

A SHARP BOUND OF THE ČEBYŠEV FUNCTIONAL FOR THE RIEMANN-STIELTJES INTEGRAL AND APPLICATIONS

S.S. DRAGOMIR

ABSTRACT. A new sharp bound of the Čebyšev functional for the Riemann-Stieltjes integral is obtained. Applications for quadrature rules including the trapezoid and mid-point rule are given.

1. INTRODUCTION

In order to generalise the classical Čebyšev functional, namely,

$$T(f, g) := \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx,$$

where f, g, fg are integrable on $[a, b]$, which has been extensively studied in the literature, see for instance the book [6], the author has introduced in [3] the following functional for Riemann-Stieltjes integrals:

$$(1.1) \quad T(f, g; u) := \frac{1}{u(b) - u(a)} \int_a^b f(t) g(t) du(t) - \frac{1}{u(b) - u(a)} \int_a^b f(t) du(t) \cdot \frac{1}{u(b) - u(a)} \int_a^b g(t) du(t),$$

provided the involved integrals exist and $u(b) \neq u(a)$.

It has been shown in [3] that

$$(1.2) \quad |T(f, g; u)| \leq \frac{1}{2} (M - m) \cdot \frac{1}{|u(b) - u(a)|} \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_{\infty} \bigvee_a^b(u),$$

provided that f, g are continuous, $m \leq f(t) \leq M$ for each $t \in [a, b]$ and u is of bounded variation on $[a, b]$ with the total variation $\bigvee_a^b(u)$. The constant $\frac{1}{2}$ is sharp in (1.2) in the sense that it cannot be replaced by a smaller quantity.

In the case that u is monotonic nondecreasing then also [3]

$$(1.3) \quad |T(f, g; u)| \leq \frac{1}{2} (M - m) \cdot \frac{1}{|u(b) - u(a)|} \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| du(t),$$

for which the constant $\frac{1}{2}$ is best possible.

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Finally, in the case where u is Lipschitzian with the constant L , and in this case we can have f and g Riemann integrable on $[a, b]$, the following result has been obtained as well [3]

$$(1.4) \quad |T(f, g; u)| \leq \frac{1}{2}L(M - m) \cdot \frac{1}{|u(b) - u(a)|} \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| dt.$$

Here $\frac{1}{2}$ is also sharp.

For other results, see [4] and [5].

The aim of the present paper is to establish a new sharp bound for the absolute value of the Čebyšev functional (1.1). Applications for the trapezoid and mid-point inequality are pointed out. A general perturbed quadrature rule and error estimates are obtained as well.

2. THE RESULTS

The following result concerning a sharp bound for the absolute value of the Čebyšev functional $T(f, g; h)$ can be stated.

Theorem 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a function of bounded variation and $g, h : [a, b] \rightarrow \mathbb{R}$ are bounded functions with $h(a) \neq h(b)$ such that the Stieltjes integrals $\int_a^b f(t)g(t)dh(t)$ and $\int_a^b g(t)dh(t)$ exist. Then:*

$$(2.1) \quad |T(f, g; h)| \leq \frac{1}{|h(b) - h(a)|} \bigvee_a^b(f) \sup_{x \in [a, b]} \left| \int_a^x g(t) dh(t) - \frac{h(x) - h(a)}{h(b) - h(a)} \int_a^b g(s) dh(s) \right|.$$

The constant $C = 1$ in the right hand side of (2.1) cannot be replaced by a smaller quantity.

Proof. We use the following result for the Riemann-Stieltjes integral obtained in [6, p. 337].

Let $u, v, w : [a, b] \rightarrow \mathbb{R}$ such that u is of bounded variation on $[a, b]$ and v, w are bounded functions with the property that the Riemann-Stieltjes integrals $\int_a^b v(t)dw(t)$ and $\int_a^b u(t)v(t)dw(t)$ exist. Then

$$(2.2) \quad \left| \int_a^b u(t)v(t)dw(t) \right| \leq \left[|u(b)| + \bigvee_a^b(u) \right] \sup_{x \in [a, b]} \left| \int_a^x v(t)dw(t) \right|.$$

We also use the representation (see also [3]):

$$(2.3) \quad T(f, g; h) = \frac{1}{h(b) - h(a)} \int_a^b [f(t) - \gamma] \times \left[g(t) - \frac{1}{h(b) - h(a)} \int_a^b g(s) dh(s) \right] dh(t),$$

which holds for any $\gamma \in \mathbb{R}$.

