

A GENERALIZATION OF VAN DER CORPUT'S INEQUALITY

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ABSTRACT. In this article, van der Corput's inequality is generalized by using the well known Euler-Maclaurin sum formula and other analytic techniques.

1. INTRODUCTION

Let $S_n = \sum_{k=1}^n \frac{1}{k}$ and $a_n \geq 0$ for $n \in \mathbb{N}$ such that $0 < \sum_{n=1}^{\infty} a_n < \infty$. The van der Corput's inequality [32] reads that

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{1/S_n} < e^{1+\gamma} \sum_{n=1}^{\infty} (n+1)a_n, \quad (1)$$

where $\gamma = 0.57721566 \dots$ stands for Euler-Mascheroni's constant. The constant $e^{1+\gamma}$ in (1) is the best possible.

Two refinements of (1) were given in [16, 19] respectively as

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{1/S_n} < e^{1+\gamma} \sum_{n=1}^{\infty} \left(n - \frac{\ln n}{4} \right) a_n \quad (2)$$

and

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{1/S_n} < e^{1+\gamma} \sum_{n=1}^{\infty} e^{-1/(4n)} \left(n - \frac{\ln n}{3n} \right) a_n. \quad (3)$$

A relation between Carleman's inequality

$$\sum_{n=1}^{\infty} \left(\prod_{i=1}^n a_i \right)^{1/n} < e \sum_{n=1}^{\infty} a_n, \quad (4)$$

see [3, 5, 12] and the references therein, where $a_n \geq 0$ for $n \in \mathbb{N}$ such that $0 < \sum_{n=1}^{\infty} a_n < \infty$ and the constant e in the right hand side of inequality (4) is the best possible, and van der Corput's inequality (1) was established in [35] and it was presented that

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k^\alpha} \right)^{1/S_n(\alpha)} < e \sum_{n=1}^{\infty} e^{\alpha n^{\alpha-1} S_n(\alpha)} a_n \quad (5)$$

2000 *Mathematics Subject Classification.* 26D15.

Key words and phrases. van der Corput's inequality, Carleman's inequality, Euler-Maclaurin sum formula, power mean inequality.

This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$.

for $\alpha \in [0, 1]$ and

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k^\alpha} \right)^{1/S_n(\alpha)} < e^{\frac{1}{1-\alpha}} \sum_{n=1}^{\infty} \left[1 - \frac{1}{2(n+1)} \right]^{\frac{\alpha}{1-\alpha}} a_n \quad (6)$$

for $\alpha \in [0, 1)$, where $S_n(\alpha) = \sum_{k=1}^n \frac{1}{k^\alpha}$.

Another extension of (1) was obtained in [37] as follows:

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/(k+\beta)} \right)^{1/T_n(\beta)} < e^{1+\gamma_1(\beta)} \sum_{n=1}^{\infty} \left(n + \frac{1}{2} + \beta \right) a_n, \quad (7)$$

where $\beta \in (-1, \infty)$, $T_n(\beta) = \sum_{k=1}^n \frac{1}{k+\beta}$ and

$$\gamma_1(\beta) = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \frac{1}{k+\beta} - \ln(n+\beta) \right]. \quad (8)$$

Applying $\beta = 0$ in (7) leads to

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{1/S_n} < e^{1+\gamma} \sum_{n=1}^{\infty} \left(n + \frac{1}{2} \right) a_n, \quad (9)$$

which improved inequality (1) clearly.

In [4], the following extension and refinement of van der Corput's inequality (1) were obtained as follows: Let $a_n \geq 0$ for $n \in \mathbb{N}$ such that $0 \leq \sum_{n=1}^{\infty} a_n < \infty$. Then

$$\sum_{n=1}^{\infty} \left[\prod_{k=1}^n a_k^{1/\sqrt{k(k+\lambda)}} \right]^{1/U_n(\lambda)} < e^{1+(1+\lambda/3)\gamma(\lambda)} \sum_{n=1}^{\infty} (n+1)^{\lambda/3} \left[1 - \frac{\ln(n+1)}{4(n+1+\lambda/2)} \right] a_n, \quad (10)$$

where $\lambda \in [0, \infty)$, $U_n(\lambda) = \sum_{k=1}^n \frac{1}{\sqrt{k(k+\lambda)}}$ and

$$\gamma(\lambda) = \lim_{n \rightarrow \infty} \left[U_n(\lambda) - 2 \ln \frac{\sqrt{n} + \sqrt{n+\lambda}}{1 + \sqrt{1+\lambda}} \right]. \quad (11)$$

In particular,

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{1/S_n} < e^{1+\gamma} \sum_{n=1}^{\infty} n \left(1 - \frac{\ln n}{3n-1/4} \right) a_n. \quad (12)$$

It is easy to see that inequality (12) refines inequalities (1), (2), (3) and (9).

