SHARPENING OF KAI-LAI ZHONG'S INEQUALITY

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Abstract. By using means of the theory of majorization, Kai-lai Zhong's Inequality is sharpened. As an application, some triangular inequalities are sharpened.

1. Introduction

Let $a_1 \geq a_2 \geq \ldots, a_n \geq 0$. If $\sum_{j=1}^{k} a_j \leq \sum_{j=1}^{k} b_j, k = 1, \ldots, n$, then

$$\sum_{j=1}^{n} a_j^2 \leq \sum_{j=1}^{n} b_j^2$$

with the equality holding only if $a_k = b_k, k = 1, \ldots, n$.

It is known as the Kai-lai Zhong's inequality[1,p.57]. In 1989, Ji Chen[2] obtained the following exponential generalization of this inequality:

Let $a_1 \geq a_2 \geq \ldots, a_n \geq 0$, $b_1 \geq b_2 \geq \ldots, b_n \geq 0$. If $\sum_{j=1}^{k} a_j \leq \sum_{j=1}^{k} b_j, k = 1, \ldots, n$, then

$$\sum_{j=1}^{n} a_j^p \leq \sum_{j=1}^{n} b_j^p, \quad (\text{ for } p > 1) \tag{1}$$

with the equality holding only if $a_k = b_k, k = 1, \ldots, n$.

In 1996, Ke Hu[3−4] given the following sharpening of the inequality in (1):

$$\sum_{j=1}^{n} a_j^p \leq \sum_{j=1}^{n} |b_j|^p \cdot \left[1 - \frac{\left(\sum_{i=1}^{n} a_1^p e_i \sum_{j=1}^{n} |b_j|^p - \sum_{i=1}^{n} a_i^p \sum_{j=1}^{n} |b_j|^p e_j\right)^2}{\left(\sum_{i=1}^{n} a_i^p \sum_{j=1}^{n} |b_j|^p\right)^2}\right]^{\frac{\theta(p)}{2}} \tag{2}$$

where $1 - e_k - e_m \geq 0$, for $k, m = 1, 2, \ldots, n$. $\theta(p) = p - 1$ for $p > 2$ and $\theta(p) = 1$ for $p < 2$.

In recent years, some further generalizations and applications about the Kai-lai Zhong's inequality have been obtained in[5-7] and the references therein. The purpose of this note is to establish a sharped Kai-lai Zhong's inequality which is very simple and clear by means of the theory of majorization. As an application, some triangular inequalities are sharpened.

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2. Definitions and Lemmas

The following definitions and lemmas on majorization will be used:

Definition.[8] Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n) \in \Re^n$. Then a is said to be majorized by b (in symbols $a \prec b$) if

$$(i) \sum_{i=1}^{k} a_{[i]} \leqslant \sum_{i=1}^{k} b_{[i]} \quad \text{for } k = 1, 2, ..., n-1, (ii) \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i,$$

where $a_{[1]} \geqslant a_{[2]} \geqslant \cdots \geqslant a_{[n]}$ and $b_{[1]} \geqslant b_{[2]} \geqslant \cdots \geqslant b_{[n]}$ are components of a and b rearranged in descending order, and a is said to strictly majors by b (written $a \prec\prec b$) if a is not permutation of b. And a is said to be weakly submajorized by b (written $a \prec_w b$) if

$$\sum_{i=1}^{k} a_{[i]} \leqslant \sum_{i=1}^{k} b_{[i]}\ ,\ \ k = 1,\ 2,\ \dots\ ,\ n.$$

Lemma 1.[8,p.7] Let $a \in \Re^n_+, b \in \Re^n$ and $\delta = \sum_{i=1}^{n}(b_i - a_i)$. If $a \prec_w b$, then

$$\left(a, \underbrace{\frac{\delta}{n}, \dots, \frac{\delta}{n}}_{n}\right) \prec \left(b, \underbrace{0, \dots, 0}_{n}\right). \tag{3}$$

Lemma 2.[8,p.48−49] Let $I \subset \Re$ be an interval, $a, b \in I^n \subset \Re^n$, and $g : I \to \Re$. Then

(i) $a \prec b$ if and only if

$$\sum_{i=1}^{n} g(a_i) \leq (\geq) \sum_{i=1}^{n} g(b_i) \tag{4}$$

holds for all convex(concave) functions g.

(ii) $a \prec\prec b$ if and only if

$$\sum_{i=1}^{n} g(a_i) < (>) \sum_{i=1}^{n} g(b_i) \tag{5}$$

holds for all strictly convex(concave) functions g.

