

BOUNDS FOR THE r -WEIGHTED GINI MEAN DIFFERENCE OF AN EMPIRICAL DISTRIBUTION

P. CERONE AND S.S. DRAGOMIR

ABSTRACT. Various bounds for the r -weighted Gini mean difference of an empirical distribution are established.

1. INTRODUCTION

The *Gini mean difference* of the sample $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ is defined by

$$G(\mathbf{a}) = \frac{1}{2n^2} \sum_{j=1}^n \sum_{i=1}^n |a_i - a_j| = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} |a_i - a_j|$$

and

$$R(\mathbf{a}) = \frac{1}{\bar{a}} G(\mathbf{a})$$

is the Gini index of \mathbf{a} , provided the sample mean \bar{a} is not zero [6, p. 257].

The Gini index of \mathbf{a} equals the Gini mean difference of the “scaled down” sample $\tilde{\mathbf{a}} = (\frac{a_1}{\bar{a}}, \dots, \frac{a_n}{\bar{a}})$ ($\bar{a} \neq 0$)

$$R(a_1, \dots, a_n) = \frac{1}{2n^2} \sum_{i=1}^n \sum_{j=1}^n \left| \frac{a_i}{\bar{a}} - \frac{a_j}{\bar{a}} \right|.$$

The following elementary properties of the Gini index for an empirical distribution of nonnegative data hold [6, p. 257]:

(i) Let $(a_1, \dots, a_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^n a_i > 0$. Then

$$0 = R(\bar{a}, \dots, \bar{a}) \leq R(a_1, \dots, a_n) \leq R\left(0, \dots, 0, \sum_{i=1}^n a_i\right) = 1 - \frac{1}{n} < 1,$$

$$R(\beta a_1, \dots, \beta a_n) = R(a_1, \dots, a_n) \quad \text{for every } \beta > 0$$

and

$$R(a_1 + \lambda, \dots, a_n + \lambda) = \frac{\bar{a}}{\bar{a} + \lambda} R(a_1, \dots, a_n) \quad \text{for } \lambda > 0.$$

(ii) R is a continuous function on \mathbb{R}_+^n .

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These and other properties have been investigated in [6], [3] and [4].

For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $\mathbf{p} = (p_1, \dots, p_n)$ a probability sequence, meaning that $p_i \geq 0$ ($i \in \{1, \dots, n\}$) and $\sum_{i=1}^n p_i = 1$, we considered in [1] the *weighted Gini mean difference* defined by formula

$$(1.1) \quad G(\mathbf{p}, \mathbf{a}) = \frac{1}{2} \sum_{j=1}^n \sum_{i=1}^n p_i p_j |a_i - a_j| = \sum_{1 \leq i < j \leq n} p_i p_j |a_i - a_j|,$$

and proved that

$$(1.2) \quad \frac{1}{2} K(\mathbf{p}, \mathbf{a}) \leq G(\mathbf{p}, \mathbf{a}) \leq \inf_{\gamma \in \mathbb{R}} \left[\sum_{i=1}^n p_i |a_i - \gamma| \right] \leq K(\mathbf{p}, \mathbf{a}),$$

where $K(\mathbf{p}, \mathbf{a})$ is the *mean absolute deviation*, namely

$$(1.3) \quad K(\mathbf{p}, \mathbf{a}) := \sum_{i=1}^n p_i \left| a_i - \sum_{j=1}^n p_j a_j \right|.$$

We have also shown that if more information on the sampling data $\mathbf{a} = (a_1, \dots, a_n)$ is available, i.e., there exists the real numbers a and A such that $a \leq a_i \leq A$ for each $i \in \{1, \dots, n\}$, then

$$(1.4) \quad G(\mathbf{p}, \mathbf{a}) \leq (A - a) \max_{J \subseteq \{1, \dots, n\}} [P_J (1 - P_J)] \quad \left(\leq \frac{1}{4} (A - a) \right),$$

where $P_J := \sum_{j \in J} p_j$. Also, we have shown that

$$(1.5) \quad G(\mathbf{p}, \mathbf{a}) \leq \sum_{i=1}^n p_i \left| a_i - \frac{A + a}{2} \right| \quad \left(\leq \frac{1}{2} (A - a) \right).$$

Notice that in general the bounds for the weighted Gini mean difference $G(\mathbf{p}, \mathbf{a})$ provided by (1.4) and (1.5) cannot be compared to conclude that one is always better than the other [1].

