APPROXIMATING THE STIELTJES INTEGRAL OF BOUNDED FUNCTIONS AND APPLICATIONS FOR THREE POINT QUADRATURE RULES

S.S. DRAGOMIR

Abstract. Sharp error estimates in approximating the Stieltjes integral with bounded integrands and bounded integrators respectively, are given. Applications for three point quadrature rules of $n-$time differentiable functions are also provided.

1. Introduction

In order to approximate the *Stieltjes integral* $\int_a^b f(t)\,du(t)$ with the simpler expression

$$\frac{1}{b-a}[u(b)-u(a)]\cdot\int_a^b f(t)\,dt, \tag{1.1}$$

S.S. Dragomir and I. Fedotov [8] introduced in 1998 the following *error functional*

$$D(f,u;a,b):=\int_a^b f(t)\,du(t)-\frac{1}{b-a}[u(b)-u(a)]\cdot\int_a^b f(t)\,dt, \tag{1.2}$$

provided that both the Stieltjes integral $\int_a^b f(t)\,du(t)$ and the *Riemann integral* $\int_a^b f(t)\,dt$ exist.

If the *integrand* f is *Riemann integrable* on $[a,b]$ and the *integrator* $u:[a,b]\to\mathbb{R}$ is $L-$*Lipschitzian*, i.e.,

$$|u(t)-u(s)|\le L|t-s| \quad \text{for each } \ t,s\in[a,b], \tag{1.3}$$

then the Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and, as pointed out in [8],

$$|D(f,u;a,b)|\le L\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt. \tag{1.4}$$

The inequality (1.4) is sharp in the sense that the multiplicative constant $C=1$ in front of L cannot be replaced by a smaller quantity. Moreover, if there exist the constants $m,M\in\mathbb{R}$ such that

$$m\le f(t)\le M \quad \text{for a.e. } \ t\in[a,b], \tag{1.5}$$

then [8]

$$|D(f,u;a,b)|\le\frac{1}{2}L(M-m)(b-a). \tag{1.6}$$

Date: February 01, 2007.

2000 *Mathematics Subject Classification.* Primary 26D15. Secondary 26A42, 41A55.

Key words and phrases. Stieltjes integral, Functions of bounded variation, Lipschitzian functions, Monotonic functions, Quadrature rules.

The constant $\frac{1}{2}$ is best possible in (1.6).

A different approach in the case of integrands of bounded variation were considered by the same authors in 2001, see [9], where they proved that

$$|D(f,u;a,b)| \leq \max_{t\in[a,b]} \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| \bigvee_a^b (u), \tag{1.7}$$

provided that f is continuous and u is of bounded variation. Here $\bigvee_a^b (u)$ denotes the total variation of u on $[a,b]$. The inequality (1.7) is also sharp.

If we assume that f is $K-$Lipschitzian, then [9]

$$|D(f,u;a,b)| \leq \frac{1}{2}K(b-a)\bigvee_a^b (u), \tag{1.8}$$

with $\frac{1}{2}$ the best possible constant in (1.8).

For various bounds on the *error functional* $D(f,u;a,b)$ where f and u belong to different classes of functions for which the Stieltjes integral exists, see [2], [5], [6] and [7] and the references therein.

The main aim of the present paper is to estimate the error of approximating the Stieltjes integral $\int_a^b f(t)\,du(t)$ with the simpler expression

$$\frac{m+M}{2}\cdot[u(b)-u(a)] \tag{1.9}$$

provided the integrand f is bounded below by m and above by M.

In the dual case, i.e., when $n \leq u(t) \leq M$ on $[a,b]$, the problem under consideration consists of approximating the same Stieltjes integral $\int_a^b f(t)\,du(t)$ with the quantity

$$\left[u(b)-\frac{n+N}{2}\right]f(b)+\left[\frac{n+N}{2}-u(a)\right]f(a). \tag{1.10}$$

Applications for the three point quadrature rule of $n-$differentiable functions are also given.

2. Inequalities for the Stieltjes Integral

The following result may be stated.

