INEQUALITIES FOR THE p-ANGULAR DISTANCE IN NORMED LINEAR SPACES

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Abstract. New upper and lower bounds for the *p-angular distance* in normed linear spaces are given. Some of the obtained upper bounds are better than the corresponding results due to L. Maligranda recently established in the paper [Simple norm inequalities, *Amer. Math. Monthly*, **113**(2006), 256-260].

1. Introduction

In the recent paper [5], L. Maligranda has considered the *p-angular distance*

$$\alpha_p[x,y] := \left\| \|x\|^{p-1} x - \|y\|^{p-1} y \right\|$$

between the vectors x and y in the normed linear space $(X, \|.\|)$ over the real or complex number field $\mathbb{K}$ and showed that

$$\alpha_p[x,y] \le \|x-y\| \tag{1.1}$$
$$\times \begin{cases} (2-p) \cdot \frac{\max\{\|x\|^p, \|y\|^p\}}{\max\{\|x\|, \|y\|\}} & \text{if } p \in (-\infty, 0) \text{ and } x, y \neq 0; \\ (2-p) \cdot \frac{1}{[\max\{\|x\|, \|y\|\}]^{1-p}} & \text{if } p \in [0,1] \text{ and } x, y \neq 0; \\ p \cdot [\max\{\|x\|, \|y\|\}]^{p-1} & \text{if } p \in (1, \infty). \end{cases}$$

The constants $2-p$ and p in (1.1) are best possible in the sense that they cannot be replaced by smaller quantities. As pointed out in [5], the inequality (1.1) for $p \in [1, \infty)$ is better than the Bourbaki inequality obtained in 1965, [1, p. 257] (see also [6, p. 516]):

$$\alpha_p[x,y] \le 3p\|x-y\|[\|x\| + \|y\|]^{p-1}, \quad x, y \in X. \tag{1.2}$$

The following result which provides a lower bound for the *p-angular distance* was stated without a proof by Gurariĭ in [3] (see also [6, p. 516]):

$$2^{-p}\|x-y\|^p \le \alpha_p[x,y] \tag{1.3}$$

where $p \in [1, \infty)$ and $x, y \in X$.

Finally, we recall the results of G.N. Hile from [3]:

$$\alpha_p[x,y] \le \frac{\|x\|^p - \|y\|^p}{\|x\| - \|y\|} \cdot \|x-y\|, \tag{1.4}$$

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for $p \in [1, \infty)$ and $x, y \in X$ with $\|x\| \neq \|y\|$, and

$$\alpha_{-p-1}[x, y] \leq \frac{\|x\|^p - \|y\|^p}{\|x\| - \|y\|} \cdot \frac{\|x - y\|}{\|x\|^p \|y\|^p}, \tag{1.5}$$

for $p \in [1, \infty)$ and $x, y \in X \setminus \{0\}$ with $\|x\| \neq \|y\|$.

The main aim of the present paper is to provide other upper and lower bounds for the *p-angular distance.* Some of the obtained upper bounds are better than the corresponding results due to Maligranda from [5].

2. Upper Bounds

We start with a lemma that provides upper bounds for the norm of the linear combination $\alpha x \pm \beta y$ where α, β are scalars while x, y are vectors in the normed linear space $(X, \|.\|)$.

Lemma 1. *For any $\alpha, \beta \in \mathbb{K}$ and $x, y \in X$ we have*

$$\|\alpha x \pm \beta y\| \leq \|x \pm y\| \max\{|\alpha|, |\beta|\} + |\alpha - \beta| \min\{\|x\|, \|y\|\} \tag{2.1}$$

and

$$\|\alpha x \pm \beta y\| \leq \|x \pm y\| \min\{|\alpha|, |\beta|\} + |\alpha - \beta| \max\{\|x\|, \|y\|\} \tag{2.2}$$

respectively.

