Ostrowski type for double integrals in terms of different norms.

In this section we combine the above two results and develop them in two dimensions to obtain a generalization of the Ostrowski inequality for n-time differentiable mappings using different types of norms.

The result presented here approximates a two-dimensional integral for $n-$time differentiable mappings via the application of function evaluations of one dimensional integrals at the boundary and an interior point.

The following result holds.

THEOREM 6.16. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a continuous mapping such that the following partial derivatives $\frac{\partial^{l+k} f(\cdot,\cdot)}{\partial x^k \partial y^l}$, $k = 0,1,...,n-1$, $l = 0,1,...,m-1$ exist and are continuous on $[a,b] \times [c,d]$. Further, for $K_n : [a,b]^2 \to \mathbb{R}$, $S_m : [c,d]^2 \to \mathbb{R}$ given by*

$$\begin{cases} K_n(x,t) := \begin{cases} \frac{(t-a)^n}{n!}, & t \in [a,x] \\ \frac{(t-b)^n}{n!}, & t \in (x,b] \end{cases} \\ S_m(y,s) := \begin{cases} \frac{(s-c)^m}{m!}, & s \in [c,y] \\ \frac{(s-d)^m}{m!}, & s \in (y,d] \end{cases} \end{cases} \tag{6.47}$$

then for all $(x,y) \in [a,b] \times [c,d]$, we have the identity:

$$\int_a^b \int_c^d f(t,s)\,ds\,dt = \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} X_k(x) Y_l(y) \frac{\partial^{l+k} f(x,y)}{\partial x^k \partial y^l} \tag{6.48}$$

$$+ (-1)^m \sum_{k=0}^{n-1} X_k(x) \int_c^d S_m(y,s) \frac{\partial^{k+m} f(x,s)}{\partial x^k \partial s^m} ds + (-1)^n \sum_{l=0}^{m-1} Y_l(y)$$

$$\times \int_a^b K_n(x,t) \frac{\partial^{n+l} f(t,y)}{\partial t^n \partial y^l} dt + (-1)^{m+n} \int_a^b \int_c^d K_n(x,t) S_m(y,s) \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} ds\,dt,$$

where

$$(6.49)\qquad \begin{cases} X_k(x) = \frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}, \\ \\ Y_l(y) = \frac{(d-y)^{l+1}+(-1)^l(y-c)^{l+1}}{(l+1)!}. \end{cases}$$

PROOF. Applying the identity (see [**11**])

$$(6.50)\qquad \int_a^b g(t)\,dt = \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]g^{(k)}(x) + (-1)^n\int_a^b P_n(x,t)\,g^{(n)}(t)\,dt,$$

where

$$P_n(x,t) = \begin{cases} \frac{(t-a)^n}{n!} & \text{if} \quad t\in[a,x], \\ \\ \frac{(t-b)^n}{n!} & \text{if} \quad t\in(x,b], \end{cases}$$

which has been used essentially in the proof of Theorem 6.15, for the partial mapping $f(\cdot,s)$, $s\in[c,d]$, we can write

$$(6.51)\qquad \int_a^b f(t,s)\,dt = \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]\frac{\partial^k f(x,s)}{\partial x^k} + (-1)^n\int_a^b K_n(x,t)\,\frac{\partial^n f(t,s)}{\partial t^n}dt$$

for every $x\in[a,b]$ and $s\in[c,d]$.

Integrating (6.51) over s on $[c,d]$, we deduce

$$(6.52)\qquad \int_a^b\int_c^d f(t,s)\,ds\,dt = \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] \times\int_c^d \frac{\partial^k f(x,s)}{\partial x^k}ds + (-1)^n\int_a^b K_n(x,t)\left(\int_c^d \frac{\partial^n f(t,s)}{\partial t^n}ds\right)dt$$

for all $x\in[a,b]$.

Applying the identity (I) again for the partial mapping $\frac{\partial^k f(x,\cdot)}{\partial x^k}$ on $[c,d]$, we obtain

$$(6.53)\qquad \int_c^d \frac{\partial^k f(x,s)}{\partial x^k}ds = \sum_{l=0}^{m-1}\left[\frac{(d-y)^{l+1}+(-1)^l(y-c)^{l+1}}{(l+1)!}\right]\frac{\partial^l}{\partial y^l}\left(\frac{\partial^k f(x,y)}{\partial x^k}\right)$$

$$+(-1)^m\int_c^d S_m(y,s)\,\frac{\partial^m}{\partial s^m}\left(\frac{\partial^k f(x,s)}{\partial x^k}\right)ds$$

$$= \sum_{l=0}^{m-1}\left[\frac{(d-y)^{l+1}+(-1)^l(y-c)^{l+1}}{(l+1)!}\right]\frac{\partial^{l+k} f(x,y)}{\partial x^k\partial y^l} + (-1)^m\int_c^d S_m(y,s)\,\frac{\partial^{k+m} f(x,s)}{\partial x^k\partial s^m}ds.$$

In addition, the identity (6.50) applied for the partial derivative $\frac{\partial^n f(t,\cdot)}{\partial t^n}$ also gives

$$\int_c^d \frac{\partial^n (t,s)}{\partial t^n} ds = \sum_{l=0}^{m-1} \left[\frac{(d-y)^{l+1} + (-1)^l (y-c)^{l+1}}{(l+1)!} \right] \frac{\partial^{n+l} f(t,y)}{\partial t^n \partial y^l} + (-1)^m \int_c^d S_m(y,s) \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} ds. \tag{6.54}$$

Using (6.53) and (6.54) and substituting into (6.52) will produce the result (6.48), and thus the theorem is proved. ∎

Corollary 6.17. With the assumptions as in Theorem 6.16, we have the representation

$$\begin{aligned}\int_a^b \int_c^d f(t,s)\, ds\, dt &= \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} X_k\left(\frac{a+b}{2}\right) Y_l\left(\frac{c+d}{2}\right) \frac{\partial^{l+k} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)}{\partial x^k \partial y^l} \\ &+ (-1)^m \sum_{k=0}^{n-1} X_k\left(\frac{a+b}{2}\right) \int_c^d \tilde{S}_m(s) \frac{\partial^{k+m} f\left(\frac{a+b}{2}, s\right)}{\partial x^k \partial s^m} ds \\ &+ (-1)^n \sum_{l=0}^{m-1} Y_l\left(\frac{c+d}{2}\right) \int_a^b \tilde{K}_n(t) \frac{\partial^{n+l} f\left(t, \frac{c+d}{2}\right)}{\partial t^n \partial y^l} dt \\ &+ (-1)^{m+n} \int_a^b \int_c^d \tilde{K}_n(t) \tilde{S}_m(s) \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} ds\, dt,\end{aligned} \tag{6.55}$$

where $X_k(\cdot)$ and $Y_l(\cdot)$ are as given in (6.48) and so

$$X_k\left(\frac{a+b}{2}\right) = \left[\frac{1+(-1)^k}{(k+1)!}\right] \frac{(b-a)^{k+1}}{2^{k+1}},$$

$$Y_l\left(\frac{c+d}{2}\right) = \left[\frac{1+(-1)^l}{(l+1)!}\right] \frac{(d-c)^{l+1}}{2^{l+1}},$$

and $\tilde{K}_n : [a,b] \to \mathbb{R}$, $\tilde{S}_m : [c,d] \to \mathbb{R}$ are given by

$$\tilde{K}_n(t) = K_n\left(\frac{a+b}{2}, t\right)$$

and

$$\tilde{S}_m(s) = S_m\left(\frac{c+d}{2}, s\right)$$

on using (6.47).

Corollary 6.18. Let f be as in Theorem 6.16. Then we have the following identity

$$\begin{aligned}\int_a^b \int_c^d f(t,s)\, ds\, dt &= \frac{1}{4} \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \left[\frac{(b-a)^{k+1}}{(k+1)!}\right] \times \left[\frac{(d-c)^{l+1}}{(l+1)!}\right] \\ &\times \frac{\partial^{l+k}}{\partial x^k \partial y^l} \left[f(a,c) + (-1)^l f(a,d) + (-1)^k f(b,c) + (-1)^{l+k} f(b,d)\right] \\ &+ \frac{1}{4}(-1)^m \sum_{k=0}^{n-1} \left[\frac{(b-a)^{k+1}}{(k+1)!}\right] \times \left[\int_c^d Y_{m-1}(s) \frac{\partial^{k+m}}{\partial x^k \partial s^m} \left[f(a,s) + (-1)^k f(b,s)\right] ds\right]\end{aligned} \tag{6.56}$$

$$+\frac{1}{4}(-1)^n\sum_{l=0}^{m-1}\left[\frac{(d-c)^{l+1}}{(l+1)!}\right]\times\left[\int_a^b X_{n-1}(t)\frac{\partial^{n+l}}{\partial t^n\partial y^l}\left[f(t,c)+(-1)^k f(t,d)\right]dt\right]$$
$$+\frac{1}{4}\int_a^b\int_c^d X_{n-1}(t)\cdot Y_{m-1}(s)\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}ds\,dt,$$

where $X_{n-1}(t)$ and $Y_{m-1}(s)$ are as given by (6.49).

Proof. By substituting $(x,y)=(a,c),(a,d),(b,c),(b,d)$ respectively and summing the resulting identities and after some simplification, we get the desired inequality (6.56). ∎

6.10. Some Integral Inequalities

We start with the following result

Theorem 6.19. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be continuous on $[a,b]\times[c,d]$, and assume that $\frac{\partial^{n+m}f}{\partial t^n\partial s^m}$ exist on $(a,b)\times(c,d)$. Then we have the inequality*

$$\left|\int_a^b\int_c^d f(t,s)\,ds\,dt-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}X_k(x)\cdot Y_l(y)\frac{\partial^{l+k}f(x,y)}{\partial x^k\partial y^l}\right. \tag{6.57}$$
$$\left.-(-1)^m\sum_{k=0}^{n-1}X_k(x)\int_c^d S(y,s)\frac{\partial^{k+m}f(x,s)}{\partial x^k\partial s^m}ds-(-1)^n\sum_{l=0}^{m-1}Y_l(y)\int_a^b K(x,t)\frac{\partial^{n+l}f(t,y)}{\partial t^n\partial y^l}dt\right|$$
$$\leq\begin{cases}\frac{1}{(n+1)!(m+1)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right]\times\left[(y-c)^{m+1}+(d-y)^{m+1}\right]\times\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty \\ \text{if }\ \frac{\partial^{n+m}f}{\partial t^n\partial s^m}\in L_\infty([a,b]\times[c,d]); \\ \frac{1}{n!m!}\left[\frac{(x-a)^{nq+1}+(b-x)^{nq+1}}{nq+1}\right]^{\frac{1}{q}}\times\left[\frac{(y-c)^{mq+1}+(d-y)^{mq+1}}{mq+1}\right]^{\frac{1}{q}}\times\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_p \\ \text{if }\ \frac{\partial^{n+m}f}{\partial t^n\partial s^m}\in L_p([a,b]\times[c,d]),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{4n!m!}\left[(x-a)^n+(b-x)^n+|(x-a)^n-(b-x)^n|\right] \\ \quad\times\left[(y-c)^m+(d-y)^m+|(y-c)^m-(d-y)^m|\right]\times\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_1 \\ \text{if }\ \frac{\partial^{n+m}f}{\partial t^n\partial s^m}\in L_1([a,b]\times[c,d])\end{cases}$$

for all $(x,y)\in[a,b]\times[c,d]$, where

$$\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty=\sup_{(t,s)\in[a,b]\times[c,d]}\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|<\infty,$$
$$\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_p=\left(\int_a^b\int_c^d\left|\frac{\partial^{n+m}}{\partial t^n\partial s^m}f(t,s)\right|^p dtds\right)^{\frac{1}{p}}<\infty.$$

PROOF. Using Theorem 6.16, we get from (6.48)

$$\begin{aligned}
&\Bigg|\int_a^b\int_c^d f(t,s)\,ds\,dt-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}X_k(x)\cdot Y_l(y)\frac{\partial^{l+k}f(x,y)}{\partial x^k\partial y^l}-(-1)^m\sum_{k=0}^{n-1}X_k(x)\\
&\quad\times\int_c^d S(y,s)\frac{\partial^{k+m}f(x,s)}{\partial x^k\partial s^m}ds-(-1)^n\sum_{l=0}^{m-1}Y_l(y)\int_a^b K(x,t)\frac{\partial^{n+l}f(t,y)}{\partial t^n\partial y^l}dt\Bigg|\\
&=\left|\int_a^b\int_c^d K_n(x,t)\,S_m(y,s)\,\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^n}ds\,dt\right|\\
&\le\int_a^b\int_c^d|K_n(x,t)\,S_m(y,s)|\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^n}\right|ds\,dt.
\end{aligned}\tag{6.58}$$

Using Hölder's inequality and properties of the modulus and integral, then we have that

$$\int_a^b\int_c^d|K_n(x,t)\,S_m(y,s)|\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|ds\,dt\tag{6.59}$$

$$\le\begin{cases}\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty\int_a^b\int_c^d|K_n(x,t)\,S_m(y,s)|\,dt\,ds\\[2ex]\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_p\left(\int_a^b\int_c^d|K_n(x,t)\,S_m(y,s)|^q\,dt\,ds\right)^{\frac{1}{q}},\\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_1\sup\limits_{(t,s)\in[a,b]\times[c,d]}|K_n(x,t)\,S_m(y,s)|\,.\end{cases}$$

Now, from (6.59) and using (6.47),

$$\begin{aligned}
&\int_a^b\int_c^d|K_n(x,t)\,S_m(y,s)|\,dt\,ds=\int_a^b|K_n(x,t)|\,dt\int_c^d|S_m(y,s)|\,ds\\
&=\left[\int_a^x\frac{(t-a)^n}{n!}dt+\int_x^b\frac{(b-t)^n}{n!}dt\right]\times\left[\int_c^y\frac{(s-c)^m}{m!}ds+\int_y^d\frac{(d-s)^m}{m!}ds\right]\\
&=\frac{\left[(x-a)^{n+1}+(b-x)^{n+1}\right]\left[(y-c)^{m+1}+(d-y)^{m+1}\right]}{(n+1)!\,(m+1)!}
\end{aligned}$$

giving the first inequality in (6.57).

Further, on using (6.47) and from (6.59)

$$\left(\int_a^b\int_c^d |K_n(x,t)\,S_m(y,s)|^q\,ds\,dt\right)^{\frac{1}{q}} = \left(\int_a^b |K_n(x,t)|^q\,dt\right)^{\frac{1}{q}}\left(\int_c^d |S_m(y,s)|^q\,ds\,dt\right)^{\frac{1}{q}}$$

$$= \frac{1}{n!m!}\left[\int_a^x (t-a)^{nq}\,dt + \int_x^b (b-t)^{nq}\,dt\right]^{\frac{1}{q}} \times \left[\int_c^y (s-c)^{mq}\,ds + \int_y^d (d-s)^{mq}\,ds\right]^{\frac{1}{q}}$$

$$= \frac{1}{n!m!}\left[\frac{(x-a)^{nq+1}+(b-x)^{nq+1}}{nq+1}\right]^{\frac{1}{q}} \times \left[\frac{(y-c)^{mq+1}+(d-y)^{mq+1}}{mq+1}\right]^{\frac{1}{q}}$$

producing the second inequality in (6.57).

Finally, from (6.47) and (6.59),

$$\sup_{(t,s)\in[a,b]\times[c,d]} |K_n(x,t)\,S_m(y,s)| = \sup_{t\in[a,b]} |K_n(x,t)| \sup_{s\in[c,d]} |S_m(y,s)|$$

$$= \max\left\{\frac{(x-a)^n}{n!},\frac{(b-x)^n}{n!}\right\} \times \max\left\{\frac{(y-c)^m}{m!},\frac{(d-y)^m}{m!}\right\}$$

$$= \frac{1}{n!m!}\left[\frac{(x-a)^n+(b-x)^n}{2} + \left|\frac{(x-a)^n-(b-x)^n}{2}\right|\right] \times \left[\frac{(y-c)^m+(d-y)^m}{2} + \left|\frac{(y-c)^m+(d-y)^m}{2}\right|\right].$$

gives the inequality in (6.57) where we have used the fact that

$$\max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{Y-X}{2}\right|.$$

Thus the theorem is now completely proved. ∎

¿From the results of Theorem 6.19 above, we have the following corollary.

Corollary 6.20. With the assumptions of Theorem 6.19, we have the inequality

$$
\begin{aligned}
(6.60)\qquad & \left|\int_a^b \int_c^d f(t,s)\,ds\,dt\right. \\
& -\sum_{k=0}^{n-1}\sum_{l=0}^{m-1} X_k\left(\frac{a+b}{2}\right) Y_l\left(\frac{c+d}{2}\right)\frac{\partial^{l+k}}{\partial x^k \partial y^l} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
& -(-1)^m \sum_{k=0}^{n-1} X_k\left(\frac{a+b}{2}\right)\int_c^d \tilde{S}_m(s)\frac{\partial^{k+m}}{\partial x^k \partial s^m} f\left(\frac{a+b}{2},s\right) ds \\
& \left. -(-1)^n \sum_{l=0}^{m-1} Y_l\left(\frac{c+d}{2}\right)\int_a^b \tilde{K}_n(t)\frac{\partial^{n+l}}{\partial t^n \partial y^l} f\left(t,\frac{c+d}{2}\right) dt\right| \\
& \leq \begin{cases}
\frac{1}{2^{n+m}(n+1)!(m+1)!}(b-a)^{n+1}(d-c)^{m+1}\times\left\|\frac{\partial^{n+m}f}{\partial t^n \partial s^m}\right\|_\infty; \\
\frac{1}{2^{n+m}n!m!}\left[\frac{(b-a)^{nq+1}(d-c)^{mq+1}}{(nq+1)(mq+1)}\right]^{\frac{1}{q}}\times\left\|\frac{\partial^{n+m}f}{\partial t^n \partial s^m}\right\|_p; \\
\frac{1}{2^{n+m}n!m!}(b-a)^n(d-c)^m\times\left\|\frac{\partial^{n+m}f}{\partial t^n \partial s^m}\right\|_1,
\end{cases}
\end{aligned}
$$

where $\|\cdot\|_p$ ($p\in[1,\infty]$) are the Lebesgue norms on $[a,b]\times[c,d]$.

Proof. Taking $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in (6.57) readily produces the result as stated. ∎

These are the tightest possible for their respective *Lebesgue* norms, because of the symmetric and convex nature of the bounds in (6.57).

Remark 6.13. For $n=m=1$ in (6.60) and $\frac{\partial^2 f}{\partial t \partial s}$ belonging to the appropriate Lebesgue spaces on $[a,b]\times[c,d]$, we have

$$
\begin{aligned}
(6.61)\quad & \left|\int_a^b\int_c^d f(t,s)\,ds\,dt-(b-a)(d-c)\,f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right. \\
& \left. +(b-a)\int_c^d \tilde{S}_1(s)\frac{\partial}{\partial s}f\left(\frac{a+b}{2},s\right)ds+(d-c)\int_a^b \tilde{K}_1(t)\frac{\partial}{\partial t}f\left(t,\frac{c+d}{2}\right)dt\right|
\end{aligned}
$$

$$
(6.62)\qquad \leq \begin{cases}
\frac{1}{16}(b-a)^2(d-c)^2\times\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty; \\
\frac{1}{4}\left[\frac{(b-a)^{q+1}(d-c)^{q+1}}{(q+1)^2}\right]^{\frac{1}{q}}\times\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_p; \\
\frac{1}{4}(b-a)(d-c)\times\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_1,
\end{cases}
$$

and thus some of the results of [**15**] and [**16**] are recaptured.

Corollary 6.21. With the assumptions on f as outlined in Theorem 6.19, we can obtain another result which is a generalization of the Trapezoid inequality

$$
\begin{aligned}
(6.63)\quad & \left| \int_a^b \int_c^d f(t,s)\,ds\,dt - \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} \frac{(b-a)^{k+1}}{(k+1)!} \cdot \frac{(d-c)^{l+1}}{(l+1)!} \right. \\
& \times \left[\frac{f(a,c) + (-1)^l f(a,d) + (-1)^k f(b,c) + (-1)^{k+l} f(b,d)}{4} \right] \\
& - (-1)^m \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \int_c^d Y_l(s) \frac{\partial^{k+m}}{\partial x^k \partial s^m} \left[\frac{f(a,s) + (-1)^k f(b,s)}{4} \right] ds \\
& \left. - (-1)^n \sum_{l=0}^{m-1} \left[\frac{(d-c)^{l+1}}{(l+1)!} \right] \int_a^b X_k(t) \frac{\partial^{l+n}}{\partial t^n \partial y^l} \left[\frac{f(t,c) + (-1)^l f(t,d)}{4} \right] dt \right| \\
\le & \begin{cases}
\kappa_{n,m} \frac{(b-a)^{n+1}(d-c)^{m+1}}{(n+1)!(m+1)!} \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty ; \\
\text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_\infty([a,b]\times[c,d]); \\
\left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_p \left(\int_a^b |T_n(a,b;t)|^q dt \right)^{\frac{1}{q}} \left(\int_c^d |T_m(c,d;s)|^q ds \right)^{\frac{1}{q}} \\
\text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_p([a,b]\times[c,d]),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\frac{(b-a)^n (d-c)^m}{4n!m!} \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_1 , \\
\text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_1([a,b]\times[c,d]).
\end{cases}
\end{aligned}
$$

where

$$
\kappa_{n,m} := \begin{cases}
1 & \text{if } n = 2r_1 \text{ and } m = 2r_2 , \\
\frac{2^n-1}{2^n} & \text{if } n = 2r_1+1 \text{ and } m = 2r_2 , \\
\frac{2^m-1}{2^m} & \text{if } n = 2r_1 \text{ and } m = 2r_2+1 , \\
\frac{(2^n-1)}{2^n} \cdot \frac{(2^m-1)}{2^m} & \text{if } n = 2r_1+1 \text{ and } m = 2r_2+1
\end{cases}
$$

PROOF. Using the identity (6.56), we find that

$$\left|\int_a^b\int_c^d f(t,s)\,ds\,dt-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{(b-a)^{k+1}}{(k+1)!}\cdot\frac{(d-c)^{l+1}}{(l+1)!}\right.$$
$$\times\frac{\partial^{l+k}}{\partial x^k\partial y^l}\left[\frac{f(a,c)+(-1)^l f(a,d)+(-1)^k f(b,c)+(-1)^{k+l}f(b,d)}{4}\right]$$
$$-(-1)^m\sum_{k=0}^{n-1}\frac{(b-a)^{k+1}}{(k+1)!}\int_c^d\frac{(s-c)^m+(s-d)^m}{m!}\times\frac{\partial^{k+m}}{\partial x^k\partial s^m}\left[\frac{f(a,s)+(-1)^k f(b,s)}{4}\right]ds$$

$$\left.-(-1)^n\sum_{l=0}^{m-1}\left[\frac{(d-c)^{l+1}}{(l+1)!}\right]\int_a^b\frac{(t-a)^n+(t-b)^n}{n!}\times\frac{\partial^{n+l}}{\partial y^l\partial t^n}\left[\frac{f(t,c)+(-1)^l f(t,d)}{4}\right]dt\right|$$
$$=\left|\int_a^b\int_c^d T_n(a,b;t)\,T_m(c,d;s)\,\frac{\partial^{n+m}f}{\partial t^n\partial s^m}dsdt\right|$$
$$\leq\begin{cases}\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty\int_a^b\int_c^d|T_n(a,b;t)\,T_m(c,d;s)|\,dt\,ds\\ \text{if } \frac{\partial^{n+m}f}{\partial t^n\partial s^m}\in L_\infty([a,b]\times[c,d]);\\ \left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_p\left(\int_a^b|T_n(a,b;t)|^q\,dt\right)^{\frac{1}{q}}\left(\int_c^d|T_m(c,d;s)|^q\,ds\right)^{\frac{1}{q}}\\ \text{if } \frac{\partial^{n+m}f}{\partial t^n\partial s^m}\in L_p([a,b]\times[c,d]),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_1\sup_{(t,s)\in[a,b]\times[c,d]}|T_n(a,b;t)\,T_m(c,d;s)|\\ \text{if } \frac{\partial^{n+m}f}{\partial t^n\partial s^m}\in L_1([a,b]\times[c,d]).\end{cases}$$

where

$$T_n(a,b;t)=\frac{1}{2}\left[\frac{(b-t)^n+(-1)^n(t-a)^n}{n!}\right],$$
$$T_m(c,d;s)=\frac{1}{2}\left[\frac{(d-s)^m+(-1)^m(s-c)^m}{m!}\right].$$

Now consider $\int_a^b|T_n(a,b;t)|\,dt$. As may be seen, explicit evaluation of the integral depends on whether n is even or odd.

(i) If n is even, put $n = 2r_1$. Therefore,

$$\begin{aligned}\int_a^b |T_n(a,b;t)|\,dt &= \frac{1}{(2r_1)!}\int_a^b \frac{(b-t)^{2r_1}+(t-a)^{2r_1}}{2}dt \\ &= \frac{1}{(2r_1)!}\cdot\frac{1}{2}\left[\frac{(b-a)^{2r_1+1}}{2r_1+1}+\frac{(b-a)^{2r_1+1}}{2r_1+1}\right] \\ &= \frac{(b-a)^{2r_1+1}}{(2r_1+1)!}=\frac{(b-a)^{n+1}}{(n+1)!}.\end{aligned}$$

Similarly,

$$\int_c^d |T_m(c,d;s)|\,ds = \frac{1}{(2r_2)!}\int_c^d \frac{(d-s)^{2r_2}+(s-c)^{2r_2}}{2}ds = \frac{(d-c)^{m+1}}{(m+1)!}.$$

(ii) Now, if n is odd, that is, $n = 2r_1+1$, then

$$T_n(a,b;t) = \frac{(b-t)^{2r_1+1}-(t-a)^{2r_1+1}}{2(2r_1+1)!}.$$

Let $g(t) = (b-t)^{2r_1+1}-(t-a)^{2r_1+1}$.

We can observe that

$$\begin{cases} g(t) < 0 \text{ for all } t \in (\frac{a+b}{2}, b] \\ g(t) = 0 \quad \text{at } t = \frac{a+b}{2} \\ g(t) > 0 \text{ for all } t \in [a, \frac{a+b}{2}). \end{cases}$$

Thus

$$\begin{aligned} & 2(2r_1+1)!\int_a^b |T_n(a,b;t)|\,dt \\ = & \left[\int_a^{\frac{a+b}{2}}\left[(b-t)^{2r_1+1}-(t-a)^{2r_1+1}\right]dt + \int_{\frac{a+b}{2}}^b \left[(t-a)^{2r_1+1}-(b-t)^{2r_1+1}\right]dt\right] \\ = & \left[2\cdot\frac{(b-a)^{2r_1+2}}{2r_1+2}-4\frac{\left(\frac{b-a}{2}\right)^{2r_1+2}}{2r_1+2}\right]\end{aligned}$$

and so

$$\begin{aligned}\int_a^b |T_n(a,b;t)|\,dt &= \frac{(b-a)^{2r_1+2}}{(2r_1+2)(2r_1+1)!}\left[1-\frac{1}{2^{2r_1+1}}\right] \\ &= \frac{(b-a)^{2r_1+2}}{(2r_1+2)!}\left[\frac{2^{2r_1+1}-1}{2^{2r_1+1}}\right] = \frac{(b-a)^{n+1}}{(n+1)!}\left[\frac{2^n-1}{2^n}\right].\end{aligned}$$

Similarly,

$$\int_c^d |T_m(c,d;s)|\,ds = \frac{(d-c)^{m+1}}{(m+1)!}\left[\frac{2^m-1}{2^m}\right].$$

and this gives the first inequality in (6.63).

Now, for the third inequality we have,

$$\sup_{t\in[a,b]} |T_n(a,b;t)| = \frac{1}{2n!} \times \begin{cases} \sup\limits_{t\in[a,b]} \left((b-t)^n + (t-a)^n\right) = \frac{(b-a)^n}{2n!} & \text{for all } n \text{ } even \\ \\ \sup\limits_{t\in[a,b]} \left|(b-t)^n - (t-a)^n\right| = \frac{(b-a)^n}{2n!} & \text{for all } n \text{ } odd \end{cases}$$

and this gives last part of the inequality in (6.63). The corollary is thus completely proved. ∎

Remark 6.14. For $n = m = 1$, we have that

$$\left|\int_a^b \int_c^d f(t,s)\,ds\,dt + \frac{(b-a)(d-c)}{4}\left[f(a,c) + f(a,d) + f(b,c) + f(b,d)\right]\right.$$
$$\left. -\frac{b-a}{2}\left[\int_c^d \left(f(a,s) + f(b,s)\right) ds\right] - \frac{d-c}{2}\left[\int_a^b \left(f(t,c) + f(t,d)\right) dt\right]\right|$$

$$\leq \begin{cases} \frac{(b-a)^2(d-c)^2}{4}\left[(x-a)^2 + (b-x)^2\right]\left[(y-c)^2 + (d-y)^2\right] \times \left\|\frac{\partial^2 f}{\partial t \partial s}\right\|_\infty \\ \\ \frac{1}{4}\left[\frac{((b-a)(d-c))^{q+1}}{(q+1)^2}\right]^{\frac{1}{q}} \times \left\|\frac{\partial^2 f}{\partial t \partial s}\right\|_p, \; p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \frac{(b-a)(d-c)}{4}\left\|\frac{\partial^2 f}{\partial t \partial s}\right\|_1. \end{cases}$$

Again, the same result was obtained by G. Hanna et al. in [**15**] and S. Dragomir et al. in [**16**].

6.11. Applications to Numerical Integration

The following application in Numerical Integration is natural to be considered.