Theorem 1. *Let $u:[a,b]\to\mathbb{R}$ be a function of bounded variation and $f:[a,b]\to\mathbb{R}$ a function such that there exists the constants $m,M\in\mathbb{R}$ with*

$$m \leq f(t) \leq M \quad \text{for each} \quad t\in[a,b], \tag{2.1}$$

and the Stieltjes integral $\int_a^b f(t)\,du(t)$ exists. Then, by defining the error functional

$$\Delta(f,u,m,M;a,b) := \int_a^b f(t)\,du(t) - \frac{m+M}{2}[u(b)-u(a)],$$

we have the bound

$$|\Delta(f,u,m,M;a,b)| \leq \frac{1}{2}(M-m)\bigvee_a^b (u). \tag{2.2}$$

The constant $\frac{1}{2}$ is best possible in (2.2) in the sense that it cannot be replaced by a smaller quantity.

Proof. Since, obviously, the function $f-\frac{m+M}{2}$ satisfies the inequality

$$\left|f(t)-\frac{m+M}{2}\right|\leq\frac{1}{2}(M-m)\quad\text{for any}\quad t\in[a,b]$$

and the Stieltjes integral $\int_a^b\left(f(t)-\frac{m+M}{2}\right)du(t)$ exists, then

$$\begin{aligned}\left|\int_a^b\left(f(t)-\frac{m+M}{2}\right)du(t)\right|&\leq\sup_{t\in[a,b]}\left|f(t)-\frac{m+M}{2}\right|\bigvee_a^b(u)\\&\leq\frac{1}{2}(M-m)\bigvee_a^b(u)\end{aligned}$$

and the inequality (2.2) is proved.

Now, assume that (2.2) holds with a positive constant C, i.e.,

$$|\Delta(f,u,m,M;a,b)|\leq C(M-m)\bigvee_a^b(u),\tag{2.3}$$

provided u is of bounded variation on $[a,b]$ and f satisfies (2.1).

If we consider the function $f_0(t):=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$ and $u_0(t)=\frac{1}{2}\left(t-\frac{a+b}{2}\right)^2$, then we observe that the Stieltjes integral $\int_a^b f_0(t)\,du_0(t)$ exists, f_0 is bounded above by $M_0=1$ and below by $m_0=-1$, u_0 is of bounded variation and

$$\bigvee_a^b(u_0)=\int_a^b|u_0'(t)|\,dt=\int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{(b-a)^2}{4}.$$

Also

$$\begin{aligned}\int_a^b f_0(t)\,du_0(t)&=\int_a^b\operatorname{sgn}\left(t-\frac{a+b}{2}\right)\left(t-\frac{a+b}{2}\right)dt\\&=\int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{(b-a)^2}{4}\end{aligned}$$

and replacing f_0 and u_0 in (2.3) produces the inequality

$$\frac{(b-a)^2}{4}\leq 2C\cdot\frac{(b-a)^2}{4}$$

which implies that $C\geq\frac{1}{2}$. ∎

The following corollary provides a natural example of functions f that can be chosen to fulfill the conditions in the above theorem.

Corollary 1. *Let $u:[a,b]\to\mathbb{R}$ be a function of bounded variation on $[a,b]$ and f a continuous function on $[a,b]$. Then*

$$\left|\widetilde{\Delta}(f,u;a,b)\right|\leq\frac{1}{2}\left[\max_{t\in[a,b]}f(t)-\min_{t\in[a,b]}f(t)\right]\bigvee_a^b(u),\tag{2.4}$$

where

$$\widetilde{\Delta}(f,u;a,b):=\int_a^b f(t)\,du(t)-\frac{\min_{t\in[a,b]}f(t)+\max_{t\in[a,b]}f(t)}{2}[u(b)-u(a)].$$

The constant $\frac{1}{2}$ is best possible.

Proof. For the sharpness of the constant, we cannot use the above example since f_0 was not continuous on $[a,b]$.

Let us now consider $u_0(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$ and $f_0(t) = \left|t - \frac{a+b}{2}\right|$. The Stieltjes integral $\int_a^b f_0(t)\,du_0(t)$ exists and

$$\begin{aligned}
&\int_a^b f_0(t)\,du_0(t) \\
&= f_0(t)\,u_0(t)\Big|_a^b - \int_a^b u_0(t)\,df_0(t) \\
&= \frac{b-a}{2} + \frac{b-a}{2} - \left[\int_a^{\frac{a+b}{2}} (-1)\,d\left(\frac{a+b}{2} - t\right) + \int_{\frac{a+b}{2}}^b (1)\,d\left(t - \frac{a+b}{2}\right)\right] \\
&= 0
\end{aligned}$$

we have then

$$\left|\widetilde{\Delta}(f_0, u_0; a, b)\right| = \frac{b-a}{2}.$$

Also

$$\frac{1}{2}\left[\max_{t\in[a,b]} f_0(t) - \min_{t\in[a,b]} f_0(t)\right]\bigvee_a^b (u_0) = \frac{b-a}{2},$$

which shows that the equality case holds in (2.4). ∎

The following result providing bounds for the Lipshitzain integrators may be stated as well:

Theorem 2. *If $u : [a,b] \to \mathbb{R}$ is $L-$Lipschitzian and $f : [a,b] \to \mathbb{R}$ is Riemann integrable and satisfies the condition (2.1), then*

$$|\Delta(f, u, m, M; a, b)| \le \frac{1}{2}(M - m)\,L\,(b-a). \tag{2.5}$$

The constant $\frac{1}{2}$ is best possible.

Proof. It is well known that if p is Riemann integrable on $[a,b]$ and v is $L-$Lipschitzian on $[a,b]$, then the Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and

$$\left|\int_a^b p(t)\,dv(t)\right| \le L\int_a^b |p(t)|\,dt. \tag{2.6}$$

Now, taking into account that $f - \frac{m+M}{2}$ is Riemann integrable, by making use of (2.6) we have

$$\begin{aligned}
\left|\int_a^b \left(f(t) - \frac{m+M}{2}\right) du(t)\right| &\le L\int_a^b \left|f(t) - \frac{m+M}{2}\right| dt \\
&\le \frac{1}{2}(M-m)\,L\,(b-a)
\end{aligned}$$

and the desired inequality (2.5) is obtained.

To prove the sharpness of the constant $\frac{1}{2}$, assume that the inequality (2.5) holds with a positive constant D, i.e.,

$$|\Delta(f, u, m, M; a, b)| \le D\,(M - m)\,L\,(b-a), \tag{2.7}$$

provided f is Riemann integrable and satisfies (2.1) while u is Lipschitz continuous with the constant $L > 0$.

Consider the functions $f_0(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$ and $u_0(t) = \left|t - \frac{a+b}{2}\right|$. It is obvious that f_0 is Riemann integrable and $M_0 = 1$, $m_0 = -1$. Since, by the triangle inequality we have

$$|u_0(t) - u_0(s)| = \left|\left|t - \frac{a+b}{2}\right| - \left|s - \frac{a+b}{2}\right|\right| \le |t-s|,$$

for any $t, s \in [a,b]$, hence u_0 is Lipschitzian with the constant $L = 1$. Now, observe that

$$\begin{aligned}\int_a^b f_0(t)\,du_0(t) &= \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) d\left(\left|t - \frac{a+b}{2}\right|\right) \\ &= \int_a^{\frac{a+b}{2}} (-1)\,d\left(\frac{a+b}{2} - t\right) + \int_{\frac{a+b}{2}}^b (1)\,d\left(t - \frac{a+b}{2}\right) \\ &= b-a,\end{aligned}$$

and introducing the above values in (2.7) we deduce

$$b - a \le 2D(b-a),$$

which implies that $D \ge \frac{1}{2}$. ∎

Corollary 2. *If f is continuous on $[a,b]$ and u is $L-$Lipschitzian, then:*

$$\left|\widetilde{\Delta}(f,u;a,b)\right| \le \frac{1}{2}\left[\max_{t\in[a,b]} f(t) - \min_{t\in[a,b]} f(t)\right] L(b-a). \tag{2.8}$$

The constant $\frac{1}{2}$ is best possible.

Proof. In order to prove the sharpness of the constant, we cannot use the example from Theorem 2 since f_0 was not continuous.

If $u_0(t) = \left|t - \frac{a+b}{2}\right|$ and f_0 is continuous, then

$$\int_a^b f_0(t)\,d\left|t - \frac{a+b}{2}\right| = \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) f_0(t)\,dt.$$

Consider now the sequence of continuous functions

$$f_{0,n}(t) = \begin{cases} -1 & \text{if } t \in \left[a, \frac{a+b}{2} - \frac{1}{n}\right]; \\ -1 + n\left(t - \frac{a+b}{2} + \frac{1}{n}\right) & \text{if } t \in \left(\frac{a+b}{2} - \frac{1}{n}, \frac{a+b}{2} + \frac{1}{n}\right); \\ 1 & \text{if } t \in \left[\frac{a+b}{2} + \frac{1}{n}, b\right], \end{cases}$$