Proof. By the triangle inequality we have

$$\|\alpha x + \beta y\| \leq |\alpha| \|x + y\| + |\alpha - \beta| \|y\|$$

and

$$\|\alpha x + \beta y\| \leq |\beta| \|x + y\| + |\alpha - \beta| \|x\|$$

which implies the following inequality that is of interest in itself

$$\begin{aligned} \|\alpha x \pm \beta y\| & \\ &\leq \min\{|\alpha| \|x \pm y\| + |\alpha - \beta| \|y\|, |\beta| \|x \pm y\| + |\alpha - \beta| \|x\|\} =: I. \end{aligned} \tag{2.3}$$

However

$$\begin{aligned} I \leq \min\{&\|x \pm y\| \max\{|\alpha|, |\beta|\} + |\alpha - \beta| \|y\|, \\ &\|x \pm y\| \max\{|\alpha|, |\beta|\} + |\alpha - \beta| \|x\|\} \\ &= \|x \pm y\| \max\{|\alpha|, |\beta|\} + |\alpha - \beta| \min\{\|x\|, \|y\|\} \end{aligned} \tag{2.4}$$

and the inequality (2.1) is proved.

Similarly,

$$\begin{aligned} I \leq \min\{&\|x \pm y\| |\alpha| + |\alpha - \beta| \max\{\|x\|, \|y\|\}, \\ &\|x \pm y\| |\beta| + |\alpha - \beta| \max\{\|x\|, \|y\|\}\} \\ &= \|x \pm y\| \min\{|\alpha|, |\beta|\} + |\alpha - \beta| \max\{\|x\|, \|y\|\} \end{aligned} \tag{2.5}$$

and the proof is complete. ∎

By adding the above inequalities one can obtain the following result:

Corollary 1. *For any $\alpha, \beta \in \mathbb{K}$ and $x, y \in X$ we have*

$$\|\alpha x \pm \beta y\| \leq \|x \pm y\| \cdot \frac{|\alpha| + |\beta|}{2} + |\alpha - \beta| \cdot \frac{\|x\| + \|y\|}{2}. \tag{2.6}$$

The following result concerning upper bounds for the p-angular distance holds:

Theorem 1. *For any two nonzero vectors x,y in the normed linear space $(X,\|.\|)$ we have*

$$\alpha_p[x,y] \tag{2.7}$$
$$\leq \begin{cases} \|x-y\|\left[\max\{\|x\|,\|y\|\}\right]^{p-1} + \left|\|x\|^{p-1}-\|y\|^{p-1}\right|\min\{\|x\|,\|y\|\} \\ \quad \text{if } p\in(1,\infty); \\ \frac{\|x-y\|}{[\min\{\|x\|,\|y\|\}]^{1-p}} + \left|\|x\|^{1-p}-\|y\|^{1-p}\right|\min\left\{\frac{\|x\|^p}{\|y\|^{1-p}},\frac{\|y\|^p}{\|x\|^{1-p}}\right\} \\ \quad \text{if } p\in[0,1]; \\ \frac{\|x-y\|}{[\min\{\|x\|,\|y\|\}]^{1-p}} + \frac{\left|\|x\|^{1-p}-\|y\|^{1-p}\right|}{\max\left\{\|x\|^{-p}\|y\|^{1-p},\|y\|^{-p}\|x\|^{1-p}\right\}} \\ \quad \text{if } p\in(-\infty,0); \end{cases}$$

and

$$\alpha_p[x,y] \tag{2.8}$$
$$\leq \begin{cases} \|x-y\|\left[\min\{\|x\|,\|y\|\}\right]^{p-1} + \left|\|x\|^{p-1}-\|y\|^{p-1}\right|\max\{\|x\|,\|y\|\} \\ \quad \text{if } p\in(1,\infty); \\ \frac{\|x-y\|}{[\max\{\|x\|,\|y\|\}]^{1-p}} + \left|\|x\|^{1-p}-\|y\|^{1-p}\right|\max\left\{\frac{\|x\|^p}{\|y\|^{1-p}},\frac{\|y\|^p}{\|x\|^{1-p}}\right\} \\ \quad \text{if } p\in[0,1]; \\ \frac{\|x-y\|}{[\max\{\|x\|,\|y\|\}]^{1-p}} + \frac{\left|\|x\|^{1-p}-\|y\|^{1-p}\right|}{\min\left\{\|x\|^{-p}\|y\|^{1-p},\|y\|^{-p}\|x\|^{1-p}\right\}} \\ \quad \text{if } p\in(-\infty,0); \end{cases}$$

respectively.