Theorem 6.22. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be as in Theorem 6.19. In addition, let I_v and J_μ be arbitrary divisions of $[a,b]$ and $[c,d]$ respectively, that is,*

$$I_v : a = \xi_0 < \xi_1 < ... < \xi_\nu = b,$$

where $x_i \in \left(\xi_i, \xi_{i+1}\right)$ for $i = 0, 1, ..., \nu - 1,$ and

$$J_\mu : c = \tau_0 < \tau_1 < ... < \tau_\mu = d,$$

with $y_j \in (\tau_j, \tau_{j+1})$ *for* $j = 0, 1, ..., \mu - 1$, *then we have the cubature formula*

$$\int_a^b \int_c^d f(t,s)\,ds\,dt = \sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\sum_{i=0}^{\nu-1}\sum_{j=0}^{\mu-1} X_k^{(i)}(x_i) Y_l^{(j)}(y_j) \frac{\partial^{i+j} f(x_i,y_j)}{\partial x^i \partial y^j} \tag{6.64}$$
$$+(-1)^m \sum_{k=0}^{n-1}\sum_{i=0}^{\nu-1}\sum_{j=0}^{\mu-1} X_k^{(i)}(x_i) \int_{\tau_j}^{\tau_{j+1}} S_m^{(j)}(y_j,s) \frac{\partial^{k+m} f(x_i,s)}{\partial x^k \partial s^m} ds$$
$$+(-1)^n \sum_{l=0}^{m-1}\sum_{i=0}^{\nu-1}\sum_{j=0}^{\mu-1} Y_l^{(j)}(y_j) \int_{\xi_i}^{\xi_{i+1}} K_n^{(i)}(x_i,t) \frac{\partial^{n+l} f(t,y_j)}{\partial t^n \partial y^l} dt$$
$$+R(f, I_v, J_\mu, x, y),$$

where the remainder term satisfies the condition

$$|R(f, I_n, J_m, x, y)|$$

$$\leq \begin{cases} \dfrac{\left\|\frac{\partial^{n+m} f}{\partial t^n \partial s^m}\right\|_\infty}{(n+1)!(m+1)!} \times \sum\limits_{i=0}^{\nu-1}\left[(x_i-\xi_i)^{n+1} + (\xi_{i+1}-x_i)^{n+1}\right] \\ \qquad \times \sum\limits_{j=0}^{\mu-1}\left[(y_j-\tau_j)^{m+1} + (\tau_{j+1}-y_j)^{m+1}\right] \quad \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_\infty([a,b]\times[c,d]); \\ \dfrac{\left\|\frac{\partial^{n+m} f}{\partial t^n \partial s^m}\right\|_p}{n!m!(nq+1)^{\frac{2}{q}}} \times \sum\limits_{i=0}^{\nu-1}\left[(x_i-\xi_i)^{nq+1} + (\xi_{i+1}-x_i)^{nq+1}\right]^{\frac{1}{q}} \\ \qquad \times \sum\limits_{j=0}^{\mu-1}\left[(y_j-\tau_j)^{mq+1} + (\tau_{j+1}-y_j)^{mq+1}\right]^{\frac{1}{q}} \\ \qquad\qquad \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_p([a,b]\times[c,d]),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{\left\|\frac{\partial^{n+m} f}{\partial t^n \partial s^m}\right\|_1}{4n!m!} \sum\limits_{i=0}^{\nu-1}\left[(x_i-\xi_i)^n + (\xi_{i+1}-x_i)^n + \left|(x_i-\xi_i)^n - (\xi_{i+1}-x_i)^n\right|\right] \\ \qquad \times \sum\limits_{j=0}^{\mu-1}\left[(y_j-\tau_j)^m + (\tau_{j+1}-y_j)^m + \left|(y_j-\tau_j)^m - (\tau_{j+1}-y_j)^m\right|\right] \\ \qquad\qquad \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_1([a,b]\times[c,d]); \end{cases}$$

where

$$X_k^{(i)}(k = 0, 1, ...n-1; i = 0, 1, ...\nu - 1),\ Y_l^{(j)}(l = 0, 1, ...m-1; j = 0, 1, ...\mu - 1)$$

and

$$K_n^{(i)}(i = 0, 1, ...\nu - 1),\ S_m^{(j)}(j = 0, 1, ...\mu - 1) \text{ are defined by}$$

$$X_k^{(i)}(x_i) := \frac{(\xi_{i+1}-x_i)^{k+1} + (-1)^k (x_i-\xi_i)^{k+1}}{(k+1)!},$$
$$Y_l^{(j)}(y_j) := \frac{(\tau_{j+1}-y_j)^{l+1} + (-1)^l (y_j-\tau_j)^{l+1}}{(l+1)!},$$
$$K_n^{(i)}(x_i,t) := \begin{cases} \frac{(t-\xi_i)^n}{n!} & ,t \in [\xi_i, x_i] \\ \frac{(t-\xi_{i+1})^n}{n!} & ,t \in (x_i, \xi_{i+1}] \end{cases}$$

and

$$S_m^{(j)}(y_j,s):=\begin{cases}\frac{(s-\tau_j)^m}{m!} & ,s\in[\tau_i,y_i]\\ \\ \frac{(s-\tau_{j+1})^m}{m!} & ,s\in(y_i,\tau_{j+1}]\end{cases}$$

The proof is obvious by Theorem 6.19 applied on the interval $\left[\xi_i,\xi_{i+1}\right]\times[\tau_j,\tau_{j+1}]$,

$(i=0,1,...\nu-1;j=0,1,...\mu-1)$, and we omit the details.

REMARK 6.15. Similar result can be obtained if we use the other results obtained in Section 6.3, but we omit the details.

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CHAPTER 7

Product Inequalities and Weighted Quadrature

by

J. ROUMELIOTIS

ABSTRACT Weighted (or product) integral inequalities are developed via Ostrowski and Grüss approaches. The inequalities provide an error estimate for weighted integrals where both the quadrature rule and error bound are given in terms of (at most) the first three moments of the weight. Rule type is distinguished and interior point, boundary point and three point rules are explored. Results for the most popular weight functions are tabulated.
Numerical experiments are provided and comparisons with other product rules of similar order are made. The methods outlined in this chapter allow for the generation of non-uniform quadrature grids with respect to any arbitrary weight employing only a small number of weight moments.

7.1. Introduction

The Ostrowski inequality [**32**] is a very fruitful starting point for the development of numerical integration rules. ¿From this point of view, one of the more important aspects of the inequality (and of Montgomery's identity) is its role in the production of bounds for the well known Newton-Cotes rules. To illustrate this point, consider the integral

$$I = \int_a^b f(x)\,dx, \tag{7.1}$$

where f is some bounded function defined on the finite interval $[a, b]$. If $b - a$ is small, we may approximate I by sampling at one point

$$I^*(x) = (b-a)f(x), \quad \text{for some} \quad a \le x \le b. \tag{7.2}$$

Using Ostrowski-type results, the error $|I - I^*(x)|$ has been tabulated for a variety of properties of f. For example, if f' exists and is bounded then we can show [**32**] that

$$|I - I^*(x)| \le (b-a)^2 \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) \|f'\|_\infty. \tag{7.3}$$

Equation (7.3) is the well known Ostrowski inequality. If f' is integrable [**20, 23**] or if f has a bounded variation [**15**] or is L-Lipschitzian [**17, 36**] then we have respectively,

$$|I - I^*(x)| \leq \begin{cases} (b-a)\left(\dfrac{1}{2} + \dfrac{\left|x - \frac{a+b}{2}\right|}{b-a}\right)\|f'\|_1, \\ (b-a)\left(\dfrac{1}{2} + \dfrac{\left|x - \frac{a+b}{2}\right|}{b-a}\right)\bigvee_a^b(f), \\ (b-a)^2\left(\dfrac{1}{4} + \dfrac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right)L. \end{cases} \tag{7.4}$$

Other generalizations to both the rule (7.2) and the bound exploiting other properties of f (for example smoothness, monotonicity and convexity) have been reported by Cerone, Dragomir, Fink, Milovanović, Pečarić and others; see [**1, 9, 10, 11, 8, 13, 16, 21, 18, 28, 29, 30, 34, 35**] and the book [**31**].

¿From a quadrature view point, the techniques described above provide explicit *a priori* error bounds in a variety of norms, but they fail to account for singular integrands and/or infinite regions. Integrals of this type are of particular importance since they arise naturally in the context of statistical estimations (for e.g. [**4, 3**]), integral equations (for e.g. [**6**], [**38**, Chapter 3]), especially as they impact mathematical models (for e.g. [**5, 40, 38**]).

In this chapter we develop weighted (or product) Ostrowski and Grüss type inequalities where the upper bound is a function of the first few derivatives of the mapping. We investigate sampling at interior points (mid-point type), boundary points (trapezoidal type) and a combination of both (three point type). The rules thus furnished provide explicit *a priori* bounds for an arbitrary mesh arrangement. The bounds can be used to produce an *optimal* mesh with the respect to an arbitrary weight as well as the construction of Kronrod type rules. This approach contrasts to that commonly used for mesh refinement, where successive *a posteriori* comparisons are made to obtain a desired accuracy.

7.2. Weight Functions

In the following sections weighted integral inequalities are developed. We assume that the weight w is integrable and non-negative. The domain may be finite or infinite and w may vanish at the boundary points. The results will be expressed in terms of the first few moments of w.

Definition 2. Let $w : (a,b) \to [0,\infty)$ be integrable, i.e. $\int_a^b w(t)\,dt < \infty$. We denote the first three moments to be m, M and N, where

(7.5)

$$m(a,b) = \int_a^b w(t)\,dt, \quad M(a,b) = \int_a^b tw(t)\,dt \quad \text{and} \quad N(a,b) = \int_a^b t^2 w(t)\,dt,$$

respectively.

We also introduce the following generic measures of w.

Definition 3. Given the conditions in Definition 2, the *mean* and *variance* on the sub-interval $(\alpha,\beta) \subseteq (a,b)$ are defined as

$$\mu(\alpha,\beta) = \frac{M(\alpha,\beta)}{m(\alpha,\beta)} \tag{7.6}$$

and

$$\sigma^2(\alpha,\beta) = \frac{N(\alpha,\beta)}{m(\alpha,\beta)} - \mu^2(\alpha,\beta), \tag{7.7}$$

respectively.

7.3. Weighted Interior Point Integral Inequalities

Mitrinović *et al.* [**31**] have reported a weighted multi-dimensional analogue of the Ostrowski inequality in the first partial derivatives of the mapping. The analysis in this chapter will be restricted to one dimension and we begin with the result in [**31**].

Theorem 7.1 ([**31**]). *Let w be as defined in Definition 2 and let $f : [a,b] \to \mathbb{R}$ be absolutely continuous and have bounded first derivative, then*

(7.8)

$$\begin{aligned}\left|\int_a^b w(t)f(t)\,dt - m(a,b)f(x)\right| &\le \|f'\|_\infty \int_a^b |x-t|w(t)\,dt \\ &= \|f'\|_\infty \left\{x\big(m(a,x)-m(x,b)\big) + M(x,b) - M(a,x)\right\}.\end{aligned}$$

Proof. Define the mapping $K(\cdot,\cdot) : [a,b]^2 \to \mathbb{R}$ by

$$K(x,t) = \begin{cases} m(a,t), & t \in [a,x], \\ m(b,t), & t \in (x,b], \end{cases} \tag{7.9}$$

where m is the zero-th moment as defined in (7.5). Integration by parts gives

$$\begin{aligned}\int_a^b K(x,t)f'(t)\,dt &= \int_a^x m(a,t)f'(t)\,dt + \int_x^b m(b,t)f'(t)\,dt \\ &= m(a,t)f(t)]_{t=a}^{x} + m(b,t)f(t)]_{t=x}^{b} - \int_a^b w(t)f(t)\,dt.\end{aligned}$$

Producing the *product Montgomery* identity

$$\int_a^b K(x,t)f'(t)\,dt = m(a,b)f(x) - \int_a^b w(t)f(t)\,dt. \tag{7.10}$$

Taking the modulus and using Hölder's inequality gives

$$\begin{aligned}\left|\int_a^b K(x,t)f'(t)\,dt\right| &\le \|f'\|_\infty \int_a^b |K(x,t)|\,dt \\ &= \|f'\|_\infty \left\{\int_a^x m(a,t)\,dt + \int_x^b m(t,b)\,dt\right\}. \qquad (7.11)\end{aligned}$$

The last result being obtained by using the fact that for fixed x, K is positive in $t \in (a, x)$ and negative in $t \in (x, b)$. Making use of (7.10), reversing the order of integration in (7.11) and evaluating the inner integrals produces the desired result (7.8). ∎

REMARK 7.1. Substituting $w = 1$ into (7.8) returns the Ostrowski inequality (7.3).

REMARK 7.2. Unlike the Ostrowski-type results outlined in Section 7.1, Theorem 7.1 is valid (depending on w) for infinite regions and singular w. For example, substituting specific weights into (7.8) produces the following weighted inequalities

$$\left| \int_0^1 f(t) \ln(1/t)\, dt - f(x) \right| \leq \|f'\|_\infty \left(x^2 \ln(1/x) + 3/2x^2 - x + 1/4 \right), \qquad x \in (0, 1],$$

$$\left| \int_0^1 \frac{f(t)}{\sqrt{t}}\, dt - 2f(x) \right| \leq \|f'\|_\infty \left(8/3x^{3/2} - 2x + 2/3 \right), \qquad x \in [0, 1],$$

$$\left| \int_0^\infty f(t) e^{-t}\, dt - f(x) \right| \leq \|f'\|_\infty \left(2e^{-x} + x - 1 \right), \qquad x \geq 0,$$

$$\left| \int_{-\infty}^\infty f(t) e^{-t^2}\, dt - \sqrt{\pi} f(x) \right| \leq \|f'\|_\infty \left(\sqrt{\pi} x \,\mathrm{erf}(x) + e^{-x^2} \right), \qquad x \in \mathbb{R}.$$

COROLLARY 7.2. The bound in (7.8) is minimized at the *median*, $x = x^*$ where

$$m(a, x^*) = m(x^*, b). \tag{7.12}$$

Thus, the following *median point* inequality holds

$$\left| \int_a^b w(t) f(t)\, dt - m(a, b) f(x^*) \right| \leq \|f'\|_\infty \big\{ M(x^*, b) - M(a, x^*) \big\}. \tag{7.13}$$

PROOF. Let F represent the bound in (7.8). That is

$$\begin{aligned} F(x) &= \int_a^b |x - t| w(t)\, dt \\ &= \int_a^x (x - t) w(t)\, dt + \int_x^b (t - x) w(t)\, dt. \end{aligned}$$

Differentiating F twice gives

$$\begin{aligned} &F'(x) = \int_a^x w(t)\, dt - \int_x^b w(t)\, dt = m(a, x) - m(x, b) \quad \text{and} \\ &F''(x) = 2w(x) \geq 0, \qquad \forall x \in (a, b). \end{aligned}$$

Inspection of the second derivative reveals that the bound is convex and hence the minimum will occur at the stationary point. From F' above, we can see that the minimum will occur at the median as in equation (7.12). ∎

Equation (7.12) can be used to find the *optimal* sampling point for the weighted inequalities in Remark 7.2. The bounds in Theorem 7.1 and Corollary 7.2 both require the second moment, M. In the next two results, we present bounds that do not rely on M but, as a result, are coarser than (7.8) and only apply for finite intervals.

Corollary 7.3. Let the conditions in Theorem 7.1 hold and let $x \in [a,b]$, where $[a,b]$ is a finite interval. The following product integral inequality holds.

$$\left| \int_a^b w(t)f(t)\,dt - m(a,b)f(x) \right| \le \|f'\|_\infty m(a,b) \left(\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right). \tag{7.14}$$

Proof. Observe that

$$\begin{aligned} \int_a^b |x-t| w(t)\,dt &\le \sup_{t\in(a,b)} |x-t| m(a,b) \\ &= \max\{x-a, b-x\} m(a,b) \\ &= \frac{m(a,b)}{2} \left((x-a) + (b-x) - |(x-a)-(b-x)| \right) \\ &= m(a,b) \left(\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right). \end{aligned}$$

Substituting the inequality above into equation (7.8) furnishes the desired result (7.14). ∎

Another estimation in terms of the $\|\cdot\|_p$ norm of w is given in the following corollary.

Corollary 7.4. Under the above assumptions for f and w and $w \in L_p[a,b]$, we have the inequality

$$\left| \int_a^b w(t)f(t)\,dt - m(a,b)f(x) \right| \le \|f'\|_\infty \|w\|_p \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{1/q}, \tag{7.15}$$

for all $x \in [a,b]$, $p > 1$ and $1/p + 1/q = 1$. The bound is minimized at the mid-point $x = (a+b)/2$.

Proof. Using Hölder's inequality we have

$$\begin{aligned} \int_a^b |x-t| w(t)\,dt &\le \|w\|_p \left(\int_a^b |x-t|^q\,dt \right)^{1/q} \\ &= \|w\|_p \left[\int_a^x (x-t)^q\,dt + \int_x^b (t-x)^q\,dt \right]^{1/q} \\ &= \|w\|_p \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{1/q}. \end{aligned}$$

To show that the bound is minimized at the mid-point, observe that $(x-a)^{q+1}$ vanishes at $x = a$ and monotonically increases while $(b-x)^{q+1}$ vanishes at $x = b$ and monotonically decreases. Thus the sum is convex and has minimum at $(x-a)^{q+1} = (b-x)^{q+1}$ or $x = (a+b)/2$. ∎

The estimation in (7.8) may be bounded using other properties of the mapping f. For example if f' is integrable we have the following theorem.

THEOREM 7.5. *Let w be as given in Definition 7.5 and let $f : [a, b] \to \mathbb{R}$ be such that $f' \in L_1(a, b)$. The following inequality holds*

$$\left| \int_a^b w(t)f(t)\,dt - m(a,b)f(x) \right| \le \frac{1}{2}\|f'\|_1 \Big\{ m(a,b) + \big| m(a,x) - m(x,b) \big| \Big\}. \tag{7.16}$$

The bound is minimized at the median.

PROOF. ¿From (7.10), we can see that

$$\begin{aligned} \left| \int_a^b w(t)f(t)\,dt - m(a,b)f(x) \right| &= \left| \int_a^b K(x,t)f'(t), dt \right| \\ &\le \sup_{t\in(a,b)} |K(x,t)| \int_a^b |f'(t)|\;dt \\ &= \max\{m(a,x), m(x,b)\}\|f'\|_1 \\ &= \frac{1}{2}\|f'\|_1 \Big\{ m(a,b) + \big| m(a,x) - m(x,b) \big| \Big\}. \end{aligned}$$

The last line being obtained from the well known result: $\max\{A, B\} = 1/2(A + B + |A - B|)$. ∎

REMARK 7.3. Substituting $w = 1$ into (7.16) returns $(7.4)_1$, the L_1 variant of the Ostrowski inequality [**20, 23**]. Recently, Peachey *et al.* [**33**] were able to show that the constant $1/2$ is the best possible in equation $(7.4)_1$. It is still an open question whether $m(a, b)$ is the best possible constant in (7.16).

Dragomir *et al.* [**19**] generalized (7.8) and developed a weighted Ostrowski-type inequality for Hölder type mappings.

THEOREM 7.6 ([**19**]). *Let w be as given in Definition 2 and let f be of $r-H-$Hölder type. That is*

$$|f(x) - f(y)| \le H|x - y|^r, \tag{7.17}$$

for all $x, y \in (a, b)$, $H > 0$ and $r \in (0, 1]$. If wf is integrable, then the following estimation of the product integral holds

$$\left| \int_a^b w(t)f(t)\,dt - m(a,b)f(x) \right| \le H \int_a^b |x-t|^r w(t)\,dt, \tag{7.18}$$

for all $a < x < b$.

PROOF. ¿From equation (7.17) it is easy to see that

$$\int_a^b w(t)|f(t) - f(x)|\,dt \le H \int_a^b |x-t|^r w(t)\,dt. \tag{7.19}$$

In addition, using the usual properties of definite integrals we can show that

$$\left| \int_a^b w(t)\big(f(t) - f(x)\big)\,dt \right| \le \int_a^b w(t)|f(t) - f(x)|\,dt. \tag{7.20}$$

Thus, combining (7.19) and (7.20) we obtain (7.18) as required. ∎

Remark 7.4. If in (7.17) $r = 1$, then f is Lipschitzian. If the Lipschitz constant is L, say, then we have

(7.21)
$$\begin{aligned}\left|\int_a^b w(t)f(t)\,dt - m(a,b)f(x)\right| &\le L\int_a^b |x-t|w(t)\,dt\\ &= L\left\{x\big(m(a,x)-m(x,b)\big)+M(x,b)-M(a,x)\right\}.\end{aligned}$$

Up to this point, we have only assumed very general properties regarding the mapping f and its derivative f'. For the remainder of this section, we will expand on this somewhat and present analogous product integral inequalities where f'' is assumed to exist. An application in numerical integration in also included.

Generalizations to higher derivatives have been made for non-product inequalities in [**23, 1, 34, 11**]. It should be possible to generalize product integral inequalities to higher derivatives of f if these techniques are combined with the material presented below.

Theorem 7.7 ([**39**]). *Let w be as given in Definition 2 and let $f : (a,b) \to \mathbb{R}$ have bounded second derivative. Then the following inequalities hold*

$$(7.22)\quad \left|\int_a^b w(t)f(t)\,dt - m(a,b)f(x) + m(a,b)\big(x-\mu(a,b)\big)f'(x)\right|$$
$$\le \begin{cases} \|f''\|_\infty \dfrac{m(a,b)}{2}\left[\big(x-\mu(a,b)\big)^2+\sigma^2(a,b)\right], & f''\in L_\infty[a,b]\\ \|f''\|_1 \dfrac{m(a,b)}{2}\left[\dfrac{m(a,x)}{m(a,b)}\big(x-\mu(a,x)\big)+\dfrac{m(x,b)}{m(a,b)}\big(\mu(x,b)-x\big)+|x-\mu(a,b)|\right], & f''\in L_1[a,b]\end{cases}$$

for all $x \in [a,b]$.

Proof. Define the mapping $K(\cdot,\cdot) : [a,b]^2 \to \mathbb{R}$ by

$$K(x,t) := \begin{cases}\int_a^t (t-u)w(u)\,du, & a\le t\le x,\\ \int_b^t (t-u)w(u)\,du, & x< t\le b.\end{cases} \tag{7.23}$$

Integrating by parts gives

$$\begin{aligned}\int_a^b K(x,t)f''(t)\,dt &= \int_a^x\int_a^t (t-u)w(u)f''(t)\,dudt + \int_x^b\int_b^t (t-u)w(u)f''(t)\,dudt\\ &= f'(x)\int_a^b (x-u)w(u)\,du\\ &\quad - \int_a^x\int_a^t (t-u)w(u)f'(t)\,dudt - \int_x^b\int_b^t (t-u)w(u)f'(t)\,dudt\\ &= \int_a^b w(t)f(t)\,dt + f'(x)\int_a^b (x-u)w(u)\,du - f(x)\int_a^b w(u)\,du\end{aligned}$$

providing the identity

$$\int_a^b K(x,t)f''(t)\,dt = \int_a^b w(t)f(t)\,dt - m(a,b)f(x) + m(a,b)\big(x-\mu(a,b)\big)f'(x) \tag{7.24}$$

that is valid for all $x \in [a,b]$.

Now taking the modulus of (7.24) we have,

$$
\begin{aligned}
(7.25)\quad & \left| \int_a^b w(t)f(t)\,dt - m(a,b)f(x) + m(a,b)\big(x-\mu(a,b)\big)f'(x) \right| \\
&= \left| \int_a^b K(x,t)f''(t)\,dt \right| \\
&\le \|f''\|_\infty \int_a^b |K(x,t)|\,dt \\
&= \|f''\|_\infty \left[\int_a^x \int_a^t (t-u)w(u)\,dudt + \int_x^b \int_b^t (t-u)w(u)\,dudt \right] \\
&= \frac{\|f''\|_\infty}{2} \int_a^b (x-t)^2 w(t)\,dt.
\end{aligned}
$$

The last line being computed by reversing the order of integration and evaluating the inner integrals. To obtain the desired result $(7.22)_1$ observe that

$$
(7.26)\qquad \int_a^b (x-t)^2 w(t)\,dt = m(a,b)\Big[\big(x-\mu(a,b)\big)^2 + \sigma^2(a,b)\Big].
$$

Closer inspection of the kernel (7.23) is required in the proof of $(7.22)_2$. Examination of the derivative

$$
\frac{d}{dt}K(x,t) = \begin{cases} \int_a^t w(u)\,du, & t \in [a,x], \\ \int_b^t w(u)\,du, & t \in (x,b], \end{cases}
$$

reveals that K increases in the interval $[a,x]$ and decreases in the interval $(x,b]$. Since K is positive, we can deduce that the maximum is attained at $t=x$. Thus from (7.24)

$$
\begin{aligned}
\left| \int_a^b K(x,t)f''(t)\,dt \right| &\le \sup_{t\in(a,b)} K(x,t)\|f''\|_1 \\
&= \max\{\int_a^x (x-t)w(t)\,dt, \int_x^b (t-x)w(t)\,dt\}\|f''\|_1.
\end{aligned}
$$

Simplifying the above expression provides the required result $(7.22)_2$. ∎

Note also that (7.22) is valid even for unbounded w or interval $[a,b]$.

The inequality $(7.22)_1$ is bounded in terms of three moments of w. The following provides a coarser upper bound using only the zero-th moment.

Corollary 7.8. Let the conditions in Theorem 7.7 hold. The following product integral inequality holds

$$\left|\int_a^b w(t)f(t)\,dt - m(a,b)f(x) + m(a,b)\big(x-\mu(a,b)\big)f'(x)\right| \le \|f''\|_\infty \frac{m(a,b)}{2}\left(\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}\right)^2. \tag{7.27}$$

Proof. To obtain (7.27) note that

$$\begin{aligned}\int_a^b (x-t)^2 w(t)\,dt &\le \sup_{t\in[a,b]}(x-t)^2 m(a,b)\\ &= \max\{(x-a)^2,(x-b)^2\}m(a,b)\\ &= \frac{1}{2}\left((x-a)^2+(x-b)^2+\left|(x-a)^2-(x-b)^2\right|\right)m(a,b)\\ &= \left(\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}\right)^2 m(a,b) \qquad (7.28)\end{aligned}$$

which upon substitution into (7.25) furnishes the result. ∎

Corollary 7.9. The following inequality for a density defined on a finite interval holds. Let w be a density (with not necessarily a unit area) and let μ and σ^2 be the mean and variance respectively. Then

$$\big(x-\mu(a,b)\big)^2+\sigma^2(a,b) \le \left(\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}\right)^2 \tag{7.29}$$

for all $x\in[a,b]$.

Proof. The result is immediately obvious when the identity (7.26) is substituted into (7.28). ∎

Remark 7.5. Substituting $x=\mu(a,b)$ and $x=\frac{a+b}{2}$ into (7.29) and adding produces the following inequality for the variance in terms of the mean of a density

$$\sigma^2(a,b) \le \frac{b-a}{2}\left(\frac{b-a}{2}+\left|\mu(a,b)-\frac{a+b}{2}\right|\right). \tag{7.30}$$

A tighter bound is obtained by using (7.29) only once. Substituting $x=\frac{a+b}{2}$ gives

$$\begin{aligned}\sigma^2(a,b) &\le \frac{(b-a)^2}{4}-\left(\frac{a+b}{2}-\mu\right)^2\\ &= (\mu-a)(b-\mu) \le \frac{(b-a)^2}{4} \qquad (7.31)\end{aligned}$$

Corollary 7.10. The inequality $(7.22)_1$ is minimized at $x=\mu(a,b)$ producing the "mean-point" inequality

$$\left|\int_a^b w(t)f(t)\,dt - m(a,b)f(\mu(a,b))\right| \le \|f''\|_\infty \frac{\sigma^2(a,b)m(a,b)}{2}. \tag{7.32}$$

PROOF. Substituting $\mu(a,b)$ for x in $(7.22)_1$ produces the desired result. Note that $x = \mu(a,b)$ not only minimizes the bound of the inequality $(7.22)_1$, but also causes the derivative term to vanish. ∎

The optimal point $\mu(a,b)$ can be interpreted in many ways. For example, this point can be viewed as that which minimizes the error variance for the probability density w (see [**4**] for an application). This point is also the Gauss node point for a one-point rule [**43**].

7.3.1. Two Interior Points.

Here a two point analogy of $(7.22)_1$ is developed where the result is extended to create an inequality with two independent parameters x_1 and x_2. This is mainly used in subsection 7.3.3 to find an optimal grid for composite weighted-quadrature rules.

THEOREM 7.11. *Let the conditions of Theorem 7.7 hold, then the following two interior point inequality is obtained*

$$\begin{aligned}(7.33)\quad \Bigg| \int_a^b w(t)f(t)\,dt - m_0(a,\xi)f(x_1) + m_0(a,\xi)\big(x_1-\mu(a,\xi)\big)f'(x_1) \\ - m_0(\xi,b)f(x_2) + m_0(\xi,b)\big(x_2-\mu(\xi,b)\big)f'(x_2)\Bigg| \\ \le \frac{\|f''\|_\infty}{2}\Bigg\{ m_0(a,\xi)\Big[\big(x_1-\mu(a,\xi)\big)^2 + \sigma^2(a,\xi)\Big] \\ + m_0(\xi,b)\Big[\big(x_2-\mu(\xi,b)\big)^2 + \sigma^2(\xi,b)\Big]\Bigg\}\end{aligned}$$

for all $a \le x_1 < \xi < x_2 \le b$.

PROOF. Define the mapping $K(\cdot,\cdot,\cdot,\cdot) : [a,b]^4 \to \mathbb{R}$ by

$$K(x_1,x_2,\xi,t) := \begin{cases} \int_a^t (t-u)w(u)\,du, & a \le t \le x_1, \\ \int_\xi^t (t-u)w(u)\,du, & x_1 < t, \xi < x_2, \\ \int_b^t (t-u)w(u)\,du, & x_2 \le t \le b. \end{cases}$$

With this kernel, the proof is almost identical to that of Theorem 7.7.

Integrating by parts produces the integral identity

$$\begin{aligned}(7.34)\quad & \int_a^b K(x_1,x_2,\xi,t)f''(t)\,dt \\ &= \int_a^b w(t)f(t)\,dt - m_0(a,\xi)f(x_1) + m_0(a,b)\big(x-\mu(a,\xi)\big)f'(x_1) \\ &\qquad - m_0(\xi,b)f(x_2) + m_0(\xi,b)\big(x-\mu(\xi,b)\big)f'(x_2).\end{aligned}$$

Re-arranging and taking bounds produces the result (7.33). ∎

COROLLARY 7.12. The optimal location of the points x_1, x_2 and ξ satisfy

$$(7.35)\qquad x_1 = \mu(a,\xi), \qquad x_2 = \mu(\xi,b), \qquad \xi = \frac{\mu(a,\xi)+\mu(\xi,b)}{2}.$$

Proof. By inspection of the right hand side of (7.33) it is obvious that choosing

$$x_1 = \mu(a,\xi) \qquad \text{and} \qquad x_2 = \mu(\xi,b) \tag{7.36}$$

minimizes this quantity. To find the optimal value for ξ write the expression in braces in (7.33) as

$$\begin{aligned}2\int_a^b |K(x_1,x_2,\xi,t)|\,dt &= m_0(a,\xi)\Big[\big(x_1-\mu(a,\xi)\big)^2+\sigma^2(a,\xi)\Big]\\ &\quad + m_0(\xi,b)\Big[\big(x_2-\mu(\xi,b)\big)^2+\sigma^2(\xi,b)\Big]\\ &= \int_a^\xi (x_1-t)^2 w(t)\,dt + \int_\xi^b (x_2-t)^2 w(t)\,dt.\end{aligned} \tag{7.37}$$

Substituting (7.36) into the right hand side of (7.37) and differentiating with respect to ξ gives

$$\frac{d}{d\xi}\int_a^b |K(\mu(a,\xi),\mu(\xi,b),\xi,t)|\,dt = \big(\mu(\xi,b)-\mu(\xi,a)\big)\left(\xi-\frac{\mu(a,\xi)+\mu(\xi,b)}{2}\right)w(\xi).$$

Assuming $w(\xi)\neq 0$, then this equation possesses only one root. A minimum exists at this root since (7.37) is convex, and so the corollary is proved. ∎

Equation (7.35) shows not only where sampling should occur within each subinterval (i.e. x_1 and x_2), but how the domain should be divided to make up these subintervals (ξ).

