III. ESTIMATES OF TRIGONOMETRIC SERIES, USEFUL IN PROBLEMS OF APPROXIMATION THEORY

3.1. SOME L^1-ESTIMATES FOR TRIGONOMETRIC SERIES WITH FOMIN'S COEFFICIENT CONDITION

Let $f(x)$ and $g(x)$ be the sums of the series (C) and (S) respectively.
It is well-known (see [2], [21], [63]) that if $\{a_n\}$ is null-quasi-convex sequence of real numbers, the series (C) is Fourier series of some $f \in L^1$ and the following estimate holds:

$$\int_0^\pi |f(x)|dx \le \frac{\pi}{2} \sum_{k=1}^{\infty} k|\Delta^2 a_{k-1}| \,. \tag{3.1}$$

The following two theorems were proved by Telyakovskii [41], [42].

Theorem 3.1 [41]. *Let* $\{a_n\} \in BV$, $a_n \to 0$, $\sum_{i=2}^{\infty} \left| \sum_{k=1}^{[i/2]} \frac{\Delta a_{i-k} - \Delta a_{i+k}}{k} \right| < \infty$, *then the following estimate holds,*

$$\int_0^\pi |f(x)|dx \le C \left(\sum_{k=0}^{\infty} |\Delta a_k| + \sum_{i=2}^{\infty} \left| \sum_{k=1}^{[i/2]} \frac{\Delta a_{i-k} - \Delta a_{i+k}}{k} \right| \right) ,$$

where C is some absolute constant.

Theorem 3.2 [42]. *Let* $\{a_n\} \in BV$, $a_n \to 0$, $a_0 = 0$,

$$\sum_{i=2}^{\infty} \left| \sum_{k=1}^{[i/2]} \frac{\Delta a_{i-k} - \Delta a_{i+k}}{k} \right| < \infty \,,$$

then the following estimate holds uniformly with respect to $s = 1, 2, 3, \ldots$ for the function g:

$$\left| \int_{\pi/(2s+1)}^{\pi} |g(x)|dx - \sum_{k=1}^{s} \frac{|a_k|}{k} \right| \le C \left(\sum_{k=0}^{\infty} |\Delta a_k| + \sum_{i=2}^{\infty} \left| \sum_{k=1}^{[i/2]} \frac{\Delta a_{i-k} - \Delta a_{i+k}}{k} \right| \right) ,$$

where C is some absolute constant.

Also, Telyakovskii [41], [43] proved the following inequality.

$$\sum_{i=2}^{\infty}\left|\sum_{k=1}^{[i/2]}\frac{\Delta a_{i-k}-\Delta a_{i+k}}{k}\right|\leq C\sum_{k=1}^{\infty}k\,|\Delta^2 a_{k-1}|\,. \tag{3.2}$$

Remark 3.1. If $\{a_k\}$ is a null-quasi-convex sequence then $\sum\limits_{k=0}^{\infty}|\Delta a_k|\leq\sum\limits_{k=1}^{\infty}k|\Delta^2 a_{k!}|$. Thus the estimate (3.1) follows from the Theorem 3.1 and the estimate (3.2) with some absolute constant C instead of $\frac{\pi}{2}$.

Theorem 3.3. *If $\{a_k\}$ is null quasi-convex sequence, $a_0 = 0$, then* (S) *will be a Fourier series iff $\sum\limits_{n=1}^{\infty}\frac{|a_n|}{n}<\infty$.*

Moreover, if $\sum\limits_{n=1}^{\infty}\frac{|a_n|}{n}<\infty$, then the following estimate holds:

$$\int_0^{\pi}\left|\sum_{k=1}^{\infty}a_k\sin kx\right|dx\leq\sum_{k=1}^{\infty}\frac{|a_k|}{k}+C\sum_{k=1}^{\infty}k|\Delta^2 a_{k-1}|\,.$$

Remark 3.2. If $\{a_n\}$ is a null-quasi-convex sequence then the estimate of the Theorem 3.3 is consequence of the Theorem 3.2 and the estimate (3.2).

Also, Telyakovskii[40] has given and direct proof of the Theorem 3.3, proving the following estimate:

$$\left|\int_{1/(s+1)}^{\pi}\left|\sum_{k=1}^{\infty}a_k\sin kx\right|dx-\sum_{k=1}^{s}\frac{|a_k|}{k}\right|\leq C\sum_{k=1}^{\infty}k|\Delta^2 a_{k-1}|\,,\quad C>0\,.$$

In [44] the indicated results on series with quasi-convex coefficients were extended to the more general case when the coefficients $\{a_k\}$ satisfy the Sidon-Telyakovskii class S. Namely, Telyakovskii proved [44] the following Theorems.

Theorem 3.4 [44]. *Let the coefficients of the series* (C) *belong to the class S. Then the series* (C) *is a Fourier series of some $f\in L^1(0,\pi)$ and the following estimate holds:*

$$\int_0^{\pi}|f(x)|dx\leq M\sum_{n=0}^{\infty}A_n\,,\quad M>0\,.$$

Theorem 3.5 [44]. *Let the coefficients of the series* (S) *belong to the class* S. *Then the following relation holds for* $p = 1, 2, 3, \ldots$

$$\int\limits_{\pi/(p+1)}^{\pi} |g(x)|dx = \sum_{n=1}^{p} \frac{|a_n|}{n} + O\left(\sum_{n=1}^{\infty} A_n\right).$$

In particular $g(x)$ *is Fourier series iff* $\sum\limits_{n=1}^{\infty} \frac{|a_n|}{n} < \infty$.

Corollary 3.1. *Let the coefficients of the series* (C) *belong to the class* $S(\delta)$. *Then the series* (C) *is a Fourier series of some* $f \in L^1(0,\pi)$ *and the following estimate holds:*

$$\int\limits_{0}^{\pi} |f(x)|dx \le M\left(\sum_{n=0}^{\infty} A_n + \sum_{n=1}^{\infty} n\delta_n\right), \quad M > 0.$$

Proof. Applying the Theorem 2.11 and Theorem 3.4, we obtain:

$$\int\limits_{0}^{\pi} |f(x)|dx \le M\sum_{n=0}^{\infty} B_n = M\left(\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty}\sum_{m=n}^{\infty} \delta_m\right) =$$

$$= M\left(\sum_{n=0}^{\infty} A_n + \sum_{n=1}^{\infty} n\delta_n\right), \quad M > 0.$$

Analogously, applying the Theorem 2.11 and Theorem 3.5, we obtain the following Corollary.

Corollary 3.2. *Let the coefficients of the series* (S) *belongs to the class* $S(\delta)$. *Then the following relation holds for* $p = 1, 2, 3, \ldots$

$$\int\limits_{\pi/(p+1)}^{\pi} |g(x)|dx = \sum_{n=1}^{p} \frac{|a_n|}{n} + O\left(\sum_{n=1}^{\infty} A_n\right) + O\left(\sum_{n=1}^{\infty} n\delta_n\right).$$

Recently, Fomin [12] proved the following estimate:

$$\sum_{i=2}^{\infty}\left|\sum_{k=1}^{[i/2]} \frac{\Delta a_{i-k} - \Delta a_{i+k}}{k}\right| \le C_p \sum_{s=0}^{\infty} 2^s\, \Delta_s^{(p)}, \tag{3.3}$$

for any $1 < p \le 2$ and any positive constant C_p depends only on p.

Lemma 3.1 [17] (Elliot). *If* $0 < q < 1$, $b_n \geq 0$ *and* $\sum\limits_{n=1}^{\infty} b_n^q < \infty$ *then the following inequality holds:*

$$\left(\frac{q}{1-q}\right)^q \sum_{n=1}^{\infty} b_n^q < \sum_{n=1}^{\infty}\left(\frac{b_n + b_{n+1} + \cdots}{n}\right)^q,$$

unless all of the b_n *are zero.*

Theorem 3.6 [58], [66]. *Let* $\{a_n\} \in F_p$, $1 < p \leq 2$, *then the series* (C) *is a Fourier series and the following inequality holds:*

$$\int\limits_0^{\pi} |f(x)|dx \leq C_p \sum_{n=1}^{\infty} \left(\frac{\sum\limits_{k=n}^{\infty} |\Delta a_k|^p}{n}\right)^{1/p}.$$

Proof. Putting $b_n = |\Delta a_n|^p$ in Lemma 3.1, where $q = \frac{1}{p}$, we get

$$\left(\frac{1}{p-1}\right)^{1/p} \sum_{n=1}^{\infty} |\Delta a_n| < \sum_{n=1}^{\infty} \left(\frac{|\Delta a_n|^p + |\Delta a_{n+1}|^p + \cdots}{n}\right)^{1/p}, \quad \text{i.e.}$$

$$\sum_{n=1}^{\infty} |\Delta a_n| < (p-1)^{1/p} \sum_{n=1}^{\infty} \left(\frac{\sum\limits_{k=n}^{\infty} |\Delta a_k|^p}{n}\right)^{1/p}.$$

On the other hand, since $U_s = \frac{1}{s}\sum\limits_{k=s}^{\infty} |\Delta a_k|^p$ is monotone decreasing sequence,

$$\sum_{s=1}^{n} 2^s \Delta_s^{(p)} \leq 2 \sum_{s=1}^{n} \left[2^{(s-1)(p-1)} \sum_{k=2^{s-1}+1}^{2^s} |\Delta a_k|^p\right]^{1/p} \leq$$

$$\leq 2 \sum_{s=1}^{n} 2^{s-1} \left[\frac{1}{2^{s-1}} \sum_{k=2^{s-1}}^{\infty} |\Delta a_k|^p\right]^{1/p} = O\left(\sum_{s=1}^{2^{n-1}} (U_s)^{1/p}\right).$$

Letting $n \to \infty$, we have:

$$\sum_{s=1}^{\infty} 2^s \Delta_s^{(p)} = O\left(\sum_{s=1}^{\infty}\left(\frac{1}{s}\sum_{k=s}^{\infty}|\Delta a_k|^p\right)^{1/p}\right).$$

Then applying Theorem 3.1, estimation (3.3) the our inequality is satisfied.
Similarly as in the proof of this theorem, applying the Theorem 3.2, we can get the following Theorem.

Theorem 3.7 [58], [66]. *Let for any $1 < p \le 2$, $\{a_n\} \in F_p$ and $a_0 = 0$ Then* (S) *will be Fourier series iff $\sum_{n=1}^{\infty}\frac{|a_n|}{n} < \infty$. Moreover if $\sum_{n=1}^{\infty}\frac{|a_n|}{n} < \infty$, then*

$$\int_0^{\pi}\left|\sum_{k=1}^{\infty} a_k \sin kx\right| dx \le \sum_{k=1}^{\infty}\frac{|a_k|}{k} + C_p \sum_{n=1}^{\infty}\left(\frac{\sum_{k=n}^{\infty}|\Delta a_k|^p}{n}\right)^{1/p}.$$

3.2. SOME RESULTS ON L^1-APPROXIMATION OF THE r-TH DERIVATE OF FOURIER SERIES

In this parth we obtain L^1-inequalities of the r-th derivatives of the series (C) and (S).
Generalizations of the Telyakovskii's inequalities [51], [52], [54] are obtained by considering the condition $\Im_r$, $r = 0, 1, 2, \ldots$ and $S_{p\alpha r}$, $1 < p \le 2$, $\alpha \ge 0$, $r \in \{0, 1, 2, \ldots [\alpha]\}$ instead of S. An equivalent form of the condition $\Im_r$, $r = 0, 1, 2, \ldots$ is given and also an extension of the Sidon's Theorem 1.3 is made.

Theorem 3.8 [51]. *Let the coefficients of the series* (C) *belong to the class $\Im_r$, $r = 0, 1, 2, \ldots$ Then the r-th derivate of the series* (C) *is a Fourier series of some $f^{(r)} \in L^1(0, \pi)$ and the following inequality holds:*

$$\int_0^{\pi}|f^{(r)}(x)|dx \le M\sum_{n=1}^{\infty} n^r A_n\,, \quad \text{where} \quad 0 < M = M(r) < \infty\,.$$

Proof 1. We have:

$$\sum_{k=1}^{n}|\Delta(k^r a_k)| \le \sum_{k=1}^{n}|(k+1)^{r+1}a_{k+1} - k^r a_{k+1}| + \sum_{k=1}^{n}|k^r a_{k+1} - k^r a_k| =$$

$$= \sum_{k=1}^{n} |\Delta(k^r) a_{k+1}| + \sum_{k=1}^{n} k^r |\Delta a_k| =$$

$$= O_r \left(\sum_{k=1}^{n} k^{r-1} |a_{k+1}| \right) + O\left(\sum_{k=1}^{n} k^r A_k \right) .$$

Applying Abel's transformation, we have:

$$\sum_{k=1}^{n} k^{r-1} |a_{k+1}| = \sum_{k=1}^{n-1} \Delta |a_{k+1}| \sum_{j=1}^{k} j^{r-1} + |a_{n+1}| \sum_{j=1}^{n} j^{r-1} \le$$

$$\le \sum_{k=1}^{n-1} |\Delta a_{k+1}| k^r + |a_{n+1}| n^r \le$$

$$\le \sum_{k=1}^{n-1} |\Delta a_{k+1}| k^r + \sum_{k=n+1}^{\infty} k^r |\Delta a_k| \le$$

$$\le \sum_{k=1}^{n-1} k^r A_k + \sum_{k=n+1}^{\infty} k^r A_k .$$

Letting $n \to \infty$, we get $\sum_{k=1}^{\infty} |\Delta(k^r a_k)| < \infty$, i.e. $\lim_{n\to\infty} S_n^{(r)}(x) = f^{(r)}(x)$.