Proof. We use the inequality (2.1) in which we choose $\alpha=\|x\|^{p-1}$ and $\beta=\|y\|^{p-1}$ to get

$$\alpha_p[x,y] \tag{2.9}$$
$$\leq \|x-y\|\min\left\{\|x\|^{p-1},\|y\|^{p-1}\right\} + \left|\|x\|^{p-1}-\|y\|^{p-1}\right|\max\{\|x\|,\|y\|\}.$$

It is well known that, for $a,b>0$ and $q\in\mathbb{R}$, we have

$$\min(\max)\{a^q,b^q\} = \begin{cases} [\min(\max)\{a,b\}]^q & \text{if } q\geq 0; \\ \frac{1}{[\max(\min)\{a,b\}]^{-q}} & \text{if } q<0. \end{cases} \tag{2.10}$$

The case when $p\in(1,\infty)$ in (2.7) is obvious from (2.9).

Now, if we assume that $p \leq 1$, then by (2.10) we can state that

$$\begin{aligned}
(2.11)\quad & \|x-y\| \min\left\{\|x\|^{p-1}, \|y\|^{p-1}\right\} + \left|\|x\|^{p-1} - \|y\|^{p-1}\right| \max\{\|x\|, \|y\|\} \\
&= \frac{\|x-y\|}{[\max\{\|x\|, \|y\|\}]^{1-p}} + \frac{\left|\|x\|^{1-p} - \|y\|^{1-p}\right|}{\|x\|^{1-p}\|y\|^{1-p}} \max\{\|x\|, \|y\|\} \\
&= \frac{\|x-y\|}{[\max\{\|x\|, \|y\|\}]^{1-p}} + \left|\|x\|^{1-p} - \|y\|^{1-p}\right| \max\left\{\frac{\|x\|^p}{\|y\|^{1-p}}, \frac{\|y\|^p}{\|x\|^{1-p}}\right\}
\end{aligned}$$

which together with (2.9) produces the second and the third part of (2.7).

The proof of (2.8) can be done in a similar way by utilising (2.2) and the details are omitted. ∎

The following coarser but perhaps more useful result can be stated as well:

Corollary 2. *For any two nonzero vectors x, y in the normed linear space $(X, \|.\|)$ we have*

$$(2.12)\quad \alpha_p[x,y] \leq \begin{cases} \|x-y\| \cdot \frac{\|x\|^{p-1}+\|y\|^{p-1}}{2} + \left|\|x\|^{p-1} - \|y\|^{p-1}\right| \cdot \frac{\|x\|+\|y\|}{2} \\ \quad \text{if } p \in [1,\infty); \\ \|x-y\| \cdot \frac{\|x\|^{1-p}+\|y\|^{1-p}}{2\|x\|^{1-p}\|y\|^{1-p}} + \left|\|x\|^{1-p} - \|y\|^{1-p}\right| \cdot \frac{\|x\|+\|y\|}{2\|x\|^{1-p}\|y\|^{1-p}} \\ \quad \text{if } p \in (-\infty, 1)\ . \end{cases}$$

3. Lower Bounds

The following lemma may be stated as well:

Lemma 2. *For any two vectors $x, y \in X$ and two scalars $\alpha, \beta \in \mathbb{K}$ we have the inequalities*

$$(3.1)\quad \|x \pm y\| \min\{|\alpha|, |\beta|\} - |\alpha - \beta| \min\{\|x\|, \|y\|\} \leq \|\alpha x \pm \beta y\|$$

and

$$(3.2)\quad \|x \pm y\| \max\{|\alpha|, |\beta|\} - |\alpha - \beta| \max\{\|x\|, \|y\|\} \leq \|\alpha x \pm \beta y\|,$$

respectively.

