ON THE HERMITE-HADAMARD TYPE INEQUALITIES FOR CONVEX FUNCTIONS AND CONVEX FUNCTIONS ON THE CO-ORDINATES IN A RECTANGLE FROM THE PLANE

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Abstract. In this paper, we establish some inequalities of Hermite-Hadamard type for convex functions and convex functions on the co-ordinates defined in a rectangle from the plane.

1. Introduction

Let I be an interval in R, $f : I \subseteq R \to R$ be a convex function and $a, b \in I$ with $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is known in the literature as Hermite-Hadamard inequality for convex function (see for example [6]).

For some results which generalize, improve and extend Hermite-Hadamard inequality (1) see $[1-18]$.

In [3], Dragomir and Buşe established the following refinements of the inequality (1):

Theorem A. Let n be a natural number, $q_i \geq 0$ $(i = 1, \cdots, n)$ and $Q_n = \sum\limits_{i=1}^{n} q_i > 0$. If I is an interval in R, $f : I \subseteq R \to R$ is convex and $a, b \in I$ with $a < b$, then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\prod_{i=1}^{n} dx_i \\ &\leq \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right)\prod_{i=1}^{n} dx_i \\ &\leq \frac{1}{b-a}\int_a^b f(x)\,dx. \end{aligned} \tag{2}$$

In [18] , Yang and Wang established the following two theorems which give some refinements of the inequality (2):

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Theorem B. Let $0 < \alpha_i < 1$ $(i = 1, \cdots, n;\ n \geq 2)$ with $\sum_{i=1}^{n} \alpha_i = 1$ and, let f be defined as in Theorem A. Then

$$
\begin{aligned}
(3)\qquad & f\left(\frac{a+b}{2}\right) \\
\leq\ & \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \prod_{i=1}^{n} dx_i \\
\leq\ & \sum_{i=1}^{n} \frac{1-\alpha_i}{n-1} \frac{1}{(b-a)^{n-1}} \left[\int_a^b \cdots \int_a^b f\left(\sum_{j=1, j\neq i}^{n} \frac{\alpha_j x_j}{1-\alpha_i}\right) \prod_{k=1, k\neq i}^{n} dx_k\right] \\
\leq\ & \frac{1}{b-a} \int_a^b f(x)\, dx.
\end{aligned}
$$

Theorem C. Let n be a natural number, $0 \leq \alpha_i \leq 1$ $(i = 1, \cdots, n)$ with $\sum_{i=1}^{n} \alpha_i = 1$ and let $\Phi : [0, 1] \to R$ be defined by

$$\Phi(t) = \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b f\left(t \sum_{i=1}^{n} \alpha_i x_i + (1-t)\frac{a+b}{2}\right) \prod_{i=1}^{n} dx_i,\ t \in [0, 1].$$

Then Φ is convex, increasing on $[0, 1]$ and, for $t \in [0, 1]$,

$$(4)\qquad f\left(\frac{a+b}{2}\right) = \Phi(0) \leq \Phi(t) \leq \Phi(1) = \int_a^b \cdots \int_a^b f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \prod_{i=1}^{n} dx_i.$$

Let $D \subseteq R^2$. In [2], a function $F : D \to R$ will be called *convex on the co-ordinates on* D if the partial mapping $F_y(x) := F(x, y)$ is convex in x for each fixed y, and the partial mapping $F_x(y) := F(x, y)$ is convex in y for each fixed x where $(x, y) \in D$. In [7], the author calls such a function convex separately with respect to each coodinate.

In [7], Lanina esatblished the following two theorems:

Theorem D. Suppose

(a) $F : D \to R$ is convex on the co-ordinates on D where $D \subseteq R^2$;

(b) $\Delta = [a, b] \times [c, d] \subseteq D$ with $a < b$ and $c < d$;

(c) n, q_i $(i = 1, \cdots n)$ and Q_n are defined as in Theorem A.

Then

$$
\begin{aligned}
(5)\qquad & F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq\ & \frac{1}{(b-a)^n (d-c)^n} \int_\Delta \cdots \int_\Delta F\left(\frac{1}{n}\sum_{i=1}^{n} x_i, \frac{1}{n}\sum_{j=1}^{n} y_j\right) \prod_{i=1}^{n} dx_i \prod_{j=1}^{n} dy_j \\
\leq\ & \frac{1}{(b-a)^n (d-c)^n} \int_\Delta \cdots \int_\Delta F\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i, \frac{1}{Q_n}\sum_{j=1}^{n} q_i y_j\right) \prod_{i=1}^{n} dx_i \prod_{j=1}^{n} dy_j \\
\leq\ & \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b F(x, y)\, dx dy.
\end{aligned}
$$

Theorem E. Let D, Δ, n, q_i $(i = 1, \cdots n)$ and Q_n be defined as in Theorem D. If $F : D \to R$ is convex, then the inequality (5) also holds.

In [2], Dragomir esatblished the following two theorems:

Theorem F. *Let* $\Delta = [a, b] \times [c, d] \subset R^2$, $F : \Delta \to R$ *be convex on the co-ordinates on* Δ *and let* $H : [0, 1]^2 \to R$ *be defined by*

$$H(t, s) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) \times dydx. \tag{6}$$

Then:

(a) *The function* H *is convex on the co-ordinates on* $[0, 1]^2$.

