SOME INEQUALITIES FOR (α, β)-NORMAL OPERATORS IN HILBERT SPACES

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Abstract. An operator T is called (α, β)-normal $(0 \leq \alpha \leq 1 \leq \beta)$ if

$$\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T.$$

In this paper, we establish various inequalities between the operator norm and its numerical radius of (α, β)-normal operators in Hilbert spaces. For this purpose, we employ some classical inequalities for vectors in inner product spaces.

1. Introduction

An operator T acting on a Hilbert space $(\mathcal{H}; \langle \cdot, \cdot \rangle)$ is called (α, β)-*normal* $(0 \leq \alpha \leq 1 \leq \beta)$ if

$$\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T.$$

Then

$$\alpha^2 \langle T^*Tx, x \rangle \leq \langle TT^*x, x \rangle \leq \beta^2 \langle T^*Tx, x \rangle ,$$

whence

$$\alpha \|Tx\| \leq \|T^*x\| \leq \beta \|Tx\|, \tag{1.1}$$

for all $x \in \mathcal{H}$. If T is invertible, then so is the bounded operator T^*T^{-1}. Hence T^*T^{-1} is bounded below and so T is (α, β)-normal for some α and β.

Normal and hyponormal operators are trivially (α, β)-normal for some appropriate values of α and β. There are however operators which are neither normal nor hyponormal. The following example of an (α, β)-normal with $\alpha = \sqrt{\frac{3-\sqrt{5}}{2}}$ and $\beta = \sqrt{\frac{3+\sqrt{5}}{2}}$ is due to M. Mirzavaziri

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in B(\mathbb{C}^2).$$

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Let $(\mathcal{H}; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. The *numerical radius* $w(T)$ of an operator T on $\mathcal{H}$ is given by

$$w(T) = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.2}$$

Obviously, by (1.2), for any $x \in \mathcal{H}$ one has

$$|\langle Tx, x\rangle| \le w(T)\|x\|^2. \tag{1.3}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(\mathcal{H})$ of all bounded linear operators. Moreover, we have

$$w(T) \le \|T\| \le 2w(T) \qquad (T \in B(\mathcal{H})).$$

For other results and historical comments on the numerical radius see [10].

In this paper, we establish various inequalities between the operator norm and its numerical radius of (α, β)-normal operators in Hilbert spaces. For this purpose, we employ some classical inequalities for vectors in inner product spaces due to Buzano, Dunkl–Williams, Dragomir–Sándor, Goldstein–Ryff–Clarke and Dragomir.

2. Inequalities Involving Numerical Radius

In this section we study some inequalities concerning the numerical radius and norm of (α, β)-normal operators. Our first result reads as follows, see also [6]:

Theorem 2.1. *Let $T \in B(\mathcal{H})$ be an (α, β)-normal operator. Then*

$$(\alpha^{2r} + \beta^{2r})\|T\|^2 \le \begin{cases} 2\beta^r w(T^2) + r^2\beta^{2r-2}\|\beta T - T^*\|^2, & \text{if } r \ge 1, \\ \\ 2\beta^r w(T^2) + \|\beta T - T^*\|^2, & \text{if } r < 1. \end{cases} \tag{2.1}$$

Proof. We use the following inequality for vectors in inner product spaces due to Goldstein, Ryff and Clarke [9]:

$$\|a\|^{2r} + \|b\|^{2r} - 2\|a\|^r\|b\|^r \cdot \frac{\mathrm{Re}\langle a, b\rangle}{\|a\|\,\|b\|} \le \begin{cases} r^2\|a\|^{2r-2}\|a-b\|^2 & \text{if } r \ge 1, \\ \\ \|b\|^{2r-2}\|a-b\|^2 & \text{if } r < 1, \end{cases} \tag{2.2}$$

provided $r \in \mathbb{R}$ and $a, b \in H$ with $\|a\| \ge \|b\|$.