From inequality $|D_n^{(r)}(x)| = O\left(\frac{n^r}{x}\right)$, (see [30]) we have that series $\sum_{k=1}^{\infty} \Delta a_k D_k^{(r)}(x)$ is uniformly convergent on any compact subset of $(0, \pi)$.

Thus the representation $f(x) = \sum_{k=0}^{\infty} \Delta a_k D_k(x)$ implies that

$$f^{(r)}(x) = \sum_{k=1}^{\infty} \Delta a_k D_k^{(r)}(x) .$$

From Lemma 1.6 and Lemma 1.10, we obtain:

$$A_N \int_0^{\pi} \left| \sum_{j=0}^{N} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx = O\big((N+1)^{r+1} A_N\big) = o(1) , \quad N \to \infty . \tag{3.4}$$

Again applying the Abel's transformation, (3.4) and Lemma 1.6, we get:

$$\int_0^{\pi} |f^{(r)}(x)| dx \le \lim_{N\to\infty} \sum_{k=0}^{N-1} (\Delta A_k) \int_0^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx$$

$$= O(1) \lim_{N\to\infty} \sum_{k=0}^{N-1} (\Delta A_k)(k+1)^{r+1} =$$

$$= O(1) \lim_{N\to\infty} \left\{ \sum_{k=0}^{N} [(k+1)^{r+1} - k^{r+1}] A_k - (N+1)^{r+1} A_N \right\} =$$

$$= O_r \left(\sum_{k=0}^{\infty} k^r A_k \right) ,$$

where O_r depends on r.

Proof 2 [69]. First we shall prove that if $\{a_n\} \in \Im_r$, then $\{n^r a_n\} \in S$.

We shall construct the sequence $\{B_k\}$ as follows:

$$B_k = k^r A_k + \sum_{i=k+1}^{\infty} [i^r - (i-1)^r] A_i \,.$$

We have:

$$B_k - B_{k+1} = k^r A_k - (k+1)^r A_{k+1} + (k+1)^r A_{k+1} - k^r A_{k+1} = k^r \Delta A_k \geq 0 \,.$$

$$\sum_{k=1}^{\infty} B_k = \sum_{k=1}^{\infty} k^r A_k + \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} [(i+1)^r - i^r] A_{i+1} =$$

$$= \sum_{k=1}^{\infty} k^r A_k + \sum_{i=1}^{\infty} \sum_{k=1}^{i} [(i+1)^r - i^r] A_{i+1} =$$

$$= \sum_{k=1}^{\infty} k^r A_k + \sum_{i=1}^{\infty} i[(i+1)^r - i^r] A_{i+1} < \sum_{k=1}^{\infty} k^r A_k + \sum_{i=1}^{\infty} [(i+1)^{r+1} - i^{r+1}] A_{i+1} =$$

$$= \sum_{k=1}^{\infty} k^r A_k + O_r \left(\sum_{i=1}^{\infty} i^r A_i \right) < \infty \,, \quad \text{i.e.} \quad \sum_{k=1}^{\infty} B_k < \infty \,.$$

Then, $\Delta(k^r a_k) = k^r a_k - (k+1)^r a_{k+1} = k^r \Delta a_k - \big((k+1)^r - k^r\big) a_{k+1}$.

The function $h(x) = (x+1)^r - x^r$ is monotone increasing on $[0, \infty)$, since $h'(x) = r[(x+1)^{r-1} - x^{r-1}] \geq 0$, for $x \geq 0$. This implies that

$$|\Delta(k^r a_k)| \leq k^r |\Delta a_k| + \big((k+1)^r - k^r\big) |a_{k+1}| \leq k^r A_k + \big((k+1)^r - k^r\big) \sum_{i=k+1}^{\infty} |\Delta a_i| \leq$$

$$\leq k^r A_k + \sum_{i=k+1}^{\infty} \big(i^r - (i-1)^r\big) |\Delta a_i| \leq k^r A_k + \sum_{i=k+1}^{\infty} \big(i^r - (i-1)^r\big) A_i = B_k \,,$$

i.e. $|\Delta(k^r A_k)| \leq B_k$. Thus $\{n^r a_n\} \in S$. Now, applying the Theorem 3.4, we obtain:

$$\int_0^\pi |f^{(r)}(x)|dx \leq M \sum_{n=0}^{\infty} B_n < M \left[\sum_{k=1}^{\infty} k^r A_k + O_r \left(\sum_{i=1}^{\infty} i^r A_i \right) \right] =$$

$$= O_r \left(\sum_{k=1}^{\infty} k^r A_k \right) ,$$

where O_r depends on r.

Theorem 3.9 [54]. *A null sequence $\{a_n\}$ belongs to the class $\Im_r$, $r = 0, 1, 2, 3, \ldots$ if and only if it can to be represent as $a_n = \sum_{k=n}^{\infty} \frac{p_k}{k} \sum_{l=n}^{k} \alpha_l$, $n \in N$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ are sequences such that $|\alpha_n| \leq 1$, for all n and*

$$\sum_{n=1}^{\infty} n^r |p_n| < \infty . \qquad (3.5)$$

Proof. Let (3.5) holds. Then

$$\Delta a_k = \alpha_k \sum_{m=k}^{\infty} \frac{p_m}{m}$$

and we denote

$$A_k = \sum_{m=k}^{\infty} \frac{|p_m|}{m} .$$

Since $|\alpha_k| \leq 1$, we get

$$|\Delta a_k| \leq |\alpha_k| \sum_{m=k}^{\infty} \frac{|p_m|}{m} \leq A_k , \quad \text{for all } k .$$

However,

$$\sum_{k=1}^{\infty} k^r A_k = \sum_{k=1}^{\infty} k^r \sum_{m=k}^{\infty} \frac{|p_m|}{m} = \sum_{m=1}^{\infty} \frac{|p_m|}{m} \sum_{k=1}^{m} k^r \leq \sum_{m=1}^{\infty} m^r |p_m| < \infty ,$$

and $A_k \downarrow 0$, i.e. $\{a_k\} \in \Im_r$.
Now, if $\{a_k\} \in \Im_r$, we put $\alpha_k = \dfrac{\Delta a_k}{A_k}$ and $p_k = k(A_k - A_{k+1})$.

Hence $|\alpha_k| \le 1$, and by Lemma 1.10 we get:

$$\sum_{k=1}^{\infty} k^r |p_k| = \sum_{k=1}^{\infty} k^{r+1}(A_k - A_{k+1}) = O\left(\sum_{k=1}^{\infty} k^r A_k\right) < \infty\,.$$

Finally,

$$a_k = \sum_{i=k}^{\infty} \Delta a_i = \sum_{i=k}^{\infty} \alpha_i A_i = \sum_{i=k}^{\infty} \alpha_i \sum_{m=i}^{\infty} \Delta A_m = \sum_{i=k}^{\infty} \alpha_i \sum_{m=i}^{\infty} \frac{p_m}{m} = \sum_{m=k}^{\infty} \frac{p_m}{m} \sum_{i=k}^{m} \alpha_i\,,$$

i.e. (3.5) holds.

Corollary 3.3 [54]. *Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ be sequences such that $|\alpha_n| \le 1$, for every n and let $\sum_{n=1}^{\infty} n^r |p_n| < \infty$, $r = 0, 1, 2, \dots$ If $a_n = \sum_{k=n}^{\infty} \frac{p_k}{k} \sum_{l=n}^{k} \alpha_l$, $n \in N$ then the r-th derivate of the series* (C) *is a Fourier series of some $f^{(r)} \in L^1$.*

Proof. The proof of this corollary follows from Theorem 3.8. and Theorem 3.9.

Lemma 3.2 [51]. *Let $\{\alpha_j\}_{j=0}^{k}$ be a sequence of real numbers. Then the following relation holds for $v = 0, 1, 2, \dots r$ and $r = 0, 1, 2, \dots$*

$$U_k = \int\limits_{\pi/(k+1)}^{\pi} \left| \sum_{j=0}^{k} \alpha_j \frac{\left(j + \frac{1}{2}\right)^v \sin\left[\left(j + \frac{1}{2}\right)x + \frac{(v+3)\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \right| dx =$$

$$= O\left((k+1)^{r-v+\frac{1}{2}} \left(\sum_{j=0}^{k} \alpha_j^2 (j+1)^{2v} \right)^{1/2} \right).$$

Proof. Applying first Cauchy-Bunjakovskii inequality, yields

$$U_k \le \left[\int\limits_{\pi/(k+1)}^{\pi} \frac{dx}{\left(\sin\left(\frac{x}{2}\right)\right)^{2(r+1-v)}} \right]^{1/2} \times$$

$$\times \left\{ \int\limits_{\pi/(k+1)}^{\pi} \left[\sum_{j=o}^{k} \alpha_j \left(j + \frac{1}{2}\right)^v \sin\left[\left(j + \frac{1}{2}\right)x + \frac{(v+3)\pi}{2}\right] \right]^2 dx \right\}^{1/2}.$$

Since

$$\int\limits_{\pi/(k+1)}^{\pi} \frac{dx}{\left(\sin\left(\frac{x}{2}\right)\right)^{2(r+1-v)}} \leq \pi^{2(r+1-v)} \int\limits_{\pi/(k+1)}^{\pi} \frac{dx}{x^{2(r+1-v)}} \leq$$

$$\leq \frac{\pi(k+1)^{2(r+1-v)-1}}{2(r+1-v)-1} \leq \pi(k+1)^{2(r+1-v)-1},$$

we have

$$U_k \leq \sqrt{\pi}[(k+1)^{2(r+1-v)-1}]^{1/2} \times$$

$$\times \left\{ \int\limits_0^{\pi} \left[\sum_{j=0}^{k} \alpha_j \left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{(v+3)\pi}{2}\right]\right]^2 dx \right\}^{1/2} \leq$$

$$\leq \sqrt{2\pi}[(k+1)^{2(r+1-v)-1}]^{1/2} \left\{ \int\limits_0^{2\pi} \left[\sum_{j=0}^{k} \alpha_j \left(j+\frac{1}{2}\right)^v \sin\left[(2j+1)t + \frac{(v+3)\pi}{2}\right]\right]^2 dt \right\}^{1/2}.$$

Then applying the Parseval's equality, we get:

$$U_k \leq \sqrt{2\pi}[(k+1)^{2(r+1-v)-1}]^{1/2} \left[\sum_{j=0}^{k} \alpha_j^2 (j+1)^{2v} \right]^{1/2}.$$

Finally,

$$U_k = O\left((k+1)^{r-v+\frac{1}{2}} \left(\sum_{j=0}^{k} \alpha_j^2 (j+1)^{2v} \right)^{1/2} \right).$$

Lemma 3.3 [54]. *Let $\{\alpha_k\}$ be a sequence of real numbers such that $|\alpha_k| \leq 1$, for all k. Then there exists a finite constant $M(r) > 0$ such that for any $n \geq 0$ and $r = 0, 1, 2, \ldots$*

$$\int\limits_{\pi/(n+1)}^{\pi} \left| \sum_{k=0}^{n} \alpha_k \overline{D}_k^{(r)}(x) \right| dx \leq M(n+1)^{r+1}.$$

Proof. Since $-\cos\left(n+\frac{1}{2}\right)x = \sin\left[\left(n+\frac{1}{2}\right)x + \frac{3\pi}{2}\right]$, by the Lemma 1.13, we get:

$$\overline{D}_n^{(r)}(x) = \sum_{k=0}^{r-1} \frac{\left(n+\frac{1}{2}\right)^k \sin\left[\left(n+\frac{1}{2}\right)x + \frac{k+3}{2}\pi\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-k}} \varphi_k(x) +$$

$$+ \frac{\left(n+\frac{1}{2}\right)^r \sin\left[\left(n+\frac{1}{2}\right)x + \frac{r+3}{2}\pi\right]}{2\sin\left(\frac{x}{2}\right)},$$

where the same φ_k denotes various analytical function of x, independent of n.

$$\int_{\pi/(n+1)}^{\pi} \left|\sum_{k=0}^{n} \alpha_k \overline{D}_k^{(r)}(x)\right| dx \leq$$

$$\leq \int_{\pi/(n+1)}^{\pi} \left|\sum_{j=0}^{n} \alpha_j \left(\sum_{v=0}^{r-1} \frac{\left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{(v+3)\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \varphi_v(x)\right)\right| dx +$$

$$+ \int_{\pi/(n+1)}^{\pi} \left|\sum_{j=0}^{n} \alpha_j \frac{\left(j+\frac{1}{2}\right)^r \sin\left[\left(j+\frac{1}{2}\right)x + \frac{(r+3)\pi}{2}\right]}{2\sin\left(\frac{x}{2}\right)}\right| dx = \lambda_n + \mu_n .$$

Since φ_v are bounded funciotns, we have:

$$\int_{\pi/(n+1)}^{\pi} \left|\sum_{j=0}^{n} \alpha_j \frac{\left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{(v+3)\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \varphi_v\right| dx \leq K\, U_n ,$$

where U_n is the integral as in the Lemma 3.2, and K is a positive constant. Applying Lemma 3.2, to the last integral, we get:

$$\int_{\pi/(n+1)}^{\pi} \left|\sum_{j=0}^{n} \alpha_j \frac{\left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{(v+3)\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \varphi_v(x)\right| dx =$$

$$= O\left((n+1)^{r-v+\frac{1}{2}} \left(\sum_{j=0}^{n} \alpha_j^2 (j+1)^{2v}\right)^{1/2}\right) =$$

$$= O\left((n+1)^{r-v+\frac{1}{2}} (n+1)^{v+\frac{1}{2}}\right) = O\big((n+1)^{r+1}\big) .$$

Since r is a finite value, we have: $\lambda_n = O\big((n+1)^{r+1}\big)$.
Similarly, we can get $\mu_n = O\big((n+1)^{r+1}\big)$.
Finally, the our inequality is satisfied.

