APPROXIMATING THE RIEMANN-STIELTJES INTEGRAL VIA A ČEBYŠEV TYPE FUNCTIONAL

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Abstract. Some new sharp upper bounds for the absolute value of the error functional $D(f,u)$ in approximating the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the quantity $[u(b)-u(a)]\cdot\frac{1}{b-a}\int_a^b f(t)\,dt$ are given.

1. Introduction

In order to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the simpler quantity

$$[u(b)-u(a)]\cdot\frac{1}{b-a}\int_a^b f(t)\,dt,$$

provided that both integrals exist, Dragomir and Fedotov introduced in [8] the following *error functional of Čebyšev type*

$$D(f;u)=\int_a^b f(t)\,du(t)-[u(b)-u(a)]\cdot\frac{1}{b-a}\int_a^b f(t)\,dt, \tag{1.1}$$

and pointed out the following sharp upper bound for $|D(f;u)|$, namely

$$|D(f;u)|\le\frac{1}{2}L(M-m)(b-a), \tag{1.2}$$

provided the *integrator* $u:[a,b]\to\mathbb{R}$ is $L-$Lipschitzian on $[a,b]$, i.e., $|u(x)-u(y)|\le L|x-y|$ for any $x,y\in[a,b]$ and the *integrand* $f:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$ and satisfies the boundedness condition

$$-\infty<m\le f(x)\le M<\infty\quad\text{for a.e. } x\in[a,b]. \tag{1.3}$$

The multiplicative constant $\frac{1}{2}$ in (1.2) is best possible in the sense that it cannot be replaced by a smaller constant.

In the follow-up paper [9], the authors provided a different bound, namely

$$|D(f;u)|\le\frac{1}{2}K(b-a)\bigvee_a^b(u), \tag{1.4}$$

provided that f is is $K-$Lipschitzian and u is of bounded variation on $[a,b]$.

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The result (1.4) was improved in [4] for the case of monotonic nondecreasing functions. We have shown in this case that

$$|D(f;u)| \leq \frac{1}{2}K(b-a)[u(b)-u(a)-K(u)] \\ \left(\leq \frac{1}{2}K(b-a)[u(b)-u(a)]\right), \tag{1.5}$$

where

$$K(u) := \frac{4}{(b-a)^2}\int_a^b u(t)\left(t-\frac{a+b}{2}\right)dt \geq 0.$$

In (1.5) the constant $\frac{1}{2}$ is best possible in both inequalities.

For other sharp bounds on the error functional $D(f;u)$, see the recent papers [6], [7], and [10]. For other inequalities for the Riemann-Stieltjes integral, see [1] and [2].

The main aim of this paper is to further investigate the error functional $D(f;u)$. Two representations are given. These are applied to obtain some inequalities for $D(f;u)$ which improve earlier results.

Applications for the classical *Čebyšev functional* $C(f,g)$, where

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt, \tag{1.6}$$

and f,g are integrable and belonging to different classes of functions, are also provided.

2. Representation Results

For a function $g:[a,b]\to\mathbb{R}$, consider the *generalised trapezoid error transform* $\Phi_g:[a,b]\to\mathbb{R}$ given by

$$\Phi_g(t) := \frac{1}{b-a}[(b-t)g(a)+(t-a)g(b)] - g(t), \quad t\in[a,b] \tag{2.1}$$

and if g is Lebesgue integrable, the *Ostrowski transform*, which is the error of approximating the function by its integral mean, defined by:

$$\Theta_g(t) := g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds, \quad t\in[a,b]. \tag{2.2}$$

We also define the kernel $Q:[a,b]^2\to\mathbb{R}$,

$$Q(t,s) := \begin{cases} t-b & \text{if } a\leq s\leq t\leq b, \\ t-a & \text{if } a\leq t<s\leq b. \end{cases} \tag{2.3}$$

The following representation result in terms of Θ_g and Q may be stated:

Lemma 1. *If $f,u:[a,b]\to\mathbb{R}$ are bounded functions and such that the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist, then we have the representation:*

$$D(f;u) = \int_a^b \Theta_f(s)\,du(s) = \frac{1}{b-a}\int_a^b\left(\int_a^b Q(t,s)\,df(t)\right)du(s). \tag{2.4}$$

Proof. We have by the definition of Q and integrating by parts in the Riemann-Stieltjes integral that

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b\left(\int_a^b Q(t,s)\,df(t)\right)du(s)\\
&=\frac{1}{b-a}\int_a^b\left[\int_a^s(t-a)\,df(t)+\int_s^b(t-b)\,df(t)\right]du(s)\\
&=\frac{1}{b-a}\int_a^b\left[f(t)(t-a)\Big|_a^s-\int_a^s f(t)\,dt+(t-b)f(t)\Big|_s^b-\int_s^b f(t)\,dt\right]du(s)\\
&=\frac{1}{b-a}\int_a^b\left[f(s)(s-a)-\int_a^s f(t)\,dt+(b-s)f(s)-\int_s^b f(t)\,dt\right]du(s)\\
&=\int_a^b\Theta_f(s)\,du(s)
\end{aligned}$$

and the second inequality is proved.

The first identity is obvious by the definition of $D(f;u)$. □

The following corollary can be stated about the representation of the *Čebyšev functional* $C(f,g)$ defined in (1.6).