Suppose that $r \ge 1$. Let $x \in H$ with $\|x\| = 1$. Noting to (1.1) and applying (2.2) for the choices $a = \beta Tx$, $b = T^*x$ we get

$$\begin{aligned} \|\beta Tx\|^{2r} + \|T^*x\|^{2r} - 2\|\beta Tx\|^{r-1}\|\,\|T^*x\|^{r-1}\,\mathrm{Re}\langle \beta Tx, T^*x\rangle \\ \le r^2\|\beta Tx\|^{2r-2}\|\beta Tx - T^*x\|^2 \end{aligned} \tag{2.3}$$

for any $x \in H$, $\|x\| = 1$ and $r \geq 1$. Using (1.1) and (2.3) we get

$$(2.4)\quad (\alpha^{2r} + \beta^{2r})\|Tx\|^{2r} \leq 2\beta^r \|Tx\|^{r-1}\|T^*x\|^{r-1}|\langle T^2x, x\rangle| + r^2\beta^{2r-2}\|Tx\|^{2r-2}\|\beta Tx - T^*x\|^2.$$

Taking the supremum in (2.4) over $x \in H$, $\|x\| = 1$, we deduce

$$(\alpha^{2r} + \beta^{2r})\|T\|^{2r} \leq 2\beta^r \|T\|^{2r-2}\|T^*\|^{r-1}w(T^2) + r^2\beta^{2r-2}\|T\|^{2r-2}\|\beta T - T^*\|^2,$$

which is the first inequality in (2.1). If $r < 1$, then one can similarly prove the second inequality in (2.1). ∎

Theorem 2.2. *Let $T \in B(\mathcal{H})$ be an (α, β)-normal operator. Then*

$$(2.5)\qquad w(T)^2 \leq \frac{1}{2}\left[\beta\|T\|^2 + w(T^2)\right].$$

Proof. The following inequality is known in the literature as the *Buzano inequality* [1]:

$$(2.6)\qquad |\langle a, e\rangle\langle e, b\rangle| \leq \frac{1}{2}(\|a\|\,\|b\| + |\langle a, b\rangle|),$$

for any a, b, e in $\mathcal{H}$ with $\|e\| = 1$.

Let $x \in H$ with $||x|| = 1$. Put $e = x, a = Tx, b = T^*x$ in (2.6) to get

$$|\langle Tx, x\rangle\langle x, T^*x\rangle| \leq \frac{1}{2}(\|Tx\|\,\|T^*x\| + |\langle Tx, T^*x\rangle|) \leq \frac{1}{2}(\beta\|Tx\|^2 + |\langle T^2x, x\rangle|).$$

Taking the supremum over $x \in H$, $\|x\| = 1$, we obtain (2.5). ∎

Theorem 2.3. *Let $T \in B(\mathcal{H})$ be an (α, β)-normal operator and $\lambda \in \mathbb{C}$. Then*

$$(2.7)\qquad \alpha\|T\|^2 \leq w(T^2) + \frac{2\beta\|T - \lambda T^*\|^2}{(1 + |\lambda|\alpha)^2}.$$

Proof. Using the *Dunkl–Williams inequality* [8]

$$\frac{1}{2}(\|a\| + \|b\|)\left\|\frac{a}{\|a\|} - \frac{b}{\|b\|}\right\| \leq \|a - b\| \qquad (a, b \in H \setminus \{0\})$$

we get

$$2 - 2\cdot\frac{\mathrm{Re}\langle a, b\rangle}{\|a\|\|b\|} = \left\|\frac{a}{\|a\|} - \frac{b}{\|b\|}\right\|^2 \leq \frac{4\|a - b\|^2}{(\|a\| + \|b\|)^2} \qquad (a, b \in H \setminus \{0\})$$

whence

$$\|a\|\|b\| \leq \frac{2\|a\|\,\|b\|\,\|a - b\|^2}{(\|a\| + \|b\|)^2} + |\langle a, b\rangle| \qquad (a, b \in H \setminus \{0\}).$$

Put $a = Tx$ and $b = \lambda T^*$ to get

$$\|Tx\|\,\|T^*x\| \leq |\langle T^2x, x\rangle| + \frac{2\|Tx\|\,\|T^*x\|\,\|Tx - \lambda T^*x\|^2}{(\|Tx\| + |\lambda|\,\|T^*x\|)^2}$$

so that

$$\alpha\|Tx\|^2 \leq |\langle T^2x, x\rangle| + \frac{2\beta\|Tx\|^2\|Tx-\lambda T^*x\|^2}{(\|Tx\|+|\lambda|\alpha\,\|Tx\|)^2} \leq |\langle T^2x, x\rangle| + \frac{2\beta\|(T-\lambda T^*)x\|^2}{(1+|\lambda|\alpha)^2}. \tag{2.8}$$

Taking the supremum in (2.8) over $x \in H$, $\|x\| = 1$, we get the desired result (2.7). ∎