7.3.2. Some Weighted Integral Inequalities. Integration with weight functions are used in countless mathematical problems. Two main areas are: (i) approximation theory and spectral analysis and (ii) statistical analysis and the theory of distributions.

In this subsection equation $(7.22)_1$ is evaluated for the more popular weight functions. The optimal point is easily identified.

7.3.2.1. *Uniform (Legendre).* Substituting $w(t)=1$ into (7.6) and (7.7) gives

$$\mu(a,b) = \frac{\int_a^b t\,dt}{\int_a^b dt} = \frac{a+b}{2} \tag{7.38}$$

and

$$\sigma^2(a,b) = \frac{\int_a^b t^2\,dt}{\int_a^b dt} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$$

respectively. Substituting into $(7.22)_1$ produces the second derivative variant of the Ostrowski inequality [**9**]

$$\begin{aligned}&\left|\int_a^b f(t)\,dt - (b-a)f(x) + (b-a)\left(x-\frac{a+b}{2}\right)f'(x)\right|\\ &\qquad\qquad \le \|f''\|_\infty \frac{b-a}{2}\left(\frac{(b-a)^2}{12}+\left(x-\frac{a+b}{2}\right)^2\right).\end{aligned}$$

7.3.2.2. *Logarithm.* This weight is present in many physical problems; the main body of which exhibit some axial symmetry. Special logarithmic rules are used extensively in the Boundary Element Method popularized by Brebbia (see for example [**6**]).

With $w(t) = \ln(1/t)$, $a = 0$, $b = 1$, (7.6) and (7.7) are

$$\mu(0,1) = \frac{\int_0^1 t\ln(1/t)\,dt}{\int_0^1 \ln(1/t)\,dt} = \frac{1}{4}$$

and

$$\sigma^2(0,1) = \frac{\int_0^1 t^2\ln(1/t)\,dt}{\int_0^1 \ln(1/t)\,dt} - \left(\frac{1}{4}\right)^2 = \frac{7}{144}$$

respectively. Substituting into $(7.22)_1$ gives

$$\left|\int_0^1 \ln(1/t)f(t)\,dt - f(x) + \left(x - \frac{1}{4}\right)f'(x)\right| \le \frac{\|f''\|_\infty}{2}\left(\frac{7}{144} + \left(x - \frac{1}{4}\right)^2\right).$$

The optimal point

$$x = \mu(0,1) = \frac{1}{4}$$

is closer to the origin than the midpoint (7.38) reflecting the strength of the log singularity.

7.3.2.3. *Jacobi.* Substituting $w(t) = 1/\sqrt{t}$, $a = 0$, $b = 1$ into (7.6) and (7.7) gives

$$\mu(0,1) = \frac{\int_0^1 \sqrt{t}\,dt}{\int_0^1 1/\sqrt{t}\,dt} = \frac{1}{3}$$

and

$$\sigma^2(0,1) = \frac{\int_0^1 t\sqrt{t}\,dt}{\int_0^1 1/\sqrt{t}\,dt} - \left(\frac{1}{3}\right)^2 = \frac{4}{45}$$

respectively. Hence, the inequality for a Jacobi weight is

$$\left|\frac{1}{2}\int_0^1 \frac{f(t)}{\sqrt{t}}\,dt - f(x) + \left(x - \frac{1}{3}\right)f'(x)\right| \le \frac{\|f''\|_\infty}{2}\left(\frac{4}{45} + \left(x - \frac{1}{3}\right)^2\right).$$

The optimal point

$$x = \mu(0,1) = \frac{1}{3}$$

is again shifted to the left of the mid-point due to the $t^{-1/2}$ singularity at the origin.

7.3.2.4. *Chebyshev.* The mean and variance for the Chebyshev weight $w(t) = 1/\sqrt{1-t^2}$, $a=-1, b=1$ are

$$\mu(-1,1) = \frac{\int_{-1}^{1} t/\sqrt{1-t^2}\,dt}{\int_{-1}^{1} 1/\sqrt{1-t^2}\,dt} = 0$$

and

$$\sigma^2(-1,1) = \frac{\int_{-1}^{1} t^2\sqrt{1-t^2}\,dt}{\int_{-1}^{1} 1/\sqrt{1-t^2}\,dt} - 0^2 = \frac{1}{2}$$

respectively. Hence, the inequality corresponding to the Chebyshev weight is

$$\left|\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}\,dt - \pi f(x) + \pi x f'(x)\right| \le \|f''\|_\infty \frac{\pi}{2}\left(\frac{1}{2} + x^2\right).$$

The optimal point

$$x = \mu(-1,1) = 0$$

is at the mid-point of the interval reflecting the symmetry of the Chebyshev weight over its interval.

7.3.2.5. *Laguerre.* The conditions in Theorem 7.7 are not violated if the integral domain is infinite. The Laguerre weight $w(t) = e^{-t}$ is defined for positive values, $t \in [0,\infty)$. The mean and variance of the Laguerre weight are

$$\mu(0,\infty) = \frac{\int_{0}^{\infty} te^{-t}\,dt}{\int_{0}^{\infty} e^{-t}\,dt} = 1$$

and

$$\sigma^2(0,\infty) = \frac{\int_{0}^{\infty} t^2e^{-t}\,dt}{\int_{0}^{\infty} e^{-t}\,dt} - 1^2 = 1$$

respectively.

The appropriate inequality is

$$\left|\int_{0}^{\infty} e^{-t} f(t)\,dt - f(x) + (x-1)f'(x)\right| \le \frac{\|f''\|_\infty}{2}\left(1 + (x-1)^2\right),$$

from which the optimal sample point of $x = 1$ may be deduced.

7.3.2.6. *Hermite.* Finally, the Hermite weight is $w(t) = e^{-t^2}$ defined over the entire real line. The mean and variance for this weight are

$$\mu(-\infty,\infty) = \frac{\int_{-\infty}^{\infty} te^{-t^2}\,dt}{\int_{-\infty}^{\infty} e^{-t^2}\,dt} = 0$$

and

$$\sigma^2(-\infty,\infty) = \frac{\int_{-\infty}^{\infty} t^2e^{-t^2}\,dt}{\int_{-\infty}^{\infty} e^{-t^2}\,dt} - 0^2 = \frac{1}{2}$$

respectively.

The inequality from Theorem 7.7 with the Hermite weight function is thus

$$\left| \int_{-\infty}^{\infty} e^{-t^2} f(t)\, dt - \sqrt{\pi} f(x) + \sqrt{\pi} x f'(x) \right| \leq \|f''\|_\infty \frac{\sqrt{\pi}}{2} \left(\frac{1}{2} + x^2 \right),$$

which results in an optimal sampling point of $x = 0$.

7.3.3. Application in Numerical Integration. Define a grid $I_n : a = \xi_0 < \xi_1 < \cdots < \xi_{n-1} < \xi_n = b$ on the interval $[a, b]$, with $x_i \in [\xi_i, \xi_{i+1}]$ for $i = 0, 1, \ldots, n-1$. The following quadrature formulae for weighted integrals are obtained.

THEOREM 7.13. *Let the conditions in Theorem 7.7 hold. The following weighted quadrature rule holds*

$$\int_a^b w(t) f(t)\, dt = A(f, \boldsymbol{\xi}, \boldsymbol{x}) + R(f, \boldsymbol{\xi}, \boldsymbol{x}) \tag{7.39}$$

where

$$A(f, \boldsymbol{\xi}, \boldsymbol{x}) = \sum_{i=0}^{n-1} \big(h_i f(x_i) - h_i (x_i - \mu_i) f'(x_i) \big)$$

and

$$|R(f, \boldsymbol{\xi}, \boldsymbol{x})| \leq \frac{\|f''\|_\infty}{2} \sum_{i=0}^{n-1} \left[(x_i - \mu_i)^2 + \sigma_i^2 \right] h_i. \tag{7.40}$$

The parameters h_i, μ_i and σ_i^2 are given by

$$h_i = m_0(\xi_i, \xi_{i+1}), \qquad \mu_i = \mu(\xi_i, \xi_{i+1}), \qquad \textit{and} \qquad \sigma_i^2 = \sigma^2(\xi_i, \xi_{i+1})$$

respectively.

PROOF. Apply Theorem 7.7 over the interval $[\xi_i, \xi_{i+1}]$ with $x = x_i$ to obtain

$$\left| \int_{\xi_i}^{\xi_{i+1}} w(t) f(t)\, dt - h_i f(x_i) + h_i (x_i - \mu_i) f'(x_i) \right| \leq \frac{\|f''\|_\infty}{2} h_i \left((x_i - \mu_i)^2 + \sigma_i^2 \right).$$

Summing over i from 0 to $n-1$ and using the triangle inequality produces the desired result. ∎

COROLLARY 7.14. The optimal location of the points x_i, $i = 0, 1, 2, \ldots, n-1$, and grid distribution I_n satisfy

$$x_i = \mu_i, \qquad i = 0, 1, \ldots, n-1 \qquad \text{and} \tag{7.41}$$

$$\xi_i = \frac{\mu_{i-1} + \mu_i}{2}, \qquad i = 1, 2, \ldots, n-1, \tag{7.42}$$

producing the composite averaged mean-point rule for weighted integrals

$$\int_a^b w(t) f(t)\, dt = \sum_{i=0}^{n-1} h_i f(x_i) + R(f, \boldsymbol{\xi}, n) \tag{7.43}$$

where the remainder is bounded by

$$|R(f, \boldsymbol{\xi}, n)| \leq \frac{\|f''\|_\infty}{2} \sum_{i=0}^{n-1} h_i \sigma_i^2 \tag{7.44}$$

n	Error (1)	Error (2)	Error (3)	Error ratio (3)	Bound ratio (3)
4	1.97(0)	2.38(0)	2.48(0)	–	–
8	3.41(-1)	2.93(-1)	2.35(-1)	10.56	3.90
16	8.63(-2)	5.68(-2)	2.62(-2)	8.97	3.95
32	2.37(-2)	1.31(-2)	4.34(-3)	6.04	3.97
64	6.58(-3)	3.20 (-3)	9.34(-4)	4.65	3.99
128	1.82(-3)	7.94(-4)	2.23(-4)	4.18	3.99
256	4.98(-4)	1.98(-4)	5.51(-5)	4.05	4.00

TABLE 7.1. The error in evaluating (7.45) under different quadrature rules. The parameter n is the number of sample points.

PROOF. The proof follows that of Corollary 7.12 where it is observed that the minimum bound (7.40) will occur at $x_i = \mu_i$. Differentiating the right hand side of (7.40) gives

$$\frac{d}{d\xi_i} \sum_{j=0}^{n-1} \left[(x_j - \mu_j)^2 + \sigma_j^2 \right] h_j = 2w(\xi_i)(x_i - x_{i-1}) \left(\xi_i - \frac{x_{i-1} + x_i}{2} \right).$$

Inspection of the second derivative at the root reveals that the stationary point is a minimum and hence the result is proved. ∎

7.3.4. Numerical Results. In this section, for illustration, the quadrature rule of subsection 7.3.3 is used on the integral

$$\int_0^1 100t \ln(1/t) \cos(4\pi t)\, dt = -1.972189325199166 \tag{7.45}$$

Equation (7.45) is evaluated using the following three rules:

(1) the composite mid-point rule, where the grid has a uniform step-size and the node is simply the mid-point of each sub-interval,
(2) the composite generalized mid-point rule (7.39). The grid, I_n, is uniform and the nodes are the mean point of each sub-interval (7.41),
(3) equation (7.43) where the grid is distributed according to (7.42) and the nodes are the sub-interval means (7.41).

Table 7.1 shows the numerical error of each method for an increasing number of sample points.

For a uniform grid, it can be seen that changing the location of the sampling point from the midpoint [method (1)] to the mean point [method (2)] roughly doubles the accuracy. Changing the grid distribution as well as the node point [method (3)] from the composite mid-point rule [method (1)] increases the accuracy by approximately an order of magnitude. It is important to note that the nodes and weights for method (3) can be easily calculated numerically using an iterative scheme. For example on a Pentium-III (550 MHz) personal computer, with $n = 64$, calculating (7.41) and (7.42) took close to 4 seconds.

Note that equations (7.41) and (7.42) are quite general in nature and only rely on the weight insofar as knowledge of the first two moments is required. This contrasts with Gaussian quadrature where for an n point rule, the first $2n$ moments are needed (or equivalently the $2n+1$ coefficients of the continued fraction expansion [**41, 42**]) to construct the appropriate orthogonal polynomial and then a root-finding procedure is called to find the abscissae [**2**]. This procedure, of course, can be greatly simplified for the more well known weight functions [**24**].

The second last column of Table 7.1 shows the ratio of the numerical errors for method (3) and the last column the ratio of the theoretical error bound (7.43)

$$\text{Bound ratio (3)} = \frac{|R(f, \boldsymbol{\xi}, n/2)|}{|R(f, \boldsymbol{\xi}, n)|}. \tag{7.46}$$

As n increases the numerical ratio approaches the theoretical one. The theoretical ratio is consistently close to 4. This value suggests an asymptotic form of the error bound

$$|R(f, \boldsymbol{\xi}, n)| \sim O\left(\frac{1}{n^2}\right) \tag{7.47}$$

for the log weight. Similar results have been obtained for the other weights of subsection 7.3.2. This is consistent with mid-point type rules and it is anticipated that developing other product rules, for example a generalized trapezoidal or Simpson's rule, will yield more accurate results. This is pursued in the next section where a generalized trapezoid quadrature rule is developed.

7.4. Weighted Boundary Point (Lobatto) Integral Inequalities

In the previous section product integral inequalities and weighted quadrature rules were developed where sampling occurred at interior points. In this section we develop analogous results where the mapping f is sampled at the boundary points.

The inequalities presented here, which are valid for mappings whose second derivative exist, will be used to develop product trapezoidal-like quadrature rules.

In addition to quadrature, these integral inequalities have been applied to the numerical analysis of first kind integral equations with symmetric kernels [**38**, Chapter 3], [**37**].

We begin by defining a weighted trapezoidal-type Peano kernel.

LEMMA 7.15. Let $x \in [a, b]$ be fixed and w be as given in Definition 2, then the Peano-type kernel, $K(\cdot, \cdot)$, given by

$$K(x, t) = \int_x^t (t-u) w(u)\, du = \begin{cases} m(t, x)\big(\mu(t, x) - t\big), & t \le x \\ m(x, t)\big(t - \mu(x, t)\big), & t > x \end{cases} \tag{7.48}$$

is convex and non-negative for $a \le x \le b$.

PROOF. Differentiation of (7.48) gives

$$\frac{d}{dt}K(x,t) = \int_x^t w(u)\,du \quad \begin{cases} < 0, & \text{if } a \le t < x, \\ = 0, & \text{if } t = x, \\ > 0, & \text{if } x < t \le b. \end{cases} \tag{7.49}$$

Equation (7.49) immediately reveals the convexity of K. In addition, note that K vanishes at its minimum $t = x$, and this property, with (7.49) suffices to prove that K is non-negative. ∎

THEOREM 7.16. *Let f be an absolutely continuous mapping defined on the finite interval $[a,b]$ whose second derivative exists and let w be a positive weight function as given in Definition 2. Then, for fixed $x \in [a,b]$ the following product-trapezoidal like inequalities hold*

(7.50)
$$\left|\int_a^b w(t)f(t)\,dt - m(a,x)\big(f(a)+(\mu(a,x)-a)f'(a)\big) - m(x,b)\big(f(b)+(\mu(x,b)-b)f'(b)\big)\right|$$
$$\le \begin{cases} \dfrac{\|f''\|_\infty}{2}\Big\{m(a,x)\,\big[(\mu(a,x)-a)^2+\sigma^2(a,x)\big] \\ \qquad + m(x,b)\,\big[(b-\mu(x,b))^2+\sigma^2(x,b)\big]\Big\}, & f'' \in L_\infty[a,b] \\ \dfrac{\|f''\|_1}{2}\Big\{m(a,x)\big[\mu(a,x)-a\big]+m(x,b)\big[b-\mu(x,b)\big] \\ \qquad + |m(a,b)\mu(a,b) - am(a,x) - bm(x,b)|\Big\}, & f'' \in L_1[a,b] \end{cases}$$

The bound in $(7.50)_1$ is minimized at the point $x = (a+b)/2$ and the bound in $(7.50)_2$ is minimized at the point x^* satisfying

$$m(a,x^*)\big(\mu(a,x^*)-a\big) = m(x^*,b)\big(b-\mu(x^*,b)\big). \tag{7.51}$$

PROOF. Integrating $I = \int_a^b K(x,t)f''(t)\,dt$ twice by parts gives

$$I = \int_a^b w(t)f(t)\,dt - m(a,x)f(a) - m(a,x)(\mu(a,x)-a)f'(a) - m(x,b)f(b) + (b-\mu(x,b))f'(b). \tag{7.52}$$

Using the well known properties of definite integrals

$$|I| \le \int_a^b |K(x,t)f''(t)|\,dt \tag{7.53}$$
$$\le \|f''\|_\infty \int_a^b K(x,t)\,dt$$
$$= \frac{\|f''\|_\infty}{2}\left\{\int_a^x (t-a)^2 w(t)\,dt + \int_x^b (t-b)^2 w(t)\,dt\right\}. \tag{7.54}$$

The last line being obtained by reversing the order of integration. Making use of equation (7.52) and substituting equations (7.5)-(7.7) into (7.54) returns $(7.50)_1$.

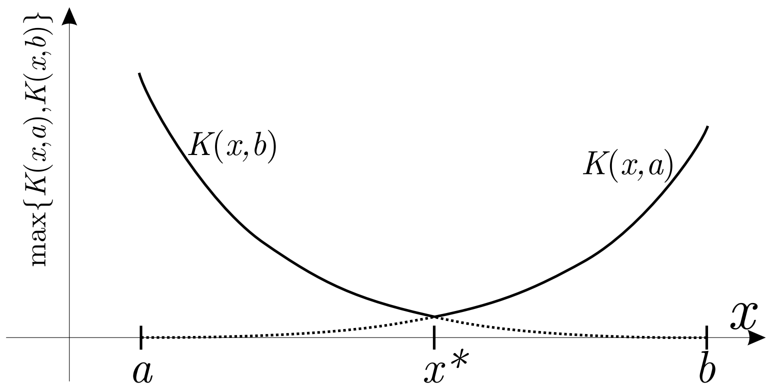

FIGURE 7.1. A schematic of the function $\max\{K(x,a),K(x,b)\}$ for $x \in [a,b]$ showing the minimum point x^*.

To find the value of x which minimizes (7.54), observe that

$$\begin{aligned}\frac{d}{dx}\int_a^b K(x,t)\,dt &= (x-a)^2 w(x) - (x-b)^2 w(x) \\ &= w(x)(b-a)\left(x - \frac{a+b}{2}\right).\end{aligned}$$

Since w is positive on (a,b), a minimum occurs at the mid-point.

To prove $(7.50)_2$, we make use of (7.53) and Lemma 7.15

$$\begin{aligned}\left|\int_a^b K(x,t) f''(t)\,dt\right| &\le \sup_{t\in(a,b)} K(x,t)\|f''\|_1 \\ &= \max\{K(x,a),K(x,b)\}\|f''\|_1 \\ &= \frac{\|f''\|_1}{2}\left[K(x,a)+K(x,b)+|K(a,x)-K(b,x)|\right].\end{aligned}$$

Simplifying the expression above produces the desired result.

To obtain the optimal point x^* note that $K(x,a)$ is increasing and $K(x,b)$ is decreasing with respect to x. Thus the minimum of $\max\{K(x,a),K(x,b)\}$ will occur at $K(x,a)=K(x,b)$ as given in equation (7.51). A schematic is shown in Figure 7.1. ∎

The product inequality may be useful as the basis for a composite weighted quadrature rule since only the first two moments are required for the rule. For the bound in $(7.50)_1$, knowledge of the third moment is needed. The following corollaries simplify (and in some cases consequently degrade) equation (7.50), in that the bound is given in terms of fewer moments. The additional constraint that $b-a$ remain finite is also imposed.

COROLLARY 7.17. Define Q to be

$$Q(x) = m(a,x)f(a) - m(a,x)(\mu(a,x)-a)f'(a) - m(x,b)f(b) + (b-\mu(x,b))f'(b).$$

Given the conditions in Theorem 7.16, the following inequalities hold

$$\left|\int_a^b w(t)f(t)\,dt - Q(x)\right| \le \frac{\|f''\|_\infty}{2}\big\{m(a,x)(x-a)(\mu(a,x)-a) + m(x,b)(b-x)(b-\mu(x,b))\big\} \tag{7.55}$$

and

$$\left|\int_a^b w(t)f(t)\,dt - Q(x)\right| \le \frac{\|f''\|_\infty}{2} m(a,b)\left(\left|x-\frac{a+b}{2}\right| + \frac{b-a}{2}\right)^2. \tag{7.56}$$

We remark that the bound in (7.55) requires the first two moments of w, which contrasts with that of (7.56) where only the first moment is needed. In addition, identifying the optimal point for the bound in (7.56) is obvious.

Proof. To prove (7.55), we appeal to equation (7.31). Applying (7.31) over the intervals $[a,x]$ and $[x,b]$ and simplifying produces (7.55).

To show that (7.56) is true, note from (7.54) that

$$\begin{aligned}\int_a^x (t-a)^2 w(t)\,dt + \int_x^b (t-b)^2 w(t)\,dt &\le m(a,b)\max\big\{(x-a)^2,(x-b)^2\big\}\\ &= \frac{1}{2}m(a,b)\big\{(a-x)^2 + (x-b)^2\\ &\quad + \left|(x-a)^2-(x-b)^2\right|\big\}\\ &= m(a,b)\left(\left|x-\frac{a+b}{2}\right| + \frac{b-a}{2}\right)^2.\end{aligned}$$

∎

Corollary 7.18. If w is bounded as well as integrable, then the following inequality holds

$$\left|\int_a^b w(t)f(t)\,dt - Q(x)\right| \le \frac{1}{2}\|f''\|_\infty\|w\|_\infty(b-a)\left[\left(x-\frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{12}\right]. \tag{7.57}$$

We remark that the bound in (7.57) does not require any moments of w but is less accurate in the sense that it is an upper-bound for (7.50).

Proof. The proof of equation (7.57) follows again from equation (7.54).

$$\begin{aligned}\int_a^x (t-a)^2 w(t)\,dt + \int_x^b (t-b)^2 w(t)\,dt &\le \frac{1}{3}\|w\|_\infty\left\{(x-a)^3-(x-b)^3\right\}\\ &= \|w\|_\infty(b-a)\left[\left(x-\frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{12}\right].\end{aligned}$$

∎

7.4.1. Development of a Product-Trapezoidal Like Quadrature Rule. Equation $(7.50)_1$ will be used as the basis for a product trapezoidal-like quadrature rule. From (7.57), we can see that the error in approximating $\int_a^b w(t)f(t)\,dt$ is $O((b-a)^3)$ (assuming finite $b-a$). Thus, to obtain an accurate estimate to the definite integral we require a composite rule where the interval $[a,b]$ is subdivided. Usually this subdivision is uniform, but in this case the grid should account for the presence of the weight.

In the next theorem, the interval $[a,b]$ is divided into two "optimal" parts: $[a,\xi],[\xi,b]$. This result is employed in the development of a weighted quadrature rule.

THEOREM 7.19 (Two intervals). *Let the conditions in Theorem 7.16 hold. The following "two interval" product integral inequality holds*

$$\begin{aligned}(7.58)\quad &\Big|\int_a^b w(t)f(t)\,dt - m(a,x_1)\big(f(a)+(\mu(a,x_1)-a)f'(a)\big)-\\ &m(x_1,x_2)\big(f(\xi)+(\mu(x_1,x_2)-\xi)f'(\xi)\big)-m(x_2,b)\big(f(b)+(\mu(x_2,b)-b)f'(b)\big)\Big|\\ &\le \frac{\|f''\|_\infty}{2}\Big\{m(a,x_1)\left[(\mu(a,x_1)-a)^2+\sigma^2(a,x_1)\right]+\\ &\quad m(x_1,x_2)\left[(\mu(x_1,x_2)-\xi)^2+\sigma^2(x_1,x_2)\right]+\\ &\quad m(x_2,b)\left[(b-\mu(x_2,b))^2+\sigma^2(x_2,b)\right]\Big\},\end{aligned}$$

for $a\le x_1\le\xi\le x_2\le b$.

The bound is minimized at the points

$$(7.59)\qquad x_1=\frac{a+\xi}{2},\quad \xi=\mu(x_1,x_2)\quad \textit{and}\quad x_2=\frac{\xi+b}{2}.$$

PROOF. Define the kernel

$$(7.60)\qquad K(x_1,x_2,\xi,t)=\begin{cases}\int_{x_1}^t (t-u)w(u)\,du, & a\le t,x_1<\xi\\ \int_{x_2}^t (t-u)w(u)\,du, & \xi\le t,x_2\le b.\end{cases}$$

Employing the same techniques as those in Theorem 7.16, namely integrating $\int_a^b K(x_1,x_2,\xi,t)f''(t)\,dt$ by parts twice and simplifying, gives

$$\begin{aligned}(7.61)\quad &\Big|\int_a^b w(t)f(t)\,dt - m(a,x_1)\big(f(a)+(\mu(a,x_1)-a)f'(a)\big)-\\ &m(x_1,x_2)\big(f(\xi)+(\mu(x_1,x2)-\xi)f'(\xi)\big)-m(x_2,b)\big(f(b)+(\mu(x_2,b)-b)f'(b)\big)\Big|\\ &=\left|\int_a^b K(x_1,x_2,\xi,t)f''(t)\,dt\right|\\ &\le\frac{\|f''\|_\infty}{2}\Big\{\int_a^{x_1}(t-a)^2w(t)\,dt+\int_{x_1}^{x_2}(t-\xi)^2w(t)\,dt+\int_{x_2}^b(t-b)^2w(t)\,dt\Big\}.\end{aligned}$$

Making use of (7.5), (7.6) and simplifying produces (7.58).

Differentiation reveals the upper bound minimum. If we let

$$I_b = \int_a^{x_1} (t-a)^2 w(t)\,dt + \int_{x_1}^{x_2} (t-\xi)^2 w(t)\,dt + \int_{x_2}^{b} (t-b)^2 w(t)\,dt,$$

then observe that

$$\frac{dI_b}{d\xi} = \int_{x_1}^{x_2} (\xi - t) w(t)\,dt = m(x_1,x_2)\big(\xi - \mu(x_1,x_2)\big),$$

$$\frac{dI_b}{dx_1} = \frac{1}{2} w(x_1)(\xi - a)\left(x_1 - \frac{a+\xi}{2}\right) \quad \text{and}$$

$$\frac{dI_b}{dx_2} = \frac{1}{2} w(x_2)(b-\xi)\left(x_2 - \frac{\xi+b}{2}\right).$$

Hence, the proof is complete. ∎

Remark 7.6. Substituting $x_1 = a$ and $x_2 = b$ in (7.58) produces the interior point inequality $(7.22)_1$.

Theorem 7.20 (Product trapezoidal-like quadrature rule). *Define a grid $I_n : a = \xi_0 < \xi_1 < \cdots < \xi_{n-1} < \xi_n = b$ on the interval $[a,b]$. The following quadrature formula for weighted integrals is obtained.*

(7.62)

$$\int_a^b w(t) f(t)\,dt = \sum_{i=0}^{n} h_i f(\xi_i) + h_0(\mu_0 - \xi_0) f'(\xi_0) - h_n(\xi_n - \mu_n) f'(\xi_n) + R(f,w,\boldsymbol{\xi},\boldsymbol{x}),$$

where

$$|R(f,w,\boldsymbol{\xi},\boldsymbol{x})| \le \frac{1}{2}\|f''\|_\infty \left[\sum_{i=0}^{n} h_i \sigma_i^2 + h_0(\mu_0-\xi_0)^2 + h_n(\xi_n-\mu_n)^2\right], \tag{7.63}$$

$$\begin{aligned} x_i &= \frac{\xi_i + \xi_{i+1}}{2}, \qquad i = 0,\ldots,n-1, \\ \xi_i &= \mu(x_{i-1}, x_i), \qquad i = 1,\ldots,n-1. \end{aligned} \tag{7.64}$$

and

$$\begin{aligned} &h_i = m(x_{i-1},x_i), \quad \sigma_i^2 = \sigma^2(x_{i-1},x_i), \qquad i = 1,\ldots,n-1, \\ &h_0 = m(\xi_0,x_0),\ h_n = m(x_{n-1},\xi_n), \\ &\mu_0 = \mu(\xi_0,x_0),\ \mu_n = \mu(x_{n-1},\xi_n), \\ &\sigma_0^2 = \sigma^2(\xi_0,x_0),\ \sigma_n^2 = \sigma^2(x_{n-1},\xi_n). \end{aligned} \tag{7.65}$$

Proof. For an $n+1$ point rule, we extend the kernel (7.60) to n branches with each branch defined over $[\xi_i, \xi_{i+1}], i = 0,1,\ldots,n-1$. Following the proof of Theorem 7.19 produces the desired quadrature rule. ∎

The grid equations (7.64) are simple to solve. Any size grid can be evaluated very quickly, by using an iterative approach, so long as the zero-th and first moment of the weight are known. In Figure 7.2 we show the node point distribution for a logarithmic weight. The clustering near the origin is obvious and is due to the presence of the singularity at this point.

FIGURE 7.2. Node point distribution produced by solving equation (7.64) for a 32 point log rule over the unit interval ($w(t) = \ln(1/t)$).

	Error			
n	Equation (7.62)	(Ratio)	Atkinson [**2**, p. 310]	Error (3) from Table 7.1
4	7.5(0)		7.48(0)	2.48(0)
8	6.76(-2)	(111)	7.03(-1)	2.35(-1)
16	2.08(-2)	(3.25)	1.27(-1)	2.62(-2)
32	4.17(-3)	(4.99)	2.78(-2)	4.34(-3)
64	9.40(-4)	(4.44)	6.59(-3)	9.34(-4)
128	2.30(-4)	(4.09)	1.61(-3)	2.23(-4)
256	5.55(-5)	(4.14)	3.99(-4)	5.51(-4)

TABLE 7.2. The error in evaluating (7.45) using different quadrature rules. n is the number of sample points.

