Refinement of an Inequality for the Generalized Logarithmic Mean *

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Abstract: In this article, we show that the generalized logarithmic mean is strictly Schur-convex function for $p > 2$ and strictly Schur-concave function for $p < 2$ on R_+^2. And then we give a refinement of an inequality for the generalized logarithmic mean inequality using a simple majoricotion relation of the vector.

Keywords: inequality, refinement, generalized logarithmic mean, strictly Schur-convex function, strictly Schur-concave function

MR(2000) Subject Classification: 26D15
Chinese Library Classification: O178.1

1 Introduction

The generalized logarithmic mean (or Stolarsky's mean) [1, p.43] of two positive numbers a and b is defined as follows

$$S_p(a,b) = \begin{cases} \left(\frac{b^p - a^p}{p(b-a)}\right)^{\frac{1}{p-1}}, \ p \neq 0, 1, \ a \neq b \\ e^{-1}\left(a^a/b^b\right)^{1/(a-b)}, \ p = 1, \ a \neq b \\ \frac{b-a}{\ln b - \ln a}, \ p = 0, \ a \neq b \\ b, \ a = b \end{cases}$$

In particular, $S_1(a,b)$ is also called the exponent mean of a and b, in symbols $E(a,b)$. $S_0(a,b)$ is also called the logarithmic mean of a and b, in symbols $L(a,b)$.

By using different methods respectively, Stolarsky K. B. [2] and Liu zheng [3] proved that, when $a \neq b$, $S_p(a,b)$ is a strictly increasing function for p.

In 1984, Yang Zhenhang [4] obtained an upper bound of $S_p(a,b)$:

$$S_p(a,b) < M_{p-1}(a,b) = \left[\frac{1}{2}\left(a^{p-1} + b^{p-1}\right)\right]^{1/(p-1)}, \ (p > 2) \tag{1}$$

In 1997, Shi Minqi and Shi Huanan [5] obtained other upper bound of $S_p(a,b)$:

$$S_p(a,b) < \frac{a+b}{p^{\frac{1}{p-1}}}, \ (p > 2) \tag{2}$$

and showed that two upper bounds in (1) and (2) can not compare.

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In 1999, Li Kanghai[6] further proved that, if $p > 2$ then

$$\frac{a+b}{2} < S_p(a,b) < \frac{a+b}{p^{\frac{1}{p-1}}}\ ,\ (a \neq b) \tag{3}$$

if $0 < p < 2$ and $p \neq 1$ then two inequalities in (3) are reversed.

In 2000, by using Hermite-Hadamard inequality [1, p.360], i.e.

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}$$

holds for all $a, b \in I$ if and if f is convex function, Elezovic N and Pecaric J [7, 8, p.384] proved the following theorem:

Theorem A. Let I be an interval with nonempty interior on R and $f$be a continuous function on I. Then

$$F(a,b) = \begin{cases} \frac{1}{b-a}\int_a^b f(t)dt,\ a,b \in I,\ a \neq b \\ f(a),\ a = b \end{cases} \tag{4}$$

is Schur-convex (Schur-concave) on I^2 if and if f is convex (concave) on I.

Remark 1. The equality in Hermite-Hadamard's inequality holds [1, p.360] only if f is the linear function, and therefore we know that if f is the nonlinear convex (concave) function on I, then $F(a,b)$ is strictly Schur-convex (Schur-concave) function on I^2.

In 2000, Qi Feng [9] established the following weighted form of inequality(4).

Theorem B. Let f be a continuous function and p a positive continuous weight on I. Then the weighted arithmetic mean of function f with weight p defined by

$$F(a,b) = \begin{cases} \frac{\int_a^b p(t)f(t)dt}{\int_a^b p(t)dt},\ a \neq b \\ f(a),\ a = b \end{cases}$$

is Schur-convex(Schur-concave) on I^2 if and only if the inequality

$$\frac{\int_a^b p(t)\, f(t)dt}{\int_a^b p(t)\, dt} \leq \frac{p(a)\, f(a) + p(b)\, f(b)}{p(a) + p(b)}$$

holds (reverses) for $(a,b) \in I^2$.

As a corollary of Theorem A, the following result is given in [7]:

Theorem C. $S_p(a,b)$ is Schur- convex for $p > 2$ and Schur-concave for $p < 2$ on R_+^2.

In this article, we show that Schur–concavity (convexity) of $S_p(a,b)$ on R_+^2 is strictly. And then we give a refinement of (3). We obtain the following results.

Theorem 1. $S_p(a,b)$ is increasing and strictly Schur-convex function for $p > 2$, and increasing and strictly Schur-concave function for $p < 2$ on R_+^2.

Theorem 2. Let $0 < c < a < b$, $u(t) = tb + (1-t)\frac{a+b}{2}$, $v(t) = ta + (1-t)\frac{a+b}{2}$, $0 < t_2 < t_1 < 1$.