Theorem 2.4. *Let $T \in B(\mathcal{H})$ be an (α, β)-normal operator and $\lambda \in \mathbb{C}\backslash\{0\}$. Then*

$$\left[\alpha^2 - \left(\frac{1}{|\lambda|} + \beta\right)^2\right] \|T\|^4 \leq w(T^2). \tag{2.9}$$

Proof. We apply the following reverse of the quadratic Schwarz inequality obtained by Dragomir in [5]

$$(0 \leq)\|a\|^2\|b\|^2 - |\langle a, b\rangle|^2 \leq \frac{1}{|\lambda|^2}\|a\|^2\|a-\lambda b\|^2 \tag{2.10}$$

provided $a, b \in H$ and $\lambda \in \mathbb{C}\backslash\{0\}$.

Set $a = Tx, b = T^*x$ in (2.10), to get

$$\alpha^2\|Tx\|^4 \leq |\langle Tx, T^*x\rangle|^2 + \frac{1}{|\lambda|^2}\|Tx\|^2\|Tx-\lambda T^*x\|^2 \leq |\langle T^2x, x\rangle|^2 + \frac{1}{|\lambda|^2}\|Tx\|^2(1+|\lambda|\beta)^2\|Tx\|^2$$

whence

$$\left[\alpha^2 - \left(\frac{1}{|\lambda|} + \beta\right)^2\right] \|Tx\|^4 \leq |\langle T^2x, x\rangle|^2. \tag{2.11}$$

Taking the supremum in (2.11) over $x \in H$, $\|x\| = 1$, we get the desired result (2.9). ∎

Theorem 2.5. *Let $T \in B(\mathcal{H})$ be an (α, β)-normal operator, $r \geq 0$ and $\lambda \in \mathbb{C}\setminus\{0\}$. If $\|\lambda T^* - T\| \leq r$ and $\frac{r}{|\lambda|} \leq \inf\{\|T^*x\| : \|x\| = 1\}$, then*

$$\alpha^2\|T\|^4 \leq w(T^2)^2 + \frac{r^2}{|\lambda|^2}\|T\|^2. \tag{2.12}$$

Proof. We use the following reverse of the Schwarz inequality obtained by Dragomir in [3] (see also [4, p. 20]):

$$(0 \leq)\, \|y\|^2\, \|a\|^2 - [\mathrm{Re}\langle y, a\rangle]^2 \leq r^2\|y\|^2, \tag{2.13}$$

provided $\|y - a\| \leq r \leq \|a\|$.

By the assumption of theorem $\|Tx-\lambda T^*x\|\leq r\leq\|\lambda T^*x\|$. Setting $a=\lambda T^*x$ and $y=Tx$, with $\|x\|=1$ in (2.13) we get

$$\|Tx\|^2\,\|\lambda T^*x\|^2\leq[\mathrm{Re}\langle Tx,\lambda T^*x\rangle]^2+r^2\|Tx\|^2$$

whence

$$\alpha^2|\lambda|^2\|Tx\|^4\leq|\lambda|^2|\langle T^2x,x\rangle|^2+r^2\|Tx\|^2. \tag{2.14}$$

Taking the supremum in (2.14) over $x\in H$, $\|x\|=1$, we get the desired result (2.12). ∎

Finally, the following result that is less restrictive for the involved parameters r and λ (from the above theorem) may be stated as well:

Theorem 2.6. *Let $T\in B(\mathcal{H})$ be an (α,β)-normal operator, $r\geq 0$ and $\lambda\in\mathbb{C}\diagdown\{0\}$. If $\|\lambda T^*-T\|\leq r$, then*

$$\alpha\|T\|^2\leq w(T^2)+\frac{r^2}{2|\lambda|}. \tag{2.15}$$

Proof. We use the following reverse of the Schwarz inequality obtained by Dragomir in [2] (see also [4, p. 27]):

$$(0\leq)\;\|y\|\,\|a\|-\mathrm{Re}\langle y,a\rangle\leq\frac{1}{2}r^2, \tag{2.16}$$

provided $\|y-a\|\leq r$.

Setting $a=\lambda T^*x$ and $y=Tx$, with $\|x\|=1$ in (2.16) we get

$$\|Tx\|\,\|\lambda T^*x\|\leq|\langle Tx,\lambda T^*x\rangle|+\frac{1}{2}r^2$$

which gives

$$\alpha\|Tx\|^2\leq|\langle T^2x,x\rangle|+\frac{1}{2}r^2.$$

Now, taking the supremum over $\|x\|=1$ in this inequality, we get the desired result (2.15) ∎

3. Inequalities Involving Norms

Our first result in this section reads as follows.

