ON THE COMPOSITION OF COMPLETELY MONOTONIC AND RELATED FUNCTIONS

SENLIN GUO

ABSTRACT. In this article, we investigate the composition of completely monotonic and related functions.

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}^+ := (0, \infty)$, I^+ is an open interval contained in $\mathbb{R}^+$, I^o is the interior of the interval $I \subset \mathbb{R}$, $\mathcal{R}(f)$ denotes the range of the function f and $C(I)$ is the class of all continuous functions on I.

We first recall some definitions.

Definition 1 ([7]). *A function f is said to be absolutely monotonic on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o and for all $n \in \mathbb{N}_0$*

$$f^{(n)}(x) \geq 0, \quad x \in I^o.$$

The class of all absolutely monotonic functions on I is denoted by $AM(I)$.

Definition 2 ([7]). *A function f is said to be completely monotonic on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o and for all $n \in \mathbb{N}_0$*

$$(-1)^n f^{(n)}(x) \geq 0, \quad x \in I^o.$$

Some mathematicians use the terminology completely monotone instead of completely monotonic.

The class of all completely monotonic functions on I is denoted by $CM(I)$.

Definition 3 ([6]). *A function f is said to be strongly completely monotonic on I^+ if, for all $n \in \mathbb{N}_0$, $(-1)^n x^{n+1} f^{(n)}(x)$ are nonnegative and decreasing on I^+.*

The class of such functions is denoted by $SCM(I^+)$.

Definition 4 ([5]). *A function f is said to be logarithmically completely monotonic on an interval I if $f > 0, f \in C(I)$, has derivatives of all orders on I^o and for $n \in \mathbb{N}$*

$$(-1)^n [\ln f(x)]^{(n)} \geq 0, \quad x \in I^o.$$

The set of all logarithmically completely monotonic functions on I is denoted by $LCM(I)$.

2000 *Mathematics Subject Classification.* Primary 26A48, 26E40; Secondary 33B15.

Key words and phrases. Composition, completely monotonic function, strongly completely monotonic function, almost strongly completely monotonic function.

Definition 5 ([1]). *A function f is said to be strongly logarithmically completely monotonic on I^+ if $f > 0$ and, for all $n \in \mathbb{N}$, $(-1)^n x^{n+1}[\ln f(x)]^{(n)}$ are nonnegative and decreasing on I^+.*

Such a function class is denoted by $SLCM(I^+)$.

Definition 6 ([1]). *A function f is said to be almost strongly completely monotonic on $\mathbb{R}^+$ if, for all $n \in \mathbb{N}$, $(-1)^n x^{n+1} f^{(n)}(x)$ are nonnegative and decreasing on $\mathbb{R}^+$.*

The class of almost strongly completely monotonic functions on $\mathbb{R}^+$ is denoted by $ASCM(\mathbb{R}^+)$.

Definition 7 ([1]). *A function f is said to be almost completely monotonic on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o and for all $n \in \mathbb{N}$*

$$(-1)^n f^{(n)}(x) \geq 0, \quad x \in I^o.$$

Let's use $ACM(I)$ to denote the set of all such functions on I.

For compositions of completely monotonic and related functions. The following two results are a version of the corresponding Theorems in [7, Chapter IV]

Theorem 8. *Suppose that $f \in AM(I_1), g \in AM(I)$ and $\mathcal{R}(g) \subset I_1$, then $f \circ g \in AM(I)$.*

Theorem 9. *Suppose that $f \in AM(I_1), g \in CM(I)$ and $\mathcal{R}(g) \subset I_1$, then $f \circ g \in CM(I)$.*

The next result, which was established in 1983 by Lorch and Newman [3, Theorem 5], is a converse of Theorem 9.

Theorem 10. *Let f be defined on $[0, \infty)$. If for each $g \in CM(\mathbb{R}^+)$, $f \circ g \in CM(\mathbb{R}^+)$, then $f \in AM(\mathbb{R}^+)$.*

The following result is a generalized form of Theorem 2 of [4].

Theorem 11. *Suppose that $f \in CM(I_1), g \in C(I), g' \in CM(I^0)$ and $\mathcal{R}(g) \subset I_1$, then $f \circ g \in CM(I)$.*

From this result we obtain that if $f \in CM(I_1)$,where $I_1 := (a, b)$ with $-\infty \leq a < b < \infty$, then $f(b - e^{-x}) \in CM(I)$. Here $I := (-\ln(b - a), \infty)$.

In 1983 Lorch and Newman [3, Theorem 4] gave an interesting result related to Theorem 11 as follows.

Theorem 12. *For each function $f \in CM(I)$, where $I := [0, \infty)$, there exists a function g on I such that $g(0) = 0, f \circ g \in CM(I)$ and $g' \notin CM(\mathbb{R}^+)$.*

In [2], the following result was established.

Theorem 13.

(1) *Suppose that $f \in ACM(I_1), g \in C(I), g' \in CM(I^0)$ and $\mathcal{R}(g) \subset I_1$, then $f \circ g \in ACM(I)$.*

(2) *Suppose that* $f \in ASCM(I_1^+), g' \in SCM(I^+)$ *and* $\mathcal{R}(g) \subset I_1^+$. *If* $2xg'(x) \geq g(x), x \in I^+$, *then* $f \circ g \in ASCM(I^+)$.

