I. CLASSES OF L^1-COVERGENCES OF FOURIER SERIES

1.1. CLASSICAL AND NEOCLASSICAL RESULTS

Denote by $L^1(T)$ Banach space of all complex, Lebesgue integrable functions on the unit circle $\mathbf{T}$. To every function $f \in L^1(T)$ corresponds the Fourier series of f

$$S(f) \sim \sum_{|n|<\infty} \hat{f}(n)e^{\text{int}}, \quad \text{where} \quad \hat{f}(n) = \frac{1}{2\pi} \int_T f(t)e^{-\text{int}}dt, \quad |n| < \infty$$

are Fourier coefficients of f.

A sequence of partial sums be denoted by

$$S_n(f) = S_n(f,t) = \sum_{|k|\le n} \hat{f}(k)e^{ikt}, \quad n = 0, 1, 2, 3, \dots ,$$

while the $(C,1)$-means (Fejer sums) of the sequances of partial sums will be written as:

$$\sigma_n(f) = \sigma_n(f,t) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(f,t), \quad n = 0, 1, 2, \dots .$$

The Dirichlet kernel is denoted by

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos\, kt = \frac{\sin\left(n + \frac{1}{2}\right)t}{2\,\sin\,\frac{t}{2}}$$

and the Fejer kernel is denoted by

$$F_n = \frac{1}{n+1} \sum_{k=0}^{n} D_k(t) = \frac{1}{2(n+1)} \left[\frac{\sin(n+1)\frac{t}{2}}{\sin\,\frac{t}{2}} \right]^2 .$$

Note that

$$\|D_n\|_1 = \frac{4}{\pi^2}\,\log + O(1), \quad n \to \infty,$$

$$\|F_n\|_1 = 1, \text{ for every}\, n, \text{ where } \|\ \|_1 \text{ denotes the } L^1(T)-\text{norm.}$$

Let

$$\tilde{D}_n(t)=\sum_{k=1}^{n}\sin\,kt=\frac{\cos\dfrac{t}{2}-\cos\Big(n+\dfrac{1}{2}\Big)t}{2\,\sin\,\dfrac{t}{2}}$$

$$\overline{D}_n(t)=-\frac{1}{2}\,\text{ctg}\,\frac{t}{2}+\tilde{D}_n(t)=-\frac{\cos\Big(n+\dfrac{1}{2}\Big)t}{2\,\sin\,\dfrac{t}{2}}\,,$$

$$\tilde{K}_n(t)=\frac{1}{n+1}\sum_{k=0}^{n}\tilde{D}_k(t)=\frac{1}{4\,\sin^2\,\dfrac{t}{2}}\left[\sin\,t-\frac{\sin(n+1)t}{n+1}\right]$$

denote the conjugate Dirichlet's kernel, modified Dirichlet's kernel and conjugate Fejer's kernel, respectively.

The Banach space in the real case will be denoted by $L^1(0,\pi)$ and norm by $\|\cdot\|$.

Let

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\,\cos nx \tag{C}$$

$$\sum_{n=1}^{\infty}a_n\,\sin nx \tag{S}$$

be the cosine and sine trigonometric series.

The partial sums of a real cosine and sine series will be denoted by $S_n(x)$ and $\tilde{S}_n(x)$ respectively.

The problem of L^1-convergence, via Fourier coefficients, consists of finding the properties of Fourier coefficients such that the cosine series (C) is a Fourier series of its sum f and $\|S_n(f)-f\|=o(1),\ n\to\infty$ if and only if $\hat{f}(n)\,\lg|n|=o(1)$, $n\to\infty$.

In general $L^1(T)$ does not admit convergence in $L^1(T)$-norm, (Zygmund [63], Vol.1. p.67) since the operator norm $\|S_n\|^{L_1(T)}$ is unbounded, i.e.

$$\|S_n\|^{L^1(T)}=\|D_n\|=\frac{4}{\pi^2}\,\log n+O(1)\,,\quad n\to\infty\,.$$

A classical result concerning the integrability and L^1-convergence of a cosine series (C) is the following well-known theorem of Young, see [61].

Theorem 1.1 (Young). *If* $\{a_n\}_{n=0}^{\infty}$ *is a convex* ($\Delta^2 a_n=\Delta(\Delta a_n)=\Delta a_n-\Delta a_{n+1}=a_n-2a_{n+1}+a_{n+2}\geq 0,\forall n$) *null sequence, then the cosine series* (C) *is* *the Fourier series of its sum f, and*

$$\|S_n(f) - f\| = o(1) \ \ n \to \infty \ \text{ iff } \ a_n \log n = o(1), \ \ n \to \infty\,. \tag{1.1}$$

The sequences $\{a_n\}$ that satisfy the condition $\sum_{n=1}^{\infty}(n+1)|\Delta^2 a_n| < \infty$ are called quasi-convex.

The next theorem of Kolmogorov extends Young's result, since every convex null sequence is also quasi-convex null sequence.

Theorem 1.2 [21] (Kolmogorov). *If $\{a_n\}$ is a quasi-convex null sequence then the cosine series* (C) *is the Fourier series of its sum f and* (1.1) *hold.*

We say that a sequence $\{a_k\}$ is of bounded variation and we write $\{a_k\} \in BV$ if $\sum_{k=0}^{\infty}|\Delta a_k| < \infty$. Several authors (Sidon, Telyakovskii, Fomin, Stanojević and others) have extended these classical rasults by answering one or both of the following two questions:

(i) For which classes of coefficients of (C), subclasses of BV, does it follows that (C) is the Fourier series of its sum function and that the sum is an integrable function?

(ii) If (C) is the Fourier series of some function $f \in L^1$ under what conditions does (1.1) hold? In other words, for which classes of Fourier coefficients of even and integrable functions does (1.1) hold.

Theorem 1.3 [35] (Sidon). *Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{P_n\}_{n=1}^{\infty}$ be the sequences such that $|\alpha_n| \le 1$, for every n and let $\sum_{n=1}^{\infty}|p_n| < \infty$, If $a_n = \sum_{k=n}^{\infty}\frac{p_k}{k}\sum_{l=n}^{k}\alpha_l$, $n = 1, 2, 3, \ldots$ then the cosine series* (C) *is the Fourier series of its sum f.*

It's obvious that Sidon's conditions [35] imply that $\{a_n\} \in BV$.

Telyakovskii [44] defined an extension of the class of the quasi-convex sequences. It's denoted by S.

The class S is defined as follows: a null sequence $\{a_n\}_{n=0}^{\infty}$ belongs to the class S if there exists a monotonically decreasing sequence $\{A_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty}A_n < \infty$ and $|\Delta a_n| \le A_n$, for all n.

Telyakovskii proved [44] that the Sidon's class is equevalent to the class S. Thus, the class S is usually called as Sidon-Telyakovskii class.

Theorem 1.4 [44] (Telyakovskii). *Let* $\{a_n\}_{n=0}^{\infty} \in S$. *Then the cosine series* (C) *is the Fourier series of its sum* f *and* (1.1) *hold.*

On the other hand, Kano [18] have extended the classical result of Kolmogorov by answering of the first question (i).

Theorem 1.5 [8] (Kano). *If* $\{a_n\}$ *is a null sequence such that* $\sum_{n=1}^{\infty} n^2 \left|\Delta^2\left(\frac{a_n}{n}\right)\right| < \infty$, *then* (C) *is a Fourier series, or equivalently it represents an integrable function.*

The following Lemma was proved by Telyakovskii in [46].

Lemma 1.1 [46]. *Condition* $\sum_{n=1}^{\infty} n^2 \left|\Delta^2\left(\frac{a_n}{n}\right)\right| < \infty$ *is equivalent to the simultaneous fulfillment of conditions* $\sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty$ *and* $\sum_{n=1}^{\infty} (n+1)|\Delta^2 a_n| < \infty$.

Remark 1.1 Using this Lemma we obtain that the Theorem 1.5 is a corollary of the Theorem 1.2.

Very later, S. Kumari and Baby Ram [22] have proved the following theorem:

Theorem 1.6 [22] (S. Kumari, B. Ram). *Let* $(k+1)^2 \left|\Delta^2\left(\frac{a_k}{k}\right)\right| \downarrow 0$. *Then*

$$h(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{1}{2}(k+1)^2 \left|\Delta^2\left(\frac{a_k}{k}\right)\right| + \sum_{v=k}^{n} (v+1)^2 \left|\Delta^2\left(\frac{a_v}{v}\right)\right| \cos kx\right]$$

exists for $x \in (0, \pi]$ *and* $h \in L(0, \pi]$ *iff* $\sum_{k=1}^{\infty} (k+1)^2 \left|\Delta^2\left(\frac{a_k}{k}\right)\right| < \infty$.

In [27] C. N. Moore generalized quasi-convexity of null sequences in the following way,

$$\sum_{n=1}^{\infty} n^k |\Delta^{k+1} a_n| < \infty\,, \quad \text{for} \quad k > 0\,, \tag{M}$$

where the order od differences is fractional and proved the corresponding integrability result.

It is well-known [7] that if $\{a_n\}$ is a null sequence satisfyng the condition (M), then $\sum_{n=1}^{\infty} n^r |\Delta^{r+1} a_n| < \infty$, for $0 \le r < k$. In particular it is of bounded variation.

N. Singh and K. M. Sharma [33] proved the following generalized theorem of Kolmogorov.

Theorem 1.7 [33] (Singh, Sharma) *Let* k *be a real number such that* $k > 0$. *If*

(i) $\lim_{n\to\infty} a_n = 0$

(ii) $\sum_{n=1}^{\infty} n^k |\Delta^{k+1} a_n| < \infty$

then for the convergence of the series (C) *in the metric* L^1 *it is necessary and sufficient that* $a_n \log n = o(1)$, $n \to \infty$.

Remark 1.2 The Theorem 1.7 is a corollary of Theorem 1.4. It suffices to show that the conditions (i) and (ii) of the Theorem 1.7 implies the Sidon-Telyakovskii type condition. Indeed, if we denote $A_n = C_k \sum_{m=n}^{\infty} m^{k-1} |\Delta^{k+1} a_m|$, where C_k is a some positive constant depend only on k, which will be later on defined.
We obtain $A_n \downarrow 0$, $n \to \infty$ and

$$\sum_{n=1}^{N} A_n = C_k \sum_{n=1}^{N} \sum_{m=n}^{N} m^{k-1} |\Delta^{k+1} a_m| + C_k \sum_{n=1}^{N} \sum_{m=N+1}^{\infty} m^{k-1} |\Delta^{k+1} a_m| =$$

$$= C_k \sum_{m=1}^{N} \sum_{n=1}^{m} m^{k-1} |\Delta^{k+1} a_m| + N C_k \sum_{m=N+1}^{\infty} m^{k-1} |\Delta^{k+1} a_m| \leq$$

$$\leq C_k \sum_{m=1}^{N} m^{k} |\Delta^{k+1} a_m| + C_k \sum_{m=N+1}^{\infty} m^{k} |\Delta^{k+1} a_m| , \quad \text{i.e.} \quad \sum_{n=1}^{\infty} A_n < \infty .$$

Now, we need the following properties for binomial coefficients $\binom{a+n}{a}$ (see [2], page 885):
a) $a > -1 \Rightarrow \binom{a+n}{a} = \dfrac{(a+1)(a+2)\ldots(a+n)}{n!} > 0$
b) $\binom{a+n}{a} \leq C_a n^a$, $C_a > 0$.
For $0 < k < 1$, we have: $\Delta a_n = \sum_{i=n}^{\infty} \binom{i-n+k-1}{i-n} \Delta^{k+1} a_i$, i.e.

$$|\Delta a_n| \leq \sum_{i=n}^{\infty} \binom{i-n+k-1}{i-n} |\Delta^{k+1} a_i| \leq C_k \sum_{i=n}^{\infty} i^{k-1} |\Delta^{k+1} a_i| = A_n .$$

The case $k \geq 1$ is evidently. Finally $\{a_n\} \in S$.
In [37] Časlav V. Stanojević and Vera B. Stanojević generalized the Telyakovskii theorem [44].
They defined a stronger class S_p, $p > 1$ as follows: a null sequence $\{a_n\}$ of real numbers belongs to the class S_p if for some $p > 1$ and some monotone sequence

$\{A_n\}$ such that $\sum_{n=1}^{\infty} A_n < \infty$ the following condition holds

$$\frac{1}{n}\sum_{k=1}^{n}\frac{|\Delta a_k|^p}{A_k^p} = O(1)\,.$$

There exists a null-sequence $\{a_n\}$ such that $\{a_n\} \in S_p$ but $\{a_n\} \notin S$.