Lemma 3.[8,p.50] Let $I \subset \Re$ be an interval, $a, b \in I^n \subset \Re^n$, and $g : I \to \Re$. If $a \prec_w b$, then

$$(g(a_1), g(a_2), \dots, g(a_n)) \prec_w (g(b_1), g(b_2), \dots, g(b_n)) \tag{6}$$

holds for all increasing convex functions g.

3. Main results and Proofs

Theorem 1. Let $a_1 \geq a_2 \geq \dots, a_n \geq 0$, $b_1 \geq b_2 \geq \dots, b_n \geq 0$, $\sum_{j=1}^{k} a_j \leq \sum_{j=1}^{k} b_j, k = 1, \dots, n$, i.e. $a \prec_w b$, and let $\delta = \sum_{j=1}^{n}(b_j - a_j)$. If $p > 1$, then

$$\sum_{j=1}^{n} a_j^p \leq \sum_{j=1}^{n} b_j^p - \frac{\delta^p}{n^{p-1}}, \tag{7}$$

if $0 < p \leq 1$, then (7) reverses, with the equality holding only if $a_j = b_j, j = 1, \dots, n$.

Proof. According to Lemma 1 and Lemma 2, it follows that Theorem 1 is holds.

4. Geometrical Application

Let $\triangle A_1A_2A_3$ be a triangle with vertices A_1, A_2, A_3, sides a_1, a_2, a_3 (with a_j opposite A_j), altitudes h_1, h_2, h_3 (with h_j from A_j), medians m_1, m_2, m_3 (with m_j from A_j), angle-bisectors w_1, w_2, w_3 (with w_j from A_j) exradii r_1, r_2, r_3 (with r_j tangent to a_j), radius of circumcircle R, radius of circle r and semi-perimeter s. And let P be an interior point of $\triangle A_1A_2A_3$ or point on sides of $\triangle A_1A_2A_3$, R_j be distance from P to the vertex A_j, j=1, 2, 3. The symbol $\sum$ denote the cyclic sum.

Lemma 4.

$$(\ln h_2h_3, \ln h_3h_1, \ln h_1h_2) \prec_w (\ln h_1r_1, \ln h_2r_2, \ln h_3r_3) \tag{8}$$

$$(\ln w_2w_3, \ln w_3w_1, \ln w_1w_2) \prec_w (\ln w_1r_1, \ln w_2r_2, \ln w_3r_3). \tag{9}$$

Proof. We prove only (9). (8) can is proved similarly. Without loss of generality, we may assume $a_1 \geq a_2 \geq a_3$. It is clear that $w_2w_3 \geq w_3w_1 \geq w_1w_2$. In order to prove (9), we need to prove :

$$w_1r_1 \geq w_2r_2 \geq w_3r_3, \tag{10}$$

$$w_2w_3 \leq w_1r_1, \tag{11}$$

$$(w_2w_3)(w_1w_2) \leq (w_1r_1)(w_2r_2), \tag{12}$$

$$(w_2w_3)(w_1w_2)(w_1w_2) \leq (w_1r_1)(w_2r_2)(w_3r_3). \tag{13}$$

From

$$w_1 = \frac{2\sqrt{a_2a_3s(s-a_1)}}{a_2+a_3}, r_1 = \sqrt{\frac{s(s-a_2)(s-a_3)}{s-a_1}}, \tag{14}$$

it is easy to see that the first inequality in (10) equivalent to

$$\sqrt{\frac{a_2(s-a_2}{a_1(s-a_1}} \geq \frac{a_2+a_3}{a_1+a_3}. \tag{15}$$

Since $a_2(s-a_2) \geq a_1(s-a_1), a_2+a_3 \geq a_1+a_3$, (15) holds, the first inequality in (10) follows immediately. The second inequality in (10) is proved similarly. From (14), it is easy to see that (11) equivalent to $(a_3+a_1)(a_1+a_2) \geq 2a_1(a_2+a_3)$, i.e. $(a_1-a_3)(a_1+a_2) \geq 0$, so (11) holds. And by $m_1^2 \leq r_1r_2, m_2^2 \leq r_2r_3, m_3^2 \leq r_3r_1$, (12) and (13) can are deduced.

