ON SOME INEQUALITIES FOR CONVEX FUNCTIONS WITH APPLICATIONS IN NORMED SPACES

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Abstract. Some inequalities for convex functions defined on convex subsets in linear spaces with applications for the $p-$mean absolute deviation of a sequence of vectors are given, in a normed linear space.

1. Introduction

Jensen's inequality is pivotal in the Theory of Inequalities because it implies at once many other classical inequalities including the Hölder, Minkowski, Beckenbach-Dresher and Young inequalities, the arithmetic mean - geometric mean inequality, the generalised triangle inequality.

Let C be a convex subset of the real linear space X and $f : C \to \mathbb{R}$ a convex function on C. If $x_i \in C$ and $p_i \in (0,1)$ with $\sum_{i=1}^n p_i = 1$, then the following well-known form of Jensen's discrete inequality holds:

$$f\left(\sum_{i=1}^n p_i x_i\right) \le \sum_{i=1}^n p_i f(x_i). \tag{1.1}$$

In [2], the authors proved, amongst other results, the following refinement of Jensen's inequality in the general setting of linear spaces:

$$\begin{aligned} &\sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \\ &\qquad \ge \max_{1\le i<j\le n}\left\{p_i f(x_i) + p_j f(x_j) - (p_i+p_j) f\left(\frac{p_i x_i + p_j x_j}{p_i+p_j}\right)\right\} \ge 0. \end{aligned} \tag{1.2}$$

In particular, if $p_i = \frac{1}{n}$, $i \in \{1,\dots,n\}$, then we get the following refinement of the unweighted Jensen inequality:

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^n f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^n x_i\right) \\ &\qquad \ge \frac{1}{n}\max_{1\le i<j\le n}\left\{f(x_i) + f(x_j) - 2f\left(\frac{x_i+x_j}{2}\right)\right\} \ge 0. \end{aligned} \tag{1.3}$$

As a natural and important application of the above result (1.2), the authors of [2] considered the case of normed linear spaces $(X, \|\cdot\|)$ and the convex function

Date: 9th January, 2007.

2000 *Mathematics Subject Classification.* 26D15, 46B05.

Key words and phrases. Convex functions, Jensen's inequality, Normed linear spaces, $p-$mean absolute deviation.

$f(x)=\|x\|^r$, $r\geq 1$, obtaining refinements of the generalised triangle inequality:

$$\sum_{i=1}^{n}p_i\|x_i\|^r-\left\|\sum_{i=1}^{n}p_ix_i\right\|^r \geq \max_{1\leq i<j\leq n}\left\{p_i\|x_i\|^r+p_j\|x_j\|^r-(p_i+p_j)^{1-r}\|p_ix_i+p_jx_j\|^r\right\}\geq 0. \tag{1.4}$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\|x_i\|^r-\left\|\frac{1}{n}\sum_{i=1}^{n}x_i\right\|^r \geq \frac{1}{n}\max_{1\leq i<j\leq n}\left\{\|x_i\|^r+\|x_j\|^r-2^{1-r}\|x_i+x_j\|^r\right\}\geq 0. \tag{1.5}$$

More recently, the second author [1] has proved the following result:

$$\begin{aligned}\max_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\}\left[\sum_{j=1}^{n}q_jf(x_j)-f\left(\sum_{j=1}^{n}q_jx_j\right)\right] &\geq \sum_{j=1}^{n}p_jf(x_j)-f\left(\sum_{j=1}^{n}p_jx_j\right)\\ &\geq \min_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\}\left[\sum_{j=1}^{n}q_jf(x_j)-f\left(\sum_{j=1}^{n}q_jx_j\right)\right],\end{aligned} \tag{1.6}$$

provided $f:C\to\mathbb{R}$ is convex on the convex subset C of the linear space X and p_i,q_i, $i\in\{1,\dots,n\}$ are probability sequences with $q_i>0$ for each $i\in\{1,\dots,n\}$.