(b) *The function* H *is increasing on the co-ordinates on* $[0, 1]^2$,

$$\sup_{(t,s)\in[0,1]^2} H(t, s) = H(1, 1) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x, y)\, dydx$$

and

$$\inf_{(t,s)\in[0,1]^2} H(t, s) = H(0, 0) = F\left(\frac{a+b}{2}, \frac{c+d}{2}\right).$$

Theorem G. *Let* $\Delta = [a, b] \times [c, d] \subset R^2$, $F : \Delta \to R$ *be convex on* Δ *and let* $h : [0, 1] \to R$ *be defined by* $h(t) = H(t, t)$ where H is defined as in (6).

Then:

(a) *The function* H *is convex on* $[0, 1]^2$.

(b) *The function* h *is convex and increasing on* $[0, 1]$,

$$\sup_{t\in[0,1]} h(t) = h(1) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x, y)\, dydx$$

and

$$\inf_{t\in[0,1]} h(t) = h(0) = F\left(\frac{a+b}{2}, \frac{c+d}{2}\right).$$

In this paper, we shall establish some inequalities for convex functions and convex functions on the co-ordinates related to Theorems A-G.

2. Main Results

In this section, we assume that m and n are natural numbers, $D \subseteq R^2$, $\Delta = [a, b] \times [c, d] \subseteq D$, $\Delta_x^n = [a, b]^n$ and $\Delta_y^m = [c, d]^m$ with $a < b$ and $c < d$. A function $H : [0, 1]^2 \to R$ will be called *increasing on the co-ordinates on* $[0, 1]^2$ if the partial mapping $H_s : [0, 1] \to R$, $H_s(t) := H(t, s)$ is *increasing* on $[0, 1]$ for each $s \in [0, 1]$, and the partial mapping $H_t : [0, 1] \to R$, $H_t(s) := H(t, s)$ is *increasing* on $[0, 1]$ for each $t \in [0, 1]$.

Theorem 1. Let $0 \le \alpha_i \le 1$ $(i = 1, \cdots, n)$ and $0 \le \beta_j \le 1$ $(j = 1, \cdots, m)$ with $\sum_{i=1}^{n} \alpha_i = 1$ and $\sum_{j=1}^{m} \beta_j = 1$. If $F : D \to R$ is convex on the co-ordinates on D, then

$$
\begin{aligned}
(7) \qquad & F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\le \; & \frac{1}{(b-a)^n (d-c)^m} \int_{\Delta_y^m} \int_{\Delta_x^n} F\left(\frac{1}{n}\sum_{i=1}^{n} x_i, \frac{1}{m}\sum_{j=1}^{m} y_j\right) \prod_{i=1}^{n} dx_i \prod_{j=1}^{m} dy_j \\
\le \; & \frac{1}{(b-a)^n (d-c)^m} \int_{\Delta_y^m} \int_{\Delta_x^n} F\left(\sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{m} \beta_j y_j\right) \prod_{i=1}^{n} dx_i \prod_{j=1}^{m} dy_j \\
\le \; & \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b F(x,y)\, dx dy.
\end{aligned}
$$

Proof. Since F is convex on the co-ordinates on D, we have

$$
\begin{aligned}
(8) \quad & \int_{\Delta_y^m} \int_{\Delta_x^n} F\left(\frac{1}{n}\sum_{i=1}^{n} x_i, \frac{1}{m}\sum_{j=1}^{m} y_j\right) \prod_{i=1}^{n} dx_i \prod_{j=1}^{m} dy_j \\
= \; & \int_{\Delta_y^m} \int_{\Delta_x^n} \frac{1}{2}\left[F\left(\frac{1}{n}\sum_{i=1}^{n} x_i, \frac{1}{m}\sum_{j=1}^{m} y_j\right) + \right. \\
& \left. F\left(\frac{1}{n}\left(a+b-x_1+\sum_{i=2}^{n} x_i\right), \frac{1}{m}\sum_{j=1}^{m} y_j\right)\right] \prod_{i=1}^{n} dx_i \prod_{j=1}^{m} dy_j \\
\ge \; & \int_{\Delta_y^m} \int_{\Delta_x^n} F\left(\frac{1}{n}\left(\frac{a+b}{2}+\sum_{i=2}^{n} x_i\right), \frac{1}{m}\sum_{j=1}^{m} y_j\right) \prod_{i=1}^{n} dx_i \prod_{j=1}^{m} dy_j \\
= \; & (b-a) \int_{\Delta_y^m} \int_{\Delta_x^{n-1}} F\left(\frac{1}{n}\left(\frac{a+b}{2}+\sum_{i=2}^{n} x_i\right), \frac{1}{m}\sum_{j=1}^{m} y_j\right) \prod_{i=2}^{n} dx_i \prod_{j=1}^{m} dy_j \\
= \; & (b-a) \int_{\Delta_y^m} \int_{\Delta_x^{n-1}} \frac{1}{2}\left[F\left(\frac{1}{n}\left(\frac{a+b}{2}+\sum_{i=2}^{n} x_i\right), \frac{1}{m}\sum_{j=1}^{m} y_j\right) + \right. \\
& \left. F\left(\frac{1}{n}\left(\frac{a+b}{2}+a+b-x_2+\sum_{i=3}^{n} x_i\right), \frac{1}{m}\sum_{j=1}^{m} y_j\right)\right] \prod_{i=2}^{n} dx_i \prod_{j=1}^{m} dy_j \\
\ge \; & (b-a) \int_{\Delta_y^m} \int_{\Delta_x^{n-1}} F\left(\frac{1}{n}\left(2\times\frac{a+b}{2}+\sum_{i=3}^{n} x_i\right), \frac{1}{m}\sum_{j=1}^{m} y_j\right) \prod_{i=2}^{n} dx_i \prod_{j=1}^{m} dy_j
\end{aligned}
$$

