SOME NEW BOUNDS FOR MATHIEU'S SERIES

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ABSTRACT. In the paper, an upper bound and two lower bounds for Mathieu's series are established, which refine to a certain extent a sharp double inequality obtained by Alzer-Brenner-Ruehr in 1998. Moreover, the very closer lower and upper bounds for $\zeta(3)$ are deduced.

1. Introduction

In 1890, Mathieu in [19] defined $S(r)$ for $r > 0$ by

$$S(r)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^2} \tag{1}$$

and conjectured that $S(r) < \frac{1}{r^2}$. We call formula (1) Mathieu's series.

There have been a lot of literature about the estimations of $S(r)$ for more than 100 years before 1998, for examples, [1, 2, 6, 7, 11, 12, 18, 29, 33, 34, 35] and the references therein. In [18], E. Makai proved

$$\frac{1}{r^2+1/2} < S(r) < \frac{1}{r^2}. \tag{2}$$

In 1998, H. Alzer, J. L. Brenner and O. G. Ruehr presented in [1] that

$$\frac{1}{r^2+\frac{1}{2\zeta(3)}} < S(r) < \frac{1}{r^2+\frac{1}{6}}, \tag{3}$$

where ζ denotes the zeta function and the constants $\frac{1}{2\zeta(3)}$ and $\frac{1}{6}$ in (3) are the best possible.

After 2000, among other things, several open problems on the estimations and integral representations of generalized Mathieu's series were posed in [14, 26, 27] by B.-N. Guo and F. Qi. Stimulated by or originated from these open problems, a lot of articles such as [3, 4, 5, 8, 9, 10, 13, 15, 20, 21, 22, 23, 24, 25, 28, 30, 31, 32] have been published in variant reputable journals by many mathematicians all over the world.

In this article, by utilizing a method and techniques used in [18], we would like to improve or refine the sharp double inequality (3) and to establish a very closer double inequality for $\zeta(3)$.

Our main results are the following four theorems.

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Theorem 1. *For $r > 0$,*

$$S(r) > \frac{1}{r^2 + \frac{1}{6} + \frac{r^2+6}{3(9r^2+8)}} = \frac{1}{r^2 + \frac{1}{2} - \frac{2(4r^2+1)}{3(9r^2+8)}}. \tag{4}$$

Remark 1. By standard argument, it is showed readily that inequality (4) is better than the left hand side inequality in (3) when $r > 2\sqrt{\frac{5\zeta(3)-6}{27-11\zeta(3)}} = 0.05\cdots$.

Theorem 2. *For $r > 0$,*

$$S(r) > \frac{1}{r^2 + \frac{1}{6} + \frac{5}{6(2r^2+3)}} = \frac{1}{r^2 + \frac{1}{2} - \frac{4r^2+1}{6(2r^2+3)}}. \tag{5}$$

Remark 2. It is not difficult to verify that inequality (5) is better than the left hand side inequality in (3) when $r > \sqrt{\frac{8\zeta(3)-9}{2[3-\zeta(3)]}} = 0.41\cdots$.

It is important to remark that inequalities (4) and (5) do not include each other, which can be proved straightforwardly.

Theorem 3. *For $r > 0$,*

$$S(r) < \frac{1}{\sqrt{r^4 + 2r^2 + 2} - 1}. \tag{6}$$

Remark 3. It is easy to deduce that inequality (6) is better than the right hand side inequality in (3) when $0 < r < \sqrt{\frac{23}{12}} = 1.38\cdots$.

Theorem 4. *For $m \in \mathbb{N}$, let $S_3(m) = \sum_{n=1}^{m} \frac{1}{n^3}$. Then*

$$\frac{1}{2m^2 + 2m + 1 - \frac{1}{6(m^2+m+3/2)}} < \zeta(3) - S_3(m) < \frac{1}{2m^2 + 2m + 1 - \frac{1}{6(m^2+m+1)}}. \tag{7}$$

Remark 4. Calculation by Mathematica 5.2 shows that

$$\zeta(3) = 1.202056903159594285399\cdots.$$

If m taking from 1 to 9, the sums of the right side term in (7) and $S_3(m)$ are

1.202247191011235955, 1.202064220183486239, 1.202057560382342322,
1.202057003155139651, 1.202056924652726768, 1.202056909039779896,
1.202056905080018071, 1.202056903877571143, 1.202056903458154800.

