MULTIDIMENSIONAL INTEGRATION VIA DIMENSION REDUCTION AND GENERATORS

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Abstract. An iterative approach is used to represent multidimensional integrals in terms of lower dimensional integrals and function evaluations. The procedure is quite general utilising one dimensional identities as the *generator* to procure multidimensional identities. Bounds are obtained from the identities. Both weighted and unweighted integrals are considered.

1. Introduction

We firstly present one-dimensional identities which may be used as *generators* for higher dimensional results.

For $f:[a,b]\to\mathbb{R}$ we define the Ostrowski and Trapezoidal functionals by

$$\mathcal{S}\left(f;c,x,d\right):=f\left(x\right)-\mathcal{M}\left(f;c,d\right) \tag{1.1}$$

and

$$T\left(f;c,x,d\right):=\left(\frac{x-c}{d-c}\right)f\left(c\right)+\left(\frac{d-x}{d-c}\right)f\left(d\right)-\mathcal{M}\left(f;c,d\right), \tag{1.2}$$

respectively, where

$$\mathcal{M}\left(f;c,d\right):=\frac{1}{d-c}\int_c^d f\left(u\right)du,\ \text{the integral mean.} \tag{1.3}$$

We note that taking $x=\frac{a+b}{2}$ and multiplication by $(b-a)$ in (1.1) and (1.2) recapture the traditional midpoint and trapezoidal rules for the evaluation of the integrals. With this in mind, the most common task is to obtain bounds on the above functionals. This task is perhaps best accomplished from identities involving the functionals. The following identities may be easily shown to hold for f of bounded variation, by an integration by parts argument of the Riemann-Stieltjes integrals and so

$$\mathcal{S}\left(f;c,x,d\right)=\int_c^d p\left(x,t,c,d\right)df\left(t\right),\quad p\left(x,t,c,d\right)=\begin{cases}\dfrac{t-c}{d-c}, & t\in[c,x]\\[2ex] \dfrac{t-d}{d-c}, & t\in(x,d]\end{cases} \tag{1.4}$$

and

$$T\left(f;c,x,d\right)=\int_c^d q\left(x,t,c,d\right)df\left(t\right),\quad q\left(x,t,c,d\right)=\frac{t-x}{d-c},\ x,t\in[c,d]. \tag{1.5}$$

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The book [13] is devoted to Ostrowski type results involving (1.1) and numerous generalisations. See also [1], [14], [17] and [19].

Further, define the three point functional $\mathfrak{T}(f;a,\alpha,x,\beta,b)$ which involves the difference between the integral mean and, a weighted combination of a function evaluated at the end points and an interior point. Namely, for $a \leq \alpha < x < \beta \leq b$,

$$\begin{aligned} (1.6) \quad & \mathfrak{T}(f;a,\alpha,x,\beta,b) \\ & := \left(\frac{\alpha-a}{b-a}\right) f(a) + \left(\frac{\beta-\alpha}{b-a}\right) f(x) + \left(\frac{b-\beta}{b-a}\right) f(b) - \mathcal{M}(f;a,b). \end{aligned}$$

Cerone and Dragomir [8] showed that for f of bounded variation, the identity

$$(1.7) \qquad \mathfrak{T}(f;a,\alpha,x,\beta,b) = \int_a^b r(x,t)\,df(t), \quad r(x,t) = \begin{cases} \dfrac{t-\alpha}{b-a}, & t \in [a,x] \\[2ex] \dfrac{t-\beta}{b-a}, & t \in (x,b] \end{cases}$$

is valid. They effectively demonstrated that the Ostrowski functional and the trapezoid functional could be recaptured as particular instances. Specifically, from (1.6) and (1.7)

$$\mathcal{S}(f;a,x,b) = \mathfrak{T}(f;a,a,x,b,b)$$

and

$$T(f;a,x,b) = \mathfrak{T}(f;a,x,x,x,b),$$

where $\mathcal{S}(f;a,x,b)$ and $T(f;a,x,b)$ are defined by (1.1) and (1.2) and satisfy identities (1.4) and (1.5) respectively.

It should be noted at this stage that

$$\begin{aligned} & (b-a)\,\mathfrak{T}\left(f;a,\frac{5a+b}{6},\frac{a+b}{2},\frac{a+5b}{6},b\right) \\ & \qquad = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \int_a^b f(x)\,dx \end{aligned}$$

is the Simpson functional.

Further, if $f(t)$ is assumed to be absolutely continuous for t over its respective interval, then $df(t) = f'(t)\,dt$ and the Riemann-Stieltjes integrals in (1.4), (1.5) and (1.7) are equivalent to Riemann integrals.

Pachpatte [20] obtains a trapezoidal type result for double integrals that involves single integrals and functional evaluation. He also investigated triple integrals, however, the techniques would be challenging to express in general dimensions.

In the current work, the Ostrowski (1.4), the generalised trapezoidal (1.5) and three point identities (1.6) – (1.7) for absolutely continuous functions are used as *generators* to produce identities involving multidimensional integrals in terms of lower dimensional integrals and function evaluations. These are used to procure bounds for $\frac{\partial^n f}{\partial t_n \dots \partial t_1} \in L_p[I^n]$, $1 \leq p \leq \infty$, where $I^n = [a_1,b_1] \times \dots \times [a_n,b_n]$. Here for $h : I^n \to \mathbb{R}$ we mean by $h \in L_p[I^n]$, the Lebesgue norms, that is,

$$(1.8) \quad \|h\|_p := \left(\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} |h(t_1,t_2,\dots,t_n)|^p \,dt_1 \dots dt_n\right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty, \text{ for } h \in L_p[I^n],$$

and

$$\|h\|_\infty := ess \sup_{\underset{\sim}{t} \in \left[\underset{\sim}{a}, \underset{\sim}{b}\right]} |h(t_1, t_2, \ldots, t_n)|, \text{ for } h \in L_\infty[I^n]. \tag{1.9}$$

In the current work, weighted *generators* are used to obtain identities involving multidimensional integrals. The identities allow *a priori* bounds on the error. Ostrowski, Trapezoid and three-point generators are utilised to procure multidimensional results. The results of Cerone [5] and [6] are recaptured if the weights are taken to be identically one.

2. Weighted Multidimensional Ostrowski Identities and Bounds from an Iterative Approach

The following theorem uses an iterative approach to extend a weighted Ostrowski functional identity to multidimensions. Firstly, we will require some notation.