In [21], inequality (12) was refined as

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{1/S_n} < e^{1+\gamma} \sum_{n=1}^{\infty} e^{-\frac{6(6n+1)\gamma-9}{(6n+1)(12n+11)}} n \left(1 - \frac{\ln n}{2n + \ln n + 11/6} \right) a_n, \quad (13)$$

where $a_n \geq 0$ for $n \in \mathbb{N}$ such that $0 < \sum_{n=1}^{\infty} n \left(1 - \frac{\ln n}{2n + \ln n + 11/6} \right) a_n < \infty$.

In [20], inequalities (1), (2), (3), (9), (12) and (13) were further refined as

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{1/S_n} < e^{1+\gamma} \sum_{n=1}^{\infty} e^{-\frac{3\gamma}{6n+4}} n \left(1 - \frac{\ln n}{2n + \ln n + 4/3} \right) a_n, \quad (14)$$

where $a_n \geq 0$ such that $0 < \sum_{n=1}^{\infty} n \left(1 - \frac{\ln n}{2n + \ln n + 4/3}\right) a_n < \infty$.

The aim of this paper is to generalize and refine van der Corput's inequality further by using Euler-Maclaurin sum formula and other analytic techniques.

Our main results are the following two theorems.

Theorem 1. *Let $a_n \geq 0$ for $n \in \mathbb{N}$ such that $0 \leq \sum_{n=1}^{\infty} a_n < \infty$. Then*

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[\prod_{k=1}^n a_k^{1/(k+\eta)^\lambda} \right]^{1/S_n(\eta, \lambda)} a_n \\ & \leq \sum_{n=1}^{\infty} \left[e \left(1 - \frac{1 + [(n+\eta+1)^\lambda - (n+\eta)^\lambda] S_n(\eta, \lambda)}{2(n+\eta+1)^\lambda S_{n+1}(\eta, \lambda)} \right) \right]^{1+T_{n, \lambda, \eta}} a_n, \end{aligned} \quad (15)$$

where $S_n(\eta, \lambda) = \sum_{k=1}^n \frac{1}{(k+\eta)^\lambda}$ and $T_{n, \lambda, \eta} = [(n+\eta+1)^\lambda - (n+\eta)^\lambda] S_n(\eta, \lambda)$ for $\eta \in (-1, \infty)$ and $\lambda \in [0, 1]$.

Remark 1. If taking $\lambda = 0$, $\eta = 0$ or $\lambda = 1$ in inequality (15) respectively, then refinements of inequalities (4), (5) or (7) are deduced respectively. This means that Theorem 1 generalizes van der Corput's inequality (1) and Carleman's inequality (4).

Corollary 1. *Let $a_n \geq 0$ for $n \in \mathbb{N}$ such that $0 \leq \sum_{n=1}^{\infty} a_n < \infty$. If $\eta \in (-1, \infty)$, then*

$$\sum_{n=1}^{\infty} \left[\prod_{k=1}^n a_k^{1/(k+\eta)} \right]^{1/S_n(\eta)} a_n \leq \sum_{n=1}^{\infty} e^{1+\gamma(\eta)} \left[n + \eta + \frac{1}{2} - \frac{3 \ln(n+\eta+1)}{16} \right] a_n, \quad (16)$$

where $\gamma(\eta) = \gamma(\eta, 1)$ and

$$\gamma(\eta, \lambda) = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \frac{1}{(k+\eta)^\lambda} - \int_1^n \frac{1}{(t+\eta)^\lambda} dt \right]. \quad (17)$$