If m taking from 1 to 9, the sums of the left side term in (7) and $S_3(m)$ are

1.201923076923076923, 1.202054794520547945, 1.202056799882886839,
1.202056893315403149, 1.202056901714344462, 1.202056902872941459,
1.202056903088695828, 1.202056903138840387, 1.202056903152657143.

These numerical computations by Mathematica 5.2 reveals that inequalities in (7) give much accurate approximations from left and right.

Corollary 1. *If $1 \le \delta < \frac{3}{2}$ and $m \ge \sqrt{\frac{3\delta^2 - \delta + \frac{1}{12}}{6-4\delta}} - 1$, then*

$$\zeta(3) < S_3(m) + \frac{1}{2m^2 + 2m + 1 - \frac{1}{6(m^2+m+\delta)}}. \tag{8}$$

Remark 5. In [17], the number $\zeta(3)$ was estimated by using Jordan's inequality and its refinements. In [16], some more general conclusions were obtained.

2. Proofs of theorems and corollary

Now we are in a position to prove our theorems and corollary.

Proof of Theorem 1. For $n \in \mathbb{N}$, let

$$w_n = n(n-1) + r^2 + \frac{1}{2} - \frac{\theta}{n^2+\gamma},$$

where $\theta = \frac{1}{3}\left(r^2 + \frac{1}{4}\right)$ and γ is a possible and undetermined positive function of r such that

$$\frac{1}{w_n} - \frac{1}{w_{n+1}} \le \frac{2n}{(n^2+r^2)^2}. \tag{9}$$

Straightforward computation yields

$$\frac{1}{w_n} - \frac{1}{w_{n+1}} = \frac{2n\left\{1 + \frac{\theta(1+1/2n)}{(n^2+\gamma)[(n+1)^2+\gamma]}\right\}}{(n^2+r^2)^2 + \frac{\theta Q(n,r,\gamma)}{(n^2+\gamma)[(n+1)^2+\gamma]}},$$

where

$$Q(n,r,\gamma) = n^4 + 4n^3 + \left(4\gamma - 2r^2 - 1\right)n^2 + \left(6\gamma - 2r^2 - 2\right)n + 3\gamma^2 + 2\left(1-r^2\right)\gamma - \frac{2r^2}{3} - \frac{5}{12}.$$

It is easy to see that if

$$\frac{1+\frac{1}{2n}}{Q(n,r,\gamma)} \le \frac{1}{(n^2+r^2)^2}, \tag{10}$$

then inequality (9) holds. Further, inequality (10) is equivalent to

$$n^4 + 4n^3 + \left(4\gamma - 2r^2 - 1\right)n^2 + \left(6\gamma - 2r^2 - 2\right)n + 3\gamma^2 + 2\left(1-r^2\right)\gamma - \frac{2r^2}{3} - \frac{5}{12} \ge \left(1 + \frac{1}{2n}\right)\left(n^2+r^2\right)^2,$$

which can be rewritten as

$$7n^3 + \left(8\gamma - 8r^2 - 2\right)n^2 + \left(12\gamma - 6r^2 - 4\right)n + 6\gamma^2 + 4\left(1-r^2\right)\gamma - 2r^4 - \frac{4r^2}{3} - \frac{5}{6} - \frac{r^4}{n} \ge 0,$$

which can be further rearranged as

$$f(n,\gamma) \triangleq (n-1)\left[7n^2 + \left(8\gamma - 8r^2 + 5\right)n + 20\gamma - 14r^2 + 1 + \frac{r^4}{n}\right] + 6\gamma^2 + 4\left(6 - r^2\right)\gamma - 3r^4 - \frac{46}{3}r^2 + \frac{1}{6} \ge 0.$$