Proof. Utilising the triangle inequality we obviously have

$$|\alpha| \|x+y\| - |\alpha - \beta| \|y\| \leq \|\alpha x + \beta y\|$$

and

$$|\beta| \|x+y\| - |\alpha - \beta| \|x\| \leq \|\alpha x + \beta y\|$$

which implies the following inequality that is of interest in itself

$$(3.3)\quad \max\{|\alpha| \|x \pm y\| - |\alpha - \beta| \|y\|, |\beta| \|x \pm y\| - |\alpha - \beta| \|x\|\} \leq \|\alpha x \pm \beta y\|,$$

and holds for any two vectors $x, y \in X$ and two scalars $\alpha, \beta \in \mathbb{K}$.

Now the proof goes like in the Lemma 1 and the details are omitted. ∎

By adding the above two inequalities we can get the following lower bound that might be more convenient for some applications:

Corollary 3. *For any two vectors $x, y \in X$ and two scalars $\alpha, \beta \in \mathbb{K}$ we have*

$$\|x \pm y\| \cdot \frac{|\alpha| + |\beta|}{2} - |\alpha - \beta| \cdot \frac{\|x\| + \|y\|}{2} \leq \|\alpha x \pm \beta y\| . \tag{3.4}$$

The following result providing lower bounds for the p-angular distance may be stated as well:

Theorem 2. *For any two nonzero vectors $x, y \in X$ we have the lower bounds for the p-angular distance:*

$$\alpha_p[x,y] \geq \begin{cases} \|x-y\| [\min\{\|x\|, \|y\|\}]^{p-1} - \left| \|x\|^{p-1} - \|y\|^{p-1} \right| \min\{\|x\|, \|y\|\} \\ \qquad \text{if } p \in (1, \infty); \\[2mm] \frac{\|x-y\|}{[\max\{\|x\|,\|y\|\}]^{1-p}} - \left| \|x\|^{1-p} - \|y\|^{1-p} \right| \min\left\{ \frac{\|x\|^p}{\|y\|^{1-p}}, \frac{\|y\|^p}{\|x\|^{1-p}} \right\} \\ \qquad \text{if } p \in [0,1]; \\[2mm] \frac{\|x-y\|}{[\max\{\|x\|,\|y\|\}]^{1-p}} - \frac{\left| \|x\|^{1-p} - \|y\|^{1-p} \right|}{\max\left\{ \|x\|^{-p}\|y\|^{1-p}, \|y\|^{-p}\|x\|^{1-p} \right\}} \\ \qquad \text{if } p \in (-\infty, 0); \end{cases} \tag{3.5}$$

and

$$\alpha_p[x,y] \geq \begin{cases} \|x-y\| [\max\{\|x\|, \|y\|\}]^{p-1} - \left| \|x\|^{p-1} - \|y\|^{p-1} \right| \max\{\|x\|, \|y\|\} \\ \qquad \text{if } p \in (1, \infty); \\[2mm] \frac{\|x-y\|}{[\min\{\|x\|,\|y\|\}]^{1-p}} - \left| \|x\|^{1-p} - \|y\|^{1-p} \right| \max\left\{ \frac{\|x\|^p}{\|y\|^{1-p}}, \frac{\|y\|^p}{\|x\|^{1-p}} \right\} \\ \qquad \text{if } p \in [0,1]; \\[2mm] \frac{\|x-y\|}{[\min\{\|x\|,\|y\|\}]^{1-p}} - \frac{\left| \|x\|^{1-p} - \|y\|^{1-p} \right|}{\min\left\{ \|x\|^{-p}\|y\|^{1-p}, \|y\|^{-p}\|x\|^{1-p} \right\}} \\ \qquad \text{if } p \in (-\infty, 0). \end{cases} \tag{3.6}$$

