SUMS OF SERIES OF ROGERS DILOGARITHM FUNCTIONS

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ABSTRACT. Some sums of series of Rogers dilogarithm functions are established by Abel's functional equation.

1. INTRODUCTION

The dilogarithm is defined [2, p.102] by the series

$$\operatorname{Li}_2(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2} \tag{1}$$

for $-1 \le x \le 1$. The Rogers dilogarithm function $L_R(x)$ is defined in [8] and [13, p. 287] for $0 \le x \le 1$ by

$$L_R(x)=\begin{cases}\operatorname{Li}_2(x)+\dfrac{1}{2}\ln x\ln(1-x), & 0<x<1,\\ 0, & x=0,\\ \dfrac{\pi^2}{6}, & x=1.\end{cases} \tag{2}$$

The function $L_R(x)$ satisfies the concise identity

$$L_R(x)+L_R(1-x)=\frac{\pi^2}{6} \tag{3}$$

for $0 \le x \le 1$, see [7, pp. 110–113], and Abel's functional equation

$$L_R(x)+L_R(y)=L_R(xy)+L_R\left(\frac{x(1-y)}{1-xy}\right)+L_R\left(\frac{y(1-x)}{1-xy}\right) \tag{4}$$

for $0<x,y<1$, see [1, pp. 189–192] and [5]. The duplication formula for $L_R(x)$ follows from Abel's functional equation (4) and is given for $0 \le x \le 1$ by

$$L_R(x)=\frac{1}{2}L_R(x^2)+L_R\left(\frac{x}{1+x}\right). \tag{5}$$

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The function $L_R(x)$ satisfies also the following identities:

$$L_R\left(\frac{1}{2}\right)=\frac{\pi^2}{12},\quad L_R(\rho)=\frac{\pi^2}{10},\quad L_R\big(\rho^2\big)=L_R(1-\rho)=\frac{\pi^2}{15},\tag{6}$$

where $\rho=\frac{\sqrt{5}-1}{2}$, and has the nice infinite series

$$\sum_{n=2}^{\infty}L_R\left(\frac{1}{n^2}\right)=\frac{\pi^2}{6}\tag{7}$$

obtained in [13, p. 298] and [14].

It is remarked that the formulas from (1) to (7) can be looked up at [18, 19].

For more information on its history, properties, identities, generalizations, applications and recent developments of the dilogarithms and Rogers dilogarithm functions, please refer to [1, pp. 189–192], [2, pp.102–107], [4, pp. 323–326], [7, pp. 110–113], [3, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] and the references therein.

The main aim of this paper is to generalize the series (7).

Our main results are the following four theorems.

Theorem 1. *For $p,q\in\mathbb{N}$ and $\alpha\ge0$,*

$$\sum_{n=0}^{\infty}L_R\left(\frac{pq}{(n+p+\alpha)(n+q+\alpha)}\right)=\sum_{n=0}^{q-1}L_R\left(\frac{p}{n+p+\alpha}\right)+\sum_{n=0}^{p-1}L_R\left(\frac{q}{n+q+\alpha}\right).\tag{8}$$

Remark 1. The series (7) is a special case of (8) for $p=q=\alpha=1$.

Theorem 2. *For $p,q\in\mathbb{N}$ and $0<\theta,\beta<1$,*

$$\sum_{n=0}^{\infty}L_R\left(\frac{\beta(1-\theta^p)(1-\theta^q)\theta^n}{(1-\beta\theta^{n+p})(1-\beta\theta^{n+q})}\right)=\sum_{n=0}^{q-1}L_R\left(\frac{\beta(1-\theta^p)\theta^n}{1-\beta\theta^{n+p}}\right)+\sum_{n=0}^{p-1}L_R\left(\frac{1-\theta^q}{1-\beta\theta^{n+q}}\right)-pL_R(1-\theta^q).\tag{9}$$

Theorem 3. *For $p,q\in\mathbb{N}$, $0<\beta\le1$ and $0<\theta<1$,*

$$\sum_{n=0}^{\infty}L_R\left(\frac{\beta(1-\theta^p)(1-\theta^q)\theta^n}{(1+\beta\theta^n)(1+\beta\theta^{n+p+q})}\right)=\sum_{n=0}^{q-1}L_R\left(\frac{\theta^p(1+\beta\theta^n)}{1+\beta\theta^{n+p}}\right)+\sum_{n=0}^{p-1}L_R\left(\frac{\beta(1-\theta^q)\theta^n}{1+\beta\theta^n}\right)-qL_R(\theta^p).\tag{10}$$

Theorem 4. *For $r > 1$,*

$$\sum_{n=0}^{\infty} \frac{1}{2^n} L_R\left(\frac{1}{r^{2^n}+1}\right) = L_R\left(\frac{1}{r}\right). \tag{11}$$

As straightforward consequences of above theorems, some sums of series of special Rogers dilogarithm functions are deduced as follows.