Example: Let us define a sequence $\{a_n\}$ as follows: let $\Delta a_n = \frac{1}{m^2}$ for $n = m^2$ and $\Delta a_n = 0$ for $n \neq m^2$. Firstly, we shall show that $\{a_n\} \notin S$. We have:

$$a_{m^2} = \sum_{i=m^2}^{\infty} \Delta a_i = \Delta a_{m^2} + \Delta a_{m^2+1} + \cdots + \Delta a_{(m+1)^2} + \cdots = \sum_{i=m}^{\infty} \Delta a_{i^2} =$$

$$= \sum_{i=m}^{\infty} \frac{1}{i^2} \to 0\,, \quad m \to \infty\,,$$

i.e. $a_n \to 0$ as $n \to \infty$. Let we denote $A_n^* = \max_{i \geq n} |\Delta a_i|$. Then $A_n^* \downarrow 0$ and $\sum_{n=1}^{\infty} A_n^* = \infty$. Really, $A_n^* = \frac{1}{m^2}$ for $(m-1)^2 + 1 \leq n \leq m^2$ and

$$\sum_{k=1}^{\infty} A_k^* = \sum_{m=1}^{\infty} \sum_{k=(m-1)^2+1}^{m^2} A_k^* = \sum_{m=1}^{\infty} \sum_{k=(m-1)^2+1}^{m^2} A_{m^2}^* =$$

$$= \sum_{m=1}^{\infty} \frac{1}{m^2}[m^2 - (m-1)^2] = \sum_{m=1}^{\infty} \frac{2m-1}{m^2} = \infty\,.$$

Therefore for arbitraty positive sequence $\{A_n\}$ such that $A_n \geq A_n^*$, we have $\sum_{n=1}^{\infty} A_n = \infty$, i.e. $\{a_n\} \notin S$.

Now, let $A_n = \frac{1}{n^{1+1/2p}}$, for all n. Then $A_n \downarrow 0$, $\sum_{n=1}^{\infty} A_n < \infty$ and for $n = m^2$, we have:

$$\frac{1}{m^2}\sum_{i=1}^{m^2}\frac{|\Delta a_i|^p}{A_i^p} = \frac{1}{m^2}\sum_{k=1}^{m}\left(\frac{|\Delta a_{k^2}|}{A_{k^2}}\right)^p = \frac{1}{m^2}\sum_{k=1}^{m}\left(\frac{\frac{1}{k^2}}{\frac{1}{k^{2+1/p}}}\right)^p =$$

$$= \frac{1}{m^2}\sum_{k=1}^{m}\left(k^{1/p}\right)^p = O(1)\,.$$

Theorem 1.8 [37] (Č. V. Stanojević and V. B. Stanojević). *Let $\{a_n\} \in S_p$, $1 < p \leq 2$. Then the cosine series* (C)*is the Fourier series of its sum f and* (1.1) *hold.*

On the other hand, Fomin [12] have extended the Sidon-Telyakovskii's class. He defined a class F_p, $1 < p \leq 2$ of Fourier coefficients as follows: a sequence $\{a_n\}$ belongs to F_p, $p > 1$ if $a_k \to 0$ as $k \to \infty$ and

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=k}^{\infty} |\Delta a_i|^p \right)^{1/p} < \infty \,. \tag{1.2}$$

We note that Fomin has given an equivalent form of the condition (1.2). Namely, he proved the following lemma.

Lemma 1.2 [12]. *A sequence $\{a_n\} \in F_p$, $p > 1$ iff $\sum_{s=1}^{\infty} 2^s \Delta_s^{(p)} < \infty$, where*

$$\Delta_s^{(p)} = \left\{ \frac{1}{2^{s-1}} \sum_{k=2^{s-1}+1}^{2^s} |\Delta a_k|^p \right\}^{1/p} .$$

In [12], Fomin note that it is easy to see that the class F_p is wider when p is closer to 1. But now we shall present the proof of this fact.

Corollary 1.1 [67] *For any $1 < r < p$ the following embedding relation holds $F_p \subset F_r$.*

Proof. By inequality $\frac{1}{r} > \frac{1}{p}$, we have $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, where $q > 0$. This equality implies that $\frac{1}{p'} + \frac{1}{q'} = 1$, where $p' = \frac{p}{r}$ and $q' = \frac{q}{r}$.
Applying the Holder's inequality, we have:

$$\sum_{k=2^s+1}^{2^{s+1}} |\Delta a_k|^r = \sum_{k=2^s+1}^{2^{s+1}} |\Delta a_k|^r \cdot 1 \leq \left(\sum_{k=2^s+1}^{2^{s+1}} |\Delta a_k|^{rp'} \right)^{1/p'} \left(\sum_{k=2^s+1}^{2^{s+1}} 1^{q'} \right)^{1/q'} =$$

$$= (2^s)^{1/q'} \left(\sum_{k=2^s+1}^{2^{s+1}} |\Delta a_k|^p \right)^{1/p'} .$$

Then

$$\sum_{s=1}^{\infty} 2^s \Delta_s^{(r)} \leq \sum_{s=1}^{\infty} 2^s \cdot 2^{-s/r} \cdot 2^{s/q'r} \left(\sum_{k=2^s+1}^{2^{s+1}} |\Delta a_k|^p \right)^{1/rp'} =$$

$$= \sum_{s=1}^{\infty} 2^s \left(\frac{1}{2^s}\right)^{1/r-1/q} \left(\sum_{k=2^s+1}^{2^{s+1}} |\Delta a_k|^p\right)^{1/p} = \sum_{s=1}^{\infty} 2^s \Delta_s^{(p)} .$$

Applying the Lemma 1.2, Fomin has proved that for the class F_p, $1 < p \le 2$ we have positive answers to both questions (i) and (ii).

Theorem 1.9 [12] (Fomin). *Let* $\{a_n\} \in F_p$, $1 < p \le 2$. *Then the cosine series* (C) *is the Fourer series of its sum* f *and* (1.1) *hold.*

Next we shall proved that S_p is a subclass of F_p, for all $p > 1$

Theorem 1.10. [67] *For every* $p > 1$ *the following embedding relation holds* $S_p \subset F_p$.

Proof. Applyng the Abel's transformation we have:

$$\sum_{k=2^{s-1}+1}^{2^s} |\Delta a_k|^p = \sum_{k=2^{s-1}+1}^{2^s} A_k^p \frac{|\Delta a_k|^p}{A_k^p} =$$

$$= \sum_{k=2^{s-1}+1}^{2^s-1} \Delta(A_k^p) \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} + A_{2^s}^p \sum_{j=1}^{2^s} \frac{|\Delta a_j|^p}{A_j^p} - A_{2^{s-1}+1}^p \sum_{j=1}^{2^{s-1}} \frac{|\Delta a_j|^p}{A_j^p} =$$

$$= \sum_{k=2^{s-1}+1}^{2^s-1} k\,\Delta(A_k^p) \left(\frac{1}{k} \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p}\right) + 2^s A_{2^s}^p \left(\frac{1}{2^s} \sum_{j=1}^{2^s} \frac{|\Delta a_j|^p}{A_j^p}\right) -$$

$$- 2^{s-1} A_{2^{s-1}+1}^p \left(\frac{1}{2^{s-1}} \sum_{j=1}^{2^{s-1}} \frac{|\Delta a_j|^p}{A_j^p}\right) =$$

$$= O(1) \left[\sum_{k=2^{s-1}+1}^{2^s-1} k\,\Delta(A_k^p) + 2^s A_{2^s}^p + 2^{s-1} A_{2^{s-1}+1}^p\right] =$$

$$= O(1) \left(\sum_{k=2^{s-1}+1}^{2^s} A_k^p + 2^{s-1} A_{2^{s-1}+1}^p - 2^s A_{2^s}^p + 2^s A_{2^s}^p + 2^{s-1} A_{2^{s-1}+1}^p\right) =$$

$$= O(1) \left(\sum_{k=2^{s-1}+1}^{2^s} A_k^p + 2^s A_{2^{s-1}+1}^p\right) = O(2^{s-1} A_{2^{s-1}}^p) .$$

Firstly applying the Fomin's lemma, then the Cauchy type theorem, we obtain

$$\sum_{s=1}^{\infty} 2^s \Delta_s^{(p)} \le O(1) \sum_{s=1}^{\infty} 2^s \left(\frac{1}{2^{s-1}}\, 2^{s-1} A_{2^{s-1}}^p\right)^{1/p} = O\left(\sum_{s=1}^{\infty} 2^{s-1} A_{2^{s-1}}\right) < \infty .$$

Very recenthly, L. Leindler [24] proved very important result. Namely, he proved that the Fomin's class F_p is a subclass of the class S_p and also he given another proof of the Theorem 1.10. Precisely he proved the following Theorem.

Theorem 1.11 (Leindler) [24]. *For all $p > 1$, the classes F_p and S_p are identical.* A still larger class that answers both questions, but expressed in terms of a condition difficult to apply, is the class $BV \cap C$, where C was defined by Garrett and Stanojević [16] as follows: a null sequences of real numbers satisfy the condition C if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ independent of n, such that

$$\int_0^\delta \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx < \varepsilon \,, \quad \text{for every } \; n \,.$$

N. Singh and K. M. Sharma [33] proved that the Garrett-Stanojević class C is a stronger than that of Moore's class (M).

Theorem 1.12 [16] (Garrett and Stanojević). *Let $\{a_n\} \in BV \cap C$. Then the series* (C) *is the Fourier series of its sum f and* (1.1) *hold.*
In [15] Garrett, Rees and Stanojević proved the following theorem.

Theorem 1.13. *For the classes S, BV and C, the following embedding relation holds*

$$S \subset BV \cap C \,.$$

Now, we shall prove the extension theorem of this theorem.

Theorem 1.14 [49]. *For the classes S_p, BV and C, the following embedding relation holds*

$$S_p \subset BV \cap C \,.$$

For the prove of this theorem we need the following lemma.

Lemma 1.3 [61] (Hausdorff-Young). *Let the sequence of complex numbers $\{c_n\} \in l^p$, $1 < p \leq 2$. Then $\{c_n\}$ is the sequence of Fourier coefficients of some $\varphi \in L^q \Big(\frac{1}{p} + \frac{1}{q} = 1 \Big)$ and*

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |\varphi(x)|^q dx \right)^{1/q} \leq \left(\sum_{n=-\infty}^{\infty} |c_n|^p \right)^{1/p} .$$

Proof of Theorem 1.14.

If suffices to show that

$$T_n = \int_0^\pi \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx = o(1)\,, \quad n \to \infty\,.$$

For each n, let k_n be the least natural number such that $n \leq 2^{k_n} - 1$.
Then T_n can be majorized by

$$T_n \leq \int_0^\pi \left| \sum_{j=n}^{2^{k_n}-1} \Delta a_j D_j(x) \right| dx + \sum_{l=k_n}^{\infty} \int_0^\pi \left| \sum_{j=2^l}^{2^{l+1}-1} \Delta a_j D_j(x) \right| dx = I_1 + I_2\,.$$

The second term is written as follows:

$$I_2 = \sum_{l=k_n}^{\infty} \left\{ \int_0^{1/2^{l+1}} + \int_{1/2^{l+1}}^{\pi} \right\} \left| \sum_{j=2^l}^{2^{l+1}-1} \Delta a_j D_j(x) \right| dx = \Sigma_1 + \Sigma_2\,.$$

For the first term, the uniform estimate $|D_n(x)| \leq n + \dfrac{1}{2}$, is applied, i.e.

$$\Sigma_1 \leq \sum_{l=k_n}^{\infty} \frac{1}{2^{l+1}} \sum_{j=2^l}^{2^{l+1}-1} |\Delta a_j| \Big(j + \frac{1}{2} \Big) \leq \sum_{l=k_n}^{\infty} \frac{1}{2^{l+1}} \sum_{j=2^l}^{2^{l+1}-1} |\Delta a_j| 2^{l+1} =$$

$$= \sum_{l=k_n}^{\infty} \sum_{j=2^l}^{2^{l+1}-1} |\Delta a_j| = \sum_{j=2^{k_n}}^{\infty} |\Delta a_j|\,.$$

By summation by parts, and by generalized arithmetic-square mean inequality, we have:

$$\sum_{i=2^{k_n}}^{\infty} |\Delta a_i| = \sum_{i=2^{k_n}}^{\infty} \frac{|\Delta a_i|}{A_i} A_i = \sum_{i=2^{k_n}}^{\infty} \Delta A_i \sum_{j=1}^{i} \frac{|\Delta a_j|}{A_j} - A_{2^{k_n}} \sum_{j=1}^{2^{k_n}-1} \frac{|\Delta a_j|}{A_j} \leq$$

$$\leq \sum_{i=2^{k_n}}^{\infty} i(\Delta A_i) \left(\frac{1}{i} \sum_{j=1}^{i} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} + 2^{k_n} A_{2^{k_n}} \left(\frac{1}{2^{k_n}} \sum_{j=1}^{2^{k_n}} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} =$$

$$= O(1) \left[\sum_{i=2^{k_n}}^{\infty} i(\Delta A_i) + 2^{k_n} A_{2^{k_n}} \right].$$

Since $\sum_{n=1}^{\infty} A_n < \infty$, and $A_n \downarrow 0$ both terms on the right-hand side of the above inequality are $o(1)$ as $n \to \infty$. Thus $\Sigma_1 = o(1)$, $n \to \infty$.