The main aim of this paper is to continue the study begun in [1] and provide various bounds for the more general r -weighted Gini mean difference that has been introduced in [1].

2. BOUNDS FOR THE r -WEIGHTED GINI MEAN DIFFERENCE

For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $\mathbf{p} = (p_1, \dots, p_n)$ a probability sequence, meaning that $p_i \geq 0$ ($i \in \{1, \dots, n\}$) and $\sum_{i=1}^n p_i = 1$, define the r -*weighted Gini mean difference*, for $r \in [1, \infty)$, by the formula [1, 291]:

$$(2.1) \quad G_r(\mathbf{p}, \mathbf{a}) := \frac{1}{2} \sum_{j=1}^n \sum_{i=1}^n p_i p_j |a_i - a_j|^r = \sum_{1 \leq i < j \leq n} p_i p_j |a_i - a_j|^r.$$

For $r = 1$ we have the *weighted Gini mean difference* $G(\mathbf{p}, \mathbf{a})$ of (1.1) which becomes, for the uniform probability distribution $\mathbf{p} = (\frac{1}{n}, \dots, \frac{1}{n})$ the *Gini mean difference*

$$G(\mathbf{a}) := \frac{1}{2n^2} \sum_{j=1}^n \sum_{i=1}^n |a_i - a_j| = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} |a_i - a_j|.$$

For the uniform probability distribution $\mathbf{p} = (\frac{1}{n}, \dots, \frac{1}{n})$ we denote

$$G_r(\mathbf{a}) := G_r(\mathbf{p}, \mathbf{a}) = \frac{1}{2n^2} \sum_{i=1}^n \sum_{j=1}^n |a_i - a_j|^r = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} |a_i - a_j|^r.$$

Now, if we define $\Delta := \{(i, j) \mid i, j \in \{1, \dots, n\}\}$, then we can simply write from (2.1)

$$(2.2) \quad G_r(\mathbf{p}, \mathbf{a}) = \frac{1}{2} \sum_{(i,j) \in \Delta} p_i p_j |a_i - a_j|^r, \quad r \geq 1.$$

The following result concerning upper and lower bounds for $G_r(\mathbf{p}, \mathbf{a})$ may be stated:

Theorem 1. *For any $p_i \in (0, 1)$, $i \in \{1, \dots, n\}$ with $\sum_{i=1}^n p_i = 1$ and $a_i \in \mathbb{R}$, $i \in \{1, \dots, n\}$, we have the inequalities*

$$(2.3) \quad \frac{1}{2} \max_{(i,j) \in \Delta} \left\{ \frac{p_i^r p_j^r + p_i p_j (1 - p_i p_j)^{r-1}}{(1 - p_i p_j)^{r-1}} |a_i - a_j|^r \right\} \leq G_r(\mathbf{p}, \mathbf{a}) \leq \frac{1}{2} \max_{(i,j) \in \Delta} |a_i - a_j|^r,$$

where $r \in (0, \infty)$.