In particular, from (1.6) the following is obtained that compares the weighted and unweighted Jensen differences:

$$\begin{aligned}n\max_{1\leq i\leq n}\{p_i\}\left[\frac{1}{n}\sum_{j=1}^{n}f(x_j)-f\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right] &\geq \sum_{j=1}^{n}p_jf(x_j)-f\left(\sum_{j=1}^{n}p_jx_j\right)\\ &\geq n\min_{1\leq i\leq n}\{p_i\}\left[\frac{1}{n}\sum_{j=1}^{n}f(x_j)-f\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right].\end{aligned} \tag{1.7}$$

The above inequalities (1.6) and (1.7) have some nice applications for the generalised triangle inequality in normed linear spaces:

$$\begin{aligned}\max_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\}\left[\sum_{j=1}^{n}q_j\|x_j\|^r-\left\|\sum_{j=1}^{n}q_jx_j\right\|^r\right] &\geq \sum_{j=1}^{n}p_j\|x_j\|^r-\left\|\sum_{j=1}^{n}p_jx_j\right\|^r\end{aligned} \tag{1.8}$$

$$\geq \min_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\}\left[\sum_{j=1}^{n} q_j \left\|x_j\right\|^r - \left\|\sum_{j=1}^{n} q_j x_j\right\|^r\right] (\geq 0),$$

and

$$\max_{1\leq i\leq n}\{p_i\}\left[\sum_{j=1}^{n}\left\|x_j\right\|^p - n^{1-p}\left\|\sum_{j=1}^{n} x_j\right\|^p\right] \tag{1.9}$$

$$\geq \sum_{j=1}^{n} p_j \left\|x_j\right\|^p - \left\|\sum_{j=1}^{n} p_j x_j\right\|^p$$

$$\geq \min_{1\leq i\leq n}\{p_i\}\left[\sum_{j=1}^{n}\left\|x_j\right\|^p - n^{1-p}\left\|\sum_{j=1}^{n} x_j\right\|^p\right] (\geq 0),$$

respectively.

In this paper some new inequalities for convex functions defined on linear spaces are given. Applications for the $p-$mean absolute deviation of a sequence of vectors in a normed linear space with given probabilities are also provided.

2. The Main Results

Theorem 1. *Let C be a convex subset in the linear space X, $f : C \to \mathbb{R}$ a convex function on C, $x_j \in C$, $p_j \in (0,1)$, $j \in \{1,\ldots,n\}$, $n \geq 2$ and $\sum_{j=1}^{n} p_j = 1$. If*

$$\sum_{j=1}^{n} p_j x_j = 0 \quad \text{and} \quad \frac{p_k}{p_k - 1}\cdot x_k \in C \quad \text{for each } k \in \{1,\ldots,n\}, \tag{2.1}$$

then,

$$\sum_{j=1}^{n} p_j f(x_j) \geq \max_{k\in\{1,\ldots,n\}}\left[p_k f(x_k) + (1-p_k) f\left(\frac{p_k}{p_k-1}\cdot x_k\right)\right] \tag{2.2}$$

$$\geq f(0).$$

In particular, if

$$\sum_{j=1}^{n} x_j = 0 \quad \text{and} \quad \frac{n}{n-1}\cdot x_k \in C \quad \text{for each } k \in \{1,\ldots,n\}, \tag{2.3}$$

then,

$$\frac{1}{n}\sum_{j=1}^{n} f(x_j) \geq \frac{1}{n}\max_{k\in\{1,\ldots,n\}}\left[f(x_k) + (n-1) f\left(\frac{n}{n-1}\cdot x_k\right)\right] \tag{2.4}$$

$$\geq f(0).$$

Proof. Firstly, since C is convex and x_k, $\frac{p_k}{p_k-1}\cdot x_k \in C$ for $k \in \{1,\ldots,n\}$, then,

$$p_k x_k + (1-p_k)\left(\frac{p_k}{p_k-1}\cdot x_k\right) = 0 \in C,$$

and by the convexity of f,

$$p_k f(x_k) + (1-p_k) f\left(\frac{p_k}{p_k-1}\cdot x_k\right) \geq f(0)$$

for each $k\in\{1,\dots,n\}$, which proves the last part of (2.2).