7.4.2. Numerical Experiment. In Table 7.2, we compare (7.62) with the interior point rule (7.39) of the previous section and a product trapezoidal type rule developed by Atkinson [**2**, p. 310]. We can see that the rule developed here compares favourably with others of similar order. The ratio shows that the rule is of order h^2.

7.4.3. Some Particular Weighted Integral Inequalities. The results of Theorem 7.19 are tabulated for some of the more popular weight functions. In each case x_1, ξ and x_2 are given by (7.59) so that the bound is minimized.

7.4.3.1. *Uniform (Legendre).*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{4}f(a) - \frac{(b-a)^2}{32}f'(a) - \frac{b-a}{2}f\left(\frac{a+b}{2}\right) - \frac{b-a}{4}f(b) + \frac{(b-a)^2}{32}f'(b)\right| \le \frac{1}{96}\|f''\|_\infty (b-a)^3 \tag{7.66}$$

7.4.3.2. *Logarithm.*

$$\left|\int_0^1 f(t)\ln(1/t)\,dt - .50911434f(0) - .039684385f'(0) - .43771782f(.38432057) - 0.53167836f(1) + 0.011076181f'(1)\right| \le 0.0077218546\|f''\|_\infty \tag{7.67}$$

7.4.3.3. *Jacobi.*

$$\left|\int_0^1 \frac{f(t)}{\sqrt{t}}\,dt - .95158029f(0) - .071805063f'(0) - .75297392f(.45275252) - 0.29544579f(1) + 0.041495030f'(1)\right| \le 0.016598012\|f''\|_\infty \tag{7.68}$$

7.4.3.4. *Chebyshev.*

(7.69)

$$\left|\int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}}\,dt - \frac{\pi}{3}f(-1) - \frac{2\pi-3\sqrt{3}}{6}f'(-1) - \frac{\pi}{3}f(0) - \frac{\pi}{3}f(1) + \frac{2\pi-3\sqrt{3}}{6}f'(1)\right| \le \frac{1}{12}\|f''\|_\infty(7\pi - 2\sqrt{3})$$

7.4.3.5. *Laguerre.* The Laguerre weight is not immediately applicable since b is infinite and hence by (7.59) so is ξ. To produce a Laguerre inequality we assume b is finite and the take the limit as $b \to \infty$. In this limit we obtain
(7.70)

$$\left|\int_0^\infty e^{-t}f(t)\,dt - (1-e^{-1})f(0) - (1-2e^{-1})f'(0) - e^{-1}f(2)\right| \le (1-2e^{-1})\|f''\|_\infty,$$

assuming that $f = o\left(e^{x/2}\right)$ as $x \to \infty$ and f'' is bounded on $[0,\infty)$.

7.5. Weighted Three Point Integral Inequalities

In the previous two sections, interior point and boundary point inequalties have been investigated. In this section we generalize and combine the previous results via a parameterization. The parameter distinguishes rule type and at its extremes will produce an interior or boundary point inequality. The inequalities are called "three point" rules. At the end of this section three point rules are constructed by a Grüss inequality.

Three point quadrature rules of Newton-Cotes type have been examined in Cerone and Dragomir [**7**] in which the error involved the behaviour of, at most, a first derivative. Riemann and Riemann-Stieltjes integrals were examined.

In the current section, weighted three point rules are investigated in which the error relies on the behaviour of the first derivative [**12**].

After developing the results in the initial subsections, composite quadrature rules are implemented and results for a log weight function are given and compared with a product-trapezoidal rule of Atkinson [**2**].

THEOREM 7.21. *([**12**]) Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative is bounded on (a,b) and denote $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Further, let a non-negative weight function $w(\cdot)$ have the properties as outlined in Definition*

2. Then for $x \in [a,b]$, $\alpha \in [a,x]$, $\beta \in (x,b]$, the following inequality holds

(7.71)
$$\left| \int_a^b w(t) f(t)\, dt - [m(\alpha,\beta) f(x) + m(a,\alpha) f(a) + m(\beta,b) f(b)] \right| \le I(\alpha,x,\beta) \|f'\|_\infty ,$$

where

$$I(\alpha,x,\beta) = \int_a^b k(x,t)\, w(t)\ dt \quad \text{and} \quad k(x,t) = \begin{cases} t-a, & t \in [a,\alpha] \\ |x-t|, & t \in (\alpha,\beta] \\ b-t, & t \in (\beta,b] \end{cases} \tag{7.72}$$

Proof. Define the mapping $K(\cdot,\cdot) : [a,b]^2 \to \mathbb{R}$ by

$$K(x,t) = \begin{cases} m(\alpha,t), & t \in [a,x] \\ m(\beta,t), & t \in (x,b] \end{cases}, \tag{7.73}$$

where $m(a,b)$ is the zeroth moment of $w(\cdot)$ over the interval $[a,b]$ and is given by $(7.5)_1$.

It should be noted that $m(c,d)$ will be non-negative for $d \ge c$.

Integration by parts gives, on using (7.73),

$$\begin{aligned} \int_a^b K(x,t) f'(t)\, dt &= \int_a^x m(\alpha,t) f'(t)\, dt + \int_x^b m(\beta,t) f'(t)\, dt \\ &= m(\alpha,t) f(t) \Big]_{t=a}^{x} + m(\beta,t) f(t) \Big]_{t=x}^{b} - \int_a^b w(t) f(t)\, dt, \end{aligned}$$

producing the identity

(7.74)
$$\int_a^b K(x,t) f'(t)\, dt = m(\alpha,\beta) f(x) + m(a,\alpha) f(a) + m(\beta,b) f(b) - \int_a^b w(t) f(t)\, dt,$$

valid for all $x \in [a,b]$.

Taking the modulus of (7.74) gives

$$\begin{aligned} &\left| \int_a^b w(t) f(t)\, dt - [m(\alpha,\beta) f(x) + m(a,\alpha) f(a) + m(\beta,b) f(b)] \right| \\ &\qquad = \left| \int_a^b K(x,t) f'(t)\, dt \right| \le \|f'\|_\infty \int_a^b |K(x,t)|\, dt . \end{aligned} \tag{7.75}$$

Now, we wish to determine $\int_a^b |K(x,t)|\, dt$. To this end notice that, from (7.73), $K(x,t)$ is a monotonically non-decreasing function of t over each of its branches. Thus, there are points $\alpha \in [a,x]$ and $\beta \in [x,b]$ such that $K(x,\alpha) = K(x,\beta) = 0$.

Thus,

(7.76)
$$\int_a^b |K(x,t)|\, dt = -\int_a^\alpha m(\alpha,t)\, dt + \int_\alpha^x m(\alpha,t)\, dt - \int_x^\beta m(\beta,t)\, dt + \int_\beta^b m(\beta,t)\, dt .$$

Integration by parts gives, for example,

$$-\int_a^\alpha m(\alpha,t)\,dt = -(t-a)\,m(\alpha,t)\Big]_{t=a}^{\alpha} + \int_a^\alpha (t-a)\,w(t)\,dt = \int_a^\alpha (t-a)\,w(t)\,dt\,.$$

A similar development for the remainder of the three integrals on the right hand side of (7.76) produces the result

$$\int_a^b |K(x,t)|\,dt = I(\alpha,x,\beta)\,, \tag{7.77}$$

where $I(\alpha,x,\beta)$ is as given by (7.72). Combining (7.75) and (7.77) produces the result (7.71) and hence the theorem is proved. ∎

It should be noted at this stage that taking $w(\cdot)\equiv 1$ reproduces the results of Cerone and Dragomir [**7**]. If $\alpha = a$ and $\beta = b$ then a weighted interior point rule is obtained. If $\alpha=\beta=x$, then a weighted rule results where the function is evaluated at the boundary points. For $\alpha = a$ or $\beta = b$ then Radau type rules are obtained while the current work will focus on Lobatto type rules allowing sampling at both ends of the boundary.

Corollary 7.22. Inequality (7.71) is minimized at $x = x^*$ where x^* satisfies

$$m(\alpha^*,x^*) = m(x^*,\beta^*)\,,\quad \alpha^* = \frac{a+x^*}{2} \quad\text{and}\quad \beta^* = \frac{x^*+b}{2}\,. \tag{7.78}$$

Proof. ¿From (7.71) - (7.72), $I(\alpha,x,\beta)$ may be written as

$$\begin{aligned} I(\alpha,x,\beta) = {} & \int_a^\alpha (t-a)\,w(t)\,dt + \int_\alpha^x (x-t)\,w(t)\,dt \\ & + \int_x^\beta (t-x)\,w(t)\,dt + \int_\beta^b (b-t)\,w(t)\,dt, \end{aligned} \tag{7.79}$$

where $\alpha\in[a,x]$ and $\beta\in(x,b]$. Equation (7.79) could equivalently be written in terms of its zeroth and first moments as given by (7.5). Differentiating (7.79) with respect to α,β and x gives

$$\frac{\partial I}{\partial\alpha} = A(\alpha,x)\,w(x)\,,\ \frac{\partial I}{\partial\beta} = B(\beta,x)\,w(x)\ \text{ and }\ \frac{\partial I}{\partial x} = m(\alpha,x) - m(x,\beta)\,, \tag{7.80}$$

where

$$A(\alpha,x) = 2\alpha-(a+x)\,,\ B(\beta,x) = 2\beta-(x+b) \tag{7.81}$$

and $m(\cdot,\cdot)$ is defined by $(7.5)_1$. An inspection of the second derivatives demonstrates that (7.79) is convex on using the fact that $w(t)$ is non-negative for $t\in(a,b)$. Thus, I is minimal at the zeros of (7.80) and so the corollary is proven. ∎

Corollary 7.22 investigates the problem of determining the optimal choice of α, x and β that produce the tightest bound. The following corollary gives coarser bounds although the bound may be easier to implement.

COROLLARY 7.23. Let the conditions be as in Theorem 7.21. Then the following inequalities hold

$$(7.82)\quad \left|\int_a^b w(t) f(t)\,dt - [m(\alpha,\beta) f(x) + m(a,\alpha) f(a) + m(\beta,b) f(b)]\right| \le \|f'\|_\infty \times \begin{cases} \|w\|_\infty \cdot K_1(x) \\ \\ \|w\|_1 \cdot K_\infty(x) \end{cases},$$

where

$$(7.83)\quad K_1(x) = \frac{1}{2}\left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right] + \left(\alpha - \frac{a+x}{2}\right)^2 + \left(\beta - \frac{x+b}{2}\right)^2$$

and

$$(7.84)\quad K_\infty(x) = \frac{1}{2}\left[\frac{b-a}{2} + \left|\alpha - \frac{a+x}{2}\right| + \left|\beta - \frac{x+b}{2}\right| + \left|x - \frac{a+b}{2} + \left|\alpha - \frac{a+x}{2}\right| - \left|\beta - \frac{x+b}{2}\right|\right|\right]$$

with $\|g\|_1 := \int_a^b |g(s)|\,ds$ meaning $g \in L_1[a,b]$, the linear space of absolutely integrable functions and $\|g\|_\infty := \sup_{t\in[a,b]} |g(t)| < \infty$.

PROOF. ¿From Theorem 7.21 and equations (7.71) - (7.72), (7.73) and (7.77) we have

$$I(\alpha,x,\beta) = \int_a^b |K(x,t)|\,w(t)\,dt = \int_a^b k(x,t)\,w(t)\,dt\,.$$

Now,

$$\int_a^b k(x,t)\,w(t)\,dt \le \begin{cases} \|w\|_\infty \int_a^b |k(x,t)|\,dt \\ \\ \|w\|_1 \sup_{t\in[a,b]} |k(x,t)| \end{cases},$$

where $k(x,t)$ is as defined in (7.72).

Some straight forward evaluation gives

$$\int_a^b |k(x,t)|\,dt = \frac{1}{2}\left[(\alpha-a)^2 + (x-\alpha)^2 + (\beta-x)^2 + (b-\beta)^2\right],$$

which may readily be shown to equal $K_1(x)$ as given by (7.83) through using the identity

$$\frac{X^2+Y^2}{2} = \left(\frac{X+Y}{2}\right)^2 + \left(\frac{X-Y}{2}\right)^2$$

three times.

Further,

$$\sup_{t\in[a,b]} |k(x,t)| = \max\{\alpha-a, x-\alpha, \beta-x, b-\beta\},$$

which can be shown to equal $K_\infty(x)$ as given by (7.84) from using the result

$$(7.85)\quad \max\{X,Y\} = \frac{X+Y}{2} + \frac{|X-Y|}{2},$$

three times.

Hence the corollary is proven. ∎

It should be noted that the tightest bounds are obtained at $x = \frac{a+b}{2}$ and $\alpha = \frac{a+x}{2}$, $\beta = \frac{x+b}{2}$. That is, at their respective midpoints. The optimal sampling scheme is independent of the weight.

Theorem 7.24. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$ (the interior of I) and $a, b \in \mathring{I}$ are such that $b > a$. If $f' \in L_1[a,b]$, then $\|f'\|_1 = \int_a^b |f'(t)|\,dt < \infty$. In addition, let a non-negative weight function $w(\cdot)$ have the properties as outlined in Definition 2. Then for $x \in [a,b]$, $\alpha \in [a,x]$ and $\beta \in (x,b]$ the following inequality holds.*

$$\left|\int_a^b w(t) f(t)\,dt - [m(\alpha,\beta) f(x) + m(a,\alpha) f(a) + m(\beta,b) f(b)]\right| \leq \theta(\alpha,x,\beta)\|f'\|_1, \tag{7.86}$$

where

$$\begin{aligned}\theta(\alpha,x,\beta) = \frac{1}{4}\big\{&m(a,b) + |m(\alpha,x) - m(a,\alpha)| + |m(\beta,b) - m(x,\beta)| \\ &+ |m(a,x) - m(x,b) + |m(\alpha,x) - m(a,\alpha)| - |m(\beta,b) - m(x,\beta)||\big\}\end{aligned} \tag{7.87}$$

and $m(a,b)$ is the zeroth moment of $w(\cdot)$ over $[a,b]$ as defined by $(7.5)_1$.

Proof. ¿From identity (7.74) we obtain, from taking the modulus

$$\theta(\alpha,x,\beta) = \sup_{t\in[a,b]} |K(x,t)|,$$

where $K(x,t)$ is as given by (7.73). As discussed in the proof of Theorem 7.21, $K(x,t)$ is a monotonic non-decreasing function of t in each of its two branches so that

$$\theta(\alpha,x,\beta) = \max\{m(a,\alpha), m(\alpha,x), m(x,\beta), m(\beta,b)\}.$$

Now, using equation (7.85) we have

$$\begin{aligned} m_1 &= \max\{m(a,\alpha), m(\alpha,x)\} = \frac{1}{2}[m(a,x) + |m(\alpha,x) - m(a,\alpha)|] \\ \text{and } m_2 &= \max\{m(x,\beta), m(\beta,b)\} = \frac{1}{2}[m(x,b) + |m(\beta,b) - m(x,\beta)|], \end{aligned}$$

to give

$$\theta(\alpha,x,\beta) = \max\{m_1,m_2\} = \frac{m_1+m_2}{2} + \left|\frac{m_1-m_2}{2}\right|$$

and hence the result (7.87) is obtained after some simplification and the theorem is proved. ∎

Remark 7.7. It should be noted that the tightest bound in (7.87) is obtained when α, x and β are taken as their respective medians. Thus, the best quadrature rule in the above sense is given by

(7.88)

$$\left|\int_a^b w(t)\,f(t)\,dt - \left[m(a,\tilde{\alpha})\,f(a) + m(\tilde{\alpha},\tilde{\beta})f(\tilde{x}) + m(\tilde{\beta},b)f(b)\right]\right| \le \frac{m(a,b)}{4}\left\|f'\right\|_1,$$

where

$$m(a,\tilde{x}) = m(\tilde{x},b),\ m(a,\tilde{\alpha}) = m(\tilde{\alpha},\tilde{x}) \text{ and } m(\tilde{\beta},b) = m(\tilde{x},\tilde{\beta}).$$

7.5.1. Development of a Quadrature Rule. The following theorem will be useful in determining the partition for composite quadrature rules. The optimal partition in terms of the partition that provides the tightest bounds will be determined. The optimal quadrature rules will result for $f' \in L_\infty[a,b]$. If $f' \in L_1[a,b]$ a similar development may be followed but will not be pursued further here.

THEOREM 7.25. *Let the conditions of Theorem 7.21 hold and let ξ partition the interval $[a,b]$ into two. Then the following inequality holds*

$$\left|\int_a^b w(t)\,f(t)\,dt - \left[m(a,\alpha_1)\,f(a) + m(\alpha_1,\beta_1)\,f(x_1) + m(\beta_1,\alpha_2)\,f(\xi)\right.\right.$$
$$\left.\left. + m(\alpha_2,\beta_2)\,f(x_2) + m(\beta_2,b)\,f(b)\right]\right| \le J(\mathbf{z},\xi)\,\left\|f'\right\|_\infty, \tag{7.89}$$

where

$$J(\mathbf{z},\xi) = J_1(\mathbf{z}_1,\xi) + J_2(\mathbf{z}_2,\xi) \tag{7.90}$$

with

$$\mathbf{z}_i^T = (\alpha_i, x_i, \beta_i),\ i = 1,2,\ \mathbf{z} = \mathbf{z}_1 \cup \mathbf{z}_2, \tag{7.91}$$

$$J_1(\mathbf{z}_1,\xi) = \int_a^\xi k_1(x_1,t)\,w(t)\,dt, \qquad J_2(\mathbf{z}_2,\xi) = \int_\xi^b k_2(x_2,t)\,w(t)\,dt$$

and

(7.92)

$$k_1(x_1,t) = \begin{cases} t-a, & t \in [a,\alpha_1] \\ |x_1 - t|, & t \in (\alpha_1,\beta_1] \\ \xi - t, & t \in (\beta_1,\xi] \end{cases},\ k_2(x_2,t) = \begin{cases} t-\xi, & t \in [\xi,\alpha_2] \\ |x_2 - t|, & t \in (\alpha_2,\beta_2] \\ b - t, & t \in (\beta_2,b] \end{cases}.$$

Further, $a \le \alpha_1 \le x_1 \le \beta_1 \le \xi$ and $\xi \le \alpha_2 \le x_2 \le \beta_2 \le b$.

PROOF. The proof follows that of Theorem 7.21. A subscript of 1 is used to denote parameters in the interval $[a,\xi]$ and 2 for parameters in $(\xi,b]$. Integration by parts of $\int_a^\xi K(x_1,t)\,f'(t)\,dt$ produces an identity similar to (7.74) with b replaced by ξ and x by x_1. Similarly for $\int_\xi^b K(x_2,t)\,f'(t)\,dt$ produces an identity like (7.74) with a replaced by ξ and x by x_2. Summing the two results produces an identity over $[a,b]$. Taking the modulus and using the triangle inequality, relying heavily on (7.72) gives the stated result after collecting the terms in order. Here on $[a,\xi]$, (α,x,β,b) are replaced by $(\alpha_1,x_1,\beta_1,\xi)$ and on $[\xi,b]$, (a,α,x,β) are replaced by $(\xi,\alpha_2,x_2,\beta_2)$. Hence the theorem is proved. ∎

Corollary 7.26. The optimal location of the parameters in Theorem 7.25 are $\alpha_1 = \alpha_1^* = \frac{a+x_1^*}{2}$, $\beta_1 = \beta_1^* = \frac{x_1^*+\xi^*}{2}$, $\alpha_2 = \alpha_2^* = \frac{\xi^*+x_2^*}{2}$, $\beta_2 = \beta_2^* = \frac{x_2^*+b}{2}$ and x_1^*, x_2^* and ξ^* satisfy the following respective equations

$$m\left(\alpha_1^*, x_1^*\right) = m\left(x_1^*, \beta_1^*\right), \; m\left(\alpha_2^*, x_2^*\right) = m\left(x_2^*, \beta_2^*\right) \text{ and } m\left(\beta_1^*, \xi^*\right) = m\left(\xi^*, \alpha_2^*\right).$$

Proof. The proof of this corollary closely follows that of Corollary 7.22. From (7.90) - (7.92), differentiation of J with respect to $(\alpha_1, x_1, \beta_1, \xi, \alpha_2, x_2, \beta_2)$ produces, on equating to zero, seven simultaneous equations.

Using the fact that the weight function is assumed to be positive, then the solution of the seven simultaneous equations give the point at which an optimal bound is produced, since an inspection of the second derivatives readily demonstrates the convexity of the function J. ∎

The results in Theorem 7.25 may be used to develop a composite quadrature rule. To this end, define a grid $I_n : a = \xi_0 < \xi_1 < \cdots < \xi_{n-1} < \xi_n = b$ on the interval $[a, b]$, with $x_i \in [\xi_i, \xi_{i+1}]$ for $i = 0, 1, \ldots, n-1$. The following quadrature formula for weighted integrals is obtained which relies only on the first two moments of the weight function.

Theorem 7.27. *Let the conditions in Theorem 7.25 hold, then following weighted quadrature rule holds*

$$\int_a^b w(t) f(t)\, dt = A(f, \boldsymbol{\xi}, \boldsymbol{x}) + R(f, \boldsymbol{\xi}, \boldsymbol{x}) \tag{7.93}$$

where

$$A(f, \boldsymbol{\xi}, \boldsymbol{x}) = \sum_{i=1}^{n} m(\alpha_i, \beta_i) f(x_i) + m(\xi_0, \alpha_1) f(\xi_0) + 2\sum_{i=1}^{n-1} m(\beta_i, \xi_i) f(\xi_i) + m(\beta_n, \xi_n) f(\xi_n) \tag{7.94}$$

and

$$|R(f, \boldsymbol{\xi}, \boldsymbol{x})| \le \|f'\|_\infty \Bigg(M(\xi_0, \xi_n) - 2\sum_{i=}^{n} \left[M(\alpha_i, \beta_i) + M(\beta_i, \xi_i)\right] + \xi_n m(\beta_n, \xi_n) - \xi_0 m(\xi_0, \alpha_1) \Bigg). \tag{7.95}$$

The parameters x_i, α_i, β_i and ξ_i satisfy

$$m(\alpha_i, x_i) = m(x_i, \beta_i), \qquad \alpha_i = \frac{\xi_{i-1} + x_i}{2}, \qquad \beta_i = \frac{x_i + \xi_i}{2} \tag{7.96}$$

for $i = 1, 2, \ldots, n$, and

$$m(\beta_i, \xi_i) = m(\xi_i, \alpha_{i+1}), \tag{7.97}$$

for $i = 1, 2, \ldots, n-1$.

Proof. Using the results of Theorems 7.21 and 7.25 over $[\xi_i, \xi_{i+1}]$ for $i = 0, 1, \ldots, n-1$ and summing readily produces the result after using Corollaries 7.22 and 7.26 to simplify. ∎

7.5.1.1. *Numerical Results.* In this section we illustrate the application of the composite quadrature rule developed in the previous section to approximate the integrals

$$\int_0^1 \frac{\ln(1/t)}{t+2}\,dt = 0.4484142069 \quad \text{and} \quad \int_0^1 e^{-1/t}\ln(1/t)\,dt = 0.05065230956 \tag{7.98}$$

The integrals are evaluated using the composite rule (7.93) and the product-trapezoidal as described in [**2**, p. 310]. The first integral, $(7.98)_1$, has been used to demonstrate the product-trapezoidal and as a result we can compare the performance with the rule developed here. Note that (7.93) is a first-order rule in that it was derived for the class of once-differentiable functions. This contrasts with the product-trapezoidal rule which is of second order. Thus, to investigate the effects of rule order, we also apply these rules to $(7.98)_2$. In contrast with $(7.98)_1$, the integrand of $(7.98)_2$ increases with the order of its derivative.

Table 7.3 shows the numerical error in evaluating (7.98) using (7.93) for an increasing number of intervals. We note that the nodes and weights of the quadrature rule are obtained by solving the $4n-1$ simultaneous equations (7.96) and (7.97). It is a simple matter to implement a numerical procedure to solve these equations iteratively with an initial uniform mesh. For example on a Pentium-90 personal computer, with $n = 32$, calculating (7.96) and (7.97) to 14 digit accuracy took close to 42 seconds.

Inspection of Table 7.3 reveals that a more accurate result is obtained for $(7.98)_1$ than for $(7.98)_2$. This is probably due to the nature of the integrands. The theoretical error ratio is consistently close to 2. This value confirms that, due to its development, the quadrature rule is at least of first order. The numerical error ratios are somewhat larger, these values suggest an asymptotic form of the error bound

$$|R(f,\boldsymbol{\xi},\boldsymbol{x})| \sim O\left(\frac{1}{n^{\gamma}}\right), \qquad \text{where} \quad \gamma \le 2. \tag{7.99}$$

n	Equation $(7.98)_1$		Equation $(7.98)_2$		Theoretical
	Relative Error	Error Ratio	Relative Error	Error Ratio	Error Ratio
2	1.64(-2)		7.27(-2)		
4	4.53(-3)	3.64	2.62(-2)	2.78	1.70
8	1.23(-3)	3.69	8.47(-3)	3.09	2.81
16	3.29(-4)	3.73	2.57(-3)	3.30	2.08
32	8.77(-5)	3.75	7.52(-4)	3.41	2.05
64	2.33(-5)	3.77	2.15(-4)	3.50	2.03

TABLE 7.3. The relative error in evaluating (7.98) using (7.93), where n is the number of intervals.

In Table 7.4 the errors in employing the product-trapezoidal rule are presented. The error ratios are consistently close to 4 which simply reflects the fact that the rule is of second order. This rule was developed by employing a linear approximation for the weighted integrand - a higher order approximation than that used here. This rule performs better than (7.93) for $(7.98)_1$ since the integrand is well behaved and its magnitude decreases as its derivatives increase. In contrast, the product-trapezoidal rule is inferior to (7.93) for $(7.98)_2$. This integrand is not well behaved and its integral is better suited to (7.93) which was developed for a more general class of function.

n	Equation $(7.98)_1$		Equation $(7.98)_2$	
	Relative Error	Error Ratio	Relative Error	Error Ratio
2	7.12(-3)		4.29(-1)	
4	1.98(-3)	3.60	8.08(-2)	5.30
8	5.17(-4)	3.83	1.90(-2)	4.25
16	1.32(-4)	3.92	4.74(-3)	4.01
32	3.33(-5)	3.96	1.18(-3)	4.00
64	8.35(-6)	3.98	2.96(-4)	4.00

TABLE 7.4. The relative error in evaluating (7.98) using the product trapezoidal rule [**2**, p. 310], where n is the number of intervals.

We note that the product-trapezoidal rule employs a uniform mesh and the behaviour of the weight function, $w(t)$, is accounted for in the quadrature rule weight. Rules of this type were explored in Subsection 7.3.4, where a one-point, second order product rule was developed. In this subsection, we showed that, for the log weight, employing a non-uniform mesh, similar to (7.97) increases accuracy by a factor of more than two for $f'' \in L_\infty[a,b]$.

Finally, we note that the rule developed here is composite in nature and identifies an "optimal" partition for an arbitrary weight. This contrasts with Gauss quadrature [**43**] which is not composite and hence provides no information as to how one should partition.

7.5.2. Application of Grüss Type Inequalities. Grüss' inequality [**25**] provides a bound for the difference between an integral of a product and a product of integrals, viz.

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)\,dx - \frac{1}{(b-a)^2} \int_a^b f(x)\,dx \int_a^b g(x)\,dx \right| \le \frac{1}{4}(\Gamma-\gamma)(\Phi-\phi), \tag{7.100}$$

where f, g are such that the integrals above exist and $\gamma \le f(x) \le \Gamma$, $\phi \le g(x) \le \Phi$.

The inequality (7.100) has been used with much success with Peano-kernel inspired applications. For example, the product integrand of (7.10) is an ideal candidate for (7.100). In addition, if one of the functions in (7.100) is explicitly known (as is the case in (7.10)) then bound may be improved [**27, 14**].

To this end, define

$$T(f,g) = \frac{1}{b-a} \int_a^b f(x)g(x)\,dx - \frac{1}{(b-a)^2} \int_a^b f(x)\,dx \int_a^b g(x)\,dx. \tag{7.101}$$

Using Korkine's identity, it has been shown that

$$|T(f,g)| \le T^{1/2}(f,f)T^{1/2}(g,g), \tag{7.102}$$

and

$$T(f,f) \le \left(\Gamma - \frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b f(x)\,dx - \gamma\right) \tag{7.103}$$

$$\le \frac{1}{4}(\Gamma-\gamma)^2. \tag{7.104}$$

Thus equations (7.102)–(7.104) provide us with the means of writing down many Grüss type inequalities; each depending on the level to which the integrands are known.

THEOREM 7.28. *Let f and w be as defined previously and let $[a,b]$ be a finite interval. The following three point Grüss type inequalities for weighted integrals hold*

$$\left|\int_a^b w(t)f(t)\,dt - m(a,b)f(x) + \left(\frac{f(b)-f(a)}{b-a}\right)m(a,b)(x-\mu(a,b))\right|$$

$$\le \frac{1}{2}(\Gamma-\gamma)\sqrt{b-a}\left\{\int_a^b K^2(x,t)\,dt - \frac{m^2(a,b)(x-\mu(a,b))^2}{b-a}\right\}^{1/2} \tag{7.105}$$

$$\le \frac{1}{2}(\Gamma-\gamma)(b-a)\left(m(a,x) - \frac{m(a,b)}{b-a}(x-\mu(a,b))\right)^{1/2} \tag{7.106}$$

$$\times\left(\frac{m(a,b)}{b-a}(x-\mu(a,b)) + m(x,b)\right)^{1/2} \tag{7.107}$$

$$\le \frac{1}{4}(\Gamma-\gamma)(b-a)m(a,b), \tag{7.108}$$

where $a \le x \le b$, $\gamma \le f'(t) \le \Gamma$ and $K(\cdot,\cdot)$ is the kernel defined in (7.9).

