

# ON SOME INEQUALITIES IN NORMED ALGEBRAS

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ABSTRACT. Some inequalities in normed algebras that provides lower and upper bounds for the norm of  $\sum_{j=1}^n a_j x_j$  are obtained. Applications for estimating the quantities  $\| \|x^{-1}\| x \pm \|y^{-1}\| y \|$  and  $\| \|y^{-1}\| x \pm \|x^{-1}\| y \|$  for invertible elements  $x, y$  in unital normed algebras are also given.

## 1. INTRODUCTION

In [1], in order to provide a generalisation of a norm inequality for  $n$  vectors in a normed linear space obtained by Pečarić and Rajić in [2], the author obtained the following result:

$$(1.1) \quad \max_{k \in \{1, \dots, n\}} \left\{ |\alpha_k| \left\| \sum_{j=1}^n x_j \right\| - \sum_{j=1}^n |\alpha_j - \alpha_k| \|x_j\| \right\} \\ \leq \left\| \sum_{j=1}^n \alpha_j x_j \right\| \leq \min_{k \in \{1, \dots, n\}} \left\{ |\alpha_k| \left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n |\alpha_j - \alpha_k| \|x_j\| \right\},$$

where  $x_j, j \in \{1, \dots, n\}$  are vectors in the normed linear space  $(X, \|\cdot\|)$  over  $\mathbb{K}$  while  $\alpha_j, j \in \{1, \dots, n\}$  are scalars in  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ).

For  $\alpha_k = \frac{1}{\|x_k\|}$ , with  $x_k \neq 0, k \in \{1, \dots, n\}$  the above inequality produces the following result established by Pečarić and Rajić in [2]:

$$(1.2) \quad \max_{k \in \{1, \dots, n\}} \left\{ \frac{1}{\|x_k\|} \left[ \left\| \sum_{j=1}^n x_j \right\| - \sum_{j=1}^n \left| \|x_j\| - \|x_k\| \right| \right] \right\} \\ \leq \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \leq \min_{k \in \{1, \dots, n\}} \left\{ \frac{1}{\|x_k\|} \left[ \left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n \left| \|x_j\| - \|x_k\| \right| \right] \right\},$$

which implies the following refinement and reverse of the generalised triangle inequality due to M. Kato et al. [3]:

$$(1.3) \quad \min_{k \in \{1, \dots, n\}} \{ \|x_k\| \} \left[ n - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \right]$$

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$$\leq \sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \leq \max_{k \in \{1, \dots, n\}} \{\|x_k\|\} \left[ n - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \right].$$

The other natural choice,  $\alpha_k = \|x_k\|$ ,  $k \in \{1, \dots, n\}$  in (1.1) produces the result

$$(1.4) \quad \max_{k \in \{1, \dots, n\}} \left\{ \|x_k\| \left\| \sum_{j=1}^n x_j \right\| - \sum_{j=1}^n \|\|x_j\| - \|x_k\|\| \|x_j\| \right\} \\ \leq \left\| \sum_{j=1}^n \|x_j\| x_j \right\| \leq \min_{k \in \{1, \dots, n\}} \left\{ \|x_k\| \left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n \|\|x_j\| - \|x_k\|\| \|x_j\| \right\},$$

which in its turn implies another refinement and reverse of the generalised triangle inequality:

$$(1.5) \quad (0 \leq) \frac{\sum_{j=1}^n \|x_j\|^2 - \left\| \sum_{j=1}^n \|x_j\| x_j \right\|}{\max_{k \in \{1, \dots, n\}} \{\|x_k\|\}} \\ \leq \sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \leq \frac{\sum_{j=1}^n \|x_j\|^2 - \left\| \sum_{j=1}^n \|x_j\| x_j \right\|}{\min_{k \in \{1, \dots, n\}} \{\|x_k\|\}},$$

provided  $x_k \neq 0$ ,  $k \in \{1, \dots, n\}$ .

In [2], the authors have shown that the case  $n = 2$  in (1.2) produces the *Maligranda-Mercer inequality*:

$$(1.6) \quad \frac{\|x - y\| - \|\|x\| - \|y\|\|}{\min \{\|x\|, \|y\|\}} \leq \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{\|x - y\| + \|\|x\| - \|y\|\|}{\max \{\|x\|, \|y\|\}},$$

for any  $x, y \in X \setminus \{0\}$ .