Now, if we choose $\gamma = f(b)$, $u(t) = f(t) - f(b)$,

$$v(t) = g(t) - \frac{1}{h(b) - h(a)} \int_a^b g(s) dh(s)$$

and $w(t) = h(t)$, $t \in [a, b]$, then we get

$$\begin{aligned} & |[h(b) - h(a)] T(f, g; h)| \\ & \leq \bigvee_a^b(f) \sup_{x \in [a, b]} \left| \int_a^x g(t) dh(t) - \frac{h(x) - h(a)}{h(b) - h(a)} \int_a^b g(s) dh(s) \right| \end{aligned}$$

and the inequality (2.1) is proved.

For the sharpness of the inequality, assume that $h(t) = t$ and $g(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, $t \in [a, b]$. Then (2.1) becomes

$$(2.4) \quad \left| \int_a^b f(t) \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt \right| \leq \bigvee_a^b(f) \sup_{x \in [a, b]} \left| \int_a^x \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt \right|,$$

provided f is of bounded variation on $[a, b]$.

Notice that, if we consider $\lambda(x)$ defined by

$$\lambda(x) := \int_a^x \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt = \begin{cases} a - x & \text{if } x \in [a, \frac{a+b}{2}]; \\ x - b & \text{if } x \in (\frac{a+b}{2}, b] \end{cases}$$

then

$$\sup_{x \in [a, b]} |\lambda(x)| = \frac{b-a}{2}.$$

Therefore, (2.4) becomes

$$(2.5) \quad \left| \int_a^b f(t) \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt \right| \leq \frac{b-a}{2} \cdot \bigvee_a^b(f).$$

Now, if in (2.5) we choose $f(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, then $\bigvee_a^b(f) = 2$,

$$\int_a^b f(t) \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt = b - a$$

and in both sides of (2.5) we get the same quantity $(b-a)$. ■

Remark 1. We observe that

$$\begin{aligned} & \int_a^x g(t) dh(t) - \frac{h(x) - h(a)}{h(b) - h(a)} \int_a^b g(s) dh(s) \\ & = \int_a^x g(t) dh(t) - \frac{h(x) - h(a)}{h(b) - h(a)} \left[\int_a^x g(s) dh(s) + \int_x^b g(s) dh(s) \right] \\ & = \frac{h(b) - h(x)}{h(b) - h(a)} \cdot \int_a^x g(s) dh(s) - \frac{h(x) - h(a)}{h(b) - h(a)} \cdot \int_x^b g(s) dh(s) \\ & = \frac{[h(b) - h(x)][h(x) - h(a)]}{h(b) - h(a)} \Delta(g, h; x, a, b), \end{aligned}$$

where $\Delta(g, h; x, a, b)$ is defined by

$$\Delta(g, h; x, a, b) = \frac{1}{h(x) - h(a)} \int_a^x g(s) dh(s) - \frac{1}{h(b) - h(x)} \int_x^b g(s) dh(s),$$

provided $h(x) \neq h(a), h(b)$ for $x \in (a, b)$.

With this notation, the inequality (2.1) becomes

$$(2.6) \quad |T(f, g; h)| \leq \frac{1}{|h(b) - h(a)|} \bigvee_a^b(f) \times \sup_{x \in [a, b]} \left\{ \left| \frac{[h(b) - h(x)][h(x) - h(a)]}{h(b) - h(a)} \right| \cdot |\Delta(g, h; x, a, b)| \right\} \leq \frac{1}{|h(b) - h(a)|} \bigvee_a^b(f) \times \sup_{x \in [a, b]} \left| \frac{[h(b) - h(x)][h(x) - h(a)]}{h(b) - h(a)} \right| \sup_{x \in [a, b]} |\Delta(g, h; x, a, b)|.$$

Now, if we assume that $h(a) < h(x) < h(b)$ for any $x \in (a, b)$ then, on utilising the elementary inequality $\alpha\beta \leq \frac{1}{4}(\alpha + \beta)^2$, $\alpha, \beta \in [0, \infty)$, we have

$$[h(b) - h(x)][h(x) - h(a)] \leq \frac{1}{4} [h(b) - h(a)]^2,$$

and from (2.5), we deduce the following simpler inequality:

$$(2.7) \quad |T(f, g; h)| \leq \frac{1}{4} \cdot \bigvee_a^b(f) \sup_{x \in [a, b]} |\Delta(g, h; x, a, b)|.$$

The constant $\frac{1}{4}$ is best possible in (2.7).