PROOF. The left hand side of each inequality is simply $|T\big(f'(\cdot),K(x,\cdot)\big)|$. Using the fact that $K(x,t)$ is bounded by

$$-m(x,b) \le K(x,t) \le m(a,x), \qquad t \in [a,b],$$

for fixed x and

$$\int_a^b K(x,t)\,dt = m(a,b)(x-\mu(a,b))$$

we can bound $|T\big(f'(\cdot),K(x,\cdot)\big)|$ by applying (7.102)–(7.104). The results are

$$|T(f'(\cdot), K(x,\cdot))| \le \frac{1}{2}(\Gamma-\gamma)T^{1/2}(K(x,\cdot),K(x,\cdot)) \tag{7.109}$$

$$\le \frac{1}{2}(\Gamma-\gamma)\left\{m(a,x)-\frac{1}{b-a}\int_a^b K(x,t)\,dt\right\}^{1/2} \tag{7.110}$$

$$\times\left\{\frac{1}{b-a}\int_a^b K(x,t)\,dt+m(x,b)\right\}^{1/2} \tag{7.111}$$

$$\le \frac{1}{2}(\Gamma-\gamma)m(a,b). \tag{7.112}$$

Simplifying readily produces (7.102)–(7.105). ∎

It is of interest to compare the left hand side of the inequalities in Theorems 7.7 and 7.28. The derivative $f'(x)$ in Theorem 7.7 is replaced with the secant slope $\frac{f(b)-f(a)}{b-a}$ in Theorem 7.28.

Corollary 7.29. The following *mean point* weighted integral inequality holds

$$\left|\int_a^b w(t)f(t)\,dt - m(a,b)f(\mu(a,b))\right| \le \frac{1}{2}(\Gamma-\gamma)\sqrt{b-a}\left\{\int_a^b K^2(\mu(a,b),t)\,dt\right\}^{1/2} \tag{7.113}$$

$$\le \frac{1}{2}(\Gamma-\gamma)(b-a)\sqrt{m(a,\mu(a,b))m(\mu(a,b),b)} \tag{7.114}$$

$$\le \frac{1}{4}(\Gamma-\gamma)(b-a)m(a,b). \tag{7.115}$$

Proof. Substitute $x=\mu(a,b)$ into the inequalities of Theorem 7.28. ∎

The mean $x=\mu(a,b)$ greatly simplifies the inequalities of Theorem 7.28, but this point does not necessarily minimise any of the upper bounds. This is done in the following subsection for particular weights.

7.5.3. Grüss-type Inequalities for Some Weight Functions.

The inequalities in Theorem 7.28 become more coarse as they become simpler to evaluate. Even so, (7.105) can be evaluated for many of the popular weight functions. In the following we tabulate the inequality (7.105) for the some weight functions. The minimum point is also identified.

7.5.3.1. *Legendre.* Substituting $w(t)=1$ into (7.105) gives

$$\left|\int_0^1 f(t)\,dt-(b-a)f(x)+\big(f(b)-f(a)\big)\left(x-\frac{a+b}{2}\right)\right| \le \frac{1}{4\sqrt{3}}(\Gamma-\gamma)(b-a)^2. \tag{7.116}$$

The above inequality is valid for all $x\in[a,b]$.

7.5.3.2. *Logarithm.* Substituting $w(t) = \ln(1/t)$, $a = 0$, $b = 1$ gives

$$\left|\int_0^1 f(t)\ln(1/t)\,dt - f(x) + \big(f(1) - f(0)\big)\left(x - \frac{1}{4}\right)\right| \le \frac{1}{72}(\Gamma - \gamma)\sqrt{87 - 648x + 648x^2 - 1296x^2\ln(x)}. \tag{7.117}$$

The bound is minimized at $x = 0.1161013$.

7.5.3.3. *Jacobi.* With the Jacobi weight, we have

$$\left|\int_0^1 \frac{f(t)}{\sqrt{t}}\,dt - 2f(x) + 2\big(f(1) - f(0)\big)\left(x - \frac{1}{3}\right)\right| \le \frac{1}{6}(\Gamma - \gamma)\sqrt{2 - 12x - 36x^2 + 48x^{3/2}}. \tag{7.118}$$

The bound is minimized at $x = 0.04465820$.

7.5.3.4. *Chebyshev.* The Chebyshev weight gives

$$\left|\int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}}\,dt - \pi f(x) - \frac{\pi}{2}\big(f(1) - f(-1)\big)x\right| \le \frac{\sqrt{2}}{4}(\Gamma - \gamma)\sqrt{8\pi x\arcsin(x) + 8\pi\sqrt{1-x^2} - 16 - 2x^2\pi^2}. \tag{7.119}$$

The bound is minimized at the boundary points $x = \pm 1$.

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CHAPTER 8

Some Inequalities for Riemann-Stieltjes Integral

by

S.S. DRAGOMIR

ABSTRACT In this chapter we present some recent results of the author concerning certain inequalities of Trapezoid type, Ostrowski type and Grüss type for Riemann-Stieltjes integrals and their natural application to the problem of approximating the Riemann-Stieltjes integral.

8.1. Introduction

Let f and u denote real-valued functions defined on a closed interval $[a,b]$ of the real line. We shall suppose that both f and u are bounded on $[a,b]$; this standing hypothesis will not be repeated. A *partition* of $[a,b]$ is a finite collection of non-overlapping intervals whose union is $[a,b]$. Usually, we describe a partition I_n by specifying a finite set of real numbers $(x_0, x_1, ..., x_n)$ such that

$$a = x_0 < x_1 < ... < x_{n-1} < x_n = b,$$

and the subintervals occurring in the partition I_n are the intervals $[x_{k-1}, x_k]$, $k = 1, ..., n$.

DEFINITION 4. If I_n is a partition of $[a,b]$, then the *Riemann-Stieltjes sum* of f with respect to u, corresponding to $I_n = (x_0, x_1, ..., x_n)$ is a real number $\sigma(I_n; f, u)$ of the form

$$\sigma(I_n; f, u) = \sum_{k=1}^{n} f(\xi_k)\{u(x_k) - u(x_{k-1})\}. \tag{8.1}$$

Here we have selected numbers ξ_k satisfying

$$x_{k-1} \le \xi_k \le x_k \quad \text{for } k = 1, 2, ..., n.$$

DEFINITION 5. We say that f is *integrable* with respect to u on $[a,b]$ if there exists a real number I such that for every number $\varepsilon > 0$ there is a partition $I_{n(\varepsilon)}$ of $[a,b]$ such that, if I_n is any refinement of $I_{n(\varepsilon)}$ and $\sigma(I_n; f, u)$ is any Riemann-Stieltjes sum corresponding to I_n, then

$$|\sigma(I_n; f, u) - I| < \varepsilon. \tag{8.2}$$

In this case the number I is uniquely determined and is denoted by

$$I = \int_a^b f\,du = \int_a^b f(t)\,du(t)\,;$$

it is called the *Riemann-Stieltjes* integral of f with respect to u over $[a,b]$. We call the function f the *integrand* and u the *integrator*. Sometimes we say that f is $u-$*integrable* if f is integrable with respect to u.

For the fundamental properties of Riemann-Stieltjes integrals related to: the Cauchy criterion for integrability, the functional properties of the integral, the integration by parts formula, the modification of the integral, the existence of the integral, the evaluation of the integral (first mean value theorem and second mean value theorem) and other properties, we refer the reader to the classical book [**5**], by R. G. Bartle.

In this chapter we point out some recent results by the author concerning certain inequalities of Trapezoid type, Ostrowski type and Grüss type for Riemann-Stieltjes integrals in terms of certain Riemann-Stieltjes sums, generalised mid-point sums, generalised trapezoidal sums, etc...

For a comprehensive study of Newton-Cotes quadrature formulae for Riemann-Stieltjes integrals and their applications to numerical evaluations of life distributions, see the paper [**73**] by M. Tortorella and the references therein.

The chapter is structured as follows:

The first section deals with the estimation of the magnitude of the difference

$$\frac{f(a)+f(b)}{2}\left[u(b)-u(a)\right] - \int_a^b f(t)\,du(t)\,,$$

where f is of Hölder type and u is of bounded variation, and vice versa.

The second section provides an error analysis for the quantity

$$f(x)\left[u(b)-u(a)\right] - \int_a^b f(t)\,du(t)\,,\quad x \in [a,b]\,,$$

which is commonly known in the literature as an Ostrowski type inequality, for the same classes of mappings.

Finally, the last section deals with Grüss type inequalities for the Riemann-Stieltjes integrals, that is, obtaining bounds for the quantity

$$\int_a^b f(t)\,du(t) - \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt.$$

All the sections contain implementation for composite quadrature formulae. A large number of references to the recent papers done by the *Research Group in Mathematical Inequalities and Applications* (RGMIA, `http://rgmia.vu.edu.au`) are included (see for example [**1**]-[**4**], [**6**]-[**64**] and [**72**]).

8.2. Some Trapezoid Like Inequalities for Riemann-Stieltjes Integral

8.2.1. Introduction. The following inequality is well known in the literature as the *"trapezoid inequality"*:

$$\left|\frac{f(a)+f(b)}{2}\cdot(b-a)-\int_a^b f(t)\,dt\right|\leq\frac{1}{12}(b-a)^3\|f''\|_\infty, \tag{8.3}$$

where the mapping $f:[a,b]\to\mathbb{R}$ is assumed to be twice differentiable on (a,b), with its second derivative $f'':(a,b)\to\mathbb{R}$ bounded on (a,b), that is, $\|f''\|_\infty := \sup_{t\in(a,b)}|f''(t)|<\infty$. The constant $\frac{1}{12}$ is sharp in (8.3) in the sense that it cannot be replaced by a smaller constant.

If $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ is a division of the interval $[a,b]$ and $h_i = x_{i+1}-x_i$, $\nu(h) := \max\{h_i | i = 0,...,n-1\}$, then the following formula, which is called the *"trapezoid quadrature formula"*

$$T(f,I_n)=\sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}\cdot h_i \tag{8.4}$$

approximates the integral $\int_a^b f(t)\,dt$ with an error of approximation $R_T(f,I_n)$ which satisfies the estimate

$$|R_T(f,I_n)|\leq\frac{1}{12}\|f''\|_\infty\sum_{i=0}^{n-1}h_i^3\leq\frac{b-a}{12}\|f''\|_\infty[\nu(h)]^2. \tag{8.5}$$

In (8.5), the constant $\frac{1}{12}$ is sharp as well.

If the second derivative does not exist or f'' is unbounded on (a,b), then we cannot apply (8.5) to obtain a bound for the approximation error. It is important, therefore, that we consider the problem of estimating $R_T(f,I_n)$ in terms of lower derivatives.

Define the following functional associated to the trapezoid inequality

$$\Psi(f;a,b):=\frac{f(a)+f(b)}{2}\cdot(b-a)-\int_a^b f(t)\,dt \tag{8.6}$$

where $f:[a,b]\to\mathbb{R}$ is an integrable mapping on $[a,b]$.

The following result is known [**34**] (see also [**60**] or [**10**]):

THEOREM 8.1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping on $[a,b]$. Then*

$$|\Psi(f;a,b)|\leq\begin{cases}\frac{(b-a)^2}{4}\|f'\|_\infty & \text{if } f'\in L_\infty[a,b];\\[2ex] \frac{(b-a)^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\|f'\|_p & \text{if } f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \frac{b-a}{2}\|f'\|_1,\end{cases} \tag{8.7}$$

where $\|\cdot\|_p$ *are the usual* $p-$*norms, i.e.,*

$$\begin{aligned}\|f'\|_\infty &: = ess \sup_{t\in[a,b]} |f'(t)|, \\ \|f'\|_p &: = \left(\int_a^b |f'(t)|^p dt\right)^{\frac{1}{p}}, \ p>1\end{aligned}$$

and

$$\|f'\|_1 := \int_a^b |f'(t)| dt,$$

respectively.

The following corollary for composite formulae holds [**34**].

Corollary 8.2. Let f be as in Theorem 8.1. Then we have the quadrature formula

$$\int_a^b f(x) dx = T(f, I_n) + R_T(f, I_n), \tag{8.8}$$

where $T(f, I_n)$ is the trapezoid rule and the remainder $R_T(f, I_n)$ satisfies the estimation

$$|R_T(f, I_n)| \leq \begin{cases} \frac{1}{4}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2 & \text{if } \quad f' \in L_\infty[a,b]; \\ \\ \frac{1}{2(q+1)^{\frac{1}{q}}}\|f'\|_p \left(\sum_{i=0}^{n-1} h_i^{q+1}\right)^{\frac{1}{q}} & \text{if } \quad f' \in L_p[a,b], \\ & \quad\quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\|f'\|_1 \nu(h). & \end{cases} \tag{8.9}$$

A more general result concerning a trapezoid inequality for functions of bounded variation has been proved in the paper [**36**] (see also [**10**]).

Theorem 8.3. *Let* $f:[a,b]\to\mathbb{R}$ *be a mapping of bounded variation on* $[a,b]$ *and denote* $\bigvee_a^b(f)$ *as its total variation on* $[a,b]$. *Then we have the inequality*

$$|\Psi(f; a, b)| \leq \frac{1}{2}(b-a)\bigvee_a^b(f). \tag{8.10}$$

The constant $\frac{1}{2}$ *is sharp in the sense that it cannot be replaced by a smaller constant.*

The following corollary which provides an upper bound for the approximation error in the trapezoid quadrature formula, for f of bounded variation, holds [**36**].

Corollary 8.4. Assume that $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$. Then we have the quadrature formula (8.8) where the reminder satisfies the estimate

$$|R_T(f, I_n)| \leq \frac{1}{2}\nu(h)\bigvee_a^b(f). \tag{8.11}$$

The constant $\frac{1}{2}$ is sharp.

For other recent results on the trapezoid inequality see the books [**60**] and [**70**] where further references are given.

8.2.2. A Trapezoid Formula for the Riemann-Stieltjes Integral. The following theorem generalizing the classical trapezoid inequality for mappings of bounded variation holds [**57**].

THEOREM 8.5. *Let $f:[a,b]\to\mathbb{K}$ $(\mathbb{K}=\mathbb{R},\mathbb{C})$ be a $p-H-$Hölder type mapping, that is, it satisfies the condition*

$$|f(x)-f(y)|\leq H|x-y|^{p} \text{ for all } x,y\in[a,b], \tag{8.12}$$

where $H>0$ and $p\in(0,1]$ are given, and $u:[a,b]\to\mathbb{K}$ is a mapping of bounded variation on $[a,b]$. Then we have the inequality:

$$|\Psi(f,u;a,b)|\leq\frac{1}{2^{p}}H(b-a)^{p}\bigvee_{a}^{b}(u), \tag{8.13}$$

where $\Psi(f,u;a,b)$ is the generalized trapezoid functional

$$\Psi(f,u;a,b):=\frac{f(a)+f(b)}{2}\cdot(u(b)-u(a))-\int_{a}^{b}f(t)\,du(t). \tag{8.14}$$

The constant $C=1$ on the right hand side of (8.13) cannot be replaced by a smaller constant.

PROOF. It is well known that if $g:[a,b]\to\mathbb{K}$ is continuous and $v:[a,b]\to\mathbb{K}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b g(t)\,dv(t)$ exists and the following inequality holds:

$$\left|\int_{a}^{b}g(t)\,dv(t)\right|\leq\sup_{t\in[a,b]}|g(t)|\bigvee_{a}^{b}(v). \tag{8.15}$$

Using this property, we have

$$\begin{aligned}
&|\Psi(f,u;a,b)| \\
=&\left|\int_{a}^{b}\left(\frac{f(a)+f(b)}{2}-f(t)\right)du(t)\right| \\
\leq&\sup_{t\in[a,b]}\left|\frac{f(a)+f(b)}{2}-f(t)\right|\bigvee_{a}^{b}u(t).
\end{aligned} \tag{8.16}$$

As f is of $p-H-$Hölder type, then we have

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-f(t)\right| &= \left|\frac{f(a)-f(t)+f(b)-f(t)}{2}\right| \\
&\leq \frac{1}{2}|f(a)-f(t)|+\frac{1}{2}|(b)-f(t)| \\
&\leq \frac{1}{2}H\left[(t-a)^{p}+(b-t)^{p}\right].
\end{aligned} \tag{8.17}$$

Now, consider the mapping $\gamma(t)=(t-a)^{p}+(b-t)^{p}$, $t\in[a,b]$, $p\in(0,1]$. Then $\gamma'(t)=p(t-a)^{p-1}-p(b-t)^{t-1}=0$ iff $t=\frac{a+b}{2}$ and $\gamma'(t)>0$ on $\left(a,\frac{a+b}{2}\right)$, $\gamma'(t)<0$ on $\left(\frac{a+b}{2},b\right)$, which shows that its maximum is realized at $t=\frac{a+b}{2}$ and $\max_{t\in[a,b]}\gamma(t)=\gamma\left(\frac{a+b}{2}\right)=2^{1-p}(b-a)^{p}$.

Consequently, by (8.17), we have

$$\sup_{t\in[a,b]}\left|\frac{f(a)+f(b)}{2}-f(t)\right|\leq H\left(\frac{b-a}{2}\right)^{p}.$$

Using (8.16), we obtain the desired inequality (8.13).

To prove the sharpness of the constant 1, assume that (8.13) holds with a constant $C>0$. That is

$$|\Psi(f,u;a,b)|\leq\frac{C}{2^{p}}H(b-a)^{p}\bigvee_{a}^{b}(u). \tag{8.18}$$

Choose $f,u:[0,1]\to\mathbb{R}$, $f(x)=x^{p}$, $p\in(0,1]$ and $u(x)=x,\ x\in[0,1]$. We observe that f is of $p-H-$Hölder type with $H=1$ and u is of bounded variation, then, by (8.18) we can obtain

$$\left|\frac{1}{2}-\frac{1}{p+1}\right|\leq\frac{C}{2^{p}},\ \text{for all }\ p\in(0,1].$$

That is,

$$C\geq\frac{1-p}{p+1}\cdot 2^{p},\ \text{ for all }\ p\in(0,1].$$

Letting $p\to 0+$, we get $C\geq 1$ and the theorem is completely proved. ∎

The following corollaries are natural consequences of (8.13):

Corollary 8.6. Let f be as above and $u:[a,b]\to\mathbb{R}$ be a monotonic mapping on $[a,b]$. Then we have

$$|\Psi(f,u;a,b)|\leq\frac{1}{2^{p}}H(b-a)^{p}|u(b)-u(a)|. \tag{8.19}$$

The proof is obvious by the above theorem, taking into account the fact that monotonic mappings are of bounded variation and for such functions u, we have $\bigvee_{a}^{b}(u)=|u(b)-u(a)|$.

Corollary 8.7. Let f be as above and $u:[a,b]\to\mathbb{K}$ be a Lipschitzian mapping with the constant $L>0$. Then

$$|\Psi(f,u;a,b)|\leq\frac{1}{2^{p}}HL(b-a)^{p+1}. \tag{8.20}$$

The proof follows by Theorem 8.5, taking into account that any Lipschitzian mapping u is of bounded variation and $\bigvee_{a}^{b}(u)\leq L(b-a)$.

Corollary 8.8. Let f be as above and $G:[a,b]\to\mathbb{R}$ be the cumulative distribution function of a certain random variable X. Then

$$\left|\frac{f(a)+f(b)}{2}-\int_{a}^{b}f(t)\,dG(t)\right|\leq\frac{1}{2^{p}}H(b-a)^{p}. \tag{8.21}$$

The proof is obvious by the above theorem, taking into account the fact that $G(b)=1$, $G(a)=0$ and $\bigvee_{a}^{b}(G)=1$.

Remark 8.1. If we assume that $g : [a,b]\,((a,b)) \to \mathbb{K}$ is continuous, then $u(x) = \int_a^x g(t)\,dt$ is differentiable, $u(b) = \int_a^b g(t)\,dt$, $u(a) = 0$, and $\bigvee_a^b(u) = \int_a^b |g(t)|\,dt$. Consequently, by (8.13), we obtain

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} \cdot \int_a^b g(t)\,dt - \int_a^b f(t)\,g(t)\,dt \right| \\ \leq\ & \frac{1}{2^p} H (b-a)^p \int_a^b |g(t)|\,dt. \end{aligned} \tag{8.22}$$

From (8.22), we get a weighted version of the trapezoid inequality,

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{\int_a^b g(t)\,dt} \cdot \int_a^b f(t)\,g(t)\,dt \right| \leq \frac{1}{2^p} H (b-a)^p, \tag{8.23}$$

provided that $g(t) \geq 0,\ t \in [a,b]$ and $\int_a^b g(t)\,dt \neq 0$.

We give now some examples of weighted trapezoid inequalities for some of the most popular weights.

Example 1. (**Legendre**) If $g(t) = 1$, $t \in [a,b]$, then we get the following trapezoid inequality for Hölder type mappings :

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \frac{1}{2^p} H (b-a)^p. \tag{8.24}$$

Example 2. (**Logarithm**) If $g(t) = \ln\left(\frac{1}{t}\right)$, $t \in (0,1]$, f is of $p-$Hölder type on $[0,1]$ and the integral $\int_0^1 f(t) \ln\left(\frac{1}{t}\right) dt$ is finite, then, by (8.23), we obtain

$$\left| \frac{f(0)+f(1)}{2} - \int_0^1 f(t) \ln\left(\frac{1}{t}\right) dt \right| \leq \frac{1}{2^p} H. \tag{8.25}$$

Example 3. **(Jacobi)** If $g(t) = \frac{1}{\sqrt{t}}, t \in (0,1]$, f is as above and the integral $\int_0^1 \frac{f(t)}{\sqrt{t}} dt$ is finite, then by (8.23), we obtain

$$\left| \frac{f(0)+f(1)}{2} - \frac{1}{2} \int_0^1 \frac{f(t)}{\sqrt{t}} dt \right| \leq \frac{1}{2^p} H. \tag{8.26}$$

Finally, we have the following:

Example 4. (**Chebychev**) If $g(t) = \frac{1}{\sqrt{1-t^2}}$, $t \in (-1,1)$, f is of $p-$Hölder type on $(-1,-1)$ and the integral $\int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}} dt$ is finite, then

$$\left| \frac{f(-1)+f(1)}{2} - \frac{1}{\pi} \int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}} dt \right| \leq H. \tag{8.27}$$

8.2.3. Approximation of the Riemann-Stieltjes Integral. Consider the partition $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ of the interval $[a,b]$, and define $h_i := x_{i+1} - x_i$ $(i = 0, ..., n-1)$, $\nu(h) := \max\{h_i | i = 0, ..., n-1\}$ and the sum

$$T_n(f,u,I_n) := \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \times [u(x_{i+1}) - u(x_i)]. \tag{8.28}$$

The following approximation of the Riemann-Stieltjes integral in terms of $T_n(f,u,I_n)$ holds [**57**].

THEOREM 8.9. *Let* $f:[a,b]\to\mathbb{K}$ *be a* $p-H-$*Hölder type mapping on* $[a,b]$ $(p\in(0,1])$ *and* $u:[a,b]\to\mathbb{K}$ *be a function of bounded variation on* $[a,b]$. *Then*

$$\int_a^b f(t)\,du(t) = T_n(f,u,I_n) + R_n(f,u,I_n), \tag{8.29}$$

where $T_n(f,u,I_n)$ *is the generalized trapezoidal formula given by* (8.28), *and the remainder* $R(f,u,I_n)$ *satisfies the estimate*

$$|R_n(f,u,I_n)| \le \frac{1}{2^p} H[\nu(h)]^p \bigvee_a^b(u). \tag{8.30}$$

PROOF. We apply Theorem 8.5 on every subinterval $[x_i,x_{i+1}]$ $(i=0,...,n-1)$ to obtain

$$\left|\frac{f(x_i)+f(x_{i+1})}{2}\times[u(x_{i+1})-u(x_i)] - \int_{x_i}^{x_{i+1}} f(t)\,du(t)\right| \le \frac{1}{2^p} H h_i^p \bigvee_{x_i}^{x_{i+1}}(u). \tag{8.31}$$

Summing the inequalities (8.31) over i from 0 to $n-1$ and using the generalized triangle inequality, we obtain

$$\begin{aligned}
&|R(f,u,I_n,\xi)| \\
\le\ & \sum_{i=0}^{n-1}\left|\frac{f(x_i)+f(x_{i+1})}{2}\times[u(x_{i+1})-u(x_i)] - \int_{x_i}^{x_{i+1}} f(t)\,du(t)\right| \\
\le\ & \frac{1}{2^p} H \sum_{i=0}^{n-1} h_i^p \bigvee_{x_i}^{x_{i+1}}(u) \le \frac{1}{2^p} H[\nu(h)]^p \sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(u) \\
=\ & \frac{1}{2^p} H[\nu(h)]^p \bigvee_a^b(u),
\end{aligned}$$

and the bound (8.30) is proved. ∎

The following corollaries have useful applications.

COROLLARY 8.10. Let f be as in Theorem 8.9 and $u:[a,b]\to\mathbb{R}$ be a monotonic mapping on $[a,b]$. Then we have the formula (8.29) and the remainder term $R_n(f,u,I_n)$ satisfies the estimate

$$|R_n(f,u,I_n)| \le \frac{1}{2^p} H[\nu(h)]^p |u(b)-u(a)|. \tag{8.32}$$

Using Corollary 8.7, we also point out the following result.

Corollary 8.11. Let f be as in Theorem 8.9 and $u:[a,b]\to\mathbb{K}$ be a Lipschitzian mapping with the constant $L>0$. Then we have the formula (8.29), for which the remainder will satisfy the bound:

$$|R_n(f,u,I_n)|\le\frac{1}{2^p}HL\sum_{i=0}^{n-1}h_i^{p+1}\le\frac{1}{2^p}HL(b-a)[\nu(h)]^p. \tag{8.33}$$

We now point out some quadrature formulae of trapezoid type for weighted integrals.

Let us assume that $g:[a,b]\to\mathbb{K}$ is continuous and $f:[a,b]\to\mathbb{K}$ is of $r-H-$Hölder type on $[a,b]$. For a given partition I_n of the interval $[a,b]$, consider the sum

$$T_n(f,g,I_n):=\sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}\times\int_{x_i}^{x_{i+1}}g(s)\,ds. \tag{8.34}$$

We can state the following corollary.

Corollary 8.12. Let $f:[a,b]\to\mathbb{K}$ be of $r-H-$Hölder type and $g:[a,b]\to\mathbb{K}$ be continuous on $[a,b]$. Then we have the formula

$$\int_a^b g(t)f(t)\,dt=T_n(f,g,I_n)+R_n(f,g,I_n), \tag{8.35}$$

where the remainder term $R_n(f,g,I_n)$ satisfies the estimate

$$|R_n(f,g,I_n)|\le\frac{1}{2^p}H[\nu(h)]^p\int_a^b|g(s)|\,ds. \tag{8.36}$$

Proof. Apply the inequality (8.22) on the intervals $[x_i,x_{i+1}]$ $(i=0,...,n-1)$ to obtain

$$\begin{aligned}&\left|\frac{f(x_i)+f(x_{i+1})}{2}\times\int_{x_i}^{x_{i+1}}g(s)\,ds-\int_{x_i}^{x_{i+1}}f(t)g(t)\,dt\right|\\ \le\ &\frac{1}{2^p}Hh_i^p\times\int_{x_i}^{x_{i+1}}|g(s)|\,ds.\end{aligned}$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality, we can state

$$\begin{aligned}&|R_n(f,g,I_n)|\\ \le\ &\sum_{i=0}^{n-1}\left|\frac{f(x_i)+f(x_{i+1})}{2}\times\int_{x_i}^{x_{i+1}}g(s)\,ds-\int_{x_i}^{x_{i+1}}f(t)g(t)\,dt\right|\\ \le\ &\frac{1}{2^p}H\sum_{i=0}^{n-1}h_i^p\times\int_{x_i}^{x_{i+1}}|g(s)|\,ds\le\frac{1}{2^p}H[\nu(h)]^p\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|g(s)|\,ds\\ =\ &\frac{1}{2^p}H[\nu(h)]^p\int_a^b|g(s)|\,ds,\end{aligned}$$

and the corollary is proved. ∎

The previous corollary allows us to obtain adaptive quadrature formulae for different weighted integrals. We point out only a few examples.

EXAMPLE 5. (**Legendre**) If $g(t) = 1$, and $t \in [a,b]$, then we get the trapezoid formula for the mapping $f : [a,b] \to \mathbb{K}$ of $p - H-$Hölder type:

$$\int_a^b f(t)\,dt = T(f, I_n) + R(f, I_n), \tag{8.37}$$

where $T(f, I_n)$ is the usual trapezoid rule

$$T(f, I_n) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \cdot h_i \tag{8.38}$$

and the remainder satisfies the estimate

$$|R(f, I_n)| \le \frac{1}{2^p} H(b-a)[\nu(h)]^p. \tag{8.39}$$

EXAMPLE 6. (**Logarithm**) If $g(t) = \ln\left(\frac{1}{t}\right)$, $t \in [a,b] \subset [0,1]$, f is of $p - H-$Hölder type and the integral $\int_a^b f(t) \ln\left(\frac{1}{t}\right) dt < \infty$, then we have the generalized trapezoid formula:

$$\int_a^b f(t) \ln\left(\frac{1}{t}\right) dt = T_L(f, I_n) + R_L(f, I_n), \tag{8.40}$$

where $T_L(f, I_n)$ is the following "Logarithm-Trapezoid" quadrature rule

$$T_L(f, I_n) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \times \left[x_i \ln\left(\frac{x_i}{e}\right) - x_{i+1} \ln\left(\frac{x_{i+1}}{e}\right)\right] \tag{8.41}$$

and the remainder term $R_L(f, I_n)$ satisfies the estimate

$$|R_L(f, I_n)| \le \frac{1}{2^p} H[\nu(h)]^p \left[a \ln\left(\frac{a}{e}\right) - b \ln\left(\frac{b}{e}\right)\right]. \tag{8.42}$$

EXAMPLE 7. (**Jacobi**) If $g(t) = \frac{1}{\sqrt{t}}$, $t \in [a,b] \subset (0, \infty)$, f is of $p - H-$Hölder type and $\int_a^b \frac{f(t)}{\sqrt{t}} dt < \infty$, then we have the generalized trapezoid formula

$$\int_a^b \frac{f(t)}{\sqrt{t}} dt = T_J(f, I_n) + R_J(f, I_n), \tag{8.43}$$

where $T_J(f, I_n)$ is the "Jacobi-Trapezoid" quadrature rule

$$T_J(f, I_n) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \times \left[\frac{1}{2}\left(\sqrt{x_{i+1}} - \sqrt{x_i}\right)\right] \tag{8.44}$$

and the remainder term $R_J(f, I_n)$ satisfies the estimate

$$|R_J(f, I_n)| \le \frac{1}{2^{p+1}} H[\nu(h)]^p \left(\sqrt{b} - \sqrt{a}\right). \tag{8.45}$$

Finally, we have

EXAMPLE 8. (**Chebychev**) If $g(t) = \frac{1}{\sqrt{1-t^2}}$, $t \in [a,b] \subset (-1,1)$, f is of $p - H-$Hölder type and $\int_a^b \frac{f(t)}{\sqrt{1-t^2}} dt < \infty$, then we have the generalized trapezoid formula

$$\int_a^b \frac{f(t)}{\sqrt{1-t^2}} dt = T_c(f, I_n) + R_c(f, I_n) \tag{8.46}$$

where $T_c(f, I_n)$ is the "Chebychev-Trapezoid" quadrature rule

$$T_c(f, I_n) := \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \times [\arcsin(x_{i+1}) - \arcsin(x_i)] \tag{8.47}$$

and the remainder term $R_c(f, I_n)$ satisfies the estimate

$$|R_c(f, I_n)| \leq \frac{1}{2^p} H [\nu(h)]^p [\arcsin(b) - \arcsin(a)]. \tag{8.48}$$

8.2.4. Another Trapezoid Like Inequality. The following theorem which complement in a sense the previous result also holds [**35**].