$$
\begin{aligned}
&= (b-a)^2 \int_{\Delta_y^m} \int_{\Delta_x^{n-2}} F\left(\frac{1}{n}\left(2\times\frac{a+b}{2}+\sum_{i=3}^{n} x_i\right), \frac{1}{m}\sum_{j=1}^{m} y_j\right) \prod_{i=3}^{n} dx_i \prod_{j=1}^{m} dy_j \\
&\quad \vdots \\
&\geq (b-a)^n \int_{\Delta_y^m} F\left(\frac{1}{n}\left(n\times\frac{a+b}{2}\right), \frac{1}{m}\sum_{j=1}^{m} y_j\right) \prod_{j=1}^{m} dy_j \\
&= (b-a)^n \int_c^d \cdots \int_c^d F\left(\frac{a+b}{2}, \frac{1}{m}\sum_{j=1}^{m} y_j\right) \prod_{j=1}^{m} dy_j
\end{aligned}
$$

Using a similar argument as the proof of the inequality (7), we have the following inequality

$$
\begin{aligned}
(9)\qquad & (b-a)^n \int_{\Delta_y^m} F\left(\frac{a+b}{2}, \frac{1}{m}\sum_{j=1}^{m} y_j\right) \prod_{j=1}^{m} dy_j \\
&\geq (b-a)^n (d-c)^m F\left(\frac{a+b}{2}, \frac{c+d}{2}\right).
\end{aligned}
$$

Multiplying (8) and (9) by $\frac{1}{(b-a)^n(d-c)^m}$, we obtain the first inequality of (7).

To proof the second inequality and third inequalities of (7), we define $\alpha_{n+t} = \alpha_t$ $(t = 1, \cdots, n-1)$ and $\beta_{m+s} = \beta_s$ $(s = 1, \cdots, m-1)$. Then, by a simple computation, we have the following two identities

$$
(10)\qquad \frac{1}{n}\sum_{i=1}^{n} x_i = \sum_{t=0}^{n-1} \frac{1}{n}\left(\sum_{l=1}^{n} \alpha_{l+t} x_l\right)
$$

and

$$
(11)\qquad \frac{1}{m}\sum_{j=1}^{m} y_j = \sum_{s=0}^{m-1} \frac{1}{m}\left(\sum_{r=1}^{m} \beta_{r+s} y_r\right)
$$

where $(x_i, y_j) \in \Delta\,(i = 1, \cdots, n; j = 1, \cdots, m)$. Since F is convex on the co-ordinates on D, using the identity (10), we obtain

$$
\begin{aligned}
(12)\qquad & \int_{\Delta_y^m} \int_{\Delta_x^n} F\left(\frac{1}{n}\sum_{i=1}^{n} x_i, \frac{1}{m}\sum_{j=1}^{m} y_j\right) \prod_{i=1}^{n} dx_i \prod_{j=1}^{m} dy_j \\
&= \int_{\Delta_y^m} \int_{\Delta_x^n} F\left(\sum_{t=0}^{n-1} \frac{1}{n}\left(\sum_{l=1}^{n} \alpha_{l+t} x_l\right), \frac{1}{m}\sum_{j=1}^{m} y_j\right) \prod_{i=1}^{n} dx_i \prod_{j=1}^{m} dy_j \\
&\leq \int_{\Delta_y^m} \left[\sum_{t=0}^{n-1} \frac{1}{n} \int_{\Delta_x^n} F\left(\sum_{l=1}^{n} \alpha_{l+t} x_l, \frac{1}{m}\sum_{j=1}^{m} y_j\right) \prod_{i=1}^{n} dx_i\right] \prod_{j=1}^{m} dy_j \\
&= \int_{\Delta_y^m} \left[\sum_{t=0}^{n-1} \frac{1}{n} \int_{\Delta_x^n} F\left(\sum_{l=1}^{n} \alpha_l x_l, \frac{1}{m}\sum_{j=1}^{m} y_j\right) \prod_{i=1}^{n} dx_i\right] \prod_{j=1}^{m} dy_j
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\Delta_y^m}\int_{\Delta_x^n} F\left(\sum_{l=1}^{n}\alpha_l x_l, \frac{1}{m}\sum_{j=1}^{m} y_j\right)\prod_{i=1}^{n} dx_i \prod_{j=1}^{m} dy_j \\
&\leq \int_{\Delta_y^m}\left[\sum_{l=1}^{n}\alpha_l \int_{\Delta_x^n} F\left(x_l, \frac{1}{m}\sum_{j=1}^{m} y_j\right)\prod_{i=1}^{n} dx_i\right]\prod_{j=1}^{m} dy_j \\
&= \int_{\Delta_y^m}\left[(b-a)^{n-1}\sum_{l=1}^{n}\alpha_l \int_a^b F\left(x_l, \frac{1}{m}\sum_{j=1}^{m} y_j\right) dx_l\right]\prod_{j=1}^{m} dy_j \\
&= (b-a)^{n-1}\int_{\Delta_y^m}\int_a^b F\left(x, \frac{1}{m}\sum_{j=1}^{m} y_j\right) dx \prod_{j=1}^{m} dy_j .
\end{aligned}
$$