Proof. Writing the inequality (3.1) for $\alpha = \|x\|^{p-1}$ and $\beta = \|y\|^{p-1}$ we get

$$\begin{aligned} & \left\| \|x\|^{p-1} x - \|y\|^{p-1} y \right\| \\ & \geq \|x-y\| \min\left\{ \|x\|^{p-1}, \|y\|^{p-1} \right\} - \left| \|x\|^{p-1} - \|y\|^{p-1} \right| \min\{\|x\|, \|y\|\}, \end{aligned} \tag{3.7}$$

which easily implies (3.5).

The second part follows from (3.2) and the details are omitted. ∎

Corollary 4. *For any two nonzero vectors x,y in the normed linear space $(X,\|.\|)$ we have*

$$\alpha_p[x,y] \geq \begin{cases} \|x-y\|\cdot\frac{\|x\|^{p-1}+\|y\|^{p-1}}{2}-\left|\|x\|^{p-1}-\|y\|^{p-1}\right|\cdot\frac{\|x\|+\|y\|}{2} \\ \text{if } p\in[1,\infty); \\ \|x-y\|\cdot\frac{\|x\|^{1-p}+\|y\|^{1-p}}{2\|x\|^{1-p}\|y\|^{1-p}}-\left|\|x\|^{1-p}-\|y\|^{1-p}\right|\cdot\frac{\|x\|+\|y\|}{2\|x\|^{1-p}\|y\|^{1-p}} \\ \text{if } p\in(-\infty,1). \end{cases} \tag{3.8}$$

4. Further Norm Inequalities

Firstly, we observe that the Corollaries 1 and 3 can be encompassed in the following result:

Proposition 1. *For any two vectors $x,y\in X$ and two scalars $\alpha,\beta\in\mathbb{K}$ we have*

$$\left|\|\alpha x\pm\beta y\|-\|x\pm y\|\cdot\frac{|\alpha|+|\beta|}{2}\right|\leq|\alpha-\beta|\cdot\frac{\|x\|+\|y\|}{2}. \tag{4.1}$$

Also the results for the p-angular distance from Corollaries 2 and 4 can be embodied in:

Proposition 2. *For any two nonzero vectors x,y in the normed linear space $(X,\|.\|)$ we have*

$$\left|\alpha_p[x,y]-\|x-y\|\cdot\frac{\|x\|^{p-1}+\|y\|^{p-1}}{2}\right| \leq\left|\|x\|^{p-1}-\|y\|^{p-1}\right|\cdot\frac{\|x\|+\|y\|}{2} \tag{4.2}$$

if $p\in[1,\infty)$ and

$$\left|\alpha_p[x,y]-\|x-y\|\cdot\frac{\|x\|^{1-p}+\|y\|^{1-p}}{2\|x\|^{1-p}\|y\|^{1-p}}\right| \leq\left|\|x\|^{1-p}-\|y\|^{1-p}\right|\cdot\frac{\|x\|+\|y\|}{2\|x\|^{1-p}\|y\|^{1-p}} \tag{4.3}$$

if $p\in(-\infty,1)$, respectively.

Now, for $s\in[-\infty,\infty]$ and $a,b>0,a\neq b$, by following [2, p. 385], we can introduce the *s-generalised logarithmic means* by

$$L^{[s]}(a,b):=\begin{cases}\left(\frac{b^{s+1}-a^{s+1}}{(s+1)(b-a)}\right)^{1/s} & \text{if } s\neq-1,0,\pm\infty; \\ \frac{b-a}{\ln b-\ln a} & \text{if } s=-1; \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} & s=0; \\ \max\{a,b\} & s=\infty; \\ \min\{a,b\} & s=-\infty.\end{cases}$$

The mapping $\mathbb{R}\ni s\to L^{[s]}(a,b)$ is strictly increasing and (see [2, p. 386])

$$\min\{a,b\}<L^{[s]}(a,b)<\max\{a,b\} \tag{4.4}$$

for any $s \in \mathbb{R}$ and $a, b > 0$, with $a \neq b$.