Direct computation reveals that

$$f\left(n, \frac{9r^2}{8}\right) = (n-1)\left[7n^2 + \left(r^2+5\right)n + \frac{17}{2}r^2 + 1 + \frac{r^4}{n}\right] + \frac{3}{32}r^4 + \frac{35}{3}r^2 + \frac{1}{6} > 0,$$

but

$$f\left(n, r^2\right) = (n-1)\left(7n^2 + 5n + 6r^2 + \frac{r^4}{n}\right) - r^4 + \frac{26}{3}r^2 + \frac{1}{6}$$

is negative if r is large enough. Consequently, if taking $\gamma=\frac{9r^2}{8}$, then inequality (9) is valid. Summing up on both sides of (9) with respect to $n=1,2,\cdots$ leads to (4). The proof of Theorem 1 is finished. □

Proof of Theorem 2. Now let us consider the sequence

$$\nu_n(r)=n(n-1)+r^2+\frac{1}{2}-\frac{\theta}{n(n-1)+\beta} \tag{11}$$

for $n\in\mathbb{N}$, where θ and β are two undetermined functions of r, in order that

$$\frac{1}{\nu_n(r)}-\frac{1}{\nu_{n+1}(r)}<\frac{2n}{(n^2+r^2)^2}. \tag{12}$$

Direct calculation yields

$$\frac{1}{\nu_n(r)}-\frac{1}{\nu_{n+1}(r)}=\frac{2n+\frac{2\theta n}{(n^2-n+\beta)(n^2+n+\beta)}}{(n^2+r^2)^2+\frac{P(n,r,\theta,\beta)}{(n^2-n+\beta)(n^2+n+\beta)}},$$

where

$$\begin{aligned}P(n,r,\theta,\beta)=\left(r^2+\frac{1}{4}-2\theta\right)n^4+\left(r^2+\frac{1}{4}\right)\beta^2-\theta\beta(2r^2+1)+\theta^2\\+\left[\left(r^2+\frac{1}{4}\right)(2\beta-1)-\theta\left(2\beta+2r^2+3\right)\right]n^2.\end{aligned}$$

Letting $r^2+\frac{1}{4}-2\theta=\theta$ and $\left(r^2+\frac{1}{4}\right)(2\beta-1)-\theta\left(2\beta+2r^2+3\right)=2\theta r^2$ gives

$$\theta=\frac{1}{3}\left(r^2+\frac{1}{4}\right)\quad\text{and}\quad\beta=r^2+\frac{3}{2}.$$

Consequently,

$$\begin{aligned}P(n,r,\theta,\beta)&=\theta n^4+2\theta r^2n^2+3\theta\beta^2-\theta\beta\left(2r^2+1\right)+\theta^2\\&=\theta\left(n^2+r^2\right)^2+\theta\left[3\beta^2-\beta\left(2r^2+1\right)+\theta-r^4\right]\\&=\theta\left(n^2+r^2\right)^2+\frac{16}{3}\theta\left(r^2+1\right).\end{aligned}$$

As a result,

$$\begin{aligned}\frac{1}{\nu_n(r)}-\frac{1}{\nu_{n+1}(r)}&=\frac{2n+\frac{2\theta n}{(n^2-n+\beta)(n^2+n+\beta)}}{(n^2+r^2)^2+\frac{\theta(n^2+r^2)^2+16\theta(r^2+1)/3}{(n^2-n+\beta)(n^2+n+\beta)}}\\&<\frac{2n+\frac{2\theta n}{(n^2-n+\beta)(n^2+n+\beta)}}{(n^2+r^2)^2+\frac{\theta(n^2+r^2)^2}{(n^2-n+\beta)(n^2+n+\beta)}}=\frac{2n}{(n^2+r^2)^2}.\end{aligned}$$

Summing up on both sides of above inequality with respect to $n\in\mathbb{N}$ leads to

$$S(r)>\frac{1}{\nu_1}=\frac{1}{r^2+\frac{1}{2}-\frac{\theta}{\beta}}=\frac{1}{r^2+\frac{1}{2}-\frac{4r^2+1}{12r^2+18}}.$$

