SUBADDITIVE AND SUPERADDITIVE PROPERTIES OF POLYGAMMA FUNCTIONS

FENG QI AND BAI-NI GUO

ABSTRACT. In this short paper, it is proved that the function $\psi^{(i)}(e^x)$ for $i \in \mathbb{N}$ is subadditive in $(\ln\theta_0, \infty)$ and superadditive in $(-\infty, \ln\theta_0)$, where $\theta_0 \in (0,1)$ is the unique root of equation $2\,\psi^{(i)}(\theta) = \psi^{(i)}(\theta^2)$.

1. Introduction

Recall [2, 3, 4] that a function f defined on an interval I is said to be subadditive on I if $f(x+y) \le f(x) + f(y)$ holds for all $x, y \in I$ such that $x + y \in I$. If $f(x+y) \ge f(x) + f(y)$, then f is called superadditive on I.

The subadditive and superadditive functions play an important role in the theory of differential equations, in the study of semi-groups, in number theory, and also in the theory of convex bodies. A lot of literature for the subadditive and superadditive functions can be found in [2, 3] and the references therein.

It is well known that the classical Euler's gamma function Γ can be defined for $x > 0$ as $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,\mathrm{d}t$. The digamma or psi function ψ is defined as the logarithmic derivative of Γ and $\psi^{(i)}$ for $i \in \mathbb{N}$ are called polygamma functions.

In [3], a subadditive property of the gamma function Γ was proved: The function $[\Gamma(x)]^\alpha$ is subadditive in $(0, \infty)$ if and only if $\frac{\ln 2}{\ln \Delta} \le \alpha \le 0$, where $\Delta = \min_{x \ge 0} \frac{\Gamma(2x)}{\Gamma(x)}$.

In [4], a subadditive property of the psi function was obtained: The function $\psi(a + e^x)$ is subadditive in $(-\infty, \infty)$ if and only if $a \ge c_0$, where c_0 is the unique positive zero of $\psi(x)$.

In this short paper, we would like to discuss the subadditive and superadditive properties of the polygamma functions $\psi^{(i)}(x)$ for $i \in \mathbb{N}$.

Our main result is the following Theorem 1.

Theorem 1. *The function* $\left|\psi^{(i)}(e^x)\right|$ *for* $i \in \mathbb{N}$ *is superadditive in* $(-\infty, \ln\theta_0)$ *or subadditive in* $(\ln\theta_0, \infty)$*, where* $\theta_0 \in (0,1)$ *is the unique root of equation* $2\left|\psi^{(i)}(\theta)\right| = \left|\psi^{(i)}(\theta^2)\right|$.

2. Proof of Theorem 1

In [5], the monotonicity of the function $x^\alpha\left|\psi^{(i)}(x+\beta)\right|$ was researched thoroughly, which is a generalization of the corresponding results in [1, 4], as follows:

(1) The function $x^\alpha\left|\psi^{(i)}(x)\right|$ in $(0, \infty)$ is strictly increasing if and only if $\alpha \ge i + 1$ and strictly decreasing if and only if $0 \le \alpha \le i$.

Date: This manuscript was finalized on 16 June 2006.
2000 *Mathematics Subject Classification.* Primary 33B15; Secondary 26D07.
Key words and phrases. subadditive function, superadditive function, polygamma function.
This paper was typeset using $\mathcal{A}_\mathcal{M}\mathcal{S}$-LaTeX.

(2) For $\beta \ge \frac{1}{2}$, the function $x^{\alpha}\left|\psi^{(i)}(x+\beta)\right|$ is strictly increasing in $[0,\infty)$ if and only if $\alpha \ge i$.
(3) Let $\delta:(0,\infty)\to\left(0,\frac{1}{2}\right)$ be defined by

$$\delta(t)=\frac{e^t(t-1)+1}{(e^t-1)^2} \tag{1}$$

for $t\in(0,\infty)$ and $\delta^{-1}:\left(0,\frac{1}{2}\right)\to(0,\infty)$ stand for the inverse function of δ. If $0<\beta<\frac{1}{2}$ and

$$\alpha \ge i+1-\left[\frac{e^{\delta^{-1}(\beta)}}{e^{\delta^{-1}(\beta)}-1}+\beta-1\right]\delta^{-1}(\beta), \tag{2}$$

then the function $x^{\alpha}\left|\psi^{(i)}(x+\beta)\right|$ is strictly increasing in $(0,\infty)$.