Let

$$\Sigma_2 = \sum_{l=k_n}^{\infty} \int\limits_{1/2^{l+1}}^{\pi} \left| \sum_{j=2^l}^{2^{l+1}-1} \frac{\Delta a_j}{A_j} \, A_j D_j(x) \right| dx \, .$$

Applying Abel's transformation, we get:

$$\int\limits_{1/2^{l+1}}^{\pi} \left| \sum_{j=2^l}^{2^{l+1}-1} \frac{\Delta a_j}{A_j} \, A_j D_j(x) \right| \le \sum_{j=2^l}^{2^{l+1}-2} \Delta A_j \int\limits_{1/2^{l+1}}^{\pi} \left| \sum_{r=1}^{j} \frac{\Delta a_r}{A_r} \, D_r(x) \right| dx +$$

$$+ A_{2^l} \int\limits_{1/2^{l+1}}^{\pi} \left| \sum_{r=1}^{2^l-1} \frac{\Delta a_r}{A_r} \, D_r(x) \right| dx + A_{2^{l+1}-1} \int\limits_{1/2^{l+1}}^{\pi} \left| \sum_{r=1}^{2^{l+1}-1} \frac{\Delta a_r}{A_r} \, D_r(x) \right| dx \, .$$

Applying the Holder type inequality, we get:

$$V_l = \int\limits_{1/2^{l+1}}^{\pi} \left| \sum_{r=1}^{2^l-1} \frac{\Delta a_r}{A_r} \, D_r(x) \right| dx = \int\limits_{1/2^{l+1}}^{\pi} \frac{1}{2 \, \sin \frac{x}{2}} \left| \sum_{r=1}^{2^l-1} \frac{\Delta a_r}{A_r} \, \sin\left(r + \frac{1}{2}\right)x \right| dx \le$$

$$\le \left[\int\limits_{1/2^{l+1}}^{\pi} \frac{dx}{\left(2 \, \sin \frac{x}{2}\right)^p} \right]^{1/p} \left[\int\limits_{1/2^{l+1}}^{\pi} \left| \sum_{r=1}^{2^l-1} \frac{\Delta a_r}{A_r} \sin\left(r + \frac{1}{2}\right)x \right|^q dx \right]^{1/q} ,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Since

$$\int\limits_{1/2^{l+1}}^{\pi} \frac{dx}{\left(2 \, \sin\left(\frac{x}{2}\right)\right)^p} \le \frac{\pi^p}{2^p} \int\limits_{1/2^{l+1}}^{\pi} \frac{dx}{x^p} \le M_p (2^{l+1})^{p-1},$$

where M_p is an absolute constant depending on p, it follows that

$$V_l \le (2^{l+1})^{1/q} (M_p)^{1/p} \left[\int\limits_0^{\pi} \left| \sum_{r=1}^{2^l} \frac{\Delta a_r}{A_r} \sin\left(r + \frac{1}{2}\right)x \right|^q dx \right]^{1/q} .$$

Applying the Hausdorff-Young inequality to the last integral we get:

$$\left[\int\limits_0^{\pi} \left| \sum_{r=1}^{2^l} \frac{\Delta a_r}{A_r} \sin\left(r + \frac{1}{2}\right)x \right|^q dx \right]^{1/q} \le B_p \left(\sum_{r=1}^{2^l-1} \frac{|\Delta a_r|^p}{A_r^p} \right)^{1/p} .$$

Thus

$$V_l \leq 2^{l+1} C_p \left(\frac{1}{2^{l+1}} \sum_{r=1}^{2^{l+1}} \frac{|\Delta a_r|^p}{A_r^p} \right)^{1/p}, \quad C_p > 0\,.$$

Then

$$\Sigma_2 \leq \sum_{l=k_n}^{\infty} \sum_{j=2^l}^{2^{l+1}-2} \Delta A_j \int_{1/2^{l+1}}^{\pi} \left| \sum_{r=1}^{j} \frac{\Delta a_r}{A_r} D_r(x) \right| dx+$$

$$+ \sum_{l=k_n}^{\infty} A_{2^l} \int_{1/2^{l+1}}^{\pi} \left| \sum_{r=1}^{2^l-1} \frac{\Delta a_r}{A_r} D_r(x) \right| dx+$$

$$+ \sum_{l=k_n}^{\infty} A_{2^{l+1}-1} \int_{1/2^{l+1}}^{\pi} \left| \sum_{r=1}^{2^{l+1}-1} \frac{\Delta a_r}{A_r} D_r(x) \right| dx =$$

$$= O_p(1) \left[\sum_{l=k_n}^{\infty} \sum_{j=2^l}^{2^{l+1}-2} j\, \Delta A_j + 4 \sum_{l=k_n}^{\infty} 2^l A_{2^l} \right].$$

Now, applying the Cauchy condensation test, we get:

$$\sum_{l=k_n}^{\infty} 2^l A_{2^l} = o(1)\,, \quad n \to \infty\,.$$

But

$$\sum_{j=2^l}^{2^{l+1}-2} j\, \Delta A_j = \sum_{j=2^l+1}^{2^{l+1}-1} A_j - 2^{l+1} A_{2^{l+1}-1} + 2^l A_{2^l} + A_{2^{l+1}-1} \leq 2^l A_{2^l} + 2^l A_{2^l} + A_{2^l}\,.$$

Thus

$$\sum_{l=k_n}^{\infty} \sum_{j=2^l}^{2^{l+1}-2} j \Delta A_j \leq 2 \sum_{l=k_n}^{\infty} 2^l A_{2^l} + \sum_{l=k_n}^{\infty} A_{2^l} = o(1)\,, \quad n \to \infty\,,$$

i.e. $\Sigma_2 = o(1)$, $n \to \infty$. Finally, $I_2 = o(1)$, $n \to \infty$.
The same method applied to I_1, yields the estimate:

$$I_1 \leq O(1) \sum_{l=2^{k_n-1}}^{2^{k_n}-1} |\Delta a_j| + O_p(1) \left(\sum_{j=2^{k_n-1}}^{2^{k_n}-2} j\, \Delta A_j + 4(2^{k_n-1} A_{2^{k_n-1}}) \right).$$

Letting $n \to \infty$, the proof of the theorem follows.

Remark 1.3. Theorem 1.8 is a corollary of Theorem 1.14 and Theorem 1,12. Thus by proving of the Theorem 1.14 we obtained a new proof of Theorem 1.8. On the other hand, Stanojević [36] proved the following inclusion connecting by the classes F_p, C and BV.

Theorem 1.15. *For all* $1 < p \leq 2$ *the following embedding relation holds*

$$F_p \subset BV \cap C\,.$$

In [15] Garrett, Rees and Stanojević defined an extension class of null sequences of bounded variation of order $m \geq 1$.

Namely, a null sequence $\{a_k\}$ belongs to the class $(BV)^{(m)}$ if for some integer $m \geq 1$, $\sum_{k=1}^{\infty}|\Delta^m a_k| < \infty$, where $\Delta^m a_k = \Delta(\Delta^{m-1}a_k) = \Delta^{m-1}a_k - \Delta^{m-1}a_{k+1}$.

For $m = 1$ the class $(BV)^1$ is the class BV.

Theorem 1.16 [15] (Garrett, Rees, Stanojević). *Let* $\{a_n\} \in (BV)^{(m)}$ *and* $a_n \log n = o(1)$, $n \to \infty$. *Then* $\|S_n - f\| = o(1)$, $n \to \infty$ *iff* $\{a_n\} \in C$

Both Fomin [11] and Stanojević [36] considered the following natural extension of the class F_p.

Let $p \geq 1$. A sequence $\{a_k\}$ belongs to C_p, if $a_k \to 0$ as $k \to \infty$ and

$$n^{p-1}\sum_{k=n}^{\infty}|\Delta a_k|^p = o(1) \quad \text{as} \quad n \to \infty\,. \tag{1.3}$$

Answering the question (ii) Fomin and Stanojević proved the following result:

Theorem 1.17 (Fomin [11], Stanojević [36]). *If* (C) *is a Fourier series of* $f \in L^1$ *and* $\{a_n\} \in C_p \cap BV$ *for some* $1 < p \leq 2$ *then* (1.1) *holds.*

Later, see [13], Fomin extended the above result by considering a still larger class:

Theorem 1.18 (Fomin). *If* (C) *is a Fourier series of* $f \in L^1$ *and for each sequence of natural numbers* $\{m_n\}$ *such that* $\frac{m_n}{n} \to 0$ *as* $n \to \infty$ *there exists* p, $1 < p \leq 2$, *independent of* $\{m_n\}$ *such that*

$$m_n^{p-1}\sum_{k=n}^{n+m_n}|\Delta a_k|^p = o(1)\,, \quad n \to \infty \tag{1.4}$$

then (1.1) *holds.*

The same statement holds for the sine series (S), i.e. the Fourier series of odd functions.

Remark 1.4. It is trivial to see that Theorem 1.17 is a corollary of Th. 1.18, that is that (1.3) implies (1.4) for each sequence of natural numbers $\{m_n\}$ such that $\frac{m_n}{n} \to 0$, $n \to \infty$.
The class C_p has an interesting subclass C_p^*.
A null sequence $\{a_k\}$ belongs to the class C_p^* if for some $1 < p \le 2$,

$$\sum_{k=1}^{\infty} k^{p-1} |\Delta a_k|^p < \infty \,.$$

The next theorem is a corollary to Theorem 1.17.

Theorem 1.19 (Fomin [11], Stanojević [36]). *Let* (C) *be a Fourier series of some* $f \in L^1(0,\pi)$ *and let* $\{a_n\} \in C_p^* \cap BV$, $1 < p \le 2$. *Then* (1.1) *hold.*
A natural extension of BV is the following class: a null-sequence $\{a_k\}$ belongs to the class P if

$$\frac{1}{n} \sum_{k=1}^{n} k |\Delta a_k| = o(1) \,, \quad n \to \infty \,.$$

Combining the class P with the condition $n\,\Delta a_n = O(1)$, Stanojević obtained a theorem for L^1-convergence of Fourier-Stiltjes series.

Theorem 1.20 [36] (Stanojević). *Let* (C) *be a Fourier-Stiltjes seris with* $\{a_k\} \in P$ *and let* $n\,\Delta a_n = O(1)$ *holds. Then it converges in* L^1 *iff* $a_n \log n = o(1)$, $n \to \infty$.
On the other hand Bojanić and Stanojević [4] defined a subclass of P as follows: a null sequence $\{a_k\}$ belongs to the class V_p if for some $p > 1$,

$$\frac{1}{n} \sum_{k=1}^{n} k^p |\Delta a_k|^p = o(1), \quad n \to \infty \,.$$

They proved the following therems.

Theorem 1.21 [4] (Bojanić-Stanojević). *If* (C) *is a Fourier series of* $f \in L^1$ *and* $\{a_k\} \in V_p$ *for some* $1 < p \le 2$ *then* (1.1) *holds.*

Theorem 1.22 [4] (Bojanić, Stanojević). *If* $\{a_k\} \in V_p \cap BV$ *for some* $1 < p \le 2$, *then* (C) *is a Fourier series iff* $\{a_k\} \in C$.
But N. Tanović-Miller [39] considered the problem of integrability of the series (C) in regard to the classes C_p, $p > 1$ and $C_1 = BV$.

Theorem 1.23 [39] (N. Tanović-Miller).
(i) *If* $\{a_k\} \in \cup\{C_p : p \geq 1\}$ *then* (C) *converges a.e. to the function*

$$f(x) = \sum_{k=0}^{\infty} \Delta a_k D_k(x)$$

and (C) *is a Fourier series iff for some* $\delta > 0$*,*

$$\int_0^{\delta} \left| \sum_{k=0}^{\infty} \Delta a_k D_k(x) \right| dx < \infty$$

in which case (C) *is the Fourier series of* f*.*
(ii) *If* $\{a_k\} \in \cup\{C_p : p > 1\}$ *then* (C) *is a Fourier series iff* $\{a_k\} \in C$*.*
These results extend the Theorem 1.22 and show that the classical question on integrability of the series (C) need not be restricted to series with coefficients of bounded variation.
J. W. Garrett and Č. V. Stanojević have obtained a theorem for L^1-convergence of Fourier series with monotone coefficients.

Theorem 1.24 [16](J. W. Garrett, Č. V. Stanojević) *Let*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{CS}$$

be the Fourier series with monotone coefficients. Then (1.1) *hold, where* S_n *is the partial sums of this series.*
On the other hand, S. A. Telyakovskii and G. A. Fomin [45] obtained a similar result for Fourier series with quasi-monotone coefficients.
A null sequence of positive numbers is called quasi-monotone if for some $\alpha \geq 0$ $\dfrac{a_n}{n^\alpha} \downarrow 0$, $n \to 0$ or equivalently $a_{n+1} \leq a_n \left(1 + \dfrac{\alpha}{n}\right)$.
Theorem 1.25 [45] (Fomin, Telyakovskii). *Let* $\{a_n\}$ *be a quasi-monotone sequence. If* (C) *is the Fourier series of its sum* f*, then* (1.1) *hold.*
The proof of sufficiency in the theorem of Fomin-Telyakovskii is simplified by Garrett-Rees-Stanojević [14] using more refined estimations of $\|S_n - \sigma_n\|$.
Telyakovskii and Fomin [45] also proved a corresponding result for the sine series, namely if $\{a_k\}$ is a quasi-monotone sequence and (S) is the Fourier series of its sum g then the same conclusion holds for the sine series.