Corollary 1. *Assume that* $f,g:[a,b]\to\mathbb{R}$ *are Riemann integrable on* $[a,b]$, *then*

$$\begin{aligned}
C(f,g)&=\frac{1}{b-a}\int_a^b\Theta_f(s)\,g(s)\,ds\\
&=\frac{1}{(b-a)^2}\int_a^b\left(\int_a^b Q(t,s)\,df(t)\right)g(s)\,ds.
\end{aligned}\tag{2.5}$$

Proof. It is well known (see for instance [3, Theorem 7.33, p. 162] that if g is Riemann integrable and $u(t)=\int_a^t g(s)\,ds$, then for any f a Riemann integrable function we have that the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and $\int_a^b f(t)\,du(t)=\int_a^b f(t)\,g(t)\,dt$. Therefore, we have $D(f;u)=(b-a)\,C(f,g)$ and

$$\int_a^b\left(\int_a^b Q(t,s)\,df(t)\right)du(s)=\int_a^b\left(\int_a^b Q(t,s)\,df(t)\right)g(s)\,ds.$$

□

The second representation of $D(f;u)$ is incorporated in

Lemma 2. *With the assumptions in Lemma 1, we have*

$$D(f;u)=\int_a^b\Phi_u(t)\,df(t)=\frac{1}{b-a}\int_a^b\left(\int_a^b Q(t,s)\,du(s)\right)df(t),\tag{2.6}$$

where Q is defined by (2.3).

Proof. By the Fubini type theorem for the Riemann-Stieltjes integral (see for instance [3, Theorem 7.41, p. 167]) we have that

$$\int_a^b\left(\int_a^b Q(t,s)\,du(s)\right)df(t)=\int_a^b\left(\int_a^b Q(t,s)\,df(t)\right)du(s),$$

and the equality between the first and the last term in (2.6) is proved.

Now, observe that

$$\int_a^b Q(t,s)\,du(s) = \int_a^t (t-b)\,du(s) + \int_t^b (t-a)\,du(s)$$
$$= (t-b)[u(t)-u(a)] + (t-a)[u(b)-u(t)]$$
$$= (b-a)\,\Phi_u(t),$$

for any $t \in [a,b]$, and then integrating over $f(t)$, we deduce the second inequality in (2.6). □

Corollary 2. *Assume that f and g are Riemann integrable on $[a,b]$, then we have*

$$C(f,g) = \frac{1}{b-a}\int_a^b \tilde{\Phi}_g(t)\,df(t) = \frac{1}{(b-a)^2}\int_a^b \left(\int_a^b Q(t,s)\,g(s)\,ds\right) df(t), \tag{2.7}$$

where

$$\tilde{\Phi}_g(t) = \Phi_{\int_a^{\cdot} g}(t) = \frac{t-a}{b-a}\int_a^b g(s)\,ds - \int_a^t g(s)\,ds, \quad t \in [a,b]. \tag{2.8}$$

3. Bounds in the Case when u is of Bounded Variation

The following lemma is of interest in itself.

Lemma 3. *If $p : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and $v : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$, then*

$$\left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b |p(t)|\,d\bigvee_a^t(v) \tag{3.1}$$
$$\le \left[\bigvee_a^b(v)\right]^{\frac{1}{q}} \left\{\int_a^b |p(t)|^p\,d\left[\bigvee_a^t(v)\right]\right\}^{\frac{1}{p}}$$
$$\le \max_{t\in[a,b]} |p(t)| \bigvee_a^b(v),$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since the Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists, then for any division $I_n : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ with the norm $v(I_n) := \max_{i\in\{0,\dots,n-1\}}(t_{i+1}-t_i) \to 0$ and for any intermediate points $\xi_i \in [t_i, t_{i+1}]$, $i \in \{0,\dots,n-1\}$ we have

$$\left|\int_a^b p(t)\,dv(t)\right| = \left|\lim_{v(I_n)\to 0}\sum_{i=0}^{n-1} p(\xi_i)[v(t_{i+1}) - v(t_i)]\right| \tag{3.2}$$
$$\le \lim_{v(I_n)\to 0}\sum_{i=0}^{n-1} |p(\xi_i)|\,|v(t_{i+1}) - v(t_i)|.$$

However,

$$|v(t_{i+1}) - v(t_i)| \le \bigvee_{t_i}^{t_{i+1}}(v) = \bigvee_a^{t_{i+1}}(v) - \bigvee_a^{t_i}(v), \tag{3.3}$$

for any $i \in \{0, \dots, n-1\}$ and by (3.3) we have

$$\left| \int_a^b p(t)\, dv(t) \right| \le \lim_{v(I_n) \to 0} \sum_{i=0}^{n-1} |p(\xi_i)| \left[\bigvee_a^{t_{i+1}} (v) - \bigvee_a^{t_i} (v) \right] = \int_a^b |p(t)|\, d\left[\bigvee_a^{t} (v) \right],$$

and the last Riemann-Stieltjes integral exists since $|p|$ is continuous and $\bigvee_a^{\cdot} (v)$ is monotonic nondecreasing.

The last part follows from the following Hölder type inequality

$$\left| \int_a^b g(t)\, dv(t) \right| \le [v(b) - v(a)]^{\frac{1}{q}} \left[\int_a^b |g(t)|^p\, dv(t) \right]^{\frac{1}{p}}, \tag{3.4}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, that holds for any continuous function $g : [a, b] \to \mathbb{R}$ and any monotonic nondecreasing function $v : [a, b] \to \mathbb{R}$. The details are omitted. □

The following result holds.

Theorem 1. *Assume that $f, u : [a, b] \to \mathbb{R}$ are of bounded variation and such that the Riemann-Stieltjes integral $\int_a^b f(t)\, du(t)$ exists. Then*

$$\begin{aligned} |D(f; u)| &\le \frac{1}{b-a} \left[\int_a^b \bigvee_a^s (f)(2s - a - b)\, d\left(\bigvee_a^s (u) \right) \right. \\ &\qquad \left. + 2\int_a^b \left(\bigvee_a^s (u) \cdot \bigvee_a^s (f) \right) ds - \bigvee_a^b (u) \int_a^b \left(\bigvee_a^s (f) \right) ds \right] \\ &\le \frac{1}{b-a} \int_a^b \bigvee_a^s (f)(2s - a - b)\, d\left(\bigvee_a^s (u) \right) \\ &\qquad + \frac{1}{b-a} \int_a^b \left(\bigvee_a^s (u) \cdot \bigvee_a^s (f) \right) ds \\ &\le \frac{1}{b-a} \int_a^b \bigvee_a^s (f)(2s - a - b)\, d\left(\bigvee_a^s (u) \right) + \bigvee_a^b (u) \cdot \bigvee_a^b (f). \end{aligned} \tag{3.5}$$