Since $\sum_{j=1}^{n}p_jx_j=0$, we have,

$$p_kx_k=-\sum_{\substack{j=1\\ j\neq k}}^{n}p_jx_j=\frac{p_k-1}{\sum\limits_{\substack{j=1\\ j\neq k}}^{n}p_j}\cdot\sum_{\substack{j=1\\ j\neq k}}^{n}p_jx_j$$

for each $k\in\{1,\dots,n\}$, which implies,

$$\frac{p_k}{p_k-1}\cdot x_k=\frac{1}{\sum\limits_{\substack{j=1\\ j\neq k}}^{n}p_j}\cdot\sum_{\substack{j=1\\ j\neq k}}^{n}p_jx_j, \tag{2.5}$$

for each $k\in\{1,\dots,n\}$.

Applying the Jensen inequality, we have from (2.5) that,

$$\begin{aligned} f\left(\frac{p_k}{p_k-1}\cdot x_k\right)&=f\left(\frac{1}{\sum\limits_{\substack{j=1\\ j\neq k}}^{n}p_j}\cdot\sum_{\substack{j=1\\ j\neq k}}^{n}p_jx_j\right)\\ &\le\frac{\sum\limits_{\substack{j=1\\ j\neq k}}^{n}p_jf(x_j)}{\sum\limits_{\substack{j=1\\ j\neq k}}^{n}p_j}=\frac{\sum_{j=1}^{n}p_jf(x_j)-p_kf(x_k)}{1-p_k},\end{aligned}$$

from which it is obvious that,

$$p_kf(x_k)+(1-p_k)f\left(\frac{p_k}{p_k-1}\cdot x_k\right)\le\sum_{j=1}^{n}p_jf(x_j) \tag{2.6}$$

for each $k\in\{1,\dots,n\}$.

Taking the maximum in (2.6) over $k\in\{1,\dots,n\}$, we deduce the first part of (2.2). ∎

The following result can be useful for applications.

Corollary 1. *Let $f:C\to\mathbb{R}$ be a convex function on the convex set C and $q_j\in(0,1)$, $j\in\{1,\dots,n\}$ with $\sum_{j=1}^{n}q_j=1$. If $v_i\in X$, $i\in\{1,\dots,n\}$ are such that,*

$$v_k-\sum_{l=1}^{n}q_lv_l,\quad \frac{q_k}{1-q_k}\left(\sum_{l=1}^{n}q_lv_l-v_k\right)\in C\quad \text{for each}\ \ k\in\{1,\dots,n\}, \tag{2.7}$$

then,

$$\begin{aligned}&\sum_{j=1}^{n}q_jf\left(v_j-\sum_{l=1}^{n}q_lv_l\right)\\ &\ge\max_{k\in\{1,\dots,n\}}\left\{q_kf\left(v_k-\sum_{l=1}^{n}q_lv_l\right)+(1-q_k)f\left[\frac{q_k}{1-q_k}\left(\sum_{l=1}^{n}q_lv_l-v_k\right)\right]\right\}\\ &\ge f(0).\end{aligned} \tag{2.8}$$

In particular, if,

$$v_k - \frac{1}{n}\sum_{l=1}^{n} v_l, \quad \frac{n}{n-1}\left(\frac{1}{n}\sum_{l=1}^{n} v_l - v_k\right) \in C \quad \text{for each } k \in \{1,\dots,n\}, \tag{2.9}$$

then,

$$\begin{aligned} &\frac{1}{n}\sum_{j=1}^{n} f\left(v_j - \sum_{l=1}^{n} v_l\right) \\ &\geq \frac{1}{n}\max_{k\in\{1,\dots,n\}}\left\{f\left(v_k - \frac{1}{n}\sum_{l=1}^{n} v_l\right) + (n-1)\, f\left[\frac{n}{n-1}\left(\frac{1}{n}\sum_{l=1}^{n} v_l - v_k\right)\right]\right\} \\ &\geq f(0). \end{aligned} \tag{2.10}$$

The proof follows by Theorem 1 on choosing $x_j = v_j - \sum_{l=1}^{n} q_l v_l$ and $p_j = q_j$, $j \in \{1,\dots,n\}$.