Remark 3.3. For $r = 0$, we get the Telyakovskii inequality, proved in [44].

Theorem 3.10 [54]. *Let the coefficients of the series $g(x)$ belong to the class $\Im_r$, $r = 0, 1, 2, ...$ Then the r-th derivate of the series* (S) *converges to a function and for $m = 1, 2, 3, , \ldots$ the following inequality holds:*

$$\int\limits_{\pi/(m+1)}^{\pi} \left| g^{(r)}(x) \right| dx \le M \left(\sum_{n=1}^{m} |a_n| \cdot n^{r-1} + \sum_{n=1}^{\infty} n^r A_n \right) , \tag{$*$}$$

where

$$0 < M = M(r) < \infty .$$

Moreover, if $\sum\limits_{n=1}^{\infty} n^{r-1} |a_n| < \infty$, then the r-th derivate of the series (S) *is a Fourier series of some $g^{(r)} \in L^1(0, \pi)$ and*

$$\int\limits_{0}^{\pi} \left| g^{(r)}(x) \right| dx \le M \left(\sum_{n=1}^{\infty} |a_n| \cdot n^{r-1} + \sum_{n=1}^{\infty} n^r A_n \right)$$

Proof. We suppose that $a_0 = 0$ and $A_0 = \max(|a_1|, A_1)$.
Applying the Abel's transformation, we have:

$$g(x) = \sum_{k=0}^{\infty} \Delta a_k \overline{D}_k(x) , \quad x \in (0, \pi] . \tag{3.6}$$

Applying the inequality of Lemma 4.3 (iii), we obtain that the series $\sum\limits_{k=0}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x)$ is uniformly convergent on any compact subset of $[\varepsilon, \pi]$, where $\varepsilon > 0$.
Thus representation (3.6) implies that

$$g^{(r)}(x) = \sum_{k=0}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x) , \quad x \in (0, \pi] .$$

Then,

$$\int\limits_{\pi/(m+1)}^{\pi} |g^{(r)}(x)| dx = \int\limits_{\pi/(m+1)}^{\pi} \left| \sum_{k=0}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x) \right| dx \le$$

$$\le \sum_{j=1}^{m} \int\limits_{\pi/(j+1)}^{\pi/j} \left| \sum_{k=0}^{j-1} \Delta a_k \overline{D}_k^{(r)}(x) \right| dx + O\left(\sum_{j=1}^{m} \int\limits_{\pi/(j+1)}^{\pi/j} \left| \sum_{k=j}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x) \right| dx \right) .$$

Let

$$I_1=\sum_{j=1}^{m}\int\limits_{\pi/(j+1)}^{\pi/j}\left|\sum_{k=0}^{j-1}\Delta a_k\overline{D}_k^{(r)}(x)\right|dx\,,\quad I_2=\sum_{j=1}^{m}\int\limits_{\pi/(j+1)}^{\pi/j}\left|\sum_{k=j}^{\infty}\Delta a_k\overline{D}_k^{(r)}(x)\right|dx\,.$$

Applyyng the well-known expansion

$$\operatorname{ctg}\frac{x}{2}=\frac{2}{x}+\sum_{n=1}^{\infty}\frac{4x}{x^2-4n^2\pi^2}$$

it is not difficult to proof the following estimate:

$$\left(\operatorname{ctg}\frac{x}{2}\right)^{(r)}=\frac{2(-1)^r r!}{x^{r+1}}+O(1)\,,\quad x\in(0,\pi]\,.$$

Thus

$$\overline{D}_n^{(r)}(x)=\frac{(-1)^{r+1}r!}{x^{r+1}}+O\big((n+1)^{r+1}\big)\,,\quad x\in(0,\pi]\,.$$

Hence,

$$I_1=r!\sum_{j=1}^{m}\left|\sum_{k=0}^{j-1}\Delta a_k\right|\int\limits_{\pi/(j+1)}^{\pi/j}\frac{dx}{x^{r+1}}+O\left(\sum_{j=1}^{m}\left[\sum_{k=0}^{j-1}|\Delta a_k|(k+1)^{r+1}\right]\int\limits_{\pi/(j+1)}^{\pi/j}dx\right)=$$

$$=O_r\left(\sum_{j=1}^{m}|a_j|j^{r-1}\right)+O\left(\sum_{j=1}^{m}\sum_{k=0}^{j-1}\frac{(k+1)^{r+1}|\Delta a_k|}{j(j+1)}\right),$$

where O_r depends on r. But

$$\begin{aligned}\sum_{j=1}^{m}\sum_{k=0}^{j-1}\frac{(k+1)^{r+1}|\Delta a_k|}{j(j+1)}&=\sum_{j=1}^{m}\frac{1}{j(j+1)}\sum_{k=0}^{j-1}(k+1)^{r+1}|\Delta a_k|\le\\&\le\sum_{k=0}^{\infty}(k+1)^{r+1}|\Delta a_k|\sum_{j=k+1}^{\infty}\frac{1}{j(j+1)}=\\&=\sum_{k=0}^{\infty}(k+1)^{r}|\Delta a_k|=\\&=|\Delta a_0|+\sum_{k=1}^{\infty}(k+1)^{r}|\Delta a_k|\le\end{aligned}$$

$$\leq |a_1| + 2^r \sum_{k=1}^{\infty} k^r |\Delta a_k| \leq$$

$$\leq \sum_{k=1}^{\infty} |\Delta a_k| + 2^r \sum_{k=1}^{\infty} k^r A_k \leq$$

$$\leq (1+2^r) \sum_{k=1}^{\infty} k^r A_k \,.$$

Thus,

$$\sum_{j=1}^{m} \sum_{k=0}^{j-1} \frac{|\Delta a_k|(k+1)^{r+1}}{j(j+1)} = O_r\left(\sum_{k=1}^{\infty} k^r A_k\right) ,$$

where O_r depends on r. Therefore,

$$I_1 = O_r\left(\sum_{j=1}^{m} |a_j| j^{r-1}\right) + O_r\left(\sum_{k=1}^{\infty} k^r A_k\right) ,$$

where O_r depends on r.

Applying the Abel's transformation yields,

$$\sum_{k=j}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x) = \sum_{k=j}^{\infty} \Delta A_k \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} \overline{D}_i^{(r)}(x) - A_j \sum_{i=0}^{j-1} \frac{\Delta a_i}{A_i} \overline{D}_i^{(r)}(x) \,.$$

Let us estimate the second integral:

$$I_2 \leq \sum_{j=1}^{m} \left[\sum_{k=j}^{\infty} (\Delta A_k) \int_{\pi/(j+1)}^{\pi} \left|\sum_{i=0}^{k} \frac{\Delta a_i}{A_i} \overline{D}_k^{(r)}(x)\right| + A_j \int_{\pi/(j+1)}^{\pi/j} \left|\sum_{i=0}^{j-1} \frac{\Delta a_i}{A_i} \overline{D}_i^{(r)}(x)\right| dx\right] .$$

Applying the Lemma 3.3, we have:

$$J_k = \int_{\pi/(j+1)}^{\pi} \left|\sum_{i=0}^{k} \frac{|\Delta a_i|}{A_i} \overline{D}_i^{(r)}(x)\right| dx = O_r\big((k+1)^{r+1}\big) , \qquad (3.7)$$

where O_r depends on r. Then by Lemma 4.3 (iii),

$$\int_{\pi/(j+1)}^{\pi/j} \left|\sum_{i=0}^{j-1} \frac{\Delta a_i}{A_i} \overline{D}_i^{(r)}(x)\right| dx =$$

$$= O\left(\int_{\pi/(j+1)}^{\pi/j} j^r \left(\sum_{i=0}^{j-1} \frac{|\Delta a_i|}{A_i}\right) \frac{dx}{x}\right) + O\left(\sum_{i=0}^{j-1} \frac{|\Delta a_i|}{A_i} \int_{\pi/(j+1)}^{\pi/j} \frac{dx}{x^{r+1}}\right) =$$

$$= O(j^r) + O_r(j^r) = O_r(j^r) , \qquad (3.8)$$

where O_r depends on r.

However, by (3.7), (3.8), Lemma 1.10, we have

$$I_2 \leq \sum_{k=1}^{\infty} (\Delta A_k)\, J_k + O_r \left(\sum_{j=1}^{\infty} j^r A_j \right) =$$

$$= O_r(1) \sum_{k=1}^{\infty} (\Delta A_k)(k+1)^{r+1} + O_r \left(\sum_{j=1}^{\infty} j^r A_j \right) =$$

$$= O_r \left(\sum_{j=1}^{\infty} j^r A_j \right) .$$

Corollary 3.4. *Let the coefficients of the series $g(x)$ satisfy the condition $\Im_r$, $r = 1, 2, 3, \ldots$ Then the following relation holds:*

$$\int_0^{\pi} |g^{(r)}(x)| dx = O_r \left(\sum_{n=1}^{\infty} n^r A_n \right) ,$$

where O_r depends on r.

Proof. By inequalities

$$\sum_{n=1}^{m} |a_n|\, n^{r-1} \leq \sum_{n=1}^{\infty} n^{r-1} \sum_{k=n}^{\infty} |\Delta a_k| \leq \sum_{n=1}^{\infty} n^{r-1} \sum_{k=n}^{\infty} A_k =$$

$$= \sum_{k=1}^{\infty} A_k \sum_{n=1}^{k} n^{r-1} \leq \sum_{k=1}^{\infty} k^r A_k \,,$$

and by Theorem 3.10, we obtain:

$$\int_{\pi/(m+1)}^{\pi} |g^{(r)}(x)| dx = O_r \left(\sum_{n=1}^{\infty} n^r A_n \right) .$$

Letting $m \to \infty$, the inequality is satisfied.

Lemma 3.4. *Let $\{\alpha_j\}_{j=0}^{k}$ be a sequence of real numbers. Then the following relation holds for $v = 0, 1, 2, \ldots r$, $\alpha \geq 0$ and $r \in \{0, 1, 2, \ldots [\alpha]\}$*

$$V_k = \int_{\pi/(k+1)}^{\pi} \left| \sum_{j=0}^{k} \alpha_j \frac{\left(j + \frac{1}{2}\right)^v \sin\left[\left(j + \frac{1}{2}\right)x + \frac{v\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \right| dx =$$

$$= O_p \left((k+1)^{1+\alpha} \left[(k+1)^{p(r-\alpha)-1} \sum_{j=0}^{k} |\alpha_j|^p \right]^{1/p} \right) ,$$

where O_p depends only on p.

Proof. Applying the Lemma 1.12, we get:

$$V_k = O_p \left[(k+1)^{r+1} \left(\frac{1}{k+1} \sum_{j=0}^{k} |\alpha_j|^p \right)^{1/p} \right] =$$

$$= O_p \left((k+1)^{1+\alpha} \left[(k+1)^{p(r-\alpha)-1} \sum_{j=0}^{k} |\alpha_j|^p \right]^{1/p} \right),$$

where O_p depends only on p.

Lemma 3.5. *Let the sequence of real numbers $\{a_n\}$ belong to the class $S_{p\alpha r}$, $1 < p \le 2$, $\alpha \ge 0$, $r \in \{0, 1, 2, \dots [\alpha]\}$. Then the following relation holds*

$$\int_0^\pi \left| \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx = O_p\big((k+1)^{\alpha+1}\big),$$

where O_p depends only on p.

Proof. We have:

$$\int_0^\pi \left| \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx = \int_0^{\pi/(k+1)} + \int_{\pi/(k+1)}^{\pi} = I_k + J_k .$$

Appliying the inequality $D_n^{(r)}(x) = O(n^{r+1})$, we obtain

$$I_k \le \gamma k^r \sum_{j=0}^{k} \frac{|\Delta a_j|}{A_j} \le \gamma (k+1)^{r+1} \left(\frac{1}{k+1} \sum_{j=0}^{k} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} =$$

$$= \gamma (k+1)^{1+\alpha} \left[(k+1)^{p(r-\alpha)-1} \sum_{j=0}^{k} \frac{|\Delta a_j|^p}{A_j^p} \right] = O\big((k+1)^{\alpha+1}\big).$$

For the second integral we applying Lemma 1.13 and Lemma 3.4 and we get:

$$J_k = O_p\big((k+1)^{1+\alpha}\big).$$

Finally the inequality is satisfied.

Lemma 3.6. *Let the sequence of real numbers* $\{a_n\}$ *belong to the class* $S_{p\alpha r}$, $1 < p \le 2$, $\alpha \ge 0$, $r \in \{0, 1, 2, \dots [\alpha]\}$. *Then the following limit holds:*

$$A_N \int_0^\pi \left| \sum_{j=0}^{N} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx = o(1)\,, \quad N \to \infty\,.$$

Proof. Applying first the Lemma 3.5, then the Lemma 1.10, yields:

$$A_N \int_0^\pi \left| \sum_{j=0}^{N} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx = O_p\big(A_N (N+1)^{1+\alpha}\big) = o(1)\,, \quad N \to \infty\,.$$

Theorem 3.11 [52]. *Let the coefficients of the series* (C) *belong to the class* $S_{p\alpha r}$, $1 < p \le 2$, $\alpha \ge 0$, $r \in \{0, 1, 2, \dots [\alpha]\}$. *Then the* r*-th derivate of the series* (C) *is a Fourier series of some* $f^{(r)} \in L^1(0,\pi)$ *and the following inequality holds:*

$$\int_0^\pi |f^{(r)}(x)| dx \le M_{p,\alpha} \sum_{n=0}^{\infty} n^\alpha A_n\,,$$

where $M_{p,\alpha}$ *is a positive constant depends on* p, α.