We notice that Maligranda proved the right inequality in [5] while Mercer proved the left inequality in [4].

We have shown in [1] that the following dual result for two vectors is also valid:

$$(1.7) \quad (0 \leq) \frac{\|x - y\|}{\min \{\|x\|, \|y\|\}} - \frac{\|\|x\| - \|y\|\|}{\max \{\|x\|, \|y\|\}} \\ \leq \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \leq \frac{\|x - y\|}{\max \{\|x\|, \|y\|\}} + \frac{\|\|x\| - \|y\|\|}{\min \{\|x\|, \|y\|\}},$$

for any  $x, y \in X \setminus \{0\}$ .

Motivated by the above results, the aim of the present paper is to establish lower and upper bounds for the norm of  $\sum_{j=1}^n a_j x_j$ , where  $a_j, x_j, j \in \{1, \dots, n\}$  are elements in a normed algebra  $(A, \|\cdot\|)$  over the real or complex number field  $\mathbb{K}$ . In the case where  $(A, \|\cdot\|)$  is a unital algebra and  $x, y$  are invertible, lower and upper bounds for the quantities

$$\|\|x^{-1}\| x \pm \|y^{-1}\| y\| \quad \text{and} \quad \|\|y^{-1}\| x \pm \|x^{-1}\| y\|$$

are provided as well.

2. INEQUALITIES FOR  $n$  PAIRS OF ELEMENTS

Let  $(A, \|\cdot\|)$  be a normed algebra over the real or complex number field  $\mathbb{K}$ .

**Theorem 1.** *If  $(a_j, x_j) \in A^2$ ,  $j \in \{1, \dots, n\}$ , then*

$$\begin{aligned}
 (2.1) \quad & \max_{k \in \{1, \dots, n\}} \left\{ \left\| a_k \left( \sum_{j=1}^n x_j \right) \right\| - \sum_{j=1}^n \|a_j - a_k\| \|x_j\| \right\} \\
 & \leq \max_{k \in \{1, \dots, n\}} \left\{ \left\| a_k \left( \sum_{j=1}^n x_j \right) \right\| - \sum_{j=1}^n \|(a_j - a_k) x_j\| \right\} \\
 & \leq \left\| \sum_{j=1}^n a_j x_j \right\| \\
 & \leq \min_{k \in \{1, \dots, n\}} \left\{ \left\| a_k \left( \sum_{j=1}^n x_j \right) \right\| + \sum_{j=1}^n \|(a_j - a_k) x_j\| \right\} \\
 & \leq \min_{k \in \{1, \dots, n\}} \left\{ \left\| a_k \left( \sum_{j=1}^n x_j \right) \right\| + \sum_{j=1}^n \|a_j - a_k\| \|x_j\| \right\}.
 \end{aligned}$$

*Proof.* Observe that for any  $k \in \{1, \dots, n\}$  we have

$$\sum_{j=1}^n a_j x_j = a_k \left( \sum_{j=1}^n x_j \right) + \sum_{j=1}^n (a_j - a_k) x_j.$$

Taking the norm and utilising the triangle inequality and the normed algebra properties, we have

$$\begin{aligned}
 \left\| \sum_{j=1}^n a_j x_j \right\| & \leq \left\| a_k \left( \sum_{j=1}^n x_j \right) \right\| + \left\| \sum_{j=1}^n (a_j - a_k) x_j \right\| \\
 & \leq \left\| a_k \left( \sum_{j=1}^n x_j \right) \right\| + \sum_{j=1}^n \|(a_j - a_k) x_j\| \\
 & \leq \left\| a_k \left( \sum_{j=1}^n x_j \right) \right\| + \sum_{j=1}^n \|a_j - a_k\| \|x_j\|,
 \end{aligned}$$

for any  $k \in \{1, \dots, n\}$ , which implies the second part in (2.1).

