ON AN INEQUALITY OF THE LUPAŞ TYPE

N.S. BARNETT AND S.S. DRAGOMIR

Abstract. An inequality of the Lupaş type for Čebyšev functionals is established.

1. Introduction

Let $L_2'[a,b]$ be the space of all real functions which are absolutely continuous on $[a,b]$, whose derivatives are square integrable in the Lebesgue sense on $[a,b]$. Consider the Čebyšev functional defined by

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx. \tag{1.1}$$

In 1970, A.M. Ostrowski [3] proved that if $f,g \in L_2'[a,b]$, then there exists a constant C with $0 < C < \frac{1}{8}(b-a)$ such that

$$|T(f,g)| \leq C\,\|f'\|_2\,\|g'\|_2\,, \tag{1.2}$$

where

$$\|f'\|_2 := \left(\int_a^b |f'(t)|^2\,dt\right)^{\frac{1}{2}},$$

is the usual Hilbertian norm in $L_2'[a,b]$.

The best possible constant in (1.2) was obtained by A. Lupaş in 1973 [2] in which he proved that,

$$|T(f,g)| \leq \frac{b-a}{\pi^2}\,\|f'\|_2\,\|g'\|_2\,. \tag{1.3}$$

There is at least a pair of functions (f_0,g_0) from $L_2'[a,b]$ for which the equality is realised in (1.3). For instance,

$$f_0(x) = A + B\sin\left(\frac{\pi}{2}\cdot\frac{a+b-2x}{b-a}\right),$$
$$g_0(x) = C + D\sin\left(\frac{\pi}{2}\cdot\frac{a+b-2x}{b-a}\right)$$

with $A,B,C,D \in \mathbb{R}$ and $\alpha \in [a,b]$.

Now, assume that $w : [a,b] \to [0,\infty)$ is a Lebesgue measurable function and consider the space $L_{2,w}'[a,b]$ of all absolutely continuous functions $f : [a,b] \to \mathbb{R}$

Date: 10th January, 2007.
2000 *Mathematics Subject Classification.* 26D15, 26D10.
Key words and phrases. Lupaş type inequality, Čebyšev functionals, Integral inequalities.

with the property that,

$$\|f'\|_{2,w} := \left(\int_a^b w(x) |f'(x)|^2 \, dx \right)^{\frac{1}{2}} < \infty. \tag{1.4}$$

One can then consider the problem of finding the best possible constant $K > 0$ such that the following inequality holds:

$$|T(f,g)| \le K \|f'\|_{2,w} \|g'\|_{2,w}, \tag{1.5}$$

for any $f', g' \in L'_w[a,b]$ with an appropriate weight w.

In this paper we show that for the weight $w : [a,b] \to [0,\infty)$

$$w(x) = (b-x)\left[(b-a)^2 + (b-a)(x-a) + (x-a)^2 \right], \quad x \in [a,b] \tag{1.6}$$

we can establish an inequality of the (1.5)-type. We also show that for some examples of elementary functions f and g, the bound in (1.5) for $|T(f,g)|$ is better than the one in (1.3). The problem of the sharpness for the obtained constant is still open.

2. The Results

Lemma 1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$, $w : [a,b] \to [0,\infty)$ defined by,*

$$w(x) = (b-x)\left[(b-a)^2 + (b-a)(x-a) + (x-a)^2 \right], \quad x \in [a,b] \tag{2.1}$$

and let $L'_{2,w}[a,b]$ be the space defined in the introduction. It follows that,

$$(0 \le)\,(b-a)\int_a^b f^2(x)\,dx - \left(\int_a^b f(x)\,dx \right)^2 \le \frac{4}{3\pi^2} \cdot \int_a^b w(x)\,[f'(x)]^2\,dx \tag{2.2}$$

for all $f \in L'_{2,w}[a,b]$.

Proof. We use the following inequality obtained by Diaz and Metcalf in [1]:

$$\int_a^b [v(x) - v(b)]^2\,dx \le 4 \cdot \frac{(b-a)^2}{\pi^2} \int_a^b [v'(x)]^2\,dx, \tag{2.3}$$

provided v is absolutely continuous on $[a,b]$. If

$$v(x) = v(b) + B \sin\left(\frac{\pi}{2} \cdot \frac{b-x}{b-a} \right), \quad B \in \mathbb{R},$$

then the equality case is realised in (2.3).

Utilising (2.3) we can write,

$$\int_a^y [f(x) - f(y)]^2\,dx \le \frac{4}{\pi^2} \cdot (b-a)^2 \int_a^y [f'(x)]^2\,dx \tag{2.4}$$

for any $y \in [a,b]$.

Integrating over y in $[a,b]$, we deduce,

$$\int_a^b \left(\int_a^y [f(x) - f(y)]^2\,dx \right) dy \le \frac{4}{\pi^2} \int_a^b (y-a)^2 \left(\int_a^y [f'(x)]^2\,dx \right) dy. \tag{2.5}$$