(I). If $p > 2$, then

$$\frac{a+b}{2} < \left(\frac{u^p(t_2) - v^p(t_2)}{pt_2(b-a)}\right)^{\frac{1}{p-1}} < \left(\frac{u^p(t_1) - v^p(t_1)}{pt_1(b-a)}\right)^{\frac{1}{p-1}}$$

$$< \left(\frac{b^p - a^p}{p(b-a)}\right)^{\frac{1}{p-1}} < \left(\frac{(a+b-c)^p - c^p}{p(a+b-2c)}\right)^{\frac{1}{p-1}} < \frac{a+b}{p^{\frac{1}{p-1}}}, \tag{5}$$

If $0 < p < 2$ and $p \neq 1$, then five inequalities in (5) are all reversed. If $p \leq 0$, then front four inequalities in (5) are all reversed (here fifth inequality in (5) is taken off).

(II). When $p = 1$, we have

$$\frac{a+b}{2} > E\left[u\left(t_2\right),\ v\left(t_2\right)\right] > E\left[u\left(t_1\right),\ v\left(t_1\right)\right]$$

$$> E(a,b) > E(a+b-c,c) > \frac{a+b}{e}. \tag{6}$$

(III). When $p = 0$, we have

$$\frac{a+b}{2} > L\left[u\left(t_2\right),\ v\left(t_2\right)\right] > L\left[u\left(t_1\right),\ v\left(t_1\right)\right]$$

$$> L(a,b) > L(a+b-c,c) > 0. \tag{7}$$

Theorem 3. Let $0 < a < b$, $u\left(t\right) = tb + (1-t)\,\frac{a+b}{2}$, $v\left(t\right) = ta + (1-t)\,\frac{a+b}{2}$, If $f(x)$ is the positive convex function on $[0,\ a+b]$, and $0 < t_2 < t_1 < 1$, then for $p > 2$, we have

$$f\left(\frac{a+b}{2}\right) \le \left(\frac{f^p\left(u\left(t_2\right)\right) - f^p\left(v\left(t_2\right)\right)}{p\left(f\left(u\left(t_2\right)\right) - f\left(v\left(t_2\right)\right)\right)}\right)^{\frac{1}{p-1}} \le \left(\frac{f^p\left(u\left(t_1\right)\right) - f^p\left(v\left(t_1\right)\right)}{p\left(f\left(u\left(t_1\right)\right) - f\left(v\left(t_1\right)\right)\right)}\right)^{\frac{1}{p-1}}$$

$$\le \left(\frac{f^p\left(b\right) - f^p\left(a\right)}{p\left(f\left(b\right) - f\left(a\right)\right)}\right)^{\frac{1}{p-1}} \le \left(\frac{f^p\left(a+b\right) - f^p\left(0\right)}{p\left(f\left(a+b\right) - f\left(0\right)\right)}\right)^{\frac{1}{p-1}}. \tag{8}$$

If $f\ (x)$ is the positive concave function on $[0,\ a+b]$, then for $p < 2$ and $p \neq 1$, four inequalities in (8) are all reversed.

2 Definitions and Lemmas

Definition 1.[10] Let x= ($x_1, x_2, \ldots, x_n$) and y = ($y_1, y_2, \ldots, y_n$) be two real n-tuples. Then x is said to be majorized by y (in symbols $x \prec y$) if

(i) $\sum\limits_{i=1}^{k} x_{[i]} \le \sum\limits_{i=1}^{k} y_{[i]}$, for $k = 1, 2 \cdots, n-1$, (ii) $\sum\limits_{i=1}^{n} x_i = \sum\limits_{i=1}^{n} y_i$,

where $x_{[1]} \ge x_{[2]} \ge \cdots \ge x_{[n]}$ and $y_{[1]} \ge y_{[2]} \ge \cdots \ge y_{[n]}$ are components of x and y rearranged in descending order. and x is said to strictly majors by y (written $x \prec\prec y$) if x is not permutation of y.

Definition 2.[10] let $\Omega \subset R^n$, φ: $\Omega \to R$.

(i) φ is said to be an increasing function on Ω if $x \le y\ on\ \Omega \Rightarrow \varphi\left(x\right) \le \varphi\left(y\right)$.

(ii) φ is said to be a Schur-convex function on Ω if $x \prec y$ on $\Omega \Rightarrow \varphi\left(x\right) \le \varphi\left(y\right)$, φ is said to be a Schur-concave function on Ω if and only $-\varphi$ is Schur- convex function, φ is said to be a strictly Schur-convex function on Ω if $x \prec\prec y\ on\ \Omega \Rightarrow \phi(x), \phi(y)$, φ is said to be a strictly Schur-concave function on Ω if and only $-\varphi$ is a strictly Schur-convex function on Ω.

Lemma 1.[10, p.64] Let $g : I \to R,\ \varphi : R^n \to R,\ \psi(x) = \varphi\left(g\left(x_1\right), g\left(x_2\right), \cdots, g\left(x_n\right)\right)$.

(a) If g is a convex function, φ is an increasing function and Schur- convex function, then ψ also is a Schur- convex function.