Theorem 2. *Let $a_n \geq 0$ for $n \in \mathbb{N}$ such that $0 \leq \sum_{n=1}^{\infty} a_n < \infty$. If $\eta \geq 0$, then*

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[\prod_{k=1}^n a_k^{1/(k+\eta)^\lambda} \right]^{1/S_n(\eta, \lambda)} a_n \\ & \leq \sum_{n=1}^{\infty} \left[e \left(1 - \frac{1}{2(n+\eta+1)} \right) \right]^{1+\lambda(k+\eta)^{\lambda-1} S_k(\eta, \lambda)} a_n. \end{aligned} \quad (18)$$

In particular, for $\eta = 0$ and $\lambda \in [0, 1)$, inequality

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k^\lambda} \right)^{1/S_n(\lambda)} < \sum_{n=1}^{\infty} \left[e \left(1 - \frac{1}{2(n+1)} \right) \right]^{\frac{1}{1-\lambda}} a_n \quad (19)$$

holds, where $S_n(\lambda) = \sum_{k=1}^n \frac{1}{k^\lambda}$.

Remark 2. Inequalities (18) and (19) improve inequalities (5) and (6), respectively.

2. LEMMAS

To prove our main results, the following lemmas are necessary.

Recall [22, 28] that a function f is called completely monotonic on an interval I if f has derivatives of all orders on I and $(-1)^k f^{(k)}(x) \geq 0$ for all $k \geq 0$ on I .

Lemma 1. *Let $\eta \in (-1, \infty)$ and $\lambda \in [0, \infty)$, the function $f(x) = \frac{1}{(x+\eta)^\lambda}$ is completely monotonic in $[1, \infty)$ and $\lim_{x \rightarrow \infty} f^{(i)}(x) = 0$ for nonnegative integer i .*

Proof. Consecutive computation yields

$$(-1)^n f^{(n)}(x) = \frac{(\lambda)_n}{(x+\eta)^{\lambda+n}} > 0,$$

where $(c)_k = c(c+1) \cdots (c+k-1)$ is Pochhammer symbol.

By induction, it is easy to verify that $\lim_{x \rightarrow \infty} f^{(i)}(x) = 0$ for any nonnegative integer i . The proof of Lemma 1 is complete. \square

Lemma 2. *Let $r < 0$ and $n \in \mathbb{N}$. For any given nonnegative integer $m \geq 0$,*

$$\frac{n+m}{n+m+1} \leq \left[\frac{\frac{1}{n+m} \sum_{i=1}^n (i+m)^r}{\frac{1}{n+m+1} \sum_{i=1}^{n+1} (i+m)^r} \right]^{1/r} = \left[\frac{\frac{1}{n+m} \sum_{i=m+1}^{n+m} i^r}{\frac{1}{n+m+1} \sum_{i=m+1}^{n+m+1} i^r} \right]^{1/r}. \quad (20)$$

Proof. For our own convenience, let $V_r(m, n) = \sum_{i=1}^n (i+m)^r = \sum_{i=m+1}^{n+m} i^r$. Then inequality (20) is equivalent to

$$\frac{V_r(m, n)}{V_r(m, n+1)} \leq \left(\frac{n+m}{n+m+1} \right)^{r+1}. \quad (21)$$

For $r \leq -1$, inequality (21) holds obviously.

For $0 > r > -1$, inequality (21) can be rewritten as

$$V_r(m, n) \leq \frac{(n+m)^{r+1}(n+m+1)^r}{(n+m+1)^{r+1} - (n+m)^{r+1}}. \quad (22)$$

When $n = 1$, it is easy to check up that inequality (22) is true. Now assume that inequality (22) is valid for some positive integer $n > 1$. By induction, it is sufficient to show that inequality (22) validates for $n+1$, that is,

$$V_r(m, n+1) \leq \frac{(n+m+1)^{r+1}(n+m+2)^r}{(n+m+2)^{r+1} - (n+m+1)^{r+1}}. \quad (23)$$

By inductive hypotheses and $V_r(m, n+1) = V_r(m, n) + (n+m+1)^r$, we have

$$\begin{aligned} V_r(m, n+1) &\leq \frac{(n+m)^{r+1}(n+m+1)^r}{(n+m+1)^{r+1} - (n+m)^{r+1}} + (n+m+1)^r \\ &= \frac{(n+m+1)^{2r+1}}{(n+m+1)^{r+1} - (n+m)^{r+1}}. \end{aligned} \quad (24)$$