A sufficient condition for h such that $h(a) < h(x) < h(b)$ for any $x \in (a, b)$ is that h is strictly increasing on $[a, b]$. The sharpness of the constant will follow from a particular case considered in Corollary 2 below.

Corollary 1. *Let $f, g, w : [a, b] \rightarrow \mathbb{R}$ be such that f is of bounded variation and the Riemann integrals $\int_a^b f(t) w(t) dt$, $\int_a^b g(t) w(t) dt$, $\int_a^b f(t) g(t) w(t) dt$ and $\int_a^b w(t) dt$ exist and $\int_a^b w(t) dt \neq 0$. Then we have the inequality*

$$(2.8) \quad \left| \frac{1}{\int_a^b w(t) dt} \int_a^b f(t) g(t) w(t) dt - \frac{1}{\int_a^b w(t) dt} \int_a^b f(t) w(t) dt \cdot \frac{1}{\int_a^b w(t) dt} \int_a^b g(t) w(t) dt \right| \leq \frac{1}{\left| \int_a^b w(t) dt \right|} \bigvee_a^b(f) \sup_{x \in [a, b]} \left| \int_a^b g(t) w(t) dt - \frac{\int_a^x w(t) dt}{\int_a^b w(t) dt} \int_a^b g(t) w(t) dt \right|.$$

The inequality is sharp.

The proof follows by Theorem 1 on choosing $h(x) = \int_a^x w(s) ds$.

Remark 2. In particular, if $w(s) > 0$ for $s \in [a, b]$, then $h(x) = \int_a^x w(s) ds$ is strictly increasing on $[a, b]$ and by (2.7) we deduce the inequality:

$$\begin{aligned}
(2.9) \quad & \left| \frac{1}{\int_a^b w(t) dt} \int_a^b f(t) g(t) w(t) dt \right. \\
& \quad \left. - \frac{1}{\int_a^b w(t) dt} \int_a^b f(t) w(t) dt \cdot \frac{1}{\int_a^b w(t) dt} \int_a^b g(t) w(t) dt \right| \\
& \leq \frac{1}{\int_a^b w(s) ds} \bigvee_a^b(f) \sup_{x \in [a, b]} \left| \int_a^x g(s) w(s) ds - \frac{\int_a^x w(s) ds}{\int_a^b w(s) ds} \int_a^b g(s) w(s) ds \right| \\
& \leq \frac{1}{4} \cdot \bigvee_a^b(f) \sup_{x \in [a, b]} \left| \frac{1}{\int_a^x w(s) ds} \int_a^x g(s) w(s) ds \right. \\
& \quad \left. - \frac{1}{\int_x^b w(s) ds} \int_x^b g(s) w(s) ds \right|.
\end{aligned}$$

The constant $\frac{1}{4}$ is best possible.

Corollary 2. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be such that f is of bounded variation and the Riemann integrals $\int_a^b g(t) dt$ and $\int_a^b f(t) g(t) dt$ exist. Then

$$\begin{aligned}
(2.10) \quad & \left| \frac{1}{b-a} \int_a^b f(t) g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt \right| \\
& \leq \frac{1}{b-a} \bigvee_a^b(f) \sup_{x \in [a, b]} \left| \int_a^x g(t) dt - \frac{x-a}{b-a} \int_a^b g(t) dt \right| \\
& \leq \frac{1}{4} \cdot \bigvee_a^b(f) \sup_{x \in (a, b)} \left| \frac{1}{x-a} \int_a^x g(s) ds - \frac{1}{b-x} \int_x^b g(s) ds \right|.
\end{aligned}$$

The constant $\frac{1}{4}$ is best possible in (2.10).

Proof. For the sharpness of the constant, consider $g(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, $t \in [a, b]$. If we denote

$$\begin{aligned}
\mu(x) := & \frac{1}{x-a} \int_a^x \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt \\
& - \frac{1}{b-x} \int_x^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt, \quad x \in (a, b),
\end{aligned}$$

then

$$\begin{aligned}
\mu(x) &= \frac{1}{x-a} \int_a^x \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt \\
& \quad - \frac{1}{b-x} \left(\int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt - \int_a^x \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt \right) \\
&= \frac{b-a}{(x-a)(b-x)} \int_a^x \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt = \frac{b-a}{(x-a)(b-x)} \cdot \lambda(x),
\end{aligned}$$

where λ has been defined in the proof of Theorem 1.