Let $I^n = \prod_{i=1}^n [a_i, b_i] = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$. Further, let $f : I^n \to \mathbb{R}$ and define operators $F_i(f)$ and $\lambda_{i,w_i}(f)$ by

$$F_i(f) := f(t_1, \ldots, t_{i-1}, x_i, t_{i+1}, \ldots, t_n) \text{ where } x_i \in [a_i, b_i] \tag{2.1}$$

and

$$\lambda_{i,w_i}(f) := \frac{1}{W_i} \int_{a_i}^{b_i} w_i(t_i) f(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n) \, dt_i, \tag{2.2}$$

where $w_i(t_i)$ are positive weight functions for $t_i \in [a_i, b_i]$, $i = 1, 2, \ldots, n$ satisfying

$$W_i = \int_{a_i}^{b_i} w_i(t_i) \, dt_i > 0. \tag{2.3}$$

That is, $F_i(f)$ evaluates $f(\cdot)$ in the ith variable at $x_i \in [a_i, b_i]$ and $\lambda_{i,w_i}(f)$ is the weighted integral mean of $f(\cdot)$ in the ith variable. Assuming that $f(\cdot)$ is absolutely continuous in the ith variable $t_i \in [a_i, b_i]$, we have

$$\mathcal{L}_{i,w_i}(f) = \frac{1}{W_i} \int_{a_i}^{b_i} P_i(x_i, t_i) \frac{\partial f}{\partial t_i} dt_i = (F_i - \lambda_{i,w_i})(f), \tag{2.4}$$

for $i = 1, 2, \ldots, n$, where

$$\frac{P_i(x_i, t_i)}{W_i} = \begin{cases} \frac{\int_{a_i}^{t_i} w_i(s) ds}{W_i}, & t_i \in [a_i, x_i] \\ \\ \frac{-\int_{t_i}^{b_i} w_i(s) ds}{W_i}, & t_i \in (x_i, b_i]. \end{cases} \tag{2.5}$$

Thus (2.4) – (2.5) is ostensibly equivalent to a weighted Montgomery identity which reduces to (1.4) for $w_i(t_i) \equiv 1$ for $f(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n)$ absolutely continuous for $t_i \in [a_i, b_i]$.

Theorem 1. *Let $f : I^n \to \mathbb{R}$ be absolutely continuous in such a manner that the partial derivatives of order one with respect to every variable exist. Then*

$$
\begin{aligned}
(2.6)\qquad E_n(f) = {} & f(x_1, x_2, \ldots, x_n) \\
& - \sum_{i=1}^{n} \frac{1}{W_i} \int_{a_i}^{b_i} w_i(t_i) f(x_1, x_2, \ldots, x_{i-1}, t_i, x_{i+1}, \ldots, x_n)\, dt_i \\
& + \sum_{i<j}^{n} \frac{1}{W_i W_j} \int_{a_j}^{b_j} \int_{a_i}^{b_i} w_i(t_i) w_j(t_j) \\
& \times f(x_1, \ldots, x_{i-1}, t_i, x_{i+1}, \ldots, t_j, \ldots, x_n)\, dt_i dt_j \\
& - \cdots - \frac{(-1)^n}{W^*} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \prod_{i=1}^{n} w_i(t_i) f(t_1, \ldots, t_n)\, dt_1 \ldots dt_n \\
:= {} & \tau_n\left(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b}\right),
\end{aligned}
$$

where

$$
(2.7)\qquad E_n(f) = \frac{1}{W^*} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \prod_{i=1}^{n} P_i(x_i, t_i) \frac{\partial^n f}{\partial t_n \ldots \partial t_1} dt_1 \ldots dt_n,
$$

$$
(2.8)\qquad W^* = \prod_{i=1}^{n} W_i,
$$

with W_i given by (2.3) and $P_i(x_i, t_i)$ is given by (2.5).

Proof. Define $E_r(f)$ by

$$
(2.9)\qquad E_r(f) = \left(\prod_{i=1}^{r} \mathcal{L}_{i,w_i}\right)(f)
$$

then from the left identity in (2.4), $E_n(f)$ is as given by (2.7). Further,

$$
(2.10)\qquad E_r(f) = \mathcal{L}_{r,w_r}(E_{r-1}(f)), \quad \text{for} \quad r = 1, 2, \ldots, n
$$

where $E_0(f) = f$.

Now, from (2.9),

$$
E_1(f) = \mathcal{L}_{1,w_1}(f) = (F_1 - \lambda_{1,w_1})(f),
$$

which is the weighted Montgomery identity for $t_1, x_1 \in [a_1, b_1]$

$$
\begin{aligned}
(2.11)\qquad E_1(f) &= \frac{1}{W_1} \int_{a_1}^{b_1} P_1(x_1, t_1) \frac{\partial f(t_1, t_2, \ldots, t_n)}{\partial t_1} dt_1 \\
&= f(x_1, t_2, \ldots, t_n) - \frac{1}{W_1} \int_{a_1}^{b_1} w_1(t_1) f(t_1, t_2, \ldots, t_n)\, dt_1.
\end{aligned}
$$

Further,

$$\begin{aligned}
E_2(f) &= \mathcal{L}_{2,w_2}(E_1(f)) = (F_2 - \lambda_{2,w_2})(E_1(f)) \\
&= F_2(E_1(f)) - \lambda_{2,w_2}(E_1(f)) \\
&= f(x_1, x_2, t_3, \dots, t_n) - \frac{1}{W_1}\int_{a_1}^{b_1} w_1(t_1) f(t_1, x_2, t_3, \dots, t_n)\, dt_1 \\
&\qquad - \frac{1}{W_2}\int_{a_2}^{b_2} w_2(t_2)\Big[f(x_1, t_2, \dots, t_n) \\
&\qquad - \frac{1}{W_1}\int_{a_1}^{b_1} w_1(t_1) f(t_1, t_2, \dots, t_n)\, dt_1\Big] dt_2 \\
&= f(x_1, x_2, t_3, \dots, t_n) - \frac{1}{W_1}\int_{a_1}^{b_1} w_1(t_1) f(t_1, x_2, t_3, \dots, t_n)\, dt_1 \\
&\qquad - \frac{1}{W_2}\int_{a_2}^{b_2} w_2(t_2) f(t_1, t_2, t_3, \dots, t_n)\, dt_2 \\
&\qquad + \frac{1}{W_1 W_2}\int_{a_2}^{b_2}\int_{a_1}^{b_1} w_1(t_1) w_2(t_2) f(t_1, t_2, \dots, t_n)\, dt_1 dt_2
\end{aligned}$$

and continuing in this manner until $r = n$ gives the result as stated in (2.6). ∎