Therefore, in order to prove inequality (23), it suffices to show the right term of (23) is not less than the term in the second line of (24), that is,

$$\frac{(n+m+1)^r}{(n+m+2)^r} \leq \frac{(n+m+1)^{r+1} - (n+m)^{r+1}}{(n+m+2)^{r+1} - (n+m+1)^{r+1}} = \frac{(n+m+\xi)^r}{(n+m+1+\xi)^r}, \quad (25)$$

where $\xi \in (0, 1)$ is deduced by using Cauchy's mean value theorem in the second term above, which sounds apparently. Thus, the proof of Lemma 2 is complete. \square

Remark 3. It is remarked that some analogies of inequality (20) in Lemma 2 have been investigated extensively in [1, 2, 6, 7, 8, 9, 10, 11, 13, 14, 15, 18, 23, 24, 25, 26, 27, 29, 30, 31, 33, 34, 39] and the reference therein.

Lemma 3 ([17, 35, 36]). *Euler-Maclaurin's formula states that*

$$\sum_{k=1}^n f(k) = \int_1^n f(x) dx + \frac{f(n) + f(1)}{2} + \int_1^n \rho_1(x) f'(x) dx, \quad (26)$$

where $\rho_1(x) = x - [x] + \frac{1}{2}$ is Bernoulli's function and $f \in C^1[1, \infty)$. Furthermore, if $(-1)^i f^{(i)}(x) > 0$ for $x \in [n, \infty)$ and $\lim_{x \rightarrow \infty} f^{(i)}(x) = 0$ for $i = 1, 2, 3$, then

$$\int_n^\infty \rho_1(x) f'(x) dx = -\frac{1}{12} f'(n) \epsilon, \quad 0 < \epsilon < 1. \quad (27)$$

Lemma 4. For $n \in \mathbb{N}$, $\eta \in (-1, \infty)$ and $\lambda \in [0, 1)$,

$$S_n(\eta, \lambda) < \frac{(n + \eta + 1)^{1-\lambda} - (1 + \eta)^{1-\lambda}}{1 - \lambda} + \gamma(\eta, \lambda), \quad (28)$$

where $S_n(\eta, \lambda)$ and $\gamma(\eta, \lambda)$ are defined in Theorem 1 and by (17) respectively. Moreover,

$$S_n(\eta, 1) < \ln(n + \eta + 1) - \ln(1 + \eta) + \gamma(\eta, 1). \quad (29)$$

Proof. It is clear that Lemma 1 allows us to apply Euler-Maclaurin's formula (26) and formula (27) to $f(x) = \frac{1}{(x+\eta)^\lambda}$. From this, if $\lambda \in (0, 1)$, it follows that

$$\begin{aligned} S_n(\eta, \lambda) &= \frac{(n + \eta + 1)^{1-\lambda} - (1 + \eta)^{1-\lambda}}{1 - \lambda} \\ &\quad + \frac{1}{2} \left[\frac{1}{(n + \eta)^\lambda} + \frac{1}{(1 + \eta)^\lambda} \right] + \int_1^n \rho_1(x) \left[\frac{1}{(x + \eta)^\lambda} \right]' dx, \\ \int_n^\infty \rho_1(x) \left[\frac{1}{(x + \eta)^\lambda} \right]' dx &= -\frac{1}{12} \left[\frac{1}{(x + \eta)^\lambda} \right]' \epsilon = \frac{\lambda \epsilon}{12(n + \eta)^{1+\lambda}}, \end{aligned}$$

where $0 < \epsilon < 1$, and

$$\gamma(\eta, \lambda) = \frac{1}{2(1 + \eta)^\lambda} + \int_1^\infty \rho_1(x) \left[\frac{1}{(x + \eta)^\lambda} \right]' dx.$$