Similarly, using the convexity of F on the co-ordinates on D and the identity (11), we obtain the following inequality

$$
\begin{aligned}
&(b-a)^{n-1}\int_c^d \cdots \int_c^d \int_a^b F\left(x, \frac{1}{m}\sum_{j=1}^{m} y_j\right) dx \prod_{j=1}^{m} dy_j \\
&= (b-a)^{n-1}\int_a^b \left[\int_c^d \cdots \int_c^d F\left(x, \frac{1}{m}\sum_{j=1}^{m} y_j\right)\prod_{j=1}^{m} dy_j\right] dx \\
&\leq (b-a)^{n-1}\int_a^b \left[(d-c)^{m-1}\int_c^d F(x,y)\, dy\right] dx \\
&= (b-a)^{n-1}(d-c)^{m-1}\int_c^d\int_a^b F(x,y)\, dxdy.
\end{aligned} \tag{13}
$$

Multiplying (12) and (13) by $\frac{1}{(b-a)^n(d-c)^m}$, we obtain the second and third inequalities of (7). This completes the proof.

Remark 1. Theorem 1 reduced to Theorem D if we choose $m = n, \beta_i = \alpha_i = \frac{q_i}{Q_n}\ (i = 1, \cdots, n)$ where $q_i\ (i = 1, \cdots, n)$ and Q_n are defined as in Theorem D.

Remark 2. Let f be defined as in Theorem A. If we choose $F(x,y) = f(x)$ $((x,y) \in \Delta)$, then Theorem 1 reduces to Theorem A.

By convexity, it is clear that all convex functions on D are convex on the co-ordinates on D. Thus, the following corollary is a simple consequence of Theorem 1.

Corollary 1. In Theorem 1, let $F : D \to R$ be convex. Then the inequality (7) also holds.

Remark 3. Corollary 1 reduced to Theorem E if we choose $m = n$, $\beta_i = \alpha_i = \frac{q_i}{Q_n}\ (i = 1, \cdots, n)$ where $q_i\ (i = 1, \cdots, n)$ and Q_n are defined as in Theorem E.

Theorem 2. Let $n, m \geq 2$, $0 < \alpha_i < 1$ $(i = 1, \cdots, n)$ and $0 < \beta_j < 1$ $(j = 1, \cdots, m)$ with $\sum_{i=1}^{n}\alpha_i = 1$ and $\sum_{j=1}^{m}\beta_j = 1$. If $F : D \to R$ is convex on the co-ordinates on D, then

$$(14)\quad F\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$\leq \frac{1}{(b-a)^{n}(d-c)^{m}}\int_{\Delta_y^m}\int_{\Delta_x^n} F\left(\sum_{i=1}^{n}\alpha_i x_i,\sum_{j=1}^{m}\beta_j y_j\right)\prod_{i=1}^{n}dx_i\prod_{j=1}^{m}dy_j$$
$$\leq \frac{1}{(b-a)^{n-1}(d-c)^{m-1}}\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{1-\beta_j}{m-1}\frac{1-\alpha_i}{n-1}\times$$
$$\left[\int_{\Delta_y^{m-1}}\int_{\Delta_x^{n-1}} F\left(\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i},\sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}\right)\prod_{k=1,k\neq i}^{n}dx_k\prod_{l=1,l\neq j}^{m}dy_j\right]$$
$$\leq \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b F(x,y)\,dxdy.$$

Proof. The first inequality of (14) follows immediately from Theorem 1. By a simple computation, we have the following two identities

$$(15)\qquad \sum_{i=1}^{n}\alpha_i x_i=\frac{1}{n-1}\sum_{i=1}^{n}\sum_{s=1,s\neq i}^{n}\alpha_s x_s=\sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i}$$

and

$$(16)\qquad \sum_{j=1}^{m}\beta_j y_j=\frac{1}{m-1}\sum_{j=1}^{m}\sum_{t=1,t\neq j}^{m}\beta_t y_t=\sum_{j=1}^{m}\frac{1-\beta_j}{m-1}\sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}$$

where $(x_i,y_j)\in\Delta$ $(i=1,\cdots,n; j=1,\cdots,m)$. Since $0<\frac{1-\alpha_i}{n-1}<1$ $(i=1,\cdots,n)$ and $\sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}=1$, by the convexity of F on the co-ordinates on D and the identity (15), we obtain

$$(17)\qquad \int_{\Delta_y^m}\int_{\Delta_x^n} F\left(\sum_{i=1}^{n}\alpha_i x_i,\sum_{j=1}^{m}\beta_j y_j\right)\prod_{i=1}^{n}dx_i\prod_{j=1}^{m}dy_j$$
$$=\int_{\Delta_y^m}\int_{\Delta_x^n} F\left(\sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i},\sum_{j=1}^{m}\beta_j y_j\right)\prod_{k=1}^{n}dx_k\prod_{j=1}^{m}dy_j$$
$$\leq\int_{\Delta_y^m}\left[\int_{\Delta_x^n}\sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}F\left(\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i},\sum_{j=1}^{m}\beta_j y_j\right)\prod_{k=1}^{n}dx_k\right]\prod_{j=1}^{m}dy$$
$$=(b-a)\int_{\Delta_y^m}\left[\sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}\left(\int_{\Delta_x^{n-1}} F\left(\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i},\sum_{j=1}^{m}\beta_j y_j\right)\times\right.\right.$$
$$\left.\left.\prod_{k=1,k\neq i}^{n}dx_k\right)\right]\prod_{j=1}^{m}dy_j$$