Theorem 1.26 [14] (Garrett-Rees-Stanojević). *Let* (CS) *be the Fourier series with quasi-monotone coefficients. Then* $\|S_n - \sigma_n\| = o(1)$*,* $n \to \infty$ *iff* $(a_n + b_n)\log n = o(1)$*,* $n \to \infty$*.*

The class P extends not only BV, but the class of quasi-monotone sequences. The next theorem is a slightly weaker form of a theorem of Telyakovskii and Fomin.

Theorem 1.27 [36] (Stanojević). *Let* (C) *be a Fourier series with quasi-monotone coefficients and let* $n\,\Delta a_n = O(1)$ *holds. Then* (1.1) *hold.*

Later, Bray and Stanojević [8] considered the question of L^1-convergence for more general Fourier series of so called asymptotically even functions.

Concerning the Fourier series of even functions one of the results in [8] can be stated as follows:

Theorem 1.28 (Bray, Stanojević). *If* (C) *is a Fourier series of* $f \in L^1$ *and for some* $1 < p \le 2$

$$\lim_{\lambda \to 1^+} \overline{\lim_{n \to \infty}} \sum_{n=k}^{[\lambda_n]} k^{p-1} |\Delta a_k|^p = 0\,,$$

then (1.1) *holds.*

Remark 1.5. Theorem 1.28 is corollary of Theorem 1.18.

1.2. GENERALIZATIONS OF THE SIDON-FOMIN'S LEMMA

Sidon [35] proved the inequality named after him in 1939 year. It is an upper estimate for the integral norm of a linear combination of trigonometric Dirichlet kernels expressed in terms of the coefficients. Since the estimate has many applications for instance in L^1-convergence problems and summation method with respects to trigonometric series, newer and newer improvements of the original inequality has been proved by several authors. Fomin [9] applying the linear method for summing of Fourier series has given another proof if this inequality. Thus the inequality is known as Sidon-Fomin's inequality.

Also, S. A. Telyakovskii in [44] has given an elegant proof of Sidon-Fomin's inequality.

Lemma 1.4 (Sidon-Fomin). *Let* $\{\alpha_k\}_{k=0}^{n}$ *be a sequence of real numbers such that*

$|\alpha_k| \le 1$ *for all* k. *Then there exists a positive constant* C *such that for any* $n \ge 0$,

$$\left\| \sum_{k=0}^{n} \alpha_k D_k(x) \right\| \le C(n+1)\,.$$

For the proof of our new result we need the following lemma.

Lemma 1.5 [63]. *If* $T_n(x)$ *is trigonometric polynomial of order* n, *then*

$$\|T_n^{(r)}\| \le n^r \|T_n\|\,.$$

This is S. Bernstein's inequality in the $L^1(0,\pi)$-metric (see [63], vol. 2, p. 11).

Lemma 1.6 [51]. *Let* $\{\alpha_k\}_{k=0}^{n}$ *be a sequence of real numbers such that* $|\alpha_k| \le 1$ *for all* k. *Then there exists a constant* $C > 0$, *such that for any* $n \ge 0$,

$$\left\| \sum_{k=0}^{n} \alpha_k D_k^{(r)}(x) \right\| \le C(n+1)^{r+1},$$

where $D_k^{(r)}(x)$, $k = 0, 1, 2, \ldots, n$ *is the* r*-th derivate of the Dirichlet's kernel.*

Proof. Since

$$\sum_{k=0}^{n} \alpha_k D_k(x) = \frac{1}{2} \sum_{i=0}^{n} \alpha_i + \sum_{k=1}^{n} \left(\sum_{i=k}^{n} \alpha_i \right) \cos kx\,,$$

we have that

$$\sum_{k=0}^{n} \alpha_k D_k(x)$$

is a cosine trigonometric polynomial of order n.
Applyyng first Bernstein's inequality, then Sidon-Fomin's lemma, yields:

$$\left\| \sum_{k=0}^{n} \alpha_k D_k^{(r)}(x) \right\| \le (n+1)^r \left\| \sum_{k=0}^{n} \alpha_k D_k(x) \right\| \le C(n+1)^{r+1}, \quad C > 0\,.$$

Lemma 1.7 [10] (Fomin-Stečkin). *Let* $1 < p \le 2$ *and* $\{\alpha_k\}_{k=0}^{n}$ *be a sequence of real numbers such that* $\sum_{k=0}^{n} \alpha_k^p \le A^p(n+1)$. *Then there exists a positive constant* C_p *depends only on* p *such that the following inequality holds:*

$$\left\| \sum_{i=0}^{n} \alpha_i D_i(x) \right\| \le C_p A(n+1)\,.$$

Lemma 1.8 [4] (Bojanić-Stanojević). *Let* $\{\alpha_k\}_{k=0}^{n}$ *be a sequence of real numbers. Then for any* $1 < p \leq 2$ *and* $n \geq 0$ *the following inequality holds:*

$$\left\|\sum_{k=0}^{n} \alpha_k D_k(x)\right\| \leq C_p(n+1)\left(\frac{1}{n+1}\sum_{k=0}^{n}|\alpha_k|^p\right)^{1/p}, \tag{1.5}$$

where the constant C_p *depends only on* p.

Remark 1.6. We note that this estimate is essentially contained (case $p = 2$) in Fomin [9].

Remark 1.7. It's casy to see that Bojanić-Stanojević type inequality is not valid for $p = 1$.
Indeed, if $\alpha_n = 1$ and $\alpha_k = 0$ $(k \neq n,\ k \in N)$ then the left side is of order $\dfrac{\log n}{n}$ while the right side is of order $\dfrac{1}{n}$ as $n \to \infty$.

Remark 1.8. Sidon-Fomin's inequality is a special case of the Bojanić-Stanojević inequality, i.e. it can easily be deduced from Lemma 1.8.
Now, we will prove a counterpart of inequality (1.5) in the case where $D_k^{(r)}$ is used instead of $D_k(x)$.

Lemma 1.9 [57]. *Let* $\{\alpha_k\}_{k=0}^{n}$ *be a sequence of real numbers. Then for any* $1 < p \leq 2$, *and* $r = 0, 1, 2, \ldots$, $n \geq 0$ *the following inequality holds:*

$$\left\|\sum_{k=0}^{n} \alpha_k D_k^{(r)}(x)\right\| \leq C_p(n+1)^{r+1}\left(\frac{1}{n+1}\sum_{k=0}^{n}|\alpha_k|^p\right)^{1/p}$$

where the constant C_p *depends only on* p.

Proof. Applying first Bernstein's inequality, then Bojanić-Stanojević inequality, yields

$$\left\|\sum_{k=0}^{n} \alpha_k D_k^{(r)}(x)\right\| \leq (n+1)^r \left\|\sum_{k=0}^{n} \alpha_k D_k(x)\right\| \leq C_p(n+1)^{r+1}\left(\frac{1}{n+1}\sum_{k=0}^{n}|\alpha_k|^p\right)^{1/p}.$$

1.3. THE EXTENSIONS ON SOME CLASSES OF FOURIER COEFFICIENTS

In this parth we shall give the extensions of the Garrett-Stanojević class C, Sidon-Telyakovskii class S and the class S_p, $p > 1$ defined by V. B. Stanojević-Č V. Stanojević respectively.

A null sequence $\{a_k\}$ belongs to the class C_r, $r = 0, 1, 2, \dots$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\int_0^\pi \left| \sum_{k=n}^\infty \Delta a_k D_k^{(r)}(x) \right| dx < \varepsilon, \quad \text{for all} \;\; n,$$

where $D_k^{(r)}(x)$ is the r-th derivate of the Dirichlet's kernel.

When $r = 0$, we denote $C_r = C$.

A null sequence $\{a_k\}$ belongs to the class $\Im_r$ $r = 0, 1, 2, \dots$ if there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum\limits_{k=1}^\infty k^r A_k < \infty$ and $|\Delta a_k| \le A_k$, for all k. When $r = 0$ it is clear that $\Im_r = S$.

A null sequence $\{a_k\}$ belongs to the class S_{pr}, $1 < p \le 2$, $r = 0, 1, 2, \dots$ if there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum\limits_{k=1}^\infty k^r A_k < \infty$ and

$$\frac{1}{n} \sum_{k=1}^n \frac{|\Delta a_k|^p}{A_k^p} = O(1).$$

When $r = 0$, we denote $S_p = S_{pr}$.

The following lemma was proved by Valee Poussin Ch.J.de la, (see [60]), but we shall present a direct proof of this lemma.

Lemma 1.10. *$A_n \downarrow 0$ with $\sum\limits_{n=1}^\infty n^r A_n < \infty$, $r \ge 0$ then $n^{r+1} A_n = o(1)$, $n \to \infty$.*

Proof 1. Let $0 < m < n$. Adding the inequalities

$$n^{r+1}\Delta A_{n-1} \geq 0$$

$$(n-1)^{r+1}\Delta A_{n-2} \geq 0$$

$$(n-2)^{r+1}\Delta A_{n-3} \geq 0$$

$$\dots \quad \dots \quad \dots \qquad \dots$$

$$(m+1)^{r+1}\Delta A_m \geq 0\,,$$

we obtain:

$$-A_n n^{r+1} + \sum_{k=m+1}^{n-1} A_k[(k+1)^{r+1} - k^{r+1}] + A_m(m+1)^{r+1} \geq 0\,.$$

The summ on the left is $o(1)$ becouse $\sum_{n=1}^{\infty} n^r A_n < \infty$.

Hence,

$$A_m(m+1)^{r+1} - A_n n^{r+1} \geq o(1)\,, \quad m, n \to \infty\,.$$

Since $m^r A_m \to 0$, this means that

$$A_m m^{r+1} - A_n n^{r+1} \geq o(1)\,, \quad m, n \to \infty\,. \tag{1.6}$$

We cannot have $\liminf_{n\to\infty} n^{r+1} A_n > 0$, since otherwise $\sum_{n=1}^{\infty} n^r A_n$ could not converge. Hence, in particular there is for each $\varepsilon > 0$ an infinite sequence of indicies m for which

$$m^{r+1} A_m < \varepsilon\,. \tag{1.7}$$

Now suppose that $\limsup_{n\to\infty} n^{r+1} A_n > 0$. Then there exists $\varepsilon > 0$, and there is an infinite sequence of indicies n such that

$$n^{r+1} A_n > 2\varepsilon > 0 \tag{1.8}$$

For each m satisfyng (1.7) take a larger n satisfyng (1.8), we get a contradiction of (1.6). Hence $\limsup_{n\to\infty} n^{r+1} A_n = 0$, i.e. $n^{r+1} A_n = o(1)$, $n \to \infty$.

Proof 2. By inequalities

$$n^{r+1} A_{2n} \leq n^r (A_{n+1} + A_{n+2} + \cdots + A_{2n}) \leq \sum_{i=n+1}^{\infty} i^r A_i\,,$$

we obtain:

$$(2n)^{r+1}A_{2n} \leq 2^{r+1} \sum_{i=n+1}^{\infty} i^r A_i = o(1), \quad n \to \infty .$$

Similarly, we can get:

$$(2n+1)^{r+1}A_{2n+1} \leq \left(2+\frac{1}{n}\right)^{r+1} \sum_{i=n+1}^{\infty} i^r A_i = o(1)\,, \quad n \to \infty .$$

Finally

$$n^{r+1}A_n = o(1)\,, \quad n \to \infty .$$

Lemma 1.11. *If $A_n \downarrow 0$ with $\sum_{n=1}^{\infty} n^r A_n < \infty$, $r \geq 0$, then $\sum_{n=1}^{\infty} n^{r+1}(\Delta A_n) < \infty$.*

Proof. By partial summation,

$$\sum_{k=1}^{n-1} k^{r+1}(\Delta A_k) = \sum_{k=1}^{n}[k^{r+1}-(k-1)^{r+1}]A_k - n^{r+1}A_n = O\left(\sum_{k=1}^{n} k^r A_k\right) - n^{r+1}A_n .$$

The series on the right converges; $n^{r+1}A_n = o(1)$, $n \to \infty$, by Lemma 1.10; so the partial sums on the left are converges as $n \to \infty$.

It is trivially to see that $\Im_{r+1} \subset \Im_r$ for all $r = 1, 2, 3, ...$ Now, let $\{a_n\}_{n=1}^{\infty} \in \Im_1$. For arbitrary real number a_0, we shall prove that sequence $\{a_n\}_{n=0}^{\infty}$ belongs to S. We define $A_0 = \max(|\Delta a_0|, A_1)$. Then $|\Delta a_0| \leq A_0$, i.e. $|\Delta a_n| \leq A_n$, for all $n \in \{0, 1, 2, ...\}$ and $\{A_n\}_{n=0}^{\infty}$ is monotonically decreasing sequence.