It is noted that

$$0<\left[\frac{e^{\delta^{-1}(\beta)}}{e^{\delta^{-1}(\beta)}-1}+\beta-1\right]\delta^{-1}(\beta)<1 \tag{3}$$

for $\beta\in(0,1)$, since $\lim_{\beta\to0^+}[\beta\delta^{-1}(\beta)]=0$.

Let

$$f_\beta(x,y)=\left|\psi^{(i)}(\beta+x)\right|+\left|\psi^{(i)}(\beta+y)\right|-\left|\psi^{(i)}(\beta+xy)\right| \tag{4}$$

for $x>0$ and $y>0$, where $\beta\ge0$ and $i\in\mathbb{N}$. In order to show Theorem 1, it is sufficient to prove the positivity or negativity of the function $f_\beta(x,y)$. Directly calculating yields

$$\begin{aligned}\frac{\partial f_\beta(x,y)}{\partial x}&=y\left|\psi^{(i+1)}(\beta+xy)\right|-\left|\psi^{(i+1)}(\beta+x)\right|\\&=\frac{1}{x}\Big[xy\left|\psi^{(i+1)}(\beta+xy)\right|-x\left|\psi^{(i+1)}(\beta+x)\right|\Big].\end{aligned} \tag{5}$$

From the monotonicity of the function $x^{\alpha}\left|\psi^{(i)}(x+\beta)\right|$ in [5] mentioned above, it follows easily that $\frac{\partial f_0(x,y)}{\partial x}\gtreqless 0$ if and only if $y\lesseqgtr 1$. This means that the function $f_0(x,y)$ is strictly increasing for $y<1$ and strictly decreasing for $y>1$ in $x\in(0,\infty)$. Since $\lim_{x\to\infty}f_0(x,y)=\left|\psi^{(i)}(y)\right|>0$, then for $y>1$ the function $f_0(x,y)$ is positive in $x\in(0,\infty)$.

For $y<1$, by the increasingly monotonicity of $f_0(x,y)$, it is deduced that

(1) if $x>1$, then $f_0(1,y)=\left|\psi^{(i)}(1)\right|<f_0(x,y)<\left|\psi^{(i)}(y)\right|$,
(2) if $x<1$, then $f_0(x,y)<f_0(1,y)=\left|\psi^{(i)}(1)\right|$,
(3) if $y<x<1$, then $f_0(y,y)<f_0(x,y)$,
(4) if $x<y<1$, then $f_0(x,x)<f_0(x,y)$.

This implies that

$$f_0(\theta,\theta)=2\left|\psi^{(i)}(\theta)\right|-\left|\psi^{(i)}(\theta^2)\right|<f_0(x,y) \tag{6}$$

for $y<1$, where $\theta<1$ with $\theta<x$ and $\theta<y$.

Since $f_0(\theta,\theta)$ is strictly increasing in $\theta\in(0,1)$ such that $f_0(1,1)=\left|\psi^{(i)}(1)\right|>0$ and $\lim_{\theta\to0^+}f_0(\theta,\theta)=-\infty$, then the function $f_0(\theta,\theta)$ has a unique zero $\theta_0\in(0,1)$ and $f_0(\theta,\theta)>0$ for $1>\theta>\theta_0$.

In conclusion, the function $f_0(x,y)$ is positive for all $x,y>\theta_0$ or negative for $0<x,y<\theta_0$. The proof of Theorem 1 is complete.

References

[1] H. Alzer, *Mean-value inequalities for the polygamma functions*, Aequationes Math. **61** (2001), no. 1, 151–161.
[2] H. Alzer and S. Koumandos, *Sub- and superadditive properties of Fejér's sine polynomial*, Bull. London Math. Soc. **38** (2006), no. 2, 261–268.
[3] H. Alzer and S. Ruscheweyh, *A subadditive property of the gamma function*, J. Math. Anal. Appl. **285** (2003), 564–577.
[4] P. Gao, *A subadditive property of the digamma function*, RGMIA Res. Rep. Coll. **8** (2005), no. 3, Art. 9; Available online at http://rgmia.vu.edu.au/v8n3.html.
[5] F. Qi, S. Guo and B.-N. Guo, *Note on a class of completely monotonic functions involving the polygamma functions*, RGMIA Res. Rep. Coll. **10** (2007), no. 1, Art. 5; Available online at http://rgmia.vu.edu.au/v10n1.html.

(F. Qi) Research Institute of Mathematical Inequality Theory, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China
E-mail address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@msn.com, qifeng618@qq.com, qifeng@hpu.edu.cn, fengqi618@member.ams.org
URL: http://rgmia.vu.edu.au/qi.html

(B.-N. Guo) School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China
E-mail address: guobaini@hpu.edu.cn