Proof. Observe that

$$\sum_{(i,j) \in \Delta} p_i p_j (a_i - a_j) = 0.$$

Then, for any fixed $(i, j) \in \Delta$ we have

$$(2.4) \quad p_i p_j (a_i - a_j) = - \sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k p_l (a_k - a_l).$$

Taking the modulus in (2.4) and utilising the Hölder discrete inequality for multiple indices and $r > 1$, $\frac{1}{r} + \frac{1}{q} = 1$ ($q = \frac{r}{r-1}$), we have successively:

$$(2.5) \quad \begin{aligned} & p_i p_j |a_i - a_j| \\ &= \left| \sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k p_l (a_k - a_l) \right| \\ &\leq \left(\sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k p_l \right)^{\frac{1}{q}} \left(\sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k p_l |a_k - a_l|^r \right)^{\frac{1}{r}} \\ &= \left(\sum_{(k,l) \in \Delta} p_k p_l - p_i p_j \right)^{\frac{1}{q}} \\ &\quad \times \left(\sum_{(k,l) \in \Delta} p_k p_l |a_k - a_l|^r - p_i p_j |a_i - a_j|^r \right)^{\frac{1}{r}} \\ &= (1 - p_i p_j)^{\frac{r-1}{r}} (2G_r(\mathbf{p}, \mathbf{a}) - p_i p_j |a_i - a_j|^r)^{\frac{1}{r}} \end{aligned}$$

for each $(i, j) \in \Delta$.

Taking the power r in (2.5) we have

$$p_i^r p_j^r |a_i - a_j|^r \leq (1 - p_i p_j)^{r-1} (2G_r(\mathbf{p}, \mathbf{a}) - p_i p_j |a_i - a_j|^r),$$

giving

$$\left[p_i^r p_j^r + p_i p_j (1 - p_i p_j)^{r-1} \right] |a_i - a_j|^r \leq 2(1 - p_i p_j)^{r-1} G_r(\mathbf{p}, \mathbf{a}),$$

so that

$$(2.6) \quad \frac{1}{2} \cdot \frac{p_i^r p_j^r + p_i p_j (1 - p_i p_j)^{r-1}}{(1 - p_i p_j)^{r-1}} |a_i - a_j|^r \leq G_r(\mathbf{p}, \mathbf{a})$$

for each $(i, j) \in \Delta$.

Taking the maximum over $(i, j) \in \Delta$ in (2.6), we deduce the first inequality in (2.3).

The second inequality is obvious on observing that

$$G_r(\mathbf{p}, \mathbf{a}) \leq \frac{1}{2} \sum_{(i,j) \in \Delta} p_i p_j \max_{(i,j) \in \Delta} |a_i - a_j|^r = \frac{1}{2} \max_{(i,j) \in \Delta} |a_i - a_j|^r.$$

The proof is complete. \blacksquare

Remark 1. *The case $r = 2$ is of interest, since*

$$G_2(\mathbf{p}, \mathbf{a}) = \frac{1}{2} \sum_{(i,j) \in \Delta} p_i p_j |a_i - a_j|^2 = \sum_{i=1}^n p_i a_i^2 - \left(\sum_{i=1}^n p_i a_i \right)^2,$$

for which we can obtain from Theorem 1 the following bounds:

$$(2.7) \quad \frac{1}{2} \max_{(i,j) \in \Delta} \left\{ \frac{p_i p_j}{1 - p_i p_j} (a_i - a_j)^2 \right\} \leq G_2(\mathbf{p}, \mathbf{a}) \leq \frac{1}{2} \max_{(i,j) \in \Delta} (a_i - a_j)^2.$$

Remark 2. *Consider the function*

$$h_r(t) := \frac{t^r + t(1-t)^{r-1}}{(1-t)^{r-1}} = t + t^r (1-t)^{1-r}$$

defined for $t \in [0, 1)$ and $r > 1$. Then

$$h_r'(t) = 1 + r t^{r-1} (1-t)^{1-r} + (r-1) t^r (1-t)^{-r}$$

which shows that h_r is strictly increasing on $[0, 1)$.