Similarly, using the convexity of F on the co-ordinates on D and the identity (16), we obtain the following inequality

$$(18)\quad (b-a)\int_{\Delta_y^m}\left[\sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}\int_{\Delta_x^{n-1}}F\left(\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i},\sum_{j=1}^{m}\beta_j y_j\right)\prod_{k=1,k\neq i}^{n}dx_k\right]\times\prod_{j=1}^{m}dy_j$$

$$=(b-a)\sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}\int_{\Delta_x^{n-1}}\left\{\int_{\Delta_y^m}F\left(\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i},\sum_{j=1}^{m}\beta_j y_j\right)\prod_{j=1}^{m}dy_j\right\}\prod_{k=1,k\neq i}^{n}dx_k$$

$$\leq\ (b-a)\sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}\int_{\Delta_x^{n-1}}\left\{(d-c)\sum_{j=1}^{m}\frac{1-\beta_j}{m-1}\times\left[\int_{\Delta_y^{m-1}}F\left(\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i},\sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}\right)\prod_{l=1,l\neq j}^{m}dy_j\right]\right\}\prod_{k=1,k\neq i}^{n}dx_k$$

$$=\ (b-a)(d-c)\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{1-\beta_j}{m-1}\frac{1-\alpha_i}{n-1}\times\left[\int_{\Delta_y^{m-1}}\int_{\Delta_x^{n-1}}F\left(\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i},\sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}\right)\prod_{k=1,k\neq i}^{n}dx_k\prod_{l=1,l\neq j}^{m}dy_j\right].$$

Since $0<\frac{\alpha_s}{1-\alpha_i}<1\ (i,s=1,\cdots,n;s\neq i)$, $\sum\limits_{s=1,s\neq i}^{n}\frac{\alpha_s}{1-\alpha_i}=1\,(i=1,\cdots,n)$ and $\sum\limits_{i=1}^{n}\frac{1-\alpha_i}{n-1}=1$, by the convexity of F on the co-ordinates on D, we obtain

$$(19)\qquad (b-a)(d-c)\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{1-\beta_j}{m-1}\frac{1-\alpha_i}{n-1}\times\left[\int_{\Delta_y^{m-1}}\int_{\Delta_x^{n-1}}F\left(\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i},\sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}\right)\prod_{k=1,k\neq i}^{n}dx_k\prod_{l=1,l\neq j}^{m}dy_j\right]$$

$$=\ (b-a)(d-c)\sum_{j=1}^{m}\frac{1-\beta_j}{m-1}\left\{\int_{\Delta_y^{m-1}}\left[\sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}\times\int_{\Delta_x^{n-1}}F\left(\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i},\sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}\right)\prod_{k=1,k\neq i}^{n}dx_k\right]\prod_{l=1,l\neq j}^{m}dy_j\right\}$$

$$\begin{aligned}
&\leq (b-a)(d-c)\sum_{j=1}^{m}\frac{1-\beta_j}{m-1}\left\{\int_{\Delta_y^{m-1}}\left[\sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}\sum_{s=1,s\neq i}^{n}\frac{\alpha_s}{1-\alpha_i}\times\right.\right.\\
&\left.\left.\int_{\Delta_x^{n-1}}F\left(x_s,\sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}\right)\prod_{k=1,k\neq i}^{n}dx_k\right)\right]\prod_{l=1,l\neq j}^{m}dy_j\right\}\\
&= (b-a)^{n-1}(d-c)\sum_{j=1}^{m}\frac{1-\beta_j}{m-1}\left\{\int_{\Delta_y^{m-1}}\left[\sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}\sum_{s=1,s\neq i}^{n}\frac{\alpha_s}{1-\alpha_i}\times\right.\right.\\
&\left.\left.\int_a^b F\left(x_s,\sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}\right)dx_s\right]\prod_{l=1,l\neq j}^{m}dy_j\right\}\\
&= (b-a)^{n-1}(d-c)\sum_{j=1}^{m}\frac{1-\beta_j}{m-1}\left\{\left(\sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}\sum_{s=1,s\neq i}^{n}\frac{\alpha_s}{1-\alpha_i}\right)\times\right.\\
&\left.\left[\int_{\Delta_y^{m-1}}\int_a^b F\left(x,\sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}\right)dx\prod_{l=1,l\neq j}^{m}dy_j\right]\right\}\\
&= (b-a)^{n-1}(d-c)\left\{\sum_{j=1}^{m}\frac{1-\beta_j}{m-1}\int_{\Delta_y^{m-1}}\int_a^b F\left(x,\sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}\right)\times\right.\\
&\left.dx\prod_{l=1,l\neq j}^{m}dy_j\right\}
\end{aligned}$$