THEOREM 8.13. *Let $f : [a, b] \to \mathbb{K}$ be a mapping of bounded variation on $[a, b]$ and $u : [a, b] \to \mathbb{K}$ be a $p - H-$Hölder type mapping, that is, it satisfies the condition:*

$$|u(x) - u(y)| \leq H |x - y|^p \text{ for all } x, y \in [a, b], \tag{8.49}$$

where $H > 0$ and $p \in (0, 1]$ are given. Then we have the inequality:

$$|\Psi(f, u; a, b)| \leq \frac{1}{2^p} H (b - a)^p \bigvee_a^b (f). \tag{8.50}$$

The constant $C = 1$ on the right hand side of (8.50) *cannot be replaced by a smaller constant.*

PROOF. It is well known that if $g : [a, b] \to \mathbb{K}$ is continuous and $v : [a, b] \to \mathbb{K}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b g(t)\, dv(t)$ exists and the following inequality holds:

$$\left| \int_a^b g(t)\, dv(t) \right| \leq \sup_{t \in [a,b]} |g(t)| \bigvee_a^b (v). \tag{8.51}$$

Using the integration by parts formula for Riemann-Stieltjes integrals, we have

$$\begin{aligned} & \int_a^b \left[u(x) - \frac{u(a) + u(b)}{2} \right] df(x) \\ = & \left[u(x) - \frac{u(a) + u(b)}{2} \right] f(x) \Big|_a^b - \int_a^b f(x)\, du(x) \\ = & \frac{f(a) + f(b)}{2} [u(b) - u(a)] - \int_a^b f(x)\, du(x). \end{aligned}$$

Consequently, if u and f are as above, then we have the equality:

$$\begin{aligned} & \int_a^b f(x)\, du(x) \\ = & \frac{f(a) + f(b)}{2} [u(b) - u(a)] - \int_a^b \left[u(x) - \frac{u(a) + u(b)}{2} \right] df(x) \end{aligned} \tag{8.52}$$

which is of importance in itself too.

Now, applying the inequality (8.51) for $g(t) = u(t) - \frac{u(a)+u(b)}{2}$ and $v(t) = f(t)$, $t \in [a,b]$, we obtain

$$|\Psi(f,u;a,b)| \leq \sup_{t\in[a,b]} \left| u(t) - \frac{u(a)+u(b)}{2} \right| \bigvee_a^b (f). \tag{8.53}$$

As u is of $p-H-$Hölder type, we have

$$\begin{aligned} \left| u(t) - \frac{u(a)+u(b)}{2} \right| &= \left| \frac{u(t)-u(a)+u(t)-u(b)}{2} \right| \\ &\leq \frac{1}{2}|u(t)-u(a)| + \frac{1}{2}|u(t)-u(b)| \\ &\leq \frac{1}{2}H\left[(t-a)^p + (b-t)^p\right]. \end{aligned}$$

Now, consider the mapping $\varphi(t) := (t-a)^p + (b-t)^p$, $t \in [a,b]$, $p \in (0,1]$. It is easy to see that its maximum is realized for $t = \frac{a+b}{2}$ and $\max_{t\in[a,b]} \varphi(t) = \varphi\left(\frac{a+b}{2}\right) = 2^{1-p}(b-a)^p$.

Consequently, we have

$$\sup_{t\in[a,b]} \left| u(t) - \frac{u(a)+u(b)}{2} \right| \leq H\left(\frac{b-a}{2}\right)^p.$$

Using (8.53), we deduce the desired inequality (8.50).

To prove the sharpness of the constant 1, assume that (8.50) holds with a constant $C > 0$, i.e.,

$$|\Psi(f,u;a,b)| \leq \frac{C}{2^p} H(b-a)^p \bigvee_a^b (f). \tag{8.54}$$

Choose $u(x) = x^p$, $p \in (0,1]$, $x \in [0,1]$ which is of $p-$Hölder type with the constant $H = 1$ and $f : [0,1] \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in [0,1), \\ 1 & \text{if} \quad x = 1 \end{cases}$$

which is of bounded variation on $[0,1]$.

Substituting in (8.54), we obtain

$$\left| \frac{1}{2} - \int_0^1 p t^{p-1} f(t)\, dt \right| \leq \frac{C}{2^p}(b-a) \bigvee_0^1 (f).$$

However,

$$\int_0^1 t^{p-1} f(t)\, dt = 0 \text{ and } \bigvee_0^1 (f) = 1,$$

and then $C \geq 2^{p-1}$ for all $p \in (0,1]$. Choosing $p = 1$, we deduce $C \geq 1$ and the theorem is completely proved. ∎

The following corollary is a natural consequence of the above Theorem 8.13.

Corollary 8.14. Let $f:[a,b]\to\mathbb{K}$ be as in Theorem 8.13 and u be an $L-$Lipschitzian mapping on $[a,b]$, that is,

$$|u(t)-u(s)|\leq L|t-s| \text{ for all } t,s\in[a,b], \tag{8.55}$$

where $L>0$ is fixed. Then we have the inequality

$$|\Psi(f,u;a,b)|\leq\frac{L}{2}(b-a)\bigvee_a^b(f). \tag{8.56}$$

Remark 8.2. If $f:[a,b]\to\mathbb{R}$ is monotonic and u is of $p-H-$Hölder type, then the inequality (8.50) becomes:

$$|\Psi(f,u;a,b)|\leq\frac{1}{2^p}H(b-a)|f(b)-f(a)|. \tag{8.57}$$

In addition, if u is $L-$Lipschitzian, then the inequality (8.56) can be replaced by

$$|\Psi(f,u;a,b)|\leq\frac{L}{2}(b-a)|f(b)-f(a)|. \tag{8.58}$$

Remark 8.3. If f is Lipschitzian with a constant $K>0$, then it is obvious that f is of bounded variation on $[a,b]$ and $\bigvee_a^b(f)\leq K(b-a)$. Consequently, the inequality (8.50) becomes

$$|\Psi(f,u;a,b)|\leq\frac{1}{2^p}HK(b-a)^{p+1}, \tag{8.59}$$

and the inequality (8.56) becomes

$$|\Psi(f,u;a,b)|\leq\frac{LK}{2}(b-a)^2. \tag{8.60}$$

We now point out some results in estimating the integral of a product.

Corollary 8.15. Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and g be continuous on $[a,b]$. Put $\|g\|_\infty:=\sup_{t\in[a,b]}|g(t)|$. Then we have the inequality:

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(s)\,ds-\int_a^b f(t)g(t)\,dt\right|\leq\frac{\|g\|_\infty}{2}(b-a)\bigvee_a^b(f). \tag{8.61}$$

Remark 8.4. Now, if in the above corollary we assume that f is monotonic, then (8.61) becomes

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\int_a^b g(s)\,ds-\int_a^b f(t)g(t)\,dt\right|\\ \leq\;&\frac{\|g\|_\infty|f(b)-f(a)|(b-a)}{2},\end{aligned} \tag{8.62}$$

and if in Corollary 8.15 we assume that f is $K-$Lipschitzian, then the inequality (8.61) becomes

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(s)\,ds-\int_a^b f(t)g(t)\,dt\right|\leq\frac{\|g\|_\infty K(b-a)^2}{2}. \tag{8.63}$$

The following corollary is also a natural consequence of Theorem 8.13.

Corollary 8.16. Let f and g be as in Corollary 8.15. Put

$$\|g\|_p := \left(\int_a^b |g(s)|^p \, ds\right)^{\frac{1}{p}} ; p > 1.$$

Then we have the inequality

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(s)\,ds - \int_a^b f(t)\,g(t)\,dt\right| \leq \frac{1}{2^{\frac{p-1}{p}}}\|g\|_p (b-a)^{\frac{p-1}{p}} \bigvee_a^b (f). \tag{8.64}$$

Proof. Consider the mapping u given by $u(t) := \int_a^t g(s)\,ds$. Then, by Hölder's integral inequality, we can state that

$$\begin{aligned} |u(t)-u(s)| &= \left|\int_s^t g(z)\,dz\right| \leq |t-s|^{\frac{1}{q}} \left|\int_s^t |g(z)|^p\,dz\right|^{\frac{1}{p}} \\ &\leq |t-s|^{\frac{p-1}{p}} \|g\|_p , \end{aligned}$$

for all $t, s \in [a,b]$, which shows that the mapping u is of $r-H-$Hölder type with $r := \frac{p-1}{p} \in (0,1)$ and $H = \|g\|_p < \infty$.
The mapping u is also differentiable on (a,b) and $u'(t) = g(t)$, $t \in (a,b)$. Therefore, by the theory of Riemann-Stieltjes integrals, we have

$$\int_a^b f(t)\,du(t) = \int_a^b f(t)\,g(t)\,dt.$$

Using (8.50), we deduce the desired inequality (8.64). ∎

We give now some examples of weighted trapezoid inequalities for some of the most popular weights.

Example 9. (**Legendre**). If $g(t) = 1, t \in [a,b]$ then by (8.61) and (8.64) we get the trapezoid inequalities

$$|\Psi(f;a,b)| \leq \frac{1}{2}(b-a)\bigvee_a^b (f) \tag{8.65}$$

and

$$|\Psi(f;a,b)| \leq \frac{1}{2^{1-1/p}}(b-a)\bigvee_a^b (f), p > 1. \tag{8.66}$$

We remark that the first inequality is better than the second one.

Example 10. (**Jacobi**). If $g(t) = \frac{1}{\sqrt{t}}, t \in (0,1]$, then obviously $\|g\|_\infty = +\infty$, so we cannot apply the inequality (8.61). If we assume that $p \in (1,2)$ then we have

$$\|g\|_p = \left[\int_0^1 \left(\frac{1}{\sqrt{t}}\right)^p dt\right]^{1/p} = \left(\frac{2}{2-p}\right)^{1/p}$$

and applying the inequality (8.64) we deduce

$$\left|\frac{f(0)+f(1)}{2}-\frac{1}{2}\int_0^1 \frac{1}{\sqrt{t}}f(t)dt\right| \leq \frac{1}{4^{(p-1)/p}}\cdot\frac{1}{(2-p)^{1/p}}\bigvee_0^1(f), \tag{8.67}$$

for all $p\in(1,2)$.

Example 11. (**Chebychev**). If $g(t)=\frac{1}{\sqrt{1-t^2}}, t\in(-1,1)$, then obviously $\|g\|_\infty = +\infty$, so we cannot apply the inequality (8.61). If we assume that $p\in(1,2)$ then we have

$$\begin{aligned} \|g\|_p &= \left[\int_{-1}^1\left(\frac{1}{\sqrt{1-t^2}}\right)^p dt\right]^{1/p} \\ &= \left[\int_{-1}^1 (t+1)^{\frac{2-p}{2}-1}(1-t)^{\frac{2-p}{2}-1}dt\right]^{1/p} \\ &= 2^{2-1/p}\left[B\left(\frac{2-p}{2},\frac{2-p}{2}\right)\right]^{1/p}. \end{aligned}$$

Applying the inequality (8.64) we deduce

$$\begin{aligned} &\left|\frac{f(-1)+f(1)}{2}-\frac{1}{\pi}\int_{-1}^1\frac{f(t)}{\sqrt{1-t^2}}dt\right| \\ &\leq \frac{2}{\pi}\left[B\left(\frac{2-p}{2},\frac{2-p}{2}\right)\right]^{1/p}\bigvee_0^1(f) \end{aligned} \tag{8.68}$$

for all $p\in(1,2)$.

8.2.5. Approximation of the Riemann-Stieltjes Integral.

Consider the partition $I_n : a=x_0<x_1<...<x_{n-1}<x_n=b$ of the interval $[a,b]$ and define $h_i := x_{i+1}-x_i \ (i=0,...,n-1)$, $\nu(h) := \max\{h_i | i\in\{0,...,n-1\}\}$ and the sum

$$T_n(f,u,I_n) := \sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}\times[u(x_{i+1})-u(x_i)]. \tag{8.69}$$

The following approximation of the Riemann-Stieltjes integral also holds [**35**] (c.f. Theorem 8.9).

Theorem 8.17. *Let $f:[a,b]\to\mathbb{K}$ be a mapping of bounded variation on $[a,b]$ and $u:[a,b]\to\mathbb{K}$ be a $p-H-$Hölder type mapping. Then we have the quadrature formula*

$$\int_a^b f(t)\,du(t) = T_n(f,u,I_n)+R_n(f,u,I_n), \tag{8.70}$$

where $T_n(f,u,I_n)$ is the generalized trapezoid formula given by (8.69) and the remainder $R_n(f,u,I_n)$ satisfies the estimate

$$|R_n(f,u,I_n)| \leq \frac{1}{2^p}H[\nu(h)]^p\bigvee_a^b(f). \tag{8.71}$$

Proof. We apply Theorem 8.13 on every subinterval $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ to obtain

$$\begin{aligned} (8.72) \qquad & \left| \frac{f(x_i) + f(x_{i+1})}{2} \cdot [u(x_{i+1}) - u(x_i)] - \int_{x_i}^{x_{i+1}} f(t)\, du(t) \right| \\ \leq\ & \frac{1}{2^p} H\, h_i^p \bigvee_{x_i}^{x_{i+1}} (f). \end{aligned}$$

Summing the inequalities (8.72) over i from 0 to $n-1$ and using the generalized triangle inequality, we obtain

$$\begin{aligned} & |R_n(f, u, I_n)| \\ \leq\ & \sum_{i=0}^{n-1} \left| \frac{f(x_i) + f(x_{i+1})}{2} [u(x_{i+1}) - u(x_i)] - \int_{x_i}^{x_{i+1}} f(t)\, du(t) \right| \\ \leq\ & \frac{1}{2^p} H \sum_{i=0}^{n-1} h_i^p \bigvee_{x_i}^{x_{i+1}} (f) \leq \frac{1}{2^p} H [\nu(h)]^p \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (f) \\ =\ & \frac{1}{2^p} H [\nu(h)]^p \bigvee_a^b (f) \end{aligned}$$

and the theorem is proved. ∎

Remark 8.5. Some particular results as in Corollaries 8.10-8.12 and Examples 5-8 can be stated as well, but we omit the details.

8.2.6. A Generalisation of the Trapezoid Inequality. The following theorem holds (see [**49**]).

Theorem 8.18. *Let $u : [a, b] \to \mathbb{R}$ be of $H - r$-Hölder type, i.e., we recall this*

$$(8.73) \qquad |u(x) - u(y)| \leq H|x - y|^r, \text{ for any } x, y \in [a, b] \text{ and some } H > 0,$$

where $r \in (0, 1]$ is given, and $f : [a, b] \to \mathbb{R}$ is of bounded variation.
Then we have the inequality:

$$\begin{aligned} (8.74) \qquad & \left| \int_a^b f(t)du(t) - [(u(b) - u(x))f(b) + (u(x) - u(a))f(a)] \right| \\ \leq\ & H \left[\frac{1}{2}(b - a) + \left| x - \frac{a+b}{2} \right| \right]^r \bigvee_a^b (f) \leq H(b - a)^r \bigvee_a^b (f) \end{aligned}$$

for any $x \in [a, b]$.
The constant $\frac{1}{2}$ is sharp in the sense that we cannot put a smaller constant instead.

Proof. Using the integration by parts formula, we may state:

$$\begin{aligned} (8.75) \qquad & \int_a^b (u(t) - u(x))df(t) \\ =\ & [u(b) - u(x)]f(b) - [u(a) - u(x)]f(a) - \int_a^b f(t)du(t). \end{aligned}$$

It is well known that if $m : [a, b] \to \mathbb{R}$ is continuous and $n : [a, b] \to \mathbb{R}$ is of bounded variation, the Riemann-Stieltjes integral $\int_a^b m(t)dn(t)$ exists, and

$$\left| \int_a^b m(t)dn(t) \right| \le \sup_{t\in[a,b]} |m(t)| \cdot \bigvee_a^b (n).$$

Thus,

$$\begin{aligned} & \left| \int_a^b (u(t) - u(x))df(t) \right| \\ \le \; & \sup_{t\in[a,b]} |u(t) - u(x)| \bigvee_a^b (f) \le H \sup_{t\in[a,b]} \{|t - x|^r\} \bigvee_a^b (f) \\ = \; & H \max\{|b - x|^r, |x - a|^r\} \bigvee_a^b (f) = H[\max(b - x, x - a)]^r \bigvee_a^b (f) \\ = \; & H \left[\frac{1}{2}(b - a) + \left| x - \frac{a + b}{2} \right| \right]^r \bigvee_a^b (f). \end{aligned}$$

Finally, as

$$\left| x - \frac{a + b}{2} \right| \le \frac{1}{2}(b - a) \text{ for any } x \in [a, b]$$

we get the last inequality in (8.74).

To prove the sharpness of the constant $\frac{1}{2}$, we assume that (8.74) holds with the constant $c > 0$, i.e.,

$$\begin{aligned} & \left| \int_a^b f(t)du(t) - [(u(b) - u(x))f(b) + (u(x) - u(a))f(a)] \right| \\ \le \; & H \left[c(b - a) + \left| x - \frac{a + b}{2} \right| \right]^r \bigvee_a^b (f). \end{aligned} \tag{8.76}$$

Choose $u(t) = t$ which is of $(1 - 1)$-Hölder type and $f : [a, b] \to \mathbb{R}$, $f(t) = 0$ if $t \in \{a, b\}$ and $f(t) = 1$ if $t \in (a, b)$, which is of bounded variation, in (8.76).

We get:

$$|b - a| \le 2 \left[c(b - a) + \left| x - \frac{a + b}{2} \right| \right], \text{ for any } x \in [a, b].$$

For $x = \frac{a+b}{2}$, we get:

$$|b - a| \le 2c(b - a), \text{ i.e. } c \ge \frac{1}{2}.$$

∎

Remark 8.6. If u is Lipschitz continuous function, i.e.

$$|u(x) - u(y)| \le L|x - y| \text{ for any } x, y \in [a, b], (\text{ and some } L > 0),$$

the inequality (8.74) becomes:

$$\left|\int_a^b f(t)du(t) - [(u(b)-u(x))f(b) + (u(x)-u(a))f(a)]\right| \tag{8.77}$$

$$\leq L \cdot \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \cdot \bigvee_a^b(f) \leq L(b-a)\bigvee_a^b(f).$$

Corollary 8.19. If f is of bounded variation on $[a,b]$ and u is absolutely continuous with $u' \in L_\infty[a,b]$ then instead of L in (8.77) we can put

$$||u'||_\infty = ess \sup_{t\in[a,b]} |u'(t)|.$$

Corollary 8.20. If $g : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ and if we choose $u(t) = \int_a^t g(s)ds$, then

$$\left|\int_a^b f(t)g(t)dt - f(b)\int_x^b g(s)ds - f(a)\int_a^x g(s)ds\right| \tag{8.78}$$

$$\leq \|g\|_\infty \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b(f) \leq \|g\|_\infty (b-a)\bigvee_a^b(f).$$

Remark 8.7. If in (8.78) we choose $x = \frac{a+b}{2}$, we get the best inequality in the class, i.e.,

$$\left|\int_a^b f(t)g(t)dt - f(b)\int_{\frac{a+b}{2}}^b g(s)ds - f(a)\int_a^{\frac{a+b}{2}} g(s)ds\right| \tag{8.79}$$

$$\leq \frac{1}{2}\|g\|_\infty (b-a)\bigvee_a^b(f).$$

8.2.7. Approximating the Riemann-Stieltjes Integral. Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ a division of $[a,b]$. Denote $h_i := x_{i+1} - x_i$, and $\nu(I_n) = \sup_{i=\overline{0,n-1}} h_i$ then construct the sums

$$S(f,u,I_n,\boldsymbol{\xi}) = \sum_{i=0}^{n-1}[u(x_{i+1}) - u(\xi_i)]f(x_{i+1}) + \sum_{i=0}^{n-1}[u(\xi_i) - u(x_i)]f(x_i), \tag{8.80}$$

where $\xi_i \in [x_i, x_{i+1}], i = \overline{0, n-1}$ and $\boldsymbol{\xi} = (\xi_0, \xi_1, \cdots \xi_{n-1})$.

We can state the following theorem concerning the approximation of Riemann-Stieltjes integral [**49**]:

Theorem 8.21. *Let f, u be as in Theorem 8.18 and $I_n, \boldsymbol{\xi}$ as defined above. Then:*

$$\int_a^b f(t)du(t) = S(f,u,I_n,\boldsymbol{\xi}) + R(f,u,I_n,\boldsymbol{\xi}) \tag{8.81}$$

when $S(f,u,I_n,\boldsymbol{\xi})$ *is defined by (8.80) and the remainder* $R(f,u,I_n,\boldsymbol{\xi})$ *satisfies the estimate:*

$$(8.82)\quad \begin{aligned} |R(f,u,I_n,\boldsymbol{\xi})| &\le H\cdot\left[\frac{1}{2}\nu(I_n)+\sup_{i=\overline{0,n-1}}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^r\bigvee_a^b(f)\\ &\le H\cdot\nu^r(I_n)\bigvee_a^b(f). \end{aligned}$$

Proof. We apply (8.74) on $[x_i,x_{i+1}]$ to get:

$$\begin{aligned} &\left|\int_{x_i}^{x_{i+1}} f(t)du(t)-[u(x_{i+1})-u(\xi_i)]f(x_{i+1})-[u(\xi_i)-u(x_i)]f(x_i)\right|\\ \le\ & H\cdot\left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^r\bigvee_{x_i}^{x_{i+1}}(f)\le H\cdot h_i^r\bigvee_{x_i}^{x_{i+1}}(f). \end{aligned}$$

Summing on i from 0 to $n-1$, and using the generalised triangle inequality we get:

$$\begin{aligned} &\left|\int_a^b f(t)du(t)-S(f,u,I_n,\boldsymbol{\xi})\right|\\ \le\ & H\cdot\sum_{i=0}^{n-1}\left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^r\cdot\bigvee_{x_i}^{x_{i+1}}(f)\\ \le\ & H\sup_{i=\overline{0,n-1}}\left[\frac{1}{2}h_i+\left|\boldsymbol{\xi}-\frac{x_i+x_{i+1}}{2}\right|\right]^r\bigvee_a^b(f)\\ \le\ & H\left[\frac{1}{2}\nu(I_n)+\sup_{i=\overline{0,n-1}}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^r\bigvee_a^b(f)\\ \le\ & H\nu^r(I_n)\bigvee_a^b(f), \end{aligned}$$

and the theorem is proved. ∎

Remark 8.8. It is obvious that if $\nu(I_n)\to 0$ then (8.81) provides an approximation for the Riemann-Stieltjes integral $\int_a^b f(t)du(t)$.

Corollary 8.22. If we consider the sum

$$\begin{aligned} &S_M(f,u,I_n)\\ =\ &\sum_{i=0}^{n-1}\left[u(x_{i+1})-u\left(\frac{x_i+x_{i+1}}{2}\right)\right]f(x_{i+1})+\sum_{i=0}^{n-1}\left[u\left(\frac{x_i+x_{i+1}}{2}\right)-u(x_i)\right]f(x_i) \end{aligned}$$

then:

$$(8.83)\quad \int_a^b f(t)du(t)=S_M(f,u,I_n)+R_M(f,u,I_n)$$

and the remainder $R_M(f,u,I_n)$ satisfies the estimate

$$(8.84)\quad |R_M(f,u,I_n)|\le\frac{1}{2^r}H\nu^r(I_n)\bigvee_a^b(f).$$

The following corollary in approximating the integral $\int_a^b f(t)g(t)dt$ holds.

COROLLARY 8.23. If f, g are as in Corollary 8.20, then

$$\int_a^b f(t)g(t)dt = P(f,g,I_n,\boldsymbol{\xi}) + R_P(f,g,I_n,\boldsymbol{\xi})$$

where

$$P(f,g,I_n,\boldsymbol{\xi}) = \sum_{i=0}^{n-1} f(x_{i+1}) \int_{\xi_i}^{x_{i+1}} g(s)ds + \sum_{i=0}^{n-1} f(x_i) \int_{x_i}^{\xi_i} g(s)ds.$$

and the remainder $R_P(f,g,I_n,\boldsymbol{\xi})$ satisfies the estimate:

$$\begin{aligned} |R_P(f,g,I_n,\boldsymbol{\xi})| &\le \|g\|_\infty \left[\frac{1}{2}\nu(I_n) + \sup_{i=\overline{0,n-1}} \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_a^b (f) \\ &\le \|g\|_\infty \nu(I_n) \bigvee_a^b (f). \end{aligned}$$

REMARK 8.9. If in the above corollary we choose $\xi_i = \frac{x_i + x_{i+1}}{2}$ $\left(i = \overline{0, n-1}\right)$ then we get the best formula in the class, i.e.,

$$P_M(f,g,I_n,\boldsymbol{\xi}) = \sum_{i=0}^{n-1} f(x_{i+1}) \int_{\frac{x_i+x_{i+1}}{2}}^{x_{i+1}} g(s)ds + \sum_{i=0}^{n-1} f(x_i) \int_{x_i}^{\frac{x_i+x_{i+1}}{2}} g(s)ds$$

and

$$R_{P_M}(f,g,I_n,\boldsymbol{\xi}) \le \frac{1}{2} \|g\|_\infty \nu(I_n) \bigvee_a^b (f).$$

8.3. Inequalities of Ostrowski Type for the Riemann-Stieltjes Integral

8.3.1. Introduction.

In 1938, A. Ostrowski proved the following integral inequality (see for example [**70**, p. 468])

THEOREM 8.24. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b), with its first derivative $f' : (a,b) \to \mathbb{R}$ bounded on (a,b), that is, $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a), \tag{8.85}$$

for all $x \in [a,b]$.
The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

For a different proof than the original one provided by Ostrowski in 1938 as well as applications for special means (identric mean, logarithmic mean, $p-$logarithmic mean, etc.) and in *Numerical Analysis* for quadrature formulae of Riemann type, see the recent paper [**66**].

In [**64**], the following version of Ostrowski's inequality for the 1-norm of the first derivatives has been given.

Theorem 8.25. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b), with its first derivative $f':(a,b)\to\mathbb{R}$ integrable on (a,b), that is, $\|f'\|_1 := \int_a^b |f'(t)|\,dt<\infty$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\|f'\|_1, \tag{8.86}$$

for all $x\in[a,b]$.
The constant $\frac{1}{2}$ is sharp.

Note that the sharpness of the constant $\frac{1}{2}$ in the class of differentiable mappings whose derivatives are integrable on (a,b) has been shown in the paper [**71**].

In [**64**], the authors applied (8.86) for special means and for quadrature formulae of Riemann type.

The following natural extension of Theorem 8.25 has been pointed out by S.S. Dragomir in [**37**].

Theorem 8.26. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $\bigvee_a^b(f)$ its total variation on $[a,b]$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\bigvee_a^b(f), \tag{8.87}$$

for all $x\in[a,b]$. The constant $\frac{1}{2}$ is sharp.

In [**37**], the author applied (8.87) for quadrature formulae of Riemann type as well as for Euler's Beta mapping.

In what follows we point out some generalizations of (8.87) for the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ where f is of Hölder type and u is of bounded variation. Applications to the problem of approximating the Riemann-Stieltjes integral in terms of Riemann-Stieltjes sums are also given.

8.3.2. Some Integral Inequalities.

The following theorem holds [**24**].

Theorem 8.27. *Let $f:[a,b]\to\mathbb{R}$ be a $p-H-$Hölder type mapping, that is, it satisfies the condition*

$$|f(x)-f(y)|\le H|x-y|^p,\ \text{for all } x,y\in[a,b]; \tag{8.88}$$

where $H>0$ and $p\in(0,1]$ are given, and $u:[a,b]\to\mathbb{R}$ is a mapping of bounded variation on $[a,b]$. Then we have the inequality

$$\begin{aligned} &\left|f(x)(u(b)-u(a))-\int_a^b f(t)\,du(t)\right| \\ \le\ & H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^p\bigvee_a^b(u), \end{aligned} \tag{8.89}$$

for all $x \in [a,b]$, *where* $\bigvee_a^b (u)$ *denotes the total variation of* u *on* $[a,b]$. *Furthermore, the constant* $\frac{1}{2}$ *is the best possible, for all* $p \in (0,1]$.

Proof. It is well known that if $g : [a,b] \to \mathbb{R}$ is continuous and $v : [a,b] \to \mathbb{R}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b g(t)\,dv(t)$ exists and the following inequality holds:

$$\left|\int_a^b g(t)\,dv(t)\right| \le \sup_{t\in[a,b]} |g(t)| \bigvee_a^b (v). \tag{8.90}$$

Using this property, we have

$$\begin{aligned} & \left|f(x)(u(b)-u(a)) - \int_a^b f(t)\,du(t)\right| \\ = & \left|\int_a^b (f(x)-f(t))\,du(t)\right| \\ \le & \sup_{t\in[a,b]} |f(x)-f(t)| \bigvee_a^b (u). \end{aligned} \tag{8.91}$$

As f is of $p-H-$Hölder type, we have

$$\begin{aligned} \sup_{t\in[a,b]} |f(x)-g(t)| &\le \sup_{t\in[a,b]} \left[H|x-t|^p\right] \\ &= H\max\{(x-a)^p, (b-x)^p\} \\ &= H[\max\{x-a, b-x\}]^p \\ &= H\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^p. \end{aligned}$$

Using (8.91), we deduce (8.89).
To prove the sharpness of the constant $\frac{1}{2}$ for any $p \in (0,1]$, assume that (8.89) holds with a constant $C > 0$, that is,

$$\begin{aligned} & \left|f(x)(u(b)-u(a)) - \int_a^b f(t)\,du(t)\right| \\ \le & H\left[C(b-a) + \left|x - \frac{a+b}{2}\right|\right]^p \bigvee_a^b (u), \end{aligned} \tag{8.92}$$

for all f, $p-H-$Hölder type mappings on $[a,b]$ and u of bounded variation on the same interval.
Choose $f(x) = x^p$ $(p \in (0,1])$, $x \in [0,1]$ and $u : [0,1] \to [0,\infty)$ given by

$$u(x) = \begin{cases} 0 \text{ if } x \in [0,1) \\ 1 \text{ if } x = 1 \end{cases}.$$

As

$$|f(x) - f(y)| = |x^p - y^p| \le |x-y|^p$$

for all $x, y \in [0,1]$, $p \in (0,1]$, it follows that f is of $p-H-$Hölder type with the constant 1.