On the other hand,

$$\sum_{n=0}^{\infty} A_n \leq A_0 + \sum_{n=1}^{\infty} nA_n .$$

If $A_0 = |\Delta a_0|$, then

$$\sum_{n=0}^{\infty} A_n = |\Delta a_0| + \sum_{n=1}^{\infty} A_n \leq |a_0| + |a_1| + \sum_{n=1}^{\infty} nA_n \leq$$

$$\leq |a_0| + \sum_{n=1}^{\infty} |\Delta a_n| + \sum_{n=1}^{\infty} nA_n \leq |a_0| + 2\sum_{n=1}^{\infty} nA_n < \infty .$$

If $A_0 = A_1$, then

$$\sum_{n=0}^{\infty} A_n = A_1 + \sum_{n=1}^{\infty} A_n \leq 2\sum_{n=1}^{\infty} nA_n < \infty .$$

Thus, $\{a_n\}_{n=0}^{\infty} \in S$, i.e. $\Im_{r+1} \subset \Im_r$, for all $r = 0, 1, 2, \ldots$. The next example verifies that the implication

$$\{a_n\} \in \Im_{r+1} \Rightarrow \{a_n\} \in \Im_r\,, \quad r = 0, 1, 2, \ldots$$

is not reversible.

Example [54]. For $n = 1, 2, 3, \ldots$ define $a_n = \sum_{k=n+1}^{\infty} \frac{1}{k^2}$. Then $a_n \to 0$ as $n \to \infty$ and for $n = 1, 2, 3, \ldots$, $\Delta a_n = \frac{1}{(n+1)^2}$. Firstly we shall show that $\{a_n\} \notin \Im_1$.
Let $\{A_n\}_{n=1}^{\infty}$ is an arbitrary positive sequence such that $A \downarrow 0$ and $\Delta a_n \le A_n$. However, $\sum_{n=1}^{\infty} nA_n \ge \sum_{n=1}^{\infty} \frac{n}{(n+1)^2}$ is divergent, i.e. $\{a_n\} \notin \Im_1$.
Now, for all $n = 0, 1, 2, \ldots$ let $A_n = \frac{1}{(n+1)^2}$. Then $A_n \downarrow 0$, $|\Delta a_n| \le A_n$ and $\sum_{n=0}^{\infty} A_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, i.e. $\{a_n\} \in S$.
Our next example will show that there exists a sequence $\{a_n\}_{n=1}^{\infty}$ such that $\{a_n\}_{n=1}^{\infty} \in \Im_r$ but $\{a_n\}_{n=1}^{\infty} \notin \Im_{r+1}$, for all $r = 1, 2, 3, \ldots$.
Namely, for all $n = 1, 2, 3, \ldots$ let $a_n = \sum_{k=n}^{\infty} \frac{1}{k^{r+2}}$. Then $a_n \to 0$ as $n \to \infty$ and for $n = 1, 2, 3, \ldots$, $\Delta a_n = \frac{1}{n^{r+2}}$. Let $\{A_n\}_{n=1}^{\infty}$ is an arbitrary positive sequence such that $A_n \downarrow 0$ and $\Delta a_n \le A_n$. However,

$$\sum_{n=1}^{\infty} n^{r+1} A_n \ge \sum_{n=1}^{\infty} n^{r+1} \frac{1}{n^{r+2}} = \sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent, i.e. $\{a_n\} \notin \Im_{r+1}$. On the other hand, for all $n = 1, 2, \ldots$ let $A_n = \frac{1}{n^{r+2}}$. Then $A_n \downarrow 0$, $|\Delta a_n| \le A_n$ and $\sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \frac{1}{n^{r+2}} < \infty$, i.e. $\{a_n\} \in \Im_r$.

Theorem 1.29 [51] *For all $r = 0, 1, 2, \ldots$ the following embedding relation holds*

$$\Im_r \subset BV \cap C_r\,.$$

Proof. It is clear that $\{a_n\} \in \Im_r$ implies $\{a_n\} \in BV$. Now for $x \ne 0$ we consider the identity:

$$\sum_{k=n}^{\infty} \Delta a_k D_k(x) = \sum_{k=n}^{\infty} (\Delta A_k) \sum_{j=0}^{k} \frac{\Delta a_j}{A_j}\, D_j(x) - A_n \sum_{j=0}^{n-1} \frac{\Delta a_j}{A_j}\, D_j(x)\,.$$

In the proof of Theorem 3.8, we shall prove that series $\sum_{k=1}^{\infty}\Delta a_k D_k^{(r)}(x)$ is uniformly convergent on any compact subset of $(0,\pi)$. This imply that

$$\int_0^{\delta}\left|\sum_{k=n}^{\infty}\Delta a_k D_k^{(r)}(x)\right|dx \le$$

$$\le \sum_{k=n}^{\infty}(\Delta A_k)\int_0^{\pi}\left|\sum_{j=0}^{k}\frac{\Delta a_j}{A_j}\,D_j^{(r)}(x)\right|dx + A_n\int_0^{\pi}\left|\sum_{j=0}^{n-1}\frac{\Delta a_j}{A_j}\,D_j^{(r)}(x)\right|dx\,.$$

Since $\left|\dfrac{\Delta a_j}{A_j}\right| \le 1$, applying the Lemma 1.6, and Lemma 1.10, we can get:

$$\int_0^{\delta}\left|\sum_{k=n}^{\infty}\Delta a_k D_k^{(r)}(x)\right|dx \le O(1)\left[\lim_{N\to\infty}\sum_{k=n}^{N-1}(\Delta A_k)(k+1)^{r+1} + A_n n^{r+1}\right] =$$

$$= O(1)\lim_{N\to\infty}\left[\sum_{k=n}^{N}[(k+1)^{r+1}-k^{r+1}]A_k - (N+1)^{r+1}A_N\right] + O(n^{r+1}A_n) =$$

$$= O\left(\sum_{k=n}^{\infty}k^r A_k\right) + o(1) = o(1)\,,\quad n\to\infty\,.$$

Next we define a new class $\Im_r^2$, $r = 0,1,2,\dots$ as follows: a null sequence $\{a_k\}$ belongs to the class $\Im_r^2$, if there exists a null monotonically decreasing sequence $\{A_k\}$ of nonnegative numbers such that $\sum_{k=1}^{\infty}k^{r+1}(\Delta A_k) < \infty$ and $|\Delta a_k| \le A_k$ for all k.

Theorem 1.30. *The class $\Im_r$ is equivalent to $\Im_r^2$, for all $r = 0,1,2,\dots$*

Proof. Let $\{a_n\}\in\Im_r$. Applying the Lemma 1.11, we get $\sum_{n=1}^{\infty}n^{r+1}(\Delta A_n) < \infty$. Now, if $\{a_n\}\in\Im_r^2$, we have:

$$n^{r+1}A_n = n^{r+1}\sum_{k=n}^{\infty}\Delta A_k \le$$

$$\le \sum_{k=n}^{\infty}k^{r+1}(\Delta A_k) = o(1),\quad n\to\infty,\quad \text{i.e.}\quad n^{r+1}A_n = o(1),\quad n\to\infty\,.$$

Then

$$\sum_{k=1}^{n} k^r A_k = \sum_{k=1}^{n-1} (\Delta A_k) \sum_{j=1}^{k} j^r + A_n \sum_{j=1}^{n} j^r = O\left(\sum_{k=1}^{n-1} k^{r+1} (\Delta A_k)\right) + O(n^{r+1} A_n)\,.$$

Letting $n \to \infty$, we obtain $\sum_{k=1}^{\infty} k^r A_k < \infty$, i.e. $\{a_n\} \in \Im_r$.

Lemma 1.12 [50]. *Let $\{\alpha_j\}_{j=1}^{k}$ be a sequence of real numbers. Then the following relation holds for $1 < p \leq 2$, $v = 0, 1, 2, \ldots, r$ and $r = 0, 1, 2, \ldots$*

$$V_k = \int_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \alpha_j \frac{\left(j + \frac{1}{2}\right)^v \sin\left[\left(j + \frac{1}{2}\right)x + \frac{v\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \right| dx =$$

$$= O_p \left[k^{r+1} \left(\frac{1}{k} \sum_{j=1}^{k} |\alpha_j|^p \right)^{1/p} \right],$$

where O_p depends only on p.

Proof. Applying first Holder inequality, yields:

$$V_k = \int_{\pi/k}^{\pi} \frac{1}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \left| \sum_{j=1}^{k} \alpha_j \left(j + \frac{1}{2}\right)^v \sin\left[\left(j + \frac{1}{2}\right)x + \frac{v\pi}{2}\right] \right| dx \leq$$

$$\leq \left[\int_{\pi/k}^{\pi} \frac{dx}{\left(\sin\left(\frac{x}{2}\right)\right)^{(r+1-v)p}} \right]^{1/p} \times$$

$$\times \left\{ \int_{0}^{\pi} \left| \sum_{j=1}^{k} \alpha_j \left(j + \frac{1}{2}\right)^v \sin\left[\left(j + \frac{1}{2}\right)x + \frac{v\pi}{2}\right] \right|^q dx \right\}^{1/q}.$$

Since

$$\int_{\pi/k}^{\pi} \frac{dx}{\left(\sin\left(\frac{x}{2}\right)\right)^{(r+1-v)p}} \leq \frac{\pi k^{(r+1-v)p-1}}{(r+1-v)p-1} \leq \frac{\pi}{p-1}\, k^{(r+1-v)p-1},$$

we have:

$$V_k \leq \left(\frac{\pi}{p-1}\right)^{1/p} (k^{(r+1-v)p-1})^{1/p} \times$$

$$\times \left\{ \int_{0}^{\pi} \left| \sum_{j=1}^{k} \alpha_j \left(j + \frac{1}{2}\right)^v \sin\left[\left(j + \frac{1}{2}\right)x + \frac{v\pi}{2}\right] \right|^q dx \right\}^{1/q}.$$

Then using the Hausdorff-Young inequality we get:

$$\left\{\int_0^\pi \left|\sum_{j=1}^k \alpha_j \Big(j+\frac{1}{2}\Big)^v \sin\Big[\Big(j+\frac{1}{2}\Big)x+\frac{v\pi}{2}\Big]\right|^q dx\right\}^{1/q} = O_p\left[\left(\sum_{j=1}^k |\alpha_j|^p j^{vp}\right)^{1/p}\right].$$

Finally,

$$V_k = O_p\left[(k^{(r+1-v)p-1})^{1/p}\left(\sum_{j=1}^k |\alpha_j|^p\, j^{vp}\right)^{1/p}\right] =$$

$$= O_p\left[(k^{(r+1)p-1})^{1/p}\left(\sum_{j=1}^k |\alpha_j|^p\right)^{1/p}\right] =$$

$$= O_p\left[k^{r+1}\left(\frac{1}{k}\sum_{j=1}^k |\alpha_j|^p\right)^{1/p}\right],$$

where O_p depends only on p.

Lemma 1.13 [30]. *Let r be a nonnegative integer and $x \in (0,\pi]$. Then*

$$D_n^{(r)}(x) = \sum_{k=0}^{r-1} \frac{\Big(n+\frac{1}{2}\Big)^k \sin\Big[\Big(n+\frac{1}{2}\Big)x+\frac{k\pi}{2}\Big]}{\Big(\sin\Big(\frac{x}{2}\Big)\Big)^{r+1-k}}\,\varphi_k(x)+$$

$$+\frac{\Big(n+\frac{1}{2}\Big)^r \sin\Big[\Big(n+\frac{1}{2}\Big)x+\frac{r\pi}{2}\Big]}{2\,\sin\Big(\frac{x}{2}\Big)},$$

where the same φ_k denotes various analytical function of x, independent of n.

Lemma 1.14. *Let the coefficients $\{a_j\}_{j=0}^k$ belong to the class S_{pr}, $1 < p \le 2$, $r = 0, 1, 2, \ldots$. Then the following inequality holds*

$$\int_0^\pi \left|\sum_{j=1}^k \frac{\Delta a_j}{A_j} D_j^{(r)}(x)\right| dx = O_p(k^{r+1}).$$

Proof. We have:

$$\int_0^\pi \left|\sum_{j=1}^k \frac{\Delta a_j}{A_j}\, D_j^{(r)}(x)\right| dx = \int_0^{\pi/k} + \int_{\pi/k}^{\pi} = I_k + J_k.$$

Applying the inequality $D_n^{(r)}(x) = O(n^{r+1})$, we have:

$$I_k \le \alpha \sum_{j=1}^{k} j^r \frac{|\Delta a_j|}{A_j} \le \alpha k^r \sum_{j=1}^{k} \frac{|\Delta a_j|}{A_j} \le$$

$$\le \alpha k^{r+1} \left(\frac{1}{k} \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} = O(k^{r+1})$$

where α is a positive constant.
Applying the Lemma 1.13, let us estimate the second integral:

$$J_k = \int_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx \le$$

$$\le \int_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} \left(\sum_{v=0}^{r-1} \frac{\left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{v\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \varphi_v(x) \right) \right| dx +$$

$$+ \int_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} \frac{\left(j+\frac{1}{2}\right)^r \sin\left[\left(j+\frac{1}{2}\right)x + \frac{r\pi}{2}\right]}{2\sin\left(\frac{x}{2}\right)} \right| dx = \lambda_k + \mu_k \,.$$

Since φ_v are bounded, we have:

$$\int_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} \frac{\left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{v\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \varphi_v \right| dx \le$$

$$\le B \int_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \alpha_j \frac{\left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{v\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \right| dx \,,$$

where B is a positive constant and $\alpha_j = \dfrac{\Delta a_j}{A_j}$, $j = 1, 2, \ldots, k$.
Applying Lemma 1.12 to the last integral, we get:

$$\int_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} \frac{\left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{v\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \varphi_v(x) \right| dx =$$

$$= O_p \left(k^{r+1} \left(\frac{1}{k} \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} \right) = O_p(k^{r+1})\,.$$