Corollary 2. *Let $f : C \to \mathbb{R}$ be a convex function on the convex set C and $x_i \in \{1,\dots,n\}$ such that, for $y_1 := x_1 - x_n$, $y_2 := x_2 - x_1, \dots, y_{n-1} := x_{n-1} - x_{n-2}, y_n := x_n - x_{n-1}$, we have y_k, $\frac{n}{1-n} y_k \in C$ for each $k \in \{1,\dots,n\}$. It follows that,*

$$\begin{aligned} &\frac{1}{n}\left[f(x_1 - x_n) + f(x_2 - x_1) + \cdots + f(x_{n-1} - x_{n-2}) + f(x_n - x_{n-1})\right] \\ &\geq \frac{1}{n}\max\left\{f(x_1 - x_n) + (n-1)\, f\left[\frac{n}{n-1}(x_1 - x_n)\right], \dots,\right. \\ &\qquad\qquad \left. f(x_n - x_{n-1}) + (n-1)\, f\left[\frac{n}{n-1}(x_n - x_{n-1})\right]\right\} \\ &\geq f(0). \end{aligned}$$

The proof is obvious by the second part of Theorem 1.

A different result is incorporated in the following.

Theorem 2. *Let C be a convex set in the linear space X and $f : C \to \mathbb{R}$ be a convex function on C. If $x_j \in C$, $p_i \in (0,1)$, $j \in \{1,\dots,n\}$ are such that $\sum_{j=1}^{n} p_j = 1$ and*

$$-x_k, \quad \frac{p_k x_k}{2 - p_k} \in C \quad \text{for each } k \in \{1,\dots,n\}, \tag{2.11}$$

then,

$$\begin{aligned} &\sum_{j=1}^{n} p_j \left[\frac{f(x_j) + f(-x_j)}{2}\right] \\ &\geq \max_{k\in\{1,\dots,n\}}\left[\frac{p_k}{2} f(-x_k) + \left(1 - \frac{p_k}{2}\right) f\left(\frac{p_k x_k}{2 - p_k}\right)\right] \\ &\geq f(0). \end{aligned} \tag{2.12}$$

In particular, if,

$$-x_k, \quad \frac{n}{2n-1} x_k \in C \quad \text{for each } k \in \{1,\dots,n\},$$

then,

$$\frac{1}{n}\sum_{j=1}^{n}\frac{f(x_j)+f(-x_j)}{2} \geq \frac{1}{n}\max_{k\in\{1,\dots,n\}}\left\{\frac{1}{2}f(-x_k)+(2n-1)f\left(\frac{2nx_k}{2n-1}\right)\right\} \geq f(0). \tag{2.13}$$

Proof. For any $k\in\{1,\dots,n\}$ we have,

$$\sum_{i=1}^{n}p_ix_i = p_kx_k + \sum_{\substack{j=1\\ j\neq k}}^{n}p_jx_j,$$

which gives,

$$\begin{aligned} p_kx_k &= \sum_{i=1}^{n}p_ix_i + \sum_{\substack{j=1\\ j\neq k}}^{n}p_j(-x_j) \\ &= \frac{\sum_{i=1}^{n}p_ix_i + \sum_{\substack{j=1\\ j\neq k}}^{n}p_j(-x_j)}{\sum_{i=1}^{n}p_i + \sum_{\substack{j=1\\ j\neq k}}^{n}p_j}\cdot\left(\sum_{i=1}^{n}p_i + \sum_{\substack{j=1\\ j\neq k}}^{n}p_j\right) \\ &= \frac{\sum_{i=1}^{n}p_ix_i + \sum_{\substack{j=1\\ j\neq k}}^{n}p_j(-x_j)}{\sum_{i=1}^{n}p_i + \sum_{\substack{j=1\\ j\neq k}}^{n}p_j}(1+1-p_k). \end{aligned}$$

This obviously implies,

$$\frac{p_kx_k}{2-p_k} = \frac{\sum_{i=1}^{n}p_ix_i + \sum_{\substack{j=1\\ j\neq k}}^{n}p_j(-x_j)}{\sum_{i=1}^{n}p_i + \sum_{\substack{j=1\\ j\neq k}}^{n}p_j} \tag{2.14}$$

for each $k\in\{1,\dots,n\}$.