The following lemma holds:

Lemma 3. *For any two nonzero vectors $x, y \in X$ we have*

$$(4.5)\quad (p-1)\left[\min\{\|x\|,\|y\|\}\right]^{p-2}\left|\|x\|-\|y\|\right| \\ \leq \left|\|x\|^{p-1}-\|y\|^{p-1}\right| \leq (p-1)\left|\|x\|-\|y\|\right|\left[\max\{\|x\|,\|y\|\}\right]^{p-2}$$

if $p \in (2,\infty)$,

$$(4.6)\quad (p-1)\frac{1}{\left[\max\{\|x\|,\|y\|\}\right]^{2-p}}\left|\|x\|-\|y\|\right| \\ \leq \left|\|x\|^{p-1}-\|y\|^{p-1}\right| \leq (p-1)\left|\|x\|-\|y\|\right|\frac{1}{\left[\min\{\|x\|,\|y\|\}\right]^{2-p}}$$

if $p \in [1,2]$, and

$$(4.7)\quad (1-p)\frac{\|x\|^{1-p}\|y\|^{1-p}}{\left[\max\{\|x\|,\|y\|\}\right]^{2-p}}\left|\|x\|-\|y\|\right| \\ \leq \left|\|x\|^{1-p}-\|y\|^{1-p}\right| \leq (1-p)\left|\|x\|-\|y\|\right|\frac{\|x\|^{1-p}\|y\|^{1-p}}{\left[\min\{\|x\|,\|y\|\}\right]^{2-p}}$$

if $p \in (-\infty,1)$, respectively.

Proof. If $x, y \in X \setminus \{0\}$ with $\|x\| = \|y\|$ then the equality case is realised in all inequalities.

If $\|x\| \neq \|y\|$, then by (4.4) we have

$$\min\{\|x\|,\|y\|\} < \left(\frac{\|x\|^{p-1}-\|y\|^{p-1}}{(p-1)(\|x\|-\|y\|)}\right)^{1/(p-2)} < \max\{\|x\|,\|y\|\}$$

for $p \neq 1, 2$. Observe also that

$$\frac{\|x\|^{p-1}-\|y\|^{p-1}}{(p-1)(\|x\|-\|y\|)} = \left|\frac{\|x\|^{p-1}-\|y\|^{p-1}}{(p-1)(\|x\|-\|y\|)}\right|$$

and the above inequality can be written as

$$(4.8)\qquad \min\{\|x\|,\|y\|\} < \left|\frac{\|x\|^{p-1}-\|y\|^{p-1}}{(p-1)(\|x\|-\|y\|)}\right|^{1/(p-2)} < \max\{\|x\|,\|y\|\}.$$

If $p > 2$, then taking the power $p-2$ in (4.8) produces the desired inequality (4.5). The inequality (4.5) remains also valid for $p = 2$.

If $p < 2$ and $p \neq 1$, then on taking the power $p-2$ in (4.8) we get the other two inequalities. The details are omitted. ∎

In the following, we discuss some upper bounds for the p-angular distance that contain as a multiplicative term the quantity $\|x-y\|$. The obtained results are compared with the inequalities of Maligranda mentioned in the introduction.

Case 1. *For $p \in [2, \infty)$, we get from Lemma 3 that*

$$\left| \|x\|^{p-1} - \|y\|^{p-1} \right| \leq (p-1) \|x-y\| \left[\max\{\|x\|, \|y\|\}\right]^{p-2}$$

for any $x, y \in X$.