Proof. Utilising the identity (2.4) and the first inequality in (3.1) we have:

$$\begin{aligned} |D(f; u)| &\le \frac{1}{b-a} \int_a^b \left| \int_a^b Q(t, s)\, df(t) \right| d\left(\bigvee_a^s (u) \right) \\ &= \frac{1}{b-a} \int_a^b \left| \int_a^s (t - a)\, df(t) + \int_s^b (t - b)\, df(t) \right| d\left(\bigvee_a^s (u) \right) \end{aligned} \tag{3.6}$$

$$\leq \frac{1}{b-a}\int_a^b \left[\left|\int_a^s (t-a)\,df(t)\right| + \left|\int_s^b (t-b)\,df(t)\right|\right] d\left(\bigvee_a^s (u)\right)$$
$$=: I.$$

Since f is of bounded variation, then by the same inequality in (3.1) we have

$$\left|\int_a^s (t-a)\,df(t)\right| \leq \int_a^s (t-a)\,d\left(\bigvee_a^t (f)\right)$$
$$= \bigvee_a^s (f)\cdot(s-a) - \int_a^s \left(\bigvee_a^t (f)\right) dt$$

and

$$\left|\int_s^b (t-b)\,df(t)\right| \leq \int_s^b (t-b)\,d\bigvee_s^t (f) = \int_s^b \left(\bigvee_s^t (f)\right) dt$$
$$= \int_s^b \left[\bigvee_a^t (f) - \bigvee_a^s (f)\right] dt$$
$$= \int_s^b \left(\bigvee_a^t (f)\right) dt - (b-s)\bigvee_a^s (f)$$

which gives that

(3.7)

$$I \leq \frac{1}{b-a}\int_a^b \left[\bigvee_a^s (f)(s-a) - \int_a^s \left(\bigvee_a^t (f)\right) dt \right.$$
$$\left. + \int_s^b \left(\bigvee_a^t (f)\right) dt - (b-s)\bigvee_a^s (f)\right] d\left(\bigvee_a^s (u)\right)$$
$$= \frac{1}{b-a}\int_a^b \left[\bigvee_a^s (f)(2s-a-b) - \int_a^s \left(\bigvee_a^t (f)\right) dt \right.$$
$$\left. + \int_s^b \left(\bigvee_a^t (f)\right) dt\right] d\left(\bigvee_a^s (u)\right)$$
$$= \frac{1}{b-a}\int_a^b (2s-a-b)\bigvee_a^s (f)\,d\left(\bigvee_a^s (u)\right)$$
$$+ \frac{1}{b-a}\int_a^b \left[\int_a^b \left(\bigvee_a^t (f)\right) dt - 2\int_a^s \left(\bigvee_a^t (f)\right) dt\right] d\left(\bigvee_a^s (u)\right)$$
$$= \frac{1}{b-a}\int_a^b (2s-a-b)\bigvee_a^s (f)\,d\left(\bigvee_a^s (u)\right) + \frac{1}{b-a}\int_a^b \left(\bigvee_a^t (f)\right) dt\cdot\bigvee_a^b (u)$$
$$- \frac{2}{b-a}\int_a^b \left(\int_a^s \left(\bigvee_a^t (f)\right) dt\right) d\left(\bigvee_a^s (u)\right).$$

However, integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned}
&\int_a^b \left(\int_a^s \left(\bigvee_a^t (f)\right) dt\right) d\left(\bigvee_a^s (u)\right) \\
&= \int_a^s \left(\bigvee_a^t (f)\right) dt \cdot \bigvee_a^s (u)\Bigg|_a^b - \int_a^b \bigvee_a^s (u) \cdot \bigvee_a^s (f)\, ds \\
&= \bigvee_a^b (u) \cdot \int_a^b \left(\bigvee_a^t (f)\right) dt - \int_a^b \bigvee_a^s (u) \cdot \bigvee_a^s (f)\, ds
\end{aligned}$$

Inserting this value in the expression of I from (3.7) we deduce the first inequality in (3.5).

The other inequalities are obvious. □

The following result may be stated as well.

Theorem 2. *If $u : [a,b] \to \mathbb{R}$ is of bounded variation and $f : [a,b] \to \mathbb{R}$ is $L-$Lipschitzian, then*

$$\begin{aligned}
|D(f;u)| &\leq L\left[\frac{1}{2}(b-a)\bigvee_a^b (u) - \frac{2}{b-a}\int_a^b \left(\bigvee_a^s (u)\right)\left(s - \frac{a+b}{2}\right) ds\right] \qquad (3.8)\\
&\leq \frac{1}{2} L (b-a) \bigvee_a^b (u).
\end{aligned}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.

Proof. It is well known that if $p : [\alpha,\beta] \to \mathbb{R}$ is $L-$Lipschitzian and $v : [\alpha,\beta] \to \mathbb{R}$ is Riemann integrable, then the Riemann-Stieltjes integral $\int_\alpha^\beta p(s)\, dv(s)$ exists and $\left|\int_\alpha^\beta p(s)\, dv(s)\right| \leq L \int_\alpha^\beta |p(s)|\, ds$. Utilising this property, we then have

$$\begin{aligned}
\left|\int_a^s (t-a)\, df(t)\right| &\leq L \int_a^s (t-a)\, dt = \frac{L}{2}(s-a)^2, \\
\left|\int_s^b (t-b)\, df(t)\right| &\leq L \int_s^b (b-t)\, dt = \frac{L}{2}(b-s)^2.
\end{aligned}$$

Therefore, by the relation (3.6) we have

$$\begin{aligned}
I &\leq \frac{L}{2(b-a)} \int_a^b \left[(b-s)^2 + (s-a)^2\right] d\left(\bigvee_a^s (u)\right) \\
&= \frac{L}{2(b-a)} \left[\left[(b-s)^2 + (s-a)^2\right] \bigvee_a^s (u)\Bigg|_a^b - 2\int_a^b \bigvee_a^s (u)(2s-a-b)\, ds\right] \\
&= \frac{L}{2(b-a)} \left[(b-a)^2 \bigvee_a^b (u) - 4\int_a^b \bigvee_a^s (u)\left(s - \frac{a+b}{2}\right) ds\right]
\end{aligned}$$

and the first inequality in (3.8) is proved.