Applying Jensen's inequality, we have from (2.14) that,

$$\begin{aligned} f\left(\frac{p_kx_k}{2-p_k}\right) &= f\left(\frac{\sum_{i=1}^{n}p_ix_i + \sum_{\substack{j=1\\ j\neq k}}^{n}p_j(-x_j)}{\sum_{i=1}^{n}p_i + \sum_{\substack{j=1\\ j\neq k}}^{n}p_j}\right) \\ &\leq \frac{\sum_{i=1}^{n}p_if(x_i) + \sum_{\substack{j=1\\ j\neq k}}^{n}p_jf(-x_j)}{1+1-p_k} \\ &= \frac{\sum_{i=1}^{n}p_i\left[f(x_i)+f(-x_i)\right]-p_kf(x_k)}{2-p_k}, \end{aligned}$$

which is clearly equivalent to,

$$\frac{p_k}{2} f(-x_k) + \left(1 - \frac{p_k}{2}\right) f\left(\frac{p_k x_k}{2 - p_k}\right) \leq \sum_{i=1}^{n} p_i \left[\frac{f(x_i) + f(-x_i)}{2}\right], \tag{2.15}$$

for each $k \in \{1, \dots, n\}$.

Taking the supremum over $k \in \{1, \dots, n\}$ in (2.15) produces the first inequality in (2.12).

By the convexity of f we also have:

$$\begin{aligned} \frac{p_k}{2} f(-x_k) + \left(1 - \frac{p_k}{2}\right) f\left(\frac{p_k x_k}{2 - p_k}\right) &\geq f\left[\frac{p_k}{2}(-x_k) + \left(1 - \frac{p_k}{2}\right)\frac{p_k x_k}{2 - p_k}\right] \\ &= f(0), \end{aligned}$$

and the last part of (2.12) is also established. ∎

3. Applications for Normed Spaces

Let $(X, \|\cdot\|)$ be a normed space over the real or complex number field $\mathbb{K}$.

For the probability sequence $\mathbf{p} = (p_1, \dots, p_n)$, $i \in \{1, \dots, n\}$, the sequence of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ and a real number $p \geq 1$, we define the $p-$*mean absolute deviation* of $\mathbf{x}$ with probability $\mathbf{p}$ by:

$$K_p(\mathbf{p}, \mathbf{x}) := \sum_{j=1}^{n} p_j \left\| x_j - \sum_{l=1}^{n} p_l x_l \right\|^p. \tag{3.1}$$

For the uniform probability $\mathbf{u} = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$ we have $K_p(\mathbf{u}, \mathbf{x}) = K_p(\mathbf{x})$, where,

$$K_p(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^{n} \left\| x_j - \frac{1}{n} \sum_{l=1}^{n} x_l \right\|. \tag{3.2}$$

The following result concerning upper and lower bounds for the $p-$mean absolute deviation can be stated:

Proposition 1. *With the above, we have,*

$$\begin{aligned} \max_{k \in \{1,\dots,n\}} \left\| x_k - \sum_{l=1}^{n} p_l x_l \right\|^p &\geq K_p(\mathbf{p}, \mathbf{x}) \\ &\geq \max_{k \in \{1,\dots,n\}} \left\{ \left[p_k + p_k^p (1 - p_k)^{1-p}\right] \left\| x_k - \sum_{l=1}^{n} p_l x_l \right\|^p \right\}, \end{aligned} \tag{3.3}$$

for any $\mathbf{x} \in X^n$, $p \geq 1$ *and* $\mathbf{p}$ *a probability sequence.*

In particular,

$$\begin{aligned} \max_{k \in \{1,\dots,n\}} \left\| x_k - \frac{1}{n} \sum_{l=1}^{n} x_l \right\|^p &\geq K_p(\mathbf{x}) \\ &\geq \frac{1}{n}\left[1 + (n-1)^{1-p}\right] \max_{k \in \{1,\dots,n\}} \left\| x_k - \frac{1}{n} \sum_{l=1}^{n} x_l \right\|^p \end{aligned} \tag{3.4}$$

for all $\mathbf{x} \in X^n$ *and* $p \geq 1$.