Therefore

$$\begin{aligned} \min_{(i,j) \in \Delta} \left\{ \frac{p_i^r p_j^r + p_i p_j (1 - p_i p_j)^{r-1}}{(1 - p_i p_j)^{r-1}} \right\} &= \min_{(i,j) \in \Delta} h_r(p_i p_j) \\ &\geq h_r \left[\min_{(i,j) \in \Delta} (p_i p_j) \right] \\ &\geq h_r \left(\min_{i \in \{1, \dots, n\}} p_i \cdot \min_{j \in \{1, \dots, n\}} p_j \right) \\ &= h_r(p_m^2) \\ &= \frac{p_m^{2r} + p_m^2 (1 - p_m^2)^{r-1}}{(1 - p_m^2)^{r-1}}, \end{aligned}$$

where $p_m := \min_{i \in \{1, \dots, n\}} p_i > 0$.

In conclusion, from Theorem 1 we can obtain a coarser but, perhaps, a more useful lower bound for the r -weighted Gini mean difference, namely:

$$(2.8) \quad G_r(\mathbf{p}, \mathbf{a}) \geq \frac{1}{2} \cdot \frac{p_m^{2r} + p_m^2 (1 - p_m^2)^{r-1}}{(1 - p_m^2)^{r-1}} \cdot \max_{(i,j) \in \Delta} |a_i - a_j|^r,$$

where p_m is defined above.

For $r = 2$, we then have:

$$(2.9) \quad G_2(\mathbf{p}, \mathbf{a}) \geq \frac{1}{2} \cdot \frac{p_m^2}{1 - p_m^2} \cdot \max_{(i,j) \in \Delta} (a_i - a_j)^2.$$

The following result for the weighted Gini mean difference can be stated:

Theorem 2. For any $p_i \in (0, 1)$, $i \in \{1, \dots, n\}$ with $\sum_{i=1}^n p_i = 1$ and $a_i \in \mathbb{R}$, $i \in \{1, \dots, n\}$, we have the bounds:

$$(2.10) \quad \frac{1}{2} \max_{(i,j) \in \Delta} \left\{ p_i p_j \left[1 + \frac{1}{\max_{(k,l) \in \Delta \setminus \{(i,j)\}} \{p_k p_l\}} \right] \cdot |a_i - a_j| \right\} \\ \leq G(\mathbf{p}, \mathbf{a}) \leq \frac{1}{2} \max_{(i,j) \in \Delta} |a_i - a_j|.$$

Proof. As in the proof of Theorem 1 we have

$$\begin{aligned} p_i p_j |a_i - a_j| &= \left| \sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k p_l (a_k - a_l) \right| \\ &\leq \max_{(k,l) \in \Delta \setminus \{(i,j)\}} \{p_k p_l\} \cdot \sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k p_l |a_k - a_l| \\ &= \max_{(k,l) \in \Delta \setminus \{(i,j)\}} \{p_k p_l\} \left[\sum_{(k,l) \in \Delta} p_k p_l |a_k - a_l| - p_i p_j |a_i - a_j| \right] \end{aligned}$$

which gives:

$$p_i p_j \left[1 + \max_{(k,l) \in \Delta \setminus \{(i,j)\}} \{p_k p_l\} \right] |a_i - a_j| \leq \max_{(k,l) \in \Delta \setminus \{(i,j)\}} \{p_k p_l\} \cdot \sum_{(k,l) \in \Delta} p_k p_l |a_k - a_l|.$$

That is

$$p_i p_j \left[\frac{1 + \max_{(k,l) \in \Delta \setminus \{(i,j)\}} \{p_k p_l\}}{\max_{(k,l) \in \Delta \setminus \{(i,j)\}} \{p_k p_l\}} \right] |a_i - a_j| \leq \sum_{(k,l) \in \Delta} p_k p_l |a_k - a_l|,$$

which, by taking the maximum over $(i, j) \in \Delta$ implies the first part of (2.10).