To prove the last part, we use the Čebyšev inequality which states that for two nondecreasing functions g and h,

$$\frac{1}{b-a}\int_a^b g(s)\,h(s)\,ds \geq \frac{1}{b-a}\int_a^b g(s)\,ds \cdot \frac{1}{b-a}\int_a^b h(s)\,ds.$$

Then

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \bigvee_a^s(u)\left(s-\frac{a+b}{2}\right)ds \\
\geq\ &\frac{1}{b-a}\int_a^b \left(\bigvee_a^s(u)\right)ds \cdot \frac{1}{b-a}\int_a^b \left(s-\frac{a+b}{2}\right)ds
\end{aligned}$$

and since $\int_a^b \left(s-\frac{a+b}{2}\right)ds = 0$, the inequality is proved.

For the sharpness of the constant, we consider the functions $f(t) = t - \frac{a+b}{2}$, $t \in [a,b]$ and $u : [a,b] \to \mathbb{R}$ defined by

$$u(t) := \begin{cases} 1 & \text{if } t = a \\ 0 & \text{if } t \in (a,b) \\ 1 & \text{if } t = b. \end{cases}$$

Then f is Lipschitzian with $L = 1$ and u is of bounded variation on $[a,b]$. We have $\bigvee_a^s(u) = 1$, $s \in (a,b)$ and $\bigvee_a^b(u) = 2$. Also,

$$\begin{aligned}
D(f;u) &= \int_a^b f(t)\,du(t) - \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt \\
&= \int_a^b f(t)\,du(t) \\
&= f(t)\,u(t)\Big|_a^b - \int_a^b u(t)\,df(t) = b-a
\end{aligned}$$

and

$$\int_a^b \left(\bigvee_a^s(u)\right)\left(s-\frac{a+b}{2}\right)ds = \int_a^b \left(s-\frac{a+b}{2}\right)ds = 0.$$

Replacing the values in (3.8) we get in all sides the same quantity $b-a$. This shows that the constant $\frac{1}{2}$ is best possible in both inequalities. □

Remark 1. *The inequality between the first and last term in (3.8) was firstly discovered by Dragomir and Fedotov in* [9] *where they also showed the sharpness of the constant* $\frac{1}{2}$.

The following result may be stated as well.

Theorem 3. *Assume that* $u : [a,b] \to \mathbb{R}$ *is of bounded variation and* $f : [a,b] \to \mathbb{R}$ *is monotonic nondecreasing and such that the Riemann-Stieltjes integral* $\int_a^b f(t)\,du(t)$

exists. Then,

$$|D(f;u)| \leq \frac{1}{b-a}\left[\int_a^b (2s-a-b) f(s)\, d\left(\bigvee_a^s (u)\right) + 2\int_a^b \left(\bigvee_a^s (u)\right) f(s)\, ds - \int_a^b f(s)\, ds \cdot \bigvee_a^b (u)\right]. \tag{3.9}$$

Proof. It is well known that if the Stieltjes integrals $\int_\alpha^\beta p(t)\, dv(t)$ and $\int_\alpha^\beta |p(t)|\, dv(t)$ exist and v is monotonic nondecreasing on $[\alpha,\beta]$, then

$$\left|\int_\alpha^\beta p(t)\, dv(t)\right| \leq \int_\alpha^\beta |p(t)|\, dv(t).$$

Utilising this property we then have

$$\left|\int_a^s (t-a)\, df(t)\right| \leq \int_a^s (t-a)\, df(t) = (s-a) f(s) - \int_a^s f(t)\, dt$$

and

$$\left|\int_s^b (t-b)\, df(t)\right| \leq \int_s^b (t-b)\, df(t) = \int_s^b f(t)\, dt - (b-s) f(s)$$

for any $s \in [a,b]$.

Utilising the relation (3.6), we obtain

$$\begin{aligned} I &\leq \frac{1}{b-a}\left[\int_a^b \left\{(s-a) f(s) - \int_a^s f(t)\, dt + \int_s^b f(t)\, dt - (b-s) f(s)\right\} d\left(\bigvee_a^s (u)\right)\right] \\ &= \frac{1}{b-a}\left[\int_a^b (2s-a-b) f(s)\, d\left(\bigvee_a^s (u)\right) + \int_a^b \left(\int_s^b f(t)\, dt\right) d\left(\bigvee_a^s (u)\right) - \int_a^b \left(\int_a^s f(t)\, dt\right) d\left(\bigvee_a^s (u)\right)\right] \\ &=: J. \end{aligned} \tag{3.10}$$

However, integrating by parts in the Riemann-Stieltjes integral, we have:

$$\begin{aligned} \int_a^b \left(\int_s^b f(t)\, dt\right) d\left(\bigvee_a^s (u)\right) &= \left(\int_s^b f(t)\, dt\right) \cdot \bigvee_a^s (u)\Bigg|_a^b - \int_a^b \left(\bigvee_a^s (u)\right) d\left(\int_s^b f(t)\, dt\right) \\ &= \int_a^b \left(\bigvee_a^s (u)\right) f(s)\, ds \end{aligned}$$

and

$$\int_a^b \left(\int_a^s f(t)\,dt\right) d\left(\bigvee_a^s (u)\right) = \left(\int_a^s f(t)\,dt\right)\cdot \bigvee_a^s (u)\Big|_a^b - \int_a^b \left(\bigvee_a^s (u)\right) d\left(\int_a^s f(t)\,dt\right)$$
$$= \int_a^b f(t)\,dt \cdot \bigvee_a^b (u) - \int_a^b \left(\bigvee_a^s (u)\right) f(s)\,ds.$$