Similarly, using the convexity of F on the co-ordinates on D, we obtain the following inequality

$$\begin{aligned}
(20)\ &(b-a)^{n-1}(d-c)\sum_{j=1}^{m}\frac{1-\beta_j}{m-1}\left[\int_{\Delta_y^{m-1}}\int_a^b F\left(x,\sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}\right)dx\prod_{l=1,l\neq j}^{m}dy_j\right]\\
&= (b-a)^{n-1}(d-c)\int_a^b\left[\sum_{j=1}^{m}\frac{1-\beta_j}{m-1}\int_{\Delta_y^{m-1}}F\left(x,\sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}\right)\prod_{l=1,l\neq j}^{m}dy_j\right]dx\\
&\leq (b-a)^{n-1}(d-c)^{m-1}\int_a^b\left[\int_c^d F(x,y)\,dy\right]dx\\
&= (b-a)^{n-1}(d-c)^{m-1}\int_c^d\int_a^b F(x,y)\,dxdy.
\end{aligned}$$

Using the inequalities $(17)-(20)$, we have

$$(21)\qquad \int_{\Delta_y^m}\int_{\Delta_x^n}F\left(\sum_{i=1}^{n}\alpha_i x_i,\sum_{j=1}^{m}\beta_j y_j\right)\prod_{i=1}^{n}dx_i\prod_{j=1}^{m}dy_j$$

$$\leq (b-a)(d-c)\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{1-\beta_j}{m-1}\frac{1-\alpha_i}{n-1}\times$$
$$\left[\int_{\Delta_y^{m-1}}\int_{\Delta_x^{n-1}}F\left(\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i},\sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}\right)\prod_{k=1,k\neq i}^{n}dx_k\prod_{l=1,l\neq j}^{m}dy_j\right]$$
$$\leq (b-a)^{n-1}(d-c)^{m-1}\int_c^d\int_a^b F(x,y)\,dxdy.$$

Multiplying (21) by $\frac{1}{(b-a)^n(d-c)^m}$, we obtain the second and third inequalities of (14). This completes the proof.

Remark 4. Let f be defined as in Theorem B. If we choose $F(x,y)=f(x)$ $((x,y)\in\Delta)$, then Theorem 2 reduces to Theorem B.

Remark 5. In Theorem 2, the inequality (14) refines the inequality (7).

Theorem 3. Let F, α_i $(i=1,\cdots,n)$ and β_j $(j=1,\cdots,m)$ be defined as in Theorem 1 and let $G:[0,1]^2\to R$ be defined by

$$\begin{aligned}(22)\qquad & G(t,s)\\ = & \frac{1}{(b-a)^n(d-c)^m}\int_{\Delta_y^m}\int_{\Delta_x^n}F\left(t\sum_{i=1}^{n}\alpha_i x_i+(1-t)\frac{a+b}{2}\right.,\\ & \left.s\sum_{j=1}^{m}\beta_j y_j+(1-s)\frac{c+d}{2}\right)\times\prod_{i=1}^{n}dx_i\prod_{j=1}^{m}dy_j.\end{aligned}$$

Then:

(*a*) *The function G is convex on the co-ordinates on $[0,1]^2$.*

(*b*) *The function G is increasing on the co-ordinates on $[0,1]^2$,*

$$\begin{aligned}&\sup_{(t,s)\in[0,1]^2}G(t,s)=G(1,1)\\ = & \frac{1}{(b-a)^n(d-c)^m}\int_{\Delta_y^m}\int_{\Delta_x^n}F\left(\sum_{i=1}^{n}\alpha_i x_i,\sum_{j=1}^{m}\beta_j y_j\right)\prod_{i=1}^{n}dx_i\prod_{j=1}^{m}dy_j\end{aligned}$$

and

$$\inf_{(t,s)\in[0,1]^2}G(t,s)=G(0,0)=F\left(\frac{a+b}{2},\frac{c+d}{2}\right).$$

Proof. (*a*) Fix $s\in[0,1]$. Since F is convex on the co-ordinates on Δ, we have for t_1, $t_2\in[0,1]$ and α, $\beta\geq 0$ with $\alpha+\beta=1$ that

$$\begin{aligned}(23)\qquad & \int_{\Delta_y^m}\int_{\Delta_x^n}F\left((\alpha t_1+\beta t_2)\sum_{i=1}^{n}\alpha_i x_i+(1-(\alpha t_1+\beta t_2))\frac{a+b}{2},\right.\\ & \left.s\sum_{j=1}^{m}\beta_j y_j+(1-s)\frac{c+d}{2}\right)\prod_{i=1}^{n}dx_i\prod_{j=1}^{m}dy_j\end{aligned}$$

$$= \int_{\Delta_y^m}\int_{\Delta_x^n} F\left(\alpha\left(t_1\sum_{i=1}^n \alpha_i x_i + (1-t_1)\frac{a+b}{2}\right) + \beta\left(t_2\sum_{i=1}^n \alpha_i x_i + (1-t_2)\frac{a+b}{2}\right),\right.$$
$$\left. s\sum_{j=1}^m \beta_j y_j + (1-s)\frac{c+d}{2}\right)\prod_{i=1}^n dx_i \prod_{j=1}^m dy_j$$

$$\leq \int_{\Delta_y^m}\int_{\Delta_x^n}\left[\alpha F\left(t_1\sum_{i=1}^n \alpha_i x_i + (1-t_1)\frac{a+b}{2}, s\sum_{j=1}^m \beta_j y_j + (1-s)\frac{c+d}{2}\right) + \right.$$
$$\left.\beta F\left(t_2\sum_{i=1}^n \alpha_i x_i + (1-t_2)\frac{a+b}{2}, s\sum_{j=1}^m \beta_j y_j + (1-s)\frac{c+d}{2}\right)\right] \times$$
$$\prod_{i=1}^n dx_i \prod_{j=1}^m dy_j.$$

Multiplying (23) by $\frac{1}{(b-a)^n (d-c)^m}$, we obtain

$$G(\alpha t_1 + \beta t_2, s) \leq \alpha G(t_1, s) + \beta G(t_2, s)$$

Similarly, if t is fixed in $[0,1]$, then for s_1, $s_2 \in [0,1]$ and α, $\beta \geq 0$ with $\alpha+\beta = 1$, we have

$$G(t, \alpha s_1 + \beta s_2) \leq \alpha G(t, s_1) + \beta G(t, s_2)$$

and the statement is proved.