Remark 1. *The result given by (2.6) may be utilised to approximate the weighted $n-$ dimensional integral in terms of lower dimensional integrals and a function evaluation $f(x_1, x_2, \dots, x_n)$ where $x_i \in [a_i, b_i]$, $i = 1, 2, \dots, n$. Specifically, there are $\binom{n}{0}$ function evaluations, $\binom{n}{1}$ single integral evaluations in each of the axes, $\binom{n}{2}$ double integral evaluations and so on, and, of course, $\binom{n}{n}$ $n-$dimensional integral evaluations. This results from the fact that from (2.9) and (2.1) – (2.4)*

$$E_n(f) = \left(\prod_{i=1}^{n} \mathcal{L}_{i,w_i}\right)(f) = \left(\prod_{i=1}^{n} (F_i - \lambda_{i,w_i})\right)(f). \tag{2.12}$$

The above procedure of utilising a one-dimensional identity as the *generator* to recursively obtain a multidimensional identity which is quite general, may be extended to utilising other one-dimensional identities.

Theorem 2. *Let $f : I^n \to \mathbb{R}$ be absolutely continuous in a manner that the partial derivatives of order one with respect to every variable exist. Then*

$$\begin{aligned}
&W^* \left|\tau_n\left(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b}\right)\right| \\
&\le \begin{cases}
\displaystyle\prod_{i=1}^{n}\left(\int_{a_i}^{b_i} |x_i - t_i|\, w_i(t_i)\, dt_i\right)\left\|\frac{\partial^n f}{\partial t_n \dots \partial t_1}\right\|_\infty, & \dfrac{\partial^n f}{\partial t_n \dots \partial t_1} \in L_\infty[I^n]; \\[2ex]
\displaystyle\left(\prod_{i=1}^{n} P_i(q)\right)^{\frac{1}{q}}\left\|\frac{\partial^n f}{\partial t_n \dots \partial t_1}\right\|_p, & \dfrac{\partial^n f}{\partial t_n \dots \partial t_1} \in L_p[I^n], \\
& p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\
\displaystyle\prod_{i=1}^{n}\theta_i\left\|\frac{\partial^n f}{\partial t_n \dots \partial t_1}\right\|_1, & \dfrac{\partial^n f}{\partial t_n \dots \partial t_1} \in L_1[I^n],
\end{cases}
\end{aligned} \tag{2.13}$$

where $\tau_n\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right)$ *is as defined by (2.6),*

$$P_i(q)=\int_{a_i}^{x_i}\left(\int_{a_i}^{t_i}w_i(s)\,ds\right)^q dt_i+\int_{x_i}^{b_i}\left(\int_{t_i}^{b_i}w_i(s)\,ds\right)^q dt_i, \tag{2.14}$$

$$\theta_i=\frac{1}{2}\int_{a_i}^{b_i}w_i(s)\,ds+\frac{1}{2}\left|\int_{a_i}^{x_i}w_i(s)\,ds-\int_{x_i}^{b_i}w_i(s)\,ds\right|. \tag{2.15}$$

Proof. From (2.6) and (2.7) we have

$$\begin{aligned}\left|\tau_n\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right)\right| &= |E_n(f)| \\ &\le \frac{1}{W^*}\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\left|\prod_{i=1}^{n}P_i(x_i,t_i)\frac{\partial^n f}{\partial t_n\ldots\partial t_1}\right|dt_1\ldots dt_n.\end{aligned} \tag{2.16}$$

Now, for $\frac{\partial^n f}{\partial t_n\ldots\partial t_1}\in L_\infty[I^n]$, we have

$$W^*\cdot|E_n(f)|\le\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\left|\prod_{i=1}^{n}P_i(x_i,t_i)\right|dt_1\ldots dt_n\left\|\frac{\partial^n f}{\partial t_n\ldots\partial t_1}\right\|_\infty, \tag{2.17}$$

where

$$\begin{aligned}\prod_{i=1}^{n}\int_{a_i}^{b_i}|P_i(x_i,t_i)|\,dt_i &= \prod_{i=1}^{n}\left[\int_{a_i}^{x_i}\int_{a_i}^{t_i}w_i(s)\,dsdt_i+\int_{x_i}^{b_i}\int_{t_i}^{b_i}w_i(s)\,dsdt_i\right] \\ &= \prod_{i=1}^{n}\int_{a_i}^{b_i}|x_i-t_i|\,w_i(t_i)\,dt_i.\end{aligned} \tag{2.18}$$

Hence, combining (2.17) and (2.18) gives the first inequality of (2.13).

Further, using the Hölder inequality, we have for $\frac{\partial^n f}{\partial t_n\ldots\partial t_1}\in L_p[I^n]$, $1\le p<\infty$,

$$W^*\cdot|E_n(f)|\le\left(\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\left|\prod_{i=1}^{n}P_i(x_i,t_i)\right|^q dt_1\ldots dt_n\right)\left\|\frac{\partial^n f}{\partial t_n\ldots\partial t_1}\right\|_p,$$

where

$$\begin{aligned}&\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\left|\prod_{i=1}^{n}P_i(x_i,t_i)\right|^q dt_1\ldots dt_n \\ &=\prod_{i=1}^{n}\int_{a_i}^{b_i}|P_i(x_i,t_i)|^q\,dt_i \\ &=\prod_{i=1}^{n}\left[\int_{a_i}^{x_i}\left(\int_{a_i}^{t_i}w_i(s)\,ds\right)^q dt_i+\int_{x_i}^{b_i}\left(\int_{t_i}^{b_i}w_i(s)\,ds\right)^q dt_i\right]\end{aligned}$$

and so the second inequality is valid on noting (2.13).