Therefore,

$$J = \frac{1}{b-a}\left[\int_a^b (2s-a-b) f(s)\, d\left(\bigvee_a^s (u)\right) + \int_a^b \left(\bigvee_a^s (u)\right) f(s)\,ds - \int_a^b f(t)\,dt \cdot \bigvee_a^b (u) + \int_a^b \left(\bigvee_a^s (u)\right) f(s)\,ds\right]$$
$$= \frac{1}{b-a}\left[\int_a^b (2s-a-b) f(s)\, d\left(\bigvee_a^s (u)\right) + 2\int_a^b \left(\bigvee_a^s (u)\right) f(s)\,ds - \int_a^b f(s)\,ds \cdot \bigvee_a^b (u)\right].$$

This together with the inequalities (3.6) and (3.10) produces the desired result (3.9). □

4. Bounds in the Case when f is of Bounded Variation

We can state the following result as well.

Theorem 4. *Assume that $f : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$. If $u : [a,b] \to \mathbb{R}$ is continuous and such that there exists the constants $L_a, L_b > 0$ and $\alpha, \beta > 0$ with the properties that:*

$$|u(t) - u(a)| \le L_a (t-a)^{\alpha}, \qquad |u(t) - u(b)| \le L_b (b-t)^{\beta} \tag{4.1}$$

for any $t \in [a,b]$, then

$$|D(f;u)| \le \frac{1}{b-a} L_a \left[\int_a^b \left(\bigvee_a^t (f)\right)(t-a)^{\alpha}\,dt - \alpha \int_a^b \left(\bigvee_a^t (f)\right)(b-t)(t-a)^{\alpha-1}\,dt\right]$$
$$+ \frac{1}{b-a} L_b \left[\beta \int_a^b \left(\bigvee_a^t (f)\right)(t-a)(b-t)^{\beta-1}\,dt - \int_a^b \left(\bigvee_a^t (f)\right)(b-t)^{\beta}\,dt\right]. \tag{4.2}$$

Proof. Utilising the identity (2.6) and the first inequality in (3.1), we have successively,

$$
\begin{aligned}
|D(f;u)| &\le \frac{1}{b-a}\int_a^b \left|\int_a^b Q(t,s)\,du(s)\right| d\left(\bigvee_a^t (f)\right) \\
&= \frac{1}{b-a}\int_a^b \left|\int_a^t Q(t,s)\,du(s) + \int_t^b Q(t,s)\,du(s)\right| d\left(\bigvee_a^t (f)\right) \\
&\le \frac{1}{b-a}\int_a^b \left[\left|\int_a^t Q(t,s)\,du(s)\right| + \left|\int_t^b Q(t,s)\,du(s)\right|\right] d\left(\bigvee_a^t (f)\right) \\
&= \frac{1}{b-a}\int_a^b \left[(b-t)\,|u(t)-u(a)| + (t-a)\,|u(b)-u(t)|\right] d\left(\bigvee_a^t (f)\right) \\
&=: P.
\end{aligned} \tag{4.3}
$$

Now, on making use of the condition (4.1), we can state that

$$
\begin{aligned}
P &\le \frac{1}{b-a}\int_a^b \left[L_a (b-t)(t-a)^{\alpha} + L_b (t-a)(b-t)^{\beta}\right] d\left(\bigvee_a^t (f)\right) \\
&= \frac{1}{b-a}\left[L_a \int_a^b (b-t)(t-a)^{\alpha}\, d\left(\bigvee_a^t (f)\right)\right. \\
&\qquad \left. + L_b \int_a^b (t-a)(b-t)^{\beta}\, d\left(\bigvee_a^t (f)\right)\right].
\end{aligned} \tag{4.4}
$$

However,

$$
\begin{aligned}
&\int_a^b (b-t)(t-a)^{\alpha}\, d\left(\bigvee_a^t (f)\right) \\
&= (b-t)(t-a)^{\alpha} \bigvee_a^t (f)\Big|_a^b - \int_a^b \left(\bigvee_a^t (f)\right) d\left[(b-t)(t-a)^{\alpha}\right] \\
&= -\int_a^b \left(\bigvee_a^t (f)\right)\left[-(t-a)^{\alpha} + \alpha (b-t)(t-a)^{\alpha-1}\right] dt \\
&= \int_a^b \left(\bigvee_a^t (f)\right)(t-a)^{\alpha}\, dt - \alpha \int_a^b \left(\bigvee_a^t (f)\right)(b-t)(t-a)^{\alpha-1}\, dt
\end{aligned}
$$

and

$$\begin{aligned}
&\int_a^b (t-a)(b-t)^\beta \, d\left(\bigvee_a^t (f)\right) \\
&= (t-a)(b-t)^\beta \bigvee_a^t (f)\Big|_a^b - \int_a^b \left(\bigvee_a^t (f)\right) d\left[(t-a)(b-t)^\beta\right] \\
&= -\int_a^b \left(\bigvee_a^t (f)\right)\left[(b-t)^\beta - \beta(t-a)(b-t)^{\beta-1}\right] dt \\
&= \beta \int_a^b \left(\bigvee_a^t (f)\right)(t-a)(b-t)^{\beta-1} dt - \int_a^b \left(\bigvee_a^t (f)\right)(b-t)^\beta dt
\end{aligned}$$

and from (4.4) we deduce the desired inequality (4.2). □

Corollary 3. *If f is as in Theorem 4 and u is of $r-H-$Hölder type, i.e.,*

$$|u(t)-u(s)| \le H|u-t|^r \quad \textit{for any } \; t,s \in [a,b], \tag{4.5}$$

where $H>0$ and $r\in(0,1)$ are given, then

$$\begin{aligned}
|D(f;u)| \le \frac{1}{b-a} H \int_a^b \left(\bigvee_a^t (f)\right)\Big\{ &(t-a)^r - (b-t)^r \\
&+ r(b-t)^{r-1}(t-a)^{r-1}\left[(t-a)^{1-r} - (b-t)^{1-r}\right]\Big\} dt.
\end{aligned} \tag{4.6}$$

Remark 2. *If $r=\frac{1}{2}$ in Corollary 3, then we obtain the inequality:*

$$\begin{aligned}
|D(f;u)| \le{}& \frac{1}{b-a} H \\
&\times \int_a^b \left(\bigvee_a^t (f)\right)\left(\sqrt{t-a}-\sqrt{b-t}\right)\left(1+\frac{1}{2\sqrt{(b-t)(t-a)}}\right) dt.
\end{aligned} \tag{4.7}$$

The following particular result may be useful for applications.