Proof. Since

$$\sum_{k=1}^{n} k^r |\Delta a_k| = \sum_{k=1}^{n-1} (\Delta A_k) \sum_{j=1}^{k} \frac{|\Delta a_j|}{A_j} j^r + A_n \sum_{j=1}^{n} \frac{|\Delta a_j|}{A_j} j^r \le$$

$$\le \sum_{k=1}^{n-1} (\Delta A_k) k^{1+\alpha} \left[k^{p(r-\alpha)-1} \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} \right]^{1/p} +$$

$$+ n^{1+\alpha} A_n \left[n^{p(r-\alpha)-1} \sum_{j=1}^{n} \frac{|\Delta a_j|^p}{A_j^p} \right] = \qquad (\triangle)$$

$$= O(1) \left[\sum_{k=1}^{n-1} (\Delta A_k) k^{1+\alpha} + n^{1+\alpha} A_n \right] = O\left(\sum_{k=1}^{n} k^\alpha A_k \right).$$

Applying the same estimates of the proof of Theorem 3.8, we obtain

$$\sum_{k=1}^{n} |\Delta(k^r a_k)| \le O_r\left(\sum_{k=1}^{n} k^{r-1} |a_{k+1}| \right) + O\left(\sum_{k=1}^{n} k^r |\Delta a_k| \right).$$

But

$$\sum_{k=1}^{n} k^{r-1}|a_{k+1}| \le \sum_{k=1}^{n-1} |\Delta a_{k+1}|k^r + \sum_{k=n+1}^{\infty} k^r|\Delta a_k|$$

implies that

$$\sum_{k=1}^{n} |\Delta(k^r a_k)| \le O_r\left(\sum_{k=1}^{n-1} |\Delta a_{k+1}|k^r\right) + O_r\left(\sum_{k=n+1}^{\infty} k^r|\Delta a_k|\right) + O\left(\sum_{k=1}^{n} k^r|\Delta a_k|\right) =$$

$$= O_r\left(\sum_{k=1}^{n} k^\alpha A_k\right) + o(1)\,, \quad n \to \infty\,.$$

Thus

$$\sum_{k=1}^{\infty} |\Delta(k^r a_k)| \le O_r\left(\sum_{k=1}^{\infty} k^\alpha A_k\right) < \infty\,, \quad \text{i.e.} \quad \lim_{n\to\infty} S_n^{(r)}(x) = f^{(r)}(x)\,.$$

Since $f^{(r)}(x) = \sum_{k=0}^{\infty} \Delta a_k D_k^{(r)}(x)$, applying Abel's transformation, Lemma 3.6 and Lemma 3.5, we obtain

$$\int_0^\pi |f^{(r)}(x)|dx = \int_0^\pi \left|\sum_{k=0}^{\infty} \Delta a_k D_k^{(r)}(x)\right| dx \le \lim_{N\to\infty} \sum_{k=0}^{N-1} (\Delta A_k) \int_0^\pi \left|\sum_{j=0}^{k} \frac{\Delta a_j}{A_j}\, D_j^{(r)}(x)\right| dx =$$

$$= O_p(1) \lim_{N\to\infty} \sum_{k=0}^{N-1} (\Delta A_k)(k+1)^{\alpha+1} =$$

$$= O_p(1) \lim_{N\to\infty} \left\{\sum_{k=0}^{N} \left[(k+1)^{\alpha+1} - k^{\alpha+1}\right] A_k - (N+1)^{\alpha+1} A_n\right\} =$$

$$= O_{p,\alpha}\left(\sum_{k=0}^{\infty} k^\alpha A_k\right)\,.$$

Finally,

$$\int_0^\pi |f^{(r)}(x)|dx \le M_{p,\alpha} \sum_{n=0}^{\infty} n^\alpha A_n\,,$$

where $M_{p,\alpha}$ depends on p and α.

Theorem 3.12 [52]. *Let the coefficients of the series* (S) *belong to the class* $S_{p\alpha r}$*,* $1 < p \le 2$*,* $\alpha \ge 0$*,* $r \in \{0, 1, 2, \dots [\alpha]\}$*. Then the* r*-th derivate of the series* (S) *converges to a function and for $m = 1, 2, 3, \ldots$ the following inequality hplds:*

$$\int\limits_{\pi/(m+1)}^{\pi} |g^{(r)}(x)|dx \le M \sum_{j=1}^{m} |a_j| j^{r-1} + O_{p,\alpha,r}\left(\sum_{k=1}^{\infty} k^{\alpha} A_k\right), \qquad (*)$$

where $0 < M = M(r) < \infty$ and $O_{p,\alpha,r}$ depends on p, r and α. Moreover, if $\sum_{n=1}^{\infty} n^{r-1}|a_n| < \infty$, then the r-th derivate of the series (S) *is a Fourier series of some $g^{(r)} \in L^1(0,\pi)$ and*

$$\int\limits_{0}^{\pi} |g^{(r)}(x)|dx \le O_r\left(\sum_{j=1}^{\infty} |a_j| j^{r-1}\right) + O_{p,\alpha,r}\left(\sum_{j=1}^{\infty} j^{\alpha} A_j\right), \qquad (**)$$

Proof. We suppose that $a_0 = 0$ and $A_0 = \max(|a_1|, A_1)$.
Applying the Lemma 4.3 (iii) and the inequality ($\triangle$) (proved in Th.3.11):

$$\sum_{k=1}^{n} k^r |\Delta a_k| = O\left(\sum_{k=1}^{n} k^{\alpha} A_k\right),$$

we have that the series $\sum_{k=1}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x)$ is uniformly convergent on any compact subset of $[\varepsilon, \pi]$, where $\varepsilon > 0$.
Thus representation (3.6) implies that

$$g^{(r)}(x) = \sum_{k=0}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x).$$

Then,

$$\int\limits_{\pi/(m+1)}^{\pi} |g^{(r)}(x)|dx \le \sum_{j=1}^{m} \int\limits_{\pi/(j+1)}^{\pi/j} \left|\sum_{k=0}^{j-1} \Delta a_k \overline{D}_k^{(r)}(x)\right| dx +$$

$$O\left(\sum_{j=1}^{m} \int\limits_{\pi/(j+1)}^{\pi/j} \left|\sum_{k=j}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x)\right| dx\right). \qquad (3.9)$$

Applying the same technique as in the proof of Theorem 3.10, we obtain

$$T = \sum_{j=1}^{m} \int\limits_{\pi/(j+1)}^{\pi/j} \left|\sum_{k=0}^{j-1} \Delta a_k \overline{D}_k^{(r)}(x)\right| dx =$$

$$= O_r\left(\sum_{j=1}^{m} |a_j| j^{r-1}\right) + O\left(\sum_{k=0}^{\infty} (k+1)^r |\Delta a_k|\right).$$

Then,

$$\sum_{k=0}^{\infty}(k+1)^r|\Delta a_k| \le |a_1| + 2^r\sum_{k=1}^{\infty}k^r|\Delta a_k| \le$$

$$\le (1+2^r)\sum_{k=1}^{\infty}k^r|\Delta a_k| = O_r\left(\sum_{k=1}^{\infty}k^\alpha A_k\right),$$

i.e.

$$T = O_r\left(\sum_{j=1}^{m}|a_j|j^{r-1}\right) + O_r\left(\sum_{k=1}^{\infty}k^\alpha A_k\right). \tag{3.10}$$

Let

$$U = \sum_{j=1}^{m}\int\limits_{\pi/(j+1)}^{\pi/j}\left|\sum_{k=j}^{\infty}\Delta a_k\overline{D}_k^{(r)}(x)\right|dx\,.$$

Applying the Abel's transformation, we have:

$$U \le \sum_{j=1}^{m}\left[\sum_{k=j}^{\infty}(\Delta A_k)\,J_k + A_jI_j\right],$$

where

$$J_k = \int\limits_{\pi/(j+1)}^{\pi}\left|\sum_{i=0}^{k}\frac{\Delta a_i}{A_i}\,\overline{D}_i^{(r)}(x)\right|dx$$

and

$$I_j = \int\limits_{\pi/(j+1)}^{\pi/j}\left|\sum_{i=0}^{j-1}\frac{\Delta a_i}{A_i}\,\overline{D}_i^{(r)}(x)\right|dx$$

Applying the Holder-Hausdorff-Young technique (see the proof of Lemma 3.5), we obtain $J_k = O_{p,r}\big((k+1)^{\alpha+1}\big)$, where $O_{p,r}$ depends on r and p. Then by Lemma 4 (iii),

$$I_j = O\left(j^r\ln\Big(1+\frac{1}{j}\Big)\left(\sum_{i=0}^{j-1}\frac{|\Delta a_i|}{A_i}\right)\right) + O\left(\sum_{i=0}^{j-1}\frac{|\Delta a_i|}{A_i}\int\limits_{\pi/(j+1)}^{\pi/j}\frac{dx}{x^{r+1}}\right) =$$

$$= O\left(j^\alpha\left(j^{p(r-\alpha)-1}\sum_{i=0}^{j-1}\frac{|\Delta a_i|^p}{A_i^p}\right)^{1/p}\right) + O_r\left(j^{r-1}\sum_{i=0}^{j-1}\frac{|\Delta a_i|}{A_i}\right) =$$

$$= O(j^\alpha) + O_r\left(j^\alpha\left(j^{p(r-\alpha)-1}\sum_{i=0}^{j-1}\frac{|\Delta a_i|^p}{A_i^p}\right)^{1/p}\right) =$$

$$= O(j^\alpha) + O_r(j^\alpha) = O_r(j^\alpha)\,.$$

Thus

$$
\begin{aligned}
U &\leq O_{p,r}(1)\sum_{k=1}^{\infty}(k+1)^{\alpha+1}(\Delta A_k) + O_r(1)\sum_{j=1}^{\infty} j^{\alpha}A_j = \\
&= O_{p,\alpha,r}(1)\sum_{k=1}^{\infty} k^{\alpha}A_k + O_r(1)\sum_{j=1}^{\infty} j^{\alpha}A_j = \\
&= O_{p,\alpha,r}\left(\sum_{k=1}^{\infty} k^{\alpha}A_k\right),
\end{aligned}
\tag{3.11}
$$

since $n^{\alpha+1}A_n = o(1)$, $n \to \infty$.

Combining the inequalities (3.9), (3.10) and (3.11), the inequality $(*)$ is satisfied. If $\sum_{n=1}^{\infty} n^{r-1}|a_n| < \infty$, by letting $m \to \infty$ in inequality $(*)$, we obtain that the r-th derivate of the series (S) is a Fourier series of some $g^{(r)} \in L^1(0,\pi)$ and the inequality $(**)$ is satisfied.

Now we consider the case $r = \alpha = 0$. Since S_p and $S_p(\delta)$, $p > 1$ are identical classes of Fourier coefficients we obtain the following corollaries.

Corollary 3.5 [48]. *Let the coeficients of the series* (C) *belong to the class $S_p(\delta)$, $1 < p \leq 2$. Then the series is a Fourier series and the following inequality holds:*

$$
\int_0^{\pi} |f(x)|dx \leq M_p\left(\sum_{n=0}^{\infty} A_n + \sum_{n=1}^{\infty} n\delta_n\right),
$$

where M_p is a positive constant depends only on p.

Corollary 3.6 [56]. *Let the coefficients of the series* (S) *belong to the class $S_p(\delta)$, $1 < p \leq 2$. Then the series converges to a function $g(x)$ and the following relation holds for $m = 1, 2, 3, \ldots$*

$$
\int_{\pi/(m+1)}^{\pi} |g(x)|dx \leq \sum_{n=1}^{m} \frac{|a_n|}{n} + O_p\left(\sum_{n=1}^{\infty} A_n + \sum_{n=1}^{\infty} n\delta_n\right),
$$

where O_p depends only on p.

3.3. NECESSARY AND SUFFICIENT CONDITIONS FOR L^1-CONVERGENCE OF THE r-TH DERIVATE OF FOURIER SERIES

Wang and Telyakovskii [23] considered the following class of sequences. A null-sequence $\{a_k\}$ belongs to the class $(BV)_r^\sigma$, $r = 0, 1, 2, 3, \ldots$, $\sigma \geq 0$ if $\sum_{k=1}^{\infty} k^r |\Delta^\sigma a_k| < \infty$. In the same paper, they have proved the following theorem.

Theorem 3.13 [23]. *Let $\rho \geq 0$, $\sigma \geq 0$. Then for all $\gamma > \sigma$, the following embedding relation holds*

$$(BV)_\rho^\sigma \subset (BV)_\rho^\gamma .$$

For $r = 0$ and $\sigma = m = 0, 1, 2, 3, \ldots$ we have a well-known class $(BV)^m$.

Corollary 3.7. *Let $\{a_n\} \in (BV)^\sigma$, $\sigma \geq 0$ and $a_n \log n = o(1)$, $n \to \infty$. Then $\|S_n - f\| = o(1)$, $n \to \infty$ iff $\{a_n\} \in C$.*

Proof. Let m is integer such that $m \geq \sigma$. Then by Theorem 3.13, (case $\rho = 0$), we have $\{a_n\} \in (BV)^m$. Applying the Theorem 1.16, the proof of the corollary is completed.

If $\sigma = 1$, we denote $(BV)_r = (BV)_r^\sigma$.