Observing that

$$\sum_{j=1}^n a_j x_j = a_k \left( \sum_{j=1}^n x_j \right) - \sum_{j=1}^n (a_k - a_j) x_j$$

and utilising the continuity of the norm, we have

$$\begin{aligned}
\left\| \sum_{j=1}^n a_j x_j \right\| &\geq \left\| a_k \left( \sum_{j=1}^n x_j \right) - \sum_{j=1}^n (a_k - a_j) x_j \right\| \\
&\geq \left\| a_k \left( \sum_{j=1}^n x_j \right) \right\| - \left\| \sum_{j=1}^n (a_k - a_j) x_j \right\| \\
&\geq \left\| a_k \left( \sum_{j=1}^n x_j \right) \right\| - \sum_{j=1}^n \|(a_k - a_j) x_j\| \\
&\geq \left\| a_k \left( \sum_{j=1}^n x_j \right) \right\| - \sum_{j=1}^n \|a_k - a_j\| \|x_j\|
\end{aligned}$$

for any  $k \in \{1, \dots, n\}$ , which implies the first part in (2.1). ■

**Remark 1.** *If there exists  $r > 0$  so that  $\|a_j - a_k\| \leq r \|a_k\|$  for any  $j, k \in \{1, \dots, n\}$ , then, by the second part of (2.1), we have*

$$(2.2) \quad \left\| \sum_{j=1}^n a_j x_j \right\| \leq \min_{k \in \{1, \dots, n\}} \{\|a_k\|\} \left[ \left\| \sum_{j=1}^n x_j \right\| + r \sum_{j=1}^n \|x_j\| \right].$$

**Corollary 1.** *If  $x_j \in A$ ,  $j \in \{1, \dots, n\}$ , then*

$$\begin{aligned}
(2.3) \quad &\max_{k \in \{1, \dots, n\}} \left\{ \left\| x_k \left( \sum_{j=1}^n x_j \right) \right\| - \sum_{j=1}^n \|x_j - x_k\| \|x_j\| \right\} \\
&\leq \max_{k \in \{1, \dots, n\}} \left\{ \left\| x_k \left( \sum_{j=1}^n x_j \right) \right\| - \sum_{j=1}^n \|(x_j - x_k) x_j\| \right\} \leq \left\| \sum_{j=1}^n x_j^2 \right\| \\
&\leq \min_{k \in \{1, \dots, n\}} \left\{ \left\| x_k \left( \sum_{j=1}^n x_j \right) \right\| + \sum_{j=1}^n \|(x_j - x_k) x_j\| \right\} \\
&\leq \min_{k \in \{1, \dots, n\}} \left\{ \left\| x_k \left( \sum_{j=1}^n x_j \right) \right\| + \sum_{j=1}^n \|x_j - x_k\| \|x_j\| \right\}.
\end{aligned}$$

**Corollary 2.** *Assume that  $A$  is a unital normed algebra. If  $x_j \in A$  are invertible for any  $j \in \{1, \dots, n\}$ , then*

$$\begin{aligned}
(2.4) \quad &\min_{k \in \{1, \dots, n\}} \|x_k^{-1}\| \left[ \sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \right] \\
&\leq \sum_{j=1}^n \|x_j^{-1}\| \|x_j\| - \left\| \sum_{j=1}^n \|x_j^{-1}\| x_j \right\| \\
&\leq \max_{k \in \{1, \dots, n\}} \|x_k^{-1}\| \left[ \sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \right].
\end{aligned}$$

*Proof.* If  $1 \in A$  is the unity, then on choosing  $a_k = \|x_k^{-1}\| \cdot 1$  in (2.1) we get

$$(2.5) \quad \begin{aligned} & \max_{k \in \{1, \dots, n\}} \left\{ \|x_k^{-1}\| \left\| \sum_{j=1}^n x_j \right\| - \sum_{j=1}^n \|x_j^{-1}\| \|x_j\| \right\} \\ & \leq \left\| \sum_{j=1}^n \|x_j^{-1}\| x_j \right\| \\ & \leq \min_{k \in \{1, \dots, n\}} \left\{ \|x_k^{-1}\| \left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n \|x_j^{-1}\| \|x_j\| \right\}. \end{aligned}$$

Now, assume that  $\min_{k \in \{1, \dots, n\}} \{\|x_k^{-1}\|\} = \|x_{k_0}^{-1}\|$ . Then

$$\begin{aligned} & \|x_{k_0}^{-1}\| \left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n \|x_j^{-1}\| \|x_j\| \\ & = -\|x_{k_0}^{-1}\| \left( \sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \right) + \sum_{j=1}^n \|x_j^{-1}\| \|x_j\|. \end{aligned}$$

Utilising the second inequality in (2.5), we deduce

$$\|x_{k_0}^{-1}\| \left( \sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \right) \leq \sum_{j=1}^n \|x_j^{-1}\| \|x_j\| - \left\| \sum_{j=1}^n \|x_j^{-1}\| x_j \right\|$$

and the first inequality in (2.4) is proved.