Corollary 4. *If f is as in Theorem 4 and $u:[a,b]\to\mathbb{R}$ is Lipschitzian with the constant $K>0$, then*

$$\begin{aligned}
|D(f;u)| &\le \frac{4}{b-a}\cdot K \int_a^b \left(t-\frac{a+b}{2}\right)\cdot \bigvee_a^t (f)\, dt \\
&\le \begin{cases}
K(b-a)\bigvee_a^b (f); \\[2ex]
\frac{2(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} K \left(\int_a^b \left[\bigvee_a^t (f)\right]^p dt\right)^{\frac{1}{p}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
2K\int_a^b \left(\bigvee_a^t (f)\right) dt.
\end{cases}
\end{aligned} \tag{4.8}$$

The multiplication constant 4 is best possible.

Proof. The first inequality follows by Theorem 4 on choosing $L_a = L_b = K$ and $\alpha = \beta = 1$.

Now, on utilising Hölder's inequality, we have

$$\int_a^b \left(t - \frac{a+b}{2}\right) \cdot \left(\bigvee_a^t (f)\right) dt \tag{4.9}$$

$$\leq \begin{cases} \sup_{t\in[a,b]} \left(\bigvee_a^t (f)\right) \int_a^b \left|t - \frac{a+b}{2}\right| dt \\ \left(\int_a^b \left[\bigvee_a^t (f)\right]^p dt\right)^{\frac{1}{p}} \left(\int_a^b \left|t - \frac{a+b}{2}\right|^q dt\right)^{\frac{1}{q}}, \quad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \int_a^b \left(\bigvee_a^t (f)\right) dt \sup_{t\in[a,b]} \left|t - \frac{a+b}{2}\right|. \end{cases}$$

However, $\sup_{t\in[a,b]} \left|t - \frac{a+b}{2}\right| = \frac{b-a}{2}$ and

$$\int_a^b \left|t - \frac{a+b}{2}\right|^q dt = 2\int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2}\right)^q dt = \frac{(b-a)^{q+1}}{2^q (q+1)}, \quad q \geq 1$$

and by (4.9) we deduce:

$$\int_a^b \left(t - \frac{a+b}{2}\right) \cdot \bigvee_a^t (f)\, dt \tag{4.10}$$

$$\leq \begin{cases} \frac{(b-a)^2}{4} \bigvee_a^t (f); \\ \frac{(b-a)^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}} \left(\int_a^b \left[\bigvee_a^t (f)\right]^p dt\right)^{\frac{1}{p}}, \quad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{b-a}{2} \int_a^b \left(\bigvee_a^t (f)\right) dt \end{cases}$$

and the second part is proved.

To prove the sharpness of the constant 4 in the first inequality in (4.8) assume that there exists $A > 0$ such that

$$|D(f;u)| \leq \frac{A}{b-a} \cdot K \int_a^b \left(t - \frac{a+b}{2}\right) \cdot \bigvee_a^t (f)\, dt, \tag{4.11}$$

provided that f is of bounded variation and u is $K-$Lipschitzian.

Let $f : [a,b] \to \mathbb{R}$,

$$f(t) = \begin{cases} 0 & \text{if } t \in \left[a, \frac{a+b}{2}\right], \\ k & \text{if } t \in \left(\frac{a+b}{2}, b\right], \end{cases}$$

with $k > 0$. Then

$$\bigvee_a^t (f) = \begin{cases} 0 & \text{if } t \in \left[a, \frac{a+b}{2}\right], \\ k & \text{if } t \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

Also, we have

$$\int_a^b \left(t - \frac{a+b}{2}\right) \cdot \bigvee_a^t (f)\, dt = \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2}\right) k dt$$
$$= \frac{k (b-a)^2}{8}.$$

Consider $u : [a, b] \to \mathbb{R}$, $u(t) = \left|t - \frac{a+b}{2}\right|$. Then u is $K-$Lipschitzian with $K = 1$. Also,

$$D(f; u) = \int_a^b f(t)\, du(t) - \frac{u(b) - u(a)}{b-a} \int_a^b f(t)\, dt$$
$$= k \int_{\frac{a+b}{2}}^b du(t) = k \left[u(b) - u\left(\frac{a+b}{2}\right)\right]$$
$$= \frac{(b-a) k}{2}.$$

Substituting these values into (4.11) produces the inequality

$$\frac{(b-a) k}{2} \leq \frac{A}{b-a} \cdot \frac{k (b-a)^2}{8},$$

which implies that $A \geq 4$. □

5. Inequalities for $(l, L)-$Lipschitzian Functions

The following simple lemma holds.

Lemma 4. *Let $u : [a, b] \to \mathbb{R}$ and $l, L \in \mathbb{R}$ with $L > l$. The following statements are equivalent:*

(i) *The function $u - \frac{l+L}{2} \cdot e$, where $e(t) = t$, $t \in [a, b]$ is $\frac{1}{2}(L - l)-$Lipschitzian;*
(ii) *We have the inequalities*

$$l \leq \frac{u(t) - u(s)}{t-s} \leq L \quad \text{for each } t, s \in [a, b],\ t \neq s;$$

(iii) *We have the inequalities*

$$l(t-s) \leq u(t) - u(s) \leq L(t-s) \quad \text{for each } t, s \in [a, b] \quad \text{with } t > s.$$

The proof is obvious and we omit the details.