Proof. The first inequality in (3.3) is obvious, the second follows by Corollary 1 applied for the convex function $f : X \to \mathbb{R}$, $f(x) = \|x\|^p$. The details are omitted. ∎

Remark 1. *The case $p=1$ produces the inequalities,*

$$\max_{k\in\{1,\dots,n\}}\left\|x_k-\sum_{l=1}^{n}p_lx_l\right\|\geq K(\mathbf{p},\mathbf{x})\geq 2\max_{k\in\{1,\dots,n\}}\left\{p_k\left\|x_k-\sum_{l=1}^{n}p_lx_l\right\|\right\}\tag{3.5}$$

and

$$\max_{k\in\{1,\dots,n\}}\left\|x_k-\frac{1}{n}\sum_{l=1}^{n}x_l\right\|\geq K(\mathbf{x})\geq\frac{2}{n}\max_{k\in\{1,\dots,n\}}\left\|x_k-\frac{1}{n}\sum_{l=1}^{n}x_l\right\|,\tag{3.6}$$

where $K(\mathbf{p},\mathbf{x})=K_1(\mathbf{p},\mathbf{x})$ and $K(\mathbf{x})=K_1(\mathbf{x})$.

If $\sigma^2(\mathbf{p},\mathbf{x})=K_2(\mathbf{p},\mathbf{x})$, where $\sigma^2(\mathbf{p},\mathbf{x})$ denotes the variance of $\mathbf{x}$ with the probability $\mathbf{p}$, then we have,

$$\max_{k\in\{1,\dots,n\}}\left\|x_k-\sum_{l=1}^{n}p_lx_l\right\|^2\geq\sigma^2(\mathbf{p},\mathbf{x})\geq\max_{k\in\{1,\dots,n\}}\left\{\frac{p_k}{p_k-1}\left\|x_k-\sum_{l=1}^{n}p_lx_l\right\|^2\right\},\tag{3.7}$$

for any $\mathbf{x}\in X^n$ and $\mathbf{p}$ a probability density.

Also, if $\sigma^2(\mathbf{x})=K_2(\mathbf{x})$,

$$\max_{k\in\{1,\dots,n\}}\left\|x_k-\frac{1}{n}\sum_{l=1}^{n}x_l\right\|\geq\sigma^2(\mathbf{x})\geq\frac{n}{n-1}\max_{k\in\{1,\dots,n\}}\left\|x_k-\frac{1}{n}x_l\right\|^2.\tag{3.8}$$

We notice that if $X=H$, H an inner product space, then $\sigma(\mathbf{p},\mathbf{x})$ and $\sigma(\mathbf{x})$ can be represented as,

$$\sigma(\mathbf{p},\mathbf{x})=\left(\sum_{j=1}^{n}p_j\|x_j\|^2-\left\|\sum_{j=1}^{n}p_jx_j\right\|^2\right)^{\frac{1}{2}},$$

while

$$\sigma(\mathbf{x})=\left(\frac{1}{n}\sum_{j=1}^{n}\|x_j\|^2-\left\|\frac{1}{n}\sum_{j=1}^{n}x_j\right\|^2\right)^{\frac{1}{2}}.$$

Since the lower bound for $K_p(\mathbf{p},\mathbf{x})$ may be difficult to use in applications, we provide the following coarse but perhaps more useful bound.

Corollary 3. *If* $p_m := \min_{k\in\{1,\dots,n\}} p_k,\ p_m \in (0,1)$ *, then*

$$K_p(\mathbf{p},\mathbf{x}) \geq \left[p_m + p_m^p(1-p_m)^{1-p}\right] \max_{k\in\{1,\dots,n\}} \left\|x_k - \sum_{l=1}^n p_l x_l\right\|^p \tag{3.9}$$

for all $\mathbf{x} \in X^n$.