(b) Since F is convex on the co-ordinates on Δ, we have, for all $(t,s) \in [0,1]^2$,

$$\int_{\Delta_y^m}\int_{\Delta_x^n} F\left(t\sum_{i=1}^n \alpha_i x_i + (1-t)\frac{a+b}{2}, s\sum_{j=1}^m \beta_j y_j + (1-s)\frac{c+d}{2}\right) \tag{24}$$
$$\times \prod_{i=1}^n dx_i \prod_{j=1}^m dy_j$$

$$= \int_{\Delta_y^m}\int_{\Delta_x^n}\frac{1}{2}\left[F\left(t\sum_{i=1}^n \alpha_i x_i + (1-t)\frac{a+b}{2}, s\sum_{j=1}^m \beta_j y_j + (1-s)\frac{c+d}{2}\right) + \right.$$
$$\left. F\left(t\left(\alpha_1(a+b-x_1) + \sum_{i=2}^n \alpha_i x_i\right) + (1-t)\frac{a+b}{2}, s\sum_{j=1}^m \beta_j y_j + (1-s)\frac{c+d}{2}\right)\right]$$
$$\times \prod_{i=1}^n dx_i \prod_{j=1}^m dy_j$$

$$\geq \int_{\Delta_y^m}\int_{\Delta_x^n} F\left(t\left(\alpha_1\cdot\frac{a+b}{2} + \sum_{i=2}^n \alpha_i x_i\right) + (1-t)\frac{a+b}{2}\right.,$$
$$\left. s\sum_{j=1}^m \beta_j y_j + (1-s)\frac{c+d}{2}\right)\prod_{i=1}^n dx_i \prod_{j=1}^m dy_j$$

$$
\begin{aligned}
&= (b-a)\int_{\Delta_y^m}\int_{\Delta_x^{n-1}} F\left(t\left(\alpha_1\cdot\frac{a+b}{2}+\sum_{i=2}^{n}\alpha_i x_i\right)+(1-t)\frac{a+b}{2}\right.,\\
&\qquad\left. s\sum_{j=1}^{m}\beta_j y_j+(1-s)\frac{c+d}{2}\right)\prod_{i=2}^{n}dx_i\prod_{j=1}^{m}dy_j\\
&\qquad\vdots\\
&\geq (b-a)^2\int_{\Delta_y^m}\int_{\Delta_x^{n-2}} F\left(t\left((\alpha_1+\alpha_2)\frac{a+b}{2}+\sum_{i=3}^{n}\alpha_i x_i\right)+(1-t)\frac{a+b}{2}\right.,\\
&\qquad\left. s\sum_{j=1}^{m}\beta_j y_j+(1-s)\frac{c+d}{2}\right)\prod_{i=3}^{n}dx_i\prod_{j=1}^{m}dy_j\\
&\qquad\vdots\\
&\geq (b-a)^n\int_{\Delta_y^m} F\left(t\left(\sum_{i=1}^{n}\alpha_i\right)\frac{a+b}{2}+(1-t)\frac{a+b}{2}, s\sum_{j=1}^{m}\beta_j y_j+(1-s)\frac{c+d}{2}\right)\\
&\qquad\times\prod_{j=1}^{m}dy_j\\
&= (b-a)^n\int_{\Delta_y^m} F\left(\frac{a+b}{2}, s\sum_{j=1}^{m}\beta_j y_j+(1-s)\frac{c+d}{2}\right)\prod_{j=1}^{m}dy_j.
\end{aligned}
$$

Multiplying (24) by $\frac{1}{(b-a)^n(d-c)^m}$, we obtain

$$G(t,s)\geq G(0,s). \tag{25}$$

Similarly, we have, for all $(t,s)\in[0,1]^2$,

$$G(t,s)\geq G(t,0). \tag{26}$$

If $0<t_1<t_2\leq 1$ and $0<s_1<s_2\leq 1$, then, for all $(t,s)\in[0,1]^2$, it follows from the convexity of G on the co-ordinates on $[0,1]^2$, (25) and (26) that we have

$$\frac{G(t_2,s)-G(t_1,s)}{t_2-t_1}\geq\frac{G(t_1,s)-G(0,s)}{t_1-0}\geq 0$$

and

$$\frac{G(t,s_2)-G(t,s_1)}{s_2-s_1}\geq\frac{G(t,s_1)-G(t,0)}{s_1-0}\geq 0$$

which show that

$$G(t_1,s)\leq G(t_2,s) \tag{27}$$

and

$$G(t,s_1)\leq G(t,s_2). \tag{28}$$

By (25) – (28), we obtain that G is increasing on the co-ordinates on $[0,1]^2$. Hence

$$\begin{aligned}&\sup_{(t,s)\in[0,1]^2} G(t,s) = G(1,1)\\ = &\frac{1}{(b-a)^n (d-c)^m} \int_{\Delta_y^m}\int_{\Delta_x^n} F\left(\sum_{i=1}^n \alpha_i x_i, \sum_{j=1}^m \beta_j y_j\right) \prod_{i=1}^n dx_i \prod_{j=1}^m dy_j\end{aligned}$$

and

$$\inf_{(t,s)\in[0,1]^2} G(t,s) = G(0,0) = F\left(\frac{a+b}{2}, \frac{c+d}{2}\right).$$

This completes the proof.