Definition 1 (see also [10])**.** *The function $u : [a, b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) from Lemma 4 is said to be $(l, L)-$Lipschitzian on $[a, b]$. If $L > 0$ and $l = -L$, then $(-L, L)-$Lipschitzian means $L-$Lipschitzian in the classical sense.*

The following result can be stated.

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation and $u : [a,b] \to \mathbb{R}$ an (l,L) −Lipschitzian function. Then*

$$|D(f;u)| \leq \frac{2}{b-a}(L-l)\int_a^b \left(t-\frac{a+b}{2}\right)\cdot \bigvee_a^t (f)\,dt \tag{5.1}$$

$$\leq \begin{cases} \frac{1}{2}(L-l)(b-a)\bigvee_a^b(f); \\ \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}(L-l)\left(\int_a^b\left[\bigvee_a^t(f)\right]^p dt\right)^{\frac{1}{p}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ (L-l)\int_a^b\left(\bigvee_a^t(f)\right)dt. \end{cases}$$

The constant 2 in the first inequality is sharp.

Proof. Observe that:

$$\begin{aligned} &D\left(f;u-\frac{l+L}{2}\cdot e\right) \\ &= \int_a^b\left(f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right)d\left[u(t)-\frac{l+L}{2}\cdot t\right] \\ &= \int_a^b\left(f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right)du(t) \\ &-\frac{l+L}{2}\int_a^b\left(f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right)dt \\ &= D(f;u). \end{aligned}$$

Now, applying Corollary 4 for the function $u-\frac{l+L}{2}e$, which is $\frac{1}{2}(L-l)$ −Lipschitzian, we get:

$$\begin{aligned} \left|D\left(f;u-\frac{l+L}{2}e\right)\right| &\leq \frac{4}{b-a}\cdot\frac{1}{2}(L-l)\int_a^b\left(t-\frac{a+b}{2}\right)\cdot\bigvee_a^t(f)\,dt \\ &= \frac{2}{b-a}(L-l)\int_a^b\left(t-\frac{a+b}{2}\right)\cdot\bigvee_a^t(f)\,dt \end{aligned}$$

and the theorem is proved. □

The second result may be stated as:

Theorem 6. *Let $u:[a,b]\to\mathbb{R}$ be a function of bounded variation. If $f:[a,b]\to\mathbb{R}$ is (ϕ,Φ) −Lipschitzian with $\Phi>\phi$, then*

$$\left|D(f;u)-\frac{\phi+\Phi}{2}\left[\frac{u(b)+u(a)}{2}(b-a)-\int_a^b u(t)\,dt\right]\right| \tag{5.2}$$
$$\leq\frac{1}{2}(\Phi-\phi)\cdot\left[\frac{1}{2}(b-a)\bigvee_a^b(u)-\frac{2}{b-a}\int_a^b\left(\bigvee_a^s(u)\right)\left(s-\frac{a+b}{2}\right)ds\right]$$
$$\leq\frac{1}{4}\cdot(\Phi-\phi)(b-a)\bigvee_a^b(u).$$

The constant $\frac{1}{2}$ in front of $(\Phi-\phi)$ and $\frac{1}{4}$ are best possible.

Proof. Observe that

$$D\left(f-\frac{\phi+\Phi}{2}\cdot e;u\right)$$
$$=\int_a^b\left[f(t)-\frac{\phi+\Phi}{2}\cdot t-\frac{1}{b-a}\int_a^b\left(f(s)-\frac{\phi+\Phi}{2}\cdot s\right)ds\right]du(t)$$
$$=\int_a^b\left[f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds-\left(\frac{\phi+\Phi}{2}t-\frac{1}{b-a}\int_a^b\frac{\phi+\Phi}{2}\cdot sds\right)\right]du(t)$$
$$=D(f;u)-\frac{\phi+\Phi}{2}\int_a^b\left(t-\frac{1}{b-a}\int_a^b sds\right)du(t)$$
$$=D(f;u)-\frac{\phi+\Phi}{2}\int_a^b\left(t-\frac{a+b}{2}\right)du(t).$$

Since, integrating by parts in the Riemann-Stieltjes integral we have:

$$\int_a^b\left(t-\frac{a+b}{2}\right)du(t)=\frac{u(b)+u(a)}{2}(b-a)-\int_a^b u(t)\,dt,$$

then

$$D\left(f-\frac{\phi+\Phi}{2}e;u\right)=D(f;u)-\frac{\phi+\Phi}{2}\left[\frac{u(b)+u(a)}{2}(b-a)-\int_a^b u(t)\,dt\right].$$

Now, on applying Theorem 2 for the function $f-\frac{\phi+\Phi}{2}e$ which is $\frac{1}{2}(L-l)$ −Lipschitzian, we deduce the desired result (5.2). □

6. Applications for the Čebyšev Functional

If we choose $u(t):=\int_a^t g(\tau)\,d\tau,\ t\in[a,b]$, where $g:[a,b]\to\mathbb{R}$ is Lebesgue integrable on $[a,b]$, then we have the equality

$$C(f;g)=\frac{1}{b-a}D(f;u).$$

Also, u is of bounded variation on any subinterval $[a,s]$, $s\in[a,b]$ and if g is continuous on $[a,b]$, then

$$\bigvee_a^s(u)=\int_a^s|g(\tau)|\,d\tau,\quad s\in[a,b].$$

If f is of bounded variation on $[a,b]$, then on utilising the inequality (3.5) we have

$$
\begin{aligned}
(6.1)\quad & |C(f;g)| \\
&\leq \frac{1}{(b-a)^2}\left[\int_a^b (2s-a-b)\,|g(s)|\bigvee_a^s(f)\,ds \right.\\
&\quad \left. +2\int_a^b\left(\int_a^s |g(\tau)|\,d\tau\right)\bigvee_a^s(f)\,ds - \int_a^b |g(\tau)|\,d\tau\cdot\int_a^b\left(\bigvee_a^s(f)\right)ds\right]\\
&\leq \frac{1}{(b-a)^2}\int_a^b (2s-a-b)\,|g(s)|\bigvee_a^s(f)\,ds \\
&\quad +\frac{1}{(b-a)^2}\int_a^b\left(\int_a^s |g(\tau)|\,d\tau\right)\bigvee_a^s(f)\,ds\\
&\leq \frac{1}{(b-a)^2}\int_a^b (2s-a-b)\,|g(s)|\bigvee_a^s(f)\,ds + \frac{1}{b-a}\int_a^b |g(\tau)|\,d\tau\cdot\bigvee_a^b(f).
\end{aligned}
$$