which coincides with $u_0(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$ on $\left[a, \frac{a+b}{2} - \frac{1}{n}\right] \cup \left[\frac{a+b}{2} + \frac{1}{n}, b\right]$ and connects the end segments of this function on $\left[\frac{a+b}{2} - \frac{1}{n}, \frac{a+b}{2} + \frac{1}{n}\right]$ respectively. Obviously

$$\begin{aligned}&\int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) f_{0,n}(t)\,dt \\ &= \int_a^{\frac{a+b}{2} - \frac{1}{n}} dt + \int_{\frac{a+b}{2} - \frac{1}{n}}^{\frac{a+b}{2} + \frac{1}{n}} \operatorname{sgn}\left(t - \frac{a+b}{2}\right) f_{0,n}(t)\,dt + \int_{\frac{a+b}{2} + \frac{1}{n}}^b dt \\ &= b - a + x_n,\end{aligned}$$

where

$$|x_n| = \left| \int_{\frac{a+b}{2}-\frac{1}{n}}^{\frac{a+b}{2}+\frac{1}{n}} \operatorname{sgn}\left(t - \frac{a+b}{2}\right) f_{0,n}(t)\,dt \right| \le \frac{2}{n}.$$

Now, if (2.8) holds with a constant $E > 0$, i.e.,

$$\left|\widetilde{\Delta}(f, u; a, b)\right| \le E \left[\max_{t\in[a,b]} f(t) - \min_{t\in[a,b]} f(t)\right] L(b-a),$$

then on choosing $f_{0,n}$ and u_0 as above, we get

$$b - a + x_n \le 2E(b-a)$$

for each $n \in \mathbb{N}$. Letting $n \to \infty$ and taking into account that $\lim_{n\to\infty} x_n = 0$, we deduce $E \ge \frac{1}{2}$, and the corollary is proved. ∎

Corollary 3. *Let $f, h : [a,b] \to \mathbb{R}$ be Riemann integrable functions, f satisfies (2.1) and $|h(t)| \le N$ for a.e. $t \in [a,b]$. Then*

$$\left| \int_a^b f(t) h(t)\,dt - \frac{m+M}{2} \int_a^b h(t)\,dt \right| \le \frac{1}{2}(M-m) N (b-a). \tag{2.9}$$

The constant $\frac{1}{2}$ is best possible.

The proof follows by (2.5) on choosing $u(t) = \int_a^t h(s)\,ds$. The details are omitted.
Finally, we can state the following result as well.

Theorem 3. *Let $u : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing function on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ a bounded function satisfying (2.1) and such that $\int_a^b f(t)\,du(t)$ exists. Then*

$$\begin{aligned} |\Delta(f, u, m, M; a, b)| &\le \int_a^b \left| f(t) - \frac{m+M}{2} \right| du(t) \\ &\le \frac{1}{2}(M-m)[u(b) - u(a)]. \end{aligned} \tag{2.10}$$

The first inequality in (2.10) is sharp. The constant $\frac{1}{2}$ is best possible.

Proof. The inequality

$$\left| \int_a^b \left(f(t) - \frac{m+M}{2} \right) du(t) \right| \le \int_a^b \left| f(t) - \frac{m+M}{2} \right| du(t)$$

follows by the definition of Stieltjes integrals.

Since

$$\left| f(t) - \frac{m+M}{2} \right| \le \frac{1}{2}(M-m) \quad \text{for each } t \in [a,b],$$

we also have that

$$\begin{aligned} \int_a^b \left| f(t) - \frac{m+M}{2} \right| du(t) &\le \frac{1}{2}(M-m) \int_a^b du(t) \\ &= \frac{1}{2}(M-m)[u(b) - u(a)] \end{aligned}$$

and the inequality (2.10) is thus proved.

Now, assume that $f_0(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, $t \in [a,b]$. Then for any continuous and monotonic nondecreasing function $u_0 : [a,b] \to \mathbb{R}$ we can state that

$$\begin{aligned}
&\Delta(f_0, u_0, m_0, M_0; a, b) \\
&= \int_a^{\frac{a+b}{2}} (-1)\, du_0(t) + \int_{\frac{a+b}{2}}^b (1)\, du_0(t) \\
&= u_0(a) + u_0(b) - 2u_0\left(\frac{a+b}{2}\right).
\end{aligned}$$

Also,

$$\int_a^b \left| f_0(t) - \frac{m_0 + M_0}{2} \right| du_0(t) = u_0(b) - u_0(a)$$

and

$$\frac{1}{2}(M_0 - m_0)[u_0(b) - u_0(a)] = u_0(b) - u_0(a),$$

which shows that the last inequality holds with equality in (1.9).