Theorem 3.1. *Let $T\in B(\mathcal{H})$ be an (α,β)-normal operator. If $p\geq 2$, then*

$$2(1+\alpha^p)\|T\|^p\leq\frac{1}{2}(\|T+T^*\|^p+\|T-T^*\|^p). \tag{3.1}$$

In general, for each $T\in B(\mathcal{H})$ and $p\geq 2$ we have

$$\left\|\frac{T^*T+TT^*}{2}\right\|^{p/2}\leq\frac{1}{4}(\|T+T^*\|^p+\|T-T^*\|^p). \tag{3.2}$$

Proof. We use the following inequality obtained by Dragomir and Sándor in [7] (see also [11, p. 544]):

$$\|a+b\|^p + \|a-b\|^p \geq 2(\|a\|^p + \|b\|^p) \tag{3.3}$$

for any $a, b \in H$ and $p \geq 2$.

Now, if we choose $a = Tx$, $b = T^*x$ in (3.3), then we get

$$\|Tx+T^*x\|^p + \|Tx-T^*x\|^p \geq 2(\|Tx\|^p + \|T^*x\|^p), \tag{3.4}$$

whence

$$\|Tx+T^*x\|^p + \|Tx-T^*x\|^p \geq 2(\|Tx\|^p + \alpha^p\|Tx\|^p), \tag{3.5}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum in (3.5) over $x \in H$, $\|x\| = 1$, we get the desired result (3.1).

Now for the general case $T \in B(\mathcal{H})$, observe that

$$\|Tx\|^p + \|T^*x\|^p = (\|Tx\|^2)^{\frac{p}{2}} + (\|T^*x\|^2)^{\frac{p}{2}} \tag{3.6}$$

and by applying the elementary inequality:

$$\frac{a^q + b^q}{2} \geq \left(\frac{a+b}{2}\right)^q, \quad a, b \geq 0 \text{ and } q \geq 1$$

we have

$$\begin{aligned}(\|Tx\|^2)^{\frac{p}{2}} + (\|T^*x\|^2)^{\frac{p}{2}} &\geq 2^{1-\frac{p}{2}}(\|Tx\|^2 + \|T^*x\|^2)^{\frac{p}{2}} \\ &= 2^{1-\frac{p}{2}}[\langle Tx, Tx\rangle + \langle T^*x, T^*x\rangle]^{\frac{p}{2}} \\ &= 2^{1-\frac{p}{2}}[\langle (T^*T + TT^*)x, x\rangle]^{\frac{p}{2}}.\end{aligned} \tag{3.7}$$

Combining (3.4) with (3.7) and (3.6) we get

$$\frac{1}{4}[\|Tx-T^*x\|^p + \|Tx+T^*x\|^p] \geq \left|\left\langle \left(\frac{T^*T+TT^*}{2}\right)x, x\right\rangle\right|^{p/2} \tag{3.8}$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$, and taking into account that

$$w\left(\frac{T^*T+TT^*}{2}\right) = \left\|\frac{T^*T+TT^*}{2}\right\|,$$

we deduce the desired result (3.2). ∎

Theorem 3.2. *Let $T \in B(\mathcal{H})$ be an (α, β)-normal operator. If $p \in (1, 2)$ and $\lambda, \mu \in \mathbb{C}$, then*

$$\begin{aligned}&[(|\lambda| + \beta|\mu|)^p + \max\{|\lambda| - |\mu|\beta, \alpha|\mu| - |\lambda|\}]\,\|T\|^p \\ &\qquad \leq \|\lambda T + \mu T^*\|^p + \|\lambda T - \mu T^*\|^p.\end{aligned} \tag{3.9}$$

Proof. We use the following inequality obtained by Dragomir and Sándor in [7] (see also [11, p. 544])

$$(\|a\|+\|b\|)^p + |\,\|a\|-\|b\|\,|^p \le ||a+b\|^p + \|a-b\|^p, \tag{3.10}$$

for any $a,b\in H$ and $p\in(1,2)$.