Wang Kunyang and Telyakovskii [23] considering the complex form of the trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k e^{ikx}$$

proved the following theorem.

Theorem 3.14 [23]. *If $\{a_n\} \in (BV)_r^\sigma$, $r = 0, 1, 2, 3, \ldots$, $\sigma \geq 0$ then the series* (C) *and* (S) *have continuous derivatives of r-th order on $(0, \pi]$.*

Lemma 3.7 [30]. $\|D_n^{(r)}\|_1 = \dfrac{4}{\pi} n^r \log n + O(n^r)$, $r = 0, 1, 2, \ldots$.

Next we shall give necessary and sufficient conditions for L^1-convergence of the r-th derivate of the series (C).

Theorem 3.15 [53]. *Let $\{a_n\} \in (BV)_r$, $r = 0, 1, 2, 3, \ldots$ and $a_n n^r \log n = o(1)$, $n \to \infty$. Then $\|S_n^{(r)} - f^{(r)}\| = o(1)$, $n \to \infty$ iff $\{a_n\} \in C_r$.*

Proof. Since $\{a_n\} \in (BV)_r$, the series $\sum\limits_{k=0}^{\infty} \Delta a_k D_k^{(r)}(x)$ is uniformly convergent on any segment $[\xi, \pi]$, $\xi > 0$. Thus, $f^{(r)}(x) = \sum\limits_{k=0}^{\infty} \Delta a_k D_k^{(r)}(x)$.

For "if" let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\int_0^\delta \left| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right| < \frac{\varepsilon}{3}, \quad \text{for all } n.$$

Then,

$$\begin{aligned}
\int_0^\pi |f^{(r)}(x) - S_n^{(r)}(x)| dx &= \int_0^\pi \left| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) - a_{n+1} D_n^{(r)}(x) \right| dx \leq \\
&\leq \int_0^\pi \left| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx + |a_{n+1}| \int_0^\pi |D_n^{(r)}(x)| dx = \\
&= \left(\int_0^\delta + \int_\delta^\pi \right) \left| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx + |a_{n+1}| \, \|D_n^{(r)}\| < \\
&< \frac{\varepsilon}{3} + \int_\delta^\pi \left| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx + \frac{\varepsilon}{3}.
\end{aligned}$$

Applying the estimate for the r-th derivate of the Dirichlet's kernel (see [30]), we obtain

$$\int_\delta^\pi \left| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx = O\left(\sum_{k=n}^{\infty} k^r |\Delta a_k| \right).$$

Hence,

$$\int_\delta^\pi \left| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx < \frac{\varepsilon}{3}.$$

We have that for sufficiently large n, $\|S_n^{(r)} - f^{(r)}\| < \varepsilon$.

For the "only if" part let $\varepsilon > 0$. Then there exists an integer N such that

$$\int_0^\pi |f^{(r)} - S_n^{(r)}| dx < \frac{\varepsilon}{4} \quad \text{if} \quad n \geq N.$$

That is

$$\int_0^\pi \left| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) - a_{n+1} D_n^{(r)}(x) \right| dx < \frac{\varepsilon}{4} \quad \text{if} \quad n \geq N.$$

Since $a_n n^r \log n \to 0$, applying the Lemma 3.7, there exists an integer M such that

$$\int_0^\pi \left| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx < \frac{\varepsilon}{2} \quad \text{if} \quad n \geq M\,.$$

Now if $\sum\limits_{k=0}^{M} k^r |\Delta a_k| = 0$, then for $n < M$,

$$\int_0^\pi \left| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx = \int_0^\pi \left| \sum_{k=M+1}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx < \frac{\varepsilon}{2} < \varepsilon\,.$$

If $\sum\limits_{k=0}^{M} k^r |\Delta a_k| \neq 0$, let $\delta = \dfrac{\varepsilon}{2M \sum\limits_{k=0}^{M} k^r |\Delta a_k|}$. For $n \geq M$, we have:

$$\int_0^\delta \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx \leq \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx < \frac{\varepsilon}{2} < \varepsilon\,.$$

For $0 \leq n < M$, we get:

$$\int_0^\delta \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx \leq \int_0^\delta \left| \sum_{k=n+1}^{M} \Delta a_k D_k^{(r)}(x) \right| dx + \int_0^\delta \left| \sum_{k=M+1}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx \leq$$

$$\leq \int_0^\delta \left(\sum_{k=n+1}^{M} k^{r+1} |\Delta a_k| \right) dx + \int_0^\pi \left| \sum_{k=M+1}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx <$$

$$< \delta M \sum_{k=0}^{M} k^r |\Delta a_k| + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

Finally, $\{a_n\} \in C_r$.

This is an extension theorem of the Garrett-Stanojević Theorem 1.12.

Applying the Theorem 1.29 and this theorem we obtain the following corollary:

Corollary 3.8. *Let* $\{a_n\} \in \Im_r$, $r = 0, 1, 2, 3, \ldots$ *Then* $\|S_n^{(r)} - f^{(r)}\| = o(1)$, $n \to \infty$ *iff* $a_n n^r \log n = o(1)$, $n \to \infty$.

On the other hand by Theorem 1.31, we get $S_{pr} \subset (BV)_r \cap C_r$, for any $1 < p \leq 2$ and $r = 0, 1, 2, 3, \ldots$ Again, applying the Theorem 3.15, we obtain:

Corollary 3.9. *Let* $1 < p \leq 2$. *If* $\{a_n\} \in S_{pr}$, $r = 0, 1, 2, 3, \ldots$, *then*

$\|S_n^{(r)} - f^{(r)}\| = o(1),\ n \to \infty$ *iff* $a_n n^r \log n = o(1),\ n \to \infty.$
Now using the Lemma 3.5, Lemma 3.6 and applyng the same technique as in the proof of the Theorem 1.31, we can get the following Theorem:

Theorem 3.16. [70]. *For any* $1 < p \leq 2$, $\alpha \geq 0$ *and* $r \in \{0, 1, 2, \dots, [\alpha]\}$ *the following embedding relations hold:*

$$S_{p\alpha r} \subset (BV)_r \cap C_r \subset BV \cap C_r \,.$$

Combining this Theorem and Theorem 3.15 we can formulated the following Theorem.

Theorem 3.17. *Let* $1 < p \leq 2$, $\alpha \geq 0$ *and* $r \in \{0, 1, 2, \dots, [\alpha]\}$. *If* $\{a_n\} \in S_{p\alpha r}$, *then* $\|S_n^{(r)} - f^{(r)}\| = o(1),\ n \to \infty$ *iff* $a_n n^r \log n = o(1),\ n \to \infty.$

Remark 3.4. This theorem was obtained by Sheng [30], but we also given a new proof of this theorem.
Denote by I_m the dyadic interval $[2^{m-1}, 2^m)$, for $m \geq 1$.
A null sequence $\{a_n\}$ belongs to the class F_{pr}, $p > 1$, $r = 0, 1, 2, \dots$ if

$$\sum_{m=1}^{\infty} 2^{m(1/q+r)} \left(\sum_{k \in I_m} |\Delta a_k|^p \right)^{1/p} < \infty \,, \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1 \,.$$

It is obvious that for $r = 0$, we obtain the Fomin's class F_p.

Theorem 3.18. *For any* $p > 1$, $\alpha \geq 0$, $r \in \{0, 1, 2, \dots, [\alpha]\}$ *the following embedding relation holds:* $S_{p\alpha r} \subset F_{pr}$.

Proof. By condition $\dfrac{1}{n^{p(\alpha - r)+1}} \displaystyle\sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1)$ and monotonicity of $\{A_k\}$ we obtain:

$$\left(\sum_{k \in I_m} |\Delta a_k|^p \right)^{1/p} \leq 2^{(m-1)\frac{1}{p}} \left(\frac{1}{2^{m-1}} \sum_{k \in I_m} \frac{|\Delta a_k|^p}{A_k^p} \right)^{1/p} A_{2^{m-1}} \leq$$

$$\leq K \cdot 2^{m/p} 2^{m(\alpha - r)} A_{2^{m-1}} \,,$$

where K is an absolute constant. Hence,

$$\sum_{m=1}^{\infty} 2^{m(1/q+r)} \left(\sum_{k \in I_m} |\Delta a_k|^p \right)^{1/p} \leq K \sum_{m=1}^{\infty} 2^{m(1/p+1/q)} 2^{m\alpha} A_{2^{m-1}} =$$

$$= K\sum_{m=1}^{\infty} 2^{m(1+\alpha)} A_{2^{m-1}} =$$

$$= K\cdot 2^{1+\alpha}\sum_{m=1}^{\infty} 2^{(m-1)(1+\alpha)} A_{2^{m-1}} < \infty\,.$$

Theorem 3.19. *Let $1 < p \le 2$, $r = 0, 1, 2, \ldots$ If $\{a_n\} \in F_{pr}$, then $\|S_n^{(r)} - f^{(r)}\| = o(1)$, iff $a_n n^r \log n = o(1)$, $n \to \infty$.*

Proof. Applying Abel's transformation on the r-th derivate of the partial sums, we obtain

$$S_n^{(r)}(x) = \sum_{k=0}^{n} \Delta a_k D_k^{(r)}(x) + a_{n+1} D_n^{(r)}(x)\,.$$

Using the inequality

$$\sum_{k=1}^{\infty} |\Delta a_k D_k^{(r)}(x)| \le \lim_{n\to\infty} \frac{M}{x} \sum_{k=1}^{n} k^r |\Delta a_k| =$$

$$= \lim_{n\to\infty} \frac{M}{x} \sum_{j=1}^{m} \left(\sum_{k=2^{j-1}}^{2^j-1} k^r |\Delta a_k| \right) \quad (\text{for } n = 2^m - 1, \text{ and } M > 0.)$$

$$\le \lim_{n\to\infty} \frac{M}{x} \sum_{j=1}^{m} 2^{j(1/q+r)} \left(\sum_{k\in I_j} |\Delta a_k|^p \right)^{1/p} < \infty$$

and

$$|a_{n+1} D_n^{(r)}(x)| \le M |a_{n+1}| \frac{n^r}{x} \le \frac{M}{x} \sum_{k=n+1}^{\infty} k^r |\Delta a_k| \to 0\,, \quad n \to \infty\,,$$

we get $\lim_{n\to\infty} S_n^{(r)}(x) = f^{(r)}(x) = \sum_{k=0}^{\infty} \Delta a_k D_k^{(r)}(x)$. This implies that

$$\|f^{(r)} - g_n^{(r)}\| = \left\| \sum_{k=n+1}^{\infty} \Delta a_k D_k^{(r)}(x) \right\|\,.$$

But, Lemma 1.9 implies that

$$\|f^{(r)} - g_n^{(r)}\| \le A_p \sum_{m=j}^{\infty} 2^{m(1/q+r)} \left(\sum_{k\in I_j} |\Delta a_k|^p \right)^{1/p} = o(1)\,, \quad n \to \infty\,,$$

by the hypothesis of the theorem; here $j = j(n)$ denotes the integer for which $2^{j-1} \leq n \leq 2^j$. Since $g_n^{(r)}$ is a polynomial, it follows that $f^{(r)} \in L^1(T)$.
Since

$$|\,\|f^{(r)} - S_n^{(r)}\| - \|a_{n+1} D_n^{(r)}\|\,| \leq \|f^{(r)} - g_n^{(r)}\| = o(1)\,, \quad n \to \infty\,,$$

by Lemma 3.7, we obtain that $\|S_n^{(r)} - f^{(r)}\| = o(1)$, $n \to \infty$ iff $a_{n+1} n^r \log n = o(1)$, $n \to \infty$.
Similarly, we can get analogical theorem for sine series (S).

Theorem 3.20. *Let* $\{a_n\} \in F_{pr}$, $1 < p \leq 2$, $r = 0, 1, 2, \ldots$ *If* $\sum_{n=1}^{\infty} n^{r-1} |a_n| < \infty$ *then the* r*-th derivate of the series* (S) *is a Fourier series of some* $g^{(r)} \in L^1$ *and* $\|\tilde{S}_n^{(r)} - g^{(r)}\| = o(1)$, $n \to \infty$, *iff* $a_{n+1} n^r \log n = o(1)$, $n \to \infty$.

IV. CONVERGENCE AND INTEGRABILITY OF THE r-TH DERIVATE OF COMPLEX TRIGONOMETRIC SERIES

4.1. ON A THEOREM OF S. S. BHATIA AND B. RAM

Let $\{c_k\colon k = 0, \pm 1, \pm 2, \ldots\}$ be a sequence of complex numbers and the partial sums of the complex trigonometric series $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ be denoted by

$$S_n(c,t) = \sum_{k=-n}^{n} c_k e^{ikt}\,, \quad t \in \mathbf{T}\,. \tag{4.1}$$

If a trigonometric series is the Fourier series of some $f \in L^1$, we shall write $c_n = \hat{f}(n)$ for all n and $S_n(c,t) = S_n(f,t) = S_n(f)$.

S. S. Bhatia and Baby Ram[32] introduced the following class $\Re^*$ of complex sequence: a null sequence $\{c_n\}$ of complex numbers belongs to the class $\Re^*$ if

$$\sum_{k=1}^{\infty} \left| \Delta \left(\frac{c_{-k} - c_k}{k} \right) \right| k \log k < \infty \quad \text{and}$$

$$\sum_{k=1}^{\infty} k^2 \left| \Delta^2 \left(\frac{c_k}{k} \right) \right| < \infty\,.$$

Let $E_n(t) = \frac{1}{2} + \sum_{k=1}^{n} e^{ikt}$ and $E_{-n}(t) = \frac{1}{2} + \sum_{k=1}^{n} e^{-ikt}$.