The second part of (2.4) can be proved in a similar manner, however, the details are omitted. ■

**Remark 2.** An equivalent form of (2.4) is:

$$(2.6) \quad \begin{aligned} & \frac{\sum_{j=1}^n \|x_j^{-1}\| \|x_j\| - \left\| \sum_{j=1}^n \|x_j^{-1}\| x_j \right\|}{\max_{k \in \{1, \dots, n\}} \|x_k^{-1}\|} \\ & \leq \sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \leq \frac{\sum_{j=1}^n \|x_j^{-1}\| \|x_j\| - \left\| \sum_{j=1}^n \|x_j^{-1}\| x_j \right\|}{\min_{k \in \{1, \dots, n\}} \|x_k^{-1}\|}, \end{aligned}$$

which provides both a refinement and a reverse inequality for the generalised triangle inequality.

### 3. INEQUALITIES FOR TWO PAIRS OF ELEMENTS

The following particular case of Theorem 1 is of interest for applications.

**Lemma 1.** If  $(a, b), (x, y) \in A^2$ , then

$$(3.1) \quad \begin{aligned} & \max \{ \|a(x \pm y)\| - \|(b-a)y\|, \|b(x \pm y)\| - \|(b-a)x\| \} \\ & \leq \|ax \pm by\| \leq \min \{ \|a(x \pm y)\| + \|(b-a)y\|, \|b(x \pm y)\| + \|(b-a)x\| \} \end{aligned}$$

or, equivalently,

$$\begin{aligned}
(3.2) \quad & \frac{1}{2} \{ \|a(x \pm y)\| + \|b(x \pm y)\| - [\|(b-a)y\| + \|(b-a)x\|] \} \\
& + \frac{1}{2} \| \|a(x \pm y)\| - \|b(x \pm y)\| + \|(b-a)y\| - \|(b-a)x\| \| \\
& \leq \|ax \pm by\| \\
& \leq \frac{1}{2} \{ \|a(x \pm y)\| + \|b(x \pm y)\| + [\|(b-a)y\| + \|(b-a)x\|] \} \\
& \quad - \frac{1}{2} \| \|a(x \pm y)\| + \|b(x \pm y)\| - \|(b-a)y\| - \|(b-a)x\| \|.
\end{aligned}$$

*Proof.* The inequality (3.1) follows from Theorem 1 for  $n = 2$ ,  $a_1 = a$ ,  $a_2 = b$ ,  $x_1 = x$  and  $x_2 = \pm y$ .

Utilising the properties of real numbers,

$$\min \{ \alpha, \beta \} = \frac{1}{2} [\alpha + \beta - |\alpha - \beta|], \quad \max \{ \alpha, \beta \} = \frac{1}{2} [\alpha + \beta + |\alpha - \beta|]; \quad \alpha, \beta \in \mathbb{R};$$

the inequality (3.1) is clearly equivalent with (3.2). ■

The following result contains some upper bounds for  $\|ax \pm by\|$  that are perhaps more useful for applications.

**Theorem 2.** *If  $(a, b), (x, y) \in A^2$ , then*

$$\begin{aligned}
(3.3) \quad & \|ax \pm by\| \leq \min \{ \|a(x \pm y)\|, \|b(x \pm y)\| \} + \|b-a\| \max \{ \|x\|, \|y\| \} \\
& \leq \|x \pm y\| \min \{ \|a\|, \|b\| \} + \|b-a\| \max \{ \|x\|, \|y\| \}
\end{aligned}$$

and

$$\begin{aligned}
(3.4) \quad & \|ax \pm by\| \leq \|x \pm y\| \max \{ \|a\|, \|b\| \} + \min \{ \|(b-a)x\|, \|(b-a)y\| \} \\
& \leq \|x \pm y\| \max \{ \|a\|, \|b\| \} + \|b-a\| \min \{ \|x\|, \|y\| \}.
\end{aligned}$$

*Proof.* Observe that  $\|(b-a)x\| \leq \|b-a\| \|x\|$  and  $\|(b-a)y\| \leq \|b-a\| \|y\|$ , and then

$$(3.5) \quad \|(b-a)x\|, \|(b-a)y\| \leq \|b-a\| \max \{ \|x\|, \|y\| \},$$

which implies that

$$\begin{aligned}
& \min \{ \|a(x \pm y)\| + \|(b-a)y\|, \|b(x \pm y)\| + \|(b-a)x\| \} \\
& \leq \min \{ \|a(x \pm y)\|, \|b(x \pm y)\| \} + \|b-a\| \max \{ \|x\|, \|y\| \} \\
& \leq \|x \pm y\| \min \{ \|a\|, \|b\| \} + \|b-a\| \max \{ \|x\|, \|y\| \}.
\end{aligned}$$

Utilising the second inequality in (3.1), we deduce (3.3).