Please note that the condition $2xg'(x) \geq g(x), x \in I^+$ in Theorem 13(2) can not be deleted even if $f \in ASCM(I_1^+)$ is replaced by a stronger condition: $f \in SCM(I_1^+)$. See Remark 2.5 of [2] for a counterexample of this.

In this article, we further investigate the composition of completely monotonic and related functions. Our main results are as follows.

Theorem 14.

(1) *Suppose that* $f \in ACM(I_1), -g \in ACM(I)$ *and* $\mathcal{R}(g) \subset I_1$, *then* $f \circ g \in ACM(I)$.
(2) *Suppose that* $f \in ACM(I_1), -g \in ASCM(I^+)$ *and* $\mathcal{R}(g) \subset I_1$. *Then* $f \circ g \in ASCM(I^+)$.

Theorem 15.

(1) *Suppose that* $f \in LCM(I_1), g \in C(I), g' \in CM(I^0)$ *and* $\mathcal{R}(g) \subset I_1$, *then* $f \circ g \in LCM(I)$.
(2) *Suppose that* $f \in SLCM(I_1^+), g' \in SCM(I^+)$ *and* $\mathcal{R}(g) \subset I_1^+$. *If* $2xg'(x) \geq g(x), x \in I^+$, *then* $f \circ g \in SLCM(I^+)$.

Remark 16. *The condition:* $2xg'(x) \geq g(x), x \in I^+$ *in Theorem 15(2) can not be deleted. For example, let* $f(x) := e^{1/x}, g(x) := \ln x$ *and* $I^+ := (e^2, \infty)$. *Then it is easy to verify that* $f \in SLCM(\mathbb{R}^+), g' \in SCM(I^+)$, *and the condition* $2xg'(x) \geq g(x), x \in I^+$ *is not satisfied. We can show that* $h(x) := f \circ g(x) = \exp(\frac{1}{\ln x}) \notin SLCM(I^+)$. *Indeed*

$$(-1)^1 x^2 [\ln h(x)]' = \frac{x}{\ln^2 x} \to \infty$$

as $x \to \infty$. *Therefore* $x/\ln^2 x$ *can not be decreasing on* I^+.

Theorem 17.

(1) *Suppose that* $f \in LCM(I_1), -g \in ACM(I)$ *and* $\mathcal{R}(g) \subset I_1$, *then* $f \circ g \in LCM(I)$.
(2) *Suppose that* $f \in LCM(I_1), -g \in ASCM(I^+)$ *and* $\mathcal{R}(g) \subset I_1$. *Then* $f \circ g \in SLCM(I^+)$.

Theorem 18. *Let* I_1 *and* I *be open interval, and* f *and* g *defined on* I_1 *and* I *respectively.*

(1) *If* $f' \in CM(I_1), g' \in CM(I)$ *and* $\mathcal{R}(g) \subset I_1$, *then* $(f \circ g)' \in CM(I)$.
(2) *If* $f' \in LCM(I_1), g' \in LCM(I)$ *and* $\mathcal{R}(g) \subset I_1$, *then* $(f \circ g)' \in LCM(I)$.

Theorem 19. *Let* f *and* g *be defined on* I_1^+ *and* I^+ *respectively. If* $f' \geq 0, f' \in ASCM(I_1^+), g' \in SCM(I^+)$, $\mathcal{R}(g) \subset I_1^+$ *and* $2xg'(x) \geq g(x), x \in I^+$, *then* $(f \circ g)' \in ASCM(I^+)$.

By using the following Lemma and similar methods as employed in [2] we can prove the main results of this article.

Lemma 20 ([1]).

(1) *$f \in LCM(I)$ if and only if $f > 0$ and $\ln f \in ACM(I)$.*
(2) *$f \in SLCM(I^+)$ if and only if $f > 0$ and $\ln f \in ASCM(I^+)$.*

References

[1] S. Guo, Some function classes connected to the class of completely monotonic functions, *RGMIA Res. Rep. Coll.* **9**(2) (2006), Art. 8 [Online: http://rgmia.vu.edu.au/v9n2.html].
[2] S. Guo, On the composition of completely monotonic functions, *RGMIA Res. Rep. Coll.* **9**(2) (2006), Art. 13 [Online: http://rgmia.vu.edu.au/v9n2.html].
[3] L. Lorch, and D.J. Newman, On the composition of completely monotonic functions completely monotonic sequences and related questions, *J. London Math. Soc.(2)* **28** (1983), 31–45.
[4] K. S. Miller and S. G. Samko, Completely monotonic functions , *Integral Transform. Spec. Funct.* **12** (2001), 389–402.
[5] F. Qi and Ch.-P. Chen, A complete monotonicity property of the gamma function, *J. Math. Anal. Appl.* **296** (2004), 603–607.
[6] S.Y. Trimble, J. Wells and F.T. Wright, Supperaddative functions and a statistical application, *SIAM J. Math. Anal.* **20** (1989), 1255–1259.
[7] D.V. Widder, *The Laplace transform*, 7th printing, Pinceton University Press, Pinceton, 1966.

Department of Mathematics, University of Manitoba, Winnipeg, MB, R3T 2N2, Canada
E-mail address: sguo@hotmail.com