By using the integration by parts formula for Riemann-Stieltjes integrals, we have:

$$\begin{aligned}\int_0^1 f(t)\,du(t) &= f(t)\,u(t)]_0^1 - \int_0^1 u(t)\,df(t)\\ &= 1-0=1\end{aligned}$$

and

$$\bigvee_0^1 (u) = 1.$$

Consequently, by (8.92), we get

$$|x^p - 1| \le \left[C + \left|x - \frac{1}{2}\right|\right]^p, \text{ for all } x \in [0,1].$$

For $x = 0$, we get $1 \le \left(C + \frac{1}{2}\right)^p$, which implies that $C \ge \frac{1}{2}$, and the theorem is completely proved. ∎

The following corollaries are natural.

Corollary 8.28. Let u be as in Theorem 8.27 and $f : [a,b] \to \mathbb{R}$ an $L-$Lipschitzian mapping on $[a,b]$, that is,

$$\text{(L)} \qquad |f(t) - f(s)| \le L\,|t-s| \text{ for all } t,s \in [a,b]$$

where $L > 0$ is fixed.
Then, for all $x \in [a,b]$, we have the inequality

$$\begin{aligned}&|\Theta(f,x;a,b)|\\ &\le L\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]\bigvee_a^b (u)\end{aligned} \tag{8.93}$$

where

$$\Theta(f,u;x,a,b) = f(x)\,(u(b)-u(a)) - \int_a^b f(t)\,du(t)$$

is the Ostrowski's functional associated to f and u as above. The constant $\frac{1}{2}$ is the best possible.

Remark 8.10. If u is monotonic on $[a,b]$ and f is of $p-H-$Hölder type, then, by (8.89) we get

$$\begin{aligned}&|\Theta(f,u;x,a,b)|\\ &\le H\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^p |u(b)-u(a)|,\ x\in[a,b],\end{aligned} \tag{8.94}$$

and if we assume that f is $L-$Lipschitzian, then (8.93) becomes

$$\begin{aligned}&|\Theta(f,u;x,a,b)|\\ &\le L\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] |u(b)-u(a)|,\ x\in[a,b].\end{aligned} \tag{8.95}$$

Remark 8.11. If u is $K-$Lipschitzian, then obviously u is of bounded variation over $[a,b]$ and $\bigvee_a^b(u) \leq L(b-a)$. Consequently, if f is of $p-H-$Hölder type, then

$$\begin{aligned} &|\Theta(f,u;x,a,b)| \\ \leq\ & HK\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^p(b-a),\ x\in[a,b] \end{aligned} \tag{8.96}$$

and it f is $L-$Lipschitzian, then

$$\begin{aligned} &|\Theta(f,u;x,a,b)| \\ \leq\ & LK\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right](b-a),\ x\in[a,b]. \end{aligned} \tag{8.97}$$

The following corollary concerning a generalization of the mid-point inequality holds:

Corollary 8.29. Let f and u be as defined in Theorem 8.27. Then we have the generalized mid-point formula

$$|\Upsilon(f,u;x,a,b)| \leq \frac{H}{2^p}(b-a)^p\bigvee_a^b(u), \tag{8.98}$$

where

$$\Upsilon(f,u;x,a,b)=f\left(\frac{a+b}{2}\right)(u(b)-u(a))-\int_a^b f(t)\,du(t)$$

is the mid point functional associated to f and u as above. In particular, if f is $L-$Lipschitzian, then

$$|\Upsilon(f,u;x,a,b)| \leq \frac{L}{2}(b-a)\bigvee_a^b(u). \tag{8.99}$$

Remark 8.12. Now, if in (8.98) and (8.99) we assume that u is monotonic, then we get the midpoint inequalities

$$|\Upsilon(f,u;x,a,b)| \leq \frac{H}{2^p}(b-a)^p|u(b)-u(a)| \tag{8.100}$$

and

$$|\Upsilon(f,u;x,a,b)| \leq \frac{L}{2}(b-a)|u(b)-u(a)| \tag{8.101}$$

respectively.
In addition, if in (8.98) and (8.99) we assume that u is $K-$Lipschitzian, then we obtain the inequalities

$$|\Upsilon(f,u;x,a,b)| \leq \frac{HK}{2^p}(b-a)^{p+1} \tag{8.102}$$

and

$$|\Upsilon(f,u;x,a,b)| \leq \frac{LK}{2}(b-a)^2. \tag{8.103}$$

The following inequalities of "rectangle type" also hold:

Corollary 8.30. Let f and u be as in Theorem 8.27. Then we have the generalized "left rectangle" inequality

$$\left| f(a)(u(b)-u(a)) - \int_a^b f(t)\,du(t) \right| \leq H(b-a)^p \bigvee_a^b (u) \tag{8.104}$$

and the "right rectangle" inequality

$$\left| f(b)(u(b)-u(a)) - \int_a^b f(t)\,du(t) \right| \leq H(b-a)^p \bigvee_a^b (u). \tag{8.105}$$

Remark 8.13. If we add the inequalities (8.104) and (8.105), then, by using the triangle inequality, we end up with the following generalized trapezoidal inequality

$$\left| \frac{f(a)+f(b)}{2}(u(b)-u(a)) - \int_a^b f(t)\,du(t) \right| \leq H(b-a)^p \bigvee_a^b (u). \tag{8.106}$$

In what follows, we point out some results for the Riemann integral of a product.

Corollary 8.31. Let $f:[a,b]\to\mathbb{R}$ be a $p-H-$Hölder type mapping and $g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. Then we have the inequality

$$\begin{aligned} &\left| f(x)\int_a^b g(s)\,ds - \int_a^b f(t)g(t)\,dt \right| \\ \leq\quad & H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^p \int_a^b |g(s)|\,ds \end{aligned} \tag{8.107}$$

for all $x\in[a,b]$.

Proof. Define the mapping $u:[a,b]\to\mathbb{R}$, $u(t)=\int_a^t g(s)\,ds$. Then u is differentiable on (a,b) and $u'(t)=g(t)$. Using the theory of the Riemann-Stieltjes integral, we have

$$\int_a^b f(t)\,du(t) = \int_a^b f(t)g(t)\,dt$$

and

$$\bigvee_a^b (u) = \int_a^b |u'(t)|\,dt = \int_a^b |g(t)|\,dt.$$

Now, from (8.89), we deduce (8.107). ∎

Remark 8.14. The best inequality we can get from (8.107) is that one for which $x=\frac{a+b}{2}$, obtaining the midpoint inequality

$$\left| f\left(\frac{a+b}{2}\right)\int_a^b g(s)\,ds - \int_a^b f(t)g(t)\,dt \right| \leq \frac{1}{2^p}H(b-a)^p \int_a^b |g(s)|\,ds. \tag{8.108}$$

We give some examples of weighted Ostrowski inequalities for some of the most popular weights.

EXAMPLE 12. (**Legendre**) If $g(t) = 1$, and $t \in [a,b]$, then we get the following Ostrowski inequality for Hölder type mappings $f : [a,b] \to \mathbb{R}$

$$\left|(b-a) f(x) - \int_a^b f(t)\,dt\right| \le H \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^p (b-a) \tag{8.109}$$

for all $x \in [a,b]$, and, in particular, the mid-point inequality

$$\left|(b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\,dt\right| \le \frac{1}{2^p} H (b-a)^{p+1}. \tag{8.110}$$

EXAMPLE 13. (**Logarithm**) If $g(t) = \ln\left(\frac{1}{t}\right)$, $t \in (0,1]$, f is of $p-$Hölder type on $[0,1]$ and the integral $\int_0^1 f(t) \ln\left(\frac{1}{t}\right) dt$ is finite, then we have

$$\left|f(x) - \int_0^1 f(t) \ln\left(\frac{1}{t}\right) dt\right| \le H \left[\frac{1}{2} + \left|x - \frac{1}{2}\right|\right]^p \tag{8.111}$$

for all $x \in [0,1]$ and, in particular,

$$\left|f\left(\frac{1}{2}\right) - \int_0^1 f(t) \ln\left(\frac{1}{t}\right) dt\right| \le \frac{1}{2^p} H. \tag{8.112}$$

EXAMPLE 14. (**Jacobi**) If $g(t) = \frac{1}{\sqrt{t}}, t \in (0,1]$, f is as above and the integral $\int_0^1 \frac{f(t)}{\sqrt{t}} dt$ is finite, then we have

$$\left|f(x) - \frac{1}{2}\int_0^1 \frac{f(t)}{\sqrt{t}} dt\right| \le H \left[\frac{1}{2} + \left|x - \frac{1}{2}\right|\right]^p \tag{8.113}$$

for all $x \in [0,1]$ and, in particular,

$$\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\int_0^1 \frac{f(t)}{\sqrt{t}} dt\right| \le \frac{1}{2^p} H. \tag{8.114}$$

Finally, we have the following:

EXAMPLE 15. (**Chebychev**) If $g(t) = \frac{1}{\sqrt{1-t^2}}$, $t \in (-1,1)$, f is of $p-$Hölder type on $(-1,-1)$ and the integral $\int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}} dt$ is finite, then

$$\left|f(x) - \frac{1}{\pi}\int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}} dt\right| \le H [1 + |x|]^p \tag{8.115}$$

for all $x \in [-1,1]$, and in particular,

$$\left|f(0) - \frac{1}{\pi}\int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}} dt\right| \le H. \tag{8.116}$$

8.3.3. Approximation of the Riemann-Stieltjes Integral. Consider $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ to be a division of the interval $[a,b]$, $h_i := x_{i+1} - x_i$ $(i = 0, ..., n-1)$ and $\nu(h) := \max\{h_i | i = 0, ..., n-1\}$. Define the general Riemann-Stieltjes sum

$$S(f, u, I_n, \xi) := \sum_{i=0}^{n-1} f(\xi_i)(u(x_{i+1}) - u(x_i)). \tag{8.117}$$

In what follows, we point out some upper bounds for the error approximation of the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by its Riemann-Stieltjes sum $S(f,u,I_n,\xi)$ [**24**].

THEOREM 8.32. *Let $u:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ a $p-H-$Hölder type mapping. Then*

$$\int_a^b f(t)\,du(t)=S(f,u,I_n,\xi)+R(f,u,I_n,\xi), \tag{8.118}$$

where $S(f,u,I_n,\xi)$ is as given in (8.117) and the remainder $R(f,u,I_n,\xi)$ satisfies the bound

$$\begin{aligned}|R(f,u,I_n,\xi)| &\le H\left[\frac{1}{2}\nu(h)+\max_{i=\overline{0,n-1}}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^p\bigvee_a^b(u)\\ &\le H[\nu(h)]^p\bigvee_a^b(u).\end{aligned} \tag{8.119}$$

PROOF. We apply Theorem 8.27 on the subintervals $[x_i,x_{i+1}]$ $(i=0,...,n-1)$ to obtain

$$\begin{aligned}&\left|f(\xi_i)(u(x_{i+1})-u(x_i))-\int_{x_i}^{x_{i+1}}f(t)\,du(t)\right|\\ &\le H\left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^p\bigvee_{x_i}^{x_{i+1}}(u),\end{aligned} \tag{8.120}$$

for all $i\in\{0,...,n-1\}$.
Summing over i from 0 to $n-1$ and using the generalized triangle inequality, we deduce

$$\begin{aligned}|R(f,u,I_n,\xi)| &\le \sum_{i=0}^{n-1}\left|f(\xi_i)(u(x_{i+1})-u(x_i))-\int_{x_i}^{x_{i+1}}f(t)\,du(t)\right|\\ &\le H\sum_{i=0}^{n-1}\left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^p\bigvee_{x_i}^{x_{i+1}}(u)\\ &\le H\sup_{i=\overline{0,n-1}}\left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^p\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(u).\end{aligned}$$

However,

$$\sup_{i=\overline{0,n-1}}\left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^p\le\left[\frac{1}{2}\nu(h)+\sup\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^p$$

and

$$\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(u)=\bigvee_a^b(u),$$

which completely proves the first inequality in (8.119).
For the second inequality, we observe that

$$\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\le\frac{1}{2}\cdot h_i,$$

for all $i \in \{0, ..., n-1\}$.
The theorem is thus proved. ∎

The following corollaries are natural.

Corollary 8.33. Let u be as in Theorem 8.32 and f an $L-$Lipschitzian mapping. Then we have the formula (8.118) and the remainder $R(f, u, I_n, \xi)$ satisfies the bound

$$\begin{aligned} |R(f,u,I_n,\xi)| &\leq L\left[\frac{1}{2}\nu(h) + \max_{i=\overline{0,n-1}}\left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right]\bigvee_a^b(u) \\ &\leq H\nu(h)\bigvee_a^b(u). \end{aligned} \tag{8.121}$$

Remark 8.15. If u is monotonic on $[a,b]$, then the error estimate (8.119) becomes

$$\begin{aligned} &|R(f,u,I_n,\xi)| \\ &\leq H\left[\frac{1}{2}\nu(h) + \max_{i=\overline{0,n-1}}\left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right]^p |u(b)-u(a)| \\ &\leq H[\nu(h)]^p |u(b)-u(a)| \end{aligned} \tag{8.122}$$

and (8.119) becomes

$$\begin{aligned} &|R(f,u,I_n,\xi)| \\ &\leq L\left[\frac{1}{2}\nu(h) + \max_{i=\overline{0,n-1}}\left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right] |u(b)-u(a)| \\ &\leq L\nu(h)|u(b)-u(a)|. \end{aligned} \tag{8.123}$$

Using Remark 8.11, we can state the following corollary.

Corollary 8.34. If $u:[a,b]\to\mathbb{R}$ is Lipschitzian with the constant K and $f:[a,b]\to\mathbb{R}$ is of $p-H-$Hölder type, then the formula (8.118) holds and the remainder $R(f,u,I_n,\xi)$ satisfies the bound

$$\begin{aligned} |R(f,u,I_n,\xi)| &\leq HK\sum_{i=0}^{n-1}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right]^p h_i \\ &\leq HK\sum_{i=0}^{n-1} h_i^{p+1} \leq HK(b-a)[\nu(h)]^p. \end{aligned} \tag{8.124}$$

In particular, if we assume that f is $L-$Lipschitzian, then

$$\begin{aligned} |R(f,u,I_n,\xi)| &\leq \frac{1}{2}LK\sum_{i=0}^{n-1} h_i^2 + LK\sum_{i=0}^{n-1}\left|\xi_i - \frac{x_i+x_{i+1}}{2}\right| h_i \\ &\leq LK\sum_{i=0}^{n-1} h_i^2 \leq LK(b-a)\nu(h). \end{aligned} \tag{8.125}$$

The best quadrature formula we can get from Theorem 8.32 is that one for which $\xi_i = \frac{x_i+x_{i+1}}{2}$ for all $i \in \{0, ..., n-1\}$. Consequently, we can state the following corollary.

COROLLARY 8.35. Let f and u be as in Theorem 8.32. Then

$$\int_a^b f(t)\,du(t) = S_M(f,u,I_n) + R_M(f,u,I_n) \tag{8.126}$$

where $S_M(f,u,I_n)$ is the generalized midpoint formula, that is;

$$S_M(f,u,I_n) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)\left(u(x_{i+1}) - u(x_i)\right)$$

and the remainder satisfies the estimate

$$|R_M(f,u,I_n)| \le \frac{H}{2^p}[\nu(h)]^p \bigvee_a^b (u). \tag{8.127}$$

In particular, if f is $L-$Lipschitzian, then we have the bound:

$$|R_M(f,u,I_n)| \le \frac{H}{2}\nu(h) \bigvee_a^b (u). \tag{8.128}$$

REMARK 8.16. If in (8.127) and (8.128) we assume that u is monotonic, then we get the inequalities

$$|R_M(f,u,I_n)| \le \frac{H}{2^p}[\nu(h)]^p |f(b) - f(a)| \tag{8.129}$$

and

$$|R_M(f,u,I_n)| \le \frac{H}{2}\nu(h) |f(b) - f(a)|. \tag{8.130}$$

The case where f is $K-$Lipschitzian is embodied in the following corollary.

COROLLARY 8.36. Let u and f be as in Corollary 8.34. Then we have the quadrature formula (8.126) and the remainder satisfies the estimate

$$|R_M(f,u,I_n)| \le \frac{HK}{2^p}\sum_{i=0}^{n-1} h_i^{p+1} \le \frac{HK}{2^p}[\nu(h)]^p. \tag{8.131}$$

In particular, if f is $L-$Lipschitzian, then we have the estimate

$$|R_M(f,u,I_n)| \le \frac{1}{2}LK\sum_{i=0}^{n-1} h_i^2 \le \frac{1}{2}LK(b-a)\nu(h). \tag{8.132}$$

8.3.4. Another Inequality of Ostrowski Type for the Riemann-Stieltjes Integral. The following result holds [**26**]:

THEOREM 8.37. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $u : [a,b] \to \mathbb{R}$ be of $r - H-$Hölder type. Then for all $x \in [a,b]$, we have the* *inequality*

$$
\begin{aligned}
(8.133)\qquad & \left|\Theta\left(f,u;x,a,b\right)\right| \\
\leq\; & H\left[\left(x-a\right)^{r}\bigvee_{a}^{x}\left(f\right)+\left(b-x\right)^{r}\bigvee_{x}^{b}\left(f\right)\right] \\
\leq\; & \begin{cases}
H\left[\left(x-a\right)^{r}+\left(b-x\right)^{r}\right]\left[\frac{1}{2}\bigvee_{a}^{b}\left(f\right)+\frac{1}{2}\left|\bigvee_{a}^{x}\left(f\right)-\bigvee_{x}^{b}\left(f\right)\right|\right]; \\
\\
H\left[\left(x-a\right)^{qr}+\left(b-x\right)^{qr}\right]^{\frac{1}{q}}\left[\left(\bigvee_{a}^{x}\left(f\right)\right)^{p}+\left(\bigvee_{x}^{b}\left(f\right)\right)^{p}\right]^{\frac{1}{p}} \\
\text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\\
H\left[\frac{1}{2}\left(b-a\right)+\left|x-\frac{a+b}{2}\right|\right]^{r}\bigvee_{a}^{b}\left(f\right)
\end{cases}
\end{aligned}
$$

where

$$\Theta\left(f,u;x,a,b\right)=\left(u\left(b\right)-u\left(a\right)\right)f\left(x\right)-\int_{a}^{b}f\left(t\right)du\left(t\right)$$

is the Ostrowski's functional associated to f and u as above.

Proof. As u is continuous and f is of bounded variation on $[a,b]$, the following Riemann-Stieltjes integrals exist and, by the integration by parts formula, we can state that

$$\int_{a}^{x}\left(u\left(t\right)-u\left(a\right)\right)df\left(t\right)=\left(u\left(x\right)-u\left(a\right)\right)f\left(x\right)-\int_{a}^{x}f\left(t\right)du\left(t\right) \tag{8.134}$$

and

$$\int_{x}^{b}\left(u\left(t\right)-u\left(b\right)\right)df\left(t\right)=\left(u\left(b\right)-u\left(x\right)\right)f\left(x\right)-\int_{x}^{b}f\left(t\right)du\left(t\right). \tag{8.135}$$

If we add the above two identities, we obtain

$$
\begin{aligned}
(8.136)\qquad & \Theta\left(f,u;x,a,b\right) \\
=\; & \int_{a}^{x}\left(u\left(t\right)-u\left(a\right)\right)df\left(t\right)+\int_{x}^{b}\left(u\left(t\right)-u\left(b\right)\right)df\left(t\right)
\end{aligned}
$$

for all $x\in[a,b]$, which is of importance in itself.
Now, using the properties of modulus, we have:

$$
\begin{aligned}
(8.137)\qquad & \left|\Theta\left(f,u;x,a,b\right)\right| \\
\leq\; & \left|\int_{a}^{x}\left(u\left(t\right)-u\left(a\right)\right)df\left(t\right)\right|+\left|\int_{x}^{b}\left(u\left(t\right)-u\left(b\right)\right)df\left(t\right)\right| \\
\leq\; & \sup_{t\in[a,x]}\left|u\left(t\right)-u\left(a\right)\right|\bigvee_{a}^{x}\left(f\right)+\sup_{t\in[x,b]}\left|u\left(t\right)-u\left(b\right)\right|\bigvee_{x}^{b}\left(f\right),
\end{aligned}
$$

and for the last inequality we have used the well-known property of Riemann-Stieltjes integrals, i.e.,

$$\left|\int_c^d p(t)\,dv(t)\right| \leq \sup_{t\in[c,d]} |p(t)| \bigvee_c^d (v), \tag{8.138}$$

provided that p is continuous on $[c,d]$ and v is of bounded variation on $[c,d]$.

Now, as u is of $r-H-$Hölder type on $[a,b]$, we can state that

$$|u(t)-u(a)| \leq H(t-a)^r,\ |u(t)-u(b)| \leq H(b-t)^r$$

and then

$$\sup_{t\in[a,x]} |u(t)-u(a)| \leq H(x-a)^r$$

and

$$\sup_{t\in[x,b]} |u(t)-u(b)| \leq H(b-x)^r.$$

Now, using (8.137), we can state that

$$|\Theta(f,u;x,a,b)| \leq H\left[(x-a)^r \bigvee_a^x (f) + (b-x)^r \bigvee_x^b (f)\right],$$

for all $x\in[a,b]$, and the first inequality in (8.133) is proved.

Further on, define the mapping $M:[a,b]\to\mathbb{R}$, given by

$$M(x) = (x-a)^r \bigvee_a^x (f) + (b-x)^r \bigvee_x^b (f).$$

It is obvious that

$$\begin{aligned} M(x) &\leq \max\left\{\bigvee_a^x (f), \bigvee_x^b (f)\right\}\left[(x-a)^r+(b-x)^r\right] \\ &= \frac{1}{2}\left[\bigvee_a^x (f)+\bigvee_x^b (f)+\left|\bigvee_a^x (f)-\bigvee_x^b (f)\right|\right]\left[(x-a)^r+(b-x)^r\right] \\ &= \left[\frac{1}{2}\bigvee_a^b (f)+\frac{1}{2}\left|\bigvee_a^x (f)-\bigvee_x^b (f)\right|\right]\left[(x-a)^r+(b-x)^r\right], \end{aligned}$$

and the first part of the second inequality is thus proved.
Using the elementary inequality of Hölder type,

$$0 \leq \alpha\beta+\gamma\delta \leq (\alpha^p+\gamma^p)^{\frac{1}{p}}(\beta^q+\delta^q)^{\frac{1}{q}},\ \alpha,\beta,\gamma,\delta\geq 0,\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,$$

we obtain

$$M(x) \leq \left[(x-a)^{qr}+(b-x)^{qr}\right]^{\frac{1}{q}}\left[\left[\bigvee_a^x (f)\right]^p+\left[\bigvee_x^b (f)\right]^p\right]^{\frac{1}{p}},$$

and the second part of the second inequality is proved.
Finally, we observe that

$$\begin{aligned} M(x) &\leq \max\{(x-a)^r,(b-x)^r\}\left[\bigvee_a^x(f)+\bigvee_x^b(f)\right] \\ &= [\max\{x-a,b-x\}]^r\bigvee_a^b(f) \\ &= \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f), \end{aligned}$$

and the last part of the second inequality is proved. ∎

The following corollaries are natural consequences of (8.133).

COROLLARY 8.38. Let f be as in Theorem 8.37 and $u:[a,b]\to\mathbb{R}$ be an $L-$Lipschitzian mapping on $[a,b]$. Then, for all $x\in[a,b]$, we have the inequality

$$\begin{aligned} & |\Theta(f,u;x,a,b)| \\ \leq\ & L\left[(x-a)\bigvee_a^x(f)+(b-x)\bigvee_x^b(f)\right] \end{aligned} \tag{8.139}$$

$$\leq\begin{cases} L\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right](b-a); \\[2ex] L\left[(x-a)^q+(b-x)^q\right]^{\frac{1}{q}}\left[\left(\bigvee_a^x(f)\right)^p+\left(\bigvee_x^b(f)\right)^p\right]^{\frac{1}{p}} \\ \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] L\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f). \end{cases}$$

REMARK 8.17. If $f:[a,b]\to\mathbb{R}$ is monotonic on $[a,b]$ and u is of $r-H-$Hölder type, then f is of bounded variation on $[a,b]$, and by (8.133) we obtain

$$\begin{aligned} & |\Theta(f,u;x,a,b)| \\ \leq\ & H\left[(x-a)^r|f(x)-f(a)|+(b-x)^r|f(b)-f(x)|\right] \end{aligned} \tag{8.140}$$

$$\leq\begin{cases} H\left[\frac{1}{2}|f(b)-f(a)|+\left|f(x)-\frac{f(a)+f(b)}{2}\right|\right]\left[(b-x)^r+(x-a)^r\right]; \\[2ex] H\left[|f(x)-f(a)|^p+|f(b)-f(x)|^p\right]^{\frac{1}{p}}\left[(b-x)^{qr}+(x-a)^{qr}\right]^{\frac{1}{q}} \\ \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] H|f(b)-f(a)|\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r, \end{cases}$$

for all $x \in [a,b]$.

In particular, if u is $L-$Lipschitzian on $[a,b]$, then from (8.139) we get

$$\begin{aligned} (8.141)\qquad & |\Theta(f,u;x,a,b)| \\ \leq\ & L\left[(x-a)|f(x)-f(a)|+(b-x)|f(b)-f(x)|\right] \\ \leq\ & \begin{cases} L\left[\frac{1}{2}|f(b)-f(a)|+\left|f(x)-\frac{f(a)+f(b)}{2}\right|\right](b-a); \\ \\ L\left[|f(x)-f(a)|^p+|f(b)-f(x)|^p\right]^{\frac{1}{p}}\left[(x-a)^q+(b-x)^q\right]^{\frac{1}{q}}, \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ L|f(b)-f(a)|\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right], \end{cases} \end{aligned}$$

for all $x \in [a,b]$.

Remark 8.18. If $f:[a,b]\to\mathbb{R}$ is Lipschitzian with a constant $K>0$, then, obviously f is of bounded variation on $[a,b]$ and $\bigvee_a^b(f)\leq K(b-a)$. Consequently, from the first inequality in (8.133), we deduce

$$|\Theta(f,u;x,a,b)|\leq HK\left[(x-a)^{r+1}+(b-x)^{r+1}\right] \tag{8.142}$$

for all $x \in [a,b]$.

If we assume that f is $L-$Lipschitzian, then from (8.141) we get

$$|\Theta(f,u;x,a,b)|\leq LK\left[\frac{1}{2}(b-a)^2+2\left(x-\frac{a+b}{2}\right)^2\right], \tag{8.143}$$

for all $x \in [a,b]$.

The following corollary concerning a generalization of the mid-point inequality holds.

Corollary 8.39. Let f and u be as in Theorem 8.37. Then we have the inequality

$$|\Upsilon(f,u;x,a,b)|\leq 2^{1-r}H\bigvee_a^b(f)(b-a)^r \tag{8.144}$$

where

$$\Upsilon(f,u;x,a,b)=f\left(\frac{a+b}{2}\right)(u(b)-u(a))-\int_a^b f(t)\,du(t)$$

is the mid point functional associated to f and u as above. In particular, if u is $L-$Lipschitzian, then

$$|\Upsilon(f,u;x,a,b)|\leq L\bigvee_a^b(f)(b-a). \tag{8.145}$$

Remark 8.19. Now, if in (8.143) and (8.144) we assume that f is monotonic, then we get the inequalities

$$|\Upsilon(f,u;x,a,b)|\leq 2^{1-r}H|f(b)-f(a)|(b-a)^r \tag{8.146}$$

and

$$|\Upsilon(f,u;x,a,b)|\leq L|f(b)-f(a)|(b-a). \tag{8.147}$$

Also, if we assume that in (8.143) and (8.144) f is $K-$Lipschitzian, then we get the inequalities

$$|\Upsilon (f, u; x, a, b)| \leq 2^{1-r} HK (b-a)^{r+1} \tag{8.148}$$

and

$$|\Upsilon (f, u; x, a, b)| \leq LK (b-a)^2 . \tag{8.149}$$

Another interesting corollary is the following one.

Corollary 8.40. Let f and u be as in Theorem 8.37. If $x_0 \in [a, b]$ is a point satisfying the property that

$$\bigvee_a^{x_0} (f) = \bigvee_{x_0}^{b} (f) \tag{E}$$

then we get the inequality

$$\begin{aligned} &\left| f(x_0)(u(b) - u(a)) - \int_a^b f(t)\, du(t) \right| \\ &\leq \frac{1}{2} H \bigvee_a^b (f) \left[(x_0 - a)^r + (b - x_0)^r \right]. \end{aligned} \tag{8.150}$$

In particular, if u is $L-$Lipschitzian on $[a, b]$, then we have

$$\left| f(x_0)(u(b) - u(a)) - \int_a^b f(t)\, du(t) \right| \leq \frac{1}{2} L (b-a) \bigvee_a^b (f). \tag{8.151}$$

Remark 8.20. If in (8.149) and (8.150), we assume that f is monotonic and x_0 is a point such that $f(x_0) = \frac{f(a)+f(b)}{2}$, then we have the inequality:

$$\begin{aligned} &\left| f(x_0)(u(b) - u(a)) - \int_a^b f(t)\, du(t) \right| \\ &\leq \frac{1}{2} H |f(b) - f(a)| \left[(x_0 - a)^r + (b - x_0)^r \right] \end{aligned} \tag{8.152}$$

and

$$\left| f(x_0)(u(b) - u(a)) - \int_a^b f(t)\, du(t) \right| \leq \frac{1}{2} L (b-a) |f(b) - f(a)| . \tag{8.153}$$

The following inequalities of "rectangle" type also hold.

Corollary 8.41. Assume that f and u are as in Theorem 8.37. Then we have the generalized left rectangle inequality:

$$\left| f(a)(u(b) - u(a)) - \int_a^b f(t)\, du(t) \right| \leq H (b-a)^r \bigvee_a^b (f). \tag{8.154}$$

We also have the right rectangle inequality

$$\left| f(b)(u(b) - u(a)) - \int_a^b f(t)\, du(t) \right| \leq H (b-a)^r \bigvee_a^b (f). \tag{8.155}$$

Remark 8.21. If we add the inequalities (8.154) and (8.155), and use the triangle inequality, we end up with the following generalized trapezoid inequality

$$\left|\frac{f(a)+f(b)}{2}\cdot(u(b)-u(a))-\int_a^b f(t)\,du(t)\right|\leq H(b-a)^r\bigvee_a^b(f). \tag{8.156}$$

Now, we point out some results for the Riemann integral of a product.