Also, since  $\|a(x \pm y)\| \leq \|a\| \|x \pm y\|$  and  $\|b(x \pm y)\| \leq \|b\| \|x \pm y\|$ , hence

$$\|a(x \pm y)\|, \|b(x \pm y)\| \leq \|x \pm y\| \max \{ \|a\|, \|b\| \},$$

which implies that

$$\begin{aligned}
& \min \{ \|a(x \pm y)\| + \|(b-a)y\|, \|b(x \pm y)\| + \|(b-a)x\| \} \\
& \leq \|x \pm y\| \max \{ \|a\|, \|b\| \} + \min \{ \|(b-a)x\|, \|(b-a)y\| \} \\
& \leq \|x \pm y\| \max \{ \|a\|, \|b\| \} + \|b-a\| \min \{ \|x\|, \|y\| \},
\end{aligned}$$

and the inequality (3.4) is also proved. ■

The following corollary may be more useful for applications.

**Corollary 3.** *If  $(a, b), (x, y) \in A^2$ , then*

$$(3.6) \quad \|ax \pm by\| \leq \|x \pm y\| \cdot \frac{\|a\| + \|b\|}{2} + \|b - a\| \cdot \frac{\|x\| + \|y\|}{2}.$$

*Proof.* Follows from Theorem 2 by adding the last inequality in (3.3) to the last inequality (3.4) and utilising the property that  $\min\{\alpha, \beta\} + \max\{\alpha, \beta\} = \alpha + \beta$ ,  $\alpha, \beta \in \mathbb{R}$ . ■

The following lower bounds for  $\|ax \pm by\|$  can be stated as well:

**Theorem 3.** *For any  $(a, b)$  and  $(x, y) \in A^2$ , we have:*

$$(3.7) \quad \begin{aligned} & \max\{\|ax\| - \|ay\|, \|bx\| - \|by\|\} - \|b - a\| \max\{\|x\|, \|y\|\} \\ & \leq \max\{\|a(x \pm y)\|, \|b(x \pm y)\|\} - \|b - a\| \max\{\|x\|, \|y\|\} \\ & \leq \|ax \pm by\| \end{aligned}$$

and

$$(3.8) \quad \begin{aligned} & \min\{\|ax\| - \|ay\|, \|bx\| - \|by\|\} - \|b - a\| \min\{\|x\|, \|y\|\} \\ & \leq \min\{\|ax\| - \|ay\|, \|bx\| - \|by\|\} - \min\{\|(b - a)x\|, \|(b - a)y\|\} \\ & \leq \|ax \pm by\|. \end{aligned}$$

*Proof.* Observe that, by (3.5) we have that

$$\begin{aligned} & \max\{\|a(x \pm y)\| - \|(b - a)y\|, \|b(x \pm y)\| - \|(b - a)x\|\} \\ & \geq \max\{\|ax \pm ay\|, \|bx \pm by\|\} - \|b - a\| \max\{\|x\|, \|y\|\} \\ & \geq \max\{\|ax\| - \|ay\|, \|bx\| - \|by\|\} - \|b - a\| \max\{\|x\|, \|y\|\} \end{aligned}$$

and on utilising the first inequality in (3.1), the inequality (3.7) is proved.

Observe also that, since

$$\|a(x \pm y)\|, \|b(x \pm y)\| \geq \min\{\|ax\| - \|ay\|, \|bx\| - \|by\|\},$$

then

$$\begin{aligned} & \max\{\|a(x \pm y)\| - \|(b - a)y\|, \|b(x \pm y)\| - \|(b - a)x\|\} \\ & \geq \min\{\|ax\| - \|ay\|, \|bx\| - \|by\|\} - \min\{\|(b - a)x\|, \|(b - a)y\|\} \\ & \geq \min\{\|ax\| - \|ay\|, \|bx\| - \|by\|\} - \|b - a\| \min\{\|x\|, \|y\|\}. \end{aligned}$$

Then, by the first inequality in (3.1), we deduce (3.8). ■

**Corollary 4.** *For any  $(a, b), (x, y) \in A^2$ , we have*

$$(3.9) \quad \frac{1}{2} \cdot [\|ax\| - \|ay\| + \|bx\| - \|by\|] - \|b - a\| \cdot \frac{\|x\| + \|y\|}{2} \leq \|ax \pm by\|.$$

The proof follows from Theorem 3 by adding (3.7) to (3.8). The details are omitted.