Corollary 1. *Let $t > 0$ and $\phi = \frac{\sqrt{5}+1}{2}$, then the following identities are valid:*

$$\sum_{n=2}^{\infty} L_R\left(\frac{2}{n(n+1)}\right) = \frac{\pi^2}{4}, \tag{12}$$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} L_R\left(\frac{1}{2^{2^n}+1}\right) = \frac{\pi^2}{12}, \tag{13}$$

$$\sum_{n=1}^{\infty} L_R\left(\frac{2}{n^2+\sqrt{5}\,n+1}\right) = \frac{\pi^2}{6} + L_R(3-\sqrt{5}\,), \tag{14}$$

$$\sum_{n=1}^{\infty} L_R\left(\frac{2}{(n+\sqrt{2}\,)(n+1+\sqrt{2}\,)}\right) = \frac{\pi^2}{6} + L_R\left(\frac{1}{2+\sqrt{2}}\right), \tag{15}$$

$$\sum_{n=1}^{\infty} L_R\left(\frac{4}{\left(2n-1+\sqrt{5}\,\right)^2}\right) = \frac{\pi^2}{5}, \tag{16}$$

$$\sum_{n=2}^{\infty} (-1)^n L_R\left(\frac{4}{n^2}\right) = \frac{\pi^2}{3} - 2L_R\left(\frac{2}{3}\right), \tag{17}$$

$$\sum_{n=1}^{\infty} L_R\left(\frac{2^n}{(2^{n+1}-1)^2}\right) = \frac{\pi^2}{12}, \tag{18}$$

$$\sum_{n=1}^{\infty} L_R\left(\frac{\phi^{n-2}}{(\phi^{n+1}-1)^2}\right) = \frac{\pi^2}{10}, \tag{19}$$

$$\sum_{n=1}^{\infty} L_R\left(\frac{2^{n-1}}{(2^{n-1}+1)(2^{n+1}+1)}\right) = \frac{3}{2} L_R\left(\frac{1}{4}\right), \tag{20}$$

$$\sum_{n=1}^{\infty} L_R\left(\frac{2^2 3^{n-1}}{(3^{n-1}+1)(3^{n+1}+1)}\right) = \frac{\pi^2}{12}, \tag{21}$$

$$\sum_{n=2}^{\infty} L_R\left(\frac{\sinh^2 t}{\sinh^2(nt)}\right) = L_R(e^{-2t}), \tag{22}$$

and

$$\sum_{n=1}^{\infty} L_R\left(\frac{\sinh^2 t}{\cosh[(n-1)t]\cosh[(n+1)t]}\right) = L_R\left(\frac{e^{-t}}{\cosh t}\right) + L_R(e^{-t}\sinh t) - L_R(e^{-2t}). \tag{23}$$

2. Proofs of theorems and corollary

Proof of Theorem 1. Let

$$x_n = \frac{p}{n+p+\alpha} \qquad \text{and} \qquad y_n = \frac{q}{n+q+\alpha}$$

for $n = 0, 1, 2, \dots$. It is clear that

$$\frac{x_n(1-y_n)}{1-x_n y_n} = \frac{p}{(n+q)+p+\alpha} = x_{n+q},$$

and

$$\frac{y_n(1-x_n)}{1-x_n y_n} = \frac{q}{(n+p)+q+\alpha} = y_{n+p}.$$

Taking $x = x_n$ and $y = y_n$ in (4) leads to

$$L_R(x_n) + L_R(y_n) = L_R\biggl(\frac{pq}{(n+p+\alpha)(n+q+\alpha)}\biggr) + L_R\bigl(x_{n+q}\bigr) + L_R\bigl(y_{n+p}\bigr)$$

for $n = 0, 1, 2, \dots$. Summing up on both sides of above equality for n from 0 to $N \ge \max\{p, q\}$ gives

$$\sum_{n=0}^{q-1} L_R(x_n) + \sum_{n=0}^{p-1} L_R(y_n) = \sum_{n=0}^{N} L_R\biggl(\frac{pq}{(n+p+\alpha)(n+q+\alpha)}\biggr) + \sum_{n=N+1-q}^{N} L_R\bigl(x_{n+q}\bigr) + \sum_{n=N+1-p}^{N} L_R\bigl(y_{n+p}\bigr).$$