The final inequality in (2.13) is obtained from (2.16) for $\frac{\partial^n f}{\partial t_n \cdots \partial t_1} \in L_1 [I^n]$, giving

$$\begin{aligned} W^* \cdot |E_n(f)| &\le \sup_{\underset{\sim}{t} \in \left[\underset{\sim}{a}, \underset{\sim}{b}\right]} \left| \prod_{i=1}^n P_i(x_i, t_i) \right| \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \left| \frac{\partial^n f}{\partial t_n \dots \partial t_1} \right| dt_1 \dots dt_n \\ &= \prod_{i=1}^n \sup_{t_i \in [a_i, b_i]} |P_i(x_i, t_i)| \left\| \frac{\partial^n f}{\partial t_n \dots \partial t_1} \right\|_1 \\ &= \prod_{i=1}^n \max \left\{ \int_{a_i}^{x_i} w_i(s)\, ds, \int_{x_i}^{b_i} w_i(s)\, ds \right\} \left\| \frac{\partial^n f}{\partial t_n \dots \partial t_1} \right\|_1 . \end{aligned}$$

On noting that $\max\{X, Y\} = \frac{X+Y}{2} + \frac{|X-Y|}{2}$ readily produces the stated result. ∎

Remark 2. *The expression for* $\tau_n \left(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b}\right)$ *may be written in a less explicit form which is perhaps more appealing. Namely,*

$$\tau_n \left(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b}\right) = f(x_1, x_2, \dots, x_n) + \sum_{k=1}^{n-1} (-1)^k \sum_k \mathcal{M}_k + (-1)^n \mathcal{M}_n, \tag{2.19}$$

where $\mathcal{M}_k$ *represents the integral means in* k *variables with the remainder being evaluated at their respective interior point and* $\sum_k \mathcal{M}_k$ *is a sum over all* $\binom{n}{k}$, $k-$*dimensional integral means. Here*

$$\mathcal{M}_n = \frac{1}{W^*} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \prod_{i=1}^n w_i(t_i) f(t_1, \dots, t_n)\, dt_1 \dots dt_n$$

and

$$\begin{aligned} \sum_1 \mathcal{M}_1 = \frac{1}{W_1} &\int_{a_1}^{b_1} w_1(t_1) f(t_1, x_2, \dots, x_n)\, dt_1 \\ &+ \frac{1}{W_2} \int_{a_2}^{b_2} w_2(t_2) f(x_1, t_2, x_3, \dots, x_n)\, dt_2 \\ &\qquad + \cdots + \frac{1}{W_n} \int_{a_n}^{b_n} w_n(t_n) f(x_1, x_2, \dots, x_{n-1}, t_n)\, dt_n. \end{aligned}$$

It should be noted that (2.19) may be written as

$$\tau_n \left(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b}\right) = \sum_{k=0}^{n} (-1)^k \sum_k \mathcal{M}_k \tag{2.20}$$

if we define the degenerate 0th integral mean $\mathcal{M}_0 = f(x_1, x_2, \dots, x_n)$.

Remark 3. *The following result was developed by Cerone* [5] *using an iterative approach from an unweighted Montgomery identity as* ***generator.***

The following theorem bounds for $\tau_n\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right)$ *were obtained where*

$$\tau_n\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right) \tag{2.21}$$
$$= f(x_1,x_2,\ldots,x_n) - \sum_{i=1}^{n}\frac{1}{d_i}\int_{a_i}^{b_i} f(x_1,x_2,\ldots,x_{i-1},t_i,x_{i+1},\ldots,x_n)\,dt_i$$
$$+\sum_{i<j}^{n}\frac{1}{d_jd_i}\int_{a_j}^{b_j}\int_{a_i}^{b_i} f(x_1,\ldots,x_{i-1},t_i,x_{i+1},\ldots,x_{j-1},t_j,x_{j+1},\ldots,x_n)\,dt_idt_j$$
$$-\cdots\cdots\cdots-\frac{(-1)^n}{D_n}\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1} f(t_1,\ldots,t_n)\,dt_1\ldots dt_n$$

where $D_n=\prod_{i=1}^{n}d_i,\;\; d_i=b_i-a_i$ *and* $\underset{\sim}{z}=(z_1,z_2,\ldots,z_n)$.

Theorem 3. *Let* $f:I^n\to\mathbb{R}$ *be absolutely continuous in a manner that the partial derivatives of order one with respect to every variable exist. Then*

$$D_n\cdot\left|\tau_n\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right)\right| \tag{2.22}$$
$$\leq\begin{cases}\prod_{i=1}^{n}P_i(1)\left\|\frac{\partial^n f}{\partial t_n\ldots\partial t_1}\right\|_\infty, & \frac{\partial^n f}{\partial t_n\ldots\partial t_1}\in L_\infty[I^n];\\[2ex] \left(\prod_{i=1}^{n}P_i(q)\right)^{\frac{1}{q}}\left\|\frac{\partial^n f}{\partial t_n\ldots\partial t_1}\right\|_p, & \frac{\partial^n f}{\partial t_n\ldots\partial t_1}\in L_p[I^n],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \prod_{i=1}^{n}\theta_i\left\|\frac{\partial^n f}{\partial t_n\ldots\partial t_1}\right\|_1, & \frac{\partial^n f}{\partial t_n\ldots\partial t_1}\in L_1[I^n],\end{cases}$$

where $\tau_n\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right)$ *is as defined by (2.21)*

$$(q+1)P_i(q)=(x_i-a_i)^{q+1}+(b_i-x_i)^{q+1}, \tag{2.23}$$

$$\theta_i=\frac{b_i-a_i}{2}+\left|x_i-\frac{a_i+b_i}{2}\right|. \tag{2.24}$$

We note that, not suprisingly, more explicit expressions for the bounds are possible in Theorem 3 (for $w_i(t_i)=1,\ i=1,2,\ldots,n$) than those of Theorem 2.

Further, the tightest bounds from Theorem 3 occur when we choose to sample at the mid-points of the respective intervals, namely, for $x_i=\frac{a_i+b_i}{2}$. This is not the case for the weighted results depicted in Theorem 2. If we let

$$m(c,d)=\int_c^d w(s)\,ds \quad\text{and}\quad M(c,d)=\int_c^d s w(s)\,ds.$$

then we observe that, for example,

$$\int_a^b|x-t|\,w(t)\,dt=\int_a^x(x-t)\,w(t)\,dt+\int_t^b(t-x)\,w(t)\,dt$$
$$=x\left[m(a,x)-m(x,b)\right]+M(x,b)-M(a,x).$$

Thus we see that the bound is simplified, although not necessarily globally minimised, at the median $x=x^*$, where $m(a,x^*)=m(x^*,b)$.

