A note on certain Euler–Mascheroni type sequences

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Abstract. Expressions of type $x_n = \sum_{k=1}^{n} \frac{1}{a_k} - \log a_n$ $(a_n > 0)$ will be called of Euler–Mascheroni type, as for $a_k \equiv k$ we obtain a sequence of approximations of the Euler–Mascheroni constant γ. The aim of this note is to solve two open problems posed by K. Kashihara [1] related to the convergence or divergence of (x_n) when $a_n = p_n$ (nth prime), and $a_n = S(n)$ (Smarandache function). An analogues result on the Smarandache ceil function is pointed out, too.

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1 Introduction

Let (a_n) be a sequence of strictly positive real numbers, and construct the new sequence (x_n) defined by

$$x_n = \sum_{k=1}^{n} \frac{1}{a_k} - \log a_n \quad (n = 1, 2, \ldots) \tag{1}$$

For $a_k = k$ $(k = 1, 2, \ldots)$ one obtains $x_n = \sum_{k=1}^{n} \frac{1}{k} - \log n$, which gives the well-known Euler sequence (or Euler–Mascheroni sequence), having as limit the Euler–Mascheroni constant γ (see [3]).

In his book K. Kashihara [1] (see p. 42) posed the problems of convergence or divergence of sequence (x_n) given by (1) for the particular cases $a_k = p_k$, the kth prime; as well as $a_k = S(k)$, the Smarandache function value. We will prove the following:

Theorem 1. *The sequence* (x_n^1) *given by*

$$(x_n^1) = \sum_{k=1}^{n} \frac{1}{p_k} - \log p_n \tag{2}$$

is divergent, being unbounded from below. The sequence (x_n^2) *given by*

$$(x_n^2) = \sum_{k=1}^{n} \frac{1}{S(k)} - \log S(n) \tag{3}$$

is divergent, being unbounded from above.

2 Proof of the Theorem

An old result of P. Chebyshev (see e.g. [2]) states that

$$\sum_{p\le x}\frac{1}{p} = \log\log x + B + \mathcal{O}(1), \tag{4}$$

where p denote primes. This means that $\left(\sum_{p\le p_n}\frac{1}{p} - \log\log p_n\right)$ is a convergent sequence. Remarking that

$$x_n^1 = \left(\sum_{p\le p_n}\frac{1}{p} - \log\log p_n\right) + \log\log p_n - \log p_n,$$

and by $\log\log p_n - \log p_n = \log\left(\frac{\log p_n}{p_n}\right)$, since $\frac{\log p_n}{p_n} \to 0$ as $n\to\infty$ we get that $x_n^1 \to -\infty$ as $n\to\infty$. This proves the first part of the theorem.

For the second part, put $n = m!$ Then, since $S(n) = \min\{k\ge 1 : n|k!\}$, we have $S(n) = m$, and $x_n^2 = \left(1+\frac{1}{2}+\cdots+\frac{1}{m} - \log m\right) + \sum_{\substack{k<n\\ k\ne l!,\, l<n}}\frac{1}{S(k)}$, because for $k = l!$, $l<m$ one has $S(k) = l$. Now, the last sum is greater than $\sum_{p<n}\frac{1}{p}$, as for primes $k = p < n$ one has $S(k) = S(p) = p$, and $p \ne l!$. It is well known that $\sum_{p=1}^{\infty}\frac{1}{p} = +\infty$, so as $m\to\infty$, clearly (x_n^2) becomes unbounded from above, since the term $1+\frac{1}{2}+\cdots+\frac{1}{m} - \log m$ is bounded.

Remarks

1) For many improvements of (4) see our monograph [2].

2) For generalized Euler-Mascheroni constants, see our paper [3].

3) The above proof shows that $S(n)$ may be replaced by any function having the properties $S(k!) = k$ and $S(p) = p$ (p prime).

4) Let $S_2(n) = \min\{m\ge 1 : n|m^2\}$ be the "Smarandache ceil function" of order 2. By defining

$$x_n^3 = \sum_{k=1}^{n}\frac{1}{S_2(k)} - \log S_2(n) \tag{5}$$

we can prove similarly that (x_n^3) is an unbounded (from above) sequence. Even, a more precise result holds true. Indeed, recently Wang Xiaoying [4] proved that

$$\sum_{n\le x}\frac{1}{S_2(n)}=\frac{3}{2\pi^2}\log^2 x+A_1\log x+A_2+\mathcal{O}(x^{-\frac{1}{4}+\varepsilon}). \tag{6}$$

Since $\sqrt{n}\le S_2(n)\le n$, we have $\log S_2(n)=\mathcal{O}(\log n)=\mathcal{O}(\log^2 n)$, so by (6) it follows that

$$\frac{x_n^3}{\log^2 n}\sim\frac{3}{2\pi^2}\text{ as } n\to\infty. \tag{7}$$

References

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