The proof of Theorem 2 is complete. □

Proof of Theorem 3. Let $u_n(r)=n(n-1)+r^2+\mu(r)$ for $n\in\mathbb{N}$, where

$$\mu(r)=\sqrt{(r^2+1)^2+1}-(r^2+1)>0. \tag{13}$$

Then

$$\frac{1}{u_n(r)} - \frac{1}{u_{n+1}(r)} = \frac{2n}{(n^2+r^2)^2 - [1-2\mu(r)]n^2 + \mu^2(r) + 2r^2\mu(r)}.$$

From (13), it is deduced that $\mu^2(r) + 2r^2\mu(r) = 1 - 2\mu(r) > 0$. Hence,

$$\frac{1}{u_n(r)} - \frac{1}{u_{n+1}(r)} = \frac{2n}{(n^2+r^2)^2 - [1-2\mu(r)](n^2-1)} \geq \frac{2n}{(n^2+r^2)^2},$$

and then

$$\sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^2} < \frac{1}{u_1} = \frac{1}{r^2+\mu(r)} = \frac{1}{\sqrt{r^4+2r^2+2}-1}.$$

The proof of Theorem 3 is complete. □

Proof of Theorem 4. Let $t_n = 2n^2 - 2n + 1 - \frac{1}{6(n^2-n+\delta)}$, where δ is a fixed positive number and $n \in \mathbb{N}$. Direct computation gives

$$\frac{1}{t_n} - \frac{1}{t_{n+1}} = \frac{4n + \frac{2n}{6(n^2-n+\delta)(n^2+n+\delta)}}{4n^4 + \frac{2n^4+(8\delta-12)n^2+6\delta^2-2\delta+1/6}{6(n^2-n+\delta)(n^2+n+\delta)}}. \tag{14}$$

If $\delta = \frac{3}{2}$, then $8\delta - 12 = 0$ and

$$\frac{1}{t_n} - \frac{1}{t_{n+1}} = \frac{4n + \frac{2n}{6(n^2-n+3/2)(n^2+n+3/2)}}{4n^4 + \frac{2n^4+32/3}{6(n^2-n+3/2)(n^2+n+3/2)}} < \frac{1}{n^3}.$$

Summing up on both sides of above inequality for n from $m+1$ to infinity produces

$$\frac{1}{t_{m+1}} = \frac{1}{2m^2+1-\frac{1}{6(m^2+m+3/2)}} < \sum_{n=m+1}^{\infty} \frac{1}{n^3}.$$

Adding $S_3(m)$ on both sides of above inequality leads to the left hand side inequality in (7).

If $\delta = 1$ and $n > 1$, then

$$\frac{1}{t_n} - \frac{1}{t_{n+1}} = \frac{4n + \frac{2n}{6(n^2-n+1)(n^2+n+1)}}{4n^4 + \frac{2n^4-[4(n^2-1)-1/6]}{6(n^2-n+1)(n^2+n+1)}} > \frac{1}{n^3}.$$

Summing up on both sides of above inequality for n from $m+1$ to infinity yields

$$\frac{1}{2m^2+2m+1-\frac{1}{2(m^2+m+1)}} > \sum_{n=m+1}^{\infty} \frac{1}{n^3}.$$

This is equivalent to the right side inequality in (7). Theorem 4 is proved. □

Proof of Corollary 1. It is easy to see that

$$2n^4 + (8\delta-12)n^2 + 6\delta^2 - 2\delta + \frac{1}{6} = 2n^4 - (12-8\delta)\left(n^2 - \frac{3\delta^2-\delta+\frac{1}{12}}{6-4\delta}\right).$$

If $1 \leq \delta < \frac{3}{2}$ and $n \geq \sqrt{\frac{3\delta^2-\delta+\frac{1}{12}}{6-4\delta}}$, from equation (14), it is deduced that

$$\frac{1}{t_n} - \frac{1}{t_{n+1}} \geq \frac{1}{n^3}.$$

By the same argument as above, when $m \geq \sqrt{\frac{3\delta^2-\delta+\frac{1}{12}}{6-4\delta}}-1$, inequality

$$\frac{1}{t_{m+1}}=\frac{1}{2m^2+2m+1-\frac{1}{6(m^2+m+\delta)}}>\sum_{n=m+1}^{\infty}\frac{1}{n^3}$$

is obtained, which is equivalent to (8). The proof of Corollary 1 is complete. □

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