3. Multidimensional Trapezoidal Identities and Bounds

The work of Section 2 used the weighted Ostrowski functional, which satisfies a weighted Montgomery identity, as a *generator* for extension to higher dimensions. We may prove, in an equivalent manner, utilising the generalised weighted trapezoidal identity as the *generator* of a higher dimensional result. We will restrict the current work to absolutely continuous functions so that the Riemann integral identity will be used. Let $f : I^n \to \mathbb{R}$ and define the operator

$$G_i(f) := \frac{A_i}{W_i} f(t_1, \ldots, t_{i-1}, a_i, t_{i+1}, \ldots, t_n) + \frac{B_i}{W_i} f(t_1, \ldots, t_{i-1}, b_i, t_{i+1}, \ldots, t_n), \tag{3.1}$$

where

$$A_i = \int_{a_i}^{x_i} w_i(s)\, ds, \quad B_i = \int_{x_i}^{b_i} w_i(s)\, ds, \quad W_i = A_i + B_i. \tag{3.2}$$

Here $W_i G_i(f)$ represents the generalised weighted trapezoid in the i^{th} variable giving the standard trapezoid when $x_i = \frac{a_i+b_i}{2}$.

Now, for $f(\cdot)$ absolutely continuous in the i^{th} variable $t_i \in [a_i, b_i]$ we have

$$\mathfrak{M}_{i,w_i}(f) = \frac{1}{W_i} \int_{a_i}^{b_i} Q_i(x_i, t_i) \frac{\partial f}{\partial t_i} dt_i = (G_i - \lambda_{i,w_i})(f), \quad i = 1, 2, \ldots, n, \tag{3.3}$$

where $G_i(f)$ and $\lambda_{i,w_i}(f)$ are as given by (3.1) – (3.2) and (2.2) respectively and,

$$\frac{Q_i(x_i, t_i)}{W_i} = \frac{\int_{x_i}^{t_i} w_i(s)\, ds}{\int_{a_i}^{b_i} w_i(s)\, ds}, \quad x_i, t_i \in [a_i, b_i]. \tag{3.4}$$

Let $c^{(0)} = (c_1, c_2, \ldots, c_n)$, where $c_i = a_i$ or b_i in the i^{th} position for $i = 1, 2, \ldots, n$. Also, let $\sigma_0\left(c^{(0)}\right)$ be the set of all such vectors which consists of 2^n possibilities. Further, let

$$\chi_k = \prod_{j=1}^{n}{}^{(k)} \frac{C_j}{W_j}, \quad k = 0, 1, \ldots, n, \tag{3.5}$$

where $C_j = A_j$ or B_j with the exception that k of the $C_j = W_j$ and so $\frac{C_j}{W_j} = 1$.

In a similar fashion, let $c^{(k)}$ be a vector taking on the fixed values a_i or b_i in the i^{th} position except for k of the positions which are variable, $t_\bullet$. Let $\mathcal{M}_k$ be $k-$dimensional weighted integral means for $f\left(c^{(k)}\right)$. Here $c^{(k)} \in \sigma_k\left(c^{(k)}\right)$ the set of all such elements, of which there are $\binom{n}{k} 2^{n-k}$.

With the above notation in place, the following theorem holds.

Theorem 4. *Let $f : I^n \to \mathbb{R}$ be absolutely continuous and be such that all partial derivatives of order one in each of the variables exist. Then*

$$\begin{aligned} R_n(f) &= \sum\nolimits_0 \chi_0 f\left(c^{(0)}\right) - \sum\nolimits_1 \chi_1 \mathcal{M}_1 + \sum\nolimits_2 \chi_2 \mathcal{M}_2 \\ &\qquad - \cdots - (-1)^{n-1} \sum\nolimits_{n-1} \chi_{n-1} \mathcal{M}_{n-1} + (-1)^n \mathcal{M}_n \\ &:= \rho_n\left(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b}\right), \end{aligned} \tag{3.6}$$

where, χ_k is as defined in (3.5), $\mathcal{M}_k$ is the $k-$dimensional integral mean for $f\left(c^{(k)}\right)$, specifically,

$$\mathcal{M}_n = \frac{1}{W^*} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \prod_{i=1}^{n} w_i\left(t_i\right) f\left(t_1, t_2, \ldots, t_n\right) dt_1 \ldots dt_n$$

and $\sum_k$ is a sum involving each of the elements of $\sigma_k\left(c^{(k)}\right)$ of which there are $\binom{n}{k} 2^{n-k}$ terms.

Further,

$$R_n\left(f\right) = \frac{1}{W^*} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \prod_{i=1}^{n} Q_i\left(x_i, t_i\right) \frac{\partial^n f}{\partial t_n \cdots \partial t_1} dt_1 \cdots dt_n, \tag{3.7}$$

and $Q_i\left(x_i, t_i\right)$ is given by (3.4), W^* by (2.8). Here, c_i is equal to either a_i or b_i in which case $C_i = A_i$ or B_i.

Proof. Let $R_r\left(f\right)$ be defined by

$$R_r\left(f\right) := \left(\prod_{i=1}^{r} \mathfrak{M}_{i,w_i}\right)\left(f\right), \tag{3.8}$$

then from the left identity in (3.3), $R_n\left(f\right)$ is as given by (3.7). Now,

$$R_r\left(f\right) = \mathfrak{M}_{r,w_r}\left(R_{r-1}\left(f\right)\right), \text{ for } r = 1, 2, \ldots, n, \tag{3.9}$$

where $R_0\left(f\right) = f$.

Thus, from (3.8)

$$R_1\left(f\right) = \mathfrak{M}_{1,w_1}\left(f\right) = \left(G_1 - \lambda_{1,w_1}\right)\left(f\right),$$

which is the generalised trapezoidal identity for $t_1, x_1 \in [a_1, b_1]$

$$\begin{aligned} R_1\left(f\right) &= \frac{1}{W_1} \int_{a_1}^{b_1} Q_1\left(x_1, t_1\right) \frac{\partial f}{\partial t_1}\left(t_1, t_2, \ldots, t_n\right) dt_1 \\ &= \frac{A_1}{W_1} f\left(a_1, t_2, \ldots, t_n\right) + \frac{B_1}{W_1} f\left(b_1, t_2, \ldots, t_n\right) \\ &\quad - \frac{1}{W_1} \int_{a_1}^{b_1} w_1\left(t_1\right) f\left(t_1, t_2, \ldots, t_n\right) dt_1 \end{aligned} \tag{3.10}$$

contains three entities; two function evaluations and one integral. Further,

$$\begin{aligned} R_2\left(f\right) &= \mathfrak{M}_{2,w_2}\left(R_1\left(f\right)\right) = \left(G_2 - \lambda_{2,w_2}\right)\left(R_1\left(f\right)\right) \\ &= G_2\left(R_1\left(f\right)\right) - \lambda_{2,w_2}\left(R_1\left(f\right)\right) \\ &= \frac{A_2}{W_2} R_1\left(f\right)\Big|_{t_2=a_2} + \frac{B_2}{W_2} R_1\left(f\right)\Big|_{t_2=b_2} - \frac{1}{W_2} \int_{a_2}^{b_2} w_2\left(t_2\right) R_1\left(f\right) dt_2 \end{aligned}$$