Then the r-th derivatives $D_n^{(r)}(t)$ and $\tilde{D}_n^{(r)}(t)$ can be written as

$$\begin{aligned} 2\,D_n^{(r)}(t) &= E_n^{(r)}(t) + E_{-n}^{(r)}(t) \\ 2\,i\tilde{D}_n^{(r)}(t) &= E_n^{(r)}(t) - E_{-n}^{(r)}(t) \end{aligned} \tag{4.2}$$

where $E_n^{(r)}(t)$ denotes the r-th derivate of $E_n(t)$.

S. S. Bhatia and Baby Ram[32] introduced the following modified sums

$$g_n(c,t) = S_n(c,t) + \frac{i}{n+1}\left[c_{n+1}E_n'(t) - c_{-(n+1)}E_{-n}'(t)\right]$$

and proved the following result.

Theorem 4.1 [32] *Let* $\{c_n\} \in \Re^*$. *Then there exists* $f(t)$ *such that*

(i) $\lim_{n\to\infty} g_n(c,t) = f(t)$ *for all* $0 < |t| \leq \pi$.

(ii) $f(t) \in L^1(T)$ *and* $\|g_n(c,t) - f(t)\|_1 = o(1)$, $n \to \infty$.

(iii) $\|S_n(f,t) - f(t)\|_1 = o(1)$ *iff* $\hat{f}(n)\log|n| = o(1)$, $|n| \to \infty$.

Now we define a new class $\Re^*(r)$, $r = 0, 1, 2, \ldots$ of complex sequence as follows: a null sequence $\{c_k\}$ of complex numbers belong to the class $\Re^*(r)$, $r = 0, 1, 2, \ldots$ if

$$\sum_{k=1}^{\infty} \left| \Delta \left(\frac{c_{-k} - c_k}{k} \right) \right| k^{r+1} \log k < \infty$$

$$\sum_{k=1}^{\infty} k^{r+2} \left| \Delta^2 \left(\frac{c_k}{k} \right) \right| < \infty \,.$$

If $r = 0$, we denote $\Re^*(r) = \Re^*$.

Č. V. Stanojević and V. B. Stanojević [37]introduced the following modified complex trigonometric sims:

$$U_n(c,t) = S_n(c,t) - \left(c_n E_n(t) + c_{-n}(t) \right).$$

The complex form of the r-th derivate of this sum, obtained by Sheng [30], is

$$U_n^{(r)}(c,t) = S_n^{(r)}(c,t) - \left(c_n E_n^{(r)}(t) + c_{-n} E_{-n}^{(r)}(t) \right)$$

B. Ram and S. Kumari [28] introduced another set of modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left(\frac{a_j}{j} \right) k \cos kx \quad \text{and}$$

$$h_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left(\frac{a_j}{j} \right) k \sin kx \,.$$

The complex form of the r-th derivate of these modified sums is

$$G_n^{(r)}(c,t) = S_n^{(r)}(c,t) + \frac{i}{n+1} \left[c_{n+1} E_n^{(r+1)}(t) - c_{-(n+1)} E_{-n}^{(r+1)}(t) \right].$$

Remark 4.1. If $|n|^r c_n \to 0$, $|n| \to \infty$, then $\|G_n^{(r)} - U_n^{(r)}\| \to 0$. Observe that by partial summation, we have

$$E_n^{(r+1)}(t) = -i \sum_{k=1}^{n} E_k^{(r)}(t) + i(n+1)E_n^{(r)}(t)$$

and similarly for $E_{-n}^{(r+1)}(t)$. Then by the formulae

$$U_{n+1}^{(r)}(c,t) = S_n^{(r)}(c,t) - c_{n+1}E_n^{(r)}(t) - c_{-(n+1)}E_{-n}^{(r)}(t)\,.$$

we obtain

$$U_{n+1}^{(r)}(c,t) - G_n^{(r)}(c,t) = -c_{n+1}\,\frac{1}{n+1}\sum_{k=1}^{n} E_k^{(r)}(t) - c_{-(n+1)}\,\frac{1}{n+1}\sum_{k=1}^{n} E_{-k}^{(r)}(t)\,.$$

Then by the well-known properties of Fejer kernels, it follows that

$$\|G_n^{(r)} - U_n^{(r)}\| \to 0\,, \quad n \to \infty\,.$$

Using the modified complex sums $G_n^{(r)}$ we shall prove the following theorem:

Theorem 4.2. *Let* $\{c_n\} \in \Re^*(r)$, $r = 0, 1, 2, \ldots$ *Then*

(i) $\lim_{n\to\infty} G_n^{(r)}(c,t) = f^{(r)}(t)$ for all $0 < |t| \le \pi$.

(ii) $f^{(r)} \in L^1(T)$ and $\|G_n^{(r)}(c,t) - f^{(r)}(t)\|_1 = o(1)$, $n \to \infty$.

(iii) $\|S_n(f,t) - f^{(r)}(t)\|_1 = o(1)$, $n \to \infty$ iff $|n|^r\,\hat{f}(n)\log|n| = o(1)$, $|n| \to \infty$.

Lemma 4.1. $\|\tilde{D}_n^{(r)}\|_1 = O(n^r \log n)$, $r = 0, 1, 2, \ldots$

Lemma 4.2 [30]. *For each non-negative integer* n, $\|c_n E_n^{(r)} + c_{-n}E_{-n}^{(r)}\|_1 = o(1)$, $n \to \infty$ *holds iff* $|n|^r c_n \log|n| = o(1)$, $|n| \to \infty$, *where* $\{c_n\}$ *is a complex sequence.*
We note that this Lemma for $r = 0$, was obtained by Bray and Stanojević in [8].

Lemma 4.3. *Let* r *be a non-negative integer. Then for all* $0 < |t| \le \pi$ *and all* $n \ge 1$ *the following estimates hold*

(i) $|E_{-n}^{(r)}(t)| \le \dfrac{4n^r\pi}{|t|}$.

(ii) $|\tilde{D}_n^{(r)}(t)| \le \dfrac{4n^r\pi}{|t|}$.

(iii) $|\tilde{D}_n^{(r)}(t)| \le \dfrac{4n^r\pi}{|t|} + O\Big(\dfrac{1}{|t|^{r+1}}\Big)$.

Proof. (i) The case $r = 0$ is trivial. Really, since $E_n(t) = D_n(t) + i\tilde{D}_n(t)$, we have

$$|E_n(t)| \le |D_n(t)| + |\tilde{D}_n(t)| \le \frac{\pi}{2|t|} + \frac{\pi}{|t|} = \frac{3\pi}{|t|} < \frac{4\pi}{|t|}$$

$$|E_{-n}(t)| = |E_n(-t)| < \frac{4\pi}{|t|}.$$

Let $r \ge 1$. Applying Abels's transformation, we have:

$$|E_n^{(r)}(t)| = i^r \sum_{k=1}^{n} k^r e^{ikt} = i^r \left[\sum_{k=1}^{n-1} \Delta(k^r)\Big(E_k(t) - \frac{1}{2}\Big) + n^r \Big(E_n(t) - \frac{1}{2}\Big)\right]$$

$$|E_n^{(r)}(t)| \le \sum_{k=1}^{n-1} [(k+1)^r - k^r]\Big(\frac{1}{2} + |E_k(t)|\Big) + n^r \Big(|E_n(t)| + \frac{1}{2}\Big) \le$$

$$\le \Big(\frac{\pi}{2|t|} + \frac{3\pi}{2|t|}\Big) \left\{\sum_{k=1}^{n-1} [(k+1)^r - k^r] + n^r\right\} = \frac{4\pi\, n^r}{|t|}.$$

Since $E_{-n}^{(r)}(t) = E_n^{(r)}(-t)$, we obtain $|E_{-n}^{(r)}(t)| \le \dfrac{4\pi n^r}{|t|}$.

(ii) Applying the inequality (i) and equation (4.2) we obtain

$$|\tilde{D}_n^{(r)}(t)| = |i\tilde{D}_n^{(r)}(t)| \le \frac{1}{2}|E_n^{(r)}(t)| + \frac{1}{2}|E_{-n}^{(r)}(t)| \le \frac{4n^r\pi}{|t|}.$$

(iii) We note that $|\Big(\operatorname{ctg} \dfrac{t}{2}\Big)^{(r)}| = O\left(\dfrac{1}{|t|^{r+1}}\right)$.

Applying the inequality (ii), we obtain

$$|\overline{D}_n^{(r)}(t)| \le |\tilde{D}_n^{(r)}(t)| + \frac{1}{2} \left|\Big(\operatorname{ctg} \frac{t}{2}\Big)^{(r)}\right| \le \frac{4n^r\pi}{|t|} + O\Big(\frac{1}{|t|^{r+1}}\Big).$$

Lemma 4.4 [32] $\|\tilde{K}_n'(t)\|_1 = O(n)$.

Lemma 4.5. $\|\tilde{K}_n^{(r)}\|_1 = O(n^r)$, $r = 0, 1, 2, \ldots$

Proof. Since $\tilde{K}_n(x) = \displaystyle\sum_{k=1}^{n} \frac{n+1-k}{n+1} \sin kx$, we have that

$$T_n(x) = \tilde{K}_n'(x) = \sum_{k=1}^{n} \frac{k(n+1-k)}{n+1} \cos kx$$

is a cosine trigonometric polynomial of order n.

Applying first Bernstein's inequality, then Lemma 4.4, yield:

$$\|\tilde{K}_n^{(r)}\|_1 = \|T_n^{(r-1)}(x)\|_1 \le n^{r-1}\|T_n(x)\|_1 = O(n^r)\,.$$

Proof of Theorem 4.2.

Applying the Abel's transformation, we have:

$$G_n^{(r)}(c,t) = S_n^{(r)}(c,t) + \frac{i}{n+1}\left[c_{n+1}E_n^{(r+1)}(t) - c_{-(n+1)}E_{-n}^{(r+1)}(t)\right] =$$

$$= 2\sum_{k=1}^{n}\Delta\Big(\frac{c_k}{k}\Big)\tilde{D}_k^{(r+1)}(t) + \sum_{k=1}^{n}\Delta\Big(\frac{c_{-k}-c_k}{k}\Big)iE_{-k}^{(r+1)}(t)\,.$$

By Lemma 4.3, we get:

$$\sum_{k=1}^{\infty}\left|\Delta\Big(\frac{c_k}{k}\Big)\tilde{D}_k^{(r+1)}\right| \le \frac{4\pi}{|t|}\sum_{k=1}^{\infty}k^{r+1}\left|\Delta\Big(\frac{c_k}{k}\Big)\right| \le$$

$$\le \frac{4\pi}{|t|}\left\{\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}k^{r+1}\left|\Delta^2\Big(\frac{c_j}{j}\Big)\right|\right\} =$$

$$= \frac{4\pi}{|t|}\left\{\sum_{j=1}^{\infty}\left(\sum_{k=1}^{j}k^{r+1}\right)\left|\Delta^2\Big(\frac{c_j}{j}\Big)\right|\right\} =$$

$$= O\left(\frac{1}{|t|}\sum_{j=1}^{\infty}j^{r+2}\left|\Delta^2\Big(\frac{c_j}{j}\Big)\right|\right) < \infty$$

and

$$\sum_{k=3}^{\infty}\left|\Delta\Big(\frac{c_{-k}-c_k}{k}\Big)E_{-k}^{(r+1)}(t)\right| \le \frac{4\pi}{|t|}\left\{\sum_{k=3}^{\infty}k^{r+1}\left|\Delta\Big(\frac{c_{-k}-c_k}{k}\Big)\right|\right\} =$$

$$= O\left(\frac{1}{|t|}\sum_{k=3}^{\infty}k^{r+1}\log k\left|\Delta\Big(\frac{c_{-k}-c_k}{k}\Big)\right|\right) < \infty\,.$$

Consequently,

$$f^{(r)}(t) = 2\sum_{k=1}^{\infty}\Delta\Big(\frac{c_k}{k}\Big)\tilde{D}_k^{(r+1)}(t) + \sum_{k=1}^{\infty}\Delta\Big(\frac{c_{-k}-c_k}{k}\Big)iE_{-k}^{(r+1)}(t)$$

exists and thus (i) follows.
Now, for $t \neq 0$, we have:

$$f^{(r)}(t) - G_n^{(r)}(c,t) =$$

$$= 2 \sum_{k=n+1}^{\infty} \Delta\Big(\frac{c_k}{k}\Big)\tilde{D}_k^{(r+1)}(t) + i \sum_{k=n+1}^{\infty} \Delta\Big(\frac{c_{-k}-c_k}{k}\Big)E_{-k}^{(r+1)}(t) =$$

$$= 2 \sum_{k=n+1}^{\infty} (k+1)\Delta^2\Big(\frac{c_k}{k}\Big)\tilde{K}_k^{(r+1)}(t) - 2(n+1)\Delta\Big(\frac{c_{n+1}}{n+1}\Big)\tilde{K}_{n+1}^{(r+1)}(t) +$$

$$+ i \sum_{k=n+1}^{\infty} \Delta\Big(\frac{c_{-k}-c_k}{k}\Big)E_{-k}^{(r+1)}(t)\,.$$