Now, if f is monotonic nondecreasing, then by (3.9) we have

$$
\begin{aligned}
(6.2)\quad |C(f;g)| \leq \frac{1}{(b-a)^2}&\left[\int_a^b (2s-a-b)\,f(s)\,|g(s)|\,ds\right.\\
&\left. +2\int_a^b\left(\int_a^s |g(\tau)|\,d\tau\right)f(s)\,ds - \int_a^b f(s)\,ds\cdot\int_a^b |g(\tau)|\,d\tau\right].
\end{aligned}
$$

The case where f is $L-$Lipschitzian provides via (3.8) a much simpler inequality:

$$
\begin{aligned}
(6.3)\quad & |C(f;g)| \\
&\leq L\left[\frac{1}{2}\int_a^b |g(\tau)|\,d\tau - \frac{2}{(b-a)^2}\int_a^b\left(\int_a^s |g(\tau)|\,d\tau\right)\left(s-\frac{a+b}{2}\right)ds\right]\\
&\leq \frac{1}{2}L\int_a^b |g(s)|\,ds.
\end{aligned}
$$

Now, if f is of bounded variation and $|g|$ is bounded above by M, i.e., $|g(t)| \leq M$ for a.e. $t\in[a,b]$, then by (4.8) we have:

$$
\begin{aligned}
(6.4)\quad |C(f;g)| &\leq \frac{4}{(b-a)^2}M\int_a^b\left(t-\frac{a+b}{2}\right)\bigvee_a^t(f)\,dt\\
&\leq \begin{cases} M\bigvee_a^b(f); \\[2ex] \frac{2(b-a)^{\frac{1}{q}-1}}{(q+1)^{\frac{1}{q}}}M\left(\int_a^b\left[\bigvee_a^t(f)\right]^p dt\right)^{\frac{1}{p}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \frac{2M}{b-a}\int_a^b\left(\bigvee_a^t(f)\right)dt. \end{cases}
\end{aligned}
$$

The constant 4 in (6.4) is best possible.

Finally, if $-\infty < \phi \leq g(t) \leq \Phi$ for a.e. $t \in [a,b]$, then $\left| g(t) - \frac{\phi+\Phi}{2} \right| \leq \frac{1}{2}(\Phi - \phi)$ and since

$$C\left(f; g - \frac{\phi + \Phi}{2}\right) = C(f;g),$$

hence, by (6.4) we deduce the inequalities

$$|C(f;g)| \leq \frac{2}{(b-a)^2}(\Phi - \phi) \int_a^b \left(t - \frac{a+b}{2}\right) \bigvee_a^t (f)\, dt \tag{6.5}$$

$$\leq \begin{cases} \frac{1}{2}(\Phi - \phi) \bigvee_a^b (f); \\[2ex] \frac{(b-a)^{\frac{1}{q}-1}}{(q+1)^{\frac{1}{q}}} (\Phi - \phi) \left(\int_a^b \left[\bigvee_a^t (f) \right]^p dt \right)^{\frac{1}{p}}, \quad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\[2ex] \frac{\Phi - \phi}{b-a} \int_a^b \left(\bigvee_a^t (f) \right) dt. \end{cases}$$

The constant 2 in the first inequality is best possible.

References

[1] G.A. ANASTASSIOU, Univariate Ostrowski inequalities, revisited. *Monatsh. Math.* **135** (2002), no. 3, 175–189.

[2] G.A. ANASTASSIOU, Chebyshev-Grüss type and comparison of integral means inequalities for the Stieltjes integral. *Panamer. Math. J.* **17** (2007), no. 3, 91–109.

[3] T. APOSTOL, *Mathematical Analysis*, Addison Wesley.

[4] S.S. DRAGOMIR, Inequalities of Grüss type for the Stieltjes integral, *Kragujevac J. Math.*, **26** (2004), 89-122.

[5] S.S. DRAGOMIR, Inequalities for Stieltjes integrals with convex integrators and applications, *Appl. Math. Lett.*, **20** (2007), 123-130.

[6] S.S. DRAGOMIR, A Sharp bound of the Cebysev functional for the Riemann-Stieltjes integral and applications, Preprint, *RGMIA Res. Rep. Coll.* **10**(2007), Supplement, Article 16. [ONLINE http://rgmia.vu.edu.au/v10(E).html].

[7] S.S. DRAGOMIR, Approximation of the Riemann-Stieltjes integral with (l, L)-Lipschitzian integrators, Preprint, *RGMIA Res. Rep. Coll.* **10**(2007), No. 3, Article 5. [ONLINE http://rgmia.vu.edu.au/v10n3.html].

[8] S.S. DRAGOMIR and I. FEDOTOV, An inequality of Grüss type for the Riemann-Stieltjes integral and applications for special means, *Tamkang J. Math.,* **29**(4) (1998), 287-292.

[9] S.S. DRAGOMIR and I. FEDOTOV, A Grüss type inequality for mappings of bounded variation and applications for numerical analysis, *Nonlinear Funct. Anal. Appl.,* **6**(3) (2001), 425-433.

[10] Z. LIU, Refinement of an inequality of Grüss type for Riemann-Stieltjes integral, *Soochow J. Math.,* **30**(4) (2004), 483-489.

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