contains nine entities. Thus,

$$
\begin{aligned}
(3.11)\quad R_2(f) = {} & \frac{A_2}{W_2}\left\{\frac{A_1}{W_1} f(a_1, a_2, t_3, \dots, t_n) + \frac{B_1}{W_1} f(b_1, a_2, t_3, \dots, t_n)\right. \\
& \left. - \frac{1}{W_1}\int_{a_1}^{b_1} w_1(t_1) f(t_1, a_2, t_3, \dots, t_n)\, dt_1\right\} \\
& + \frac{B_2}{W_2}\left\{\frac{A_1}{W_1} f(a_1, b_2, t_3, \dots, t_n) + \frac{B_1}{W_1} f(b_1, b_2, t_3, \dots, t_n)\right. \\
& \left. - \frac{1}{W_1}\int_{a_1}^{b_1} w_1(t_1) f(t_1, b_2, t_3, \dots, t_n)\, dt_1\right\} \\
& - \frac{1}{W_2}\int_{a_2}^{b_2} w_2(t_2)\left\{\frac{A_1}{W_1} f(a_1, t_2, \dots, t_n) + \frac{B_1}{W_1} f(b_1, t_2, t_3, \dots, t_n)\right. \\
& \left. - \frac{1}{W_1}\int_{a_1}^{b_1} w_1(t_1) f(t_1, t_2, \dots, t_n)\, dt_1\right\} dt_2 \\
= {} & \frac{A_2}{W_2}\frac{A_1}{W_1} f(a_1, a_2, t_3, \dots, t_n) + \frac{A_2}{W_2}\frac{B_1}{W_1} f(b_1, a_2, t_3, \dots, t_n) \\
& + \frac{B_2}{W_2}\frac{A_1}{W_1} f(a_1, b_2, t_3, \dots, t_n) + \frac{B_2}{W_2}\frac{B_1}{W_1} f(b_1, b_2, t_3, \dots, t_n) \\
& - \frac{A_2}{W_2}\cdot\frac{1}{W_1}\int_{a_1}^{b_1} w_1(t_1) f(t_1, a_2, t_3, \dots, t_n)\, dt_1 \\
& - \frac{B_2}{W_2}\cdot\frac{1}{W_1}\int_{a_1}^{b_1} w_1(t_1) f(t_1, b_2, t_3, \dots, t_n)\, dt_1 \\
& - \frac{A_1}{W_1}\cdot\frac{1}{W_2}\int_{a_2}^{b_2} w_2(t_2) f(a_1, t_2, \dots, t_n)\, dt_2 \\
& - \frac{B_1}{W_1}\cdot\frac{1}{W_2}\int_{a_2}^{b_2} w_2(t_2) f(b_1, t_2, t_3, \dots, t_n)\, dt_2 \\
& + \frac{1}{W_2 W_1}\int_{a_2}^{b_2}\int_{a_1}^{b_1} w_1(t_1) w_2(t_2) f(t_1, t_2, \dots, t_n)\, dt_1 dt_2.
\end{aligned}
$$

From the 3^2 entities of $R_2(f)$ there are 2^2 function evaluations, 2×2 single integrals and one double integral

$$
\begin{aligned}
R_3(f) &= \mathfrak{M}_{3,w_3}(R_2(f)) = (G_3 - \lambda_{3,w_3})(R_2(f)) = G_3(R_2(f)) - \lambda_{3,w_3}(R_2(f)) \\
&= \frac{A_3}{W_3} R_2(f)\Big|_{t_3=a_3} + \frac{B_3}{W_3} R_2(f)\Big|_{t_3=b_3} - \frac{1}{W_3}\int_{a_3}^{b_3} w_3(f_3) R_2(f)\, dt_3.
\end{aligned}
$$

This will produce 3^2 entities with $\binom{3}{0} 2^3$ function evaluations, $\binom{3}{1} 2^2$ single integrals, $\binom{3}{2} 2^1$ double integrals and $\binom{3}{0} 2^0$ triple integrals. The 2 occurs since evaluation is at either the a_i or the b_i.

Continuing in this manner we obtain the result as stated where there are 3^n entities for $R_n(f)$, with $\binom{n}{0} 2^n$ function evaluations only, $\binom{n}{1} 2^{n-1}$ single integrals, $\binom{n}{2} 2^{n-2}$ double integrals, ... , $\binom{n}{n-1} 2$, $(n-1)^{\text{th}}$ integrals and one n-dimensional integral. ∎

The following theorem gives bounds for $\rho_n\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right)$, as defined in (3.6) for $\frac{\partial^n f}{\partial t_n \ldots \partial t_1} \in L_p[I^n]$, $1 \le p \le \infty$ with the usual the Lebesgue norms. Following the proof of Theorem 2 closely, it may be shown that

Theorem 5. *Let $f : I^n \to \mathbb{R}$ be absolutely continuous in a manner such that all partial derivatives of order one with respect to every variable exists. Then*

$$W^* \cdot \left|\rho_n\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right)\right| \le \begin{cases} \prod_{i=1}^{n}\left(\int_{a_i}^{b_i}|x_i - t_i|\, w_i(t_i)\,dt_i\right)\left\|\frac{\partial^n f}{\partial t_n \cdots \partial t_1}\right\|_\infty, & \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \in L_\infty[I^n]; \\ \left(\prod_{i=1}^{n} Q_i(q)\right)^{\frac{1}{q}}\left\|\frac{\partial^n f}{\partial t_n \cdots \partial t_1}\right\|_p, & \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \in L_p[I^n], \\ & p > 1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \prod_{i=1}^{n}\phi_i\left\|\frac{\partial^n f}{\partial t_n \cdots \partial t_1}\right\|_1, & \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \in L_1[I^n]; \end{cases} \tag{3.12}$$

where $\rho_n\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right)$ is as defined by (2.13),

$$Q_i(q) = \prod_{i=1}^{n}\left[\int_{a_i}^{x_i}\left(\int_{a_i}^{t_i} w_i(t_i)\,dt_i\right)^q dt_i + \int_{x_i}^{b_i}\left(\int_{t_i}^{b_i} w_i(t_i)\,dt_i\right)^q dt_i\right] \tag{3.13}$$

$$\theta_i = \frac{1}{2}\int_{a_i}^{b_i} w_i(s)\,ds + \frac{1}{2}\left|\int_{a_i}^{x_i} w_i(s)\,ds - \int_{x_i}^{b_i} w_i(s)\,ds\right|. \tag{3.14}$$