Finally, to have equality in the first part of (2.10) it is sufficient selecting u_0 to vanish in $\left[a, \frac{a+b}{2}\right]$ and being continuous and monotonic nondecreasing on $\left[\frac{a+b}{2}, b\right]$. In this situation we get in all terms of (2.10) the same quantity $u_0(b)$. ∎

Corollary 4. *If f is continuous on $[a,b]$ and u is monotonic nondecreasing, then*

$$\begin{aligned}
(2.11) \qquad \left|\widetilde{\Delta}(f, u; a, b)\right| &\le \int_a^b \left| f(t) - \frac{\min_{t\in[a,b]} f(t) + \max_{t\in[a,b]} f(t)}{2} \right| du(t) \\
&\le \frac{1}{2}\left[\max_{t\in[a,b]} f(t) - \min_{t\in[a,b]} f(t)\right][u(b) - u(a)].
\end{aligned}$$

To prove the sharpness of the inequality we use the functions $f_0(t) = \left|t - \frac{a+b}{2}\right|$ and $u_0(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$ which produce in all terms of (2.11) the quantity $\frac{b-a}{2}$.

Corollary 5. *If f, w are Riemann integrable on $[a,b]$ and f satisfies (2.1) while w is nonnegative, then*

$$\begin{aligned}
(2.12) \qquad \left| \int_a^b f(t)\, w(t)\, dt - \frac{m+M}{2} \int_a^b w(t)\, dt \right| &\le \int_a^b \left| f(t) - \frac{m+M}{2} \right| w(t)\, dt \\
&\le \frac{1}{2}(M - m) \int_a^b w(t)\, dt.
\end{aligned}$$

The dual case, i.e., when the integrator is bounded below and above, is incorporated in the following result.

Theorem 4. *Assume that u is Riemann integrable on $[a,b]$ and*

$$(2.13) \qquad -\infty < n \le u(t) \le N < \infty \quad \textit{for a.e.} \quad t \in [a,b].$$

Define the error functional of generalised trapezoid type

$$\nabla(f, u, n, N; a, b) := \left[u(b) - \frac{n+N}{2}\right] f(b) + \left[\frac{n+N}{2} - u(a)\right] f(a) - \int_a^b f(t)\, du(t).$$

(i) *If f is of bounded variation and such that the Stieltjes integral $\int_a^b f(t)\,du(t)$ exists, then*

$$|\nabla(f,u,n,N;a,b)| \leq \frac{1}{2}(N-n)\bigvee_a^b(f). \tag{2.14}$$

The constant $\frac{1}{2}$ is best possible in (2.14).

(ii) *If f is $K-$Lipschitzian on $[a,b]$, then*

$$|\nabla(f,u,n,N;a,b)| \leq \frac{1}{2}(N-n)K(b-a). \tag{2.15}$$

The constant $\frac{1}{2}$ is best possible in (2.15).

(iii) *If f is monotonic nondecreasing on $[a,b]$ such that the Stieltjes integrals, $\int_a^b f(t)\,du(t)$, $\int_a^b \left|u(t)-\frac{n+N}{2}\right| df(t)$ exist, then*

$$\begin{aligned} |\nabla(f,u,n,N;a,b)| &\leq \int_a^b \left|u(t)-\frac{n+N}{2}\right| df(t) \\ &\leq \frac{1}{2}(N-n)[f(b)-f(a)]. \end{aligned} \tag{2.16}$$

The first inequality is sharp and the constant $\frac{1}{2}$ is best possible in (2.16).

Proof. The proof follows by Theorems 1 – 3 on utilising the integral identity:

$$\begin{aligned} &\left[u(b)-\frac{n+N}{2}\right]f(b)+\left[\frac{n+N}{2}-u(a)\right]f(a)-\int_a^b f(t)\,du(t) \\ &= \int_a^b \left[u(t)-\frac{n+N}{2}\right]df(t) \end{aligned}$$

and the details are omitted. ∎

Remark 1. *The above inequalities also hold for continuous functions $u:[a,b]\to\mathbb{R}$ when n is replaced by $\min_{t\in[a,b]} u(t)$ and N is replaced by $\max_{t\in[a,b]} u(t)$. The details are left to the interested reader.*