The second part is obvious. ■

Remark 3. Since

$$\max_{(k,l) \in \Delta \setminus \{(i,j)\}} \{p_k p_l\} \leq \max_{(k,l) \in \Delta} \{p_k p_l\} = p_M^2,$$

where $p_M := \max_{k \in \{1, \dots, n\}} p_k$, hence

$$1 + \frac{1}{\max_{(k,l) \in \Delta \setminus \{(i,j)\}} \{p_k p_l\}} \geq 1 + \frac{1}{p_M^2}$$

and we get from Theorem 2 the following lower bounds for $G(\mathbf{p}, \mathbf{a})$

$$(2.11) \quad \begin{aligned} G(\mathbf{p}, \mathbf{a}) &\geq \frac{1}{2} \left(\frac{p_M^2 + 1}{p_M^2} \right) \max_{(i,j) \in \Delta} \{p_i p_j |a_i - a_j|\} \\ &\geq \frac{1}{2} p_m^2 \left(\frac{p_M^2 + 1}{p_M^2} \right) \max_{(i,j) \in \Delta} |a_i - a_j|, \end{aligned}$$

where $p_m := \min_{k \in \{1, \dots, n\}} p_k$ and $p_M := \max_{k \in \{1, \dots, n\}} p_k$.

3. RELATED RESULTS

The following result is due to Izumino and Pečarić [5] (see also [2, p. 174 - 175]):

Lemma 1. *Let f be a convex even function defined on $[m - M, M - m]$ ($0 < m < M$) with $f(0) = 0$. Then for each n -tuple $x = (x_1, \dots, x_n)$ satisfying the condition $m \leq x_k \leq M$ ($k = 1, \dots, n$) and for each positive weight $q = (q_1, \dots, q_n)$ we have*

$$(3.1) \quad \begin{aligned} \sum_{1 \leq i < j \leq n} q_i q_j f(x_i - x_j) &\leq f(M - m) \max_{J \subseteq \{1, \dots, n\}} [Q_J (1 - Q_J)] \\ &\leq \frac{1}{4} f(M - m), \end{aligned}$$

where $Q_j := \sum_{j \in J} q_j$.

The following result holds concerning upper bounds for the r -weighted Gini mean difference when some information on the size of the elements a_i , $i \in \{1, \dots, n\}$ are available.

Theorem 3. *For any $p_i \in (0, 1)$, $i \in \{1, \dots, n\}$ with $\sum_{i=1}^n p_i = 1$ and $a_i \in \mathbb{R}$, $i \in \{1, \dots, n\}$ with the property that*

$$(3.2) \quad -\infty < a \leq a_i \leq A < \infty \quad \text{for each } i \in \{1, \dots, n\},$$

we have the inequality:

$$(3.3) \quad G_r(\mathbf{p}, \mathbf{a}) \leq (A - a)^r \max_{J \subseteq \{1, \dots, n\}} [P_J (1 - P_J)] \quad \left(\leq \frac{1}{4} (A - a)^r \right),$$

for $r \geq 1$.

Proof. Without loss of generality, we may assume that $a \geq 0$.

Now, if we apply Lemma 1 for $f(x) = |x|^r$, $x_i = a_i$ and $q_i = p_i$, $i \in \{1, \dots, n\}$, we get

$$G_r(\mathbf{p}, \mathbf{a}) = \frac{1}{2} \sum_{i,j=1}^n p_i p_j |a_i - a_j|^r \leq |A - a|^r \max_{J \subseteq \{1, \dots, n\}} [P_J (1 - P_J)]$$

and the result is proved. ■

Finally, the following result that provides a connection between

$$G_2(\mathbf{p}, \mathbf{a}) = \sum_{i=1}^n p_i a_i^2 - \left(\sum_{i=1}^n p_i a_i \right)^2,$$

and

$$G_2(\mathbf{a}) = \frac{1}{n} \sum_{i=1}^n a_i^2 - \left(\frac{1}{n} \sum_{i=1}^n a_i \right)^2,$$

can be stated.