Remark 4. *The bound for the unweighted result for $\left|\rho_n\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right)\right|$ may be shown to be in agreement with that for $\left|\tau_n\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right)\right|$ given in Theorem 3. It was shown in Cerone* [7] *that the bounds in terms of the Lebesgue norms on $|S(f;c,x,d)|$ and $|T(f;c,x,d)|$, as defined in (1.1) and (1.2), are the same. This result carries across to weighted functionals which when used as **generators** would produce the same bounds since the kernels $P_i(x_i,t_i)$ from (2.5) and $Q_i(x_i,t_i)$ from (3.5) produce the same norms.*

4. Three Point Identities and their Bounds

It may be noticed from (1.4) – (1.5) that the three point functional can be expressed in terms of a combination of Trapezoidal functionals, see [12] and [14] for treatment of three point weighted quadrature rules.

In particular, for $f(\cdot)$ absolutely continuous and using the results of Section 3 for weighted trapezoidal functionals as generators, we have the identity

$$\begin{aligned} \Psi_n\left(\underset{\sim}{a},\underset{\sim}{\alpha},\underset{\sim}{x},\underset{\sim}{\beta},\underset{\sim}{b}\right) &:= \rho_n\left(\underset{\sim}{a},\underset{\sim}{\alpha},\underset{\sim}{x}\right) + \rho_n\left(\underset{\sim}{x},\underset{\sim}{\beta},\underset{\sim}{b}\right) \\ &= \int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\prod_{i=1}^{n} r_i(x_i,t_i)\,\frac{\partial^n f}{\partial t_n \cdots \partial t_1}\,dt_1 \cdots dt_n, \end{aligned} \tag{4.1}$$

where

$$\frac{r_i(x_i,t_i)}{W_i}=\begin{cases}\dfrac{\int_{\alpha_i}^{t_i} w_i(t_i)\,dt_i}{W_i}, & t_i\in[a_i,x_i]\\[2ex] \dfrac{\int_{\beta_i}^{t_i} w_i(t_i)\,dt_i}{W_i}, & t_i\in(x_i,b_i]\end{cases}. \tag{4.2}$$

It is important to obtain an identity for the three point rule since the bounds are tighter than using the bounds of the two trapezoidal rules as this would entail using the triangle inequality. We notice that $\Psi_n\left(\underset{\sim}{a},\underset{\sim}{\alpha},\underset{\sim}{x},\underset{\sim}{\beta},\underset{\sim}{b}\right)$ in (4.1) is not expressed explicitly. This may be accomplished by returning to (3.6) or else we may use the *generator* methodology utilised to obtain the results in Section 2 and 3.

Let $f:I^n\to\mathbb{R}$ and define the operator

$$\begin{aligned}H_i(f):=\frac{\nu_i^{(a)}}{W_i}f(t_1,\dots,t_{i-1},a_i,t_{i+1},\dots,t_n)&\\ +\frac{\nu_i^{(x)}}{W_i}f(t_1,\dots,t_{i-1},x_i,t_{i+1},\dots,t_n)&\\ +\frac{\nu_i^{(b)}}{W_i}f(t_1,\dots,t_{i-1},b_i,t_{i+1},\dots,t_n)&,\end{aligned}\tag{4.3}$$

where,

$$\begin{aligned}\nu_i^{(a)}&=\int_{a_i}^{\alpha_i}w_i(t_i)\,dt_i,\quad \nu_i^{(x)}=\int_{\alpha_i}^{\beta_i}w_i(t_i)\,dt_i,\\ \nu_i^{(b)}&=\int_{\beta_i}^{b_i}w_i(t_i)\,dt_i,\quad W_i=\int_{a_i}^{b_i}w_i(t_i)\,dt_i.\end{aligned}\tag{4.4}$$

Then, for $f(\cdot)$ absolutely continuous in the i^{th} variable $t_i\in[a_i,b_i]$ we have

$$\mathfrak{N}_i(t)=\frac{1}{W_i}\int_{a_i}^{b_i}r_i(x_i,t_i)\frac{\partial f}{\partial t_i}dt_i=(H_i-\lambda_{i,w_i})(f),\quad i=1,2,\dots,n. \tag{4.5}$$

where $H_i(f)$ and $\lambda_{i,w_i}(f)$ are as given by (4.3) – (4.4) and (2.2) respectively.

If we now follow the work of the previous section and let $c^{(0)}=(c_1,c_2,\dots,c_n)$, where now $c_i=a_i,x_i$ or b_i in the i^{th} partition for $i=1,2,\dots,n$. Then $\sigma_0\left(c^{(0)}\right)$ which is the set of all such vectors consists of 3^n possibilities. Further, let χ_n be as in (3.5) where now, $C_j=\nu_j^{(a)}$ or $\nu_j^{(x)}$ or $\nu_j^{(b)}$ are as defined by (4.4) with the exception that k of the $C_j=W_j$ and so $\frac{C_j}{W_j}=1$.

Further, $c^{(k)}$ is a vector taking on fixed values of either a_i,x_i or b_i in the i^{th} position except for k of the positions which are variable, $t_\bullet$. Let $\mathcal{M}_k$ be a $k-$dimensional weighted integral means for $f\left(c^{(k)}\right)$, then the following theorem holds.

Theorem 6. *Let $f : I^n \to \mathbb{R}$ be absolutely continuous and be such that all partial derivatives of order one in each of the variables exist. Then*

$$\begin{aligned} B_n(f) &= \sum\nolimits_0 \chi_0 f\left(c^{(0)}\right) - \sum\nolimits_1 \chi_1 \mathcal{M}_1 + \sum\nolimits_2 \chi_2 \mathcal{M}_2 \\ &\quad - \cdots - (-1)^{n-1} \sum\nolimits_{n-1} \chi_{n-1} \mathcal{M}_{n-1} + (-1)^n \mathcal{M}_n \\ &:= \Psi_n\left(\underset{\sim}{a}, \underset{\sim}{\alpha}, \underset{\sim}{x}, \underset{\sim}{\beta}, \underset{\sim}{b}\right), \end{aligned} \tag{4.6}$$

where

$$\chi_k = \prod_{j=1}^{n}{}^{(k)} \frac{C_j}{W_j}, \quad k = 0, 1, \ldots, n, \tag{4.7}$$

with $C_j = \nu_j^{(a)}$, $\nu_j^{(x)}$ or $\nu_j^{(b)}$ as defined in (4.4) except for k of the $C_j = W_j$ giving $\frac{C_j}{W_j} = 1$, $\mathcal{M}_k$ is the $k-$dimensional weighted integral mean of $f\left(c^{(k)}\right)$ and specifically,

$$\mathcal{M}_n = \frac{1}{W^*} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \prod_{i=1}^{n} w_i(t_i) f\left(c^{(n)}\right) dt_1 \ldots dt_n, \; c^{(n)} = (t_1, t_2, \ldots, t_n). \tag{4.8}$$

Finally, $\sum_k$ is a sum over all $\binom{n}{k} 3^{n-k}$ terms and $B_n(f)$ is as defined in (4.1).