Then,

$$\|f^{(r)}(t) - G_n^{(r)}(c,t)\|_1 \le 2 \sum_{k=n+1}^{\infty} (k+1)\left|\Delta^2\Big(\frac{c_k}{k}\Big)\right| \int_{-\pi}^{\pi} |\tilde{K}_k^{(r+1)}(t)|dt +$$

$$+ 2(n+1)\left|\Delta\Big(\frac{c_{n+1}}{n+1}\Big)\right| \int_{-\pi}^{\pi} \left|\tilde{K}_{n+1}^{(r+1)}(t)\right| dt + \sum_{k=n+1}^{\infty} \left|\Delta\Big(\frac{c_{-k}-c_k}{k}\Big)\right| \int_{-\pi}^{\pi} |E_{-k}^{(r+1)}(t)|dt\,.$$

Applying Lemma 4.5, Lemma 3,7 and Lemma 4.1, we have:

$$\|f^{(r)}(t) - G_n^{(r)}(c,t)\|_1 = O\left(\sum_{k=n+1}^{\infty} (k+1)^{r+2}\left|\Delta^2\Big(\frac{c_k}{k}\Big)\right|\right) +$$

$$+ O\left((n+1)^{r+2}\left|\Delta\Big(\frac{c_{n+1}}{n+1}\Big)\right|\right) + O\left(\sum_{k=n+1}^{\infty}\left|\Delta\Big(\frac{c_{-k}-c_k}{k}\Big)\right| k^{r+1}\log k\right)\,.$$

But

$$\left|\Delta\Big(\frac{c_{n+1}}{n+1}\Big)\right| = \left|\sum_{k=n+1}^{\infty}\Delta^2\Big(\frac{c_k}{k}\Big)\right| \le \sum_{k=n+1}^{\infty}\frac{k^{r+2}}{k^{r+2}}\left|\Delta^2\Big(\frac{c_k}{k}\Big)\right| \le$$

$$\le \frac{1}{(n+1)^{r+2}}\sum_{k=n+1}^{\infty} k^{r+2}\left|\Delta^2\Big(\frac{c_k}{k}\Big)\right| = o\Big(\frac{1}{(n+1)^{r+2}}\Big), \quad n\to\infty\,.$$

Hence, $\|f^{(r)}(t) - G_n^{(r)}(c,t)\|_1 = o(1)$, $n \to \infty$ by the hypothesis of the theorem. Since $G_n^{(r)}(c,t)$ is a polynomial, it follows that $f^{(r)} \in L^1(T)$.

The proof of (iii) follows from the estimate

$$\left| \|f^{(r)} - S_n^{(r)}(f)\|_1 - \left\| \frac{i}{n+1} \big(\hat{f}(n+1)E_n^{(r+1)} - \hat{f}\big(-(n+1)\big)E_{-n}^{(r+1)}\big) \right\|_1 \right| \le$$

$$\le \|f^{(r)} - G_n^{(r)}(c,t)\|_1 = o(1)\,, \quad n \to \infty$$

and from Lemma 4.2.

Considering the sums $U_n^{(r)}$ instead of $G_n^{(r)}$ and in view of the proceding Remark 4.1, statement (ii) in Theorem 4.2 can be replaced by:

(ii') $f^{(r)} \in L^1(T)$ and $\|U_n^{(r)}(c,t) - f^{(r)}(t)\|_1 = o(1)$, $n \to \infty$.

Thus we have the following results:

Theorem 4.3. *Under the hypothesis of Theorem* 4.2*, statements* (i), (ii') *and* (iii) *hold.*

4.2. ON A THEOREM OF P. L. ULJANOV

The function $\varphi(x)$ is called A-integrability on $[a,b]$ if

a) $mE\{|\varphi(x)| > n\} = o\left(\frac{1}{n}\right)$

b) the following limit exists $\lim\limits_{n\to\infty} \int\limits_a^b \big[\varphi(x)\big]_n dx = I$, where

$$\big[\varphi(x)\big]_n = \begin{cases} n\colon \varphi(x) > n \\ \varphi(x)\colon |\varphi(x)| \le n \\ -n\colon \varphi(x) < -n\,. \end{cases}$$

The number I is called A-integral of function $\varphi(x)$ for all $x \in [a,b]$.

As an application of A-integrals, P. L. Uljanov [59] obtained an interesting result concerning the integrability of $|f|^p$ and $|g|^p$, for any $0 < p < 1$, where

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx$$

$$g(x) = \sum_{k=1}^{\infty} a_k \sin kx$$

and $\{a_n\}$ is a null-sequence of bounded variation. Namely he proved the following theorem.

Theorem 4.4 [59]. *Let* $\{a_n\} \in BV$. *Then for any* $0 < p < 1$,

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |f(x) - S_n(x)|^p dx = 0 \text{ and } \lim_{n\to\infty} \int_{-\pi}^{\pi} |g(x) - \tilde{S}_n(x)|^p dx = 0\,. \tag{4.4}$$

It is obvious that this Theorem holds when the coefficients $\{a_n\}$ belong to the classes S, F_q, S_q, $S_{q\alpha}$ (case $r = 0$), $q > 1$, $\alpha \geq 0$.
Next, we shall define a new L^p, $0 < p < 1$, integrability class as follows.
A null sequence $\{a_n\}$ belongs to the class $H_{q\alpha}$, $0 < q \leq 1$, $\alpha \geq 0$ if there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} k^\alpha A_k < \infty$ and $\frac{1}{n^{q\alpha+q}} \sum_{k=1}^{n} \frac{|\Delta a_k|^q}{A_k^q} = O(1)\,.$

Theorem 4.5. *For any* $0 < q \leq 1$ *and any* $\alpha \geq 0$ *the class* $H_{q\alpha}$ *is a subclass of* BV.

Proof. Applying the Abel's transformation and well-known inequality

$$\left(\sum b_i\right)^q \leq \left(\sum b_i^q\right)\,, \text{ for } b_i \geq 0\,, \text{ and } 0 < q \leq 1\,, \tag{4.5}$$

we obtain:

$$\sum_{k=1}^{n} |\Delta a_k| = \sum_{k=1}^{n-1} k^{\alpha+1} (\Delta A_k) \Big(\frac{1}{k^{\alpha+1}} \sum_{j=1}^{k} \frac{|\Delta a_j|}{A_j} \Big) + + n^{\alpha+1} A_n \Big(\frac{1}{n^{\alpha+1}} \sum_{j=1}^{n} \frac{|\Delta a_j|}{A_j} \Big) \leq$$

$$\leq \sum_{k=1}^{n-1} k^{\alpha+1} (\Delta A_k) \Big(\frac{1}{k^{q\alpha+q}} \sum_{j=1}^{k} \frac{|\Delta a_j|^q}{A_j^q} \Big)^{1/q} + n^{\alpha+1} A_n \Big(\frac{1}{n^{q\alpha+q}} \sum_{j=1}^{n} \frac{|\Delta a_j|^q}{A_j^q} \Big)^{1/q} =$$

$$= O_q(1) \left[\sum_{k=1}^{n-1} k^{\alpha+1} (\Delta A_k) + n^{\alpha+1} A_n \right]\,.$$

Now, letting $n \to \infty$ and applying Lemma 1.10 and Lemma 1.11, we obtain $\{a_n\} \in BV$.
Combining this theorem and Theorem 4.5, we can formulated the following corollary.

Corollary 4.1. *Let* $\{a_n\} \in H_{q\alpha}$, $0 < q \leq 1$ $\alpha \geq 0$. *Then for any* $0 < p < 1$, *the limits* (4.4) *hold.*

In this parth, I shall prove a version of Uljanov's theorem and extend it to the r-th derivate of the complex series:

$$\sum_{|n|<\infty} c_n e^{int}\,, \quad t \in \mathbf{T}\,,$$

where $\{c_n\}$ is a null sequence of complex numbers such that for $r = 0, 1, 2, \ldots$

$$\sum_{|k|<\infty} k^r |\Delta c_k| < \infty\,. \tag{4.6}$$

The class of null sequence of complex numbers such that (4.6) holds we denote by $(BV)^*_r$. For $r = 0$, we have $(BV)^* = (BV)^*_r$, i.e. it is a class of null sequence of complex numbers of bounded variation.

Theorem 4.6. *Let $\{c_n\} \in (BV)^*_r$, $r = 0, 1, 2, \ldots$ Then the point-wise limit $f^{(r)}$ of the r-th derivate of sums* (1.1) *exists in $\mathbf{T} \setminus \{0\}$ and for any $0 < p < 1$,*

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \left| f^{(r)}(t) - S_n^{(r)}(t) \right|^p dt = 0\,. \tag{4.7}$$

Proof. If $t \neq 0$, we obtain

$$\sum_{k=0}^{n} c_k (e^{ikt})^{(r)} = \sum_{k=1}^{n-1} \Delta c_k E_k^{(r)}(t) + c_n E_n^{(r)}(t)\,.$$

Applying the inequality (4.3) and

$$c_n n^r \leq \sum_{k=n}^{\infty} k^r |\Delta c_k| \to 0\,, \quad n \to \infty\,,$$

we get that $\sum_{k=0}^{\infty} c_k (e^{ikt})^{(r)}$ exists a.e.

Similarly $\sum_{k=-\infty}^{-1} c_k (e^{ikt})^{(r)}$ convergences a.e. and so $\lim_{n\to\infty} S_n^{(r)}(t) = f^{(r)}(t)$ exists in $\mathbf{T} \setminus \{0\}$. It is obvious that for $t \neq 0$,

$$f(t) - S_n(t) = \sum_{|j|\geq n+1} \Delta c_j E_j(t)\,.$$

By inequality (4.3) the series $\sum_{|j|\geq n+1} \Delta c_j E_j^{(r)}(t)$ is uniformly convergent on any compact subset of $\mathbf{T}\setminus\{0\}$. Consequently,

$$f^{(r)}(t) - S_n^{(r)}(t) = \sum_{|j|\geq n+1} \Delta c_j E_j^{(r)}(t)\,.$$

Finally we,

$$\int_{-\pi}^{\pi} |f^{(r)}(t) - S_n^{(r)}(t)|^p = \int_{-\pi}^{\pi} \left| \sum_{|j|\geq n+1} \Delta c_j E_j^{(r)}(t) \right|^p dt =$$

$$= O\left(\left(\sum_{|j|\geq n+1} j^r |\Delta c_j|\right)^p\right) \int_{-\pi}^{\pi} \frac{dt}{|t|^p} \to 0 \text{ as } n\to\infty\,.$$

Let us replace the conditions $\Im_r$, S_{pr}, F_{pr}, $S_{p\alpha r}$ by the conditions $\Im_r^*$, S_{pr}^*, F_{pr}^*, $S_{p\alpha r}^*$ when the coefficients are sequence of complex numbers.
It is obvious that $\Im_r^* \subset (BV)_r^*$, $S_{pr}^* \subset (BV)_r^*$, $F_{pr}^* \subset (BV)_r^*$, $S_{p\alpha r}^* \subset (BV)_r^*$. Applying these inclusions we obtain the following corollaryes of the Theorem 4.6.

Corollary 4.2. *Let $\{c_n\} \in \Im_r^*$, $r = 0,1,2,\dots$ Then the point-wise $f^{(r)}$ of the r-th derivate of sums* (4.1) *exists in $\boldsymbol{T}\setminus\{0\}$ and for any $0 < p < 1$, the limit* (4.7) *holds.*

Corollary 4.3. *Let $\{c_n\} \in S_{qr}^*$, $q > 1$, $r = 0,1,2,\dots$ Then the point-wise limit $f^{(r)}$ of the r-th derivate of sums* (4.1) *exists in $\boldsymbol{T}\setminus\{0\}$ and for any $0 < p < 1$, the limit* (4.7) *holds.*

Corollary 4.4. *Let $\{c_n\} \in F_{qr}^*$, $q > 1$, $r = 0,1,2,\dots$ Then the point-wise limit $f^{(r)}$ of the r-th derivate of sums* (4.1) *exists in $\boldsymbol{T}\setminus\{0\}$ and for any $0 < p < 1$, the limit* (4.7) *holds.*

Corollary 4.5. *Let $\{c_n\} \in S_{q\alpha r}^*$, $q > 1$, $\alpha \geq 0$, $r = \{0,1,2,\dots[\alpha]\}$. Then the point-wise limit $f^{(r)}$ of the r-th derivate of sums* (4.1) *exists in $\boldsymbol{T}\setminus\{0\}$ and for any $0 < p < 1$, the limit* (4.7) *holds.*
Now, we shall define a new subclass of $(BV)_r^*$. Namely, a null sequence $\{c_k\}$ of complex numbers belongs to the class $H_{q\alpha r}^*$, $0 < q \leq 1$, $\alpha \geq 0$, $r \in \{0,1,2,\dots[\alpha]\}$ if there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} k^\alpha A_k < \infty$ and $\frac{1}{n^{q(\alpha-r)+q}} \sum_{k=1}^{n} \frac{|\Delta c_k|^q}{A_k^q} = O(1)$.