Utilising the first branch of the inequalities (2.7) and (2.8) we can state that

$$\alpha_p[x,y] \leq \|x-y\| \times \begin{cases} [\max\{\|x\|,\|y\|\}]^{p-2} [\max\{\|x\|,\|y\|\} + (p-1)\min\{\|x\|,\|y\|\}] \\ \\ [\min\{\|x\|,\|y\|\}]^{p-1} + (p-1)[\max\{\|x\|,\|y\|\}]^{p-1} \end{cases} \left(\leq p\|x-y\| [\max\{\|x\|,\|y\|\}]^{p-1}\right), \tag{4.9}$$

for any $x, y \in X$.

We observe that both inequalities in (4.9) are better than Maligranda's result in (1.1) for $p \in [2, \infty)$.

Case 2. *For $p \in [1, 2)$, we get from Lemma 3 that*

$$\left| \|x\|^{p-1} - \|y\|^{p-1} \right| \leq (p-1) \frac{\|x-y\|}{[\min\{\|x\|,\|y\|\}]^{2-p}}$$

for any $x, y \in X \setminus \{0\}$.

Utilising the first branch of the inequalities (2.7) and (2.8) we can state that

$$\alpha_p[x,y] \leq \|x-y\| \times \begin{cases} [\max\{\|x\|,\|y\|\}]^{p-1} + (p-1)[\min\{\|x\|,\|y\|\}]^{p-1} \\ \\ [\min\{\|x\|,\|y\|\}]^{p-1} + (p-1)\frac{\max\{\|x\|,\|y\|\}}{[\min\{\|x\|,\|y\|\}]^{2-p}} \end{cases} \tag{4.10}$$

for any $x, y \in X \setminus \{0\}$.

Due to the fact that the second term in the first branch of (4.10) is smaller than $(p-1)[\max\{\|x\|,\|y\|\}]^{p-1}$ it follows that the first inequality in (4.10) is better than Maligranda's result in (1.1) for $p \in [1, 2)$.

Now, let us denote

$$B_1(x,y) := [\min\{\|x\|,\|y\|\}]^{p-1} + (p-1)\frac{\max\{\|x\|,\|y\|\}}{[\min\{\|x\|,\|y\|\}]^{2-p}}$$

and

$$B_2(x,y) := p \cdot [\max\{\|x\|,\|y\|\}]^{p-1}$$

with $x, y \in X \setminus \{0\}$ and $p \in [1, 2)$.

If we consider the difference $\Delta_p(x,y) := B_1(x,y) - B_2(x,y)$ and plot it for $p = 3/2$ and (x, y) pairs of real numbers in the box $[0.2, 1.5] \times [0.2, 1.5]$ (see Figure 1), then we can conclude that neither of the bounds $\|x-y\| B_1(x,y)$ and $\|x-y\| B_2(x,y)$ for $\alpha_p[x,y]$ is always better.

Case 3. *For $p \in [0, 1)$, we get from Lemma 3 that*

$$\left| \|x\|^{1-p} - \|y\|^{1-p} \right| \leq (1-p) \|x-y\| \frac{\|x\|^{1-p} \|y\|^{1-p}}{[\min\{\|x\|,\|y\|\}]^{2-p}}$$

for any $x, y \in X \setminus \{0\}$.

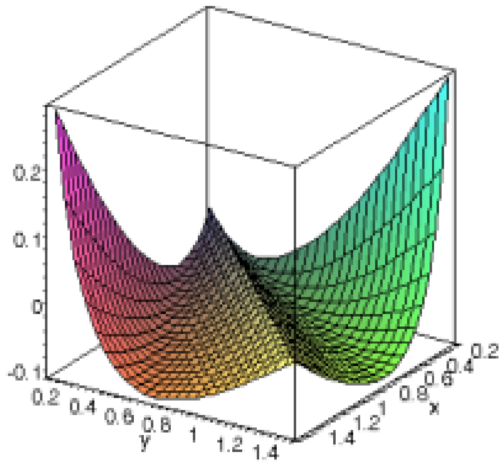

FIGURE 1. Plot of $\Delta_p(x,y)$ for $p=3/2$.