Letting $N \to \infty$ yields

$$\lim_{N\to\infty} \sum_{n=N+1-q}^{N} L_R\bigl(x_{n+q}\bigr) = \lim_{N\to\infty} \sum_{n=N+1-p}^{N} L_R\bigl(y_{n+p}\bigr) = 0.$$

The proof of Theorem 1 is complete. □

Proof of Theorem 2. Now let us consider the sequences

$$x_n = \frac{\beta(1-\theta^p)\theta^n}{1-\beta\theta^{n+p}} \qquad \text{and} \qquad y_n = \frac{1-\theta^q}{1-\beta\theta^{n+q}}$$

for $n = 0, 1, 2, \dots$. It is obvious that $0 < x_n < 1$ and $0 < y_n < 1$. Straightforward computation gives

$$\frac{x_n(1-y_n)}{1-x_n y_n} = \frac{\beta(1-\theta^p)\theta^{n+q}}{1-\beta\theta^{(n+q)+p}} = x_{n+q}$$

and

$$\frac{y_n(1-x_n)}{1-x_n y_n} = \frac{1-\theta^q}{1-\beta\theta^{(n+p)+q}} = y_{n+p}.$$

Using identity (4) again gives

$$L_R(x_n) + L_R(y_n) = L_R(x_n y_n) + L_R\big(x_{n+q}\big) + L_R\big(y_{n+p}\big).$$

Summing up for n from 0 to $N \geq \max\{p, q\}$ leads to

$$\sum_{n=0}^{q-1} L_R(x_n) + \sum_{n=0}^{p-1} L_R(y_n) = \sum_{n=0}^{N} L_R(x_n y_n) + \sum_{n=N+1-q}^{N} L_R\big(x_{n+q}\big) + \sum_{n=N+1-p}^{N} L_R\big(y_{n+p}\big).$$

Since $\lim_{n\to\infty} L_R(x_n) = 0$ and $\lim_{n\to\infty} L_R(y_n) = L_R(1-\theta^q)$, if taking $N \to \infty$ in above identity, then formula (9) follows. The proof of Theorem 3 is finished. $\square$

Proof of Theorem 3. Let

$$x_n = \frac{\theta^p(1+\beta\theta^n)}{1+\beta\theta^{n+p}} \qquad \text{and} \qquad y_n = \frac{\beta(1-\theta^q)\theta^n}{1+\beta\theta^n}$$

for $n = 0, 1, 2, \ldots$. It is apparent that $0 < x_n, y_n < 1$. Direct calculation reveals

$$x_n y_n = \frac{\beta(1-\theta^q)\theta^{n+p}}{1+\beta\theta^{n+p}} = y_{n+p}$$

and

$$\frac{x_n(1-y_n)}{1-x_n y_n} = \frac{\theta^p(1+\beta\theta^{n+q})}{1+\beta\theta^{(n+q)+p}} = x_{n+q}$$

with

$$\frac{y_n(1-x_n)}{1-x_n y_n} = \frac{\beta(1-\theta^p)(1-\theta^q)\theta^n}{(1+\theta^n)(1+\theta^{n+p+q})} \triangleq z_n.$$

From identity (4), it follows that

$$L_R(x_n) + L_R(y_n) = L_R\big(y_{n+p}\big) + L_R\big(x_{n+q}\big) + L_R(z_n).$$

Therefore, for $N \geq \max\{p, q\}$,

$$\sum_{n=0}^{q-1} L_R(x_n) + \sum_{n=0}^{p-1} L_R(y_n) = \sum_{n=0}^{N} L_R(z_n) + \sum_{n=N+1-q}^{N} L_R(x_{n+q}) + \sum_{n=N+1-p}^{N} L_R(y_{n+p}).$$

Since $\lim_{n\to\infty} L_R(x_n) = L_R(\theta^p)$ and $\lim_{n\to\infty} L_R(y_n) = 0$, then formula (10) is deduced by taking $N \to \infty$. Theorem 3 is proved. $\square$