Theorem 4. *If $p_i \in (0, 1)$ for $i \in \{1, \dots, n\}$ with $\sum_{i=1}^n p_i = 1$, then for any $a_i \in \mathbb{R}$ $i \in \{1, \dots, n\}$ we have the inequality:*

$$(3.4) \quad G_2(\mathbf{p}, \mathbf{a}) \leq n^2 \left[1 - \frac{(\sum_{i=1}^n p_i^3)^2}{(\sum_{i=1}^n p_i^2)^2} \right] G_2(\mathbf{a}).$$

Proof. Utilising the Cauchy-Bunyakovsky-Schwarz inequality, we have that:

$$(3.5) \quad \begin{aligned} & p_i p_j |a_i - a_j| \\ &= \left| \sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k p_l (a_k - a_l) \right| \\ &\leq \left(\sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k^2 p_l^2 \right)^{\frac{1}{2}} \left(\sum_{(k,l) \in \Delta \setminus \{(i,j)\}} |a_k - a_l|^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{(k,l) \in \Delta} p_k^2 p_l^2 - p_i^2 p_j^2 \right)^{\frac{1}{2}} \left(\sum_{(k,l) \in \Delta} |a_k - a_l|^2 - |a_i - a_j|^2 \right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{k=1}^n p_k^2 \right)^2 - p_i^2 p_j^2 \right]^{\frac{1}{2}} \left(\sum_{(k,l) \in \Delta} |a_k - a_l|^2 - |a_i - a_j|^2 \right)^{\frac{1}{2}} \end{aligned}$$

The square of (3.5) produces

$$p_i^2 p_j^2 |a_i - a_j|^2 \leq \left[\left(\sum_{k=1}^n p_k^2 \right)^2 - p_i^2 p_j^2 \right] \left[\sum_{(k,l) \in \Delta} |a_k - a_l|^2 - |a_i - a_j|^2 \right],$$

giving

$$\begin{aligned} & \left[p_i^2 p_j^2 + \left(\sum_{k=1}^n p_k^2 \right)^2 - p_i^2 p_j^2 \right] |a_i - a_j|^2 \\ & \leq \left[\left(\sum_{k=1}^n p_k^2 \right)^2 - p_i^2 p_j^2 \right] \sum_{(k,l) \in \Delta} |a_k - a_l|^2 \end{aligned}$$

from which we get

$$(3.6) \quad |a_i - a_j|^2 \leq \left[1 - \frac{p_i^2 p_j^2}{(\sum_{k=1}^n p_k^2)^2} \right] \sum_{(k,l) \in \Delta} |a_k - a_l|^2.$$

Now, if we multiply (3.6) with $p_i p_j \geq 0$ and sum over $(i, j) \in \Delta$ then we get

$$(3.7) \quad G_2(\mathbf{p}, \mathbf{a}) \leq n^2 \left[1 - \frac{(\sum_{i=1}^n p_i^3)^2}{(\sum_{i=1}^n p_i^2)^2} \right] G_2(\mathbf{a}),$$

and the result is proved. ■

Remark 4. It is obvious, by the definition of $G_r(\mathbf{p}, \mathbf{a})$ in (2.2) that for $r = 2$

$$(3.8) \quad G_2(\mathbf{p}, \mathbf{a}) = \frac{1}{2} \sum_{(i,j) \in \Delta} p_i p_j |a_i - a_j|^2 \leq \frac{1}{2} \max_{(i,j) \in \Delta} \{p_i p_j\} \sum_{(i,j) \in \Delta} |a_i - a_j|^2 \\ = n^2 \max_{(i,j) \in \Delta} \{p_i p_j\} G_2(\mathbf{a}).$$