Corollary 8.42. Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and g be continuous on $[a,b]$. Put $\|g\|_\infty := \sup_{t\in[a,b]}|g(t)|$. Then we have the inequality

$$\begin{aligned}
&\left|f(x)\int_a^b g(s)\,ds-\int_a^b f(t)g(t)\,dt\right| \\
\leq\ & \|g\|_\infty\left[(x-a)\bigvee_a^x(f)+(b-x)\bigvee_x^b(f)\right] \\
\leq\ & \begin{cases}
\|g\|_\infty(b-a)\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]; \\
\\
\|g\|_\infty\left[(x-a)^q+(b-x)^q\right]^{\frac{1}{q}}\left[\left(\bigvee_a^x(f)\right)^p+\left(\bigvee_x^b(f)\right)^p\right]^{\frac{1}{p}} \\
\text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\\
\|g\|_\infty\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f),
\end{cases}
\end{aligned} \tag{8.157}$$

for all $x\in[a,b]$.

Similar results for f monotonic of f Lipschitzian with a constant $K>0$ apply, but we omit the details.

Finally, we have

Corollary 8.43. Let f and g be as in Corollary 8.42. Put

$$\|g\|_p := \left(\int_a^b |g(s)|^p\,ds\right)^{\frac{1}{p}},\ p>1.$$

Then we have the inequality

$$\begin{aligned}
&\left|f(x)\int_a^b g(s)\,ds-\int_a^b f(t)g(t)\,dt\right| \\
\leq\ & \|g\|_p\left[(x-a)^{\frac{p-1}{p}}\bigvee_a^x(f)+(b-x)^{\frac{p-1}{p}}\bigvee_x^b(f)\right]
\end{aligned} \tag{8.158}$$

$$\leq \begin{cases} \|g\|_p \left[(x-a)^{\frac{p-1}{p}} + (b-x)^{\frac{p-1}{p}}\right] \left[\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^x(f) - \bigvee_x^b(f)\right|\right]; \\ \\ \|g\|_p \left[(x-a)^{\frac{\alpha(p-1)}{p}} + (b-x)^{\frac{\alpha(p-1)}{p}}\right]^{\frac{1}{\alpha}} \left[\left(\bigvee_a^x(f)\right)^\beta + \left(\bigvee_x^b(f)\right)^\beta\right]^{\frac{1}{\beta}}, \\ \text{where } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \\ \|g\|_p \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^{\frac{p-1}{p}} \bigvee_a^b(f), \end{cases}$$

for all $x \in [a,b]$.

Similar results for f monotonic and f Lipschitzian with a constant $K > 0$ apply, but we omit the details.

Corollary 8.44. Let f be of bounded variation on $[a,b]$ and $g \in L_1[a,b]$. Put $\|g\|_1 := \int_a^b |g(t)|\,dt$. Then we have the inequality

$$\begin{aligned} &\left| f(x)\int_a^b g(s)\,ds - \int_a^b f(t)\,g(t)\,dt \right| \\ &\leq \sup_{t\in[a,x]}\left|\int_a^t g(s)\,ds\right| \bigvee_a^x(f) + \sup_{t\in[x,b]}\left|\int_t^b g(s)\,ds\right| \bigvee_x^b(f) \\ &\leq \int_a^x |g(s)|\,ds \cdot \bigvee_a^x(f) + \int_x^b |g(s)|\,ds \cdot \bigvee_x^b(f) \\ &\leq \|g\|_1 \left[\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^x(f) - \bigvee_x^b(f)\right|\right], \end{aligned} \tag{8.159}$$

for all $x \in [a,b]$.

Proof. Put in inequality (8.137) $u(t) = \int_a^t g(s)\,ds$, to obtain

$$\begin{aligned} &\left| f(x)\int_a^b g(s)\,ds - \int_a^b f(t)\,g(t)\,dt \right| \\ &\leq \sup_{t\in[a,x]}\left|\int_a^t g(s)\,ds\right| \bigvee_a^x(f) + \sup_{t\in[x,b]}\left|\int_a^t g(s)\,ds - \int_a^b g(s)\,ds\right| \bigvee_x^b(f) \\ &= \sup_{t\in[a,x]}\left|\int_a^t g(s)\,ds\right| \bigvee_a^x(f) + \sup_{t\in[x,b]}\left|\int_t^b g(s)\,ds\right| \bigvee_x^b(f) \\ &\leq \sup_{t\in[a,x]}\int_a^t |g(s)|\,ds \cdot \bigvee_a^x(f) + \sup_{t\in[x,b]}\int_t^b |g(s)|\,ds \cdot \bigvee_x^b(f) \end{aligned}$$

$$
\begin{aligned}
&= \int_a^x |g(s)|\,ds \cdot \bigvee_a^x (f) + \int_x^b |g(s)|\,ds \cdot \bigvee_x^b (f) \\
&= \max\left\{ \bigvee_a^x (f), \bigvee_x^b (f) \right\} \left[\int_a^x |g(s)|\,ds + \int_x^b |g(s)|\,ds \right] \\
&= \|g\|_1 \left[\frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right],
\end{aligned}
$$

and the corollary is proved. ∎

8.3.5. Approximation of the Riemann-Stieltjes Integral. Consider $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ a division of the interval $[a,b]$, $h_i := x_{i+1} - x_i$ $(i = 0, ..., n-1)$ and $\nu(h) := \max\{h_i | i = 0, ..., n-1\}$. Define the general Riemann-Stieltjes sum as

$$
S(f, u, I_n, \xi) := \sum_{i=0}^{n-1} f(\xi_i)(u(x_{i+1}) - u(x_i)). \tag{8.160}
$$

In what follows, we point out some upper bounds for the error of the approximation of the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by its Riemann-Stieltjes sum $S(f, u, I_n, \xi)$.

Every inequality in Theorem 8.37 can be used in pointing out an upper bound for that error. However, we feel that the last inequality can give a much simpler and nicer one. Therefore, we will focus our attention to that one alone and its consequences [**26**].

THEOREM 8.45. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $u : [a,b] \to \mathbb{R}$ be of $r - H-$Hölder type on $[a,b]$. Then*

$$
\int_a^b f(t)\,du(t) = S(f, u, I_n, \xi) + R(f, u, I_n, \xi) \tag{8.161}
$$

where $S(f, u, I_n, \xi)$ is as given in (8.160) *and the remainder $R(f, u, I_n, \xi)$ satisfies the bound*

$$
|R(f, u, I_n, \xi)| \le H \left[\frac{1}{2}\nu(h) + \max_{i=\overline{0,n-1}} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right]^r \bigvee_a^b (f). \tag{8.162}
$$

PROOF. We apply Theorem 8.37 on the subintervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ to get

$$
\begin{aligned}
& \left| f(\xi_i)(u(x_{i+1}) - u(x_i)) - \int_{x_i}^{x_{i+1}} f(t)\,du(t) \right| \\
&\le H \left[\frac{1}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right]^r \bigvee_{x_i}^{x_{i+1}} (f)
\end{aligned} \tag{8.163}
$$

for all $i \in \{0, ..., n-1\}$. Summing over i from 0 to $n-1$ and using the generalized triangle inequality, we deduce

$$\begin{aligned} |R(f,u,I_n,\xi)| &\leq \sum_{i=0}^{n-1} \left| f(\xi_i)(u(x_{i+1}) - u(x_i)) - \int_{x_i}^{x_{i+1}} f(t)\,du(t) \right| \\ &\leq H \sum_{i=0}^{n-1} \left[\frac{1}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right]^r \bigvee_{x_i}^{x_{i+1}} (f) \\ &\leq H \sup_{i=\overline{0,n-1}} \left[\frac{1}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right]^r \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (f). \end{aligned}$$

However,

$$\sup_{i=\overline{0,n-1}} \left[\frac{1}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right]^r \leq \left[\frac{1}{2} \nu(h) + \sup_{i=\overline{0,n-1}} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right]^r$$

and

$$\sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (f) = \bigvee_a^b (f),$$

which completely proves the first inequality in (8.162).
For the second inequality, we observe that

$$\left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \leq \frac{1}{2} \cdot h_i,$$

for all $i \in \{0, ..., n-1\}$.
The theorem is thus proved. ∎

The following corollaries are a natural consequence of Theorem 8.45.

Corollary 8.46. Let f be as in Theorem 8.45 and u be an $L-$Lipschitzian mapping. Then we have the formula (8.161), where the remainder $R(f,u,I_n,\xi)$ satisfies the bound

$$\begin{aligned} &|R_n(f,u,I_n,\xi)| \\ \leq\ & L \left[\frac{1}{2} \nu(h) + \max_{i=\overline{0,n-1}} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \bigvee_a^b (f) \\ \leq\ & H \nu(h) \bigvee_a^b (f). \end{aligned} \tag{8.164}$$

Remark 8.22. If f is monotonic on $[a,b]$, then the error estimate (8.162) becomes

$$\begin{aligned} &|R_n(f,u,I_n,\xi)| \\ \leq\ & H \left[\frac{1}{2} \nu(h) + \max_{i=\overline{0,n-1}} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] |f(b) - f(a)| \\ \leq\ & H [\nu(h)]^r |f(b) - f(a)| \end{aligned} \tag{8.165}$$

and (8.164) becomes

$$\begin{aligned} &|R_n(f,u,I_n,\xi)| \\ \leq\ & L\left[\frac{1}{2}\nu(h) + \max_{i=\overline{0,n-1}}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]|f(b) - f(a)| \\ \leq\ & L\nu(h)\,|f(b) - f(a)|. \end{aligned} \tag{8.166}$$

Using Remark 8.18, we can state the following corollary.

Corollary 8.47. Let u be as in Theorem 8.45 and $f:[a,b]\to\mathbb{R}$ be a $K-$Lipschitzian mapping on $[a,b]$. Then the approximation formula (8.161) holds and the remainder $R(f,u,I_n,\xi)$ satisfies the bound

$$\begin{aligned} |R_n(f,u,I_n,\xi)| &\leq HK\sum_{i=0}^{n-1}\left[(\xi_i - x_i)^{r+1} + (x_{i+1} - \xi_i)^{r+1}\right] \\ &\leq HK\sum_{i=0}^{n-1} h_i^{r+1} \leq HK(b-a)[\nu(h)]^r. \end{aligned} \tag{8.167}$$

In particular, if u is $L-$Lipschitzian, we get

$$\begin{aligned} |R_n(f,u,I_n,\xi)| &\leq LK\left[\frac{1}{2}\sum_{i=0}^{n-1} h_i^2 + 2\sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right] \\ &\leq LK\sum_{i=0}^{n-1} h_i^2 \leq LK(b-a)\,\nu(h). \end{aligned} \tag{8.168}$$

We have the mention that the best inequality we can get in Theorem 8.45 is that one for which $\xi_i = \frac{x_i + x_{i+1}}{2}$ for all $i\in\{0,...,n-1\}$. Consequently, we can state the following corollary:

Corollary 8.48. Let f and u be as in Theorem 8.45. Then

$$\int_a^b f(t)\,du(t) = S_M(f,u,I_n) + R_M(f,u,I_n) \tag{8.169}$$

where $S_M(f,u,I_n)$ is the generalized mid-point formula. That is,

$$S_M(f,u,I_n) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)(u(x_{i+1}) - u(x_i))$$

and the remainder $R_M(f,u,I_n)$ satisfies the estimate

$$|R_M(f,u,I_n)| \leq \frac{H}{2^r}[\nu(h)]^r \bigvee_a^b (f). \tag{8.170}$$

In particular, if u is $L-$Lipschitzian, then we have the bound:

$$|R_M(f,u,I_n)| \leq \frac{H}{2}\nu(h) \bigvee_a^b (f). \tag{8.171}$$

REMARK 8.23. If in (8.170) and (8.171) we assume that f is monotonic, then we get the inequalities

$$|R_M(f,u,I_n)| \leq \frac{H}{2^r}[\nu(h)]^r |f(b)-f(a)| \tag{8.172}$$

and

$$|R_M(f,u,I_n)| \leq \frac{H}{2}\nu(h)|f(b)-f(a)|. \tag{8.173}$$

The case where f is $K-$Lipschitzian is embodied in the following corollary.

COROLLARY 8.49. Let u and f be as in Corollary 8.47. Then we have the quadrature formula (8.169) where the remainder satisfies the estimate

$$|R_M(f,u,I_n)| \leq \frac{HK}{2^r}\sum_{i=0}^{n-1} h_i^{r+1} \leq \frac{HK}{2^r}(b-a)[\nu(h)]^r. \tag{8.174}$$

In particular, if u is Lipschitzian, then we have the estimate

$$|R_M(f,u,I_n)| \leq \frac{1}{2}LK\sum_{i=0}^{n-1} h_i^2 \leq \frac{1}{2}LK(b-a)\nu(h). \tag{8.175}$$

8.4. Some Inequalities of Grüss Type for Riemann-Stieltjes Integral

8.4.1. Introduction. In 1935, G. Grüss [**68**] proved an inequality which establishes a connection between the integral of the product of two functions and the product of the integrals. Namely, he has shown that:

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right|\\ \leq\ &\frac{1}{4}(\Phi-\phi)(\Gamma-\gamma),\end{aligned}$$

provided f and g are two integrable functions on $[a,b]$ and satisfy the condition:

$$\phi \leq f(x) \leq \Phi \text{ and } \gamma \leq g(x) \leq \Gamma$$

for all $x \in [a,b]$.

The constant $\frac{1}{4}$ is the best possible one and is achieved for

$$f(x) = g(x) = sgn\left(x - \frac{a+b}{2}\right).$$

In the recent paper [**56**], S.S. Dragomir and I. Fedotov proved the following results of Grüss type for Riemann-Stieltjes integral:

THEOREM 8.50. *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is L-Lipschitzian on $[a,b]$, i.e.,*

$$|u(x)-u(y)| \leq L|x-y| \tag{8.176}$$

for all $x,y\in[a,b]$, f is Riemann integrable on $[a,b]$ and there exists the real numbers m,M so that

$$m \leq f(x) \leq M, \tag{8.177}$$

for all $x \in [a,b]$. *Then we have the inequality*

$$\left|\int_a^b f(x)du(x) - \frac{u(b)-u(a)}{b-a} \times \int_a^b f(t)dt\right| \leq \frac{1}{2}L(M-m)(b-a), \tag{8.178}$$

and the constant $\frac{1}{2}$ *is sharp, in the sense it can not be replaced by a smaller one.*

For other results related to Grüss inequality, generalizations in inner product spaces, for positive linear functionals, discrete versions, determinantal versions, etc., see Chapter X of the book [**69**] and the papers [**22**]-[**67**], where further references are given.

In this section we point out an inequality of Grüss type for Lipschitzian mappings and functions of bounded variation as well as its applications in numerical integration for the Riemann-Stieltjes integral.

8.4.2. Integral Inequalities. The following result of Grüss type holds [**55**].

THEOREM 8.51. *Let* $f,u : [a,b] \to \mathbb{R}$ *be such that* u *is* L*-Lipschitzian on* $[a,b]$, *and* f *is a function of bounded variation on* $[a,b]$. *Then the following inequality holds*

$$\left|\int_a^b u(x)\,df(x) - \frac{f(b)-f(a)}{b-a} \times \int_a^b u(t)\,dt\right| \leq \frac{L}{2}(b-a)\bigvee_a^b(f). \tag{8.179}$$

The constant $\frac{1}{2}$ *is sharp, in the sense that it cannot be replaced by a smaller one.*

PROOF. As f is a function of bounded variation on $[a,b]$ and u is continuous on $[a,b]$, we have

$$\begin{aligned} & \left|\int_a^b u(x)\,df(x) - \frac{f(b)-f(a)}{b-a}\int_a^b u(t)\,dt\right| \\ = & \left|\int_a^b \left(u(x) - \frac{1}{b-a}\int_a^b u(t)\,dt\right)df(x)\right| \\ \leq & \sup_{x\in[a,b]}\left|u(x) - \frac{1}{b-a}\int_a^b u(t)\,dt\right|\bigvee_a^b(f) \end{aligned}$$

$$= \frac{1}{b-a}\sup_{x\in[a,b]}\left|\int_a^b [u(x)-u(t)]\,dt\right|\bigvee_a^b(f). \tag{8.180}$$

Using the fact that u is $L-$Lipschitzian on $[a,b]$, we can state, for any $x \in [a,b]$, that:

$$\begin{aligned} \left|\int_a^b [u(x)-u(t)]\,dt\right| & \leq \int_a^b |u(x)-u(t)|\,dt \leq L\int_a^b |x-t|\,dt \\ & = \frac{L}{2}\left[(x-a)^2 + (x-b)^2\right] \end{aligned}$$

$$= L\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)^2 \le \frac{L}{2}(b-a)^2, \tag{8.181}$$

and by (8.180)-(8.181) we get:

$$\sup_{x\in[a,b]}\left|u(x) - \frac{1}{b-a}\int_a^b u(t)\,dt\right| \le \frac{L(b-a)}{2} \tag{8.182}$$

whence we obtain (8.179).
To prove the sharpness of the inequality (8.179), let us choose

$$u(x) = x - \frac{a+b}{2} \quad\text{and}\quad f(x) = \begin{cases} 1 & \text{if} \quad x = a \\ 0 & \text{if} \quad a < x < b \\ 1 & \text{if} \quad x = b \end{cases}.$$

Then u is Lipschitzian with $L = 1$, and f is of bounded variation. Also we have

$$\begin{aligned} &\int_a^b u(x)\,df(x) - \frac{f(b)-f(a)}{b-a}\times\int_a^b u(t)\,dt \\ = &\int_a^b \left(x - \frac{a+b}{2}\right)df(x) \\ = &\left(x - \frac{a+b}{2}\right)f(x)\Big|_a^b - \int_a^b f(x)\,dx = b - a. \end{aligned}$$

On the other hand, the right hand side of (8.179) is equal to $b-a$, and hence the sharpness of the constant is proved. ∎

The following corollaries hold:

COROLLARY 8.52. Let $f : [a,b] \to \mathbb{R}$ be as above and $u : [a,b] \to \mathbb{R}$ be a differentiable mapping with a bounded derivative on $[a,b]$. Then we have the inequality:

$$\left|\int_a^b u(x)\,df(x) - \frac{f(b)-f(a)}{b-a}\times\int_a^b u(t)\,dt\right| \le \frac{\|u'\|_\infty}{2}(b-a)\bigvee_a^b(f). \tag{8.183}$$

The inequality (8.183) is sharp in the sense that the constant $\frac{1}{2}$ cannot be replaced by a smaller one.

COROLLARY 8.53. Let u be as above and $f : [a,b] \to \mathbb{R}$ be a differentiable mapping whose derivative is integrable, i.e.,

$$\|f'\|_1 = \int_a^b |f'(t)|\,dt < \infty.$$

Then we have the inequality:

$$\begin{aligned} &\left|\int_a^b u(x)\,f'(x)\,dx - \frac{f(b)-f(a)}{b-a}\times\int_a^b u(t)\,dt\right| \\ \le\ &\frac{\|u'\|_\infty\|f'\|_1}{2}(b-a). \end{aligned} \tag{8.184}$$

Remark 8.24. If we assume that $g : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and if we set $f(x) = \int_a^x g(x)\,dx$, then from (8.184) we get the following Grüss type inequality for the Riemann integral:

$$\left|\frac{1}{b-a}\int_a^b u(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b u(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt\right| \leq \frac{\|u'\|_\infty \|g\|_1}{2}(b-a). \tag{8.185}$$

Corollary 8.54. Assume that $u : [a, b] \to \mathbb{R}$ is differentiable on (a, b), $u(a) \neq u(b)$, and $u' : [a, b] \to \mathbb{R}$ is bounded on (a, b). Then we have the trapezoid inequality:

$$\left|\frac{u(a)+u(b)}{2}\cdot(b-a) - \int_a^b u(t)\,dt\right| \leq \frac{\|u'\|_\infty \|u'\|_1}{2|u(b)-u(a)|}\cdot(b-a)^2. \tag{8.186}$$

Proof. If we choose in Corollary 8.53, $f(x) = u(x)$, we get

$$\left|\int_a^b u(x)\,u'(x)\,dx - \frac{u(b)-u(a)}{b-a} \times \int_a^b u(t)\,dt\right| \leq \frac{\|u'\|_\infty \|u'\|_1}{2}\cdot(b-a). \tag{8.187}$$

Now (8.186) follows from (8.187). ∎

8.4.3. A Numerical Quadrature Formula for the Riemann-Stieltjes Integral. In what follows, we shall apply Theorem 8.51 to approximate the Riemann-Stieltjes integral $\int_a^b u(x)\,df(x)$ in terms of the Riemann integral $\int_a^b u(t)\,dt$ (see also [**55**]).

Theorem 8.55. *Let $f, u : [a, b] \to \mathbb{R}$ be as in Theorem 8.51 and*

$$I_n = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}$$

a partition of $[a, b]$. Denote $h_i = x_{i+1} - x_i$, $i = 0, 1, \ldots, n-1$. Then we have:

$$\int_a^b u(x)\,df(x) = A_n(u, f, I_n) + R_n(u, f, I_n) \tag{8.188}$$

where

$$A_n(u, f, I_n) = \sum_{i=0}^{n-1} \frac{f(x_{i+1}) - f(x_i)}{h_i} \times \int_{x_i}^{x_{i+1}} u(t)\,dt \tag{8.189}$$

and the remainder term $R_n(u, f, I_n)$ satisfies the estimation

$$|R_n(u, f, I_n)| \leq \frac{L}{2}\nu(h) \bigvee_a^b (f) \tag{8.190}$$

where $\nu(h) = \max_{i=0,\ldots,n-1}\{h_i\}$.

PROOF. Applying Theorem 8.51 on the interval $[x_i, x_{i+1}]$, $i = 0, 1, ..., n-1$ we get

$$\left|\int_{x_i}^{x_{i+1}} u(x)\,df(x) - \frac{f(x_{i+1}) - f(x_i)}{h_i} \times \int_{x_i}^{x_{i+1}} u(t)\,dt\right| \le \frac{L}{2} \cdot h_i \bigvee_{x_i}^{x_{i+1}} (f).$$

Summing over i from 0 to $n-1$ and using the triangle inequality we obtain

$$\begin{aligned}\left|\int_a^b u(x)\,df(x) - A_n(u, f, I_n)\right| &\le \frac{L}{2}\sum_{i=0}^{n-1} h_i \bigvee_{x_i}^{x_{i+1}} (f) \le \frac{L}{2} \cdot \nu(h) \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (f) \\ &= \frac{L}{2} \cdot \nu(h) \bigvee_a^b (f)\end{aligned}$$

and the corollary is proved. ∎

REMARK 8.25. Consider the equidistant partition I_n given by

$$I_n : \; x_i = a + i \cdot \frac{b-a}{n}, \; i = 0, ..., n;$$

and define

$$\begin{aligned}&A_n(u, f, I_n) \\ = \;& \frac{n}{b-a}\sum_{i=0}^{n-1}\left[f\left(a + (i+1)\cdot\frac{b-a}{n}\right) - f\left(a + i\cdot\frac{b-a}{n}\right)\right] \times \int_{a+i\cdot\frac{b-a}{n}}^{a+(i+1)\cdot\frac{b-a}{n}} u(t)\,dt.\end{aligned}$$

Then we have

$$\int_a^b u(x)\,df(x) = A_n(u, f) + R_n(u, f),$$

where the remainder $R_n(u, f)$ satisfies the estimation

$$|R_n(u, f)| \le \frac{L(b-a)}{2n} \cdot \bigvee_a^b (f).$$

Thus, if we want to approximate the integral $\int_a^b u(x)\,df(x)$ by the sum $A_n(u, f, I_n)$ with an error of magnitude less than ε we need at least

$$n_0 = \left[\frac{L(b-a)}{2\varepsilon} \cdot \bigvee_a^b (f)\right] + 1 \in \mathbb{N}$$

points.

COROLLARY 8.56. Assume that u and f are as in Corollary 8.52. If I_n is as above, then (8.188) holds and the remainder term $R_n(u, f, I_n)$ satisfies the estimation

$$|R_n(u, f, I_n)| \le \frac{\|u'\|_\infty}{2} \cdot \nu(h) \bigvee_a^b (f). \tag{8.191}$$

COROLLARY 8.57. Assume that u and f are as in Corollary 8.53. Then (8.188) holds and the remainder term $R_n(u, f, I_n)$ satisfies the estimation

$$|R_n(u, f, I_n)| \le \frac{\|u'\|_\infty \|f'\|_1}{2} \cdot \nu(h). \tag{8.192}$$

8.4.4. Quadrature Methods for the Riemann-Stieltjes Integral of Continuous Mappings. Let $I_n = \{a = x_0 < x_1 < ... < x_{n-1} < x_n = b\}$ be a partition of $[a,b]$ and denote $h_i = x_{i+1} - x_i$, $i = 0, 1, ...n-1$.

We start to the following lemma which is of interest in itself [**55**].

Lemma 8.58. Let f be a function from $C[x_i, x_{i+1}]$, i.e., f is continuous on $[x_i, x_{i+1}]$, and let u be a function of bounded variation on the same interval. The following inequality holds:

$$\left|\int_{x_i}^{x_{i+1}} f(x)\,du(x) - [u(x_{i+1}) - u(x_i)]\,\bar{f}_i\right| \le \omega[f, h_i] \bigvee_{x_i}^{x_{i+1}} (u) \tag{8.193}$$

where $\bar{f}_i = \dfrac{1}{h_i}\int_{x_i}^{x_{i+1}} f(x)\,dx,$ $h_i = x_{i+1} - x_i,$ and $\omega[f, h_i] = \sup\limits_{|x-t|\le\delta} |f(x) - f(t)|,$ is the modulus of continuity.

Proof. Since $f \in C[x_i, x_{i+1}]$ and u is a function of bounded variation, by the well known property of such couple of functions we have

$$\left|\int_{x_i}^{x_{i+1}} f(x)\,du(x)\right| \le \|f\|_{C[x_i,x_{i+1}]} \bigvee_{x_i}^{x_{i+1}} (u).$$

Therefore,

$$\begin{aligned}
& \left|\int_{x_i}^{x_{i+1}} \left[f(x) - \bar{f}_i\right] du(x)\right| \\
= & \left|\int_{x_i}^{x_{i+1}} f(x)\,du(x) - [u(x_{i+1}) - u(x_i)]\,\bar{f}_i\right| \\
\le & \left\|f(x) - \bar{f}_i\right\|_{C[x_i,x_{i+1}]} \bigvee_{x_i}^{x_{i+1}} (u) \\
= & \sup_{x\in[x_i,x_{i+1}]} \left|f(x) - \bar{f}_i\right| \bigvee_{x_i}^{x_{i+1}} (u) \\
= & \frac{1}{h_i} \sup_{x\in[x_i,x_{i+1}]} \left|h_i f(x) - h_i \bar{f}_i\right| \bigvee_{x_i}^{x_{i+1}} (u) \\
= & \frac{1}{h_i} \sup_{x\in[x_i,x_{i+1}]} \left|\int_{x_i}^{x_{i+1}} f(x)\,dt - \int_{x_i}^{x_{i+1}} f(t)\,dt\right| \bigvee_{x_i}^{x_{i+1}} (u) \\
\le & \frac{1}{h_i} \sup_{x\in[x_i,x_{i+1}]} \left|\int_{x_i}^{x_{i+1}} [f(x) - f(t)]\,dt\right| \bigvee_{x_i}^{x_{i+1}} (u) \\
\le & \sup_{x\in[x_i,x_{i+1}],t\in[x_i,x_{i+1}]} |f(x) - f(t)| \bigvee_{x_i}^{x_{i+1}} (u) \\
= & \omega[f, h_i] \bigvee_{x_i}^{x_{i+1}} (u),
\end{aligned}$$

and the lemma is proved. ∎

Let $f \in C[a,b]$. The dual to $C[a,b]$ is the space of functions of bounded variation, the general form of the functional on $C[a,b]$ is $I(f) = \int_a^b f(x)\,du(x)$ where u belongs to the space of functions of bounded variation.

That is why in the theory of quadrature methods for continuous functions the case of the integrals of such a type is the most interesting. For the integral with continuous integrand $\int_a^b f(x)\,w(x)\,dx$ $(w(x)>0)$ we introduce the error functional for the quadrature rule (with the weights s_{nk} and nodes x_k)

$$I(f) \equiv \int_a^b f(t)\,w(t)\,dt \approx -\sum_{k=0}^{n} f(x_k)\,s_{nk} \equiv I_n(f)$$

by the formula

$$E_n(f) = I(f) - I_n(f).$$

Here is the more general result for the composite quadrature rules for the functions from $C[a,b]$, [**55**].

THEOREM 8.59. *Let a and b be finite real numbers and let $f,u : [a,b] \to \mathbb{R}$ be such that $f \in C[a,b]$ and u is a function of bounded variation on $[a,b]$. If $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ is a division of $[a,b]$ such that $|h_i| < \delta$, for all $i = 0,1,\ldots,n-1$ where $h_i = x_{i+1} - x_i$, then the following estimation for the Error functional of the Riemann-Stieltjes quadrature rule is true*

$$|E_n^{comp}(f)| \le \omega[f,\delta] \cdot \bigvee_a^b (u), \tag{8.194}$$

where

$$E_n^{comp}(f) = \int_a^b f(x)\,du(x) - \sum_{i=0}^{n-1} \frac{u(x_{i+1}) - u(x_i)}{h_i} \times \int_{x_i}^{x_{i+1}} f(x)\,dx,$$

and $\omega[f,\delta]$ is the modulus of continuity of f with respect to δ.

PROOF. For a given division of $[a,b]$ as above, we have

$$\int_a^b f(x)\,du(x) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\,du(x).$$

Then by Lemma 8.58, we can write successively:

$$\begin{aligned}
& \left|E_n^{comp}(f)\right| \\
= & \left|\int_a^b f(x)\,du(x) - \sum_{i=0}^{n-1} \frac{u(x_{i+1}) - u(x_i)}{h_i} \times \int_{x_i}^{x_{i+1}} f(x)\,dx\right| \\
= & \left|\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\,du(x) - \sum_{i=0}^{n-1} \frac{u(x_{i+1}) - u(x_i)}{h_i} \times \int_{x_i}^{x_{i+1}} f(x)\,dx\right| \\
\leq & \sum_{i=0}^{n-1} \left|\int_{x_i}^{x_{i+1}} f(x)\,du(x) - \frac{u(x_{i+1}) - u(x_i)}{h_i} \int_{x_i}^{x_{i+1}} f(x)\,dx\right| \\
\leq & \sum_{i=0}^{n-1} \omega[f, h_i] \bigvee_{x_i}^{x_{i+1}} (u) \leq \omega[f, \delta] \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (u) \\
= & \ \omega[f, \delta] \bigvee_a^b (u),
\end{aligned}$$

and the theorem is proved. ∎

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