Proof of Theorem 4. Applying (5) to $x = x_n = \frac{1}{r^{2^n}+1}$ for $n = 0, 1, 2, \dots$ gives

$$L_R(x_n) = \frac{1}{2}L_R(x_{n+1}) + L_R\left(\frac{1}{r^{2^n}+1}\right)$$

and

$$\frac{1}{2^n}L_R(x_n) = \frac{1}{2^{n+1}}L_R(x_{n+1}) + \frac{1}{2^n}L_R\left(\frac{1}{r^{2^n}+1}\right)$$

for $n = 0, 1, 2, \dots$. Summing up for n from 0 to ∞ yields

$$\sum_{n=0}^{\infty}\frac{1}{2^n}L_R\left(\frac{1}{r^{2^n}+1}\right) = L_R(x_0) = L_R\left(\frac{1}{r}\right).$$

The proof of Theorem 4 is complete. □

Proof of Corollary 1. Taking $p = 2$, $q = 1$ and $\alpha = 1, \frac{\sqrt{5}-1}{2}, \sqrt{2}$ in (8) and simplifying by employing (3) and (6) leads to the identities (12), (14) and (15) respectively.

Identity (13) is a direct consequence of (11) for $r = 2$.

Letting $p = q = 1$ and $\alpha = \frac{\sqrt{5}-1}{2}$ in (8) yields (16).

It is easy to see that

$$\sum_{n=2}^{\infty}(-1)^n L_R\left(\frac{4}{n^2}\right) = \sum_{n=1}^{\infty}L_R\left(\frac{1}{n^2}\right) - \sum_{n=1}^{\infty}L_R\left(\frac{1}{(n+1/2)^2}\right).$$

Combining this with (8) for $p = q = 1$ and $\alpha = \frac{1}{2}$ leads to (17).

Identities (18) and (19) are special cases of (9) for $p = q = 1$, $\beta = \theta = \frac{1}{2}$ and $\beta = \theta = \frac{1}{\phi} = \frac{\sqrt{5}-1}{2}$, respectively.

Applying $p = q = \beta = 1$ and $\theta = \frac{1}{2}$ in (10) gives

$$\sum_{n=1}^{\infty}L_R\left(\frac{2^{n-1}}{(2^{n-1}+1)(2^{n+1}+1)}\right) = L_R\left(\frac{2}{3}\right) + L_R\left(\frac{1}{4}\right) - L_R\left(\frac{1}{2}\right).$$

Taking $x = \frac{1}{2}$ in identity (5) yields

$$L_R\left(\frac{2}{3}\right) - L_R\left(\frac{1}{2}\right) = \frac{1}{2}L_R\left(\frac{1}{4}\right).$$

Thus, identity (20) is obtained.

Identity (21) is a direct consequence of (10) for $p = q = \beta = 1$ and $\theta = \frac{1}{3}$.

Taking $\theta = e^{-2t}$ and $\beta = e^{-2b}$ in (9) and (10) and simplifying gives

$$\sum_{n=0}^{\infty}L_R\left(\frac{\sinh(pt)\sinh(qt)}{\sinh((n+p)t+b)\sinh((n+q)t+b)}\right) = \sum_{n=0}^{q-1}L_R\left(\frac{e^{-(nt+b)}\sinh(pt)}{\sinh((n+p)t+b)}\right) + \sum_{n=0}^{p-1}L_R\left(\frac{e^{(nt+b)}\sinh(qt)}{\sinh((n+q)t+b)}\right) - pL_R\big(1-e^{2qt}\big) \quad (24)$$

for $t > 0$ and $b > 0$ and

$$\sum_{n=0}^{\infty} L_R\left(\frac{\sinh(pt)\sinh(qt)}{\cosh(nt+b)\cosh((n+p+q)t+b)}\right) = \sum_{n=0}^{q-1} L_R\left(\frac{e^{-pt}\cosh(nt+b)}{\cosh((n+p)t+b)}\right) + \sum_{n=0}^{p-1} L_R\left(\frac{e^{-(n+q)t-b}\sinh(qt)}{\cosh(nt+b)}\right) - qL_R\left(e^{-2t}\right) \quad (25)$$

for $t > 0$ and $b \geq 0$. Identities (22) and (23) are special cases of (24) and (25) for $p = q = 1$, $b = t$ and $b = 0$, respectively. □

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