Then, it is natural to ask when comparing (3.7) and (3.8) the question, when is the bound

$$B_1(\mathbf{p}) := 1 - \frac{(\sum_{i=1}^n p_i^3)^2}{(\sum_{i=1}^n p_i^2)^2}$$

better than

$$B_2(\mathbf{p}) := \max_{(i,j) \in \Delta} \{p_i p_j\}.$$

If we take $n = 2$ and $p_1 = p$, $p_2 = 1 - p$, $p \in (0, 1)$ then

$$B_1(p) = 1 - \left[\frac{p^3 + (1-p)^3}{p^2 + (1-p)^2} \right]^2$$

and

$$B_2(p) = \max \{p^2, p(1-p), (1-p)^2\}.$$

The variation of the bounds $B_1(p)$ and $B_2(p)$ are depicted in Figure 1 and Figure 2, respectively. The plot of the difference $D(p) := B_1(p) - B_2(p)$ shows that one bound is not always better than the other (see Figure 3).

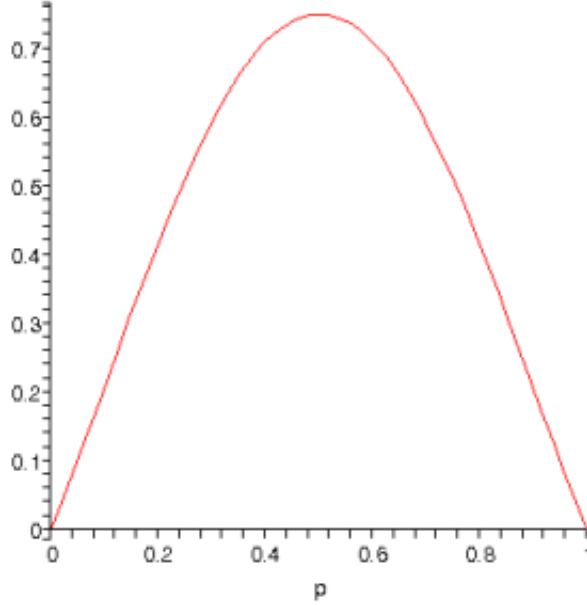
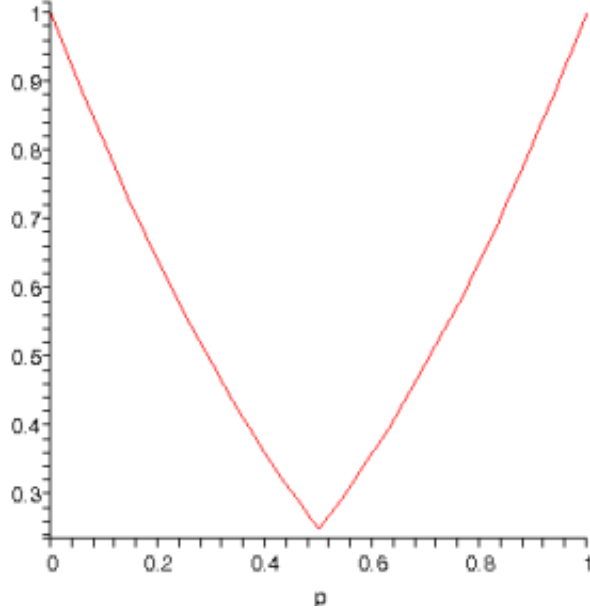


FIGURE 1. The plot of $B_1(p)$.

FIGURE 2. The plot of $B_2(p)$.

Finally, the following result in comparing the weighted Gini mean difference $G(\mathbf{p}, \mathbf{a})$ with the unweighted means $G_r(\mathbf{a})$ may be stated:

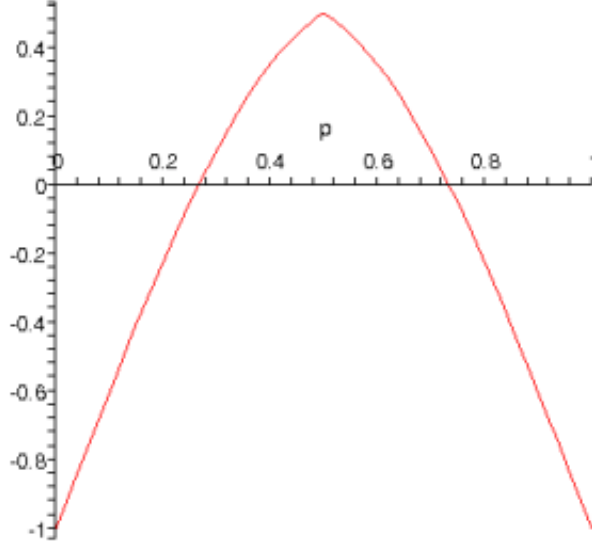
Theorem 5. *If $p_i \in (0, 1)$ for $i \in \{1, \dots, n\}$ with $\sum_{i=1}^n p_i = 1$, and $q, r > 1$ with $\frac{1}{q} + \frac{1}{r} = 1$, then for any $a_i \in \mathbb{R}$ $i \in \{1, \dots, n\}$ we have the inequality:*

$$(3.9) \quad G(\mathbf{p}, \mathbf{a}) \leq 2^{1/r-2} n^{2/r+2} \left(\sum_{i=1}^n p_i^q \right)^{2/q} [G_r(\mathbf{a})]^{1/r}.$$

Proof. We use Hölder's inequality for double sums to get

$$(3.10) \quad \begin{aligned} p_i p_j |a_i - a_j| &= \left| \sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k p_l (a_k - a_l) \right| \\ &\leq \left(\sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k^q p_l^q \right)^{1/q} \left(\sum_{(k,l) \in \Delta \setminus \{(i,j)\}} |a_k - a_l|^r \right)^{1/r} \\ &\leq \left(\sum_{(k,l) \in \Delta} p_k^q p_l^q - p_i^q p_j^q \right)^{1/q} \left(\sum_{(k,l) \in \Delta} |a_k - a_l|^r - |a_i - a_j|^r \right)^{1/r} \\ &= \left[\left(\sum_{k=1}^n p_k^q \right)^2 - p_i^q p_j^q \right]^{1/q} (2n^2 G_r(\mathbf{a}) - |a_i - a_j|^r)^{1/r} \end{aligned}$$

for each $(i, j) \in \Delta$.

FIGURE 3. The plot of the difference $D_1(p)$.

Utilising the elementary inequality

$$(\alpha^r - \beta^r)^{1/r} (\gamma^q - \delta^q)^{1/q} \leq \alpha\gamma - \beta\delta$$

provided $\alpha \geq \beta, \gamma \geq \delta$ and $q, r > 1$ with $\frac{1}{q} + \frac{1}{r} = 1$, we can get that

$$p_i p_j |a_i - a_j| \leq \left(\sum_{i=1}^n p_i^q \right)^{2/q} [2n^2 G_r(\mathbf{a})]^{1/r} - p_i p_j |a_i - a_j|$$

which gives

$$(3.11) \quad 2p_i p_j |a_i - a_j| \leq \left(\sum_{i=1}^n p_i^q \right)^{2/q} [2n^2 G_r(\mathbf{a})]^{1/r},$$

for each $(i, j) \in \Delta$.

Summing in the inequality (3.11) over $(i, j) \in \Delta$ we deduce the desired result (3.9). ■

Remark 5. The particular case $q = r = 2$ provides the following simple inequality

$$(3.12) \quad G(\mathbf{p}, \mathbf{a}) \leq 2^{-3/2} n^3 \left(\sum_{i=1}^n p_i^2 \right) [G_2(\mathbf{a})]^{1/2}.$$

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SCHOOL OF COMPUTER SCIENCE AND MATHEMATICS, VICTORIA UNIVERSITY, PO Box 14428,
MELBOURNE CITY, VIC 8001, AUSTRALIA.

E-mail address: `pietro.cerone@vu.edu.au`

URL: `http://rgmia.vu.edu.au/cerone`

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.vu.edu.au/dragomir`