Since r is a finite value, we have: $\lambda_k = O_p(k^{r+1})$. Similarly, we can get: $\mu_k = O_p(k^{r+1})$.
Hence

$$\int_0^\pi \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx = O(k^{r+1}) + O_p(k^{r+1}) = O_p(k^{r+1})\,.$$

Lemma 1.15. *Let the coefficients $\{a_j\}_{j=0}^{k}$ belong to the class S_{pr}, $1 < p \le 2$, $r = 0, 1, 2, \ldots$ Then*

$$A_n \int_0^\pi \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx = o(1)\,, \quad n \to \infty\,.$$

Proof. Applying first the Lemma 1,14, then Lemma 1.10, we obtain

$$A_n \int_0^\pi \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx = O_p(n^{r+1} A_n) = o(1)\,, \quad n \to \infty\,.$$

Theorem 1.31 [50]. *For each $1 < p \le 2$ and $r = 0, 1, 2, \ldots$ the following embedding relationholds*

$$S_{pr} \subset BV \cap C_r\,.$$

Proof. We have:

$$\sum_{k=1}^{n} |\Delta a_k| \le \sum_{k=1}^{n} k^r |\Delta a_k| = \sum_{k=1}^{n-1} (\Delta A_k) \sum_{j=1}^{k} \frac{|\Delta a_j|}{A_j}\, j^r + A_n \sum_{j=1}^{n} \frac{|\Delta a_j|}{A_j}\, j^r \le$$

$$\le \sum_{k=1}^{n-1} k^r (\Delta A_k) \left(\sum_{j=1}^{k} \frac{|\Delta a_j|}{A_j} \right) + n^r A_n \sum_{j=1}^{n} \frac{|\Delta a_j|}{A_j} \le$$

$$\le \sum_{k=1}^{n-1} k^{r+1} (\Delta A_k) \left(\frac{1}{k} \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} + n^{r+1} A_n \left(\frac{1}{n} \sum_{j=1}^{n} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} =$$

$$= O(1) \left[\sum_{k=1}^{n-1} k^{r+1} (\Delta A_k) + n^{r+1} A_n \right] = O\left(\sum_{k=1}^{n} k^r A_k \right)\,.$$

Letting $n \to \infty$, we get $\{a_n\} \in BV$.

Then applying Abel's transformation, Lemma 1.15, Lemma 1.14 and Lemma 1.11 we have:

$$\int_0^{\delta}\left|\sum_{k=n}^{\infty}\Delta a_k D_k^{(r)}(x)\right|dx \le \sum_{k=n}^{\infty}(\Delta A_k)\int_0^{\pi}\left|\sum_{j=1}^{k}\frac{\Delta a_j}{A_j}\,D_j^{(r)}(x)\right|dx + o(1)$$

$$= O_p(1)\left[\sum_{k=n}^{\infty}k^{r+1}(\Delta A_k)\right] = o(1)\,, \quad n\to\infty\,.$$

II. CLASSIFICATION ON QUASI-MONOTONE SEQUENCES AND ITS APPLICATIONS FOR L^1-CONVERGENCE OF TRIGONOMETRIC SERIES

2.1. REMARKS ON TRIGONOMETRIC SERIES WITH QUASI-MONOTONE COEFFICIENTS

Quasi-monotone sequences are known to share many of the numbers properties of decreasing sequences: for example Vallee Poussin's [60] theorem: $\sum_{n=1}^{\infty} a_n < \infty \Rightarrow na_n \to 0$ (see also in [38]), the Cauchy condensation test for convergence and a number of theorems about the trigonometric series.
Some proofs of the convergence theorems about the trigonometric series are based on the use of modified cosine sums defined by Rees-Stanojević [29] as follows:

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\left[\left(\sum_{i=k}^{n} \Delta a_i\right)\cos kx\right] = S_n(x) - a_{n+1}D_n(x)\,.$$

Marzug [25] proved the following theorem for L^1-convergence of trigonometric series with quasi-monotone coefficients.

Theorem 2.1 [25] (Marzug). *Let $\{a_k\}$ be a nonnegative quasi-monotone sequence tending to zero, $\sum_{k=1}^{\infty}\frac{a_k}{k} < \infty$ and $\sum_{k=1}^{\infty}(k+1)[|\Delta a_k| - \Delta a_k] < \infty$. Then $\lim_{n\to\infty} g_n(x) = g(x) \in L^1[-\pi,\pi]$ iff $\sum_{n=1}^{\infty} a_n < \infty$.*

N. Singh and K. M. Sharma [33] defined a class of L^1-convergence as follows. Namely, a sequence $\{a_k\}$ belongs to the class S' if $a_k \to 0$ as $k \to \infty$ and there exists a sequence $\{A_k\}$ such that $\{A_k\}$ is quasi-monotone, $\sum_{k=1}^{\infty} A_k < \infty$, and $|\Delta a_k| \le A_k$, for all k. They proved the following theorem.

Theorem 2.2 [33]. *Let $\{a_k\} \in S'$, then $g_n(x)$ convergens to $g(x)$ in L^1-metric.*

Let the null quasi-monotone sequence be denoted by $A_n \Downarrow 0$.
For convience the following notations are used:

$$M_\alpha = \{A_n \colon A_n \downarrow 0 \quad \text{and} \quad \sum_{n=1}^{\infty} n^\alpha A_n < \infty\,, \quad \text{for some} \quad \alpha \geq 0\}$$

$$M'_\alpha = \{A_n \colon A_n \Downarrow 0 \quad \text{and} \quad \sum_{n=1}^{\infty} n^\alpha A_n < \infty\,, \quad \text{for some} \quad \alpha \geq 0\}$$

$$S_{p\alpha r} = \{a_n \colon a_n \to 0\,, \quad n \to \infty\,, \quad \text{for some} \quad \alpha \geq 0, \quad r \in \{0,1,2,\dots[\alpha]\}\,,$$

$$\frac{1}{n^{p(\alpha-r)+1}} \sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1)\,, \quad p > 1 \quad \text{and} \quad A_n \in M_\alpha\}$$

$$S'_{p\alpha r} = \{a_n \colon a_n \to 0\,, \quad n \to \infty\,, \quad \text{for some} \quad \alpha \geq 0, \quad r \in \{0,1,2,\dots[\alpha]\}\,,$$

$$\frac{1}{n^{p(\alpha-r)+1}} \sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1)\,, \quad p > 1 \quad \text{and} \quad A_n \in M'_\alpha\}\,.$$

We note that the classes M'_α and $S'_{p\alpha r}$ were defined by Sheng [30].

Theorem 2.3 [68], [70]. *The classes M_α and M'_α are identical.*

Proof. It is obvious that $M_\alpha \subset M'_\alpha$. For the proof of the inclusion $M'_\alpha \subset M_\alpha$, we use an idea obtained by Telyakovskii [47], i.e. we construct the sequence:

$$B_k = A_k + \beta \sum_{m=k}^{\infty} \frac{A_m}{m}\,, \quad \text{for some} \quad \beta \geq 0\,, \quad \text{where} \quad A_n \in M'_\alpha\,. \tag{2.1}$$

We have:

$$B_k - B_{k+1} = \Delta B_k = \Delta A_k + \beta\,\frac{A_k}{k} \geq 0\,, \quad \text{i.e.} \quad B_k \downarrow 0 \quad \text{as} \quad k \to \infty \quad \text{and}$$

$$\sum_{k=1}^{\infty} k^\alpha B_k = \sum_{k=1}^{\infty} k^\alpha A_k + \sum_{k=1}^{\infty} \beta k^\alpha \sum_{m=k}^{\infty} \frac{A_m}{m} \leq \sum_{k=1}^{\infty} k^\alpha A_k + \beta \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} m^{\alpha-1} A_m$$

$$= \sum_{k=1}^{\infty} k^\alpha A_k + \beta \sum_{m=1}^{\infty} \sum_{n=1}^{m} m^{\alpha-1} A_m = \sum_{k=1}^{\infty} k^\alpha A_k + \beta \sum_{m=1}^{\infty} m^\alpha A_m < \infty\,.$$

Thus $M_\alpha \equiv M'_\alpha$.

Theorem 2.4 [68], [70]. *The classes $S_{p\alpha r}$ and $S'_{p\alpha r}$ are identical.*

Proof. It obvious that $S_{p\alpha r} \subset S'_{p\alpha r}$. Let $\{a_n\} \in S'_{p\alpha r}$. It suffices to show that the sequence (2.1) satisfyes the condition

$$\frac{1}{n^{p(\alpha-r)+1}} \sum_{k=1}^{n} \frac{|\Delta a_k|^p}{B_k^p} = O(1)\,.$$

Clearly,

$$\frac{1}{n^{p(\alpha-r)+1}} \sum_{k=1}^{n} \frac{|\Delta a_k|^p}{B_k^p} \le \frac{1}{n^{p(\alpha-r)+1}} \sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1)\,, \quad \text{i.e} \quad S_{p\alpha r} \equiv S'_{p\alpha r}\,.$$

Fomin [13], applying the following two theorems have given a new proof of the Theorem 1.25.

Theorem 2.5 [13]. *If* (C) *converges in mean, then for any sequence of natural numbers* $\{m_n\}$ *such that* $m_n \le n$, $n = 1, 2, 3, \ldots$ *the following limit holds:*

$$\lim_{n\to\infty} \sum_{k=1}^{m_n} \frac{a_{n+k}}{k} = 0\,.$$

Theorem 2.6 [13]. *Let* $\{m_n\}$ *be a sequence of natural numbers such that* $\lim\limits_{n\to\infty} \dfrac{m_n}{n} = 0$. *If the following limit holds:*

$$\lim_{n\to\infty} \left[\sum_{k=1}^{m_n-1} |\Delta a_{n+k}| \log(k+1) + |a_{n+m_n}| \log(m_n+1) \right] = 0\,,$$

then the series (C) *converges in mean.*

2.2. TRIGONOMETRIC SERIES WITH δ-QUASI-MONOTONE COEFFICIENTS

As an extension of the quasi-monotonic sequence, R. P. Boas [3] defined δ-quasi-monotonic sequence as follows.

A sequence $\{a_n\}$ is called δ-quasi-monotonic if $a_n \to 0$, $a_n > 0$ ultimately and $\Delta a_n \ge -\delta_n$, where δ_n is sequence of positive numbers. A quasi-monotonic sequence with $a_n \to 0$, is one that is δ-quasi-monotonic with $\delta_n = \alpha \dfrac{a_n}{n}$.

Boas [3] proved the following lemmas about the δ-quasi-monotonic sequences.

Lemma 2.1. *If* $\{a_n\}$ *is* δ*-quasi-monotonic with* $\sum_{n=1}^{\infty} n\delta_n < \infty$ *then the convergence of* $\sum_{n=1}^{\infty} a_n$ *implies that* $na_n = o(1),\ n \to \infty$.

Remark 2.1. This lemma includes the corresponding result for classical quasi-monotone; indeed if $\{a_n\}$ is quasi-monotonic we have
$\sum_{n=1}^{\infty} n\delta_n = \sum_{n=1}^{\infty} n\alpha \frac{a_n}{n} = \alpha \sum_{n=1}^{\infty} a_n < \infty$, which is assumed convergent in the hypothesis.

Lemma 2.2. *Let* $\{a_n\}$ *be a* δ*-quasi-monotonic with* $\sum_{n=1}^{\infty} n\delta_n < \infty$. *If* $\sum_{n=1}^{\infty} a_n < \infty$, *then* $\sum_{n=1}^{\infty} (n+1)|\Delta a_n| < \infty$.

Ahmad and Zahid in [1] proved the following theorem.

Theorem 2.7 [1]. *Let* (CS) *be a Fourier series with* δ*-quasi-monotone coefficients with* $\sum_{n=1}^{\infty} n\delta_n < \infty$. *Then* $\|S_n - \sigma_n\| = o(1),\ n \to \infty$ *iff* $(a_n + b_n)\log n = o(1)$, $n \to \infty$.

Applying the Theorem 2.5, and 2.6, for the series (C) we shall present a new proof of this theorem, rewritten as follows:

Theorem 2.8. *Let* (C) *be a Fourier series with* δ*-quasi-monotone coefficients with* $\sum_{n=1}^{\infty} n\delta_n < \infty$. *Then the series* (C) *converges in mean iff* $a_n \log n = o(1),\ n \to \infty$.