3. Applications for Three Point Quadrature Rules

In [1] (see also [10, p. 223]) P. Cerone and S.S. Dragomir established the following three point quadrature rule for $n-$times differentiable functions:

$$\begin{aligned} \int_a^b f(t)\,dt = \sum_{k=1}^{n}\frac{1}{k!}\Big\{&(1-\gamma)^k\left[(b-x)^k+(-1)^{k-1}(x-a)^k\right]f^{(k-1)}(x) \\ &+\gamma^k\left[(x-a)^k f^{(k-1)}(a)+(-1)^{k-1}(b-x)^k f^{(k-1)}(b)\right]\Big\} \\ &+(-1)^n\int_a^b C_n(x,t)\,f^{(n)}(t)\,dt, \end{aligned} \tag{3.1}$$

where

$$C_n(x,t)=\begin{cases} \frac{[t-(\gamma x+(1-\gamma)a)]^n}{n!} & \text{if } t\in[a,x]; \\ \frac{[t-(\gamma x+(1-\gamma)b)]^n}{n!} & \text{if } t\in(x,b], \end{cases} \tag{3.2}$$

and $\gamma\in[0,1]$, $x\in(a,b)$.

This representation comprises amongst others the interior point quadrature rule obtained by Cerone et al. [3] in 1999 for $\gamma = 0$ and the trapezoid quadrature rule obtained by Cerone et al. [4] in 2000 for $\gamma = 1$.

Consider the function:

$$K_n(x,t) := (-1)^n \begin{cases} \frac{[t-(\gamma x+(1-\gamma)a)]^{n+1}}{(n+1)!} & \text{if } t \in [a,x]; \\ \frac{[t-(\gamma x+(1-\gamma)b)]^{n+1}}{(n+1)!} & \text{if } t \in (x,b]. \end{cases} \tag{3.3}$$

The function $K_n(x,\cdot) : [a,b] \to \mathbb{R}$, for each fixed $x \in [a,b]$, is of bounded variation and

$$\begin{aligned} \bigvee_a^b (K_n(x,\cdot)) &= \int_a^x \left| \frac{dK_n(x,t)}{dt} \right| dt + \int_x^b \left| \frac{dK_n(x,t)}{dt} \right| dt \\ &= \int_a^x \frac{|t-(\gamma x+(1-\gamma)a)|^n}{n!} dt + \int_x^b \frac{|\gamma x+(1-\gamma)b-t|^n}{n!} dt. \end{aligned}$$

We have

$$\begin{aligned} I_1 &= \int_a^x \frac{|t-(\gamma x+(1-\gamma)a)|^n}{n!} dt \\ &= \int_a^{\gamma x+(1-\gamma)a} \frac{[\gamma x+(1-\gamma)a-t]^n}{n!} dt + \int_{\gamma x+(1-\gamma)a}^x \frac{[t-(\gamma x+(1-\gamma)a)]^n}{n!} dt \\ &= -\left[\left. \frac{[\gamma x+(1-\gamma)a-t]^{n+1}}{(n+1)!} \right|_a^{\gamma x+(1-\gamma)a} \right] + \left. \frac{[t-(\gamma x+(1-\gamma)a)]^{n+1}}{(n+1)!} \right|_{\gamma x+(1-\gamma)a}^x \\ &= \frac{\gamma^{n+1}(x-a)^{n+1}}{(n+1)!} + \frac{(1-\gamma)^{n+1}(x-a)^{n+1}}{(n+1)!} \\ &= \frac{1}{(n+1)!}(x-a)^{n+1}\left[\gamma^{n+1}+(1-\gamma)^{n+1}\right] \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_x^b \frac{|\gamma x+(1-\gamma)b-t|^n}{n!} dt \\ &= \int_x^{\gamma x+(1-\gamma)b} \frac{[\gamma x+(1-\gamma)b-t]^n}{n!} dt + \int_{\gamma x+(1-\gamma)b}^b \frac{[t-(\gamma x+(1-\gamma)b)]^n}{n!} dt \\ &= -\left[\left. \frac{[\gamma x+(1-\gamma)b-t]^{n+1}}{(n+1)!} \right|_x^{\gamma x+(1-\gamma)b} \right] + \left. \frac{[t-(\gamma x+(1-\gamma)b)]^{n+1}}{(n+1)!} \right|_{\gamma x+(1-\gamma)b}^b \\ &= \frac{(1-\gamma)^{n+1}(b-x)^{n+1}}{(n+1)!} + \frac{\gamma^{n+1}(b-x)^{n+1}}{(n+1)!} \\ &= \frac{1}{(n+1)!}(b-x)^{n+1}\left[\gamma^{n+1}+(1-\gamma)^{n+1}\right]. \end{aligned}$$