(b) If g is a concave function, φ is an increasing function and Schur- concave function, then ψ also is a Schur- concave function.

Lemma 2.[8, p.374, 11] Let interval $I \subset R$ and f is the continuous function on I. If f is the increasing (descending) function on I, then $F(a,b)$ is the increasing (descending) function on I^2.

3 Proof of Theorems

Proof of Theorem 1. Let $Q(a,b)=\frac{1}{b-a}\int_a^b t^{p-1}dt$, it is easy to see that, when $p\neq 1$, for $a,b\in R_+,\ a\neq b$, we have

$$S_p(a,b)=[Q(a,b)]^{\frac{1}{p-1}}.$$

Now we consider four cases:

Case 1. When $p>2$, since $f:t\to t^{p-1}$ is the nonlinear convex function on $R_+=[0,+\infty)$, from Remark 1 we know that $Q(a,b)$ is strictly Schur-convex on R_+^2. Furthermore since $g:t\to t^{\frac{1}{p-1}}$ is the strictly increasing function on R_+, $S_p(a,b)$ is also strictly Schur-convex on R_+^2.

Case 2. When $1<p<2$, since $f:t\to t^{p-1}$ is the nonlinear concave function on R_+, from Remark 1 we know that $Q(a,b)$ is strictly Schur-concave on R_+^2. Furthermore since $g:t\to t^{\frac{1}{p-1}}$ is strictly increasing function on R_+, $S_p(a,b)$ is also strictly Schur-concave on R_+^2.

Case 3. When $p<1$, since $f:t\to t^{p-1}$ is the nonlinear convex function on R_+, from Remark 1 we know that $Q(a,b)$is strictly Schur-convex on R_+^2. Furthermore since $g:t\to t^{\frac{1}{p-1}}$ is strictly descending function on R_+, $S_p(a,b)$ is strictly Schur-concave on R_+^2.

Case 4. When $p=1$, note that $E(a,b)=\exp\left\{\frac{1}{b-a}\int_a^b \ln x dx\right\}$, and by analogous discussing with Case 1-3, we can know that $E(a,b)$ is the strictly Schur- concave on R_+^2.

Combining Lemma 2, the proof of Theorem 1 is complete.

Proof of Theorem 2. First we show that for $0\le t_2<t_1\le 1$,

$$(u(t_2),\ v(t_2))\prec\prec(u(t_1),\ v(t_1)). \tag{9}$$

In fact, since $a\le b$, $u(t_2)\ge\ v(t_2),\ u(t_1)\ge\ v(t_1)$, and

$$u(t_1)-u(t_2)=(t_1-t_2)\left(b-\frac{a+b}{2}\right)\ge 0,$$

i.e. $u(t_1)\ge u(t_2)$. Furthermore it is clear that $a+b=u(t_2)+\ v(t_2)=u(t_1)+\ v(t_1)$, and so (9) holds.

Note that

$$(u(0),\ v(0))=\left(\frac{a+b}{2},\ \frac{a+b}{2}\right),\quad (u(1),\ v(1))=(a,\ b),$$

and

$$(a,\ b)\prec\prec(a+b-c,\ c),$$

for $0<t_2<t_1<1$, we have

$$\left(\frac{a+b}{2},\ \frac{a+b}{2}\right)\prec\prec(u(t_2),\ v(t_2))\prec\prec(u(t_1),\ v(t_1))$$

$$\prec\prec(a,\ b)\prec\prec(a+b-c,\ c), \tag{10}$$

and that for $p>2$, from Theorem 1 we have

$$S_p\left(\frac{a+b}{2},\ \frac{a+b}{2}\right)<S_p(u(t_2),\ v(t_2))<S_p(u(t_1),\ v(t_1))$$

$$<S_p(a,\ b)<S_p(a+b-c,\ c).$$

Notice that

$S_p\left(a+b-c,\ c\right)=\left(\frac{(a+b-c)^p-c^p}{p(a+b-2c)}\right)^{\frac{1}{p-1}}\to\frac{a+b}{p^{\frac{1}{p-1}}},(c\to 0),$

fifth inequality in (5) also holds. If $0<p<2$ and $p\neq 1$, then five inequalities in (5) are all reversed. If $p\leq 0$, then front four inequalities in (5) are all reversed (here fifth inequality in (5) is taken off).

When $p=1$, Notice that using L'Hospital's law, we can prove that

$E(a+b-c,c)\to\frac{a+b}{e},(c\to 0),$

And then similarly, we can prove that (6) and (7) holds.

Proof of Theorem 3. Since $f(x)$ is the positive convex function on $[0,\ a+b]$, from (a) in Lemma 1 we know that $S_p\left(f\left(a\right),f\left(b\right)\right)$ is Schur-convex function. And combining (10), we get (8). If $f(x)$ is the positive concave function on $[0,\ a+b]$, from (b) in Lemma 1 we know that four inequalities in (8) are all reversed.

Acknowledgements. The author is indebted to professor XU Tie-quan and the referee for their many helpful and valuable comments and suggestions.

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