Therefore,

$$\begin{aligned} S_n(\eta, \lambda) &= \frac{(n + \eta)^{1-\lambda} - (1 + \eta)^{1-\lambda}}{1 - \lambda} + \frac{1}{2(n + \eta)^\lambda} + \gamma(\eta, \lambda) - \frac{\lambda \epsilon}{12(n + \eta)^{1+\lambda}} \\ &< \frac{(n + \eta)^{1-\lambda} - (1 + \eta)^{1-\lambda}}{1 - \lambda} + \frac{1}{2(n + \eta)^\lambda} + \gamma(\eta, \lambda), \end{aligned}$$

and then

$$\begin{aligned} S_n(\eta, \lambda) &= \sum_{k=1}^{n+1} \frac{1}{(k + \eta)^\lambda} - \frac{1}{(n + \eta + 1)^\lambda} \\ &< \frac{(n + \eta + 1)^{1-\lambda} - (1 + \eta)^{1-\lambda}}{1 - \lambda} - \frac{1}{2(n + \eta + 1)^\lambda} + \gamma(\eta, \lambda) \\ &< \frac{(n + \eta + 1)^{1-\lambda} - (1 + \eta)^{1-\lambda}}{1 - \lambda} + \gamma(\eta, \lambda), \end{aligned}$$

Inequality (28) follows for $\lambda \in (0, 1)$.

For $\lambda = 0$, inequality (28) holds apparently.

Since

$$\lim_{\lambda \rightarrow 1} \frac{(n + \eta)^{1-\lambda} - (1 + \eta)^{1-\lambda}}{1 - \lambda} = \ln(n + \eta) - \ln(1 + \eta)$$

and $\frac{1}{x} < 2 \ln(1 + \frac{1}{x})$ with $x \geq 1$, then

$$\begin{aligned} S_n(\eta, 1) &= \sum_{k=1}^n \frac{1}{k + \eta} \\ &< \ln(n + \eta) - \ln(1 + \eta) + \frac{1}{2(n + \eta)} + \gamma(\eta, 1) \\ &< \ln(n + \eta + 1) - \ln(1 + \eta) + \gamma(\eta, 1). \end{aligned}$$

The proof of Lemma 2 is complete. \square

Lemma 5. For $x \in [0, \infty)$ and $\alpha \in [0, \infty)$,

$$\left[1 - \frac{1}{2(x + 1 + \alpha)}\right]^{1 + \ln(x + \alpha + 1)} < 1 - \frac{1}{2(n + \alpha + 1)} - \frac{3 \ln(x + \alpha + 1)}{16(x + 1 + \alpha)}. \quad (30)$$

Proof. By $(1 - \frac{1}{t})^{-t} > e$ for $t > 1$, $e^t < 1 + t + \frac{t^2}{2}$ for $t < 0$ and $x + \alpha > \ln(x + \alpha)$ with $\alpha \geq 0$, it follows that

$$\begin{aligned} \left[1 - \frac{1}{2(x + 1 + \alpha)}\right]^{1 + \ln(x + \alpha + 1)} &< \left[1 - \frac{1}{2(x + 1 + \alpha)}\right] \left(\frac{1}{e}\right)^{\frac{\ln(x + \alpha + 1)}{2(x + 1 + \alpha)}} \\ &< \left[1 - \frac{1}{2(x + 1 + \alpha)}\right] \left[1 - \frac{\ln(x + \alpha + 1)}{2(x + 1 + \alpha)} + \frac{\ln^2(x + \alpha + 1)}{8(x + 1 + \alpha)^2}\right] \\ &< \left[1 - \frac{1}{2(x + 1 + \alpha)}\right] \left[1 - \frac{3 \ln(x + \alpha + 1)}{8(x + \alpha + 1)}\right] \\ &< 1 - \frac{1}{2(x + \alpha + 1)} - \frac{3 \ln(x + \alpha + 1)}{16(x + \alpha + 1)}. \end{aligned}$$

The proof of Lemma 5 is complete. \square

Lemma 6. For $k \in \mathbb{N}$, $\eta \in [0, \infty)$ and $\lambda \in [0, 1]$, we have

$$\begin{aligned} B_k(\eta, \lambda) &\triangleq \left[\frac{(k + \eta + 1)^\lambda S_{k+1}(\eta, \lambda)}{(k + \eta)^\lambda S_k(\lambda)} \right]^{(k + \eta)^\lambda S_k(\lambda)} \\ &\leq \left[e \left(1 - \frac{1}{2(k + 1 + \eta)} \right) \right]^{1 + \lambda(k + \eta)^{\lambda - 1} S_k(\eta, \lambda)}. \end{aligned} \quad (31)$$