Therefore

$$\bigvee_a^b (K_n(x,\cdot)) = \frac{1}{(n+1)!}\left[\gamma^{n+1}+(1-\gamma)^{n+1}\right]\left[(b-x)^{n+1}+(x-a)^{n+1}\right]. \tag{3.4}$$

We also have

$$\int_a^b f^{(n)}(t)\,d(K_n(x,t)) = \int_a^x f^{(n)}(t)\,d\left[(-1)^n \frac{[t-(\gamma x+(1-\gamma)a)]^{n+1}}{(n+1)!}\right] + \int_x^b f^{(n)}(t)\,d\left[(-1)^n \frac{[t-(\gamma x+(1-\gamma)b)]^{n+1}}{(n+1)!}\right] = (-1)^n \int_a^b C_n(t,x)\,f^{(n)}(t)\,dt, \tag{3.5}$$

with $C_n(t,x)$ defined by (3.2).

We can state the following result in approximating the Riemann integral $\int_a^b f(x)\,dx$ of $n-$times differentiable functions f in terms of three point quadrature rules.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a function such that for $n\geq 1$ the derivative $f^{(n-1)}$ is absolutely continuous and there exists the real constants γ_n,Γ_n such that*

$$\gamma_n \leq f^{(n)}(t) \leq \Gamma_n \quad \textit{for a.e.} \quad t\in[a,b]. \tag{3.6}$$

Then

$$\int_a^b f(t)\,dt = \sum_{k=1}^n \frac{1}{k!}\Big\{(1-\gamma)^k\left[(b-x)^k+(-1)^{k-1}(x-a)^k\right]f^{(k-1)}(x) + \gamma^k\left[(x-a)^k f^{(k-1)}(a)+(-1)^{k-1}(b-x)^k f^{(k-1)}(b)\right]\Big\} + \frac{\gamma_n+\Gamma_n}{2}\left[(-1)^n(b-x)^{n+1}+(x-a)^{n+1}\right]\frac{\gamma^{n+1}}{(n+1)!}+R_n, \tag{3.7}$$

and the error R_n satisfies the bound

$$|R_n| \leq \frac{1}{2}(\Gamma_n-\gamma_n)\frac{1}{(n+1)!}\left[\gamma^{n+1}+(1-\gamma)^{n+1}\right]\left[(b-x)^{n+1}+(x-a)^{n+1}\right] \tag{3.8}$$

for $\gamma\in[0,1]$ and $x\in[a,b]$.

Proof. We apply Theorem 1 for the functions $f^{(n)}$ and $K(x,\cdot)$ to get:

$$\left|\int_a^b f^{(n)}(t)\,dK_n(x,t)-\frac{\gamma_n+\Gamma_n}{2}\left[K_n(x,b)-K_n(x,a)\right]\right| \leq \frac{1}{2}(\Gamma_n-\gamma_n)\bigvee_a^b(K_n(x,\cdot))$$

for $x\in[a,b]$.

Since

$$K_n(x,b) = (-1)^n\frac{[b-(\gamma x+(1-\gamma)b)]^{n+1}}{(n+1)!} = (-1)^n\frac{\gamma^{n+1}(b-x)^{n+1}}{(n+1)!}$$

and

$$K_n(x,a) = (-1)^n \frac{[a-(\gamma x+(1-\gamma)a)]^{n+1}}{(n+1)!}$$
$$= (-1)^n \frac{[\gamma(a-x)]^{n+1}}{(n+1)!}$$
$$= -\frac{\gamma^{n+1}(x-a)^{n+1}}{(n+1)!},$$

hence by (3.4) and (3.5) we deduce:

$$\left|(-1)^n \int_a^b C_n(t,x) f^{(n)}(t)\,dt - \frac{\gamma_n+\Gamma_n}{2}\left[(-1)^n \frac{\gamma^{n+1}(b-x)^{n+1}}{(n+1)!} + \frac{\gamma^{n+1}(x-a)^{n+1}}{(n+1)!}\right]\right| \le \frac{1}{2}(\Gamma_n-\gamma_n)\frac{1}{(n+1)!}\left[\gamma^{n+1}+(1-\gamma)^{n+1}\right]\left[(b-x)^{n+1}+(x-a)^{n+1}\right].. \tag{3.9}$$