However,

$$
\begin{aligned}
I &:= \int_a^b \left(\int_a^y [f(x) - f(y)]^2 \, dx \right) dy \\
&= \int_a^b \left[\int_a^y f^2(x) \, dx - 2f(y) \int_a^y f(x) \, dx + f^2(y)(y-a) \right] dy \\
&= \left[y \int_a^y f^2(x) \, dx \right] \Bigg|_a^b - \int_a^b y f^2(y) \, dy \\
&\qquad - 2 \cdot \frac{1}{2} \left(\int_a^y f(x) \, dx \right)^2 \Bigg|_a^b + \int_a^b f^2(y)(y-a) \, dy \\
&= b \int_a^b f^2(x) \, dx - \int_a^b y f^2(y) \, dy - \left(\int_a^b f(x) \, dx \right)^2 + \int_a^b f^2(y)(y-a) \, dy \\
&= (b-a) \int_a^b f^2(x) \, dx - \left(\int_a^b f(x) \, dx \right)^2
\end{aligned}
$$

and

$$
\begin{aligned}
J &:= \int_a^b (y-a)^2 \left(\int_a^y [f'(x)]^2 \, dx \right) dy \\
&= \frac{(y-a)^3}{3} \int_a^y [f'(x)]^2 \, dx \Bigg|_a^b - \frac{1}{3} \int_a^b (y-a)^3 [f'(y)]^2 \, dy \\
&= \frac{1}{3} \int_a^b \left[(b-a)^3 - (y-a)^3 \right] [f'(y)]^2 \, dy \\
&= \frac{1}{3} \int_a^b w(x) [f'(x)]^2 \, dx.
\end{aligned}
$$

Replacing I and J in (2.5), we deduce the desired result in (2.2). ∎

Returning to the Čebyšev functional $T(f,g)$, we are now able to state the following result:

Theorem 1. *For any $f, g \in L'_{2,w}[a,b]$ with w defined as in (2.1), we have:*

$$
|T(f,g)| \leq \frac{4}{3\pi^2} \cdot \frac{1}{(b-a)^2} \|f'\|_{2,w} \|g'\|_{2,w} . \tag{2.6}
$$

Proof. We know that (see for instance [3]),

$$
|T(f,g)|^2 \leq T(f,f) \, T(g,g) \tag{2.7}
$$

for any $f, g \in L_2[a,b]$.

On utilising Lemma 1,

$$
T(f,f) \leq \frac{4}{3\pi^2} \cdot \frac{1}{(b-a)^2} \|f'\|_{2,w} \tag{2.8}
$$

and

$$
T(g,g) \leq \frac{4}{3\pi^2} \cdot \frac{1}{(b-a)^2} \|g'\|_{2,w} , \tag{2.9}
$$

for any $f, g \in L'_{2,w}[a,b]$.

Making use of (2.7) – (2.9) we deduce the desired result (2.6). ∎

Open Problem. It is an open problem whether or not the constant $\frac{4}{3\pi^2}$ is the best possible in (2.2) and (2.6) respectively.

3. Numerical Experiments

If we consider $a = 0$, $b = 1$ in the above, then,

$$(0 \leq) \int_0^1 f^2(x)\,dx - \left(\int_0^1 f(x)\,dx\right)^2 \leq \frac{1}{\pi^2} \cdot \int_0^1 [f'(x)]^2\,dx \tag{3.1}$$

and

$$(0 \leq) \int_0^1 f^2(x)\,dx - \left(\int_0^1 f(x)\,dx\right)^2 \leq \frac{4}{3\pi^2} \cdot \int_0^1 \left(1 - x^3\right)[f'(x)]^2\,dx. \tag{3.2}$$

Consider now, $f_r : [0,1] \to [0,\infty)$, $f_r(x) = x^r$, $r \in (1/2, \infty)$ and define

$$I_1(r) := \int_0^1 [f_r'(x)]^2\,dx = \frac{r^2}{2r-1}$$

and

$$I_2(r) := \frac{4}{3} \cdot \int_0^1 \left(1 - x^3\right)[f_r'(x)]^2\,dx = \frac{4r^2}{(2r-1)(2r+2)}.$$

The plot of the difference $\Delta(r) := I_2(r) - I_1(r)$ on $(1/2, \infty)$ shows that on the interval $(1/2, 1]$ the bound provided by (3.1) is better than the bound provided by (3.2) for the quantity

$$\int_0^1 f^2(x)\,dx - \left(\int_0^1 f(x)\,dx\right)^2,$$

while for $r \in (1, \infty)$ the conclusion is reversed.

References

[1] J.B. DIAZ and F.T. METCALF, Variations of Wirtinger inequality, in "*Inequalities*", Ed. Oved Shisha, Avademic Press, 1967, pp. 79-103.

[2] A. LUPAŞ, The best constant in an integral inequality, *Mathematica* (Cluj), **15** (**38**) (1973), No. 2, 219-222.

[3] A.M. OSTROWSKI, On an integral inequality, *Aequationes Math.,* **4** (1970), 358-373.

School of Computer Science and Mathematics, Victoria University, PO Box 14428, Melbourne City, VIC 8001, Australia.

E-mail address: neil.barnett@vu.edu.au

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.vu.edu.au/dragomir

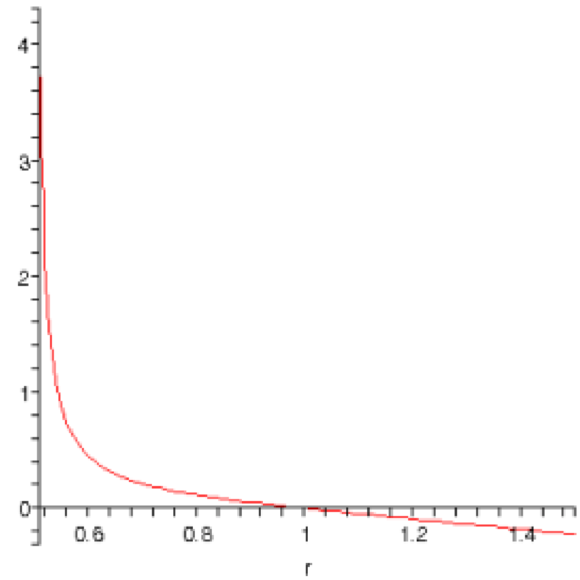

Figure 1. Plot of the difference $\Delta(r)$