Proof. For $k \in \mathbb{N}$,

$$B_k(\eta, \lambda) = \left\{ 1 + \frac{1 + [(k + \eta + 1)^\lambda - (k + \eta)^\lambda] S_k(\eta, \lambda)}{(k + \eta)^\lambda S_k(\lambda)} \right\}^{(k + \eta)^\lambda S_k(\eta, \lambda)} \triangleq C_k^{h(k)},$$

where

$$C_k = \left[1 + \frac{1}{g(k)} \right]^{g(k)}, \quad g(k) = \frac{(k + \eta)^\lambda S_k(\eta, \lambda)}{h(k)}$$

and

$$h(k, \lambda) = 1 + [(k + \eta + 1)^\lambda - (k + \eta)^\lambda] S_k(\eta, \lambda).$$

By Lemma 2, it is easy to see that

$$g(k, \lambda) + 1 = \frac{(k + \eta + 1)^\lambda S_{k+1}(\eta, \lambda)}{1 + [(k + \eta + 1)^\lambda - (k + \eta)^\lambda] S_k(\eta, \lambda)} \leq k + \eta + 1. \quad (32)$$

By

$$\left(1 + \frac{1}{x}\right)^x < e \left[1 - \frac{1}{2(x+1)}\right] \quad (33)$$

in [38] and inequality (32), it is deduced that

$$C_k = \left[1 + \frac{1}{g(k)}\right]^{g(k)} \leq e \left\{1 - \frac{1}{2[g(k) + 1]}\right\} \leq e \left[1 - \frac{1}{2(k + 1 + \eta)}\right]. \quad (34)$$

For $\lambda \in [0, 1]$, by using Bernoulli's inequality, we have

$$h(k, \lambda) \leq 1 + (k + \eta)^\lambda \left[\left(1 + \frac{1}{k + \eta}\right)^\lambda - 1 \right] S_k(\eta, \lambda) \leq 1 + \lambda(k + \eta)^{\lambda-1} S_k(\eta, \lambda). \quad (35)$$

Hence, from inequalities (34) and (35), it is showed that

$$\begin{aligned} B_k(\eta, \lambda) &\leq \left\{ e \left[1 - \frac{1}{2(k + 1 + \eta)}\right] \right\}^{h(k, \lambda)} \\ &\leq \left\{ e \left[1 - \frac{1}{2(k + 1 + \eta)}\right] \right\}^{1 + \lambda(k + \eta)^{\lambda-1} S_k(\eta, \lambda)}. \end{aligned}$$

The proof of Lemma 6 is complete. \square

3. PROOFS OF THEOREMS

Proof of Theorem 1. Setting $c_k > 0$ for $1 \leq k \leq n$ and letting

$$\left[\prod_{k=1}^n c_k^{1/(k+\eta)^\lambda} \right]^{-1/S_n(\eta, \lambda)} = \frac{1}{(n + \eta + 1)^\lambda S_{n+1}(\eta, \lambda)},$$

then

$$c_k = \frac{[(k + \eta + 1)^\lambda S_{k+1}(\eta, \lambda)]^{(k+\eta)^\lambda S_k(\eta, \lambda)}}{[(k + \eta)^\lambda S_k(\eta, \lambda)]^{(k+\eta)^\lambda S_{k-1}(\eta, \lambda)}}. \quad (36)$$

Using the discrete weighted arithmetic-geometric mean inequality and (36) and interchanging the order of summation yields