Therefore,

$$\sup_{x \in [a, b]} |\mu(x)| = (b-a) \sup_{x \in [a, b]} \delta(x)$$

where

$$\delta(x) = \begin{cases} \frac{1}{b-x} & \text{if } x \in [a, \frac{a+b}{2}); \\ \frac{1}{x-a} & \text{if } x \in (\frac{a+b}{2}, b]. \end{cases}$$

Since $\sup_{x \in [a, b]} \delta(x) = 2$, the inequality (2.10) becomes, for g given above,

$$(2.11) \quad \left| \int_a^b f(t) \operatorname{sgn} \left(t - \frac{a+b}{2} \right) dt \right| \leq \frac{1}{2} \bigvee_a^b(f),$$

for any function f of bounded variation on $[a, b]$.

If in this inequality we choose $f(t) = \operatorname{sgn} \left(t - \frac{a+b}{2} \right)$, then we obtain in both sides of (2.11) the same quantity $(b-a)$. ■

3. APPLICATIONS FOR THE TRAPEZOID RULE

The following result concerning the error estimate for the trapezoid rule can be stated:

Proposition 1. *Assume that $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous and has the derivative $f' : [a, b] \rightarrow \mathbb{R}$ of bounded variation on $[a, b]$. Then*

$$(3.1) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} \right| \leq \frac{1}{8} (b-a) \bigvee_a^b(f').$$

The constant $\frac{1}{8}$ is best possible.

Proof. We use the identity (see for instance [1])

$$(3.2) \quad \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt = \frac{1}{b-a} \int_a^b \left(t - \frac{a+b}{2} \right) f'(t) dt.$$

If we apply the inequality (2.10), then we can write that

$$(3.3) \quad \left| \frac{1}{b-a} \int_a^b f'(t) \left(t - \frac{a+b}{2} \right) dt - \frac{1}{b-a} \int_a^b f'(t) dt \cdot \frac{1}{b-a} \int_a^b \left(t - \frac{a+b}{2} \right) dt \right| \\ \leq \frac{1}{b-a} \bigvee_a^b(f') \sup_{x \in [a, b]} \left| \int_a^x \left(t - \frac{a+b}{2} \right) dt - \frac{x-a}{b-a} \int_a^b \left(t - \frac{a+b}{2} \right) dt \right|.$$

Since

$$\int_a^b \left(t - \frac{a+b}{2} \right) dt = 0, \quad \int_a^x \left(t - \frac{a+b}{2} \right) dt = \frac{1}{2} \left[\left(x - \frac{a+b}{2} \right)^2 - \left(\frac{b-a}{2} \right)^2 \right]$$

and

$$\begin{aligned} \sup_{x \in [a, b]} \left| \int_a^x \left(t - \frac{a+b}{2} \right) dt \right| &= \frac{1}{2} \sup_{x \in [a, b]} \left| \left(x - \frac{a+b}{2} \right)^2 - \left(\frac{b-a}{2} \right)^2 \right| \\ &= \frac{(b-a)^2}{8}, \end{aligned}$$

hence, by (3.2) and (3.3) we deduce (3.1).

For the sharpness of the constant we choose $f(t) = \left| t - \frac{a+b}{2} \right|$. For this function, we have

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(t) dt &= \frac{1}{b-a} \int_a^b \left| t - \frac{a+b}{2} \right| dt = \frac{b-a}{4}, \\ \frac{f(a) + f(b)}{2} &= \frac{b-a}{2}, \\ f'(t) &= \begin{cases} -1 & \text{if } x \in [a, \frac{a+b}{2}); \\ 1 & \text{if } x \in (\frac{a+b}{2}, b] \end{cases} \end{aligned}$$

and $V_a^b(f') = 2$.

If we replace the above quantities in (3.1), we get the same result $\frac{b-a}{4}$ in both sides. ■

The following result can be stated as well.