The proof of Lemma 4 is now completed. (This proof Due to Jian Liu)

Theorem 2. For $\triangle A_1A_2A_3$, if $p > 1$, then

$$\sum m_j^p \leq \sum a_j^p - \frac{(\sum a_j - \sum m_j)^p}{3^{p-1}}, \tag{16}$$

$$\sum R_j^p \leq \sum a_j^p - \frac{(\sum a_j - \sum R_j)^p}{3^{p-1}}, \tag{17}$$

$$\sum m_j^p \leq \sum r_j^p - \frac{(\sum r_j - \sum m_j)^p}{3^{p-1}}, \tag{18}$$

$$\left(\frac{\sqrt{3}}{2}\right)^p \sum a_j^p \leq \sum r_j^p - \frac{(\sum r_j - \frac{\sqrt{3}}{2}\sum a_j)^p}{3^{p-1}}, \tag{19}$$

if $0 < p \le 1$, then inequalities in (16)-(19) are all reverses.

Proof. Notice that

$$m_1 < \frac{1}{2}(a_1 + a_2) \le a_1, m_2 < \frac{1}{2}(a_2 + a_3) \le a_2, m_3 < \frac{1}{2}(a_3 + a_1) \le a_3,$$

it is easy to check that $(m_1, m_2, m_3) \prec_w (a_1, a_2, a_3)$, and then by Theorem 1, (16) is proved. It is easy to check that $(R_1, R_2, R_3) \prec_w (a_1, a_2, a_3)$, and then by Theorem 1, (17) is proved. By the following majorization in [10, p.205], (18) and (19) can are proved respectively:

$$(m_1, m_2, m_3) \prec_w (r_1, r_2, r_3)$$

and

$$\left(\frac{\sqrt{3}}{2}a_1, \frac{\sqrt{3}}{2}a_2, \frac{\sqrt{3}}{2}a_3\right) \prec_w (r_1, r_2, r_3).$$

Remark 1. (17) is sharpening of a result due to Zhen-ping An[9], (18) is sharpening of a result due to Ji Chen[1,p.236], and (19) is too sharpening of a known result (see [1, p.226]).

Theorem 3. For $\triangle A_1A_2A_3$, if $p > 1$, then

$$\sum h_2^p h_3^p \le \sum h_1^p r_1^p - \frac{(\sum h_1 r_1 - \sum h_2 h_3)^p}{3^{p-1}}, \tag{20}$$

$$\sum w_2^p w_3^p \le \sum w_1^p r_1^p - \frac{(\sum w_1 r_1 - \sum w_2 w_3)^p}{3^{p-1}}, \tag{21}$$

if $< p \le 1$, then inequalities in (20) and (21) are all reverses.

Proof. Notice that $g(x) = e^x$ be increasing convex function, by Lemma 3, from (8) and (9) it follows

$$(h_2h_3, h_3h_1, h_1h_2) \prec_w (h_1r_1, h_2r_2, h_3r_3)$$

and

$$(w_2w_3, w_3w_1, w_1w_2) \prec_w (w_1r_1, w_2r_2, w_3r_3)$$

respectively, and then by Theorem 1, (20) and (21) are proved.

In order to prove the following conjecture proposed by Jian Liu in 2000:

$$\sum \frac{1}{a_1^k} \le \frac{1}{3^{\frac{k}{2}}}\left(\frac{1}{R^k} + \frac{1}{2^{k-1}r^k}\right), \quad (\text{ for } 0 < k \le 1), \tag{22}$$

in [11], Lin-bo Situ proved that

$$\left(\sqrt{3}\cos\frac{A}{2}, \sqrt{3}\cos A, \sqrt{3}\cos A\right) \prec_w \left(2\sin A\cos\frac{A}{2}, 1 + \sin\frac{A}{2}, 1 + \sin\frac{A}{2}\right), \tag{23}$$

where $A_1 \ge \frac{\pi}{3}$, so by (23), we can obtain the following result.

Theorem 4. For $\triangle A_1A_2A_3$ with $A_1 \ge \frac{\pi}{3}$, if $p > 1$, then

$$\begin{aligned} &3^{\frac{p}{2}}\left(\sqrt{3}\cos^p\frac{A}{2} + \sin^p A\right) \\ &\le 3^{1-\frac{p}{2}}\left[2\left(1 + \sin A\cos\frac{A}{2} + 2\sin\frac{A}{2}\right) - \cos\frac{A}{2} + 2\sin A\right]^p, \end{aligned} \tag{24}$$

if $0 < p \le 1$, then inequalities in (24) is reverses.

Remark 2. (24) is sharpening of a result in [11].

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