Proof. Let $\|S_n - f\| = o(1),\ n \to \infty$. We have:

$$\frac{a_{2n-1}}{n} \geq \frac{a_{2n}}{n} - \frac{\delta_{2n-1}}{n}$$

$$\begin{array}{l}
\frac{a_{2n-2}}{n-1} \geq \frac{a_{2n-1}}{n-1} - \frac{\delta_{2n-2}}{n-1} \geq \frac{a_{2n}}{n-1} - \frac{\delta_{2n-1}}{n-1} - \frac{\delta_{2n-2}}{n-1} \\
\frac{a_{2n-3}}{n-2} \geq \frac{a_{2n-2}}{n-2} - \frac{\delta_{2n-3}}{n-2} \geq \frac{a_{2n}}{n-2} - \frac{\delta_{2n-1}}{n-2} - \frac{\delta_{2n-2}}{n-2} - \frac{\delta_{2n-3}}{n-2} \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
\frac{a_n}{1} \geq \frac{a_{n+1}}{1} - \frac{\delta_n}{1} \geq \frac{a_{2n}}{1} - \frac{\delta_{2n-1}}{1} - \frac{\delta_{2n-2}}{1} - \cdots - \frac{\delta_n}{1}.
\end{array}$$

Adding these inequalities, we obtain:

$$\sum_{k=1}^{n} \frac{a_{n+k-1}}{k} \geq \Big(1+\frac{1}{2}+\cdots+\frac{1}{n}\Big)a_{2n} - \Big(1+\frac{1}{2}+\cdots+\frac{1}{n}\Big)\delta_{2n-1} -$$

$$-\Big(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\Big)\delta_{2n-2} - \cdots - \Big(1+\frac{1}{2}\Big)\delta_{n+1} - \frac{\delta_n}{1} >$$

$$> \Big(1+\frac{1}{2}+\cdots+\frac{1}{n}\Big)a_{2n} - \Big(1+\frac{1}{2}+\cdots+\frac{1}{n}\Big)\sum_{k=n}^{2n-1}\delta_k$$

By inequalities, $\log n < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \leq n$, $n \in \mathbf{N}$, we obtain

$$a_n + \sum_{k=2}^{n} \frac{a_{n+k}}{k} > a_{2n}\log n - \sum_{k=n}^{2n-1} k\delta_k\,, \quad \text{i.e.}$$

$$a_{2n}\log n < a_n + \sum_{k=1}^{n}\frac{a_{n+k}}{k} + \sum_{k=n}^{\infty} k\delta_k$$

Letting $n \to \infty$ and applying Theorem 2.5 we get $a_n \log n = o(1)$, $n \to \infty$.
"Only if": Let $a_n \log n = o(1)$, $n \to \infty$. Applying Theorem 2.6, it suffices to show that

$$A_n = \sum_{k=1}^{m_n-1} |\Delta a_{k+n}| \log(k+1) = o(1)\,, \quad n \to \infty\,.$$

Indeed,

$$A_n \leq (\log m_n) \sum_{k=1}^{m_n-1} |\Delta a_{k+n}| \leq$$

$$\leq (\log m_n)\left(\sum_{k=1}^{m_n-1} \Delta a_{k+n} + 2\sum_{k=1}^{m_n-1}\delta_{k+n}\right) =$$

$$= (\log m_n)(a_{n+1} - a_{n+m_n-1}) + 2(\log m_n)\sum_{k=1}^{m_n-1}\delta_{k+n} =$$

$$= O(a_{n+1}\log n) + O\left(\sum_{i=n+1}^{\infty} i\delta_i\right) = o(1)\,, \quad n \to \infty\,.$$

This generalizies a theorem of J. W. Garrett, C. S. Rees and Č. V. Stanojević of a [14]by replacing the condition of quasi-monotonicity with the condition of δ-quasi-monotonicity.

On the other hand, S. M. Mazhar [26] defined a class $S(\delta)$.
A null sequence $\{a_n\}$ belongs to the class $S(\delta)$ if there exists a sequence $\{A_n\}$ such that $\{A_n\}$ is δ-quasi-monotone, $\sum_{n=1}^{\infty} n\delta_n < \infty$, $\sum_{n=1}^{\infty} A_n < \infty$ and $|\Delta a_n| \le A_n$, for all n.
Later, Husein Bor [6] showed that condition $\{a_n\} \in S(\delta)$ is a sufficient for the integrability of the limit $g(x) = \lim_{n\to\infty} g_n(x)$.

Theorem 2.9 [6] (H. Bor). *Let $\{a_n\}$ be a sequence belongs to the class $S(\delta)$. Then*

$$\frac{1}{x}\sum_{k=1}^{\infty} \Delta a_k \sin\Big(k+\frac{1}{2}\Big)x = \frac{h(x)}{x}$$

converges for $x \in (0,\pi]$ and $\frac{h(x)}{x} \in L(0,\pi]$.

But in [48] we defined a new class of positive sequences. Namely, we say that a null-sequence $\{a_k\}$ belongs to the class $S_p(\delta)$, $p>1$ if there exists a sequence of numbers $\{A_k\}$ such that

a) $\{A_k\}$ is δ-quasi-monotone and $\sum_{k=1}^{\infty} k\delta_k < \infty$.

b) $\sum_{k=1}^{\infty} A_k < \infty$,

c) $\frac{1}{n+1}\sum_{k=0}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1)$.

Thus, in view of the above definitions it is obvious that $S(\delta) \subset S_p(\delta)$.
Applying the Holder-Hausdorff-Young technique (similarly as in the proof of Theorem 1.14) we can get the following Lemma.

Lemma 2.3 [55]. *Let the coefficients $\{a_j\}_{j=0}^{k}$ belong to the class $S_p(\delta)$, $1<p\le 2$. Then the following inequality holds*

$$\int_0^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x) \right| dx = O_p(k+1)\,,$$

where O_p depends only on p.

Lemma 2.4. [55]. *Let the coefficients $\{a_j\}_{j=0}^{n}$ belong to the class $S_p(\delta)$, $1<p\le 2$. Then*

$$A_n \int_0^{\pi} \left| \sum_{j=0}^{n} \frac{\Delta a_j}{A_j} D_j(x) \right| dx = o(1)\,, \quad n\to\infty\,,$$

Proof. Applying first the Lemma 2.3, then Lemma 2.1, yelds

$$A_n \int_0^\pi \left| \sum_{j=0}^{n} \frac{\Delta a_j}{A_j} D_j(x) \right| dx = O_p\big((n+1)A_n\big) = o(1), \quad n \to \infty .$$

Theorem 2.10 [55]. *Let* $\{a_k\} \in S_p(\delta)$, $1 < p \le 2$. *Then* (C) *is a Fourier series of some* $f \in L^1(0,\pi)$ *and* $\|S_n - f\| = o(1)$, $n \to \infty$ *if and only if* $a_n \log n = o(1)$, $n \to \infty$.

Proof. By summation by parts, and by generalized arithmetic-square mean inequality, we have:

$$\sum_{k=1}^{n} |\Delta a_k| = \sum_{k=1}^{n} A_k \frac{|\Delta a_k|}{A_k} \le$$

$$\le \sum_{k=1}^{n-1} (k+1)|\Delta A_k| \left(\frac{1}{k+1} \sum_{j=0}^{k} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} + (n+1)A_n \left(\frac{1}{n+1} \sum_{j=0}^{n} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} =$$

$$= O(1) \left[\sum_{k=1}^{n-1} (k+1)|\Delta A_k| + (n+1)A_n \right] .$$

Application of Lemma 2.1 and Lemma 2.2 yields, $\sum_{n=1}^{\infty} |\Delta a_n| < \infty$, i.e. $S_n(x)$ converges to $f(x)$, for $x \ne 0$. Using Abel's transformation, we obtain:

$$f(x) = \sum_{k=0}^{\infty} \Delta a_k D_k(x) ,$$

by the fact that $\lim_{n \to \infty} a_n D_n(x) = 0$, if $x \ne 0$. Then,

$$\|S_n - f\| = \|g_n(x) - f(x) + a_{n+1} D_n(x)\| ,$$

where $g_n(x)$ is the Rees-Stanojević sums.
Using Abel's transformation, we have:

$$g_n(x) = S_n(x) - a_{n+1} D_n(x) = \sum_{k=0}^{n} \Delta a_k D_k(x) .$$

Since $\sum\limits_{n=1}^{\infty}|\Delta a_n| < \infty$, the series $\sum\limits_{k=0}^{\infty}\Delta a_k D_k(x)$ converges. Hence $\lim\limits_{n\to\infty} g_n(x)$ exists for $x \neq 0$. Then,

$$\|f(x) - g_n(x)\| = \left\| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right\| = \frac{1}{\pi}\int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx\,.$$

Application of Abel's transformation, Lemma 2.4, Lemma 2.3 and Lemma 2.2 yields:

$$\int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx \leq$$

$$\leq \sum_{k=n+1}^{\infty} |\Delta A_k| \int_0^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x) \right| dx + o(1) = o(1)\,, \quad n \to \infty\,.$$

Hence, $\|f(x) - g_n(x)\| = o(1)$, $n \to \infty$.

"If": Let $\|S_n - f\| = o(1)$, $n \to \infty$. Since $\|D_n(x)\| = O(\log n)$, by the estimate

$$\|a_{n+1}D_n(x)\| = \|S_n - g_n\| \leq \|S_n - f\| + \|f - g_n\| = o(1) + o(1)\,, \quad n \to \infty\,,$$

we have, $a_n \log n = o(1)$, $n \to \infty$.

"Only if": Let $a_n \log n = o(1)$, $n \to \infty$. Then,

$$\|S_n - f\| \leq \|g_n - f\| + \|a_{n+1}D_n(x)\| = o(1) + a_{n+1}O(\log n) = o(1)\,, n \to \infty\,.$$

Corollary 2.1. *Let the sequence $\{a_n\}$ belongs to the class $S_p(\delta)$, $1 < p \leq 2$. Then*

$$\frac{1}{x}\sum_{k=1}^{\infty} \Delta a_k \sin\Big(k + \frac{1}{2}\Big)x = \frac{h(x)}{x}$$

converges for $x \in (0, \pi]$ and $\dfrac{h(x)}{x} \in L(0, \pi]$,

Proof. Since

$$2\,\sin\,\frac{x}{2}\,f(x) = a_0\,\sin\,\frac{x}{2} + \sum_{k=1}^{\infty} a_k \Big(2\,\sin\,\frac{x}{2}\,\cos kx\Big) =$$

$$= a_0\,\sin\,\frac{x}{2} + \sum_{k=1}^{\infty} a_k \Big[\sin\,\Big(k + \frac{1}{2}\Big)x - \Big(k - \frac{1}{2}\Big)x\Big] =$$

$$= (a_0 - a_1)\sin\,\frac{x}{2} + (a_1 - a_2)\,\sin\,\frac{3x}{2} + (a_2 - a_3)\sin\,\frac{5x}{2} + \cdots$$

$$= \sum_{k=1}^{\infty} \Delta a_k \sin(2k+1)\,\frac{x}{2} = h(x)\,,$$

by Teorem 2.10, proof is obvious.

2.3. ON THE EQUIVALENCE OF CLASSES OF FOURIER COEFFICIENTS

In [5] Bor Husein considered the following class $S^2(\delta)$. A sequence $\{a_k\}$ belongs to the class $S^2(\delta)$ if $a_k \to 0$ as $k \to \infty$, there exists a sequence of numbers $\{A_k\}$ such that it is δ-quasi-monotone, $\sum_{k=1}^{\infty} k\delta_k < \infty$, $\sum_{k=1}^{\infty} k|\Delta A_k| < \infty$ and $|\Delta a_k| \le A_k$, for all k. Also, he proved the Theorem 2.2 and 2.9, considering the class $S^2(\delta)$ instead of S' and $S(\delta)$.

Very recently, Telyakovskii [47] and Leindler [64] proved that these classes together with Sidon-Telyakovskii class S are equivalent.

Now, we shall present the proof of S. A. Telyakovskii.

Theorem 2.11. *The following classes S, S', $S(\delta)$ and $S^2(\delta)$ are equivalent.*

Proof. First we shall prove that classes $S(\delta)$ and $S^2(\delta)$ are equivalent.

Let $\{a_n\} \in S(\delta)$. It suffices to show that $\sum_{n=1}^{\infty} n|\Delta A_n| < \infty$. The last condition is satisfied by Lemma 2.2.

If $\{a_n\} \in S^2(\delta)$, then

$$nA_n = n\sum_{k=n}^{\infty} \Delta A_k \le \sum_{k=n}^{\infty} k|\Delta A_k| = o(1)\,, \quad n \to \infty\,, \quad \text{i.e. } nA_n = o(1)\,, \quad n \to \infty\,.$$

But

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n-1} k\,\Delta A_k + nA_n \le \sum_{k=1}^{n-1} k|\Delta A_k| + nA_n\,,$$

and this implies that $\sum_{n=1}^{\infty} A_n < \infty$ i.e. $\{a_n\} \in S(\delta)$.

Next we shall prove that the classes S and $S(\delta)$ are equivalent. It is obvious that $S \subset S(\delta)$. If $\{a_n\} \in S(\delta)$, we construct the sequence

$$B_k = A_k + \sum_{m=k}^{\infty} \delta_m\,.$$

Now we have: $B_k - B_{k+1} = \Delta A_k + \delta_k \ge 0$, i.e. $B_k \downarrow 0$ as $k \to \infty$.