Proposition 2. *If $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$, then*

$$\begin{aligned} (3.4) \quad & \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} \right| \\ & \leq \sup_{x \in [a, b]} \left| f(x) - f(a) - (x-a) \cdot \frac{f(b) - f(a)}{b-a} \right| \\ & \leq \frac{1}{4} (b-a) \cdot \sup_{x \in (a, b)} \left| \frac{f(x) - f(a)}{x-a} - \frac{f(b) - f(x)}{b-x} \right|. \end{aligned}$$

Proof. Applying the inequality (2.10), we can also write that

$$\begin{aligned} (3.5) \quad & \left| \frac{1}{b-a} \int_a^b f'(t) \left(t - \frac{a+b}{2} \right) dt \right. \\ & \quad \left. - \frac{1}{b-a} \int_a^b f'(t) dt \cdot \frac{1}{b-a} \int_a^b \left(t - \frac{a+b}{2} \right) dt \right| \\ & \leq \frac{1}{b-a} V_a^b \left(\cdot - \frac{a+b}{2} \right) \cdot \sup_{x \in [a, b]} \left| \int_a^x f'(t) dt - \frac{x-a}{b-a} \int_a^b f'(t) dt \right| \\ & \leq \frac{1}{4} V_a^b \left(\cdot - \frac{a+b}{2} \right) \cdot \sup_{x \in [a, b]} \left| \frac{\int_a^x f'(t) dt}{x-a} - \frac{\int_x^b f'(t) dt}{b-x} \right|, \end{aligned}$$

which, together with the identity (3.2) produces the desired inequality (3.4). ■

For other results on the trapezoid rule, see [1].

4. APPLICATIONS FOR THE MIDPOINT RULE

The following result concerning the error estimates for the midpoint rule can be stated.

Proposition 3. *Assume that $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous and has the derivative $f' : [a, b] \rightarrow \mathbb{R}$ of bounded variation on $[a, b]$. Then*

$$(4.1) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{8} (b-a) \bigvee_a^b(f').$$

The constant $\frac{1}{8}$ is best possible.

Proof. We use the identity (see for instance [2]):

$$(4.2) \quad f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt = \frac{1}{b-a} \int_a^b p(t) f'(t) dt$$

where $p : [a, b] \rightarrow \mathbb{R}$ is given by

$$p(t) = \begin{cases} t-a, & \text{if } t \in [a, \frac{a+b}{2}]; \\ t-b, & \text{if } t \in (\frac{a+b}{2}, b]. \end{cases}$$

If we apply the inequality (2.10), we can write that

$$(4.3) \quad \left| \frac{1}{b-a} \int_a^b f'(t) p(t) dt - \frac{1}{b-a} \int_a^b f'(t) dt \cdot \frac{1}{b-a} \int_a^b p(t) dt \right| \\ \leq \frac{1}{b-a} \bigvee_a^b(f') \sup_{x \in [a, b]} \left| \int_a^x p(t) dt - \frac{x-a}{b-a} \int_a^b p(t) dt \right|.$$

We notice that

$$\int_a^b p(t) dt = 0$$

and

$$\delta(x) := \int_a^x p(t) dt \\ = \begin{cases} \int_a^x (t-a) dt & \text{if } t \in [a, \frac{a+b}{2}]; \\ \int_a^{\frac{a+b}{2}} (t-a) dt + \int_{\frac{a+b}{2}}^x (t-b) dt & \text{if } t \in (\frac{a+b}{2}, b]; \end{cases} \\ = \begin{cases} \frac{1}{2} (x-a)^2 & \text{if } t \in [a, \frac{a+b}{2}]; \\ \frac{1}{2} (b-x)^2 & \text{if } t \in (\frac{a+b}{2}, b]; \end{cases}$$

for $x \in [a, b]$.

Since

$$\sup_{x \in [a, b]} |\delta(x)| = \frac{1}{8} (b-a)^2,$$

then by (4.2) and (4.3), we deduce (4.1).

For the sharpness of the constant $\frac{1}{8}$, observe that for the absolutely continuous function $f(t) = |t - \frac{a+b}{2}|$, we get in both sides of (4.1) the same quantity $\frac{b-a}{4}$. ■

The following result can be stated as well.