Proof. For $p \geq 1$, consider the function $h_p : [0,1) \to \mathbb{R},\ h_p(t) := t + t^p(1-t)^{1-p}$. The function h_p is differentiable on $[0,1)$ and

$$h_p'(t) = 1 + pt^{p-1}(1-t)^{1-p} + (p-1)t^p(1-t)^{-p} > 0,$$

for any $t \in [0,1)$, showing that h_p is strictly increasing on $[0,1)$. It follows that,

$$\min_{k\in\{1,\dots,n\}} \left[p_k + p_k^p(1-p_k)^{1-p}\right] = p_m + p_m^p(1-p_m)^{1-p},$$

which together with (3.3) provides the desired bound (3.9). ∎

Remark 2. *In particular, we have,*

$$K(\mathbf{p},\mathbf{x}) \geq 2p_m \max_{k\in\{1,\dots,n\}} \left\|x_k - \sum_{l=1}^n p_l x_l\right\| \tag{3.10}$$

and

$$\sigma^2(\mathbf{p},\mathbf{x}) \geq \frac{p_m}{1-p_m} \max_{k\in\{1,\dots,n\}} \left\|x_k - \sum_{l=1}^n p_l x_l\right\|^2. \tag{3.11}$$

From a different perspective, we can state the following inequalities as well.

Proposition 2. *Let* $(X, \|\cdot\|)$ *be a normed linear space,* $\mathbf{x} = (x_1,\dots,x_n) \in X^n$, $p \geq 1$ *and* $p_i \in (0,1)$ *with* $\sum_{i=1}^n p_i = 1$, *then,*

$$\sum_{i=1}^n p_i \|x_i\|^p \geq \frac{1}{2} \max_{k\in\{1,\dots,n\}} \left[p_k + p_k^p(2-p_k)^{1-p}\right] \|x_k\|^p, \tag{3.12}$$

for any $\mathbf{x},\mathbf{p}$ *and* p *as above.*

In particular,

$$\frac{1}{n}\sum_{i=1}^n \|x_i\|^p \geq \frac{1}{2n}\left[1 + (2n-1)^{1-p}\right] \max_{k\in\{1,\dots,n\}} \{\|x_k\|^p\} \tag{3.13}$$

for any $\mathbf{x} \in X^n$.

The proof is obvious by Theorem 2 applied for the convex function $f : X \to \mathbb{R}_+$, $f(x) = \|x\|^p$. The details are omitted.

Remark 3. *The case* $p = 2$ *gives the simple inequalities:*

$$\sum_{i=1}^n p_i \|x_i\|^2 \geq \max_{k\in\{1,\dots,n\}} \left[\frac{p_k}{2-p_k}\|x_k\|^2\right] \tag{3.14}$$

and

$$\frac{1}{n}\sum_{i=1}^n \|x_i\|^2 \geq \frac{1}{2n-1} \max_{k\in\{1,\dots,n\}} \|x_k\|^2. \tag{3.15}$$

As pointed out before, for applications, the lower bound for the quantity $\sum_{i=1}^{n} p_i \|x_i\|^2$ may not be as useful as one where the p_k's and x_k's are separate. This can be achieved, however, by the following coarser result:

Corollary 4. *If* $p_m := \min_{k\in\{1,\dots,n\}} p_k,\ p_m \in (0,1)$, *then,*

$$\sum_{i=1}^{n} p_i \|x_i\|^p \geq \frac{1}{2}\left[p_m + p_m^p (2-p_m)^{1-p}\right] \max_{k\in\{1,\dots,n\}} \|x_k\|^p, \tag{3.16}$$

for any $\mathbf{x} \in X^n$.

Proof. Consider the function $g_p : [0,1) \to \mathbb{R}$, $g_p(t) = t + t^p (2-t)^{1-p}$ which is differentiable on $[0,1)$ and

$$g_p'(t) = 1 + pt^{p-1}(2-t)^{1-p} + (p-1)t^p (2-t)^{-p} > 0$$

for any $t \in [0,1)$, showing that g_p is strictly increasing on $[0,1)$. Therefore,

$$\min_{k\in\{1,\dots,n\}} \left[p_k + p_k^p (2-p_k)^{1-p}\right] = p_m + p_m^p (2-p_m)^{1-p},$$

which, together with (3.12), provides the desired result (3.16). ∎

Remark 4. *In particular,*

$$\sum_{i=1}^{n} p_i \|x_i\|^2 \geq \frac{p_m}{2-p_m} \max_{k\in\{1,\dots,n\}} \|x_k\|^2, \tag{3.17}$$

for any $\mathbf{x} \in X^n$.

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