Put $a=\lambda Tx$, $b=\mu T^*x$ in (3.10) to obtain

$$(\|\lambda Tx\|+\|\mu T^*x\|)^p + |\|\lambda Tx\|-\|\mu T^*x\||^p \le \|\lambda Tx+\mu T^*x\|^p + \|\lambda Tx-\mu T^*x\|^p,$$

whence

$$(|\lambda|+|\mu|\alpha)^p\|Tx\|^p + (\max\{|\lambda|-|\mu|\beta, \alpha|\mu|-|\lambda|\})\,\|Tx\|^p \le \|\lambda Tx+\mu T^*x\|^p + \|\lambda Tx-\mu T^*x\|^p, \tag{3.11}$$

for any $x\in H$, $\|x\|=1$.

Taking the supremum in (3.11) over $x\in H$, $\|x\|=1$, we get the desired result (3.9). ∎

4. Other Inequalities for General Operators

Finally, we present two results holding in the general case of bounded linear operators in Hilbert spaces:

Theorem 4.1. *Let $T,S\in B(\mathcal{H})$. Then*

$$|\,\|T^*T+S^*S\|-\|T+S\|^2| \le 2w(S^*T), \tag{4.1}$$

in particular,

$$|\,\|T^*T+|\lambda|TT^*\|-\|T+\lambda T^*\|^2| \le 2|\lambda|w(T^2). \tag{4.2}$$

Proof. We have

$$\|Tx\pm Sx\|^2 = \|Tx\|^2 \pm 2\mathrm{Re}\langle Tx,Sx\rangle + \|Sx\|^2 \tag{4.3}$$

for any $x\in H$. Hence

$$\|Tx+Sx\|^2 \le \langle (T^*T+S^*S)x,x\rangle + 2|\langle (S^*T)x,x\rangle|.$$

Taking the supremum over $x\in H$, $\|x\|=1$, we get

$$\|T+S\|^2 \le w(T^*T+S^*S) + 2w(S^*T) = \|T^*T+S^*S\| + 2w(S^*T),$$

It follows from (4.3) that

$$\begin{aligned}\langle (T^*T + S^*S)x, x\rangle &= \|Tx\|^2 + \|Sx\|^2 \\ &= 2\mathrm{Re}\langle Tx, Sx\rangle + \|Tx - Sx\|^2 \\ &\le 2|\langle Tx, Sx\rangle| + \|Tx - Sx\|^2.\end{aligned}$$

Replacing S by $-S$ in the later equality and taking the supremum over $x \in H$, $\|x\| = 1$, we get

$$\|T^*T + S^*S\| = w(T^*T + S^*S) \le 2w(S^*T) + \|T + S\|^2.$$

The desired inequality (4.1) follows from (1.2) and (1.3). The last inequality can be obtained by putting $S = \lambda T^*$ in (4.1). ∎

Theorem 4.2. *Let $T, S \in B(\mathcal{H})$, and $p, q > 0$. Then*

$$w(T + S) \le \left[w\left(\frac{p+q}{p} \cdot T^*T + \frac{p+q}{q} \cdot S^*S \right) \right]^{1/2}. \tag{4.4}$$

In particular,

$$w(T + \lambda T^*) \le \left[w\left(\frac{p+q}{p} \cdot T^*T + \frac{p+q}{q} |\lambda|^2 TT^* \right) \right]^{1/2}.$$

Proof. Utilizing the following elementary inequality

$$\frac{(a+b)^2}{p+q} \le \frac{a^2}{p} + \frac{b^2}{q},$$

holding for any real numbers a, b and for the positive numbers p, q, we get

$$\begin{aligned}|\langle (T+S)x, x\rangle|^2 &\le (p+q)\frac{(|\langle Tx, x\rangle| + |\langle Sx, x\rangle|)^2}{p+q} \\ &\le (p+q)\left(\frac{|\langle Tx, x\rangle|^2}{p} + \frac{|\langle Sx, x\rangle|^2}{q} \right) \\ &\le (p+q)\left(\frac{\|Tx\|^2}{p} + \frac{\|Sx\|^2}{q} \right) \\ &\le (p+q)\left(\left\langle \frac{T^*T}{p}x, x \right\rangle + \left\langle \frac{S^*S}{q}x, x \right\rangle \right) \\ &\le \left\langle \left(\frac{p+q}{p} T^*T + \frac{p+q}{q} S^*S \right) x, x \right\rangle.\end{aligned}$$

Putting $S = \lambda T^*$ in (4.4), we get the last desired inequality. ∎

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