#### 4. APPLICATIONS FOR TWO INVERTIBLE ELEMENTS

In this section we assume that  $A$  is a unital algebra with the unity 1. The following results provide some upper bounds for the quantity  $\| \|x^{-1}\|x \pm \|y^{-1}\|y \|$ , where  $x$  and  $y$  are invertible in  $A$ .

**Proposition 1.** *If  $(x, y) \in A^2$  are invertible, then*

$$(4.1) \quad \begin{aligned} & \left| \|x^{-1}\| x \pm \|y^{-1}\| y \right| \\ & \leq \|x \pm y\| \min \{ \|x^{-1}\|, \|y^{-1}\| \} + \left| \|x^{-1}\| - \|y^{-1}\| \right| \max \{ \|x\|, \|y\| \} \end{aligned}$$

and

$$(4.2) \quad \begin{aligned} & \left| \|x^{-1}\| x \pm \|y^{-1}\| y \right| \\ & \leq \|x \pm y\| \max \{ \|x^{-1}\|, \|y^{-1}\| \} + \left| \|x^{-1}\| - \|y^{-1}\| \right| \min \{ \|x\|, \|y\| \}. \end{aligned}$$

*Proof.* Follows by Theorem 2 on choosing  $a = \|x^{-1}\| \cdot 1$  and  $b = \|y^{-1}\| \cdot 1$ . ■

**Corollary 5.** *With the above assumption for  $x$  and  $y$ , we have*

$$(4.3) \quad \begin{aligned} & \left| \|x^{-1}\| x \pm \|y^{-1}\| y \right| \\ & \leq \|x \pm y\| \cdot \frac{\|x^{-1}\| + \|y^{-1}\|}{2} + \left| \|x^{-1}\| - \|y^{-1}\| \right| \cdot \frac{\|x\| + \|y\|}{2}. \end{aligned}$$

Lower bounds for  $\left| \|x^{-1}\| x \pm \|y^{-1}\| y \right|$  are provided below:

**Proposition 2.** *If  $(x, y) \in A^2$  are invertible, then*

$$(4.4) \quad \begin{aligned} & \|x \pm y\| \max \{ \|x^{-1}\|, \|y^{-1}\| \} - \left| \|x^{-1}\| - \|y^{-1}\| \right| \max \{ \|x\|, \|y\| \} \\ & \leq \left| \|x^{-1}\| x \pm \|y^{-1}\| y \right| \end{aligned}$$

and

$$(4.5) \quad \begin{aligned} & \|x \pm y\| \min \{ \|x^{-1}\|, \|y^{-1}\| \} - \left| \|x^{-1}\| - \|y^{-1}\| \right| \min \{ \|x\|, \|y\| \} \\ & \leq \left| \|x^{-1}\| x \pm \|y^{-1}\| y \right|. \end{aligned}$$

*Proof.* The first inequality in (4.4) follows from the second inequality in (3.7) on choosing  $a = \|x^{-1}\| \cdot 1$  and  $b = \|y^{-1}\| \cdot 1$ .

We know from the proof of Theorem 3 that

$$(4.6) \quad \max \{ \|a(x \pm y)\| - \|(b-a)y\|, \|b(x \pm y)\| - \|(b-a)x\| \} \leq \|ax \pm by\|.$$