On the other hand

$$\sum_{k=1}^{\infty} B_k = \sum_{k=1}^{\infty} A_k + \sum_{k=1}^{\infty}\sum_{m=k}^{\infty} \delta_m = \sum_{k=1}^{\infty} A_k + \sum_{m=1}^{\infty}\sum_{k=1}^{m} \delta_m = \sum_{k=1}^{\infty} A_k + \sum_{m=1}^{\infty} m\delta_m < \infty\,,$$

and $|\Delta a_n| \le A_n < B_n$, for all n, i.e. $\{a_n\} \in S$. Now we have:

$$S \subset S' \subset S(\delta) \subset S\,.$$

Consequently,

$$S \equiv S' \equiv S(\delta) \equiv S^2(\delta)\,.$$

Applying this result, inequality

$$\frac{1}{n}\sum_{k=1}^{n} \frac{|\Delta a_k|^p}{B_k^p} \le \frac{1}{n}\sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1)\,,$$

and also Theorem 2.4, we can get the following corollary.

Corollary 2.2. *For all $p > 1$, the classes S_p, S'_p (case $\alpha = r = 0$) and $S_p(\delta)$ are equivalent.*

Remark 2.2. If $c_n \equiv a_n$ is real and even sequence ($c_n = c_{-n} = a_n$, $n = 0, 1, 2, \ldots$) then the Theorem 1.8 of Č. V. Stanojević and V. B. Stanojević, the Sheng's theorem (see **III**, 3.3, Theorem 3.17) and Theorem 2.10 are equivalent.

2.4. TRIGONOMETRIC SERIES WITH REGULARLY QUASI-MONOTONE COEFFICIENTS

A positive measurable funkction $L(u)$ is said to be *slowly varying* in the sence of Karamata [19] if for every $\lambda > 0$, $\lim\limits_{u\to\infty} \dfrac{L(\lambda u)}{L(u)} = 1$.
A basic property of slowly varying functions is the asymptotic relation [19]:

$$u^{\alpha} \max_{u \le s < \infty} s^{-\alpha} L(s) \sim L(u)\,, \quad u \to \infty\,, \quad \text{for any} \quad \alpha > 0\,.$$

Slowly varying sequences are defined analogously: a positive sequence $\{l_n\}$ is said to be slowly varying if, for every $\lambda > 0$, $\lim\limits_{n\to\infty} \dfrac{l_{[\lambda n]}}{l_n} = 1$.
The class of *slowly varying sequences* is denoted by $SV(N)$.

A non-decreasing sequence $\{r_n\}$ of positive numbers is *regularly varying*, i.e. $\{r_n\} \in (RV)(N)$ in the sence of J. Karamata [20], if for some $\alpha \geq 0$

$$\lim_{n\to\infty} \frac{r_{[\lambda n]}}{r_n} = \lambda^\alpha , \quad \lambda > 1 .$$

Regularly varying sequences are characterized [20] in form as follows:
$\{r_n\} \in (RV)(N)$ if and only if $r_n = n^\alpha l_n$, for some $\alpha > 0$ and some $\{l_n\} \in (SV)(N)$. On the other hand, a sequence $\{a_n\}$ is called a *regularly quasi-monotone*, or briefly, written as $\{a_n\} \in RQM$, if for some $\{r_n\} \in (RV)(N)$, $\dfrac{a_n}{r_n} \downarrow 0$. It is obvious that the class of quasi-monotone sequences is a subclass of the class RQM. The next theorem is generalization of the Valle Poissin's theorem (see II. 2.1, p.28).

Theorem 2.12. *If $\{a_n\} \in RQM$, $\sum_{n=1}^{\infty} a_n < \infty$ then $na_n = o(1)$, $n \to \infty$.*

Proof. We have.

$$a_{2n-1} \geq \Big(1 + \frac{\alpha}{2n-1}\Big)^{-1} a_{2n} \frac{l_{2n-1}}{l_{2n}} ,$$

$$a_{2n-2} \geq \Big(1 + \frac{\alpha}{2n-2}\Big)^{-1} a_{2n-1} \frac{l_{2n-2}}{l_{2n-1}} \geq \Big(1 + \frac{\alpha}{2n-2}\Big)^{-2} a_{2n} \frac{l_{2n-2}}{l_{2n}} ,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$a_n \geq \Big(1 + \frac{\alpha}{n}\Big)^{-n} a_{2n} \frac{l_n}{l_{2n}} .$$

Adding these inequalities, we obtain:

$$\sum_{v=n}^{2n-1} a_v \geq \frac{a_{2n}}{l_{2n}} \sum_{v=n}^{2n-1} l_v \Big(1 + \frac{\alpha}{v}\Big)^{-(2n-v)} .$$

But

$$\Big(1 + \frac{\alpha}{v}\Big)^{2n-v} \leq \Big(1 + \frac{\alpha}{n}\Big)^{2n-v} \leq \Big(1 + \frac{\alpha}{n}\Big)^{n} ,$$

implies that

$$\sum_{v=n}^{2n-1} a_v \geq \frac{a_{2n}}{l_{2n}} \sum_{v=n}^{2n-1} l_v \Big(1 + \frac{\alpha}{n}\Big)^{-n} .$$

Thus

$$a_{2n} \frac{\sum_{v=n}^{2n-1} l_v}{l_{2n}} \leq \Big(1 + \frac{\alpha}{n}\Big)^{n} \sum_{v=n}^{2n-1} a_v \leq e^\alpha \sum_{v=n}^{2n-1} a_v .$$

The asymptotic relation (2.2) $l_k \sim k^\beta \left[\sup_{n\geq k} n^{-\beta} l_n\right]$, $k\to\infty$ gives for large n,

$$\begin{aligned}\sum_{v=n}^{2n-1} l_v \approx \sum_{v=n}^{2n-1} v^\beta \left[\sup_{m\geq v}\ m^{-\beta} l_m\right] \geq \left[\sup_{m\geq 2n-1}\ m^{-\beta} l_m\right] \geq \sum_{v=n}^{2n-1} v^\beta \geq \\ \geq n^\beta \left[\sup_{m\geq 2n-1}\ m^{-\beta} l_m\right] \sum_{v=n}^{2n-1} 1 = \\ = \left(\frac{n}{2n-1}\right)^\beta (2n-1)^\beta \left[\sup_{m\geq 2n-1}\ m^{-\beta} l_m\right] n \sim \\ \sim \frac{1}{2^\beta}\, n l_{2n-1}\,,\end{aligned}$$

for some $\beta > 0$. Consequently,

$$n a_{2n} \frac{l_{2n-1}}{l_{2n}} \leq e^\alpha 2^\beta \sum_{v=n}^{\infty} a_v\,.$$

Letting $n\to\infty$, we obtain $na_n = o(1)$, $n\to\infty$.
Sheng Shuyun [31] proved the following results for L^1-approximation of trigonometric series with regulary quasi-monotone coefficients.

Theorem 2.13. *Let the sequence of coefficients* $\{a_n\}$, $\{b_n\} \in RQM$ *in* (CS). *Then there exists two positive constans* C_1 *and* C_2, *such that*

$$C_1 \sum_{v=n+1}^{2n-1} \frac{a_v + b_v}{v-n} \leq \|S_n(x) - \tau_n(x)\|\,,$$

$$\begin{aligned}\|S_n(x) - \tau_n(x)\| \leq & C_2 \Big\{ \sum_{v=n+1}^{2n-1} (a_v + b_v)\Big(\frac{l_{v+1}}{(v-n)l_v} + \varepsilon_v\Big) + \\ & + \frac{1}{n}\sum_{v=n+1}^{2n-1} (a_v + b_v)\, \log(v-n+1)\Big\}\end{aligned}$$

where, $\tau_n(x) = \frac{1}{n}\sum_{k=n}^{2n-1} S_k(x)$ *and* $\varepsilon_v = \frac{l_{v+1}}{l_v} - 1$.
The next theorem generalizies Theorem 1.25.

Theorem 2.14. *Let* (C) *be a Fourier series with* $\{a_n\} \in RQM$. *Then* (C) *converges in mean iff* $a_n \log n = o(1)$, $n\to\infty$.

Proof. For the necessity we applying the Theorem 2.5, i.e. $\sum_{k=1}^{n}\frac{a_{n+k}}{k} = o(1)$, $n \to \infty$. From the fact $\{a_n\} \in RQM$ we obtain the inequalities:

$$\sum_{k=1}^{n}\frac{a_{n+k}}{k} = \sum_{k=1}^{n}\frac{a_{n+k}}{(n+k)^{\alpha}l_{n+k}}\,\frac{(n+k)^{\alpha}l_{n+k}}{k} \geq$$

$$\geq \frac{a_{2n}}{(2n)^{\alpha}l_{2n}}\sum_{k=1}^{n}\frac{(n+k)^{\alpha}l_{n+k}}{k} \geq$$

$$\geq \left(\frac{n+1}{2n}\right)^{\alpha}\frac{a_{2n}}{l_{2n}}\sum_{k=1}^{n}\frac{l_{n+k}}{k}\,.$$

Applying the asymptotic relation (2.2) for large n, we have:

$$\sum_{k=1}^{n}\frac{l_{n+k}}{k} = \sum_{k=n+1}^{2n}\frac{l_k}{k-n} \approx \sum_{k=n+1}^{2n}\frac{k^{\beta}\sup\limits_{m\geq k} m^{-\beta}l_m}{k-n} \geq$$

$$\geq \left(\sup_{m\geq 2n}\ m^{-\beta}l_m\right)\sum_{k=n+1}^{2n}\frac{k^{\beta}}{k-n} \geq$$

$$\geq (n+1)^{\beta}\left(\sup_{m\geq 2n}\ m^{-\beta}l_m\right)\sum_{k=1}^{n}\frac{1}{k} =$$

$$= \left(\frac{n+1}{2n}\right)^{\beta}\left[(2n)^{\beta}\sup_{m\geq 2n}\ m^{-\beta}l_m\right]\sum_{k=1}^{n}\frac{1}{k} \approx \frac{1}{2^{\beta}}\,l_{2n}\,\log n\,.$$

Letting $n \to \infty$ in inequality

$$a_{2n}\log n < 2^{\beta}\left(\frac{2n}{n+1}\right)^{\alpha}\sum_{k=1}^{n}\frac{a_{n+k}}{k}\,,$$

the proof of necessity is complete.

For sufficiency, we shall applying the Theorem 2.6. From the monotonicity of the sequence $\frac{a_n}{r_n}$, we get:

$$A_n = \sum_{k=1}^{m_n-1}|\Delta a_{k+n}|\log(k+1) = \sum_{i=n+1}^{n+m_n-1}|\Delta a_i|\log(i-n+1) =$$

$$\sum_{i=n+1}^{n+m_n-1} \left| r_i \Delta\Big(\frac{a_i}{r_i}\Big) + \frac{a_{i+1}}{r_{i+1}}(r_i - r_{i+1}) \right| \log(i-n+1) \le$$

$$\le r_{n+m_n-1} \log m_n \sum_{i=n+1}^{n+m_n-1} \Delta\Big(\frac{a_i}{r_i}\Big) + \sum_{i=n+1}^{n+m_n-1} \frac{a_{i+1}}{r_{i+1}}(r_{i+1} - r_i) \log(i+1) =$$

$$= r_{n+m_n-1} \log m_n \Big(\frac{a_{n+1}}{r_{n+1}} - \frac{a_{n+m_n-1}}{r_{n+m_n-1}}\Big) +$$

$$+ \max_{n+1 \le i \le n+m_n-1} \big(a_{i+1} \log(i+1)\big) \sum_{i=n+1}^{n+m_n-1} \Big(1 - \frac{r_i}{r_{i+1}}\Big).$$

Since

$$\sum_{i=n+1}^{n+m_n-1} \Big(1 - \frac{r_i}{r_{i+1}}\Big) \le \log \prod_{i=n+1}^{n+m_n-1} \frac{r_{i+1}}{r_i} = \log \frac{r_{n+m_n}}{r_{n+1}} \le \frac{r_{n+m_n}}{r_n},$$

we obtain

$$A_n \le \frac{r_{n+m_n-1}}{r_n} \frac{\log m_n}{\log n} \big(a_{n+1} \log(n+1)\big) + \frac{r_{n+m_n}}{r_n} \max_{n+1 \le i \le n+m_n-1} \big[a_{i+1} \log(i+1)\big].$$

The hypothesis $a_n \log n = o(1)$, $n \to \infty$ and $\{r_n\} \in (RV)(N)$ implies that the first and second term on the right side are $o(1)$, $n \to \infty$.
Finaly, $A_n = o(1)$, $n \to \infty$, i.e. the series (C) converges in mean.

Remark 2.3. The proof of necessity of this theorem, we can simplified using the monotonicity of the sequence $\{r_n\}$ and fact that $\left\{\frac{a_n}{r_n}\right\} \downarrow$.
We have

$$\sum_{k=1}^{n} \frac{a_{n+k}}{k} = \sum_{k=1}^{n} \frac{a_{n+k}}{r_{n+k}} \frac{r_{n+k}}{k} \ge \frac{a_{2n}}{r_{2n}} r_n \sum_{k=1}^{n} \frac{1}{k} \approx (a_{2n} \log n) \frac{r_n}{r_{2n}}.$$

Taking $n \to \infty$ in the inequality $a_{2n} \log n \le \frac{r_{2n}}{r_n} \sum_{k=1}^{n} \frac{a_{n+k}}{k}$, the proof of the necessity is obvious.