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left[\prod_{k=1}^n a_k^{1/(k+\eta)\lambda} \right]^{1/S_n(\eta,\lambda)} \\
&= \sum_{n=1}^{\infty} \left[\prod_{k=1}^n (c_k a_k)^{1/(k+\eta)\lambda} \right]^{1/S_n(\eta,\lambda)} \left[\prod_{k=1}^n c_k^{1/(k+\eta)\lambda} \right]^{-1/S_n(\eta,\lambda)} \\
&\leq \sum_{n=1}^{\infty} \sum_{k=1}^n \frac{1}{(k+\eta)^\lambda S_n(\eta,\lambda)} c_k a_k \frac{1}{(k+\eta+1)^\lambda S_{n+1}(\eta,\lambda)} \\
&= \sum_{k=1}^{\infty} \frac{1}{(k+\eta)^\lambda} c_k a_k \sum_{n=k}^{\infty} \frac{1}{(k+\eta+1)^\lambda S_n(\eta,\lambda) S_{n+1}(\eta,\lambda)} \\
&= \sum_{k=1}^{\infty} \frac{1}{(k+\eta)^\lambda} c_k a_k \sum_{n=k}^{\infty} \left[\frac{1}{S_n(\eta,\lambda)} - \frac{1}{S_{n+1}(\eta,\lambda)} \right] \\
&= \sum_{k=1}^{\infty} \frac{1}{(k+\eta)^\lambda} c_k a_k \frac{1}{S_k(\eta,\lambda)} \\
&= \sum_{k=1}^{\infty} \left[\frac{(k+\eta+1)^\lambda S_{k+1}(\eta,\lambda)}{(k+\eta)^\lambda S_k(\eta,\lambda)} \right]^{(k+\eta)^\lambda S_k(\eta,\lambda)} a_k.
\end{aligned} \tag{37}$$

Applying (33) and the left side of inequality (32) in the final line of (37) gives inequality (15). The proof of Theorem 1 is complete. \square

Proof of Corollary 1.

$$\begin{aligned}
B_k(\eta) &= \left[\frac{(k+\eta+1) S_{k+1}(\eta)}{(k+\eta) S_k(\eta)} \right]^{(k+\eta) S_k(\eta)} \\
&= \left\{ \left[1 + \frac{1}{(k+\eta) S_k(\eta)/(S_k(\eta)+1)} \right]^{(k+\eta) S_k(\eta)/(S_k(\eta)+1)} \right\}^{S_k(\eta)+1} \\
&< \left\{ e \left[1 - \frac{S_k(\eta)+1}{2(k+\eta+1) S_k(\eta)} \right] \right\}^{S_k(\eta)+1} \\
&< \left\{ e \left[1 - \frac{1}{2(k+\eta+1)} \right] \right\}^{1+\ln(k+\eta+1)+\gamma(\eta)} \\
&< e^{1+\gamma(\eta)} (k+\eta+1) \left[1 - \frac{1}{2(k+\eta+1)} \right]^{1+\ln(k+\eta+1)} \\
&< e^{1+\gamma(\eta)} \left[k + \eta + \frac{1}{2} - \frac{3 \ln(k+\eta+1)}{16} \right]
\end{aligned} \tag{38}$$

Taking $\lambda = 1$ in inequality (37) yields

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/(k+\eta)} \right)^{1/S_n(\eta)} &\leq \sum_{n=1}^{\infty} \left[\frac{(k+\eta+1) S_{k+1}(\eta)}{(k+\eta) S_k(\eta)} \right]^{(k+\eta) S_k(\eta)} a_n \\
&< \sum_{n=1}^{\infty} e^{1+\gamma(\eta)} \left[n + \eta + \frac{1}{2} - \frac{3 \ln(n+\eta+1)}{16} \right] a_n
\end{aligned} \tag{39}$$

The proof of Corollary 1 is complete. \square

Proof of Theorem 2. Applying Lemma 6 in (37) gives inequality (18) clearly.

For $\lambda = 0$, since $S_n(0) = n$, inequality (19) is reduced to Carleman's inequality

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k \right)^n < \sum_{n=1}^{\infty} e \left[1 - \frac{1}{2(n+1)} \right] a_n. \quad (40)$$

For $\lambda \in (0, 1)$,

$$S_n(\eta, \lambda) = \sum_{k=1}^n \frac{1}{(k+\eta)^\lambda} < \int_0^n \frac{1}{(t+\eta)^\lambda} dt = \frac{(n+\eta)^{1-\lambda}}{1-\lambda} - \frac{\eta^{1-\lambda}}{1-\lambda}. \quad (41)$$

Taking $\eta = 0$ in (18) and (41) yields inequality (41). The proof of Theorem 2 is complete. \square

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