If in this inequality we choose  $a = \|x^{-1}\| \cdot 1$  and  $b = \|y^{-1}\| \cdot 1$ , then we get

$$\begin{aligned} & \left| \|x^{-1}\| x \pm \|y^{-1}\| y \right| \\ & \geq \max \{ \|x^{-1}\| \|x \pm y\| - \left| \|x^{-1}\| - \|y^{-1}\| \right| \|y\|, \|y^{-1}\| \|x \pm y\| - \left| \|x^{-1}\| - \|y^{-1}\| \right| \|x\| \} \\ & \geq \|x \pm y\| \min \{ \|x^{-1}\|, \|y^{-1}\| \} - \left| \|x^{-1}\| - \|y^{-1}\| \right| \min \{ \|x\|, \|y\| \} \end{aligned}$$

and the inequality (4.5) is obtained. ■

**Corollary 6.** *If  $(x, y) \in A^2$  are invertible, then*

$$(4.7) \quad \begin{aligned} & \|x \pm y\| \cdot \frac{\|x^{-1}\| + \|y^{-1}\|}{2} - \left| \|x^{-1}\| - \|y^{-1}\| \right| \cdot \frac{\|x\| + \|y\|}{2} \\ & \leq \left| \|x^{-1}\| x \pm \|y^{-1}\| y \right|. \end{aligned}$$



**Remark 3.** We observe that the inequalities (4.3) and (4.7) are in fact equivalent with:

$$(4.8) \quad \left| \left| \|x^{-1}\| x \pm \|y^{-1}\| y \right\| - \|x \pm y\| \cdot \frac{\|x^{-1}\| + \|y^{-1}\|}{2} \right| \\ \leq \left| \|x^{-1}\| - \|y^{-1}\| \right| \cdot \frac{\|x\| + \|y\|}{2}.$$

Now we consider the dual expansion  $\left| \|y^{-1}\| x \pm \|x^{-1}\| y \right|$ , for which the following upper bounds can be stated.

**Proposition 3.** *If  $(x, y)$  are invertible in  $A$ , then*

$$(4.9) \quad \left| \|y^{-1}\| x \pm \|x^{-1}\| y \right| \\ \leq \|x \pm y\| \min \{ \|x^{-1}\|, \|y^{-1}\| \} + \left| \|x^{-1}\| - \|y^{-1}\| \right| \max \{ \|x\|, \|y\| \}$$

and

$$(4.10) \quad \left| \|y^{-1}\| x \pm \|x^{-1}\| y \right| \\ \leq \|x \pm y\| \max \{ \|x^{-1}\|, \|y^{-1}\| \} + \left| \|x^{-1}\| - \|y^{-1}\| \right| \min \{ \|x\|, \|y\| \}.$$

In particular,

$$(4.11) \quad \left| \|y^{-1}\| x \pm \|x^{-1}\| y \right| \\ \leq \|x \pm y\| \cdot \frac{\|x^{-1}\| + \|y^{-1}\|}{2} + \left| \|x^{-1}\| - \|y^{-1}\| \right| \cdot \frac{\|x\| + \|y\|}{2}.$$

The proof follows from Theorem 2 on choosing  $a = \|y^{-1}\| \cdot 1$  and  $b = \|x^{-1}\| \cdot 1$ . The lower bounds for the quantity  $\left| \|y^{-1}\| x \pm \|x^{-1}\| y \right|$  are incorporated in:

**Proposition 4.** *If  $(x, y)$  are invertible in  $A$ , then*

$$(4.12) \quad \|x \pm y\| \max \{ \|x^{-1}\|, \|y^{-1}\| \} - \left| \|x^{-1}\| - \|y^{-1}\| \right| \max \{ \|x\|, \|y\| \} \\ \leq \left| \|y^{-1}\| x \pm \|x^{-1}\| y \right|$$

and

$$(4.13) \quad \|x \pm y\| \min \{ \|x^{-1}\|, \|y^{-1}\| \} - \left| \|x^{-1}\| - \|y^{-1}\| \right| \min \{ \|x\|, \|y\| \} \\ \leq \left| \|y^{-1}\| x \pm \|x^{-1}\| y \right|.$$

In particular,

$$(4.14) \quad \|x \pm y\| \cdot \frac{\|x^{-1}\| + \|y^{-1}\|}{2} - \left| \|x^{-1}\| - \|y^{-1}\| \right| \cdot \frac{\|x\| + \|y\|}{2} \\ \leq \left| \|y^{-1}\| x \pm \|x^{-1}\| y \right|.$$

**Remark 4.** We observe that the inequalities (4.11) and (4.14) are equivalent with

$$(4.15) \quad \left| \left| \|y^{-1}\| x \pm \|x^{-1}\| y \right\| - \|x \pm y\| \cdot \frac{\|x^{-1}\| + \|y^{-1}\|}{2} \right| \\ \leq \left| \|x^{-1}\| - \|y^{-1}\| \right| \cdot \frac{\|x\| + \|y\|}{2}.$$

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