Utilising the second branch of the inequalities (2.7) and (2.8), and performing the necessary calculations, we get that

$$\alpha_p[x,y] \le \|x-y\| \times \begin{cases} \frac{2-p}{[\min\{\|x\|,\|y\|\}]^{1-p}}; \\ \frac{1}{[\max\{\|x\|,\|y\|\}]^{1-p}} + (1-p)\frac{\max\{\|x\|,\|y\|\}}{[\min\{\|x\|,\|y\|\}]^{2-p}}; \end{cases} \tag{4.11}$$

for any $x,y\in X\setminus\{0\}$.

We notice that Maligranda's result from the second branch of (1.1) is better than the both inequalities in (4.11).

Case 4. *For $p\in(-\infty,0)$, we get from Lemma 3 that*

$$\left|\|x\|^{1-p}-\|y\|^{1-p}\right| \le (1-p)\|x-y\|\frac{\|x\|^{1-p}\|y\|^{1-p}}{[\min\{\|x\|,\|y\|\}]^{2-p}}$$

for any $x,y\in X\setminus\{0\}$.

Utilising the third branch of the inequalities (2.7) and (2.8) and performing the necessary calculations we get that

$$\alpha_p[x,y] \le \|x-y\| \times \begin{cases} \frac{2-p}{[\min\{\|x\|,\|y\|\}]^{1-p}}; \\ \frac{1}{[\max\{\|x\|,\|y\|\}]^{1-p}} + (1-p)\frac{\max\{\|x\|,\|y\|\}}{[\min\{\|x\|,\|y\|\}]^{2-p}}; \end{cases} \tag{4.12}$$

for any $x,y\in X\setminus\{0\}$.

Now, observe that Maligranda's first inequality in (1.1) can be written as

$$\alpha_p[x,y] \le \frac{(2-p)\|x-y\|}{\max\{\|x\|,\|y\|\}[\min\{\|x\|,\|y\|\}]^{-p}}, \tag{4.13}$$

for any $x,y\in X\setminus\{0\}$, and is better than the first inequality in (4.12).

Finally, consider

$$C_1(x,y) := \frac{1}{[\max\{\|x\|,\|y\|\}]^{1-p}} + (1-p)\frac{\max\{\|x\|,\|y\|\}}{[\min\{\|x\|,\|y\|\}]^{2-p}}$$

and

$$C_2(x,y) := \frac{(2-p)}{\max\{\|x\|,\|y\|\}[\min\{\|x\|,\|y\|\}]^{-p}},$$

where $x,y \in X \setminus \{0\}$ *and* $p \in (-\infty,0)$. *If* $\Gamma_p(x,y) := C_1(x,y) - C_2(x,y)$ *then several numerical experiments conducted for* $(x,y) \in \mathbb{R}^2$ *and* $p \in (-\infty,0)$ *have lead us to conjecture that Maligranda's first inequality in (1.1) is better than the second inequality in (4.12). However, we do not have an analytic proof even in the case of real numbers. The plot depicted in Figure 2 shows the behavior of* $\Gamma_p(x,y)$ *for* $p=-3$ *and* $(x,y) \in [0.5,1.5]\times[0.5,1.5]$.

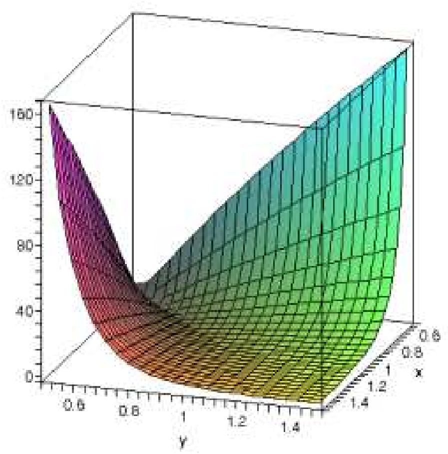

FIGURE 2. Plot of $\Gamma_p(x,y)$ for $p=-3$.

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