The following theorem gives bounds for the $\Psi_n\left(\underset{\sim}{a}, \underset{\sim}{\alpha}, \underset{\sim}{x}, \underset{\sim}{\beta}, \underset{\sim}{b}\right)$ as given in either (4.6) or (4.1).

Theorem 7. *Let the conditions of Theorem 6 continue to hold. Then,*

$$\begin{aligned} & W^* \left|\Psi_n\left(\underset{\sim}{a}, \underset{\sim}{\alpha}, \underset{\sim}{x}, \underset{\sim}{\beta}, \underset{\sim}{b}\right)\right| \\ & \leq \begin{cases} \prod\limits_{i=1}^{n} S_i(1) \left\|\frac{\partial^n f}{\partial t_n \cdots \partial t_1}\right\|_\infty, & \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \in L_\infty[I^n]; \\ \left(\prod\limits_{i=1}^{n} S_i(q)\right)^{\frac{1}{q}} \left\|\frac{\partial^n f}{\partial t_n \cdots \partial t_1}\right\|_p, & \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \in L_\infty[I^n], \\ & p > 1, \; \frac{1}{p} + \frac{1}{q} = 1 \\ \prod\limits_{i=1}^{n} \zeta_i \left\|\frac{\partial^n f}{\partial t_n \cdots \partial t_1}\right\|_1, & \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \in L_1[I^n]; \end{cases} \end{aligned} \tag{4.9}$$

where $\|h\|_p$, $1 \leq p < \infty$ and $\|h\|_\infty$ are defined by (1.8) and (1.9), $\Psi_n\left(\underset{\sim}{a}, \underset{\sim}{\alpha}, \underset{\sim}{x}, \underset{\sim}{\beta}, \underset{\sim}{b}\right)$ is defined by (4.1) or, explicitly, by (4.6),

$$S_i(q) = \int_{a_i}^{b_i} |r_i(x_i, t_i)|^q \, dt_i \quad \text{and} \quad \zeta_i = \sup_{t_i \in [a_i, b_i]} |r_i(x_i, t_i)|. \tag{4.10}$$

Proof. (Sketch) From (4.1) – (4.2) and (4.6) we have

$$\begin{aligned} \Psi_n\left(\underset{\sim}{a}, \underset{\sim}{\alpha}, \underset{\sim}{x}, \underset{\sim}{\beta}, \underset{\sim}{b}\right) &= |B_n(f)| \\ &\leq \frac{1}{W^*} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \left|\prod_{i=1}^{n} r_i(x_i, t_i) \frac{\partial^n f}{\partial t_n \ldots \partial t_1}\right| dt_1 \ldots dt_n. \end{aligned} \tag{4.11}$$

Now, for $\frac{\partial^n f}{\partial t_n \dots \partial t_1} \in L_\infty [I^n]$, then

$$W^* |B_n(f)| \le \left\| \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \right\|_\infty \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \left| \prod_{i=1}^n r_i(x_i, t_i) \right| dt_1 \dots dt_n, \tag{4.12}$$

where $\|h\|_\infty$ is defined by (1.9).

Moreover, using the Hölder inequality for multiple integrals, we have, from (4.11) for $\frac{\partial^n f}{\partial t_n \dots \partial t_1} \in L_p [I^n]$, $1 < p < \infty$,

$$W^* |B_n(f)| \le \left\| \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \right\|_p \left(\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \left| \prod_{i=1}^n r_i(x_i, t_i) \right|^q dt_1 \dots dt_n \right)^{\frac{1}{q}}, \tag{4.13}$$

where

$$\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \left| \prod_{i=1}^n r_i(x_i, t_i) \right|^q dt_1 \dots dt_n = \prod_{i=1}^n \int_{a_i}^{b_i} |r_i(x_i, t_i)|^q \, dt_i.$$

Finally, for $\frac{\partial^n f}{\partial t_n \dots \partial t_1} \in L_1 [I^n]$, we have from (4.11)

$$\begin{aligned} W^* |B_n(f)| &\le \sup_{\underset{\sim}{t} \in \left[\underset{\sim}{a}, \underset{\sim}{b}\right]} \left| \prod_{i=1}^n r_i(x_i, t_i) \right| \left\| \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \right\|_1 \\ &= \prod_{i=1}^n \sup_{t_i \in [a_i, b_i]} |r_i(x_i, t_i)| \left\| \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \right\|_1, \end{aligned}$$

giving the third bound. ∎

Remark 5. *The bounds obtained above in Theorem 7 are the product of the bounds for the one dimensional integral results. These were studied extensively in Cerone and Dragomir* [9]*. It should further be noted that the three point results of the current section recaptures the generalised trapezoidal results of the previous section if we take $\alpha_i = \beta_i = x_i$. In addition, the Ostrowski type results of Cerone* [5] *are recaptured if we take $\alpha_i = a_i$ and $\beta_i = b_i$ and the weight functions $w_i(t_i) \equiv 1$, $i = 1, \dots, n$. The results of Cerone* [6] *would be recaptured if $w_i(t_i) \equiv 1$ from Theorem 6 and the results in (4.10) may be simplified further.*

5. Concluding Remarks

Weighted rules of Ostrowski, Trapezoidal and Three-point type have been investigated in the current work as generators for multidimensional integration. This results in product form weight functions in the multidimensional integral. The procedure developed in [5] and [6] may also be used to include higher order formulae involving the behaviour of higher derivatives for its bounds. Multidimensional results based on an m branched Peano kernel producing function evaluations at $m+1$ points are also possible using the methodology. Finally, we are not restricted to using the same identity in each of the directions but may use different ones as long as we are able to justify this.

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