Remark 6. Let f be defined as in Theorem F. If we choose $m = n = 1$, then Theorem 3 reduces to Theorem F.

Theorem 4. Let α_i $(i = 1, \cdots, n)$ and β_j $(j = 1, \cdots, m)$ be defined as in Theorem 1 and let $F : \Delta \to R$ be convex. Then:

(a) G is convex on $[0,1]^2$ where G is defined as in (22).

(b) Define $g : [0,1] \to R$ by $g(t) := G(t,t)$. Then g is convex, increasing on $[0,1]$,

$$\begin{aligned}&\sup_{t\in[0,1]} g(t) = g(1)\\ = &\frac{1}{(b-a)^n (d-c)^m} \int_{\Delta_y^m}\int_{\Delta_x^n} F\left(\sum_{i=1}^n \alpha_i x_i, \sum_{j=1}^m \beta_j y_j\right) \prod_{i=1}^n dx_i \prod_{j=1}^m dy_j\end{aligned} \tag{29}$$

and

$$\inf_{t\in[0,1]} g(t) = g(0) = F\left(\frac{a+b}{2}, \frac{c+d}{2}\right). \tag{30}$$

Proof.(a) Since F is convex, we have for (t_1, s_1), $(t_2, s_2) \in [0,1]^2$ and α, $\beta \geq 0$ with $\alpha + \beta = 1$ that

$$\begin{aligned}
& G(\alpha(t_1, s_1) + \beta(t_2, s_2)) \\
= \; & G(\alpha t_1 + \beta t_2, \alpha s_1 + \beta s_2) \\
= \; & \frac{1}{(b-a)^n (d-c)^m} \int_{\Delta_y^m} \int_{\Delta_x^n} F\Bigg((\alpha t_1 + \beta t_2) \sum_{i=1}^n \alpha_i x_i + (1 - (\alpha t_1 + \beta t_2)) \frac{a+b}{2}, \\
& (\alpha s_1 + \beta s_2) \sum_{j=1}^m \beta_j y_j + (1 - (\alpha s_1 + \beta s_2)) \frac{c+d}{2} \Bigg) \prod_{i=1}^n dx_i \prod_{j=1}^m dy_j \\
= \; & \frac{1}{(b-a)^n (d-c)^m} \int_{\Delta_y^m} \int_{\Delta_x^n} F\Bigg(\alpha \left(t_1 \sum_{i=1}^n \alpha_i x_i + (1-t_1) \frac{a+b}{2}, s_1 \sum_{j=1}^m \beta_j y_j + (1-s_1) \frac{c+d}{2} \right) \\
& + \beta \left(t_2 \sum_{i=1}^n \alpha_i x_i + (1-t_2) \frac{a+b}{2}, s_2 \sum_{j=1}^m \beta_j y_j + (1-s_2) \frac{c+d}{2} \right) \Bigg) \prod_{i=1}^n dx_i \prod_{j=1}^m dy_j \\
\leq \; & \frac{1}{(b-a)^n (d-c)^m} \int_{\Delta_y^m} \int_{\Delta_x^n} \Bigg[\alpha F \left(t_1 \sum_{i=1}^n \alpha_i x_i + (1-t_1) \frac{a+b}{2}, s_1 \sum_{j=1}^m \beta_j y_j + (1-s_1) \frac{c+d}{2} \right) \\
& + \beta F \left(t_2 \sum_{i=1}^n \alpha_i x_i + (1-t_2) \frac{a+b}{2}, s_2 \sum_{j=1}^m \beta_j y_j + (1-s_2) \frac{c+d}{2} \right) \Bigg] \prod_{i=1}^n dx_i \prod_{j=1}^m dy_j \\
= \; & \alpha G(t_1, s_1) + \beta G(t_2, s_2),
\end{aligned}$$

which shows that G is convex on $[0,1]^2$.

(b) Let t_1, $t_2 \in [0,1]$ and α, $\beta \geq 0$ with $\alpha + \beta = 1$. Then

$$\begin{aligned}
& g(\alpha t_1 + \beta t_2) \\
= \; & G(\alpha(t_1, t_1) + \beta(t_2, t_2)) \\
\leq \; & \alpha G(t_1, t_1) + \beta G(t_2, t_2) \\
= \; & \alpha g(t_1) + \beta g(t_2)
\end{aligned}$$

which shows that g is convex on $[0,1]$. By Theorem 3, we have that, for $0 \leq t_1 < t_2 \leq 1$,

$$g(t_1) = G(t_1, t_1) \leq G(t_2, t_1) \leq G(t_2, t_2) = g(t_2)$$

which show that g is increasing on $[0,1]$. Since g is increasing on $[0,1]$, (29) and (30) hold. This completes the proof.

Remark 7. Let f be defined as in Theorem G. If we choose $m = n = 1$, then Theorem 4 reduces to Theorem G.

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