Theorem 4.7. *For any $0 < q \le 1$, $\alpha \ge 0$, $r \in \{0,1,2,\dots[\alpha]\}$ the following embedding relation holds $H^*_{q\alpha r} \subseteq (BV)^*_r$.*

Proof. Let $\{c_n\} \in H^*_{q\alpha r}$. Applying the Abel's transformation and inequality (4.5), we obtain:

$$\sum_{k=1}^{n} k^r|\Delta c_k| = \sum_{k=1}^{n-1} k^{\alpha+1}(\Delta A_k)\Big(\frac{1}{k^{\alpha+1}}\sum_{j=1}^{k} j^r\frac{|\Delta c_j|}{A_j}\Big) + n^{\alpha+1}A_n\Big(\frac{1}{n^{\alpha+1}}\sum_{j=1}^{n} j^r\frac{|\Delta c_j|}{A_j}\Big) \le$$

$$\le \sum_{k=1}^{n-1} k^{\alpha+1}(\Delta A_k)\Big(\frac{1}{k^{\alpha-r+1}}\sum_{j=1}^{k}\frac{|\Delta c_j|}{A_j}\Big) + n^{\alpha+1}A_n\Big(\frac{1}{n^{\alpha-r+1}}\sum_{j=1}^{n}\frac{|\Delta c_j|}{A_j}\Big) \le$$

$$\le \sum_{k=1}^{n-1} k^{\alpha+1}(\Delta A_k)\Big(\frac{1}{k^{q(\alpha-r)+q}}\sum_{j=1}^{k}\frac{|\Delta c_j|^q}{A_j^q}\Big)^{1/q} +$$

$$+ n^{\alpha+1}A_n\Big(\frac{1}{n^{q(\alpha-r)+q}}\sum_{j=1}^{n}\frac{|\Delta c_j|^q}{A_j^q}\Big)^{1/q} =$$

$$= O_q(1)\left[\sum_{k=1}^{n-1} k^{\alpha+1}(\Delta A_k) + n^{\alpha+1}A_n\right].$$

Letting $n \to \infty$, and applying the Lemma 1.10 and Lemma 1.11, we obtain $\{c_n\} \in (BV)^*_r$.

Corollary 4.6. *Let $\{c_n\} \in H^*_{q\alpha r}$, $0 < q \le 1$, $\alpha \ge 0$, $r \in \{0,1,2,\dots[\alpha]\}$. Then the point-wise limit $f^{(r)}$ of the r-th derivate of sum (4.1) exists in $\boldsymbol{T} \setminus \{0\}$ and for any $0 < p < 1$, the limit (4.7) holds.*

REFERENCES

[1] Z. U. Ahmad and S. Zahid Ali Zaini, *A note on L^1-convergence of Fourier series with δ-quasi-monotone coefficients*, Comm. Fac. Sci. Univ. Ankara SerA1-Math **30** (1981), no. 1,1-5

[2] N. K. Bari, *Trigonometric series*, Fizmatgiz, Moscow, 1961

[3] R. P. Boas, *Quasi-positive sequence and trigonometric series*, Proc. London Math. Soc. **14**A (1965), 38-48

[4] R. Bojanić and Č. V. Stanojević, *A class of L^1-convergence*, Trans. Amer. Math. Soc. **269** (1982), 677-683

[5] H. Bor, *On integrability of Rees-Stanojević sums*, Bull. Inst. Math. Acad. Sinica. **13** (1985), no. 4, 357-361

[6] H. Bor, *Integrability of Rees-Stanojević sums*, Tamkang Journal of Math. **15** (1984), 157-160

[7] L. S. Bosanquet, *Note on convergence and summability factors* (III), Proc. London Math. Soc. (1949), 482-496

[8] W. O. Bray and Č. V. Stanojevic: *Tauberian L^1-convergence classes of Fourier series*, Transs. Amer. Math. Soc. **275** (1983), 59-69.

[9] G. A. Fomin, *On linear method for summing Fourier seres*, Mt. Sb. **66** (107) (1954), 133-152 (Russian)

[10] Г. А. Фомин, *О линейных методах суммирования рядов Фурье, подобных методу Бернштейна-Рогозинского*, Известия академии наук СССР, **31** (2), (1967) 335-348

[11] Г. А. Фомин, *О некоторых условияш сходимости рядов Фурье в метрике L*, Мат. Заметки **21** (1977), 587-592

[12] G. A. Fomin, *A class of trigonometric series*, Mat. Zametki, **23** (1978), 117-124. (Russian)

[13] G. A. Fomin: *On the convergence in mean of Fourier series*, Mat. Sb. **38** (1981), 231-245 (Russian)

[14] J. W. Garrett, C. S. Rees and Č. V. Stanojević, *On L-convergence of Fourier series with quasi-monotone coefficients*, Proc. Amer. Math. Soc. **72** (1978),

535-538.

[15] J. W. Garrett, C. S. Rees and Č. V. Stanojević, *L^1-convergence of Fourier series with coefficients of bounded variation*, Proc. Amer. Math. Soc. **80** (1980), 423-430

[16] J. W. Garrett and Č.V. Stanojević, *Necessary and sufficient conditions for L^1-convergence of trigonometric series*, Proc. Amer. Math. Soc. **60** (1976), 68-72

[17] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, New York (1934)

[18] T. Kano, *Coefficients of some trigonometric series*, Jour. Fac. Shinshu Univ. **3** (1968), 153-162

[19] J. Karamata, *Sur un mode de croissance reguliere des fonctions*, Mathematica (Cluj) **4** (1930), 38-53

[20] J. Karamata, *Sur certains "Tauberian theorems" de M. M. Hardy et Littlewood*, Mathematica (Cluj) **3** (1930), 33-48

[21] A. N. Kolmogorov, *Sur l'ordre de grandeur des coefficients de la serie de Fourier-Lebesgue*, Bull. Acad. Polon. SerA. Sci. Math. (1923), 83-86

[22] S. Kumari and B. Ram, *Convergence and integrability of certain trigonometric sums*, Tamkang Journal of Math. **19** (1988), 31-36

[23] К. Ван, С. А. Теляаковский, *Дифференциальные свойства сумм одного класса тригонометрических рядов*, ВЕСТН. МОСК. УН-ТА. СЕР. 1 МАТ-МЕХ. (1999) но. 1, 24-29

[24] L. Leindler, *On the equivalence of classes of Fourier coefficients, Mathematical Inequalities* & Applications, **3** (1) (2000), 45-50

[25] H. Mayer Marzuq, *Necessary and sufficient conditions for integrability of certain cosine sums*, Notices Amer. Math. Soc. **22** (1975), A-513

[26] S. M. Mazhar, *On generalized quasi-convex sequence and its application*, Ind. Journal of Pure and Appl. Math **8** (1977), 784-790

[27] C. N. Moore, *On the use of Cesaro means indetermining criteria for Fourier constants*, Soc. Amer. Assoc. Advan. Sc., June 21, 1933

[28] B. Ram and S. Kumari, *On L^1-convergence of certain trigonometric sums*, Indian J. Pure Appl. Math. **20** (1989), 908-914

[29] C. S. Rees, Č. V. Stanojević, *Necessary and sufficient condition for integrability of certain cosine sums*, J. Math. Anal. Appl. **43** (1973), 579-586

[30] S. Sheng, *The extension of the theorems of Č. V. Stanojević and V. B. Stanojević*, Proc. Amer. Math. Soc. **110** (1990), 895-904

[31] S. Sheng, *Some results on L^1-approximation*, Approx. Theory & Appl. 7 (1991), no. 3, 1-9

[32] S. Singh Bhatia and B. Ram, *On L^1-convergence of modified complex trigonometric sums*, Proc. Indian. Acad. Sci. **105**, 1995 (193-199)

[33] N. Singh and K. M. Sharma, *Convergence of certain cosine sums in a metric space L*, Proc. Amer. Math. Soc. **72** (1978), 117-120

[34] N. Singh and K. M. Sharma, *L^1-convergence of modified cosine sums with generalized quasi-convex coefficients*, Journal of Math. Analysis and Appl. **136**, (1988), 189-200

[35] S. Sidon, *Hinreichende Bedingungen fur den Fourier charakter einer Trigonometrischen Reihe*, J. London. Math. Soc. **14** (1939), 158-160

[36] Č.V. Stanojević, *Classes of L^1-convergence of Fourier and Fourier-Stiltjes series*, Proc. Amer. Math. Soc. **82** (1981), 209-215

[37] Č. V. Stanojevic and V. B. Stanojević: *Generalizations of the Sidon-Telyakovskii theorem*, Proc. Amer. Math. Soc. **101** (1987), 679-684.

[38] O. Szasz, *Quasi-monotone series*, Amer. J. Math. **70** (1948), 203-206

[39] N. Tanović-Miller, *On a paper of Bojanić and Stanojević*, Rendiconti del Circolo Matematico di Palermo, **34** (1985), 310-324

[40] S. A. Telyakovskii, *Some estimates for trigonometric series with quasi-convex coefficients*, Mat. Sb. **63** (105), (1964), 426-444 (Russian)

[41] S. A. Telyakovskii, *Integrability conditions for trigonometrical series and their application to the study of linear summation methods of Fourier series*, Izv. Akad. Nauk. SSSR Ser. Mat. **28** (1964), 1209-1238 (Russian)

[42] S. A. Telyakovskii, *An asymptotic estimate of the integral of the absolute value of a function given by sine series*, Sibirsk. Mat. Ž. **8** (1967), 1416-1422 (Russian)

[43] S. A. Telyakovskii, *An estimate, useful in problems of approximation theory of the norm of a function by means of its Fourier coefficients*, Trudy Mat. Inst. Steklov**109** (1971), 73-109 (Russian)

[44] S. A. Telyakovskii, *On a sufficient condition of Sidon for the integrability of trigonometric series*, Mat. Zametki **14** (1973), 742-748 (Russian)

[45] S. A. Telyakovskii and G. A. Fomin, *On convergence in L^1 metric of Fourier series with quasi-monotone coefficients*, Trudy Mat. Inst. Acad. Sci. USSR, **134** (1975), 310-313 (Russian)

[46] S. A. Telyakovskii, *On the integrability of sine series*, Trudy Mat. Inst. Steklov. **163** (1984), 229-233 (Russian)

[47] С. А. Теляаковский, *Замачение об условиях интегрирыемости тригонометрических рядов*, ВЕСТН. МОСК. УН-ТА, серия 1, математика-механика (**4**) (2000), 58-60.

[48] Ž. Tomovski, *An application of the Hausdorff-Young inequality*, Math. Ineq.& Appl. **1** (4) (1998), 527-532

[49] Ž. Tomovski, *A note on some classes of Fourier coefficients*, Math. Ineq & Appl. **2** (1) (1999), 15-18

[50] Ž. Tomovski, *An extension of the Garrett-Stanojević class*, Approx. Theory & Appl., **16** (1) (2000), 46-51. A corrected version is available in the RGMIA Research report collection, **3** (4), Article 3, (2000)
URL: http: //melba.vu.edu.au/∼rgmia/v3n4.html

[51] Ž. Tomovski, *An extension of the Sidon-Fomin type inequality and its applications*, Math. Ineq. & Applications **4** (2) (2001), in press

[52] Ž. Tomovski, *New generalizations of the Telyakovskii inequalities*, VESTNIK Russian University of People's Friendship, Moscow (to appear)

[53] Ž. Tomovski, *Necessary and sufficient condition for L^1-convergence of complex trigonometric series*, VESTNIK, Russian University of People's Friendship, Moscow, **7** (1) (2000), p.139-145

[54] Ž. Tomovski, *Some results on L^1-approximation of the r-th derivate of Fourier series*, RGMIA, Research Report Collection 2(5), Article 11 (1999), 709-717, (Melburne, Australia), electronically version:
http: //melba.vu.edu.au/∼rgmia/v2n5.html; (to appear in JIPAM).

[55] Ž. Tomovski, *Necessary and sufficient condition for L^1convergence of cosine trigonometric series with δ-quasi monotone coefficients*, Mathematical Communications, **(4)** N.2 (1999), 219-224.

[56] Ž. Tomovski, *Generalization theorem on convergence and integrability for sine*

series, Math. Ineq. & Appl. **3** (3) (2000), 369-375

[57] Ž. Tomovski, *On a Bojanić-Stanojević Type inequality and its Applications*, Journal of Ineq. in Pure and Appl. Math. **1** (2), Article 13 (2000) http://jipam.vu.edu.au

[58] Ž. Tomovski, *Two new L^1 estimates for trigonometric series with Fomin's coefficient condition*, RGMIA, Research Report Collection, **2** (6), Article 5, 1999 (Melbourne, Australia); http://melba. vu. edu. au/∼rgmia/v2n6. html

[59] P. L. Uljanov, *Application of A-integration on a class of trigonometric series*, Math. Sb. **35** (77) (1954), 469-490 (Russian)

[60] Ch.-J de la Vall'ee Poussin, *Cours d'Analyse Infinit'esimale*, Paris (1936) (Russian)

[61] W. H. Young, *On the Fourier series of bounded functions*, Proc. London Math. Soc. **2** (12) (1913), 47-70

[62] S. Zahid Ali Zenei, *Integrability of trigonometric series*, Tamkang J. Math. **21** (1990), 295-301

[63] A. Zygmund, *Trigonometric Series*, Univ, Press, Cambridge (1959).

Supplementary references

[64] L. Leindler, *On the equivalence of classes of numerical sequences*, Analysis Mathematica, **26** (2000), 227-234

[65] L. Leindler, *On the utility of Telyakovskii's class*, JIPAM (accepted)

[66] Ž. Tomovski, *Two new L^1-estimates for trigonometric series with Fomin's coefficient condition*, Contributions, Sec. Math. Tech. Sci., MANU, XX, 1-2 (1999), 39-44

[67] Ž. Tomovski, *Some classes of L^1-convergence of Fourier series*, Mathematical Bulletin **24** (L) (2000), 37-46

[68] Ž. Tomovski, *Remartks on some classes of Fourier coefficients*, Analysis Mathematica, (Szeged-Moscow) (accepted)

[69] Ž. Tomovski, *A note on two inequalities of Telyakovskii type*, RGMIA, Research Report Collection, **4** (1), Article 7 (2001) (Melbourne, Australia): http: //melba.vu.edu.au/∼rgmia/v4n1.html (to appear in Math. Ineq. & Appl., Zagreb)

[70] Ž. Tomovski, *Some classes of L^1-convergence of Fourier series*, Journal of