Finally, on utilising the identity (3.1) we deduce from (3.9) the representation (3.7) and the estimate (3.8). ∎

Remark 2. *The above approximation of the integral* $\int_a^b f(t)\,dt$ *contains some particular cases of interest.*

If $\lambda = 0$, *then we have*

$$\int_a^b f(t)\,dt = \sum_{k=1}^{n} \frac{1}{k!}\left[(b-x)^k + (-1)^{k-1}(x-a)^k\right] f^{(k-1)}(x) + T_n, \tag{3.10}$$

with

$$|T_n| \le \frac{1}{2}(\Gamma_n-\gamma_n)\frac{1}{(n+1)!}\left[(b-x)^{n+1}+(x-a)^{n+1}\right].$$

If $\lambda = \frac{1}{2}$, *then we have*

$$\int_a^b f(t)\,dt = \sum_{k=1}^{n} \frac{1}{2^k k!}\left\{\left[(b-x)^k + (-1)^{k-1}(x-a)^k\right] f^{(k-1)}(x) + \left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b)\right]\right\} + \frac{\gamma_n+\Gamma_n}{2^{n+2}(n+1)!}\left[(-1)^n(b-x)^{n+1}+(x-a)^{n+1}\right] + M_n, \tag{3.11}$$

with

$$|M_n| \le \frac{1}{2^{n+1}(n+1)!}(\Gamma_n-\gamma_n)\left[(b-x)^{n+1}+(x-a)^{n+1}\right].$$

Finally, if $\lambda = 1$, *then we have*

$$\int_a^b f(t)\,dt = \sum_{k=1}^{n} \frac{1}{k!}\left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b)\right] + \frac{\gamma_n+\Gamma_n}{2(n+1)!}\left[(-1)^n(b-x)^{n+1}+(x-a)^{n+1}\right] + Q_n, \tag{3.12}$$

with

$$|Q_n| \leq \frac{1}{2(n+1)!}(\Gamma_n - \gamma_n)\left[(b-x)^{n+1} + (x-a)^{n+1}\right].$$

References

[1] P. CERONE and S.S. DRAGOMIR, Three point identities and inequalities for $n-$time differentiable functions, *SUT J. of Math.* (Japan), **36**(2) (2000), 351-383.

[2] P. CERONE and S.S. DRAGOMIR, Approximation of the Stieltjes integral and application in numerical integration, *Applications of Math.,* **51**(1) (2006), 37-47.

[3] P. CERONE, S.S. DRAGOMIR and J. ROUMELIOTIS, Some Ostrowski type inequalities for $n-$time differentiable mappings and applications, *Demonstratio Math.*, **32**(2) (1999), 697-712.

[4] P. CERONE, S.S. DRAGOMIR, J. ROUMELIOTIS and J. SŬNDE, A new generalisation of the trapezoid formula for $n-$time differentiable mappings and applications, *Demonstratio Math.*, **33**(4) (2000), 719-736.

[5] S.S. DRAGOMIR, Inequalities of Grüss type for the Stieltjes integral, *Kragujevac J. Math.*, **26** (2004), 89-122.

[6] S.S. DRAGOMIR, A generalisation of Cerone's identity and applications, *Oxford Tamsui J. Math.*, (in press), Preprint *RGMIA Res. Rep. Coll.* **8**(2005), No. 2. Artcile 19.[ONLINE: http://rgmia.vu.edu.au/v8n2.html].

[7] S.S. DRAGOMIR, Inequalities for Stieltjes integrals with convex integrators and applications, *Appl. Math. Lett.*, **20** (2007), 123-130.

[8] S.S. DRAGOMIR and I. FEDOTOV, An inequality of Grüss type for the Riemann-Stieltjes integral and applications for special means, *Tamkang J. Math.,* **29**(4) (1998), 287-292.

[9] S.S. DRAGOMIR and I. FEDOTOV, A Grüss type inequality for mappings of bounded variation and applications for numerical analysis, *Nonlinear Funct. Anal. Appl.,* **6**(3) (2001), 425-433.

[10] S.S. DRAGOMIR and Th. M. RASSIAS (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration,* Kluwer Academic Publishers, Dordrecht, 504 pp./ 2002.

School of Computer Science and Mathematics, Victoria University, PO Box 14428, Melbourne City, VIC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.vu.edu.au/dragomir