Proposition 4. *If $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$, then:*

$$(4.4) \quad \begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \sup_{x \in [a, b]} \left| f(x) - f(a) - (x-a) \cdot \frac{f(b) - f(a)}{b-a} \right| \\ & \leq \frac{1}{4} (b-a) \cdot \sup_{x \in (a, b)} \left| \frac{f(x) - f(a)}{x-a} - \frac{f(b) - f(x)}{b-x} \right|. \end{aligned}$$

Proof. Applying the inequality (2.10), we can write:

$$(4.5) \quad \begin{aligned} & \left| \frac{1}{b-a} \int_a^b p(t) f'(t) dt - \frac{1}{b-a} \int_a^b p(t) dt \cdot \frac{1}{b-a} \int_a^b f'(t) dt \right| \\ & \leq \frac{1}{b-a} \bigvee_a^b(p) \sup_{x \in [a, b]} \left| \int_a^x f'(t) dt - \frac{x-a}{b-a} \int_a^b f'(t) dt \right| \\ & \leq \frac{1}{4} \bigvee_a^b(p) \sup_{x \in [a, b]} \left| \frac{\int_a^x f'(t) dt}{x-a} - \frac{\int_x^b f'(t) dt}{b-x} \right|, \end{aligned}$$

and since $\bigvee_a^b(p) = b-a$, we deduce from (4.5) the desired inequality (4.4). ■

For other results on the midpoint rule, see [2].

5. APPLICATIONS FOR GENERAL QUADRATURE RULES

Let $h : [a, b] \rightarrow \mathbb{R}$ be a Riemann integrable function. Suppose that h is n -time differentiable and that there exists the division $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ and the weights $\alpha_0, \dots, \alpha_n$ such that

$$(5.1) \quad \int_a^b h(t) dt = \sum_{i=0}^n \alpha_i h(x_i) + \int_a^b K_n(t) h^{(n)}(t) dt,$$

where $K_n : [a, b] \rightarrow \mathbb{R}$ is the *Peano kernel* associated with the quadrature rule $A(h) := \sum_{i=0}^n \alpha_i h(x_i)$.

Utilising the inequality (2.10), we can produce a “perturbed quadrature rule” by approximating the error terms $\int_a^b K_n(t) h^{(n)}(t) dt$ as follows.

Proposition 5. *With the above assumptions and if $h^{(n)}$ is of bounded variation, then*

$$(5.2) \quad \int_a^b h(t) dt = \sum_{i=0}^n \alpha_i h(x_i) + \frac{h^{(n-1)}(b) - h^{(n-1)}(a)}{b-a} \cdot \int_a^b K_n(t) dt + E_n(h)$$

and the error term $E_n(h)$ satisfies the bound

$$(5.3) \quad \begin{aligned} |E_n(h)| & \leq \bigvee_a^b(h^{(n)}) \sup_{x \in [a, b]} \left| \int_a^x K_n(t) dt - \frac{x-a}{b-a} \int_a^b K_n(t) dt \right| \\ & \leq \frac{1}{4} \cdot (b-a) \bigvee_a^b(h^{(n)}) \sup_{x \in (a, b)} \left| \frac{\int_a^x K_n(t) dt}{x-a} - \frac{\int_x^b K_n(t) dt}{b-x} \right|. \end{aligned}$$

The proof is obvious by (2.5) on choosing $f = h^{(n)}$ and $g = K_n$.

The second natural possibility is incorporated in

Proposition 6. *With the above assumption and if K_n is of bounded variation on $[a, b]$, then the representation (5.2) holds and the error term $E_n(h)$ satisfies the bounds*

$$(5.4) \quad |E_n(h)| \leq \int_a^b (K_n) \sup_{x \in [a, b]} \left| h^{(n-1)}(x) - h^{(n-1)}(a) - (x-a) \cdot \frac{h^{(n-1)}(b) - h^{(n-1)}(a)}{b-a} \right| \\ \leq \frac{1}{4} \cdot \int_a^b (K_n) \sup_{x \in (a, b)} \left| \frac{h^{(n-1)}(x) - h^{(n-1)}(a)}{x-a} - \frac{h^{(n-1)}(b) - h^{(n-1)}(x)}{b-x} \right|.$$

The proof follows by the inequality (2.10) on choosing $f = K_n$ and $g = h^{(n)}$.

Remark 3. *As noted in the previous section, in practical applications and for a large number of quadrature rules, the Peano kernel K_n is available and the involved quantities in the error estimates (5.3) and (5.4) can be completely specified. In some cases, the new perturbed rules provide a better approximation than the original one. The details are left to the interested reader.*

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SCHOOL OF COMPUTER SCIENCE AND MATHEMATICS, VICTORIA UNIVERSITY, PO Box 14428, MELBOURNE CITY MC., VIC 8001, AUSTRALIA.

E-mail address: sever.dragomir@vu.edu.au

URL: <http://rgmia.vu.edu.au/dragomir>